Perturbative Study of Concentric 5-Branes in Plane-Wave Matrix Model

and Development of Mathematical Formalism for Wigner J-Symbols

by

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Abstract

Expansion and computation of quantities around various $(k \times N) \times (k \times N)$ vacuua in BMN theory results in introduction of Matrix spherical harmonics and various traces of their products. In this thesis the formalism to deal with these harmonics is developed and a few bosonic diagrams that arise in computation of $Tr(X^a X^a)$ are computed as an example. It is then seen that the large N limit is dominated by a planar subclass of these diagrams with a particular topology. This suggests a stronger constraint than that of 't Hooft.

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I would also like to thank Dr. Ariel Zhitnitsky, who first introduced me the joy of theoretical physics.

Dedication

To my family.

Chapter 1

Introduction

String theory is a theory that seeks to combine all the fundamental forces in nature in one fundamental theory. These forces are Electro-magnetism, the Weak force, the Strong force and gravity. The first three have been successfully combined in what is called the Standard Model. The Standard Model is essentially a gauge theory with $SU(3) \times SU(2) \times U(1)$ symmetry.

To deal with gravity, one has the theory of General Relativity that explains gravity in large scales quite well. But this theory is classical in nature and is bound to fail in small scales in which quantum effects come into play. One example that such situation arises is in study of black holes.

In other words the real challenge is coming up with a quantum theory of gravity. Numerous attempts to incorporate gravity into the Standard Model, as a simple gauge theory, have failed thus far. So far string theory is the only theory that has provided a consistent theory of quantum gravity.

But string theory is more than a mere consistent quantum gravity theory. Contrary to the Standard Model, string theory does not have any free parameters. This means that all properties of the nature including that of the elementary particles should be derived from basic principles. In the Standard Model one has free parameters whose values are set by experiments and are not derived from fundamental principles.

In string theory it is assumed that the elementary particles are not really particles but rather different modes of string like extended objects called fundamental string.

This theory was first proposed as a theory to explain the strong force, but it turned out that Quantum Chromodynamics or QCD (which is now part of the Standard Model) does a much better job. Naturally string theory was abandoned in favour of QCD.

It was once again revived as a consistent theory of quantum gravity, by Green and Schwarz. This time the size and scale of the strings were considered to be much smaller than of the strings that were supposed to explain the strong force. This revival of string theory is commonly referred to as the *first superstring revolution.* The fact that one has the prefix super in front of the word *string* is that in order to have a consistent string theory free of anomalies, and also having the possibility to have fermionic strings (that can lead to explaining fermions), one needs to introduce a new symmetry to string action, namely the supersymmetry.

Coming up with a consistent string theory, free of anomalies, is not an easy job. After the first string revolution people came up with only five such theories: *Type I*, *Type IIA*, *Type IIB* and two theories called *Heterotic* superstring theories which are denoted by *HE* and *HO*. Yet the question remained that which one of these theories was the true description of nature.

Closely related to the concept of string are D-branes. D-branes or Dp-branes are objects in string theory that emerge when dealing with open strings. Open strings (which are oriented) start and terminate on D-branes. The D at the beginning of brane stands for Dirichlet, and has roots in Dirichlet boundary condition that is imposed to string ends.

D-branes have gained much importance in recent years and are considered important objects in their own merit and are related to other fundamental objects that appeared after the second superstring revolution, explained shortly. For example, by arranging D-branes in space and therefore constricting where Open strings can live (or rather start and end) , one can generate different gauge theories. It is also possible to write an action which describes a particular Dp-Brane.

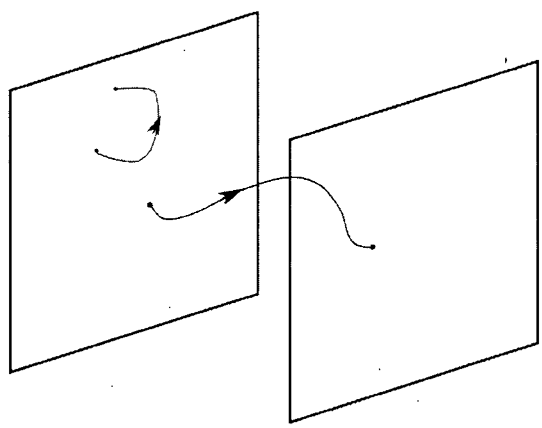


Figure 1.1: Open strings and D-Branes

In 1995 Ed. Witten and others showed that the five different superstring theories are related to one another through some dualities and they all seem to be limits of a more fundamental theory, that he named the M-theory. This is commonly referred to as the *the second superstring revolution.*

Since the second superstring revolution, M-theory has emerged as the primary candidate for the (yet unknown) theory of everything combining standard model and the general theory of relativity. Nobody completely understands or has fully formulated M-theory yet. When formulated, M-theory would combine all five superstring theories and the eleven-dimensional supergravity. As mentioned earlier, it was shown that various string theories are related to each-other, supergravity and M-theory through a series of dualities and limits.

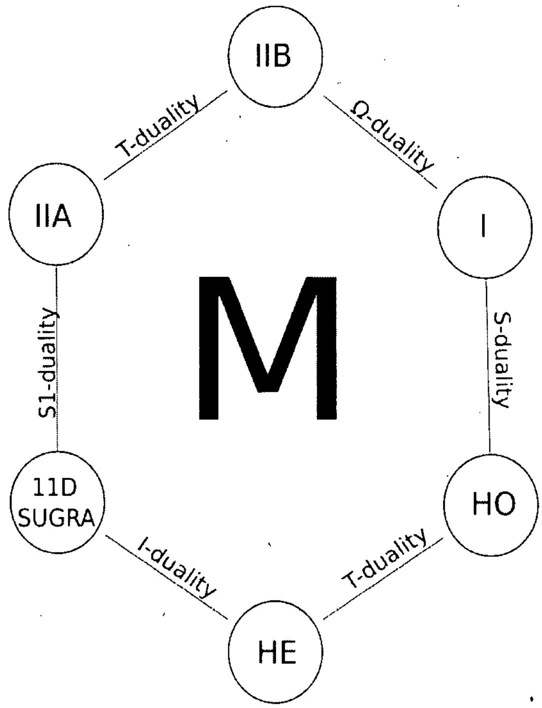


Figure 1.2: Dualities between various string theories

M-theory can be thought of in a number of ways.

- It can be understood as a quantized version of the 11D supergravity. The problem is that nobody really knows how to quantize supergravity.
- It can be understood as strong coupling limit of type IIA string theory. More precisely, the weak coupling of M-theory when the 11th dimension is compactified (S^1 Topology) results in 10D IIA string theory. The smaller the coupling the smaller the radius of the circle is.
- It can also be understood as the strong coupling limit of HE. This

time the 11th dimension is a line interval (rather than a circle), and gets smaller as the coupling gets smaller.

M-theory contains two kinds of extended objects: supermembranes (2-branes) and 5-branes. P-branes (where p is the dimension) in M-theory are related to Dp-Branes in string theory. For example a 2-brane can be thought as a string in type IIA string theory (if it is wrapped around the eleventh dimension) or as a D2-brane (if it is not wrapped around the eleventh dimension).

1.1 Matrix Model

Matrix theory first appeared as light-front quantization of membrane theory, in which infinitely many degrees of freedom of the membrane were replaced with $N \times N$ (finitely many) degrees of freedom of matrices. The theory approaches continuous membrane theory for $N \to \infty$. The Hamiltonian is

$$H_0 = R \ \mathrm{Tr}\left(\frac{1}{2}\Pi_A^2 - \frac{1}{4}[X_A, X_B]^2 - \frac{1}{2}\Psi^\top \gamma^A [X_A, \Psi]\right) \quad A, B = 1, \cdots, 9. \tag{1.1}$$

X and Ψ are Hermitian $N \times N$ matrices. Π_A is the matrix of the canonically conjugate momenta.

One remarkable fact is that the same $U(N)$ matrix theory arises as the regularization of a theory describing a membrane of any genus g. Although the matrix action arises from regularizing a single membrane, it contains more structure than the smooth membrane theory that one is approximating. For example, matrix configuration with large values of N can approximate any system of multiple membranes with arbitrary topologies, whereas the original continuous membrane theory clearly describes only one membrane. The greater structure of Matrix theory has to do with its second-quantized nature.

1.2 Matrix Theory and M-theory

In 1996, motivated by developments in D-Branes and dualities, Banks, Fischer, Shenker and Susskind (BFSS) conjectured that large N limit of the above supersymmetric matrix quantum mechanics model should describe all of M-theory in a light-front coordinate system. Susskind later argued that the finite N matrix quantum mechanics theory should be equivalent to the discrete light-front quantized sector of M-theory with N units of compact momentum.

It should be emphasized that although the matrix model of BFSS conjecture is precisely the same matrix model that is obtained from supermembrane theory, the starting point and the derivation of the two were completely different. BFSS matrix model comes from the low-energy Lagrangian for the system of N type IIA D0-branes, which is the matrix quantum mechanics Lagrangian arising from the dimensional reduction to $0+1$ dimensions of the 10D super Yang-Mills Lagrangian.

BFSS model is a very difficult model to study for a variety of reasons. It has flat directions in the potential. Furthermore it is difficult to distinguish single-particle and multi-particle states. It doesn't have a tunable coupling constant either, so perturbation theory or any other similar weak interaction approximation does not make sense for it.

1.3 Plane-Wave Matrix Model

A massive deformation of BFSS model was proposed by Berenstein, Maldacena, and Nastase with the Hamiltonian:

$$\begin{aligned} H = H_0 \quad + \quad & \frac{R}{2}\mathrm{Tr}\Big(\sum_{i=1}^{3}\left(\frac{\mu}{3R}\right)^2 X_i^2 + \sum_{a=4}^{9}\left(\frac{\mu}{6R}\right)^2 X_a^2 \\ & \qquad + i\frac{\mu}{4R}\Psi^\top\gamma^{123}\Psi + i\frac{2\mu}{3R}\epsilon^{ijk}X_iX_jX_k\Big) \,. \end{aligned} \tag{1.2}$$

It was shown that this model, in its large N limit, describes M-theory on maximally supersymmetric pp-wave:

$$\begin{aligned} ds^2 &= -2dx^+dx^- + \sum_{A=1}^{9} dx^A dx^A - \left(\sum_{i=1}^{3}\frac{\mu^2}{9}x^ix^i + \sum_{a=4}^{9}\frac{\mu^2}{36}x^ax^a\right)dx^+dx^+ \\ F_{123+} &= \mu \end{aligned} \tag{1.3}$$

As a matter of fact it can be shown that this matrix model can be derived as regularized theory of light-cone gauge supermembranes in the maximally supersymmetric pp-wave background.

This model for a non-zero μ is much nicer than BFSS and allows further study and perturbative expansion. The focus from hereon will be on this model. Plane-wave matrix model is sometimes called BMN matrix model.

As mentioned earlier, M-theory has 5D extended objects known as 5-branes. In this thesis these 5-brane objects for pp-wave matrix model will be studied in more detail and some perturbative calculations will be performed.

Before examining BMN model any further, it is necessary to develop some math formalism that becomes useful in later analysis.

Chapter 2

Math Machinery

2.1 Matrix Spherical Harmonics

It is known from quantum mechanics and basic group theory that spherical harmonics form a complete basis for expanding scalar functions on a sphere. A formalism exists through which the concept of spherical harmonics can be generalized to expand any-spin spinor field on a sphere. Although in principle one can expand different components of say a vector field that lives on a sphere, with regular spherical harmonic, this expansion will not behave nicely under rotation. It is desired that the eigenfunctions behave similarly to the expanded spinor function under rotation. Below one formalism is explained which is based after Edmonds[4]. For another formalism see [1].

The formal way to generate any-spin spherical harmonics is

$$Y_{JlM}(\theta,\phi) = \sum_{m,q} Y_{l,m}\hat{e}_q \langle s,m;s,q|J,M\rangle \tag{2.1}$$

where $Y_{l,m}$ is a regular spherical harmonic, $\langle s,m;s,q|J,Mi\rangle$is the Clebsch-Gordan coefficient, s is the desired spin and $\hat{e}_q$ is an appropriate basis for unit spinors.

For example vector spherical harmonics can be generated as

$$\vec{Y}_{JlM}(\theta,\phi) = \sum_{m,q} Y_{l,m}\hat{e}_q \langle s,m;s,q|J,M\rangle \tag{2.2}$$

and the basis is defined as follows.

$$\begin{aligned}
\hat{e}_{+1} &= \frac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y) \\
\hat{e}_0 &= \hat{e}_z \\
\hat{e}_{-1} &= \frac{1}{\sqrt{2}}(\hat{e}_x - i\hat{e}_y)
\end{aligned}$$

2.1.1 Matrix Scalar Spherical Harmonics

The concept of continuous Spherical Harmonics can be generalized to matrices. In this case an $N \times N$ matrix A can be expanded as $A = C^{lm}Y_{lm}$ where repetition of index implies summation and matrix indices are suppressed and Y_{lm}'s are Matrix Spherical Harmonics.

Matrix Spherical Harmonics satisfy the following equations:

$$\begin{aligned} [J_3, Y_{jm}] &= mY_{jm}, & [J^+, Y_{jm}] &= \sqrt{(j-m)(j+1+m)}Y_{j\,m+1} \\ [J^i,[J^i,Y_{jm}]] &= j(j+1)Y_{jm} & [J^-, Y_{jm}] &= \sqrt{(j+m)(j+1-m)}Y_{j\,m-1} \end{aligned} \quad (2.3)$$

and $(Y_{jm})^\dagger = (-1)^m Y_{j,-m}$. We also choose the normalization

$$\mathrm{Tr}(Y^\dagger_{j'm'}Y_{jm}) = N\delta_{j'\,j}\delta_{m'\,m}. \quad (2.4)$$

Here Y's are $N \times N$ matrices and J^i's are $N \times N$ generators of $SU(2)$ in the irreducible representation.

It is possible, though not much useful for the purposes here, to construct these matrices explicitly. For an extensive treatment refer to [5]

For illustration purposes the explicit example of 3×3 matrix spherical harmonics will be provided shortly. The reason that the 2×2 matrices are not discussed is that it turns out that in 2×2 case matrix spherical harmonics are proportional to σ^+, σ^-, and σ^0 respectively and of-course the identity which is Y_{00}. This behaviour is expected because one knows that Pauli matrices together with the identity matrix form a complete basis in 2×2 case.

$$Y_{00} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad Y_{1-1} = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 \end{pmatrix} \quad Y_{10} = \begin{pmatrix} \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{3}{2}} \end{pmatrix}$$

$$Y_{11} = \begin{pmatrix} 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & -\sqrt{\frac{3}{2}} \\ 0 & 0 & 0 \end{pmatrix} \quad Y_{2-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \end{pmatrix} \quad Y_{2-1} = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & -\sqrt{\frac{3}{2}} & 0 \end{pmatrix}$$

$$Y_{20} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} \quad Y_{21} = \begin{pmatrix} 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} \\ 0 & 0 & 0 \end{pmatrix} \quad Y_{22} = \begin{pmatrix} 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similar to the case of regular scalar spherical harmonics, matrix scalar spherical harmonics can be generalized to higher-spin spherical harmonics.

2.2 Matrix Vector Spherical Harmonics

Analogous to the regular spherical harmonics case matrix vector spherical harmonics can be defined as

$$\vec{Y}_{JlM} = \sum_{m,q} Y_{l,m} \hat{e}_q \langle l, m; 1, q | J, M \rangle \tag{2.5}$$

These matrices reduce to regular spherical harmonics in large N limit.[4]

As Edmonds states as a result of angular momentum addition rules l is not independent of J. For any J, l can assume one of the three values of $J+1$, J and $J-1$. Below some properties of Vector Spherical Harmonics are listed:

$$J^2 \vec{Y}_{JlM} = J(J+1)\vec{Y}_{JlM} \tag{2.6}$$

$$J_z \vec{Y}_{JlM} = M\vec{Y}_{JlM} \tag{2.7}$$

where J^i is angular momentum operator.

The vector spherical harmonics, as defined in (2.5), are solutions of the eigenvalue problem

$$Y^i_{jlm} + i\epsilon^{ijk}[J^j, Y^k_{jlm}] = \lambda Y^i_{jlm} \tag{2.8}$$

where the eigenvalue corresponding to $j = l+1$ is $l+1$, to $j = l-1$ is $-l$ and to $j = l$ vanishes. Under hermitian conjugation, the vector spherical harmonics transform as $(\vec{Y}_{J,l,M})^\dagger = (-1)^{J+l+M+1}\vec{Y}_{J,l,-M}$. As in the case of normal vector spherical harmonics, the action of J on the scalar matrix spherical harmonics can be written as

$$[\vec{J}, Y_{jm}] = \sqrt{j(j+1)}\vec{Y}_{jjm}. \tag{2.9}$$

One can also check that

$$[J^i, (Y_{jlm})_i] = \delta_{j,l}\sqrt{j(j+1)}Y_{jm}, \tag{2.10}$$

The following property is also useful:

$$[J^i, Y^j_{JlM}] = \sqrt{l(l+1)} \sum_{n,m} Y_{lm} \langle l, n; 1, j | J, M \rangle \langle l, m; 1, i | l, n \rangle \tag{2.11}$$

2.2.1 Matrix Spinor Spherical Harmonics

In a similar fashion spin half Matrix spherical harmonics can be defined as:

$$\mathcal{Y}_{JlM} = \sum_{m,q} Y_{l,m}\chi^q \langle l, m; \tfrac{1}{2}, q | J, M \rangle \tag{2.12}$$

The spinor spherical harmonics, given in (2.12), are solutions of the eigenvalue problem

$$\vec{\sigma} \cdot [\vec{J}, \mathcal{Y}_{jlm}] = \lambda \mathcal{Y}_{jlm} \tag{2.13}$$

where the eigenvalues in the cases of $j = l + 1/2$ and $j = l - 1/2$ are l and $-(l+1)$ respectively.

The following normalization property is satisfied:

$$\mathrm{Tr}((\mathcal{Y}_{J_1 l_1 M_1})^\dagger \mathcal{Y}_{J_2 l_2 M_2}) = N \delta_{J_1,J_2} \delta_{l_1,l_2} \delta_{M_1,M_2}. \tag{2.14}$$

2.3 Wigner Symbols

2.3.1 Wigner 3j-symbols

Wigner 3j-symbols or Wigner coefficients arise in couplings of three angular momenta. Therefore as one expects, they will appear when one is calculating traces over spherical harmonics, that soon becomes necessary.

3j-symbols have six parameters and are usually shown as $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ where m_i and j_i can be integers or half integers.

The selection rules are as follows:

$$\begin{aligned} m_i &\in \{-|j_i| \cdots + |j_i|\} \\ m_1 + m_2 + m_3 &= 0 \\ |j_a - j_b| \leq j_c &\leq j_a + j_b \end{aligned} \tag{2.15}$$

These rules imply that $\sum j_i$ is an integer. If these rules are not satisfied the value is 0.

3j-symbols are related to the more renowned Clebsch-Gordan coefficients as follows:

$$\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle = (-1)^{j_1 - j_2 + m_3} \sqrt{2j_3 + 1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \tag{2.16}$$

One way to calculate 3j-symbols is using the Racah formula:

$$\begin{pmatrix} nj_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{(j_1-j_2-m_3)} \sqrt{\Delta(j_1j_2j_3)} \sqrt{(j1+m_1)!(j_1-m_1)!}$$
$$\sqrt{(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!}$$
$$\sum_t \frac{(-1)^t}{t!(j_3-j_3+t+m_1)!(j_3-j_1+t-m_2)!(j_1+j_2-j_3-t)!(j_2-t+m_2)!} \tag{2.17}$$

$\Delta(j_1j_2j_3)$ is called triangle coefficient as is defined as follows:

$$\Delta(j_1j_2j_3) \equiv \frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+j_3+1)!}$$

t in (2.17) runs over all integers for which all factorials have non-negative arguments.

3j-symbols have the following symmetries:

If we define P as a permutation of columns then:

for even P:

$$P(\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{2.18}$$

for odd P:

$$P(\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}) = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{2.19}$$

Another useful equation is:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} \tag{2.20}$$

It should be noted that 3j-symbols obey a number of orthogonality relation as follows:

$$\sum_{j,m} (2j_1+1) \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{pmatrix}$$
$$= \delta_{m_1,m_1'} \delta_{m_2,m_2'} \sum_{m_1,m_2} (2j_1+1) \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{pmatrix}$$
$$= \delta_{j,j'} \delta_{m,m'} \tag{2.21}$$

where $\delta_{a,b}$ is the Kronecker delta

It is often necessary to sum over a number of m's in the product of several 3j-symbols. Since the results are usually expressed in terms of higher j symbols the corresponding identities are introduced after defining higher j symbols.

2.3.2 Wigner 6j-symbols

Wigner 6j-symbols or just 6j-symbols are quantities that arise in couplings of four angular momenta. They are denoted as

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ J_1 & J_2 & J_3 \end{Bmatrix}$$

where J's (and j's) are integers or half integers. For the purposes of this thesis it is easiest and enough to define them in terms of Wigner 3j-symbols:

$$\sum_{m6=-j6}^{j6} \sum_{m5=-j5}^{j5} \sum_{m4=-j4}^{j4} \begin{pmatrix} j5 & j1 & j6 \\ m5 & -m1 & -m6 \end{pmatrix} \begin{pmatrix} j6 & j2 & j4 \\ m6 & -m2 & -m4 \end{pmatrix} (-1)^{j4+j5+j6-m4-m5-m6} = \begin{pmatrix} j1 & j2 & j3 \\ m1 & m2 & m3 \end{pmatrix} \begin{Bmatrix} j1 & j2 & j3 \\ j4 & j5 & j6 \end{Bmatrix} \quad (2.22)$$

For the 6-j symbol $\begin{Bmatrix} j_1 & j_2 & j_3 \\ J_1 & J_2 & J_3 \end{Bmatrix}$ the triads (j_1, j_2, j_3) (j_1, J_2, J_3) (J_1, j_2, J_3) and (J_1, J_2, J_3) must satisfy two conditions:

1. Each triad should satisfy the triangular inequality:
2. The sum of the elements of each triad should be an integer. (so either no half integers or two half integer)

If the conditions are not satisfies the value of 6-j symbol is 0.
6j-symbols are invariant under permutation of columns.

2.3.3 Wigner 9j-symbols

Wigner 9-j symbols arise in the coupling of four angular momenta. They are denoted by: $\begin{Bmatrix} j_1 & j2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix}$ For the purposes of this thesis it is easier and enough to write them in terms of Wigner 3j-symbols or Wigner 6j symbols.

In terms of 3j-symbols:

$$\begin{pmatrix} J_{13} & J_{24} & J \\ M_{13} & M_{24} & M \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} =$$
$$\sum_{allm's} \begin{pmatrix} j_1 & j_2 & J_{12} \\ m_1 & m_2 & M_{12} \end{pmatrix} \begin{pmatrix} j_3 & j_4 & J_{34} \\ m_3 & m_4 & M_{34} \end{pmatrix} \begin{pmatrix} j_1 & j_3 & J_{13} \\ m_1 & m_3 & M_{13} \end{pmatrix}$$
$$\begin{pmatrix} j_2 & j_4 & J_{24} \\ m_2 & m_4 & M_{24} \end{pmatrix} \begin{pmatrix} J_{12} & J_{34} & J \\ M_{12} & M_{34} & M \end{pmatrix} \quad (2.23)$$

In terms of 6j symbols:

$$\begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} = \sum_g (-1)^{2g}(2g+1) \begin{Bmatrix} j_1 & j_2 & J_{12} \\ J_{34} & J & g \end{Bmatrix}$$
$$\begin{Bmatrix} j_3 & j_4 & J_{34} \\ j_2 & g & J_{24} \end{Bmatrix} \begin{Bmatrix} J_{13} & J_{24} & J \\ g & j_1 & j_3 \end{Bmatrix} \quad (2.24)$$

A 9j-symbol is invariant under the reflection through one of the diagonals. If one exchanges two rows or columns the 9j-symbol is multiplied by $(-1)^{sumofallninej's}$ 9j-symbols also satisfy the following orthogonality condition:

$$\sum_{J_{13},J_{24}} (2J_{13}+1)(2J_{24}+1) \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_3 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} =$$
$$\frac{\delta_{J_{12},J'_{12}} \delta_{J_{34},J'_{34}}}{(2J_{12}+1)(2J_{34}+1)} \quad (2.25)$$

2.3.4 Wigner Symbols Summation Identities

In calculating the traces of spherical harmonics one is usually encountered with the problem of summing over m's for a product of several 3j-symbols. This can be done either by some diagrammatic scheme or explicitly by applying a number of identities

For discussion of two of these diagrammatic schemes see [11] and [8].

Below a number of identities that become handy are listed.

$$\sum_{m1=-j1}^{j1}(-1)^{j1-m1}\begin{pmatrix} j1 & j1 & j3 \\ m1 & -m1 & 0 \end{pmatrix} = \sqrt{2j1+1}\delta_{j3,0}; 2j1 \in N \quad (2.26)$$

$$\sum_{m2=-j2}^{j2}\sum_{m3=-j3}^{j3}\begin{pmatrix} j1 & j2 & j3 \\ m1 & m2 & m3 \end{pmatrix}\begin{pmatrix} j4 & j2 & j4 \\ m4 & m2 & m3 \end{pmatrix} = \frac{\delta_{j1,j3}\delta_{m1,m4}}{2j1+1} \quad (2.27)$$

$$\sum_{m6=-j6}^{j6}\sum_{m5=-j5}^{j5}\sum_{m4=-j4}^{j4}(-1)^{j4+j5+j6-m4-m5-m6}\begin{pmatrix} j5 & j1 & j6 \\ m5 & -m1 & -m6 \end{pmatrix}$$
$$\begin{pmatrix} j6 & j2 & j4 \\ m6 & -m2 & -m4 \end{pmatrix} = \begin{pmatrix} j1 & j2 & j3 \\ m1 & m2 & m3 \end{pmatrix}\begin{Bmatrix} j1 & j2 & j3 \\ j4 & j5 & j6 \end{Bmatrix}$$

$$\sum_{m1=-j1}^{j1}\sum_{m2=-j2}^{j2}\sum_{m4=-j4}^{j4}\sum_{m5=-j5}^{j5}\sum_{m6=-j6}^{j6}\begin{pmatrix} j2 & j3 & j1 \\ m2 & -m3 & m1 \end{pmatrix}\begin{pmatrix} j2 & j3 & j1 \\ m2 & -m3 & m1 \end{pmatrix}$$
$$(-1)^{j1+j2+j3+j4+j5+j6-m1-m2-m3-m4-m5-m6}\begin{pmatrix} j1 & j5 & j6 \\ -m1 & m5 & m6 \end{pmatrix} \quad (2.28)$$

2.4 Traces over Spherical Harmonics

In this section the formalism introduced above will be put in action and various traces over spherical harmonics are computed. The traces that are computed here appear in computing various diagrams in matrix theory. The orthogonality relation of scalar matrix spherical harmonics was introduced in (2.25). The next most basic equation would be trace of three scalar spherical harmonics. This expression is taken directly from [5]

$$\mathrm{Tr}(Y_{j_1m_1}Y_{j_2m_2}Y_{j_3m_3}) = N^{3/2}\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}$$
$$(-1)^{N-1+j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \quad (2.29)$$

Having that in hand, and with the aid of the introduced identities computation of other traces are conceptually easy:

In principle any monstrous looking trace over any number of different spin spherical harmonics can be dealt with in three steps. First one gets rid of non-zero spin spherical harmonics by writing them in terms of scalar spherical harmonics multiplied by some coefficient involving 3j-symbols (CG coefficients written in terms of 3j-symbols) and summation over some indices.

In next step each $(Y_{l_1m_1}Y_{l_2m_2})_{ab}$ term can be expanded in terms of $C^{lm}_{ab}Y_{lm}$ with repeated index indicating summation. This produces more coefficients but less number of spherical harmonics.

At the end of the day one is left with a number of Wigner j symbols multiplied together and summed over a number of indices. Then one can simplify them by invoking the above identities.

Needless to say that various tricks can be used to shorten the path, and the heuristic way is not necessarily the easiest or the shortest way.

These traces have been computed and the results are quoted below.

Before quoting the results, an elaborate calculation is performed:

$$\begin{aligned}
\mathrm{Tr}(\vec{Y}_{j_1l_1m_1}\cdot\vec{Y}_{j_2l_2m_2}Y_{j_3m_3}) &= \sum_{M_1,M_2,q_1,q_2} (-1)^{l_1+m1+l_2+m_2}\,\mathrm{Tr}(Y_{l_1M_1}Y_{l_2M_2}Y_{l_3M_3}) \\
&\quad \begin{pmatrix} l_1 & 1 & j_1 \\ M_1 & q_1 & -m_1 \end{pmatrix}\begin{pmatrix} l_2 & 1 & j_2 \\ M_2 & q_2 & -m_2 \end{pmatrix}\sqrt{(2j_1+1)(2j_2+1)}e^i_{q_1}e^i_{q_2} \\
&= \sum_{M_1,M_2,q_1,q_2} N^{3/2}\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}(-1)^{N-1+j_3+m_1+m_2} \\
&\quad \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} l_1 & 1 & j_1 \\ M_1 & q_1 & -m_1 \end{pmatrix}\begin{pmatrix} l_2 & 1 & j_2 \\ M_2 & q_2 & -m_2 \end{pmatrix} \\
&\quad \begin{Bmatrix} j_1 & j_2 & j_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix}\sqrt{(2j_1+1)(2j_2+1)}\delta q_1 - q_2 \qquad (2.30)
\end{aligned}$$

Now using the third 3j summation identity and some phase simplification: Once gets:

$$\begin{aligned}
&\mathrm{Tr}(\vec{Y}_{j_1l_1m_1}\cdot\vec{Y}_{j_2l_2m_2}Y_{j_3m_3}) = \\
&N^{3/2}\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2l_1+1)(2l_2+1)}(-1)^{N-1+j_1+l_1} \\
&\qquad \begin{Bmatrix} l_1 & l_2 & j_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_2 & l_1 & 1 \end{Bmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \qquad (2.31)
\end{aligned}$$

The key identities are listed below:

$$\begin{aligned}
&\mathrm{Tr}(Y_{j_1m_1}Y_{j_2m_2}Y_{j_3m_3}) = \\
&\qquad N^{3/2}\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}(-1)^{N-1+j_1+j_2+j_3} \\
&\qquad\qquad \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \qquad (2.32)
\end{aligned}$$

$$\mathrm{Tr}(Y_{j_1m_1}\vec{Y}_{j_2l_2m_2}\cdot[\vec{J},Y_{j_3m_3}]) =
N^{3/2}(2j_3+1)\sqrt{j_3(j_3+1)(2j_2+1)(2j_1+1)(2l_2+1)}(-1)^{N-1+j_2+l_2}
\begin{Bmatrix} j_1 & l_2 & j_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ 1 & j_3 & l_2 \end{Bmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{2.33}$$

$$\mathrm{Tr}([\vec{J},Y_{j_1m_1}]\cdot\vec{Y}_{j_2l_2m_2}Y_{j_3m_3}) =
N^{3/2}(2j_1+1)\sqrt{j_1(j_1+1)(2j_2+1)(2j_3+1)(2l_2+1)}(-1)^{N-1}
\begin{Bmatrix} j_1 & l_2 & j_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_2 & j_1 & 1 \end{Bmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{2.34}$$

$$\mathrm{Tr}(\epsilon_{ijk}Y^i_{j_1l_1m_1}Y^j_{j_2l_2m_2}Y^k_{j_3l_3m_3}) = i\sqrt{6}(-1)^{N+j_1+j_2+j_3}N^{3/2}
\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2l_1+1)(2l_2+1)(2l_3+1)}
\begin{Bmatrix} l_1 & l_2 & l_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 1 \\ l_1 & l_2 & l_3 \\ j_1 & j_2 & j_3 \end{Bmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{2.35}$$

$$\mathrm{Tr}(\vec{Y}_{j_1l_1m_1}\cdot\vec{Y}_{j_2l_2m_2}Y_{j_3m_3}) =
N^{3/2}\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2l_1+1)(2l_2+1)}(-1)^{N-1+j_1+l_1}
\begin{Bmatrix} l_1 & l_2 & j_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_2 & l_1 & 1 \end{Bmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{2.36}$$

$$\mathrm{Tr}(\mathcal{Y}^\dagger_{j_1l_1m_1}Y_{j_2m_2}\mathcal{Y}_{j_3l_3m_3}) = N^{3/2}(-1)^{N+j_1+l_1+m_1+j_3+l_3+\frac{1}{2}}
\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2l_1+1)(2l_3+1)}
\begin{Bmatrix} l_1 & j_2 & l_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_3 & \frac{1}{2} & l_1 \end{Bmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & m_2 & m_3 \end{pmatrix} \tag{2.37}$$

$$\mathrm{Tr}(\mathcal{Y}^\dagger_{j_1l_1m_1}\mathcal{Y}_{j_2l_2m_2}Y_{j_3m_3}) =
N^{3/2}(-1)^{N+m_1+\frac{1}{2}}(2j_1+1)(2j_2+1)(2j_3+1)(2l_1+1)(2l_2+1)
\begin{Bmatrix} l_1 & l_2 & j_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_2 & l_1 & \frac{1}{2} \end{Bmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & m_2 & m_3 \end{pmatrix} \tag{2.38}$$

$$\mathrm{Tr}(\mathcal{Y}^{\dagger}_{j_1l_1m_1}\vec{\sigma}\cdot\vec{Y}_{j_2l_2m_2}\mathcal{Y}_{j_3l_3m_3}) = N^{3/2}(-1)^{N+l_1+m_1+j_2+j_3}$$
$$\sqrt{6(2j_1+1)(2j_2+1)(2j_3+1)(2l_1+1)(2l_2+1)(2l_3+1)}$$
$$\begin{Bmatrix} l_1 & l_2 & l_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ l_1 & l_2 & l_3 \\ j_1 & j_2 & j_3 \end{Bmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & m_2 & m_3 \end{pmatrix} \quad (2.39)$$

$$\mathrm{Tr}(\mathcal{Y}^{\dagger}_{j_1l_1m_1}\vec{\sigma}\mathcal{Y}_{j_3l_3m_3}\cdot\vec{Y}_{j_2l_2m_2}) = (-1)^{l_1+l_2+l_3}$$
$$\mathrm{Tr}(\mathcal{Y}^{\dagger}_{j_1l_1m_1}\vec{\sigma}\cdot\vec{Y}_{j_2l_2m_2}\mathcal{Y}_{j_3l_3m_3}) \quad (2.40)$$

$$\mathrm{Tr}(\epsilon_{\alpha\beta}\mathcal{Y}^{\alpha}_{j_1l_1m_1}\mathcal{Y}^{\beta}_{j_2l_2m_2}Y_{j_3m_3}) =$$
$$N^{3/2}(-1)^{N+j_1+l_1+\frac{1}{2}}\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2l_1+1)(2l_2+1)}$$
$$\begin{Bmatrix} l_1 & l_2 & j_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_2 & l_1 & \frac{1}{2} \end{Bmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad (2.41)$$

$$\mathrm{Tr}(\epsilon_{\alpha\beta}\mathcal{Y}^{\alpha\dagger}_{j_1l_1m_1}\mathcal{Y}^{\beta\dagger}_{j_2l_2m_2}Y_{j_3m_3}) = N^{3/2}(-1)^{N+j_2+l_2+\frac{1}{2}-m_1-m_2}$$
$$\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2l_1+1)(2l_2+1)}$$
$$\begin{Bmatrix} l_1 & l_2 & j_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_2 & l_1 & \frac{1}{2} \end{Bmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & m_3 \end{pmatrix} \quad (2.42)$$

$$\mathrm{Tr}(Y_{j_1m_1}Y_{j_2m_2}Y_{j_3m_3}Y_{j_4m_4}) =$$
$$\frac{1}{N}\sum_{j_5,m_5}(-1)^{m_5}\,\mathrm{Tr}(Y_{j_1m_1}Y_{j_2m_2}Y_{j_5m_5})\,\mathrm{Tr}(Y_{j_5,-m_5}Y_{j_3m_3}Y_{j_4m_4}) \quad (2.43)$$

$$\mathrm{Tr}(\vec{Y}_{j_1l_1m_1}\cdot\vec{Y}_{j_2l_1m_2}Y_{j_3m_3}Y_{j_4m_4}) =$$
$$\frac{1}{N}\sum_{j_5m_5}(-1)^{m_5}\,\mathrm{Tr}(\vec{Y}_{j_1l_1m_1}\cdot\vec{Y}_{j_2l_2m_2}Y_{j_5m_5})\,\mathrm{Tr}(Y_{j_5,-m_5}Y_{j_3m_3}Y_{j_4m_4}) \quad (2.44)$$

$$\mathrm{Tr}(\vec{Y}_{j_1l_1m_1}Y_{j_2m_2}\cdot\vec{Y}_{j_3l_3m_3}Y_{j_4m_4}) = -\frac{1}{N}\sum_{j_5l_5m_5}(-1)^{j_5+l_5+m_5}\,\mathrm{Tr}(\vec{Y}_{j_5l_5m_5}\cdot\vec{Y}_{j_1l_1m_1}Y_{j_2m_2})\,\mathrm{Tr}(\vec{Y}_{j_5l_5,-m_5}\cdot\vec{Y}_{j_3l_3m_3}Y_{j_4m_4}) \tag{2.45}$$

$$\mathrm{Tr}([J^i,Y^j_{j_1l_1m_1}]Y^i_{j_2l_2m_2}Y^j_{j_3l_3m_3}) = N^{3/2}(-1)^{N-1+l_1+j_2+j_3}(2l_1+1)\sqrt{l_1(l_1+1)(2j_1+1)(2j_2+1)(2j_3+1)(2l_2+1)(2l_3+1)}\begin{Bmatrix} l_1 & l_2 & l_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2}\end{Bmatrix}\begin{Bmatrix} l_1 & l_2 & l_3 \\ j_2 & l_1 & 1\end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_3 & 1 & l_1\end{Bmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} \tag{2.46}$$

$$\mathrm{Tr}(\vec{Y}_{j_1l_1m_1}\cdot\vec{Y}_{j_2l_2m_2}\vec{Y}_{j_3l_3m_3}\cdot\vec{Y}_{j_4l_4m_4}) = \frac{1}{N}\sum_{j_5,m_5}(-1)^{m_5}\,\mathrm{Tr}(\vec{Y}_{j_1l_1m_1}\cdot\vec{Y}_{j_2l_2m_2}Y_{j_5m_5})\,\mathrm{Tr}(\vec{Y}_{j_3l_3m_3}\cdot\vec{Y}_{j_4l_4m_4}Y_{j_5,-m_5}) \tag{2.47}$$

$$\begin{aligned}\mathrm{Tr}(Y^i_{j_1l_1m_1}Y^j_{j_2l_2m_2}Y^i_{j_3l_3m_3}Y^j_{j_4l_4m_4}) = {}& N^2(-1)^{j_1+j_2+j_3+j_4}\sqrt{(2l_3+1)(2l_4+1)} \\ & \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2l_1+1)(2l_2+1)} \\ & \sum_{j_5,l_5,j_6,m_6}(-1)^{l_5+j_6-m_6}(2j_5+1)(2l_5+1)(2j_6+1) \\ & \begin{Bmatrix} l_1 & l_2 & l_5 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2}\end{Bmatrix}\begin{Bmatrix} l_3 & l_4 & l_5 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2}\end{Bmatrix} \\ & \begin{Bmatrix} l_1 & l_2 & l_5 \\ j_5 & 1 & j_1\end{Bmatrix}\begin{Bmatrix} l_3 & l_4 & l_5 \\ j_5 & 1 & j_3\end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_6 \\ 1 & j_5 & l_2\end{Bmatrix} \\ & \begin{Bmatrix} j_3 & j_4 & j_6 \\ 1 & j_5 & l_4\end{Bmatrix}\begin{pmatrix} j_1 & j_2 & j_6 \\ m_1 & m_2 & -m_6\end{pmatrix}\begin{pmatrix} j_3 & j_4 & j_6 \\ m_3 & m_4 & m_6\end{pmatrix}\end{aligned} \tag{2.48}$$

Chapter 3

Further Study of BMN

One can write down the BMN action in a convenient form which makes $SO(3) \times SO(6)$ symmetries manifest. It means that we divide the scalars into two categories X^i with $i = 1, 2, 3$ and X^a with $4 \leq a \leq 9$. We also write the fermions as $\psi_{I\alpha}$ where I is a fundamental index of $SU(4) \sim SO(6)$ and α is a fundamental index of $SU(2) \sim SO(3)$. These fermions can be related to real 16 component spinor Ψ. For a complete treatment see [3].

$$
\begin{aligned}
\mathcal{L} \;=\; & \mathrm{Tr}\left(\frac{1}{2R} D_0 X^i D_0 X^i + \frac{1}{2R} D_0 X^a D_0 X^a + i\psi^{\dagger I\alpha} D_0 \psi_{I\alpha}\right) \\
& + R\ \mathrm{Tr}\left(-\frac{1}{2}\left(\frac{\mu}{3R}\right)^2 (X^i)^2 - \frac{1}{2}\left(\frac{\mu}{6R}\right)^2 (X^a)^2 - \frac{\mu}{4R}\psi^{\dagger I\alpha}\psi_{I\alpha} - \frac{i\mu}{3R}\epsilon_{ijk} X^i X^j X^k \right. \\
& \quad - \psi^{\dagger I\alpha}\sigma^{i\ \beta}_{\alpha}[X^i, \psi_{I\beta}] + \frac{1}{2}\epsilon_{\alpha\beta}\psi^{\dagger\alpha I} g^a_{IJ}[X^a, \psi^{\dagger\beta J}] - \frac{1}{2}\epsilon^{\alpha\beta}\psi_{\alpha I}(g^{a\dagger})^{IJ}[X^a, \psi_{\alpha J}] \\
& \quad \left. + \frac{1}{4}[X^i, X^j]^2 + \frac{1}{4}[X^a, X^b]^2 + \frac{1}{2}[X^i, X^a]^2 \right)
\end{aligned}
\tag{3.1}
$$

σ^i are the familiar Pauli matrices. g^a_{IJ} relate the antisymmetric product of two $SU(4)$ fundamentals to an $SO(6)$ fundamental.

As mentioned earlier all matrices are $N \times N$ Hermitian matrices. Furthermore one has a Gauss law constrained for physical states being that these states have to be invariant under $U(N)$ similarity transformation $M \to UMU^{-1}$.

3.1 Vacuum Solution

As discussed in [3] one can see that the bosonic part of the potential can be written as a sum of positive definite terms:

$$
V = \frac{R}{2}\,\mathrm{Tr}\left[\left(\frac{\mu}{3R}X^i + i\epsilon^{ijk}X^jX^k\right)^2 + \frac{1}{2}(i[X^a, X^b])^2 + (i[X^a, X^i])^2 + \left(\frac{\mu}{6R}\right)^2 (X^a)^2\right]
$$

For vacuum solution all the terms should be zero, independently. It therefore follows that $X^a = 0$ and

$$X^i = \frac{\mu}{3R} J^i$$

where J^i form some representation of $SU(2)$ algebra, obeying the commutation rule

$$[J^i, J^j] = i\epsilon^{ijk} J^k$$

A full solution thus corresponds to any representation of $SU(2)$ or more elaborately any block-diagonal matrix where each block forms an irreducible representation of $SU(2)$ algebra. For $N \times N$ matrices X^i the number of possible solutions is the number of partitions of N into sums of positive integers. The order of partitioning (which number comes first) does not matter and is taken care by the gauge condition.

As discussed in [6] a solution labelled by a partition $\{N_1, \ldots, N_k\}$ can be interpreted in two ways. Either a number of membranes or a number of five-branes.

3.1.1 Membrane Interpretation

A general vacuum corresponding to a partition $N = N_1 + \cdots N_m$ can be interpreted as m concentric fuzzy sphere membranes with radii being $r_i \sim \frac{N_i \mu}{6R}$

3.1.2 Fivebrane Interpretation

A vacuum state can also be thought as a collection of fivebranes where the number of fivebranes is equal to the size of the largest irreducible representation, and the momentum M_n carried by the nth fivebrane is equal to the number of irreducible representations with size greater than or equal to n.

As discussed in [6] membrane interpretation is more appropriate when one keeps the ratio of the block sizes fixed and takes N to infinity. Fivebrane interpretation is more appropriate when one keeps the sizes of the blocks fixed and takes the number of blocks to infinity.

3.2 Expansion About the Vacuua

The action can be expanded about each of the vacuua.

$$X^i = \frac{\mu}{3R} J^i + Y^i \tag{3.2}$$

for any desired J.

One can rescale the variables for making the quadratic part of the action look much nicer as follows:

$$X^i \to \sqrt{\frac{R}{\mu}} Y^i,\ X^a \to \sqrt{\frac{R}{\mu}} X^a,\ t \to \frac{1}{\mu} t \tag{3.3}$$

The quadratic part of the resultant action looks like:

$$S = S_2^Y + S_2^X + S_2^\psi$$

where

$$\begin{aligned}
S_2^Y &= \mathrm{Tr}\left(\frac{1}{2}\dot{Y}^i\dot{Y}^i - \frac{1}{2}(\frac{1}{3})^2(Y^i + i\epsilon^{ijk}[J^j, Y^k])^2\right) \\
S_2^X &= \mathrm{Tr}\left(\frac{1}{2}(\dot{X}^a)^2 - \frac{1}{2}(\frac{1}{3})^2(\frac{1}{4}(X^a)^2 - [J^i, X^a]^2)\right) \\
S_2^\psi &= \mathrm{Tr}\left(i\psi^\dagger\dot{\psi} - \frac{1}{4}\psi^\dagger\psi - \frac{1}{3}\sigma^i\psi^\dagger[J^i, \psi_\beta]\right)
\end{aligned} \tag{3.4}$$

are quadratic actions for X, Y, and ψ, and the interaction (cubic and quartic) terms are S_3 and S_4 defined as

$$\begin{aligned}
S_3 &= \left(\frac{R}{\mu}\right)^{\frac{3}{2}} \mathrm{Tr}\left(\frac{1}{3}[J^i, X^a][Y^i, X^a] + \frac{1}{3}[J^i, Y^j][Y^i, Y^j] - \frac{i}{3}\epsilon^{ijk}Y^iY^jY^k\right. \\
&\qquad \left. -\psi^\dagger\sigma^i[Y^i, \psi] + \frac{1}{2}\epsilon\psi^\dagger \mathrm{g}^a[X^a, \psi^\dagger] - \frac{1}{2}\epsilon\psi(\mathrm{g}^\dagger)^a[X^a, \psi]\right) \\
S_4 &= \left(\frac{R}{\mu}\right)^3 \mathrm{Tr}\left(\frac{1}{4}[Y^i, Y^j]^2 + \frac{1}{4}[X^a, X^b]^2 + \frac{1}{2}[Y^i, X^a]^2\right).
\end{aligned} \tag{3.5}$$

Also note that the gauge field A is set to zero. Of course for full treatment regular derivatives should be replaced with covariant ones for Y's.

It is seen that by taking $\frac{\mu}{R}$ to infinity one is suppressing the interaction terms. Therefore the theory can be solved exactly and the spectrum can be obtained in the limit $\left(\frac{\mu}{R}\right) \to \infty$

3.3 Exact Spectrum in the Large μ Limit

For fixed N and R and sending μ to infinity one can proceed and find the complete spectrum of the action in such limit.

The energy spectrum can be found once the quadratic terms are diagonalized. The task is accomplished if one expands the fluctuation matrices in a basis which behaves nicely under the commutator part of the quadratic part of the action. With the machinery introduced earlier and knowing the behaviour of matrix spherical harmonics under the commutator it is not hard to see that this can be done once the matrices are expanded in terms of matrix spherical harmonics. It should be noted that the correct family of (scalar, vector or spinor) spherical harmonics should be chosen to expand different matrices. The choice comes from the form of the commutator of the quadratic part. One needs a family which behaves nicely under the corresponding commutator.

It is seen that the commutator part of the quadratic part of the action of X^i's is suitable for vector spherical harmonics. Spinor spherical harmonics are suitable for ψ's. $SO(6)$ bosons seem a bit complicated. After noting that the potential part of the corresponding quadratic part can be written as

$$V^X = \frac{1}{2}\frac{1}{3^2}\,\mathrm{Tr}\left(X^a\Big(\frac{1}{4}X^a + [J^i,[J^i,X^a]]\Big)\right)$$

it becomes apparent that scalar spherical harmonics should be used for expansion.

Below the steps that were sketched are actually performed:

As mentioned earlier we consider expansions around particular form of vacuua namely k copies (direct sum) of an $N \times N$ irreducible representation of $SU(2)$.

One can expand the bosons in the $SO(6)$ directions in terms of the matrix spherical harmonics as

$$X^a = \frac{1}{\sqrt{N}}\sum_{j,m} x^a_{jm} Y_{jm} \tag{3.6}$$

x^a_{jm} is as $k \times k$ matrix. X^a is Hermitian, so $(x^a_{jm})^\dagger = (-1)^m x^a_{j,-m}$. Similarly for the bosons in the $SO(3)$ directions we have

$$Y^i = \frac{1}{\sqrt{N}}\sum_{jlm} \alpha_{jlm} Y^i_{jlm} \tag{3.7}$$

where the $\vec{Y}_{jlm}$ are matrix vector spherical harmonics, which are defined in (2.5)

$$\vec{Y}_{JlM} = \sum_{m,q} Y_{l,m}\hat{e}_q \langle l,m;1,q|J,M\rangle \tag{3.8}$$

where $l = j \pm 1$ ($l = j$ we eliminate with our gauge choice).

The fermions can also be expanded, in this case in terms of spinor spherical harmonics, as

$$\psi = \frac{1}{\sqrt{N}} \sum_{jlm} \psi_{jlm} \mathcal{Y}_{jlm} \tag{3.9}$$

where

$$\mathcal{Y}_{JlM} = \sum_{m,q} Y_{l,m} \chi^q \langle l, m; \tfrac{1}{2}; q | J, M \rangle \tag{3.10}$$

where χ^q are eigenstates of σ^0.

After expanding our fields in terms of the matrix spherical harmonics we find that the quadratic terms in the action have the form

$$\begin{aligned} S_2^Y &= \sum_{jlm} \mathrm{Tr}\left(\frac{1}{2}|D_0 \alpha_{jlm}|^2 - \frac{1}{2}(\frac{1}{3}\lambda_j)^2 |\alpha_{jlm}|^2\right) \\ S_2^X &= \mathrm{Tr}\left(\frac{1}{2}|D_0 x^a_{jm}|^2 - \frac{1}{2}(\frac{1}{3}\lambda_j)^2 |x^a_{jm}|^2\right) \\ S_2^\psi &= \mathrm{Tr}\left(i\psi^\dagger_{jlm} D_0 \psi_{jlm} - (\frac{1}{4} - \frac{1}{3}\lambda_j)\psi^\dagger_{jlm}\psi_{jlm}\right). \end{aligned} \tag{3.11}$$

where λ_i are the eigenvalues of the eigen-equation of corresponding spherical harmonics.

As it can be seen for the modes $(j = l)$ the resulting mass (eigenvalue) is zero. It was shown in [3] that these modes are non-physical modes. These modes can be eliminated by the gauge choice.

3.4 t' Hooft Limit

3.4.1 Some Intro Material

In field theory to do perturbative calculations one usually expands he action in powers of the coupling constant. 't Hooft realized that for a rank N gauge theory, the expansion parameter is not really g^2_{YM}, rather it is a combination of the rank N and Yang-Mills coupling which is now known as 't Hooft coupling λ:

$$\lambda = g^2_{YM} N \tag{3.12}$$

't Hooft also realized that one can categorize Feynman diagrams based on their topology (genus number g) of the surface that they can be drawn on without edges overlap. The lowest g is zero which corresponds to a planar diagram or a diagram that can be drawn on a sphere without edges overlapping. A diagram of $g = 1$ can cannot be drawn on a sphere, with the condition specified and instead can be drawn on a torus. Each genus contributes a factor of $1/N^g$ to the overall diagram pre-factor. This comes from summing over indices of $N \times N$ matrices and is most easily seen in double line notation which is introduced later on. For a nice little proof refer to [7]

From this it is seen that for large N planar diagrams dominate, because they are not suppressed by factors of $(1/N)$.

3.4.2 Double-Line Notation

If one has matrices as the fundamental degrees of freedom then one can write the propagator with two lines with each line having a direction that shows the follow of the index and the arrows point in opposite direction.

Looking at the way indices of the matrices appear in the in the propagator (far indices are coupled together through Kronecker δ and near indices coupled together) double line notation is easily justifiable.

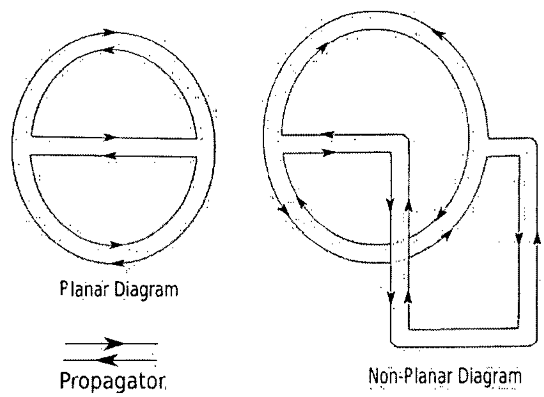


Figure 3.1: Planar and non-planar diagrams in t' Hooft double line notation

Each index loop contributes a factor of N. As it can be seen in fig.3.1 the planar diagram has three index loops and the non- planar ($g = 1$) only one. So indeed the results agree with 't Hooft's claim and the non-planar diagram is suppressed by a factor of $1/N^2$.

Chapter 4

Some Explicit Calculations

Having the machinery and infrastructure it is now desired to perform some explicit calculations and study the behaviour of different types of diagram for large N limit. By 't Hooft analysis we already know that planar diagrams dominate so the goal is to classify the behaviour of different planar diagrams.

We are explicitly interested in behaviour of a subclass of diagrams that arise in computation of quantities like $\left\langle \mathrm{Tr}\,(X^a)^2 \right\rangle$. Before doing so it is necessary to carefully write down the Lagrangian in Lorentzian gauge, along with ghosts arising from gauge fixing and the gauge field.

$$\begin{aligned}
S_2^Y &= \mathrm{Tr}\left(\frac{1}{2}\dot{Y}^i\dot{Y}^i - \frac{1}{2}(\frac{1}{3})^2(Y^i + i\epsilon^{ijk}[J^j, Y^k])^2\right) \\
S_2^X &= \mathrm{Tr}\left(\frac{1}{2}(\dot{X}^a)^2 - \frac{1}{2}(\frac{1}{3})^2(\frac{1}{4}(X^a)^2 - [J^i, X^a]^2)\right) \\
S_2^\psi &= \mathrm{Tr}\left(i\psi^\dagger\dot{\psi} - \frac{1}{4}\psi^\dagger\psi - \frac{1}{3}\sigma^i\psi^\dagger[J^i, \psi_\beta]\right) \\
S_2^A &= \mathrm{Tr}\left(-\frac{1}{2}\frac{1}{9}[A, J^i]^2\right) \\
S_2^\eta &= \mathrm{Tr}\left(-\frac{1}{9}[J^i, \bar{\eta}][J^i, \eta]\right)
\end{aligned} \tag{4.1}$$

The cubic part of the interaction becomes:

$$\begin{aligned}
S_3 = \left(\frac{R}{\mu}\right)^{\frac{3}{2}} \mathrm{Tr}\Bigg(& \frac{1}{3}[J^i, X^a][Y^i, X^a] + \frac{1}{3}[J^i, Y^j][Y^i, Y^j] - \frac{i}{3}\epsilon^{ijk}Y^iY^jY^k \\
& -\psi^\dagger\sigma^i[Y^i, \psi] + \frac{1}{2}\epsilon\psi^\dagger \mathrm{g}^a[X^a, \psi^\dagger] - \frac{1}{2}\epsilon\psi(\mathrm{g}^\dagger)^a[X^a, \psi] \\
& -i\dot{Y}^i[A, Y^i] - \frac{1}{3}[A, J^i][A, Y^i] - i\dot{X}^a[A, X^a] \\
& \frac{1}{3}[J^i, \bar{\eta}][Y^i, \eta] + \psi^\dagger[A, \psi]\Bigg).
\end{aligned} \tag{4.2}$$

The quartic part is:

$$S_4 = \left(\frac{R}{\mu}\right)^3 \mathrm{Tr}\left(\frac{1}{4}[Y^i, Y^j]^2 + \frac{1}{4}[X^a, X^b]^2 + \frac{1}{2}[Y^i, X^a]^2\right). \tag{4.3}$$

Using the procedure described above and expanding the spherical harmonics exactly as followed we get the following expression for the propagators:

$$
\begin{aligned}
\left\langle \left(Y_{jlm}\right)^p_q(t)\left(Y_{j'l'm'}\right)^{\dagger s}_{\ r}(0)\right\rangle &= \delta_{jj'}\delta_{ll'}\delta_{mm'}\delta^p_s\delta^r_q\Delta_{m_y}(t) \\
\left\langle \left(X^a_{jlm}\right)^p_q(t)\left(X^b_{j'l'm'}\right)^{\dagger s}_{\ r}(0)\right\rangle &= \delta_{ab}\delta_{jj'}\delta_{ll'}\delta_{mm'}\delta^p_s\delta^r_q\Delta_{m_x}(t) \\
\left\langle \left(\psi_{jlm}\right)(t)\left(\psi_{j'l'm'}\right)^{\dagger}(0)\right\rangle &= \delta_{jj'}\delta_{ll'}\delta_{mm'}\Delta^F_{m_y}(t)
\end{aligned}
\tag{4.4}
$$

where

$$m_x = \frac{2j+1}{6} \tag{4.5}$$

$$m_y = \frac{j+l+1}{6} \tag{4.6}$$

$$m_\psi = \frac{4(j-l)(4l-2j+1)+3}{12} \tag{4.7}$$

$$
\begin{aligned}
\Delta_m(t) &= \frac{1}{2m}\left(\theta(t)e^{-imt}+\theta(-t)e^{imt}\right) \\
&= \frac{1}{2\pi}\int dk\ \frac{ie^{-ikt}}{k^2-m^2+i\epsilon}
\end{aligned}
\tag{4.8}
$$

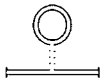
$$
\begin{aligned}
\Delta^F_m &= \begin{cases} e^{-imt}\theta(t) & m>0 \\ -e^{-imt}\theta(-t) & m<0 \end{cases} \\
&= \frac{1}{2\pi}\int dk\ \frac{i(k+m)e^{-ikt}}{k^2-m^2+i\epsilon}
\end{aligned}
\tag{4.9}
$$

We explicitly evaluate some of the diagrams:

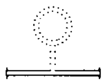
$$
\begin{aligned}
&\rightarrow \frac{15\lambda ki}{N}\left(\frac{R}{\mu}\right)^3\sum_{\{j,m\}}\int dt\Delta^X_{\frac{2j_1+1}{6}}(t)\Delta^X_{\frac{2j_1+1}{6}}(t)\Delta^X_{\frac{2j_2+1}{6}}(0) \\
&(-1)^{m_1+m_2}\,\mathrm{Tr}(Y_{j_1-m_1}Y_{j_1m_1}Y_{j_2-m_2}Y_{j_2m_2})
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
&\rightarrow \frac{6\lambda ki}{N}\left(\frac{R}{\mu}\right)^3\sum_{\{j,m\}}\int dt\Delta^Y_{\frac{j_1+l_1+1}{6}}(0)\Delta^X_{\frac{2j_2+1}{6}}(t)\Delta^X_{\frac{2j_2+1}{6}}(t) \\
&(-1)^{j_1+l_1+m_1+m_2}\,\mathrm{Tr}(Y^i_{j_1l_1-m_1}Y^i_{j_1l_1m_1}Y_{j_2-m_2}Y_{j_2m_2})
\end{aligned}
\tag{4.11}
$$

$$\rightarrow \frac{-4\lambda k}{3N}\left(\frac{R}{\mu}\right)^3 \sum_{\{j,l,m\}} \int dt \int dt' (\Delta^X_{\frac{2j_1+1}{6}}(t)\Delta^X_{\frac{2j_1+1}{6}}(t')\Delta^X_{\frac{2j_3+1}{6}}(t-t')$$
$$\Delta^Y_{\frac{j_2+l_2+1}{6}}(t-t'))(-1)^{j_2+l_2+m_1+m_2+m_3+1}\,\mathrm{Tr}([\vec{J},Y_{j_1-m_1}]\cdot\vec{Y}_{j_2l_2-m_2}Y_{j_3-m_3})$$
$$\mathrm{Tr}([\vec{J},Y_{j_3m_3}]\cdot\vec{Y}_{j_2l_2m_2}Y_{j_1m_1}) \tag{4.12}$$



$$\rightarrow \frac{-8\lambda ki}{N}\left(\frac{R}{\mu}\right)^3 \sum_{\{j,l,m\}} \int dt \int dt' \Delta^X_{\frac{2j_1+1}{6}}(t)\Delta^X_{\frac{2j_1+1}{6}}(t)\Delta^X_{\frac{2j_2+1}{6}}(0)\Delta^Y_{\frac{j_3+l_3+1}{6}}(t'-t)$$
$$\mathrm{Tr}(Y^\dagger_{j_1m_1}[\vec{J},Y_{j_1m_1}]\cdot\vec{Y}^\dagger_{j_3l_3m_3})\,\mathrm{Tr}(Y^\dagger_{j_2m_2}[\vec{J},Y_{j_2m_2}]\cdot\vec{Y}_{j_3l_3m_3}) \tag{4.13}$$



$$\rightarrow \frac{-4\lambda ki}{3N}\left(\frac{R}{\mu}\right)^3 \sum_{\{j,l,m\}} \int dt \int dt' \Delta^X_{\frac{2j_1+1}{6}}(t)\Delta^X_{\frac{2j_1+1}{6}}(t)\Delta^Y_{\frac{j_2+l_2+1}{6}}(0)\Delta^Y_{\frac{j_3+l_3+1}{6}}(t'-t)$$
$$\Big[2\,\mathrm{Tr}([\vec{J},Y^\dagger_{j_1m_1}]\cdot\vec{Y}^\dagger_{j_2l_2m_2}Y_{j_1m_1})\Big(\mathrm{Tr}([J^i,Y^j_{j_2l_2m_2}]Y^{\dagger i}_{j_3l_3m_3}Y^j_{j_3l_3m_3})+$$
$$\mathrm{Tr}([J^i,Y^j_{j_3l_3m_3}]Y^i_{j_2l_2m_2}Y^{\dagger j}_{j_3l_3m_3}) + \mathrm{Tr}([J^i,Y^{\dagger j}_{j_3l_3m_3}]Y^i_{j_3l_3m_3}Y^j_{j_2l_2m_2})\Big)+$$
$$3\,\mathrm{Tr}([\vec{J},Y^\dagger_{j_1m_1}]\cdot\vec{Y}^\dagger_{j_2l_2m_2}Y_{j_1m_1})\,\mathrm{Tr}(\epsilon_{ijk}Y^i_{j_2l_2m_2}Y^{\dagger j}_{j_3l_3m_3}Y^k_{j_3l_3m_3})\Big] \tag{4.14}$$

$$\tag{4.15}$$

Before going any further and expand the expressions that are obtained for the diagrams so far, the integrals are needed to be evaluated. The final expression for these are quoted below. However before that some convention is introduced: Since the general form of $\Delta^X_{m_x}$ and $\Delta^Y_{m_y}$ is the same Δ_m is used for both of them.

$$\Delta_m(0) = \frac{1}{2m} \tag{4.16}$$

$$\int dt\Delta_m(t) = \frac{i}{m^2} \tag{4.17}$$

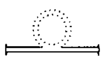
$$\int dt\Delta_m(t)\Delta_m(t) = \frac{-i}{4m^3} \tag{4.18}$$

$$\int dt \int dt' (\Delta_m(t)\Delta_m(t')$$
$$\Delta_{m_2}(t-t')\Delta_{m_3}(t-t')) = \frac{-(2m+m_2+m_3)}{8m^3m_2m_3(m+m_2+m_3)^2} \tag{4.19}$$

$$\int dt \int d' (\Delta_m(t)\Delta_m(t')$$

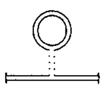
$$\Delta^F_{m_2}(t-t')\Delta^F_{m_3}(t-t')) \quad = \quad \frac{-(2m+m_2+m_3)}{2m^2(m+m_2+m_3)^2} \tag{4.20}$$

$$\rightarrow \quad 2430k\lambda N \sum_{\{j\}} (-1)^{j_1+j_2}(2j_1+1)^{\frac{-3}{2}}(2j_2+1)^{\frac{1}{2}}$$
$$\begin{Bmatrix} j_1 & j_1 & 0 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \begin{Bmatrix} 0 & j_2 & j_2 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \tag{4.21}$$

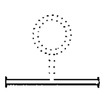


$$\rightarrow \quad 972k\lambda N \sum_{\{j\}} (-1)^{j_1+j_2+2l_1+1} \frac{(2j_1+1)^{\frac{3}{2}}(2l_1+1)}{(j_1+l_1+1)(2j_2+1)^{\frac{3}{2}}}$$
$$\begin{Bmatrix} l_1 & l_1 & 0 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \begin{Bmatrix} 0 & j_2 & j_2 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \begin{Bmatrix} j_1 & j_1 & 0 \\ l_1 & l_1 & 1 \end{Bmatrix} \tag{4.22}$$

$$\rightarrow \quad 108k\lambda N \sum_{\{j,l\}} \sqrt{j_1(j_1+1)(2j_2+1)(j_3)(j_3+1)(2j_3+1)}$$
$$\frac{(2j_2+1)(2l_2+1)}{j_2+l_2+1} \left(\frac{2j_1+1}{6}\right)\left(\frac{2j_3+1}{6}\right)\left(\frac{j_2+l_2+1}{6}\right)$$
$$\frac{-(2\frac{2j_1+1}{6} + \frac{j_2+l_2+1}{6} + \frac{2j_3+1}{6})}{8\left(\frac{2j_1+1^3}{6}\right)\left(\frac{j_2+l_2+1}{6}\right)\left(\frac{2j_3+1}{6}\right)\left(\frac{2j_1+1}{6} + \frac{j_2+l_2+1}{6} + \frac{2j_3+1}{6}\right)^2}$$
$$\begin{Bmatrix} j_1 & l_2 & j_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix}^2 \begin{Bmatrix} j_1 & j_2 & j_3 \\ 1 & j_3 & l_2 \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_2 & j_1 & 1 \end{Bmatrix} \tag{4.23}$$
$$\frac{-(2\frac{2j_1+1}{6} + \frac{4(j_2-l_2)(4l_2-2j_2+1)+3}{12} + \frac{4(j_3-l_3)(4l_3-2j_3+1)+3}{12})}{2(\frac{2j_1+1}{6})^2(\frac{2j_1+1}{6} + \frac{4(j_2-l_2)(4l_2-2j_2+1)+3}{12} + \frac{4(j_3-l_3)(4l_3-2j_3+1)+3}{12})^2} \tag{4.24}$$



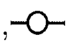
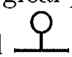
$$\rightarrow \quad 139968k\lambda N \sum_{\{j\}} \frac{(2j_2+1)\sqrt{j_2(j_2+1)j_1(j_1+1)}}{2j_1+1}$$
$$(-1)^{1-j_1-j_2} \begin{Bmatrix} j_1 & j_1 & 1 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \begin{Bmatrix} j_2 & j_2 & 1 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix}$$
$$\begin{Bmatrix} 0 & 1 & 1 \\ j_1 & j_1 & j_1 \end{Bmatrix} \begin{Bmatrix} 0 & 1 & 1 \\ j_2 & j_2 & j_2 \end{Bmatrix} \tag{4.25}$$



$$\rightarrow \quad -1944k\lambda N \sum_{\{j,l\}} \frac{(2j_3+1)^{\frac{3}{2}}}{(2j_1+1)(j_3+l_3+1)}$$
$$2\begin{Bmatrix} j_1 & 1 & j_1 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \begin{Bmatrix} 1 & l_3 & l_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \begin{Bmatrix} j_1 & 0 & j_1 \\ 1 & j_1 & 1 \end{Bmatrix}$$

$$
\begin{aligned}
&(4\sqrt{6}(-1)^{1-j_1+l_3}(2l_3+1)\sqrt{j_1(j_1+1)} \\
&\begin{Bmatrix} 1 & l_3 & l_3 \\ j_3 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} 0 & j_3 & j_3 \\ l_3 & 1 & 1 \end{Bmatrix} \\
&+4(-1)^{-j_1+j_3}\sqrt{j_1(j_1+1)l_3(l_3+1)(2l_3+1)} \\
&\begin{Bmatrix} l_3 & 1 & l_3 \\ 0 & l_3 & 1 \end{Bmatrix} \begin{Bmatrix} j_3 & 0 & j_3 \\ l_3 & 1 & l_3 \end{Bmatrix} \\
&+4(-1)^{-j_1+j_3}(2l_3+1)^{\frac{3}{2}}\sqrt{j_1(j_1+1)l_3(l_3+1)} \\
&\begin{Bmatrix} l_3 & l_3 & 1 \\ j_3 & l_3 & 1 \end{Bmatrix} \begin{Bmatrix} j_3 & j_3 & 0 \\ 1 & 1 & l_3 \end{Bmatrix} \\
&+6\sqrt{6}I(-1)^{l_3-j_1}(2l_3+1)^2\sqrt{j_1(j_1+1)} \\
&\begin{Bmatrix} 1 & 1 & 1 \\ 1 & l_3 & l_3 \\ 0 & j_3 & j_3 \end{Bmatrix})
\end{aligned}
\tag{4.26}
$$

4.1 Large N Behaviour

From a topological point of view, there are three different types of diagrams: ,  and .

The behaviour of these diagram can be analyzed as N changes. Before applying brute force and evaluating the diagrams for different N's, one may wish to have some heuristic estimate of the behaviour.

The matrices are k blocks of $N \times N$ sub-matrices. Before bringing the results of spherical harmonics into play k and N factors come into the game as follows:

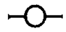
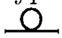
- $\frac{1}{k \times N}$ from expansion parameter of Lagrangian
- $(k \times N)^2$ for the two index loops.
- $\frac{1}{\sqrt{N}}$ for each field expansion in terms of spherical harmonics stemming from our expansion conventions described in precious section

By looking at the traces of spherical harmonics one sees that for each Y, there is $N^{\frac{1}{2}}$ contribution which effectively cancels out the contribution coming form the field expansion parameters as one expects. So at the end of the day one has a $\frac{k}{N}$ factor at front. multiplied by big sum over j's and l's of a product of j-symbols and factors.

So naively all diagrams have apparent same N dependence, and only differ in the summation arguments including the j-symbols.

Below the log-log graph of the value of the diagram as a function of N (or number of M5 branes) is plotted and the slope of best fitted line is provided.

All results are calculated with Maple symbolically and only at the final stage the result is converted to floating numbers.

Obviously, the modification to the exact integer dependence of N is because of the summation. For  type diagrams the sum slightly suppresses the effect of N and for the case of  type diagrams the sum increases the N dependence power slightly. For  type diagrams, however the sum contributes slightly more than a full order of N and therefore for large N's one can ignore other diagrams with different topologies in favour of these ones.

It is rather interesting that  type diagrams are zero for the special case of $N = 1$, or having only one fivebrane.

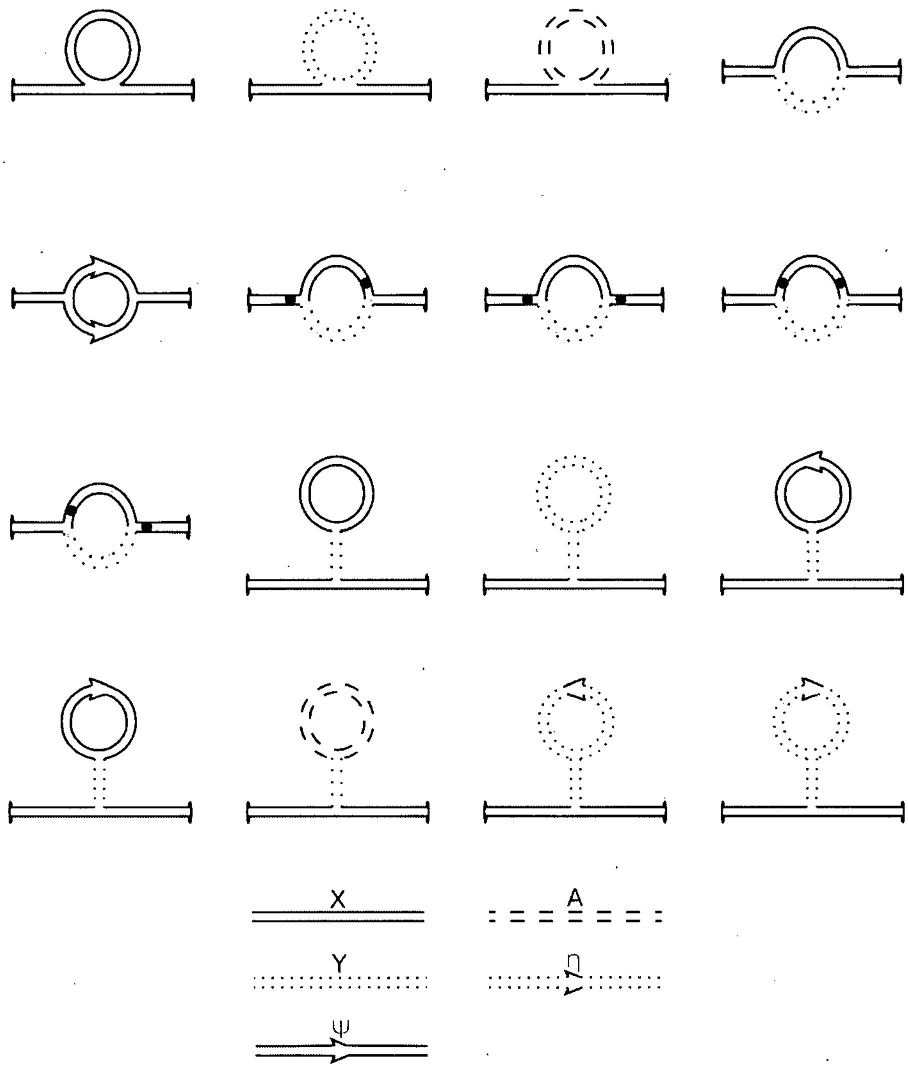
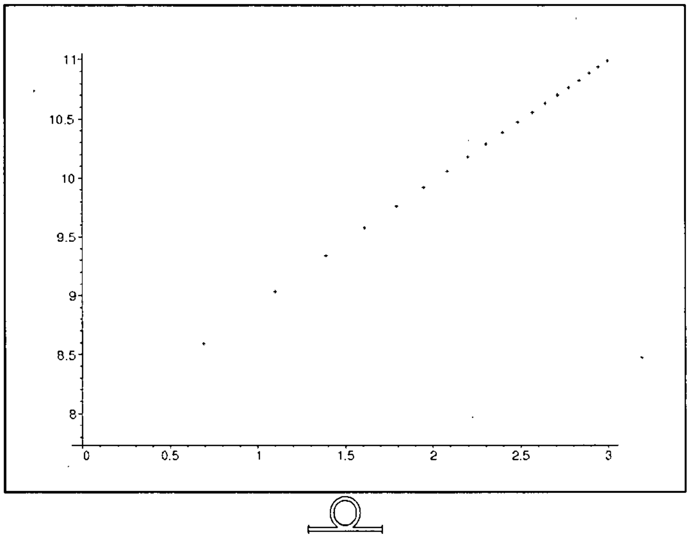
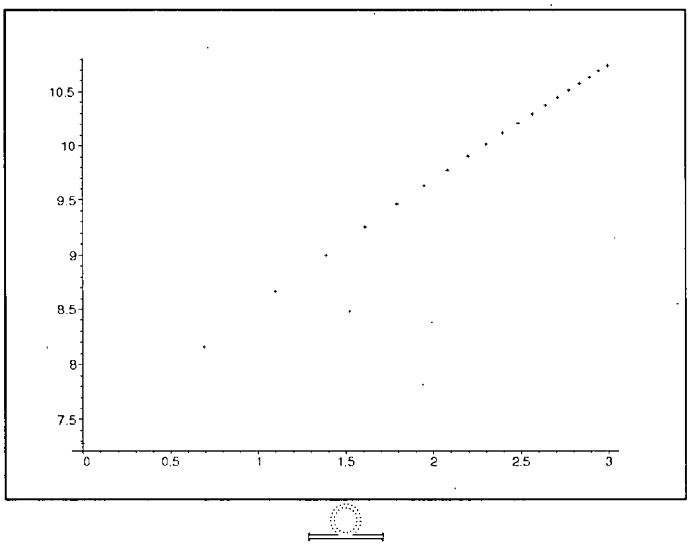


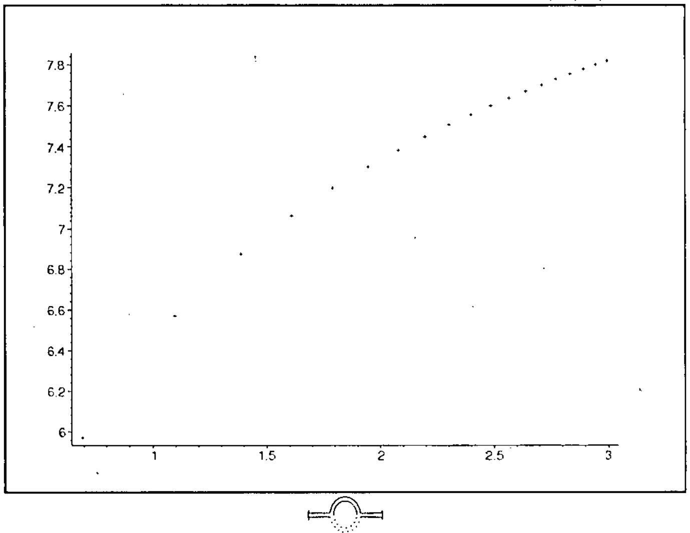
Figure 4.1: All possible second order diagrams



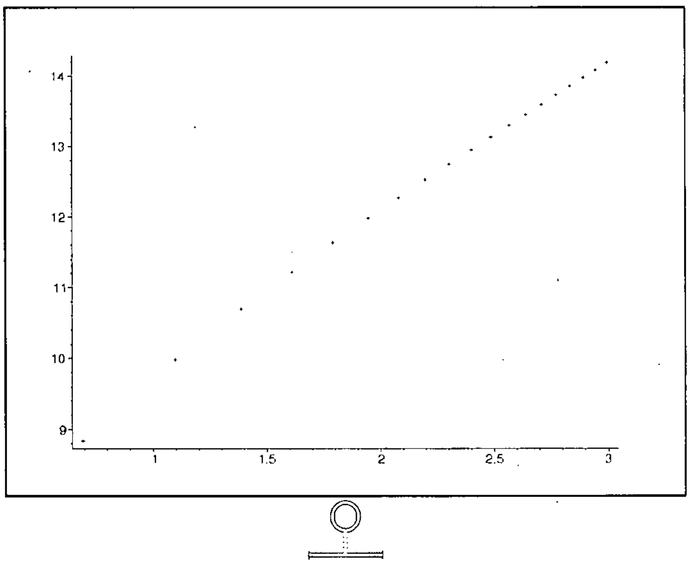
$$y = 7.857627816 + 1.052319183x$$



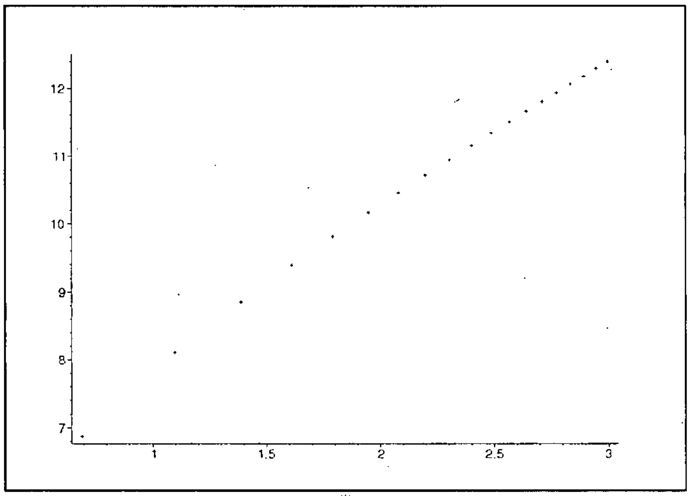
$$y = 7.386301167 + 1.134267388x$$



$$y = 5.802976127 + .7113185375x$$



$$y = 7.498191559 + 2.259245284x$$



$$y = 5.571100960 + 2.310262782x$$

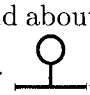
Chapter 5

Conclusion and Wrap Up

In this thesis two subjects related to BMN matrix theory were explored. It was seen that expansion and computation of quantities around various vacuua in BMN theory results in introduction of Matrix spherical harmonics and various traces of their products. The formalism regarding such expansions and computation of these traces was developed and all different kinds of traces that may appear in such computation were computed and presented. In the next section as an example, a few bosonic diagrams that arise in computation of $Tr(X^a X^a)$ were computed.

Having the formalism to compute such diagrams, one can ask a variety of questions. One question would be the relation between the topology of a diagram and its behaviour as a function of the number of five-branes (N). Thanks to the work of 't Hooft, it is already known that such a relation is indeed present in a weaker sense. As explained above when the sizes of matrices ($k \times N$) get larger planar diagrams dominate because of factors of N that appear due to traces over internal indices.

By keeping k (number of two-branes) constant and only increasing N one is indeed increasing the size of matrices, so again the planar diagrams dominate over non-planar ones. But the N dependence of our diagrams are much more subtle than their k dependence. It was shown that, naively looking at the diagrams, their explicit N dependence is the same and can be obtained by heuristic arguments. The actual N dependence can not be obtained easily as it is apparent from the expressions. Therefore the value of diagrams were computed numerically and the dependence was obtained by means of line fitting. It was shown that a certain subclass of the diagrams have a much greater N dependence: Depending on N quadratically as opposed to linearly which is expected from naive heuristic arguments.

Since these results were obtained numerically not much can be said about other non-computed diagrams and the full merit of dominance of -shape diagrams.

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