MUON DECAY IN AN SU(2) $\times$ U(1) GAUGE THEORY WITH SPINOR AND VECTOR HIGGS FIELDS AND MASSIVE MAJORANA NEUTRINOS

by

JOHN DWAYNE HAMILTON

B.E., The University of Saskatchewan, 1965

M.Sc., The University of British Columbia, 1967

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Department of PHYSICS

The University of British Columbia
1956 Main Mall
Vancouver, Canada
V6T 1Y3

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ABSTRACT

This thesis investigates the implications for muon decay of a gauge theory that incorporates modifications to the standard electro-weak theory, the Weinberg-Salam model. The Higgs sector of the original model is broadened to include an iso-vector or triplet Higgs field. Such an addition naturally provides for massive Majorana-type neutrinos and permits the decay modes $\mu \rightarrow e\gamma$ and $\mu \rightarrow 3e$ which would not otherwise be allowed. The particles most responsible for these decay modes are new, charged, physical, scalar Higgs particles not present in the Weinberg-Salam model. The decay rates found, although considerably more favorable to muon decay than the simple addition of neutrino mass to the standard theory, are found to be much below the present limits of experimental detectability. The lengthy calculations required, including the derivation and utilization of the relevant Feynman diagrams, are displayed in some detail.

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TABLE OF CONTENTS

LIST OF TABLES

page

LIST OF FIGURES

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I. INTRODUCTION AND SUMMARY

I.1 INTRODUCTION: FIFTY YEARS OF WEAK INTERACTIONS

The recent explosive growth in our experimental and theoretical knowledge of weak interaction physics could hardly be imagined fifty years ago when it began with the discovery of the neutron and its relatively slow β-decay into a proton, an electron and a conjectured spin-$\frac{1}{2}$, relatively massless, neutral particle, the neutrino. Fermi (1934) was the first to attempt a mathematical description of the process, with a rather phenomenological interaction Lagrangian that accomplished little more than the simple description of the then-known experimental facts.

More than two decades had to pass before the discovery of parity non-conservation (Lee and Yang, 1956) provided the clue for a significant modification of the original Fermi theory. It was observed that, in the high energy limit at least, only electrons and neutrinos of left-handed helicity (and their right-handed anti-particles) participated in weak interactions, right-handed electrons not at all. This was expressed mathematically in the so-called "V-A" ("vector minus axial-vector") theory (Feynman and Gell-Mann, 1958; Marshak and Sudarshan, 1958), which was shortly thereafter extended to the hadrons, or strongly interacting particles. Experimental support for the theory, at the few GeV level and lesser energies, started to mount. With essentially the same coupling constants, neutron, pion and muon decays could be described -- a quantitative success known as "universality" -- but dramatic confirmation for the mathematical description was given by pion decay, in which the rate of decay into muons rather than the seemingly more likely, but helicity-suppressed, electrons was quite precisely described by the V-A theory, as were some of the properties of the much later discovered τ-particle (Perkins, 1982, Sec. 6.8, 6.14).

On the other hand, well-known weaknesses of the theory, such as its nonrenormalizability, or mathematical inability to describe higher order processes, and its violation of basic conservation of probability (or "unitarity") at high energies (a few hundred GeV) strongly resisted attempts to perfect the theory, even efforts that attempted to incorporate the heavy, charged bosons that were postulated to mediate weak interactions. These efforts went unrewarded, for the most part, until the invention of the non-abelian gauge theory now known as the Weinberg(1967)-Salam(1968) model, which is the point of departure for Chap. II.

The story of the weak interaction to the time of the Weinberg-Salam theory and its relation to the strong interaction is a long and fascinating one, with many successes and failures on the theoretical side and enormous challenges on the experimental. It has been well told in many places, most recently in Taylor (1976), Itzykson and Zuber (1980), Veltman

(1980), Lee (1981), Cheng and Li (1984), and many earlier accounts. (The brief and superficial account given here hardly begins to do it justice.)

The Weinberg-Salam theory became the standard model of the weak interactions with proof of its renormalizability by t'Hooft (1971). Experimental support began to accumulate shortly thereafter. Neutral currents, so-called, or interactions mediated by a massive, neutral boson, were discovered in electron-neutrino and nucleon-neutrino scattering, as were helicity asymmetries in electron-deuteron scattering, in which the helicity-dependent weak interaction became experimentally noticeable. Even contributions of the weak interaction to electron-electron scattering, until recently wholely within the purview of quantum electrodynamics, became detectable. Accounts of these experiments in the mid to late 1970's and early 1980's and their quantitative agreement with the Weinberg-Salam theory can be found in, for example, Perkins (1982, Sec. 8.8). Now, recently, the long-sought particles required by the theory to mediate the weak interaction have been detected, with masses in rather dramatic agreement with the theory (as to be discussed in Chap. II).

So why, one might wonder, would anyone want to modify or extend the basic Weinberg-Salam theory, the remarkable success that it is?

There are both experimental and theoretical reasons, a few of which we now discuss.

Reports of measurements that indicate a non-zero mass for the neutrino (Lubimov, 1980) have serious implications for weak interaction theories because the Weinberg-Salam model in particular assumed at the outset that all neutrinos were massless. There is no unique and straightforward way to amend the model to accommodate species of neutrinos with various masses, the subject to which Chap. III is devoted.

Further, recent reports (e.g., Rohlf, 1985) of anomalous events at the colliders in which the heavy gauge bosons ($W^{\pm}, Z^{0}$) were first detected seem to indicate that the Weinberg-Salam model cannot be the whole story, that new particles might perhaps exist that come as a surprise to everyone.

Finally, the time is rapidly approaching in which the new generation of particle accelerators, with center of mass energies in the TeV range, will likely test all particle theories severely, and conceivably show that much new physics exists in this energy realm (Salam, 1982; Weinberg, 1984; Eichten *et al.*, 1984). It is not unreasonable for theory to attempt to predict just what new physics this might entail.

On the theoretical side, possibly the existence of three (so far) generations or families of particles poses the greatest challenge, especially for the description of neutrinos. Possibly the mysterious neutrinos would be not quite so mysterious if they did possess a variety of masses, like their charged leptonic cousins, and one would like to be able to describe such a hierarchy in the most economical way without disturbing too much the basic Weinberg-Salam model which has come to be considered as basically correct and therefore rather fixed.

A related concern is with what is called the Higgs sector of the theory, a category of spinless particles whose existence seems to be required to make gauge theories work in general and the Weinberg-Salam model in particular. The new generation of particle accelerators and colliders should create such particles if they really exist (still an open question, perhaps: Veltman, 1980), and if more than the one such particle required by the Weinberg-Salam model exists severe constraints would be placed on modifications to the model, and on the grand unified theories that would have to incorporate them in a more general setting.

Finally, and what is most relevant for this thesis, new detection techniques increase by a few orders of magnitude the possibility of observing rare (if at all possible) particle decay modes, in particular the decay of a muon into an electron and a photon (the non-observation of which is responsible for the idea of separate conservation of electron lepton number, muon lepton number, etc.) or into an electron and electron-positron pair. It is most interesting that such decays are not independent of the question of neutrino mass, that the former seems to imply the latter (but massive neutrinos do not inevitably lead to muon decay in the ways mentioned).

Muon decays and massive neutrinos, along with the least disruptive changes to the Weinberg-Salam model that make them possible, are the primary topics of this thesis.

I.2 OVERVIEW AND SUMMARY

There are many ways in which the basic Weinberg-Salam theory can be modified or extended to encompass new physical effects without much changing the model's predictions in the realms that have now been experimentally confirmed. Almost everyone's first preoccupation would be to provide for massive neutrinos and there are a number of ways in which this can be accomplished, almost always by broadening the Higgs sector of the original model (see Sec. II.3 for examples and references).

A generalization long known to accomplish the generation of neutrino mass is to augment the standard doublet or iso-spinor Higgs field with a triplet or iso-vector Higgs field, such a field being a natural representation of the gauge group ($SU(2)\times U(1)$) that seems most suitable for the unification of the electromagnetic and weak interactions.

There is one reason in particular why the vector Higgs field is such a natural supplement to an iso-spinor field. The original complex doublet consists of a charged pair of scalar fields, and an uncharged pair. Although all but one neutral particle of this group is unphysical in the sense that gauges exist in which these particles are absent, it is convenient for now, and especially later when a gauge is chosen to facilitate calculations rather than to dispose of unphysical particles, to consider all four. A triplet Higgs field introduces six new scalar particles, but the point is that they are not independent of the original four -- they blend with them, so to speak. The blend results in an unphysical charged pair, an unphysical neutral scalar particle, a physical neutral scalar particle -- all just as in the original Weinberg-Salam model -- plus another two neutral scalar particles, a new, physical, singly-charged pair of scalar particles and a new, physical, doubly-charged pair. Thus the experimental prospects become enriched: rather than a single, neutral particle to search for, one has three, and in addition, two pairs of charged scalar particles although, unfortunately, all of unknown mass except that one scalar particle is massless. In general, the modified theory is no improvement in this respect.

An independent concern is with the generation of neutrino mass. It will be shown in Chap. III that the triplet Higgs field automatically generates neutrino mass (assuming, of course, that the appropriate coupling constants do not vanish), of the Majorana type, thus sparing us the unpleasant task (because of its obviously arbitrary nature) of having to add *ad hoc* terms to the Lagrangian specifically designed to give mass to the neutrino fields. No further arbitrary terms need be considered.

The $SU(2)\times U(1)$ gauge theory that results seems first to have been considered in detail by Pal and Wolfenstein (1982) who applied it to the problem of massive neutrino decay. This thesis will employ the model to explore its consequences for muon decay, to see if decay modes such as a radiative decay, or a decay into a triplet of electrons (using the term "electron" generically), may possibly come within the range of detectability in comparison with the usual muon decay into an electron and a neutrino pair.

One should note that very general analyses of virtually all simple additions to the Higgs sector have been published in recent years (see the references cited early in Sec. II.3), with results for muon decay, for example, that vary over many orders of magnitude in their predictions. Such generality, however, does not consider the many interesting results that exist within each detailed model, and overlooks the possibility that within a specific model interesting physical predictions may be made, as is possibly the case with the model to be examined in detail here, although our results in no way contradict more general ones. General and specific analyses of gauge theoretic models can certainly co-exist.

The natural starting point for the next chapter is a brief elucidation of the Weinberg-Salam model. This will make clear the notation and sign conventions to be used and display, for later reference, those segments of the theory that will be modified. The triplet or iso-vector Higgs field will be introduced next, and it will be shown how it augments and changes the original Weinberg-Salam theory's predictions, such as, for example, the masses of the gauge bosons and the number of new, scalar Higgs particles. These six new scalar particles, along with the original four, must be grouped into physical and unphysical particles. The $SU(2)\times U(1)$ gauge invariance of the new Higgs field will then be demonstrated in detail.

Before actual calculations can proceed, however, a specific gauge must be chosen. The popular "renormalizable gauge" (or R_ξ gauge, as it is frequently designated) will be explained, and used to show how it cleverly handles the problem of the unphysical Higgs particles and certain of their interaction terms that must be eliminated. Briefly, the gauge gives a gauge-dependent mass to such particles, but otherwise enables one to treat them as if they were real. All trace of these unphysical particles vanishes from the final results, as required, but they are indispensable during the intermediate stages to maintain the theory's gauge invariance.

Finally, Chap. II will conclude with a consideration of the C, P, and T-invariance properties of the theory. There will be no new results here, as is to be expected.

Chapter III will discuss the problem of neutrino mass in general and the two types of neutrinos that can exist. As no new terms beyond the iso-vector Higgs field and its interaction with the lepton sector are to be added to the Weinberg-Salam

Lagrangian, it will be shown how a specific neutrino type -- massive Majorana neutrinos -- emerges from the theory.

The major tools used in detailed calculations with quantum field theories for the past thirty years have been the so-called "Feynman rules" of the particular theory under investigation. Such techniques display the interaction terms of a theory in their most basic and elementary way, enabling one to then select in a relatively simple and straightforward fashion only those terms needed for a specific interaction process, a selection immeasurably aided by the "Feynman diagrams" that accompany and give pictoral representation to the interaction terms. Chapter IV will list the rules and diagrams that follow from the gauge theory described in Chap. II, III, but only those that will be needed in this thesis, a fraction of all that could be listed.

The real calculations of this thesis will begin in Chap. V, which considers the model's implications for the radiative decay of the muon. The calculations are rather lengthy but the final result is not. We will find that, with the exception of the new, physical charged Higgs bosons, the Majorana neutrinos used here do not change the basic result of Cheng and Li (1980b, 1984) concerning this decay (where they employed Dirac neutrinos), namely that it is some 40 (!) or so orders of magnitude less likely than the usual decay mode. But our overall result is quite different. The vector Higgs model contains an interesting peculiarity: the terms involving the physical, charged Higgs particle are actually and remarkably independent of the neutrino masses, enabling us to show that the radiative decay discussed here may actually be some 25 or more orders of magnitude greater than the pessimistic result obtained by Cheng and Li, but still, it seems, some 10 or so orders of magnitude below current detectability.

The other decay mode permitted by our gauge model is the three-electron decay, to which Chap. VI is devoted. This decay mode is possible in three different ways, all, it seems, of the same order of magnitude, and, like the $\mu \rightarrow e\gamma$ decay, appear to be about 10 orders of magnitude below current detectability. It turns out, though, that in each case the $\mu \rightarrow 3e$ decay is more likely than the radiative decay. Numerical estimates of the various decay rates are made and discussed in Chap. VII.

The mathematical notation used throughout this thesis is for the most part conventional, but differs from the usual in at least one notable respect. A manifestly covariant, matrix representation-free, geometric formalism for the Dirac algebra and the SU(2) or Pauli algebra is employed freely, although it seems not to be widely known. It is briefly described in App. B, and at more length and in greater detail in Hamilton (1984a,b) and references contained therein. With this formalism one need not concern oneself with any matrix representaion of the Dirac algebra, which is especially convenient with the C, P, and T transformations which are expressed here in a manifestly covariant way, in marked contrast to the usual representation-dependent transformations (as explained in Sec. II.6 and App. C). For this reason the C, P, and T transformations are considered in some detail.

The matrix-free Dirac algebra is especially useful in Chap. V and App. E in which the few terms with the correct form for the spinor amplitude describing a radiative transition could be extracted with some ease from among the multitude of terms available. Further, the usual "trace" method of evaluating spinor amplitudes is replaced by one in which the scalar component of the algebra is used, which provides, for the first time, a matrix and representation-free method of evaluating spinor amplitudes, a method that is quick, simple and powerful. The basic results are explained in App. B and are used extensively in App. E and F in evaluating the lengthy spinor amplitudes.

In its 3-vector, or SU(2), or Pauli algebraic form, the notation enables one to succinctly express various SU(2) gauge invariant Lagrangian terms for both the spinor and vector Higgs fields, especially the gauge fixing terms of Sec. II.6.

The reader is therefore encouraged to consider App. A and B before embarking on a reading of the main text, which follows next, in which the 3 and 4-vector algebraic notation is used freely and without further explanation.

II. THE SU(2)×U(1) GAUGE THEORY WITH ISO-SPINOR AND ISO-VECTOR HIGGS FIELDS

II.1 QED AND SIGN CONVENTIONS FOR CHARGES AND FIELDS

Quantum electrodynamics (QED) is the quintessential gauge theory, and, having been developed for over thirty years in its modern form, is the quantum field theory after which one hopes to model others. The basic idea of QED as an abelian U(1), or phase invariant, gauge theory is quite simple, as gauge theories go, particularly when confined to the electrodynamic interaction of spin-$\frac{1}{2}$ fermions.

The free particle Dirac Lagrangian for spin-$\frac{1}{2}$, charged particles (with the notation of App. A),

$$\mathcal{L} = \bar{\psi}(i\partial - m)\psi \ , \tag{2.1.1}$$

is coupled minimally (it is said) to the electromagnetic field if the momentum operator $p \equiv i\partial$ changes in the presence of an electromagnetic field as

$$p \rightarrow p + eA \tag{2.1.2a}$$

or

$$\partial \rightarrow \partial - ieA \ , \tag{2.1.2b}$$

where the convention employed here, unfortunately not the usual one (Bjorken and Drell, 1964, Chap. 1; Itzykson and Zuber, 1980, Chap. 2), is $e > 0$, so e is the charge on the positron or proton. In (2.1.2) A is the electromagnetic vector potential, related to the electromagnetic bivector field F (App. B) by

$$F = \partial \wedge A \ , \tag{2.1.3a}$$

or, in component form,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{2.1.3b}$$

With a kinetic term for the electromagnetic field the Lagrangian of QED becomes

$$\mathcal{L} = \bar{\psi}(i\partial - m + eA)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{2.1.4}$$

which is invariant under the change of gauge

$$\psi(x) \rightarrow \exp[ie\alpha(x)]\ \psi(x) \tag{2.1.5a}$$

$$\bar{\psi} \rightarrow \bar{\psi}\ \exp[-ie\alpha(x)] \tag{2.1.5b}$$

$$A(x) \rightarrow A(x) + \partial\alpha(x) \ , \tag{2.1.5c}$$

where $\alpha(x)$ is some continuous and differentiable function of space-time. In practice a "gauge fixing" term (Nash, 1978) must be added to the QED Lagrangian because no Green function exists for the electromagnetic field equations as they follow from (2.1.4). This will be considered later in Sec. II.6 in a more general context.

The interaction Lagrangian that follows from (2.1.4) is

$$\mathcal{L}_I = e\bar{\psi}A\psi \equiv - j_\mu A^\mu = - \mathcal{H}_I \ , \tag{2.1.6}$$

where the electromagnetic current is

$$j^\mu = -e\ \bar{\psi}\gamma^\mu\psi \ , \tag{2.1.7}$$

and $\mathcal{H}_I$ is the interaction Hamiltonian (density). The signs are such that $j \cdot A \sim j_0 A_0$ is a positive energy density for like charges.

The usual expansion of a spinor field in terms of Fock-space creation and annihilation operators is (Bjorken and Drell, 1965, Chap. 13)

$$\psi(x) = (2\pi)^{-3/2} \int d^3p \sqrt{m/\epsilon} \, [e^{-ip\cdot x} a_r(p) \, u_r(p) + e^{ip\cdot x} b_r^*(p) \, v_r(p)] , \qquad (2.1.8)$$

where $\epsilon \equiv p_0$, a^* and b^* are the creation operators for electrons and positrons, respectively, and u, v are basis spinors. Thus the field ψ annihilates negatively charged electrons, creates positively charged positrons, while $\bar{\psi}$ creates electrons and annihilates positrons. The electric charge that results from (2.1.7) and (2.1.8) is

$$Q = \int d^3x \; j^0 = -e \int d^3p \; (a_r^* a_r - b_r^* b_r) \qquad (2.1.9)$$

(where normal ordering of the Fock-space operators has been employed). These results fix our electromagnetic sign conventions, enable us to identify any charged currents and also provide a check on the signs that follow when charged scalar and vector fields are considered.

The remaining sign conventions in the field Lagrangian (such as the negative sign in front of the electromagnetic field term) are fixed by requiring the energy density of free fields to be positive. Thus the Lagrangian for a real, massive scalar field is

$$\mathcal{L} = \tfrac{1}{2} \; \partial\phi\cdot\partial\phi - \tfrac{1}{2} \; m^2 \; \phi^2 \; ; \qquad (2.1.10)$$

for a complex, massive scalar field is

$$\mathcal{L} = \partial\phi^*\cdot\partial\phi - m^2 \; \phi^*\phi \qquad (2.1.11)$$

(which could be written as two real fields $\phi_1 = (1/\sqrt{2})(\phi + \phi^*)$ and $\phi_2 = (-i/\sqrt{2})(\phi - \phi^*)$ of the same mass); for a real, massive vector field is

$$\mathcal{L} = -\tfrac{1}{4}\, F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2}\, m^2\, A_\mu A^\mu \; ; \tag{2.1.12}$$

and for a complex, massive vector field is

$$\mathcal{L} = -\tfrac{1}{2}\, F^*_{\mu\nu} F^{\mu\nu} + m^2\, A^*_\mu A^\mu \; . \tag{2.1.13}$$

In each case the canonical energy-momentum tensor is (Jauch and Rohrlich, 1976, Chap. 1)

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\, \partial^\nu \phi_i - \eta^{\mu\nu}\, \mathcal{L} \tag{2.1.14}$$

(where the i generically represents field components), which satisfies the conservation law

$$\partial_\mu T^{\mu\nu} = 0 \; , \tag{2.1.15}$$

and in each case the (free) field energy

$$P^0 = \int d^3x\; T^{00} \tag{2.1.16}$$

is positive. The Lagrangians (2.1.10) to (2.1.13) will also enable us to identify the masses of any massive scalar or vector fields.

II.2 THE WEINBERG-SALAM MODEL

The standard model of the electromagnetic and weak interactions was formulated in 1967-1968 by Weinberg (1967) and Salam (1968) as an SU(2) × U(1) gauge theory in analogy to QED and the non-abelian gauge theory of Yang and Mills (1954). The model is reviewed in detail in Abers and Lee (1973), Itzykson and Zuber (1980), Langacker (1981), Cheng and Li (1984) and numerous other places.

The particle content of the model, which is almost always formulated first in a gauge theory, is as follows: All particles are initially assumed to be massless, so that it makes sense to consider the left-handed electron and left-handed (electron) neutrino (assumed not to have a right-handed counterpart) as an SU(2) iso-spinor or doublet

$$\Psi = \Psi_L = \binom{\nu}{e}_L \equiv \Sigma_- \binom{\nu}{e} , \qquad (2.2.1)$$

where $\Sigma_\pm \equiv \frac{1}{2}[1 \pm (-i)\ \gamma_5]$ are the chirality projection operators, which happen to be the same as spin projection operators in the case of massless fields. In (2.2.1) ν and e stand for their respective spinor fields. The remaining right-handed electron ($e_R \equiv \Sigma_+ e$) is an SU(2) singlet. There is an SU(2) gauge field $\vec{A} = \vec{A}^\mu\ \gamma_\mu$ and a U(1) gauge field $a = a^\mu\ \gamma_\mu$, analogous to, but not the same as, the electromagnetic field.

The Lagrangian, in analogy with QED in (2.1.4), is

$$\begin{aligned} \mathcal{L}(\Psi,\vec{A},a) = {} & i\ \bar{\Psi}\Sigma_+(\partial - \tfrac{1}{2}\ ig\vec{A} - \tfrac{1}{2}\ ig' Y_L a)\Psi \\ & + i\ \bar{e}\Sigma_-(\partial \qquad\qquad - \tfrac{1}{2}\ ig' Y_R a)e \\ & - \tfrac{1}{4}\ \vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} - \tfrac{1}{4}\ f_{\mu\nu}f^{\mu\nu} , \end{aligned} \qquad (2.2.2)$$

where g, g′ are as yet unknown coupling constants, $Y_{R,L}$ are numbers referred to as the "weak hypercharge", and where

$$\vec{F}_{\mu\nu} \equiv \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + g\ \vec{A}_\mu \times \vec{A}_\nu \qquad (2.2.3a)$$

$$f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu \qquad (2.2.3b)$$

$$\bar{\Psi} \equiv \widehat{\underbrace{\bar{\nu}\quad \bar{e}}} \ . \qquad (2.2.3c)$$

The first factor $\frac{1}{2}$ in (2.2.2) is because vectors are $\vec{A} = A_i\sigma_i$ but ($\frac{1}{2}\ \sigma_i$) are the SU(2) generators; the second is for later convenience. The fields (A_i, a) are taken to be real. This Lagrangian is now augmented by terms for the Higgs field

$$\Phi \equiv \begin{pmatrix} \phi_+ \\ \phi_o \end{pmatrix} , \tag{2.2.4a}$$

an SU(2) spinor or doublet the components of which are complex scalar fields, charged in the sense described by their subscripts. One defines

$$\Phi^\dagger \equiv \overbrace{\underbrace{\phi_+^* \quad \phi_o^*}} = \overbrace{\underbrace{\phi_- \quad \phi_o^*}} , \tag{2.2.4b}$$

so the Lagrangian terms for this field are

$$\begin{aligned} \mathcal{L}(\Phi) = & [(\partial - \tfrac{1}{2}\, ig\vec{A} - \tfrac{1}{2}\, ig' Y_\phi a)\ \Phi]^\dagger \\ & \times [(\partial - \tfrac{1}{2}\, ig\vec{A} - \tfrac{1}{2}\, ig' Y_\phi a)\ \Phi] - V(\Phi) , \end{aligned} \tag{2.2.5}$$

where the self-interaction term is frequently taken to be

$$V(\Phi) = -\mu\ \Phi^\dagger\Phi + \lambda\ (\Phi^\dagger\Phi)^2 ; \tag{2.2.6}$$

however its specific form is not of importance at the present time. It will be considered in Sec. II.4 in connection with the iso-vector Higgs field.

Finally, the Lagrangian is augmented by a Yukawa coupling, an interaction among the scalar and spinor fields:

$$\mathcal{L}(\mathrm{Yuk}) = -\beta_e\ (\bar{\Psi}\Sigma_+\Phi e + \bar{e}\Sigma_-\Phi^\dagger\Psi). \tag{2.2.7}$$

Additional *families* of fermions are readily considered -- one merely adds terms for the muon [$\Psi_\mu = \Sigma_-(\nu_\mu/\mu)$] or tau [$\Psi_\tau = \Sigma_-(\nu_\tau/\tau)$] exactly as was done for the electron family.

The complete Lagrangian is U(1) invariant if

$$\begin{aligned} \Psi_L \to \exp[iY_L\alpha(x)]\ \Psi_L , \quad & e_R \to \exp[iY_R\alpha(x)]\ e_R \\ \Phi \to & \exp[iY_\phi\alpha(x)]\ \Phi \\ a \to & a + (2/g')\ \partial\alpha(x) , \end{aligned} \tag{2.2.8}$$

and from the Yukawa coupling one finds

$$-Y_L + Y_\Phi + Y_R = 0. \tag{2.2.9}$$

The weak hypercharge Y is fixed by defining

$$Q = T_3 + \tfrac{1}{2} Y I , \tag{2.2.10}$$

where Q, T_3, I are SU(2) operators (or matrices) with Q being the charge, T_3 the third (or diagonal) component of the SU(2) generators and I is a unit matrix. One has

$$Y = 2 \, (Q)_{av} , \tag{2.2.11}$$

where $(\cdots)_{av}$ denotes average, so Y for a multiplet is twice the mean electric charge:

$$Y_L = -1, \quad Y_\Phi = +1, \quad Y_R = -2 . \tag{2.2.12}$$

It was for this reason that we began initially with ϕ_+, rather than ϕ_-. The SU(2) invariance of the Lagrangian is considered in detail in Sec. II.5.

The main trick of modern gauge theory -- the Higgs mechanism -- is to suppose that the neutral component of the Higgs field has a non-zero vacuum expectation value ($v/\sqrt{2}$), denoted by $\langle\cdots\rangle$:

$$\Phi \equiv \begin{pmatrix} \phi_+ \\ \phi_0 + v/\sqrt{2} \end{pmatrix} \tag{2.2.13}$$

$$\langle\Phi\rangle \equiv \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} .$$

This destroys the SU(2) and U(1) gauge invariance but leaves a further U(1) gauge invariance which can be interpreted as the electromagnetic gauge invariance. Further, and most important, many of the previously massless particles acquire a mass. The neutrino remains massless but the electron acquires a mass

$$m_e = \beta_e \, v/\sqrt{2} \tag{2.2.14}$$

from (2.2.7), and the real, physical Higgs field $\mathrm{Re}(\Phi_0)$ acquires a mass from $V(\Phi)$, which we need not consider at this time. With the definitions

$$W_{\pm} \equiv (1/\sqrt{2})\ (A_1 \mp i\ A_2) \tag{2.2.15a}$$

$$Z \equiv A_3 \cos\theta_W - a \sin\theta_W \tag{2.2.15b}$$

$$-A \equiv A_3 \sin\theta_W + a \cos\theta_W\ , \tag{2.2.15c}$$

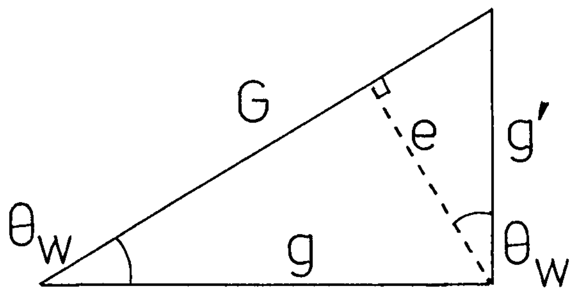

Fig. 1: The Weinberg angle θ_W

---

where the angle θ_W is defined in Fig. 1, with

$$G^2 \equiv g^2 + g'^2 \tag{2.2.16a}$$

$$e \equiv g \sin\theta_W = g' \cos\theta_W = G \cos\theta_W \sin\theta_W\ , \tag{2.2.16b}$$

physical vector fields -- the gauge bosons -- are introduced, which, from (2.2.5) and (2.2.13) can be shown to have the masses

$$M_W^2 = \tfrac{1}{4}\ v^2\ g^2 \tag{2.2.17a}$$

$$M_Z^2 = \tfrac{1}{4}\ v^2\ G^2 \tag{2.2.17b}$$

$$M_A^2 = 0\ . \tag{2.2.17c}$$

The massless field A is the usual electromagnetic vector field.

From (2.2.2) and (2.2.7) the leptonic part of the Lagrangian can be shown to be

$$\mathcal{L}(\text{lepton}) = i\bar{\nu}\Sigma_+\partial\nu + \bar{e}(i\partial - m)e + e\,\bar{e}Ae \tag{2.2.18}$$
$$+ (g/\surd 2)(\bar{\nu}\Sigma_+W_+e + \bar{e}\Sigma_+W_-\nu) + \tfrac{1}{2}G\bar{\nu}\Sigma_+Z\nu$$
$$+ G\sin^2\theta_W\ \bar{e}\Sigma_-Ze - \tfrac{1}{2}G(\cos^2\theta_W - \sin^2\theta_W)\ \bar{e}\Sigma_+Ze$$
$$- \beta_e(\bar{\nu}\Sigma_+\Phi_+e + \bar{e}\Sigma_-\Phi_-\nu + \bar{e}\Sigma_+\Phi_o e + \bar{e}\Sigma_-\Phi_o^* e)\ .$$

The connection with the older Fermi four-point interaction Lagrangian (or V-A theory),

$$\mathcal{L}_I = -(G_F/\surd 2)[\bar{\nu}(1 - i\gamma_5)\gamma_\mu e]\ [\bar{e}(1 - i\gamma_5)\gamma^\mu\nu]\ , \tag{2.2.19}$$

is in the fourth term of (2.2.18), which, with ν or e energies being much smaller then M_W, is effectively

$$G_F/\surd 2 \approx g^2/8M_W^2\ , \tag{2.2.20}$$

because the W propagator is essentially $(\eta^{\mu\nu}/M_W^2)$ at low energies. With

$$G_F \approx 10^{-5}/M_p^2 \tag{2.2.21}$$

from experiment (the μ lifetime, essentially), where M_p is the proton mass, the W mass can be estimated as

$$M_W^2 = \frac{\surd 2\ \ e^2}{8G_F\sin^2\theta_W} \approx \left(\frac{40\ M_p}{\sin\theta_W}\right)^2\ . \tag{2.2.22}$$

Its recently measured value was (Arnison *et al.*, 1983a,c; Banner *et al.*, 1983)

$$M_W \approx 83 \pm 3\ \text{GeV}\ , \tag{2.2.23a}$$

with (Arnison *et al.*, 1983b; Bagnaia *et al.*, 1983)

$$M_Z \approx 93 \pm 2\ \text{GeV}\ , \tag{2.2.23b}$$

and

$$\sin^2\theta_W \approx .23 \ , \tag{2.2.23c}$$

with the important theoretical relation

$$M_W^2 \ / \ M_Z^2 = g^2/(g^2 + g'^2) = \cos^2\theta_W \tag{2.2.24}$$

being rather well verified (Marciano and Sirlin, 1984; Schewe, 1984; Eichten *et al.*, 1984; Reutens *et al.*, 1985). The only unknown now remaining in the original Weinberg-Salam model is the mass of the (physical) Higgs particle: there is no present hint as to its value, if, indeed, such a particle actually exists.

The interactions among the gauge bosons, their self-interactions and the interactions between the gauge bosons and the Higgs particles are obtained by expanding the Lagrangian in terms of these fields. Unfortunately, the resulting Lagrangian is extremely lengthy and extremely complicated, a testament to the intrinsically complicated structure of non-abelian gauge theories (the Weinberg-Salam model being essentially the simplest, realistic model). Among those terms that result from an expansion of the Higgs part of the Lagrangian are

$$\tfrac{1}{2}ivg(\partial_\mu\phi_+W_-^\mu - \partial_\mu\phi_-W_+^\mu) - \tfrac{1}{2}ivGZ^\mu(1/\surd 2)\partial_\mu(\phi_o - \phi_o^*) \tag{2.2.25}$$

which identifies $\phi_\pm$, $(1/\surd 2)(\phi_o - \phi_o^*)$ as *unphysical*, by which is meant that such fields can be transformed away by a suitable gauge transformation. In practice, however, such terms are eliminated by the gauge fixing terms to be considered later, in Sec. II.6. Thus the four-component Higgs field contains only one real, scalar field $(1/\surd 2)(\phi_o + \phi_o^*)$, of undetermined mass as mentioned above. Another term in the expansion is

$$-ieA^{\mu}(\phi_- \partial_{\mu}\phi_+ - \phi_+ \partial_{\mu}\phi_-) \ , \tag{2.2.26}$$

which identifies

$$j = ie\ (\phi_- \partial\phi_+ - \phi_+ \partial\phi_-) \tag{2.2.27}$$

as the (unphysical) current due to the charged Higgs fields, as (2.1.6) shows. Further discussion of the Higgs fields will be postponed until after the next section because the new Higgs fields to be introduced there mix in a natural way with those of this section, a mix that generates physical and unphysical charged and neutral particles.

The interactions among the gauge bosons alone (that is, without the Higgs fields) follow from the seemingly compact kinetic term in the original Weinberg-Salam Lagrangian (2.2.2):

$$\mathcal{L}(W,A,Z) = -\tfrac{1}{4}\ \vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} - \tfrac{1}{4}\ f_{\mu\nu}f^{\mu\nu} \ , \tag{2.2.28}$$

with $\vec{F}$ and f being given in (2.2.3). The extreme nonlinearity and innate complexity of these terms becomes evident when they are expanded in terms of the $W_{\pm}$, A and Z fields, defined in (2.2.15). We ignore fourth order terms, or those of order g^2: these are the lengthy and complicated terms and will not be needed in subsequent calculations. Thus the product of the two $(\vec{A}_{\mu}\times\vec{A}_{\nu})$ terms in (2.2.3a) is to be neglected. The kinetic terms that follow from (2.2.28) are, as is to be expected,

$$\mathcal{L}(W,A,Z\text{:kinetic}) = -\tfrac{1}{2}(\partial_{\mu}W_{\nu}^{-}-\partial_{\nu}W_{\mu}^{-})(\partial^{\mu}W_{+}^{\nu}-\partial^{\nu}W_{+}^{\mu}) \tag{2.2.29}$$
$$- \tfrac{1}{4}(\partial_{\mu}Z_{\nu}-\partial_{\nu}Z_{\mu})(\partial^{\mu}Z^{\nu}-\partial^{\nu}Z^{\mu}) - \tfrac{1}{4}(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu})(\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}).$$

The remaining terms of (2.2.28) describe interactions among the gauge bosons -- interactions, because they depend on the gauge coupling constant g. The final result of reducing (2.2.28) is, except for the terms of order g^2 and the kinetic terms of (2.2.29),

$$
\begin{aligned}
\mathcal{L}(W,A,Z\!:\!\mathrm{int}) = {} & \tfrac{1}{2}ie[(A^{\mu}W^{\nu}_{-}-A^{\nu}W^{\mu}_{-})(\partial_{\mu}W^{+}_{\nu}-\partial_{\nu}W^{+}_{\mu}) \\
& - (A^{\mu}W^{\nu}_{+}-A^{\nu}W^{\mu}_{+})(\partial_{\mu}W^{-}_{\nu}-\partial_{\nu}W^{-}_{\nu}) + (\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu})(W^{\mu}_{-}W^{\nu}_{+}-W^{\nu}_{-}W^{\mu}_{+})] \\
& - \tfrac{1}{2}ig\cos\theta_{W}[(Z^{\mu}W^{\nu}_{-}-Z^{\nu}W^{\mu}_{-})(\partial_{\mu}W^{+}_{\nu}-\partial_{\nu}W^{+}_{\mu}) \\
& - (Z^{\mu}W^{\nu}_{+}-Z^{\nu}W^{\mu}_{+})(\partial_{\mu}W^{-}_{\nu}-\partial_{\nu}W^{+}_{\mu}) + (\partial_{\mu}Z_{\nu}-\partial_{\nu}Z_{\mu})(W^{\mu}_{-}W^{\nu}_{+}-W^{\nu}_{-}W^{\mu}_{+})] \; .
\end{aligned}
\tag{2.2.30}
$$

From the first term of this interaction Lagrangian the charged current can be identified using the definition (2.1.6):

$$
j_{\mu} = -ie[(\partial_{\mu}W^{+}_{\nu}-\partial_{\nu}W^{+}_{\mu})W^{\nu}_{-} - W^{\nu}_{+}(\partial_{\mu}W^{-}_{\nu}-\partial_{\nu}W^{-}_{\mu})] \; . \tag{2.2.31}
$$

This expression correctly gives the total electric charge

$$
Q = -e \int d^{3}k \; (a^{*}_{-}a_{-} - a^{*}_{+}a_{+}) \tag{2.2.32}
$$

when expanded in terms of Fock-space creation and annihilation operators, where $a^{*}_{\pm}$ creates a $W_{\pm}$. This is an independent check on the correctness of the W-A interaction term in (2.2.30).

II.3 <u>THE ISO-VECTOR (TRIPLET) HIGGS FIELD</u>

The Weinberg-Salam gauge theory can be modified or generalized in a number of ways, most notably through an enrichment of the Higgs sector of the theory (Lee and Shrock, 1977; Cheng and Li, 1980a, 1984; Langacker, 1981; Marshak, Riazuddin and Mohapatra, 1981). Since the original Weinberg-Salam model is generally regarded as being basically correct, and therefore fixed, any changes one would make to the theory are determined by further assumptions regarding the masses or interactions of the leptons. It is not difficult, for example, to add simple terms to the Weinberg-Salam Lagrangian to give the neutrinos a mass or to mix the various lepton families. Further, one could add SU(2) singlet Higgs fields, additional SU(2) iso-spinor or doublet Higgs fields, or Higgs fields that constitute other SU(2) representations, such as the iso-vector or triplet representation to be considered presently. Neutrino mass terms will be considered in detail in the next chapter.

There are two basic reasons for considering the Higgs triplet as the most "natural" generalization of the Weinberg-Salam model. First, an unwanted proliferation of Higgs particles is kept to a minimum because the iso-spinor and vector fields combine in a number of cases (neutral and singly charged particles) to form linear combinations of these fields. Second, a mass term -- a Majorana mass term -- is readily constructed from a Yukawa-type interaction between the already existing leptons and the Higgs vector field without adding explicit mass terms for the neutrino. This item, too, is discussed in detail in the next chapter.

Under a general SU(2) transformation a vector field $\vec{H}$ transforms as

$$\vec{H} \rightarrow U\vec{H}U^{-1} , \tag{2.3.1}$$

which is shown in the next section, from which it follows that the "gauge derivative"

$$D\vec{H} \equiv \partial\vec{H} - \tfrac{1}{2}ig(\vec{A}\vec{H}-\vec{H}\vec{A}) - \tfrac{1}{2}ig'Y_H\partial\vec{H} \tag{2.3.2}$$

transforms under the SU(2) transformation (2.3.1) as

$$D\vec{H} \rightarrow U\, D\vec{H}\, U^{-1} \ . \tag{2.3.3}$$

The second term of (2.3.2) could be written $(-ig\vec{A}\wedge\vec{H})$ or $(g\vec{A}\times\vec{H})$, in the notation of App. B, so is just an SU(2) 3-vector. Thus the kinetic or field part to be added to the original Weinberg-Salam Lagrangian is

$$\mathcal{L}(\vec{H}) = S[(D_\mu\vec{H})^\dagger(D^\mu\vec{H})] = (D_\mu\vec{H})_i^\dagger(D^\mu\vec{H})_i \ , \tag{2.3.4}$$

where $S[\cdots]$ refers to the scalar part of the SU(2) or Pauli algebra, as discussed in App. B. In a basis (σ_i), where

$$\vec{H} = H_i\ \sigma_i \ , \tag{2.3.5}$$

the derivative part of (2.3.4) is

$$S[(\partial_\mu\vec{H})^\dagger(\partial^\mu\vec{H})] = \partial_\mu H_i^*\ \partial^\mu H_i \ , \tag{2.3.6}$$

the correct form for each of the three complex scalar fields H_i. If the Pauli matrices are used to represent the basis vectors one obtains

$$\vec{H} = \begin{pmatrix} H_3 & H_1 - iH_2 \\ H_1 + iH_2 & -H_3 \end{pmatrix} = \begin{pmatrix} H_+ & \sqrt{2}H_{++} \\ \sqrt{2}H_0 & -H_+ \end{pmatrix} \ , \tag{2.3.7}$$

where the charge assignments have been made on the basis of hindsight and the $\sqrt{2}$ factors are added as a required normalization because (2.3.6) becomes

$$\partial_\mu H_0^*\partial^\mu H_0 + \partial_\mu H_+^*\partial^\mu H_+ + \partial_\mu H_{++}^*\partial^\mu H_{++} \ . \tag{2.3.8}$$

The weak hypercharge assignment (2.2.11) gives

$$Y_H = +2 \ . \tag{2.3.9}$$

Now, in analogy with the original Higgs doublet one assumes that

$$\langle H_0 \rangle \equiv v_H/\sqrt{2} \ , \tag{2.3.10}$$

so that (2.3.4) makes a contribution to the W and Z masses when it is expanded in terms of the W, A and Z fields. Using $\langle\vec{H}\rangle$ in place of $\vec{H}$ in (2.3.4) results in

$$\mathcal{L}(\langle\vec{H}\rangle) = \tfrac{1}{2}v_H^2 g^2 W^-_\mu W^\mu_+ + \tfrac{1}{2}v_H^2 G^2 Z_\mu Z^\mu \ , \tag{2.3.11}$$

so the W and Z masses become, with the result (2.2.17) from the doublet Higgs fields

$$M_W^2 = \tfrac{1}{4}v^2 g^2 + \tfrac{1}{2}v_H^2 g^2 \tag{2.3.12a}$$

$$M_Z^2 = \tfrac{1}{4}v^2 G^2 + v_H^2 G^2 \ . \tag{2.3.12b}$$

Experimental verification of (2.2.24), however, implies that

$$v_H << v \ , \tag{2.3.13}$$

(Arnison *et al.*, 1983a,c; Bagnaia *et al.*, 1983; Marciano and Sirlin, 1984; Reutens *et al.*, 1985) an inequality that will later be of some importance.

In principle and in general a complicated interaction between the Φ and $\vec{H}$ fields should, in addition to $V(\Phi)$ of (2.2.6), and a $V(\vec{H})$ term, be added to the general Lagrangian and, when "minimized", should determine the masses of the physical Higgs particles and establish which are the physical, and which the unphysical, particles. This is the subject of the next section. In the sequel we will use (2.3.13) and will need a charged Higgs particle of unknown mass.

From (2.3.4) there follows a term

$$(1/\sqrt{2})iv_H g(W^\mu_- \partial_\mu H_+ - W^\mu_+ \partial_\mu H_-) - (1/\sqrt{2})iv_H G Z^\mu \partial_\mu (H_0 - H_0^*) \ , \tag{2.3.14}$$

which, with (2.2.25), identifies

$$S_\pm \equiv \phi_\pm \cos\theta + H_\pm \sin\theta \ , \tag{2.3.15a}$$

with

$$\tan\theta \equiv \sqrt{2}\, v_H/v \ , \tag{2.3.15b}$$

as an unphysical, charged Higgs field, unphysical in the sense of the remarks following (2.2.25). The orthogonal combination

$$B_\pm \equiv -\Phi_\pm \sin\theta + H_\pm \cos\theta \tag{2.3.16}$$

turns out to be a physical field of considerable importance to this thesis. From (2.2.25) and (2.3.14) one sees that the combination

$$(1/\sqrt{2})[v(\Phi_0-\Phi_0^*) + 2v_H(H_0-H_0^*)] \tag{2.3.17}$$

is also unphysical. This illustrates the remark made earlier, that the vector Higgs field has a natural relationship with the spinor Higgs field.

Besides the gauge fixing terms to be considered later, in Sec. II.5, there remains the Yukawa-type coupling between the leptons and the vector Higgs fields. The only SU(2) and U(1) gauge invariant interaction is, as shown in Sec. II.5,

$$\mathcal{L}(\text{Yuk}-\vec{H}) = -\Sigma_{ij}\beta_{ij}[\bar{\Psi}_i^c\Sigma_-(i\sigma_2\vec{H})\Psi_j + \bar{\Psi}_j\Sigma_+(i\sigma_2\vec{H})^\dagger\Psi_i^c] \ , \tag{2.3.18}$$

where $i,j = 1,2,\cdots = e,\mu,\cdots$, ψ^c is the charge conjugate of the spinor field ψ (App. C), and where $(i\sigma_2\vec{H})$ is needed rather than $\vec{H}$ alone because in the usual matrix representation for the (σ_i),

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2.3.19a}$$

and

$$\sigma_2\ U^*\ \sigma_2 = U \ , \tag{2.3.19b}$$

where U is an SU(2) transformation. The factor i is to make $(i\sigma_2)$ real, as are the β_{ij} coupling coefficients because of CPT invariance. When expanded using (2.3.7) and (2.3.10) the interaction term becomes

$$
\begin{aligned}
\mathcal{L}(\mathrm{Yuk}-\vec{H}) = & -\Sigma_{ij}\beta_{ij}v_H(\bar{\nu}_i^c\Sigma_-\nu_j + \bar{\nu}_j\Sigma_+\nu_i^c) \\
& -\Sigma_{ij}\beta_{ij}[\bar{\nu}_i^c\Sigma_-(\sqrt{2}H_0\nu_j - H_+\ell_j) + \bar{\ell}_i^c\Sigma_-(-H_+\nu_j - \sqrt{2}H_{++}\ell_j) \\
& + \bar{\nu}_j\Sigma_+(\sqrt{2}H_0^*\nu_i^c - H_-\ell_i^c) + \bar{\ell}_j\Sigma_+(-H_-\nu_i^c - \sqrt{2}H_{--}\ell_i^c)] ,
\end{aligned}
\tag{2.3.20}
$$

where $\ell_1, \ell_2, \cdots = e, \mu, \cdots$, which justifies the charge assignments of the components of the vector Higgs field $\vec{H}$. This important set of interaction terms will be considered in detail in the next two sections and in Chap. III. The first term is the important Majorana mass term.

II.4 THE PHYSICAL AND UNPHYSICAL HIGGS FIELDS

It has already become apparent, as in (2.2.25) and (2.3.14), that not all of the ten Higgs fields so far introduced, nor arbitrary linear combinations of them, can be physical, or the source of real particles one might encounter in an actual experiment. Just as in the original Weinberg-Salam model there must be three unphysical fields that could, in principle, be gauge-transformed away so as to provide for the extra longitudinal states needed to make the $W_\pm$, Z_0 fields massive.

Not all of the physical and unphysical fields can be identified as readily as those obtained from (2.2.25) and

(2.3.14). In particular, the neutral, physical Higgs fields can only be established in the following manner.

To the general Lagrangian must be added an interaction term $V(\Phi,\vec{H})$ that describes the self-interactions among the fields and the other interactions among them, and one that is SU(2)×U(1) gauge invariant. A general such potential term we will take to be

$$V(\Phi,\vec{H}) = \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 + \mu_H^2 S[\vec{H}^\dagger\vec{H}] - \lambda_H(S[\vec{H}^\dagger\vec{H}])^2$$
$$- \lambda'_H S[\vec{H}^\dagger\vec{H}\vec{H}^\dagger\vec{H}] + a\Phi^\dagger\Phi S[\vec{H}^\dagger\vec{H}] + bS[\vec{H}^\dagger\vec{H}\Phi\Phi^\dagger] , \quad (2.4.1)$$

where the global symmetry $\Phi \to -\Phi$, $\vec{H} \to -\vec{H}$ has been imposed, as is conventional, to simplify the types of terms needed, where $S[\cdots] = \frac{1}{2}\mathrm{Tr}[\cdots]$ is the scalar part of the Pauli algebra described in App. B, and where the coefficients are generally unknown. Other terms, such as $S[\vec{H}^\dagger\vec{H}\vec{H}\vec{H}^\dagger]$ or $S[\vec{H}\vec{H}^\dagger\Phi\Phi^\dagger]$ could have been added but make no changes in principle to the final result. This potential and its implications seem to have been first considered by Gelmini and Roncadelli (1981) and subsequently considered in further detail by Georgi, Glashow and Nussinov (1981) who, in particular, modify the potential above so as to have $V(\langle\Phi\rangle,\langle\vec{H}\rangle) = 0$, but which is not necessary here because our intention is only to identify the physical states.

The iso-spinor Higgs field Φ is written as

$$\Phi = \phi_+\chi_+ + (\phi_0+v/\sqrt{2})\chi_- \quad (2.4.2)$$

in terms of the Pauli spinors $\chi_\pm$ (App. B), with $\langle\phi_0\rangle = 0$, and the iso-vector Higgs field is written as in (2.3.5) as

$$\vec{H} = (H_0/\sqrt{2} + \tfrac{1}{2}v_H)(\sigma_1 - i\sigma_2) + (H_{++}/\sqrt{2})(\sigma_1 + i\sigma_2) + H_+\sigma_3 \ , \qquad (2.4.3)$$

with $\langle H_0 \rangle = 0$. Now rather than "minimize" the potential $V(\Phi,\vec{H})$ it is sufficient for the purposes required here to adjust the coefficients of (2.4.1) so as to eliminate linear terms in the fields. On expanding (2.4.1) to second order in the fields one obtains

$$\mu^2 v - \lambda v^3 + \tfrac{1}{2}avv_H^2 \equiv 0 \qquad (2.4.4)$$

as the coefficient of $(\phi_0+\phi_0^*)/\sqrt{2}$, and

$$\mu_H^2 v_H - \lambda_H v_H^3 - 2\lambda_H' v_H^3 + \tfrac{1}{2}av^2 v_H \equiv 0 \qquad (2.4.5)$$

as the coefficient of $(H_0+H_0^*)/\sqrt{2}$. The coefficients of $\phi_0\phi_0^*$ and $H_0H_0^*$ are, respectively, proportional to (2.4.4) and (2.4.5) and thus also vanish.

The coefficient of $H_{--}H_{++}$ gives the mass of the doubly-charged Higgs field:

$$\begin{aligned} -M^2(H_{\pm\pm}) &\equiv \mu_H^2 - \lambda_H v_H^2 + \tfrac{1}{2}av^2 + \tfrac{1}{2}bv^2 \\ &= \tfrac{1}{2}bv^2 + 2\lambda_H' v_H^2 \ . \end{aligned} \qquad (2.4.6)$$

What remain are terms for the singly charged Higgs fields $(\phi_\pm, H_\pm)$ and the real parts of ϕ_0 and H_0, namely $(\phi_0+\phi_0^*)/\sqrt{2}$ and $(H_0+H_0^*)/\sqrt{2}$. After (2.4.4) and (2.4.5) have been employed, the coefficient of $\phi_+\phi_-$ is $\tfrac{1}{2}bv_H^2$, of H_+H_- is $\tfrac{1}{4}bv^2$, and of $(H_+\phi_- + H_-\phi_+)$ is $-(1/2\sqrt{2})bvv_H$. Thus these fields are linear combinations of fields

$$\begin{aligned} \phi_\pm &\equiv S_\pm \cos\theta - B_\pm \sin\theta \\ H_\pm &\equiv S_\pm \sin\theta + B_\pm \cos\theta \ , \end{aligned} \qquad (2.4.7)$$

where

$$\cos\theta = v/(v^2+2v_H^2)^{1/2} \quad , \quad \sin\theta = \sqrt{2}v_H/(v^2+2v_H^2)^{1/2} \quad , \tag{2.4.8}$$

with mass terms

$$-M^2(S_{\pm}) = 0 \tag{2.4.9}$$

$$-M^2(B_{\pm}) = (\tfrac{1}{4}bv^4+bv^2v_H^2+bv_H^4)/(v^2+2v_H^2)$$

$$\approx \tfrac{1}{4}bv^2 \quad ,$$

provided $v_H \ll v$, which gives in such a case

$$M(H_{\pm\pm}) \approx \sqrt{2}\ M(B_{\pm}) \quad , \tag{2.4.10}$$

as noted by Georgi, Glashow and Nussinov (1981), but which is probably not justified because the coefficients in $V(\Phi,\vec{H})$ are quite unknown, even if $v_H \ll v$. Thus $S_{\pm}$ is unphysical and $B_{\pm}$ is physical: these are precisely the fields identified earlier, in (2.3.15).

For the physical, neutral, scalar Higgs fields that arise from $\phi_1 \equiv (\phi_0+\phi_0^*)/\sqrt{2}$ and $H_1 \equiv (H_0+H_0^*)/\sqrt{2}$, $V(\Phi,\vec{H})$ gives the terms

$$-(\mu^2+\tfrac{1}{2}av_H^2)\phi_1^2 - (\mu_H^2+\tfrac{1}{2}av^2)H_1^2 + avv_H\phi_1 H_1 \quad , \tag{2.4.11}$$

where (2.4.4) and (2.4.5) have been used, which is set equal to

$$-\tfrac{1}{2}M_\phi^2\phi^2 - \tfrac{1}{2}M_\omega^2\omega^2 \tag{2.4.12}$$

where

$$\phi_1 \equiv \phi\cos\alpha + \omega\sin\alpha \tag{2.4.13}$$

$$H_1 \equiv -\phi\sin\alpha + \omega\cos\alpha \quad ,$$

where ϕ and ω are the physical fields, and one finds

$$\tan 2\alpha = (avv_H)/[\lambda v^2-(\lambda_H+2\lambda_H')v_H^2] \tag{2.4.14}$$

$$\approx (a/\lambda)(v_H/v) \quad ,$$

provided $v_H \ll v$, and

$$\tfrac{1}{2}M_\phi^2 = \lambda v^2\cos^2\alpha + (\lambda_H+2\lambda_H')v_H^2\sin^2\alpha + avv_H\sin\alpha\cos\alpha \quad , \tag{2.4.15a}$$

$$\tfrac{1}{2}M_\omega^2 = \lambda v^2 \sin^2\alpha + (\lambda_H + 2\lambda'_H) v_H^2 \cos^2\alpha - a v v_H \sin\alpha \cos\alpha \, , \tag{2.4.15b}$$

so that, again if $v_H << v$,

$$M_\Phi >> M_\omega \tag{2.4.16}$$

and

$$\alpha \approx \tfrac{1}{2}(a/\lambda)(v_H/v) << 1 \tag{2.4.17}$$

(provided $a \sim \lambda$) and so $\Phi \approx \phi_1$, $\omega \approx H_1$. Note that ϕ_1 was the only physical field in the original Weinberg-Salam model.

If we now define $\phi_2 \equiv -i(\phi_0-\phi_0^*)/\surd 2$ and $H_2 \equiv -i(H_0-H_0^*)/\surd 2$, then (2.3.17) gives

$$\Phi' \equiv \phi_2 \cos\theta' + H_2 \sin\theta' \tag{2.4.18}$$

as the third unphysical field, where

$$\cos\theta' = v/(v^2+4v_H^2)^{1/2} \, , \quad \sin\theta' = 2v_H/(v^2+4v_H^2)^{1/2} \, , \tag{2.4.19}$$

while the orthogonal combination

$$\omega' \equiv -\phi_2 \sin\theta' + H_2 \cos\theta' \tag{2.4.20}$$

is physical but *massless*, since from $V(\Phi,\vec{H})$ the coefficient of ω'^2 is 0. This massless Goldstone boson cannot be avoided (Gelmini and Roncadelli, 1981; Georgi, Glashow and Nussinov, 1981): it is referred to as a *majoron*, and is of considerable current interest (Gelmini, Nussinov and Roncadelli, 1982; Dugan *et al.*, 1985; Glashow and Manohar, 1985).

The physical and unphysical Higgs bosons, and their relation to the original Higgs fields, are listed in Table I.

The presence of a massless Higgs particle is a signal that a symmetry of the theory has been broken by the presence of the non-zero vacuum expectation value of the iso-vector Higgs field, a consequence known as Goldstone's theorem (Bernstein, 1974; Cheng and Li, 1984, Sec. 5.3). The global symmetry violated here is related to lepton-number conservation.

As can be seen in (2.2.2) and (2.2.7), written out in detail in (2.2.18), the original Weinberg-Salam model separately conserves electron lepton number, muon lepton number, etc., because of the independent global symmetries

$$(e,\nu_e) \to e^{i\theta_1} (e,\nu_e) \qquad (2.4.21)$$
$$(\mu,\nu_\mu) \to e^{i\theta_2} (\mu,\nu_\mu) \ ,$$

where θ_1, θ_2, etc., are constants. From Noether's theorem, then (Jauch and Rohrlich, 1976, Chap. 1),

$$\partial_\nu \Sigma_\psi \left(\int d^4x \ \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi)} \ \delta\psi \right) = 0 \ , \qquad (2.4.22)$$

where ψ represents the various fields in the Lagrangian, we have the conserved electron lepton number

$$L_e = \int d^3x \ (\bar{e}\gamma^0 e + \bar{\nu}_e \Sigma_+ \gamma^0 \nu_e) \ , \qquad (2.4.23)$$

with similar expressions for the other lepton families.

The iso-vector Higgs field changes all this. From (2.3.20) it can be seen that, before spontaneous symmetry breaking (i.e., before $H_0 \to \langle H_0 \rangle = v_H/\sqrt{2}$), there is the *one* global symmetry

$$(\text{all leptons}) \to e^{i\theta} (\text{all leptons}) \qquad (2.4.24)$$

(with $\bar{\ell} \to e^{-i\theta}\bar{\ell}$, $\ell^c \to e^{-i\theta}\ell^c$, etc.) *provided* one also imposes

$$(H_{++}, H_+, H_0) \to e^{-2i\theta} (H_{++}, H_+, H_0) \ , \qquad (2.4.25)$$

which means that the iso-vector Higgs field must also be given a lepton number (which is reasonable because we now have the possibility of processes such as $e^- + e^- \to H_{--}$). If N(e) stands for the number of particles of type e, then what is conserved is

$$N(e^-) - N(e^+) + N(\mu^-) - N(\mu^+) + \cdots + N(\Sigma_- \nu_e) - N(\Sigma_+ \nu_e^c) + \cdots$$
$$- 2N(H_{++} + H_+ + H_0) + 2N(H_{--} + H_- + H_0^*) \ . \qquad (2.4.26)$$

From the first term of (2.3.20), however, if $v_H \neq 0$, even this one global symmetry is violated, because, for example, processes like $\Sigma_- \nu_e \to \Sigma_+ \nu_\mu^c$ are ostensibly permitted. Thus the peculiar violation of this global lepton-number conservation is by the Majorana neutrino mass term.

TABLE I: THE PHYSICAL AND UNPHYSICAL HIGGS FIELDS

Original Fields	Physical Fields	Unphysical Fields
$\Phi_1, \Phi_2, \Phi_\pm$	$H_{\pm\pm}$	
$H_1, H_2, H_\pm, H_{\pm\pm}$	$B_\pm \equiv \Phi_\pm \sin\theta + H_\pm \cos\theta$	$S_\pm \equiv \Phi_\pm \cos\theta - H_\pm \sin\theta$
	$(\tan\theta = \sqrt{2} v_H/v)$	
$\Phi_1 \equiv (\Phi_0 + \Phi_0^*)/\sqrt{2}$	$\Phi \equiv \Phi_1 \cos\alpha - H_1 \sin\alpha$	
$H_1 \equiv (H_0 + H_0^*)/\sqrt{2}$	$\omega \equiv \Phi_1 \sin\alpha + H_1 \cos\alpha$	
	$[\tan 2\alpha \approx (a/\lambda)(v_H/v)]$	
$\Phi_2 \equiv -i(\Phi_0 - \Phi_0^*)/\sqrt{2}$	$\omega' \equiv -\Phi_2 \sin\theta' + H_2 \cos\theta'$	$\Phi' \equiv \Phi_2 \cos\theta' + H_2 \sin\theta'$
$H_2 \equiv -i(H_0 - H_0^*)/\sqrt{2}$	$(\tan\theta' = 2v_H/v)$	
	$[\text{mass}(\omega')=0]$	

II.5 GAUGE INVARIANCE OF THE THEORY

The invariance properties of non-abelian gauge theories have been understood since the first such models were formulated by Yang and Mills (1954) and have been extensively reviewed since (in, for example, Abers and Lee, 1973; Itzykson and Zuber, 1980, Chap. 12). The special case of an SU(2) gauge theory can, however, be appreciably simplified by using the Pauli algebra described in App. B because elementary 3-vector algebra can be used, rather than the group theoretical techniques required for the general case.

The U(1) invariance of the original Weinberg-Salam Lagrangian (2.2.2) is quite readily demonstrated when the U(1) transformation (2.2.8) is applied, while for the vector Higgs field one needs

$$\vec{H} \to \exp[iY_H\alpha(x)]\ \vec{H} \tag{2.5.1}$$

and

$$\Psi^c \to \exp[-iY\alpha(x)]\ \Psi^c \tag{2.5.2a}$$

if

$$\Psi \to \exp[iY\alpha(x)]\ \Psi\ , \tag{2.5.2b}$$

and the Yukawa interaction (2.3.18) is also seen to be U(1) invariant.

In the case of the SU(2) transformation one has

$$\Psi_L \to U\Psi_L \,, \qquad e_R \to e_R \,, \qquad \Phi \to U\Phi \tag{2.5.3}$$
$$\vec{A} \to U \vec{A} U^{-1} - (2i/g)\ \partial U\ U^{-1}$$

which results in, as the references cited above show,

$$(\partial - \tfrac{1}{2}ig\vec{A})\Psi \to U(\partial - \tfrac{1}{2}ig\vec{A})\Psi \tag{2.5.4}$$
$$\vec{F}_{\mu\nu} \equiv \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu \to U\ \vec{F}_{\mu\nu}\ U^{-1} \,,$$

so the Weinberg-Salam Lagrangian is seen to be SU(2) gauge invariant as well. These last relations are more readily demonstrated when

$$\vec{a}\times\vec{b} = -i\vec{a}\wedge\vec{b} = -\tfrac{1}{2}i(\vec{a}\vec{b}-\vec{b}\vec{a}) \tag{2.5.5}$$

is used, as described in App. B.

The SU(2) gauge derivative in the vector or triplet case is not (Pal and Wolfenstein, 1982, Eq. (44))

$$D\vec{H} = (\partial - \tfrac{1}{2}ig\vec{A})\vec{H}$$

but

$$D\vec{H} = \partial\vec{H} - \tfrac{1}{2}ig(\vec{A}\vec{H}-\vec{H}\vec{A}) = \partial\vec{H} + g\vec{A}\times\vec{H} \,, \tag{2.5.6}$$

as stated in (2.3.2), because a 3-vector transforms as

$$\vec{H} \to U\ \vec{H}\ U^{-1} \,, \tag{2.5.7}$$

not as

$$\vec{H} \to U\ \vec{H} \,.$$

To show this one uses (2.5.3) and (2.5.6) to obtain

$$D\vec{H} \to \partial(U\vec{H}U^{-1}) - ig[U\vec{A}U^{-1}-(2i/g)\partial UU^{-1}]\wedge U\vec{H}U^{-1} \tag{2.5.8}$$
$$= U\partial\vec{H}U^{-1}+\partial UU^{-1}U\vec{H}U^{-1}-U\vec{H}U^{-1}\partial UU^{-1}-igU\vec{A}U^{-1}\wedge U\vec{H}U^{-1}-2\partial UU^{-1}\wedge U\vec{H}U^{-1}$$
$$= U(\partial\vec{H}-ig\vec{A}\wedge\vec{H})U^{-1} = U\ D\vec{H}\ U^{-1}$$

as required, where

$$\partial(U^{-1}) = -U^{-1}\ \partial U\ U^{-1} \tag{2.5.9}$$

has been employed. Thus $S[(D_\mu\vec{H})^\dagger(D^\mu\vec{H})]$ is SU(2) invariant since

$$S[\vec{A}\vec{A}] = \vec{A}\cdot\vec{A} = \vec{B}\cdot\vec{B} \qquad (2.5.10a)$$

if

$$\vec{B} = U\,\vec{A}\,U^{-1}\ . \qquad (2.5.10b)$$

The Yukawa term (2.3.18) is also SU(2) gauge invariant because if

$$\Psi \to U\Psi\ , \quad \bar{\Psi} \to \bar{\Psi}U^{-1} \qquad (2.5.11a)$$

then

$$\Psi^c = \Psi^* \to U^*\Psi^* = U^*\Psi^c \qquad (2.5.11b)$$
$$\bar{\Psi}^c \to \bar{\Psi}^c(U^{-1})^*$$

and so

$$\bar{\Psi}^c\sigma_2\vec{H}\Psi \to \bar{\Psi}^c(U^{-1})^*\sigma_2 U\vec{H}U^{-1}U\Psi \qquad (2.5.12)$$
$$= \bar{\Psi}^c\sigma_2 U^{-1}U\vec{H}\Psi = \bar{\Psi}^c\sigma_2\vec{H}\Psi\ ,$$

as is also required, where (2.3.19) has been used.

Thus the U(1) and SU(2) gauge invariance of the additional Higgs vector terms has been demonstrated. This gauge invariance, however, exists only initially. It is broken not only by the Higgs mechanism (the vacuum is not invariant) but also by the specific gauge fixing terms which are about to be considered.

II.6 GAUGE FIXING TERMS: THE GENERAL R_ξ GAUGE

Although the gauge invariance of the model to be employed here has been demonstrated, it is not possible to do any meaningful calculations using it. There are at least two reasons for this. No Green function or propagator can be constructed for any massless vector field (such as the photon field) because the differential operator on the field that follows from the Lagrangian has no inverse -- a definite gauge must be selected (Abers and Lee, 1973; Nash, 1978; see also Sec. IV.3). Secondly, there are interaction terms such as (2.2.25) or (2.3.14) and (2.3.17) that are quite absurd because a vector particle could never decay into a scalar particle. Such scalar Higgs fields are unphysical because they can be eliminated by a judicious choice of gauge.

It is a great convenience that both problems can be eliminated together (t'Hooft, 1971), in the so-called renormalizable gauge, designated R_ξ, where ξ is the gauge parameter about to be defined. This is accomplished by adding terms to the original Lagrangian -- gauge fixing terms -- that exactly cancel the unwanted two-particle terms. Although the unphysical fields are not eliminated entirely (unless one uses what is referred to as the unitary gauge, in which $\xi \to \infty$) they can be conveniently manipulated as though they were real scalar particles, the masses of which are dependent on the gauge parameter ξ. Of course, at the completion of any calculation concerning real, physical processes all trace of these fictitious particles must have disappeared: the gauge parameter ξ must cancel everywhere.

The two gauge fixing terms to be added to the complete Lagrangian are

$$\mathcal{L}(\text{gauge fix 1}) = -(2\xi)^{-1} \left(\partial_\mu \vec{A}^\mu - ig\xi V[\langle\Phi\rangle\Phi^\dagger - \Phi\langle\Phi^\dagger\rangle + \langle\vec{H}\rangle\vec{H}^\dagger - \vec{H}\langle\vec{H}^\dagger\rangle]\right)^2 \tag{2.6.1}$$

and

$$\mathcal{L}(\text{gauge fix 2}) = -(2\xi)^{-1} \left(\partial_\mu a^\mu - ig'Y_\Phi \xi S[\langle\Phi\rangle\Phi^\dagger - \Phi\langle\Phi^\dagger\rangle] - ig'\tfrac{1}{2}Y_H \xi S[\langle\vec{H}\rangle\vec{H}^\dagger - \vec{H}\langle\vec{H}^\dagger\rangle]\right)^2, \tag{2.6.2}$$

which have been written in terms of the Pauli algebra of App. B. Here $V[\cdots]$ means the 3-vector part of whatever exists inside the square brackets, and $S[\cdots]$ refers to the scalar part. The gauge fixing terms have not been written this way before but they clearly exhibit a striking similarity not only between the SU(2) gauge fixing term of (2.6.1) and the U(1) gauge fixing term of (2.6.2), but also show a similarity in structure between terms for the original spinor Higgs field Φ and the newly added Higgs field $\vec{H}$.

With the field Φ written in terms of 2-component Pauli spinors $(\chi_\pm)$ (App. B)

$$\Phi = \phi_+\chi_+ + \phi_0\chi_- \tag{2.6.3a}$$

$$\Phi^\dagger = \phi_-\chi_+^\dagger + \phi_0^*\chi_-^\dagger \tag{2.6.3b}$$

$$\langle\Phi\rangle = (v/\sqrt{2})\,\chi_-\,, \tag{2.6.3c}$$

and the vector Higgs written explicitly as in (2.3.7),

$$\vec{H} = H_i\sigma_i = (1/\sqrt{2})(H_0+H_{++})\sigma_1 + (-i/\sqrt{2})(H_0-H_{++})\sigma_2 + H_+\sigma_3 \tag{2.6.4a}$$

$$\langle\vec{H}\rangle = \tfrac{1}{2}v_H\ (\sigma_1 - i\sigma_2)\ , \tag{2.6.4b}$$

and using the important spinor-vector connection (App. B)

$$x_\pm x_\pm^\dagger = \tfrac{1}{2}(1 \pm \sigma_3) \tag{2.6.5a}$$

$$x_- x_+^\dagger = \tfrac{1}{2}(\sigma_1 - i\sigma_2) \tag{2.6.5b}$$

$$x_+ x_-^\dagger = \tfrac{1}{2}(\sigma_1 + i\sigma_2) \tag{2.6.5c}$$

results in, after a lengthy manipulation,

$$\mathcal{L}(gf1) = -(2\xi)^{-1}\{\partial_\mu \vec{A}^\mu - ig\xi[(M_W/\surd 2g)(\sigma_1(S_- - S_+) - i\sigma_2(S_- + S_+)) + \sigma_3((v/2\surd 2)(\phi_0 - \phi_0^*) + (v_H/\surd 2)(H_0 - H_0^*))]\}^2 \tag{2.6.6}$$

$$\mathcal{L}(gf2) = -(2\xi)^{-1}\{\partial_\mu a^\mu + ig'\xi[(v/2\surd 2)(\phi_0 - \phi_0^*) + (v_H/\surd 2)(H_0 - H_0^*)]\}^2\ , \tag{2.6.7}$$

where $\frac{1}{2}Y_H = Y_\phi = 1$ has been used. The expansion of (2.6.6) and (2.6.7) is

$$\begin{aligned}\mathcal{L}(gf1) = & -(2\xi)^{-1}\partial_\mu A_i^\mu \partial_\nu A_i^\nu - iM_W(W_+^\mu \partial_\mu S_- - W_-^\mu \partial_\mu S_+) - \xi M_W^2 S_+ S_- \\ & - igA_3^\mu[(v/2\surd 2)\partial_\mu(\phi_0 - \phi_0^*) + (v_H/\surd 2)\partial_\mu(H_0 - H_0^*)] \\ & + \tfrac{1}{2}g^2\xi[(v/2\surd 2)(\phi_0 - \phi_0^*) + (v_H/\surd 2)(H_0 - H_0^*)]^2\end{aligned} \tag{2.6.8}$$

and

$$\begin{aligned}\mathcal{L}(gf2) = & -(2\xi)^{-1}\partial_\mu a^\mu \partial_\nu a^\nu \\ & + ig' a^\mu[(v/2\surd 2)\partial_\mu(\phi_0 - \phi_0^*) + (v_H/\surd 2)\partial_\mu(H_0 - H_0^*)] \\ & + \tfrac{1}{2}g'^2\xi[(v/2\surd 2)(\phi_0 - \phi_0^*) + (v_H/\surd 2)(H_0 - H_0^*)]^2\ ,\end{aligned} \tag{2.6.9}$$

with the following consequences. The first terms of each of these expressions fix the gauge for the gauge bosons; the second term of (2.6.8) and the combination of the fourth term of (2.6.8) with the second term of (2.6.9) exactly cancel the unwanted interaction terms (2.2.25) and (2.3.14); the third term of (2.6.8) gives the mass of the unphysical field $S_\pm$ as

$$M_S^2 = \xi\ M_W^2\ , \tag{2.6.10}$$

and the last terms of (2.6.8) and (2.6.9) give the mass term for the unphysical, neutral Higgs particle as $M_\Phi^2 = \xi M_Z^2$. One should note that the unphysical masses depend on the (arbitrary) gauge parameter ξ.

With the gauge fixing terms of this section and the Lagrangian of Sec. II.2, II.3, we now have the complete gauge theory with both iso-spinor and vector Higgs fields. Before the many interaction terms can be turned into practical "Feynman rules" and applied to the physical process under investigation here -- muon decay -- the neutrino terms must be reformulated and properly interpreted to make sense as those for a realistic particle. First, however, the important discrete symmetries of the theory -- the C, P and T transformations -- will be briefly considered.

II.7 C, P AND T INVARIANCE PROPERTIES OF THE THEORY

The overthrow of parity conservation by Lee and Yang (1956) has demonstrated the importance of the discrete symmetry operations of charge conjugation(C), parity (P) and time reversal (or reversal of motion) (T). Not all theories, especially those constructed to describe weak interaction physics, need be invariant under these transformations. In this section we will see that, like what has come to be known about weak interactions since Lee and Yang, the interactions described in this chapter are neither C nor P invariant, but are CP, T and CPT invariant.

The C, P and T transformations for quantum spinor fields that are to be used here are described in detail in App. C. In that they do not depend on any specific representation of the Dirac matrices they are somewhat novel. The conventional transformations are described in, for example, Bjorken and Drell (1965, Chap. 15). The transformations we use are

$$\psi^{p}(t,\vec{r}) \equiv P\ \psi(t,\vec{r})\ P^{-1} \tag{2.7.1}$$

$$\equiv \gamma_0\ \psi(t,-\vec{r})\ ,$$

$$\psi^{c}(x) \equiv C\ \psi(x)\ C^{-1} \tag{2.7.2}$$

$$\equiv \psi^{*}(x)\ ,$$

$$\psi^{t}(t,\vec{r}) \equiv T\ \psi(t,\vec{r})\ T^{-1} \tag{2.7.3}$$

$$\equiv \gamma_0\ \psi(-t,\vec{r})\ ,$$

$$\psi^{cpt}(x) \equiv \Theta\ \psi(x)\ \Theta^{-1} \tag{2.7.4}$$

$$\equiv \psi^{*}(-x)\ ,$$

where P, C, T and Θ≡TPC are unitary operators in the Fock-space of creation and annihilation operators, and where T and Θ are anti-linear operators. It will be essential in applying these transformations to use the important definitions and identities described in App. B, which will be identified when needed. We take (*) to denote the complex conjugate of scalars and the hermitian conjugate of Fock-space operators.

All 4-vector fields will be assumed to transform as does the electromagnetic vector potential A, as described in App. C:

$$A^{p}(t,\vec{r}) \equiv \gamma_0 \; A(t,-\vec{r}) \; \gamma_0 \qquad (2.7.5a)$$

or, in component form

$$A^{\mu p}(t,\vec{r}) = (A^{0}(t,-\vec{r}),\; -A^{i}(t,-\vec{r})) \;, \qquad (2.7.5b)$$

$$A^{c}(x) \equiv -A^{*}(x) \qquad (2.7.6)$$

or $-A(x)$, if A is a real field,

$$A^{t}(t,\vec{r}) \equiv \gamma_0 \; A(-t,\vec{r}) \; \gamma_0 \qquad (2.7.7)$$

$$A^{cpt}(x) \equiv -A^{*}(-x) \qquad (2.7.8)$$

or $-A(-x)$, if A is real. It is important to note that the Higgs field $\vec{H}$ is not of this type: it consists of a set of three (complex) scalar fields.

Scalar fields will be taken to transform as

$$\phi^{p}(t,\vec{r}) \equiv \phi(t,-\vec{r}) \qquad (2.7.9)$$

$$\phi^{c}(x) \equiv \phi^{*}(x) \qquad (2.7.10)$$

$$\phi^{t}(t,\vec{r}) \equiv \phi(-t,\vec{r}) \qquad (2.7.11)$$

$$\phi^{cpt}(x) \equiv \phi^{*}(-x) \;. \qquad (2.7.12)$$

A charged scalar field, such as ϕ_-, satisfies $\phi_-^{c} = \phi_-^{*} = \phi_+$.

Of the kinetic terms in the Lagrangian only those for the spinor fields, such as the first two terms of (2.2.18), are not trivial, so we consider these exclusively. A term such as

$$\mathcal{L}(\psi) = i\bar{\psi}\Sigma_+ \partial\psi \;, \qquad (2.7.13)$$

which can represent either neutrino or electron (or muon, etc.) fields, has the following transformation properties. Under P we have

$$P\, i\bar{\psi}\Sigma_+\partial\psi\, P^{-1} = i\bar{\psi}^P\Sigma_+\partial\psi^P \tag{2.7.14}$$

$$= i\bar{\psi}(-\vec{r})\gamma_0\Sigma_+\partial\gamma_0\psi(-\vec{r})$$

$$= i\bar{\psi}(-\vec{r})\Sigma_-\gamma_0\gamma^\nu\gamma_0\partial_\nu\psi(-\vec{r}) \;,$$

and as the transformation $\vec{r} \to -\vec{r}$ is permitted in the action integral, this becomes

$$\mathcal{L}(\psi) \to i\bar{\psi}\Sigma_-\partial\psi \;. \tag{2.7.15}$$

Thus the chirality projection operators prevent such a term from being P-invariant; if absent, as they are for an electron, for example, the term would be P-invariant.

Under C we have

$$C\, i\bar{\psi}\Sigma_+\partial\psi\, C^{-1} = i\bar{\psi}^c\Sigma_+\partial\psi^c \tag{2.7.16}$$

$$= i\bar{\psi}^*\Sigma_+\gamma^\nu\partial_\nu\psi^* = -i\partial_\nu\bar{\psi}\gamma_5\gamma^\nu\Sigma_+\gamma_5\psi$$

$$= i\bar{\psi}\gamma^\nu\Sigma_+\partial_\nu\psi - i\partial_\nu(\bar{\psi}\gamma^\nu\Sigma_+\psi)$$

$$= i\bar{\psi}\Sigma_-\partial\psi + \text{surface term} \;,$$

where the important and crucial identity (B.38) has been used, and where the surface term disappears in the action integral. Again, we see that such a term is not invariant if the projection operators $\Sigma_\pm$ are present. This term is, however, CP-invariant:

$$(CP)\, i\bar{\psi}\Sigma_+\partial\psi\, (CP)^{-1} = \bar{\psi}^*(-\vec{r})\gamma_0\Sigma_+\gamma^\nu\gamma_0\partial_\nu\psi^*(-\vec{r}) \tag{2.7.17}$$

$$= -i\partial_\nu\bar{\psi}(-\vec{r})\gamma_5\gamma_0\gamma^\nu\Sigma_+\gamma_0\gamma_5\psi(-\vec{r})$$

$$= i\bar{\psi}(-\vec{r})\Sigma_+\gamma_0\gamma^\nu\gamma_0\partial_\nu\psi(-\vec{r}) + \text{surface term}$$

$$\to i\bar{\psi}\Sigma_+\partial\psi$$

when the change $\vec{r} \to -\vec{r}$ is made.

The CPT and T transformations are similar: we illustrate with the former, remembering that both are anti-linear transformations. We have

$$\Theta\, i\bar{\psi}\Sigma_+\partial\psi\, \Theta^{-1} = -i\Theta\bar{\psi}\Theta^{-1}\Sigma_-\partial\Theta\psi\Theta^{-1} \tag{2.7.18}$$

$$= -i\bar{\psi}^*(-x)\Sigma_-\gamma^\nu\partial_\nu\psi^*(-x) = i\partial_\nu\bar{\psi}(-x)\gamma_5\gamma^\nu\Sigma_-\gamma_5\psi(-x)$$

$$= -i\bar{\psi}(-x)\Sigma_+\gamma^\nu\partial_\nu\psi(-x) + \text{surface term}$$

$$\rightarrow i\bar{\psi}\Sigma_+\partial\psi\ ,$$

when the permitted transformation $x \rightarrow -x$ is made in the action integral.

Next to be considered are spinor-boson and boson-boson interactions. The former consists of two types: for the (vector) gauge bosons $(W_\pm, A, Z_0)$ we have interaction terms like

$$\mathcal{L}(\psi_1,\psi_2,A) \sim \bar{\psi}_1\Sigma_+A\psi_2 + \bar{\psi}_2\Sigma_+A^*\psi_1\ , \tag{2.7.19}$$

apart from real constants, and for the (scalar) Higgs bosons terms like

$$\mathcal{L}(\psi_1,\psi_2,\phi) \sim \bar{\psi}_1\Sigma_+\phi\psi_2 + \bar{\psi}_2\Sigma_-\phi^*\psi_1\ . \tag{2.7.20}$$

Note that only Σ_+ is to be found in the first, while both of $\Sigma_\pm$ exist in the second. Considering the first term only we have

$$P\ \mathcal{L}(\psi_1,\psi_2,A)\ P^{-1} \sim \bar{\psi}_1^P\Sigma_+A^P\psi_2^P \tag{2.7.21}$$

$$= [\bar{\psi}_1\gamma_0\Sigma_+(\gamma_0A\gamma_0)\gamma_0\psi_2](-\vec{r}) = (\bar{\psi}_1\Sigma_-A\psi_2)(-\vec{r})$$

$$\rightarrow (\bar{\psi}_1\Sigma_-A\psi_2)(\vec{r})\ ,$$

so is not invariant. For the C transformation we have

$$C\ \mathcal{L}(\psi_1,\psi_2,A)\ C^{-1} \sim \bar{\psi}_1^C\Sigma_+A^C\psi_2^C \tag{2.7.22}$$

$$= \bar{\psi}_1^*\Sigma_+(-A^*)\psi_2^* = \bar{\psi}_2\gamma_5A^*\Sigma_+\gamma_5\psi_1$$

$$= \bar{\psi}_2\Sigma_-A^*\psi_1\ ,$$

where (B.38) has again been used. Thus these terms are not C-invariant either. They are, however, CP-invariant:

$$(CP)\ \mathcal{L}(\psi_1,\psi_2,A)\ (CP)^{-1} \sim [\bar{\psi}_1^* \gamma_0 \Sigma_+ (-\gamma_0 A \gamma_0) \gamma_0 \psi_2^*](-\vec{r}) \qquad (2.7.23)$$
$$= -(\bar{\psi}_1^* \Sigma_- A^* \psi_2^*)(-\vec{r}) = (\bar{\psi}_2 \gamma_5 A^* \Sigma_- \gamma_5 \psi_1)(-\vec{r})$$
$$\rightarrow (\bar{\psi}_2 \Sigma_+ A^* \psi_1)(\vec{r}) \ .$$

These terms are also T-invariant:

$$T\ \mathcal{L}(\psi_1,\psi_2,A)\ T^{-1} \sim T\bar{\psi}_1 T^{-1} T\Sigma_+ A T^{-1} T\psi_2 T^{-1} \qquad (2.7.24)$$
$$= [\bar{\psi}_1 \gamma_0 \Sigma_- (\gamma_0 A \gamma_0) \gamma_0 \psi_2](-t) = (\bar{\psi}_1 \Sigma_+ A \psi_2)(-t)$$
$$\rightarrow (\bar{\psi}_1 \Sigma_+ A \psi_2)(t) \ ,$$

and CPT-invariant:

$$\Theta\ \mathcal{L}(\psi_1,\psi_2,A)\ \Theta^{-1} \sim \Theta\bar{\psi}_1 \Theta^{-1} \Theta\Sigma_+ A \Theta^{-1} \Theta\psi_2 \Theta^{-1} \qquad (2.7.25)$$
$$= [\bar{\psi}_1^* \Sigma_- (-A^*) \psi_2^*](-x) = (\bar{\psi}_2 \gamma_5 A^* \Sigma_- \gamma_5 \psi_1)(-x)$$
$$\rightarrow (\bar{\psi}_2 \Sigma_+ A^* \psi_1)(x) \ ,$$

where the essential anti-linearity of T and $\Theta \equiv TPC$ should be noted.

For the scalar Higgs bosons we have, again considering only the first term,

$$P\ \mathcal{L}(\psi_1,\psi_2,\Phi)\ P^{-1} \sim (\bar{\psi}_1 \gamma_0 \Sigma_+ \Phi \gamma_0 \psi_2)(-\vec{r}) \qquad (2.7.26)$$
$$\rightarrow (\bar{\psi}_1 \Sigma_- \Phi \psi_2)(\vec{r}) \ ,$$

again not invariant,

$$C\ \mathcal{L}(\psi_1,\psi_2,\Phi)\ C^{-1} \sim \bar{\psi}_1^* \Sigma_+ \Phi^* \psi_2^* \qquad (2.7.27)$$
$$= -\bar{\psi}_2 \gamma_5 \Sigma_+ \Phi^* \gamma_5 \psi_1 = \bar{\psi}_2 \Sigma_+ \Phi^* \psi_1 \ ,$$

so is not C-invariant either, but

$$(PC)\ \mathcal{L}(\psi_1,\psi_2,\Phi)\ (PC)^{-1} \sim (\bar{\psi}_1^* \gamma_0 \Sigma_+ \Phi^* \gamma_0 \psi_2^*)(-\vec{r}) \qquad (2.7.28)$$
$$= -(\bar{\psi}_2 \gamma_5 \Sigma_- \Phi^* \gamma_5 \psi_1)(-\vec{r})$$
$$\rightarrow (\bar{\psi}_2 \Sigma_- \Phi^* \psi_1)(\vec{r}) \ ,$$

so is CP-invariant. We also have

$$T\ \mathcal{L}(\psi_1,\psi_2,\phi)\ T^{-1} \sim \bar{\psi}_1(-t)\gamma_0 T\Sigma_+\phi T^{-1}\gamma_0\psi_2(-t) \tag{2.7.29}$$
$$= (\bar{\psi}_1\gamma_0\Sigma_-\gamma_0\phi\psi_2)(-t)$$
$$\rightarrow (\bar{\psi}_1\Sigma_+\phi\psi_2)(t)\ ,$$

and

$$\Theta\ \mathcal{L}(\psi_1,\psi_2,\phi)\ \Theta^{-1} \sim \bar{\psi}_1^*(-x)\Theta\Sigma_+\phi\Theta^{-1}\psi_2^*(-x) \tag{2.7.30}$$
$$= (\bar{\psi}_1^*\Sigma_-\phi^*\psi_2^*)(-x) = -(\bar{\psi}_2\gamma_5\Sigma_-\phi^*\gamma_5\psi_1)(-x)$$
$$\rightarrow (\bar{\psi}_2\Sigma_-\phi^*\psi_1)(x)\ ,$$

so both aré invariant as expected.

Of the boson-boson interactions only two types are of interest. The A-W-Higgs scalar(H) (physical or unphysical) interaction is

$$\mathcal{L}(AWH) = -eM_W(A\cdot W_-H_+ + A\cdot W_+H_-)\ , \tag{2.7.31}$$

and is easily shown to be C, P and T-invariant. The other type is the electromagnetic interaction between the photon A and the W or Higgs currents. The former current is given in (2.2.31):

$$j^\mu(W) = -ie[(\partial^\mu W_+^\nu - \partial^\nu W_+^\mu)W_\nu^- - W_\nu^+(\partial^\mu W_-^\nu - \partial^\nu W_-^\mu)]\ . \tag{2.7.32}$$

Examination of the first term only shows the transformation properties

$$P\ j^\mu(W)\ P^{-1} = ie[\gamma_0 W_-(-\vec{r})\gamma_0]\cdot\partial^\mu[\gamma_0 W_+(-\vec{r})\gamma_0] + \cdots \tag{2.7.33}$$
$$= (j^0,\ -j^i)(-\vec{r})$$

(due to $\vec{r} \rightarrow -\vec{r}$ in ∂^μ),

$$C\ j^\mu(W)\ C^{-1} = ie\ W_\nu^{-*}\partial^\mu W_+^{\nu*} + \cdots \tag{2.7.34}$$
$$= ie\ W_\nu^+\partial^\mu W_-^\nu + \cdots$$
$$= -j^\mu(W)\ ,$$

and

$$T\, j^{\mu}(W)\, T^{-1} = -ie[\gamma_0 W_-(-t)\gamma_0]\cdot\partial^{\mu}[\gamma_0 W_+(-t)\gamma_0] + \cdots \qquad (2.7.35)$$
$$= (j^o,\ -j^i)(-t)$$

(due to $t \to -t$ in ∂^{μ}), all of which demonstrate C, P and T invariance. The electric current due to the Higgs bosons, given in (2.2.27),

$$j^{\mu}(\phi) = ie(\phi_-\partial^{\mu}\phi_+ - \phi_+\partial^{\mu}\phi_-)\ , \qquad (2.7.36)$$

is rather easily shown to have similar transformation properties, again confirming C, P and T invariance.

Thus in general we may conclude that the theory is C and P invariant except where the chirality projection operators $\Sigma_{\pm}$ occur, in which case only CP invariance can be demonstrated. There are, however, no exceptions to T and CPT invariance, the latter being required quite generally of any local, Lorentz-invariant field theory (Bjorken and Drell, 1965, Sec. 15.14).

III. DIRAC AND MAJORANA NEUTRINOS

III.1 NEUTRINO MASS

Since its conception a half century ago the neutrino has been assumed to be massless, or at least to have a very small mass, of the order, at most, of a few eV. During the past decade, and even earlier, a number of analyses have been published concerning the consequences of a non-zero neutrino mass (Case, 1967; Bilenky and Pontecorvo, 1978; Cheng and Li, 1980a; Langacker, 1981; Frampton and Vogel, 1982; Li and Wilczek, 1982). At least one modern experiment (Lubimov, 1980; Stockdale, 1984, Ref. 1) has claimed a mass of the order of 30 eV, while others (Simpson, 1982; Kirsten *et al.*, 1983; Avignone *et al.*, 1983) still claim to have observed no measurable mass in the few eV range, or question the measured mass (Simpson, 1984).

A number of interesting physical consequences follow from the conjectured existence of massive neutrinos: neutrino oscillations (Bilenky and Pontecorvo, 1978; Cheng and Li, 1980a; Barger, Langacker and Leveille, 1980; Frampton and Vogel, 1982), or the spontaneous change of neutrino type, such as to electron type from muon type, and so on; neutrino decay (Cheng and Li, 1980b; Pal and Wolfenstein, 1982), in which the heavier neutrinos would decay into the lighter varieties, such as, for example, the radiative decay of a muon neutrino into an electron neutrino; a total neutrino mass of cosmological significance (Frampton and Vogel, 1982), as the total mass of the neutrinos in the universe would, in all likelihood, have an impact on the "missing mass problem" and the overall topology of the universe; new ways in which the various leptons interact with neutrinos of other families, one of the concerns of this thesis. Such physical consequences are, at best, at the very edge of detectability, and only through various assumptions and models can the possibilities be explored and experimental limits approached. Indeed, it was the very undetectability of certain of these process (especially the radiative decay $\mu\to e\gamma$) that gave rise to the concept of not only conservation of lepton number, but separately of electron lepton number, muon lepton number, etc. It must be remembered, however, that at the present time there is no conclusive evidence that any of these conservation laws is not really valid.

It so happens that there are at least two theoretical descriptions of massive neutrinos to be considered, which result, in a number of cases, in quite different predictions being made. First, the Dirac neutrino, a sort of light, neutral electron with two spin states for each particle and anti-particle, and the best understood of the two because its mathematical description so resembles that of the electron. Second, a single, two-state neutral particle referred to as a Majorana neutrino, one that is, indifferently, its own anti-particle or that has no anti-particle counterpart, like the photon. These particles, their descriptions and interactions, their role in the gauge theory of the previous chapter, are, in turn, the subject of the rest of this chapter.

III.2 DIRAC NEUTRINOS

To assure the existence of a Dirac-type neutrino mass one must arrange that the Lagrangian contain the terms necessary to resemble those of a general Dirac-type particle:

$$\mathcal{L} = \bar{\psi}(i\partial - m)\psi \qquad (3.2.1)$$

$$= i\bar{\psi}\Sigma_{+}\partial\psi + i\bar{\psi}\Sigma_{-}\partial\psi - m\bar{\psi}(\Sigma_{+}+\Sigma_{-})\psi \ .$$

The leptonic part of the Weinberg-Salam Lagrangian (2.2.2) contains, as far as the neutrino is concerned, only the first term of (3.2.1), or $i\bar{\nu}\Sigma_{+}\partial\nu$. Therefore an SU(2) singlet $\nu_R \equiv \Sigma_{+}\nu$ must be added to the original particle content, with the following aesthetically displeasing aspect. Its weak hypercharge, according to the rule (2.2.11), must be zero, so that this particle, at least in the pre-Higgs mechanism stage, participates in no interactions whatever with the gauge bosons, and only weakly with the gauge bosons and other fermions -- much more weakly than its left-handed partner -- after symmetry breaking.

An SU(2)×U(1) gauge invariant Yukawa-type interaction term that can be added to the original Lagrangian, in analogy with (2.2.7), is, summing over lepton families,

$$-\Sigma_\ell \ \beta_{\nu\ell} \ (\bar{\Psi}_\ell \Sigma_+ \Phi' \nu_\ell + \bar{\nu}_\ell \Sigma_- \Phi'^{\dagger} \Psi_\ell) \ , \tag{3.2.2}$$

where

$$\Phi' \equiv i \ \sigma_2 \ \Phi^* = \begin{pmatrix} \phi_0^* \\ -\phi_- \end{pmatrix} \ , \tag{3.2.3}$$

which is needed rather than Φ to have the hypercharge in (3.2.2) sum to zero, which expresses the U(1) gauge invariance as in (2.2.9). After symmetry breaking where

$$\Phi' \rightarrow \begin{pmatrix} \phi_0^* + v/\surd 2 \\ -\phi_- \end{pmatrix} \tag{3.2.4a}$$

$$\langle \Phi' \rangle = \begin{pmatrix} v/\surd 2 \\ 0 \end{pmatrix} \ , \tag{3.2.4b}$$

one obtains a term of the form

$$-\Sigma_\ell \ \beta_{\nu\ell} \ (v/\surd 2) \ (\bar{\nu}_\ell \Sigma_+ \nu_\ell + \bar{\nu}_\ell \Sigma_- \nu_\ell) \ , \tag{3.2.5}$$

so that the type ℓ neutrino has a mass

$$m(\nu_\ell) = \beta_{\nu\ell} \ v/\surd 2 \ , \tag{3.2.6}$$

that, like the coupling constants $\beta_{\nu\ell}$, is totally arbitrary. There is no connection whatever between the mass of a lepton ℓ and its associated neutrino ν_ℓ, another aesthetic shortcoming of this construction.

Implicit in the model so far has been the assumption that the neutrino states ($\nu_{1,2,\ldots} = \nu_{e,\mu,\ldots}$) in the Lagrangian -- the so-called weak eigenstates -- are also mass eigenstates; that is, each of the fields ν_ℓ has been supposed to have a definite mass as opposed to being linear superpositions of fields of various but definite masses. The most general Yukawa term that generates a (Dirac) mass for neutrinos is a generalization of (3.2.2):

$$-\Sigma_{ij}\ \beta_{ij}\ (\bar{\Psi}_i \Sigma_+ \Phi' \nu_j + \bar{\nu}_j \Sigma_- \Phi'^{+} \Psi_i)\ , \tag{3.2.7}$$

which, after symmetry breaking, contains the mass term

$$-\Sigma_{ij}\ \beta_{ij}\ (v/\sqrt{2})\ (\bar{\nu}_i \Sigma_+ \nu_j + \bar{\nu}_j \Sigma_- \nu_i) \tag{3.2.8}$$

$$= -\Sigma_{ij}\ \beta_{ij}\ (v/\sqrt{2})\ (\bar{\nu}_i \nu_j)\ ,$$

where $\beta_{ij} \equiv \beta_{ji}$ (so that both chirality states are to have the same mass), which can be diagonalized by an orthogonal transformaton to

$$-\Sigma_i\ m_i\ \bar{\nu}_i \nu_i\ , \tag{3.2.9}$$

where the m_i are the eigenvalues of the mass matrix $(\beta_{ij} v/\sqrt{2})$, assumed positive (Cheng and Li, 1980a). If any m_i were to be negative a transformation $m_i \rightarrow -m_i$, $\nu_i \rightarrow \gamma_5 \nu_i$ would be performed, because $\bar{\nu}\partial\nu \rightarrow \bar{\nu}\partial\nu$ while $\bar{\nu}\nu \rightarrow -\bar{\nu}\nu$. The orthogonal transformation is

$$\bar{\nu}_k = \Sigma_\alpha U_{\alpha k}\ \nu_\alpha \tag{3.2.10}$$

where

$$\Sigma_{k\ell} U_{\alpha k}\ (\beta_{k\ell}\ v/\sqrt{2})\ U_{\beta\ell} = \delta_{\alpha\beta}\ m_\alpha\ , \tag{3.2.11}$$

with Greek subscripts denoting mass eigenstates, so that the charged lepton-neutrino terms in the Lagrangian that follow from (2.2.18) and (3.2.7) are, respectively,

$$\Sigma_{\alpha\ell}[(g/\sqrt{2})U_{\alpha\ell}(\bar{\nu}_\alpha \Sigma_+ W_+ \ell + \bar{\ell}\Sigma_+ W_- \nu_\alpha) \tag{3.2.12}$$

$$+\ \beta_\ell U_{\alpha\ell}(\bar{\nu}_\alpha \Sigma_+ \Phi_+ \ell + \bar{\ell}\Sigma_- \Phi_- \nu_\alpha)]$$

(with $\beta_\ell v/\sqrt{2} = M_\ell$) and

$$\Sigma_{\ell j\alpha} \beta_{\ell j} U_{\alpha j}(\bar{\ell}\Sigma_+ \Phi_- \nu_\alpha + \bar{\nu}_\alpha \Sigma_- \Phi_+ \ell)\ , \tag{3.2.13}$$

with

$$\Sigma_j \beta_{\ell j} U_{\alpha j} v/\sqrt{2} = m_\alpha U_{\alpha\ell}\ , \tag{3.2.14}$$

from (3.2.11). Thus if the off-diagonal elements of the orthogonal matrix do not vanish there is a mixing among the various neutrino types; a given neutrino of fixed mass interacts with several leptons (and vice versa), and the probability of processes such as neutrino decay arises. No longer would there be any separate conservation of electron lepton number, muon lepton number, etc.

III.3 MAJORANA NEUTRINOS

One can understand from the preceding section that the generalization of the original Weinberg-Salam Lagrangian to incorporate massive neutrinos of the Dirac type is not altogether satisfactory because of the number and type of new terms that must be added and the lack of restriction on any of the new parameters. It so happens that neutrino mass can be generated without the addition of any of these objectionable terms provided an iso-vector Higgs field, with its Yukawa interaction (2.3.18), is introduced (Cheng and Li, 1980a, 1984; Marshak, Riazuddin and Mohapatra, 1981; Schechter and Valle, 1980). The particle that results -- referred to as a Majorana neutrino -- is a single two-state particle without anti-particle (or, if preferred, is its own anti-particle).

After symmetry breaking there results the first term of (2.3.20) (in this section one is to sum over repeated subscripts):

$$-\beta_{ij} v_H (\bar{\nu}^c_i \Sigma_- \nu_j + \bar{\nu}_j \Sigma_+ \nu^c_i) \ , \qquad (3.3.1)$$

where, by construction, $\beta_{ij} = \beta_{ji}$, and the β's are real, by CPT invariance. The mass matrix, $v_H \beta_{ij} \equiv \frac{1}{2} M_{ij}$, can, as in the previous section, be diagonalized by an orthogonal transformation:

$$U \ M \ U^T \equiv M_{diag} \ , \qquad (3.3.2a)$$

or

$$U_{ik} \ M_{k\ell} \ U_{j\ell} = \delta_{ij} m_i \eta_i \ , \qquad (3.3.2b)$$

where the m_i are all positive and $\eta_i = \pm 1$, where the eigenvalues of M are $m_i \eta_i$. In the case of two families, for example, it follows from (3.3.2a) that, since $U^T = U^{-1}$,

$$\begin{aligned} Tr(UMU^T) &= Tr(M) = M_{11} + M_{22} \\ &= Tr(M_{diag}) = \eta_1 m_1 + \eta_2 m_2 \end{aligned} \qquad (3.3.3)$$

and

$$\begin{aligned} Det(UMU^T) &= Det(M) = M_{11}M_{22} - M_{12}M_{21} \\ &= Det(M_{diag}) = (\eta_1 m_1)(\eta_2 m_2) \ , \end{aligned} \qquad (3.3.4)$$

which is readily solved for the masses m_1, m_2 in terms of the original β coefficients:

$$\begin{aligned} m_{1,2} &= \tfrac{1}{2} | Tr(M) \pm [Tr^2(M) - 4 \ Det(M)]^{1/2} | \\ &= v_H | (\beta_{11} + \beta_{22}) \pm [(\beta_{11} - \beta_{22})^2 + 4\beta_{12}\beta_{21}]^{1/2} | \ . \end{aligned} \qquad (3.3.5)$$

It is neither convenient nor necessary to consider more general examples.

One defines the Majorana field χ (Pal and Wolfenstein, 1982) by

$$\chi_i \equiv (1/\surd 2) \ U_{ij} \ (\Sigma_- \nu_j + \eta_i \Sigma_+ \nu^c_j) \qquad (3.3.6a)$$

$$\bar{\chi}_i = (1/\sqrt{2})\, U_{ij}\, (\bar{\nu}_j\Sigma_+ + \eta_i\bar{\nu}^c_j\Sigma_-)\ , \tag{3.3.6b}$$

but where the reference just cited does not have the (necessary) $(1/\sqrt{2})$ factor. We have

$$\begin{aligned} m_a\bar{\chi}_a\chi_a &= \tfrac{1}{2}\, U_{ai}U_{aj}m_a\eta_a\, (\bar{\nu}_i\Sigma_+\nu^c_j + \bar{\nu}^c_i\Sigma_-\nu_j) \\ &= \tfrac{1}{2}\, M_{ij}\, (\bar{\nu}_j\Sigma_+\nu^c_i + \bar{\nu}^c_i\Sigma_-\nu_j) \end{aligned} \tag{3.3.7}$$

which is (3.3.1), the mass term, where, from (3.3.2b) follows

$$M_{ab} = U_{ia}U_{ib}m_i\eta_i \tag{3.3.8}$$

and

$$\bar{\nu}_i\Sigma_+\nu^c_j = \bar{\nu}_i\Sigma_+\nu^*_j = -\bar{\nu}_j\gamma_5\Sigma_+\gamma_5\nu^*_i = \bar{\nu}_j\Sigma_+\nu^c_i\ , \tag{3.3.9}$$

where (B.38) has been applied. We also have

$$\begin{aligned} i\bar{\chi}_a\partial\chi_a &= \tfrac{1}{2}\, iU_{ai}U_{aj}(\bar{\nu}_i\Sigma_+\partial\nu_j + \eta_a\eta_a\bar{\nu}^c_i\Sigma_-\partial\nu^c_j) \\ &= \tfrac{1}{2}\, i(\bar{\nu}_i\Sigma_+\partial\nu_i + \bar{\nu}^c_i\Sigma_-\partial\nu^c_i) \end{aligned} \tag{3.3.10}$$

but the last term becomes, again using (B.38),

$$\bar{\nu}^c_i\Sigma_-\partial\nu^c_i = -\partial_\nu\bar{\nu}_i\gamma_5\gamma^\nu\Sigma_-\gamma_5\nu_i = \bar{\nu}_i\Sigma_+\partial\nu_i + \text{surface term}, \tag{3.3.11}$$

so that

$$i\bar{\chi}_a\partial\chi_a = i\bar{\nu}_i\Sigma_+\partial\nu_i \tag{3.3.12}$$

in the action integral. Thus the pure neutrino part of the Lagrangian becomes

$$\mathcal{L}(\chi) = i\bar{\chi}_a\partial\chi_a - m_a\bar{\chi}_a\chi_a\ , \tag{3.3.13}$$

as required. What remains now is to replace the ν-fields in the interaction terms with the χ-fields.

It is important to note that, to within a phase, the Majorana field χ is self-conjugate:

$$\chi^c_i = \chi^*_i = \eta_i\ \chi_i\ , \tag{3.3.14}$$

a fact that follows easily from its definition (3.3.6a). The usual expansion of a free spinor field (2.1.8)

$$\psi = (2\pi)^{-3/2} \int d^3p \sqrt{m/\epsilon}\, (e^{-ip\cdot x} a_r u_r + e^{ip\cdot x} b_r^* v_r) \qquad (3.3.15)$$

yields, when (3.3.14) is applied,

$$b_r = a_r , \qquad (3.3.16)$$

to within a phase, so that the field ψ describes but one two-state particle, as claimed. In order to obtain the correct free-field energy-momentum

$$P^\mu = \int d^3p\, p^\mu\, a_r^*(p) a_r(p) \qquad (3.3.17)$$

from the canonical prescriptions (2.1.14) and (2.1.16) a factor of $(1/\sqrt{2})$ must be added to (3.3.15):

$$\chi_i = (1/\sqrt{2})(2\pi)^{-3/2} \int d^3p \sqrt{m_i/\epsilon}\, (e^{-ip\cdot x} a_r u_r + \eta_i e^{ip\cdot x} a_r^* v_r). \qquad (3.3.18)$$

This factor of $(1/\sqrt{2})$ means that the χ-field propagator (Sec. IV.3) is one-half of the usual for spin-$\frac{1}{2}$ particles. It is very important to note that the χ-field both creates and annihilates its appropriate Majorana particle.

From the definitions (3.3.6a) and (3.3.6b) follow

$$\Sigma_- \nu_j = \sqrt{2}\, U_{ij} \Sigma_- \chi_i \qquad (3.3.19a)$$

$$\bar{\nu}_j \Sigma_+ = \sqrt{2}\, U_{ij}\, \bar{\chi}_i \Sigma_+ \qquad (3.3.19b)$$

$$\Sigma_+ \nu_j^c = \sqrt{2}\, U_{ij} \eta_i \Sigma_+ \chi_i \qquad (3.3.19c)$$

$$\bar{\nu}_j^c \Sigma_- = \sqrt{2}\, U_{ij} \eta_i\, \bar{\chi}_i \Sigma_- , \qquad (3.3.19d)$$

which can now be used to eliminate all reference to the ν-fields in the original interaction Lagrangian. The charged lepton-neutrino-W interaction from (2.2.18) becomes

$$\mathcal{L}(\ell\chi W) = g U_{i\ell}\, (\bar{\chi}_i \Sigma_+ W_+ \ell + \bar{\ell} \Sigma_+ W_- \chi_i) , \qquad (3.3.20)$$

while the charged lepton-neutrino-charged Higgs interaction becomes

$$\mathcal{L}(\ell\chi\Phi_\pm) = -\sqrt{2}\, U_{i\ell} \beta_\ell\, (\bar{\chi}_i \Sigma_+ \Phi_+ \ell + \bar{\ell} \Sigma_- \Phi_- \chi_i) \qquad (3.3.21)$$

and, from (2.3.20),

$$\mathcal{L}(\ell\times H) = 2\sqrt{2}\, U_{ai}\beta_{i\ell}\eta_a\, (\bar{x}_a\Sigma_- H_+\ell + \bar{\ell}\Sigma_+ H_- x_a) \;. \qquad (3.3.22)$$

These last two terms can be combined into interactions with the unphysical and physical Higgs fields, S and B of (2.3.15), (2.3.16), respectively, into the final form

$$\mathcal{L}(\ell\times S) = -(g/M_W)U_{i\ell}[\bar{x}_i(M_\ell\Sigma_+ - m_i\Sigma_-)S_+\ell + \bar{\ell}(M_\ell\Sigma_- - m_i\Sigma_+)S_- x_i] \qquad (3.3.23)$$

and

$$\mathcal{L}(\ell\times B) = (g/M_W)U_{i\ell}[\bar{x}_i(M_\ell\tan\theta\Sigma_+ + m_i\cot\theta\Sigma_-)B_+\ell + \bar{\ell}(M_\ell\tan\theta\Sigma_- + m_i\cot\theta\Sigma_+)B_- x_i], \qquad (3.3.24)$$

where the β_i, β_{ij}, v, v_H factors have been written in terms of the charged lepton and neutrino masses (M_i and m_i, respectively, from (2.2.14) and (3.3.2b)) and the coupling constant g from (2.3.12a). These interaction terms will be the basis of all future work.

IV. THE FEYNMAN RULES OF THE THEORY

IV.1 LEPTON-BOSON INTERACTIONS: SUMMARY

In this section all of the terms in the Lagrangian describing the interactions between the leptons and the gauge bosons and between the leptons and the Higgs bosons will be listed and summarized. Such terms will be needed to construct the "Feynman rules" for the vertices of the theory, or at least those consisting of two leptons and one gauge boson or Higgs boson relevant for the central problem of this thesis (Chap. V, VI), which is the calculation of the rate of decay of a muon into an electron plus a photon, or an electron plus an electron-positron pair.

There are at least two good reasons why such a construction must be done from the beginning. Each gauge-theoretic model has its own particular interaction terms and those for the specific model used here, namely the model with a vector Higgs field and massive Majorana neutrinos, have not been listed before. Second, in constructing all of the rules to be used here a consistent sign convention can be used throughout, the same for all interactions, so that any ambiguity of phase with vertices published elsewhere can be avoided.

The interactions among the leptons and the gauge bosons ($W_{\pm}$, A, Z) are given in (2.2.18) which is modified to include a sum over the lepton families, and the neutrino fields given in Chap. II are modified to become Majorana neutrinos, as given in (3.3.19). These changes result in the interaction terms

$$\begin{aligned}\mathcal{L}(\ell,x;W,A,Z) = {} & \Sigma_{\ell i} gU_{i\ell}(\bar{x}_i\Sigma_+W_+\ell + \bar{\ell}\Sigma_+W_-x_i) \\ & + \Sigma_\ell[e\bar{\ell}A\ell + G\sin^2\theta_W\bar{\ell}\Sigma_-Z\ell - \tfrac{1}{2}G(\cos^2\theta_W - \sin^2\theta_W)\bar{\ell}\Sigma_+Z\ell] \\ & + \Sigma_{\ell ij} GU_{i\ell}U_{j\ell}\bar{x}_i\Sigma_+Z\Sigma_-x_j \ .\end{aligned} \tag{4.1.1}$$

The orthogonality of the matrix U can be exploited to rewrite the last term as

$$\Sigma_i G\bar{x}_i\Sigma_+Zx_i \ , \tag{4.1.2}$$

but this term will not be needed in what follows.

The most important term of (4.1.1) for the needs of this thesis, except for the electromagnetic interaction, is the first, which shows the coupling of the various charged leptons (e, μ, $\cdots$) to the varieties of Majorana neutrinos x_i ($i=1,2,\cdots$), a coupling governed by the unknown components (U_{ij}) of the orthogonal transformation that diagonalizes the Yukawa coupling matrix of (2.3.18). One is tempted to presume that the values of U_{ij} are greatest when $i=j$ and decrease as the difference between i and j increases but, unfortunately, there is no concrete evidence for such an assumption.

The interactions among the leptons and the Higgs fields are all that remain to be considered. The basic interaction terms for the iso-spinor Higgs field were given in (2.2.18), or, when generalized to include a sum over lepton families,

$$\begin{aligned}\mathcal{L}(\nu,\ell,\Phi) = {} & -\Sigma_\ell\beta_\ell(\bar{\nu}_\ell\Sigma_+\Phi_+\ell + \bar{\ell}\Sigma_-\Phi_-\nu_\ell \\ & + \bar{\ell}\Sigma_+\Phi_0\ell + \bar{\ell}\Sigma_-\Phi_0^*\ell) \ ,\end{aligned} \tag{4.1.3}$$

along with (2.3.20) for the vector Higgs field. As mentioned earlier in Chap. II the fields $\Phi_\pm$ and $H_\pm$ are replaced by a physical Higgs field $B_\pm$ and an unphysical field $S_\pm$ defined in (2.3.15) and (2.3.16), and the neutrino ν-fields are to be replaced by the Majorana χ-fields of the previous chapter. The result was displayed earlier, in (3.3.23) and (3.3.24):

$$\mathcal{L}(\ell,\chi,S) = -\Sigma_{i\ell}(g/M_W)U_{i\ell}[\bar{\chi}_i(M_\ell\Sigma_+ - m_i\Sigma_-)S_+\ell + \bar{\ell}(M_\ell\Sigma_- - m_i\Sigma_+)S_-\chi_i] \ , \tag{4.1.4}$$

$$\mathcal{L}(\ell,\chi,B) = \Sigma_{i\ell}(g/M_W)U_{i\ell}[\bar{\chi}_i(M_\ell\tan\theta\Sigma_+ + m_i\cot\theta\Sigma_-)B_+\ell + \bar{\ell}(M_\ell\tan\theta\Sigma_- + m_i\cot\theta\Sigma_+)B_-\chi_i] \ . \tag{4.1.5}$$

The doubly charged Higgs field makes a contribution given in (2.3.20):

$$\mathcal{L}(\ell,H_{\pm\pm}) = \Sigma_{ij}\sqrt{2}\beta_{ij}(\bar{\ell}_i^c\Sigma_-H_{++}\ell_j + \bar{\ell}_j\Sigma_+H_{--}\ell_i^c) \ . \tag{4.1.6}$$

Both (4.1.3) and (2.3.20) contain interaction terms between the leptons and neutral Higgs particles, which, like their singly charged counterparts, must first be recast into physical and unphysical fields, as was done in Sec. II.4. From (4.1.3) and using $\Phi_0 = \Phi_1 + i\Phi_2$ we have

$$\begin{aligned} -\Sigma_\ell\beta_\ell(\bar{\ell}\Sigma_+\Phi_0\ell + \bar{\ell}\Sigma_-\Phi_0^*\ell) &= -\Sigma_\ell\beta_\ell\bar{\ell}(\Phi_1 + \gamma_5\Phi_2)\ell \\ &\approx -\Sigma_\ell\beta_\ell\bar{\ell}\Phi\ell \end{aligned} \tag{4.1.7}$$

from Table I, which is by far the leading contribution.

The lepton-singly charged, physical Higgs boson interaction of (4.1.5) contains something very interesting, something not considered before in its possible contribution to muon decay: the factor $m_i\cot\theta$. From (2.3.15b) and (3.3.2) one sees that

$$(m_i)(\cot\theta) \sim (v_H\beta_{ij})(v/v_H) \tag{4.1.8}$$

so the factor is actually *independent* of v_H and therefore of the neutrino masses as well. Indeed, if the Yukawa coupling constants β_{ij} and β_ℓ are of the same order of magnitude, this factor is of the order of magnitude of a charged lepton mass, and makes, therefore, a contribution potentially much greater than any that depend on neutrino masses. This interaction term will, in what follows, make a most interesting contribution to muon decay, both the radiative decay of Chap. V and the triple electron decay of Chap. VI.

IV.2 BOSON-BOSON INTERACTIONS: SUMMARY

The last topic for consideration before we can construct the Feynman rules needed in the sequel is a list of the relevant interaction terms among the gauge bosons, and among the Higgs and gauge bosons. The list to be given here will only be to the order of perturbation theory needed in subsequent calculations which, generally, is trilinear in the fields.

The interactions among the gauge bosons are an integral part of the original Weinberg-Salam model, and were given and explained in Sec. II.2. These terms are independent of the Higgs fields. Thus, to an order trilinear in the fields, we have the interaction term of (2.2.30):

$$\mathcal{L}(W,A,Z) = (i/2)e[(A^\mu W_-^\nu - A^\nu W_-^\mu)(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+) \tag{4.2.1}$$
$$- (A^\mu W_+^\nu - A^\nu W_+^\mu)(\partial_\mu W_\nu^- - \partial_\nu W_\mu^-) + (\partial_\mu A_\nu - \partial_\nu A_\mu)(W_-^\mu W_+^\nu - W_-^\nu W_+^\mu)]$$
$$- (i/2)g\cos\theta_W[(Z^\mu W_-^\nu - Z^\nu W_-^\mu)(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)$$
$$- (Z^\mu W_+^\nu - Z^\nu W_+^\mu)(\partial_\mu W_\nu^- - \partial_\nu W_\mu^-) + (\partial_\mu Z_\nu - \partial_\nu Z_\mu)(W_-^\mu W_+^\nu - W_-^\nu W_+^\mu)] .$$

The electromagnetic interaction of the Higgs bosons was described by (2.2.26) for the spinor Higgs field, with similar terms following from the vector Higgs Lagrangian (2.3.4):

$$\mathcal{L}(A,\Phi,H) = -ieA^\mu(\Phi_-\partial_\mu\Phi_+ - \Phi_+\partial_\mu\Phi_-) \tag{4.2.2}$$
$$- ieA^\mu(H_-\partial_\mu H_+ - H_+\partial_\mu H_-) - ieA^\mu(H_{--}\partial_\mu H_{++} - H_{++}\partial_\mu H_{--}) .$$

In terms of the physical field (B) and unphysical field (S) of (2.3.16) and (2.3.15), respectively, the first two terms of (4.2.2) become, not surprisingly,

$$\mathcal{L}(A,B) = -ieA^\mu(B_-\partial_\mu B_+ - B_+\partial_\mu B_-) \tag{4.2.3a}$$
$$\mathcal{L}(A,S) = -ieA^\mu(S_-\partial_\mu S_+ - S_+\partial_\mu S_-) . \tag{4.2.3b}$$

Finally, from the spinor Higgs Lagrangian (2.2.5) and the vector Higgs Lagrangian (2.3.4) arise interactions among the charged Higgs, the W and the electromagnetic field:

$$\mathcal{L}(W,A,\Phi_\pm,H_\pm) = (1/\sqrt{2})gg' a_\mu[W_-^\mu(v_H H_+ + (v/\sqrt{2})\Phi_+) \tag{4.2.4}$$
$$+ W_+^\mu(v_H H_- + (v/\sqrt{2})\Phi_-)] ,$$

which, with the definition of the unphysical field S of (2.3.15), becomes

$$\mathcal{L}(W,A,S) = -M_W eA_\mu(W_-^\mu S_+ + W_+^\mu S_-) , \tag{4.2.5}$$

where (2.3.12a) was used, and (2.2.15c) was used to obtain A_μ from the U(1) gauge field a_μ. Similar interactions with the neutral gauge boson Z have been ignored, not being needed in the sequel.

It is important to note that there is no interaction similar to (4.2.5) for the physical Higgs boson $B_{\pm}$: this interaction is a result of the gauge fixing term (2.6.8) and is crucial for the gauge invariance of the theory, as the Feynman diagrams of Sec. V.2 will display.

There are many more interaction terms within the gauge theory of Chap.II than have been listed in this and the previous section, interactions with the neutral physical Higgs fields and, for example, interactions among neutrinos. But we have listed all those that will be needed in what follows and they are now to be used to construct the Feynman rules of the theory.

IV.3 PROPAGATORS

The Feynman rules of the theory, considered in this and the subsequent section, comprise a consistent set of "diagrams" representing elementary interaction processes that are pieced together in such a way as to generate with some ease, to the order of perturbation theory required, the exact interaction terms needed for a specific process involving given initial and final particle states. The diagrams consist of two basic types: *propagators*, so called, which are essentially Green functions for the virtual motion of a particle from one space-time event to another, and *vertices*, which represent specific interaction terms when three or more particles are involved at, either entering or leaving, some event.

This section will list the various propagators required. There is absolutely nothing new in this -- a standard technique since the inception of modern quantum electrodynamics more than thirty years ago -- but by calculating them all according to a definite scheme we can be assured that the relative signs among them will be correct. Since a common phase factor for them is irrelevant we need not be concerned here, nor in the next section, about whether the signs obtained are the same as those listed elsewhere. For the propagators they will be, for the vertices they will not.

Propagators are considered in detail in, for example, Bjorken and Drell (1964, 1965), Itzykson and Zuber (1980) and Cheng and Li (1984). It is extremely quick and convenient to use the procedure developed in the last work cited: a propagator is i times the Fourier transform of the inverse of the differential operator obtained from the kinetic part of the free field Lagrangian. If, for example,

$$\mathcal{L}(\text{free}) = \tfrac{1}{2}\ \Phi_i(x)\ A_{ij}(x)\ \Phi_j(x)\ , \tag{4.3.1}$$

where the factor $\frac{1}{2}$ appears only for real scalar and real vector fields, one then defines

$$A_{ij}(x)\ A^{-1}_{jk}(x-y) \equiv \delta_{ik}\ \delta^4(x-y) \tag{4.3.2}$$

and

$$A^{-1}_{ij}(x) = (2\pi)^{-4} \int d^4k\ e^{-ik\cdot x}\ A^{-1}_{ij}(k)\ , \tag{4.3.3}$$

and then the propagator is said to be

$$\Delta_{ij}(k) \equiv i\, A^{-1}_{ij}(k) \ , \tag{4.3.4}$$

where k is to be interpreted as the particle's 4-momentum.

For a scalar field Φ we have

$$\begin{aligned} \mathcal{L}(\text{free}) &= \tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \tfrac{1}{2}m^2\Phi^2 \\ &= -\tfrac{1}{2}\Phi(\partial^2 + m^2)\Phi + \text{surface term} \end{aligned} \tag{4.3.5}$$

and

$$-(\partial^2 + m^2)(2\pi)^{-4}\int d^4k\, e^{-ik\cdot(x-y)}\, A^{-1}(k) \equiv \delta^4(x-y) \ , \tag{4.3.6}$$

from which we obtain

$$-(-k^2 + m^2)\, A^{-1}(k) = 1 \ , \tag{4.3.7}$$

because of the representation of the delta function

$$\delta^4(x) = (2\pi)^{-4}\int d^4k\, e^{-ik\cdot x} \ , \tag{4.3.8}$$

so the propagator for a real or complex scalar field (the factor $\frac{1}{2}$ in (4.3.5) is irrelevant) is

$$\Delta(k) = i/(k^2 - m^2) \ , \tag{4.3.9}$$

as is listed in every textbook on quantum field theory.

For the spinor field we have

$$\mathcal{L}(\text{free}) = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \tag{4.3.10}$$

so that

$$\begin{aligned} (i\partial\!\!\!/-m)(2\pi)^{-4}\int d^4k\, e^{-ik\cdot(x-y)}\, A^{-1}(k) &\equiv \delta^4(x-y) \\ &= (2\pi)^{-4}\int d^4k\, e^{-ik\cdot(x-y)}\, (k\!\!\!/-m)\, A^{-1}(k) \ , \end{aligned} \tag{4.3.11}$$

so the propagator for spin-$\frac{1}{2}$ fermions (electrons, muons, Dirac neutrinos, etc., but *not* Majorana neutrinos -- see below) is

$$\Delta(k) = i/(k\!\!\!/ - m) \ , \tag{4.3.12}$$

the standard result.

For a massive, complex (or real, for which a factor of $\frac{1}{2}$

must be added) vector field, with the important and necessary gauge fixing term of (2.6.8) or (2.6.9), we have

$$\mathcal{L}(\text{free}) = -\tfrac{1}{2}F^{*}_{\mu\nu}F^{\mu\nu} + m^2 A^{*}_{\mu}A^{\mu} - \xi^{-1}\partial^{\mu}A^{*}_{\mu}\partial^{\nu}A_{\nu} \ , \tag{4.3.13a}$$

where

$$F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{4.3.13b}$$

as usual, and we obtain

$$\mathcal{L} = A^{*}_{\mu}[\eta^{\mu\nu}(\partial^2+m^2) + (\xi^{-1}-1)\partial^{\mu}\partial^{\nu}]A_{\nu} + \text{surface term} \ , \tag{4.3.14}$$

which results in

$$\begin{aligned}
&[\eta^{\mu\nu}(\partial^2+m^2) + (\xi^{-1}-1)\partial^{\mu}\partial^{\nu}](2\pi)^{-4}\int d^4k \ e^{-ik\cdot(x-y)} \ A^{-1}_{\nu\lambda}(k) \qquad (4.3.15)\\
&\quad \equiv \delta^{\mu}_{\lambda} \ \delta^4(x-y)\\
&\quad = (2\pi)^{-4}\int d^4k \ e^{-ik\cdot(x-y)} \ [\eta^{\mu\nu}(-k^2+m^2) - (\xi^{-1}-1)k^{\mu}k^{\nu}]A^{-1}_{\nu\lambda}(k) \ .
\end{aligned}$$

The solution for iA^{-1} is the propagator Δ:

$$\begin{aligned}
\Delta_{\mu\nu}(k) &= -i\eta_{\mu\nu}(k^2-m^2)^{-1}\\
&\quad + i(\xi^{-1}-1)k_{\mu}k_{\nu}(k^2-m^2)^{-1}[(\xi^{-1}-1)k^2+(k^2-m^2)]^{-1} \ . \qquad (4.3.16)
\end{aligned}$$

Note the limits: in the "unitary gauge" ($\xi^{-1} = 0$) we have

$$\Delta_{\mu\nu}(k) = -i\eta_{\mu\nu}(k^2-m^2)^{-1} + i(k_{\mu}k_{\nu}/m^2)(k^2-m^2)^{-1} , \tag{4.3.17}$$

and in the massless case (photon propagator) we have

$$\Delta_{\mu\nu}(k) = -i\eta_{\mu\nu}(k^2)^{-1} + i(1-\xi)k_{\mu}k_{\nu}(k^2)^{-2} , \tag{4.3.18}$$

where one can *not* now have $\xi^{-1} = 0$. The Feynman (or t'Hooft-Feynman) gauge has $\xi \equiv 1$. It is very useful to note that (4.3.16) can be written

$$\begin{aligned}
\Delta_{\mu\nu}(k) &= -i\eta_{\mu\nu}(k^2-m^2)^{-1} + i(k_{\mu}k_{\nu}/m^2)(k^2-m^2)^{-1} \qquad (4.3.19)\\
&\quad - i(k_{\mu}k_{\nu}/m^2)(k^2-\xi m^2)^{-1} \ .
\end{aligned}$$

The prescriptions used above contain no hint of the origin of propagators and their use. They actually arise from an integration over internal (or virtual) lines in Feynman diagrams, an integral that contains a vacuum expectation value of a time-ordered series of field operators. For example, in the scalar field case one has (Bjorken and Drell, 1965)

$$\Delta(k) = \int d^4k\ e^{-ik\cdot x}\ \langle 0|T[\Phi(x)\Phi(0)]|0\rangle \qquad (4.3.20)$$
$$= i\ /\ (k^2 - m^2 + i\epsilon)\ ,$$

where $|0\rangle$ is the vacuum state, T the time ordering operator, and $\epsilon>0$ a prescription describing how the integral is to be carried out to avoid poles in the complex t or x_0-plane, which arises from

$$\theta(t) \equiv \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases} \qquad (4.3.21)$$
$$= (2\pi i)^{-1} \int d\alpha\ e^{i\alpha t}\ (\alpha - i\epsilon)^{-1}\ ,$$

and which gives meaning to the propagator as it is integrated around its poles in the complex k_0-plane.

Although the procedure (4.3.20) gives the propagator for properly normalized fields (which destroy the particle and create the anti-particle) the Majorana field is unique among spinor fields -- it both creates and destroys its particle. As a result an extra factor of $1/\sqrt{2}$ must be added to the field's momentum expansion as explained in (3.3.18), so that (4.3.20) implies that, in such a case, the Majorana fermion propagator is

$$\Delta(k) = \tfrac{1}{2}\ i\ /\ (k-m)\ , \qquad (4.3.22)$$

the last of the propagators that will be needed.

IV.4 VERTICES

While the particle propagators depend only on the particle type (scalar, spinor, etc.) and are the same for all theories the rules for the vertices depend totally on the specific interactions proscribed by the theory. Indeed, in a practical sense, the collection of vertices *is* the theory. There being, it seems, no standard method of establishing the vertices the approach to be employed here will construct the transition amplitude directly from the interaction Lagrangian, from which the vertices will be extracted.

For an initial state $|i\rangle$ and final state $|f\rangle$ the complete scattering amplitude in terms of the interaction Lagrangian $\mathcal{L}_I$ is (Itzykson and Zuber, 1980, Chap. 4)

$$S_{i\to f} \equiv \langle f|S|i\rangle = i\int d^4x\ \langle f|\mathcal{L}_I|i\rangle \equiv A\ (2\pi)^4\ \delta^4(\sum p_f - \sum p_i) \qquad (4.4.1)$$

where

$$S \approx 1 - i\int d^4x\ \mathcal{H}_I \qquad (4.4.2)$$

(in the first approximation) is the "scattering operator", A the invariant amplitude, $\mathcal{H}_I = -\mathcal{L}_I$ is the interaction Hamiltonian density, where $|f\rangle \neq |i\rangle$ has been assumed, and where the energy-momentum conserving δ-function has been separated out.

For example, the one vertex of spinor QED (Fig. 2) follows from (2.1.6):

$$\mathcal{L}_I(ee\gamma) = e\ \bar{\psi}A\psi\ . \qquad (4.4.3)$$

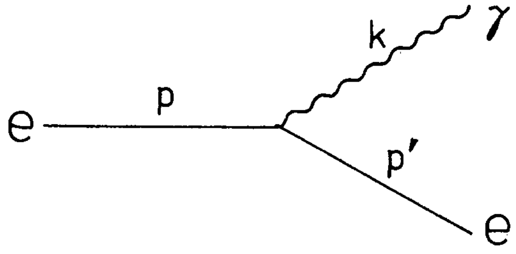

Fig. 2: The QED vertex

The unrenormalized fields are (traditionally one replaces the normalization constants when the transition rate or cross-section is calculated)

$$\psi = \int d^3p \; e^{-ip\cdot x} \; a_r(p) \; u_r(p) \tag{4.4.4}$$

(for electrons only) and

$$A_\mu = \int d^3k \; \epsilon_\mu^{(\alpha)} \; [e^{-ik\cdot x} \; c_{(\alpha)}(k) + e^{ik\cdot x} \; c^*_{(\alpha)}(k)] \tag{4.4.5}$$

for the electromagnetic field, and for the initial and final states

$$\begin{aligned} |i\rangle &= a_r^*(P) \; |0\rangle \\ |f\rangle &= a^*_{r'}(P') \; c^*_{(\alpha)}(K) \; |0\rangle \; , \end{aligned} \tag{4.4.6}$$

with the commutation relations

$$\begin{aligned} \{a_r(p), \; a^*_{r'}(p')\} &= \delta_{rr'} \;\; \delta^3(\vec{p}-\vec{p}\,') \\ [c_{(\alpha)}(k), \; c^*_{(\alpha')}(k')] &= \delta_{\alpha\alpha'} \;\; \delta^3(\vec{k}-\vec{k}') \; , \end{aligned} \tag{4.4.7}$$

one finds

$$\begin{aligned} A(2\pi)^4 \delta^4(P'+K-P) = ie(2\pi)^4 \delta^4(P'+K-P) \\ \times \; \bar{u}_{r'}(P')\gamma^\mu u_r(P) \; \epsilon_\mu^{(\alpha)}(K) \; . \end{aligned} \tag{4.4.8}$$

The QED vertex V is A with the external spinors and vectors deleted:

$$V(\mathrm{QED}) = ie\gamma_\mu \; . \tag{4.4.9}$$

This is the negative of that usually listed (in, for example, Bjorken and Drell, 1964, App. B), but we have decreed earlier (Sec. II.1) that e>0, the opposite of the usual convention. The result (4.4.9) follows for any combination of directions, but this is not a general feature of gauge theories: the directions do matter on occasion.

Our task is now straightforward: we must proceed systematically through the lepton-boson and boson-boson interactions listed in the first two sections of this chapter to extract the vertices as was done above for quantum electrodynamics.

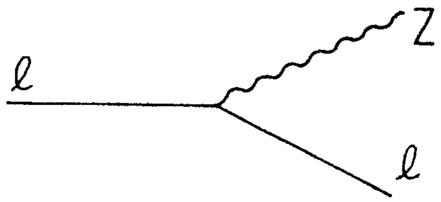

Fig. 3: The charged lepton-Z vertex

The charged lepton(ℓ)-Z boson interaction is contained in (4.1.1):

$$\mathcal{L}_I(\ell\ell Z) = \sum_\ell [G\sin^2\theta_W \ \bar{\ell}\Sigma_- Z\ell - \tfrac{1}{2}G(\cos^2\theta_W - \sin^2\theta_W)\bar{\ell}\Sigma_+ Z\ell] \quad (4.4.10)$$
$$= \sum_\ell (G\sin^2\theta_W \ \bar{\ell}Z\ell - \tfrac{1}{2}G \ \bar{\ell}Z\Sigma_-\ell) \ .$$

The Z boson can be written exactly as was the photon in (4.4.5) and exactly as above one finds the vertex to be (Fig. 3)

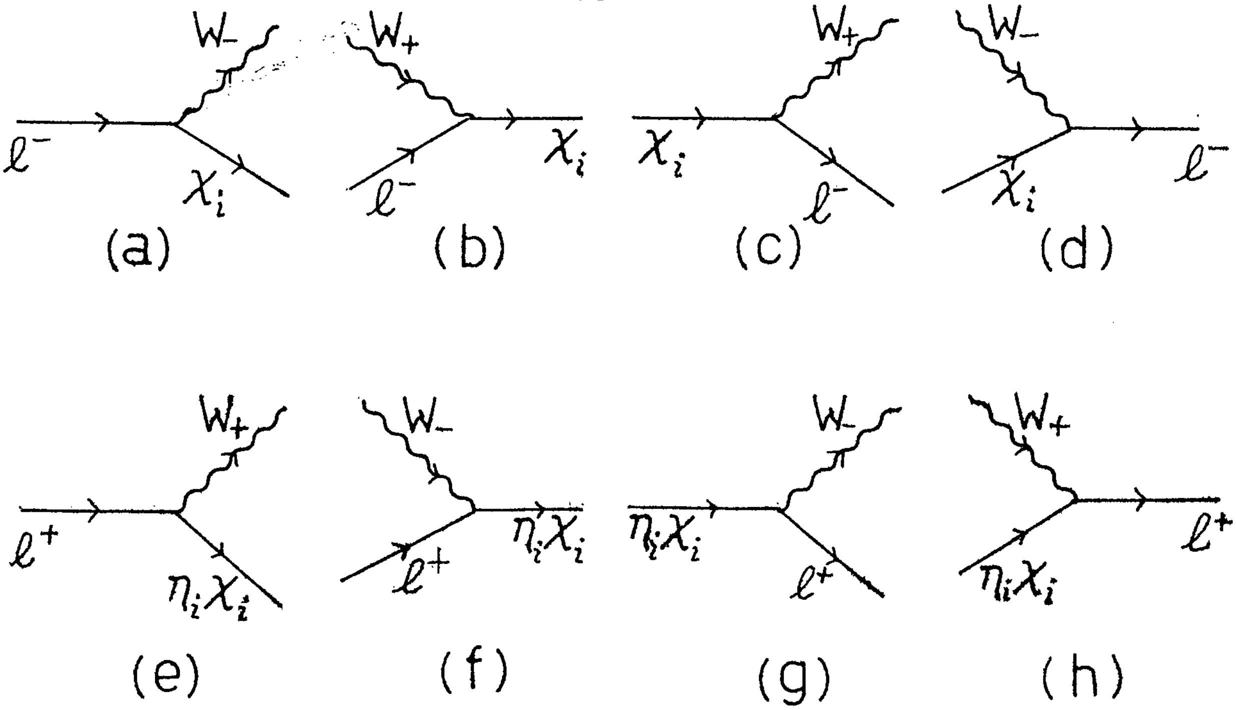

Fig. 4: Charged lepton-neutrino-W vertices

$$V(\ell\ell Z) = iG\gamma_\mu \,(\sin^2\theta_W - \tfrac{1}{2}\Sigma_-) \qquad (4.4.11)$$

for all directions.

The charged lepton(ℓ)-Majorana neutrino(χ)-W boson interaction is described by the first term of (4.1.1):

$$\mathcal{L}_I(\ell\chi W) = \Sigma_{i\ell}\, gU_{i\ell}\, (\bar{\chi}_i\Sigma_+W_+\ell + \bar{\ell}\Sigma_+W_-\chi_i) \ . \qquad (4.4.12)$$

Unforunately, this is somewhat complicated by the phase ambiguity in (3.3.18), as $\eta_i = \pm1$. For cases (a)-(d) in Fig. 4 one finds

$$V(\ell\chi W) = igU_{i\ell}\gamma_\mu\Sigma_- \ , \qquad (4.4.13)$$

with the same, times η_i, for cases (e)-(h) in Fig. 4. In μ^- decay diagrams (a) and (d) of Fig. 4 are needed, which have no η_i factor, while for μ^+ decay diagrams (e) and (h) are needed, with two such factors, but which results in $(\eta_i)^2 \equiv +1$. Thus the factor causes no particular difficulty.

The last interaction described in (4.1.1) or (4.1.2) is the x-Z interaction, but as it will not be needed in what follows it will not be considered further.

The lepton-Higgs boson interactions were given in (4.1.3) to (4.1.6). The interactions with the neutral Higgs particle are not of major importance here and so will not be pursued beyond (4.1.7). However, the interactions with the charged Higgs bosons -- the unphysical $S_{\pm}$, the physical $B_{\pm}$ and the doubly charged $H_{\pm\pm}$ -- are crucial for the theory. Here the sequence of events makes a difference: the vertices for an initial electron and final neutrino are not the same as those for as initial neutrino and final electron.

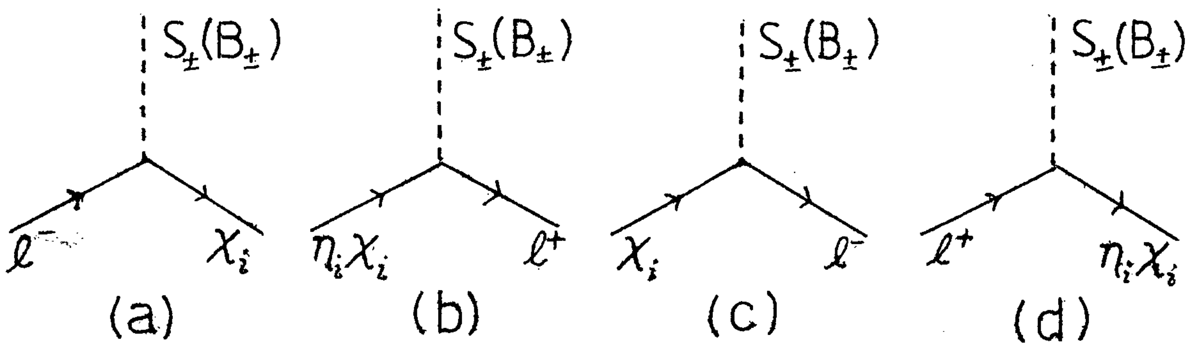

Fig. 5: Charged lepton-neutrino-charged Higgs vertices

For the unphysical Higgs particle we use (4.1.4):

$$\mathcal{L}_I(\ell\chi S) = -(g/M_W)\Sigma_{i\ell}U_{i\ell}[\bar{\chi}_i(M_\ell\Sigma_+ - m_i\Sigma_-)S_+\ell + \bar{\ell}(M_\ell\Sigma_- - m_i\Sigma_+)S_-\chi_i] \ , \tag{4.4.14}$$

where, again, M_ℓ and m_i are the charged lepton and neutrino masses, respectively. The first term of (4.4.14) gives diagrams (a) and (b) of Fig. 5, and one has

$$V(\ell^- \to \chi S) = -i(g/M_W)U_{i\ell}(M_\ell\Sigma_+ - m_i\Sigma_-) \tag{4.4.15}$$

(with the factor η_i if Fig. 5(b) is being used), and similarly, the second term of (4.4.14) gives diagrams (c) and (d) of Fig. 5, with

$$V(\chi \to \ell^- S) = -i(g/M_W)U_{i\ell}(M_\ell\Sigma_- - m_i\Sigma_+) \tag{4.4.16}$$

(with η_i again, for Fig. 5(d)), where the momentum expansion

$$\phi_- = \int d^3k \ (e^{-ik\cdot x} \ a_-(k) + e^{ik\cdot x} \ a_+^*(k)) \ , \tag{4.4.17}$$

with $\phi_+ = \phi_-^*$, is used for all charged scalar fields.

For the physical Higgs boson we use (4.1.5):

$$\mathcal{L}_I(\ell\chi B) = (g/M_W)\Sigma_{i\ell}U_{i\ell}[\bar{\chi}_i(M_\ell\tan\theta\Sigma_+ + m_i\cot\theta\Sigma_-)B_+\ell + \bar{\ell}(M_\ell\tan\theta\Sigma_- + m_i\cot\theta\Sigma_+)B_-\chi_i] \ . \tag{4.4.18}$$

Figure 5 can be used for $B_\pm$ as readily as for $S_\pm$. The first term of (4.4.18) gives diagrams (a) and (b) of Fig. 5 and the vertex

$$V(\ell^- \to \chi B) = i(g/M_W)U_{i\ell}(M_\ell\tan\theta\Sigma_+ + m_i\cot\theta\Sigma_-) \tag{4.4.19}$$

(with a factor η_i for diagram (b)) and the second term of (4.4.18) gives diagrams (c) and (d) and the vertex

$$V(\chi \to \ell^- B) = i(g/M_W)U_{i\ell}(M_\ell\tan\theta\Sigma_- + m_i\cot\theta\Sigma_+) \tag{4.4.20}$$

(with a factor η_i for diagram (d)).

We will also need the lepton-doubly charged Higgs vertices, the interaction being given by (4.1.6):

$$\mathcal{L}_I(\ell H_{\pm\pm}) = \Sigma_{ij}\sqrt{2}\beta_{ij}\ (\bar{\ell}^c_i \Sigma_- H_{++} \ell_j + \bar{\ell}_j \Sigma_+ H_{--} \ell^c_i) \ . \qquad (4.4.21)$$

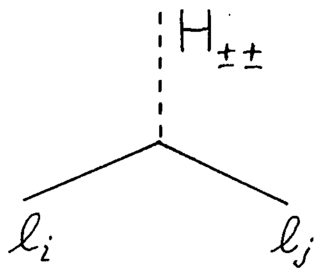

Fig. 6: The charged lepton-doubly charged Higgs vertex

The vertex is (Fig. 6)

$$V(\ell_i \ell_j H_{\pm\pm}) = i\sqrt{2}\beta_{ij}\Sigma_\pm \ , \qquad (4.4.22)$$

where Σ_+ is used if an H_{--} is destroyed (or H_{++} created) and Σ_- is used if an H_{--} is created (or H_{++} destroyed) as (4.4.21) indicates.

This completes the list of relevant lepton vertices -- that is, those that will be needed in the calculation of muon decay -- and we must now list the rather more complicated triple-boson vertices.

The coupling between the $W_\pm$ gauge bosons and the electromagnetic field A is given in the first term of (4.2.1):

$$\mathcal{L}_I(WWA) = \tfrac{1}{2}ie[(A^\mu W^\nu_- - A^\nu W^\mu_-)(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu) \qquad (4.4.23)$$
$$- (A^\mu W^\nu_+ - A^\nu W^\mu_+)(\partial_\mu W^-_\nu - \partial_\nu W^-_\mu) + (\partial_\mu A_\nu - \partial_\nu A_\mu)(W^\mu_- W^\nu_+ - W^\nu_- W^\mu_+) \ .$$

The case of a W_-, for example, radiating a photon would be described by

$$A^\mu = \int d^3k\ e^{ik\cdot x}\ c^*(k)\ \epsilon^\mu(k) \qquad (4.4.24a)$$

$$W^\mu_- = \int d^3p\ e^{-ip\cdot x}\ a_-(p)\ E^\mu(p) \qquad (4.4.24b)$$

$$W_-^{\mu *} = W_+^{\mu} = \int d^3p' \; e^{ip'\cdot x} \; a_-^*(p') \; E^{\mu}(p') \;, \tag{4.4.24c}$$

where ϵ and E are polarization vectors, and states

$$|i\rangle = a_-^*(P) \; |0\rangle \tag{4.4.25a}$$

$$|f\rangle = a_-^*(P') \; c^*(K) \; |0\rangle \;, \tag{4.4.25b}$$

(ignoring polarization indices) and all this, when put into (4.4.1) results in

$$(2\pi)^4 \delta^4(P'+K-P)A = -\tfrac{1}{2}ie(2\pi)^4 \delta^4(K+P'-P)\epsilon^{\alpha}(K)E^{\beta}(P)E^{\gamma}(P') \tag{4.4.26}$$
$$\times \; [(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}-\delta^{\nu}_{\alpha}\delta^{\mu}_{\beta})(P'_{\mu}\eta_{\nu\gamma}-P'_{\nu}\eta_{\mu\gamma}) + (\delta^{\mu}_{\alpha}\delta^{\nu}_{\gamma}-\delta^{\nu}_{\alpha}\delta^{\mu}_{\gamma})(P_{\mu}\eta_{\beta\nu}-P_{\nu}\eta_{\beta\mu})$$
$$+ \; (\delta^{\mu}_{\beta}\delta^{\nu}_{\gamma}-\delta^{\nu}_{\beta}\delta^{\mu}_{\gamma})(K_{\mu}\eta_{\alpha\nu}-K_{\nu}\eta_{\alpha\mu})] \;.$$

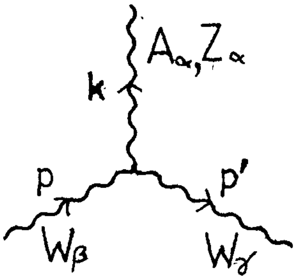

Fig. 7: The W-W-A,Z vertex

When the polarization vectors are dropped, the WWA vertex is obtained:

$$V(WWA) = -ie[(P'_{\alpha}+P_{\alpha})\eta_{\beta\gamma} + (K_{\beta}-P'_{\beta})\eta_{\alpha\gamma} + (-P_{\gamma}-K_{\gamma})\eta_{\alpha\beta}] \;, \tag{4.4.27}$$

where, as in Fig. 7, the momenta are positive in the directions indicated. This result is in agreement with Cheng and Li (1984, App. B). The WWZ vertex, described by the second term of (4.2.1), is virtually identical (Fig. 7):

$$V(WWZ) = ig\cos\theta_W[(P'_{\alpha}+P_{\alpha})\eta_{\beta\gamma} + (K_{\beta}-P'_{\beta})\eta_{\alpha\gamma} + (-P_{\gamma}-K_{\gamma})\eta_{\alpha\beta}] \tag{4.4.28}$$

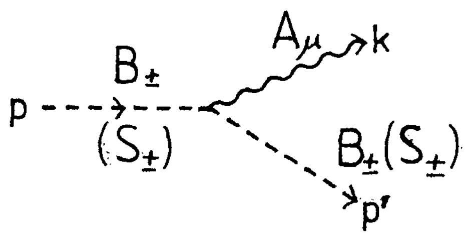

Fig. 8: The charged Higgs-photon vertex

The charged Higgs-electromagnetic coupling is given by (4.2.3) as

$$\mathcal{L}_I(AB) = -ieA^\mu(B_-\partial_\mu B_+ - B_+\partial_\mu B_-) \tag{4.4.29}$$

for the physical Higgs field, and

$$\mathcal{L}_I(AS) = -ieA^\mu(S_-\partial_\mu S_+ - S_+\partial_\mu S_-) \tag{4.4.30}$$

for the unphysical Higgs field. With the field B being given as in (4.4.17) and the electromagnetic field as in (4.4.5) we have

$$|i\rangle = a_-^*(P)\ |0\rangle \tag{4.4.31a}$$

$$|f\rangle = a_-^*(P')\ c_{(\alpha)}^*(K)\ |0\rangle\ , \tag{4.4.31b}$$

and one obtains

$$(2\pi)^4\delta^4(P'+K-P)A = ie(2\pi)^4\delta^4(P'+K-P)\ \epsilon^\mu_{(\alpha)}(K)\ (P'_\mu+P_\mu), \tag{4.4.32}$$

so the vertex, for the directions of Fig. 8, is

$$V(AB) = ie\ (P'_\mu + P_\mu)\ , \tag{4.4.33}$$

and the same applies to the S field:

$$V(AS) = ie\ (P'_\mu + P_\mu)\ . \tag{4.4.34}$$

Fig. 9: The W-A-S vertex

Finally, for the W-A-S vertex (Fig. 9) described by (4.2.5)

$$\mathcal{L}_I(WAS) = -eM_W\, A_\mu(W^\mu_- S_+ + W^\mu_+ S_-) \ , \tag{4.4.35}$$

we have, simply,

$$V(WAS) = -ieM_W \eta_{\mu\nu} \tag{4.4.36}$$

regardless of momentum direction.

The vertices calculated here that are relevant for muon decay are collected and displayed in App. G.

Now, finally, the task of exploring the implications of our gauge theory for muon decay can begin.

IV.5 CHANGES TO WEINBERG-SALAM MODEL AND THE HADRONIC SECTOR

It is important to know to what extent the modifications to the original Weinberg-Salam theory introduced above change the theory's predictions. With the Feynman rules of the previous two sections we are in a position to answer this question, in so far as we are able to estimate the many unknown parameters in the changes brought about by the iso-vector Higgs field.

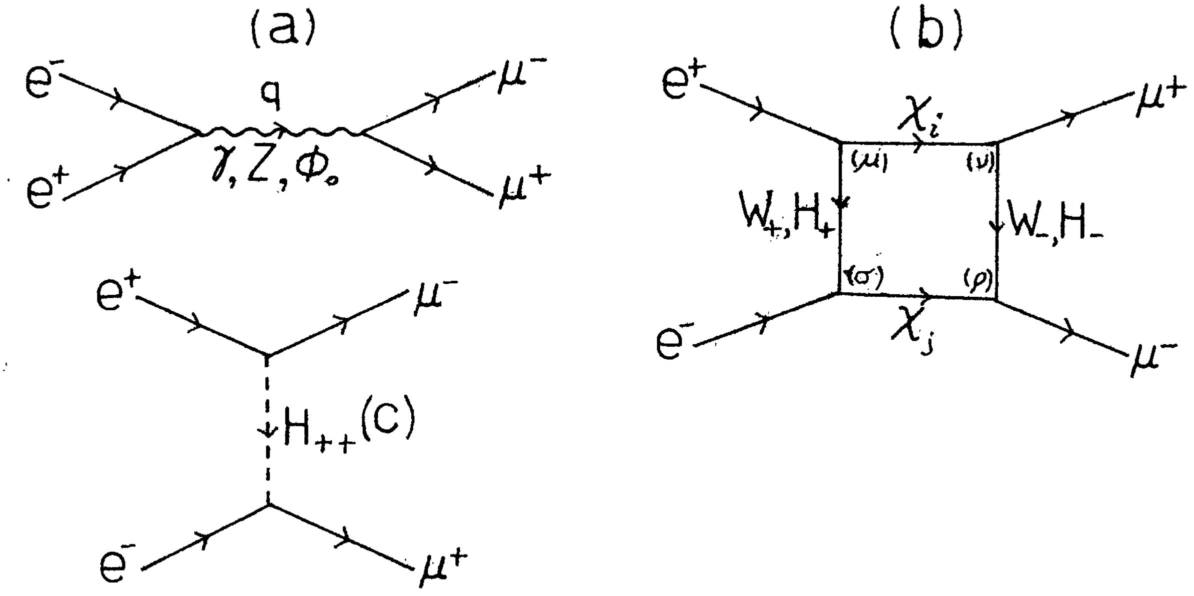

Fig. 10: Feynman diagrams for 2e → 2μ

As an example we will consider the reaction 2e → 2μ which can happen in a number of ways (Fig. 10). Diagram (a) is possible in the Weinberg-Salam theory, while (b) and (c) are only made possible by the changes to the theory made earlier. In Fig. 10, Φ_0 and $H_\pm$ stand for neutral and singly charged

Higgs particles, whether physical or unphysical.

The basic amplitude from QED for Fig. 10(a) with the intermediate γ is, ignoring phases,

$$A(2e \to \gamma \to 2\mu) \sim (e^2/q^2)\ (\bar{v}_e \gamma_\nu u_e)(\bar{u}_\mu \gamma^\nu v_\mu)\ . \qquad (4.5.1)$$

From (4.4.11), the Z's contribution is

$$A(2e \to Z \to 2\mu) \sim [G^2/(q^2 - M_Z^2)]\ [\bar{v}_e \gamma_\nu (\sin^2\theta_W - \tfrac{1}{2}\Sigma_-) u_e] \qquad (4.5.2)$$
$$\times\ [\bar{u}_\mu \gamma^\nu (\sin^2\theta_W - \tfrac{1}{2}\Sigma_-) v_\mu]\ ,$$

which is comparable to (4.5.1) only at very high energies, such as $q^2 \sim M_Z^2$, or higher.

The neutral Higgs contribution is, from (4.1.7),

$$A(2e \to \Phi_0 \to 2\mu) \sim [\beta_e \beta_\mu/(q^2 - M_\Phi^2)]\ \bar{v}_e \Sigma_- u_e\ \bar{u}_\mu \Sigma_+ v_\mu \qquad (4.5.3)$$

and as

$$\beta_e \beta_\mu = g^2\ m_e M_\mu / M_W^2 \sim 10^{-8} g^2\ , \qquad (4.5.4)$$

this is seen to be quite negligible at all energies.

The first new term arises from Fig. 10(b), which for the W is, from (4.4.12),

$$A(2e \to W\chi \to 2\mu) \sim \Sigma_{ij} \int d^4k\ g^2 U_{i1} U_{i2}\ g^2 U_{j1} U_{j2}\ \bar{v}_e \Sigma_+ \gamma_\mu \left(\frac{1}{p_i - k - m_i}\right) \Sigma_+ \gamma_\nu v_\mu$$
$$\times \left(\frac{\eta^{\mu\sigma}}{k^2 - M_W^2}\right)\left(\frac{\eta^{\nu\rho}}{k'^2 - M_W^2}\right)\ \bar{u}_\mu \Sigma_+ \gamma_\rho \left(\frac{1}{p_j - k' - m_j}\right) \Sigma_+ \gamma_\sigma u_e \qquad (4.5.5)$$

which is totally negligible because, not only are there two W propagators, at least two of $(U_{i1} U_{i2} U_{j1} U_{j2})$ would be expected to be small, being off-diagonal. For the Higgs particles one has $g \to g(M_\ell/M_W)$, where M_ℓ is a lepton mass, but in general the amplitudes would be expected to be comparable to that for the W particle because unphysical Higgs particles are needed to keep the overall result gauge invariant. After all, Fig. 10(b) is a fourth order process, while (a) and (c) are of second order.

For Fig. 10(c) we have, from (4.4.21),

$$A(2e \to H_{\pm\pm} \to 2\mu) \sim [(\beta_{12})^2/(k^2-M_H^2)]\ \bar{v}_e \Sigma_+ v_\mu\ \bar{v}_e \Sigma_- v_\mu \qquad (4.5.6)$$

and, in a speculative and probably generous approximation to be made and discussed later, we imagine the Yukawa coupling constants to be comparable, so $\beta_{ij} \sim \beta_i \approx g(M_\ell/M_W)$, in which case this is of order $(m_e M_\mu/M_W^2) \sim 10^{-8}$ times the amplitude for the Z particle, assuming $M_H \sim M_{W,Z}$.

Thus in this example, and others, the new effects introduced along with the vector Higgs fields are comparable to the effects induced by the original Higgs particles, which is to say, very small indeed.

The hadronic sector cannot be totally ignored here as it might bear on weak interaction processes that are related to those of this thesis, particularly the $\mu \to 3e$ decay of Chap. VI. Its primary connection to the Weinberg-Salam model is through the SU(2) spinor or doublet $\Psi_q \equiv \binom{u}{d}_L \equiv \Sigma_-\binom{u}{d}$, and singlets $u_R \equiv \Sigma_+ u$, d_R, where the u and d quarks have electric charges $q_u = \frac{2}{3}e$ and $q_d = -\frac{1}{3}e$, analogous to the electron family's $\Sigma_-\binom{\nu}{e} \equiv \binom{\nu}{e}_L$, e_R. Their weak hypercharges, according to the rule (2.2.11), are $Y(\Psi_q) = \frac{1}{3}$, $Y(u_R) = \frac{4}{3}$ and $Y(d_R) = -\frac{2}{3}$. The quarks' interaction with the Higgs sector is through the Yukawa coupling

$$\begin{aligned}\mathcal{L}(\text{Yuk-q}) = &-\beta_d(\bar{\Psi}_q \Sigma_+ \Phi d + \bar{d}\Sigma_- \Phi^\dagger \Psi_q) \qquad (4.5.7)\\ &- \beta_u(\bar{\Psi}\Sigma_+ \Phi' u + \bar{u}\Sigma_- \Phi'^\dagger \Psi_q)\ ,\end{aligned}$$

with Φ and $\Phi' \equiv i\sigma_2\Phi^*$ as in Sec. II.2, III.2. Because of the quarks' fractional hypercharge there can be *no* Yukawa coupling with the vector Higgs field $\vec{H}$. Thus the only quark-Higgs coupling follows from (4.5.7), which, expanded in terms of the basic fields, is

$$\begin{aligned} \mathcal{L}(\text{Yuk-q}) = & -\beta_d(v/\sqrt{2})\bar{d}d - \beta_u(v/\sqrt{2})\bar{u}u \\ & - \beta_d(\bar{u}\Sigma_+\phi_+d + \bar{d}\Sigma_-\phi_-u + \bar{d}\Sigma_+\phi_0 d + \bar{d}\Sigma_-\phi_0^* d) \\ & - \beta_u(\bar{u}\Sigma_+\phi_0^* u + \bar{u}\Sigma_-\phi_0 u - \bar{d}\Sigma_+\phi_- u - \bar{u}\Sigma_-\phi_+ d) \ . \end{aligned} \tag{4.5.8}$$

Thus the quark masses are

$$m_d = \beta_d \ v/\sqrt{2} \ , \quad m_u = \beta_u \ v/\sqrt{2} \ , \tag{4.5.9}$$

so that

$$\beta_{u,d} \approx g \ (m_{u,d} \ / \ \sqrt{2}M_W) \tag{4.5.10}$$

and the rest of (4.5.8) is used for the quark couplings to the physical and unphysical Higgs fields, and, through them, to the lepton sector.

Thus the quarks are coupled to all of the Higgs fields *except* the doubly charged field $H_{\pm\pm}$. These interactions will be utilized later (Chap. VI) when the question of processes similar to $\mu \to 3e$ is considered.

V. THE DECAY $\mu \to e\gamma$

V.1 RADIATIVE DECAYS IN GENERAL

A complete list of the Feynman diagrams allowed by the gauge theory investigated here with one muon in the initial state and an electron and a photon in the final state would be rather lengthy. The diagrams would be of diverse magnitudes (because, for example, $M_\mu >> m_e$), and, indeed, many would diverge, a characteristic of perturbation methods in quantum field theory. All divergent diagrams must, however, cancel because of the gauge theory's guaranteed renormalizability (Ityzkson and Zuber, 1980, Chap. 8; Cheng and Li, 1984, Chap. 6). In the particular case of a radiative transition it is fortunately possible to extract those finite terms, all of the same order of magnitude, rather directly, without bothering at all with the multitude of terms that become ultimately irrelevant (Nieves, 1982; Cheng and Li, 1984, Sec. 13.3).

The electromagnetic transition amplitude for $\mu \to e\gamma$ (Fig. 11) must have the form

$$A \sim \epsilon^{\nu}_{(\alpha)}(q)\ \bar{u}_e(p')\ J_\nu\ u_\mu(p) \tag{5.1.1}$$

where ϵ is the polarization vector of the photon of 4-momentum q (q^2=0 for real photons) and polarization α (=1,2), and where the operator or multivector J_ν (App. B) is unknown. As the electromagnetic interaction for an electric current j is j·A the current has the form

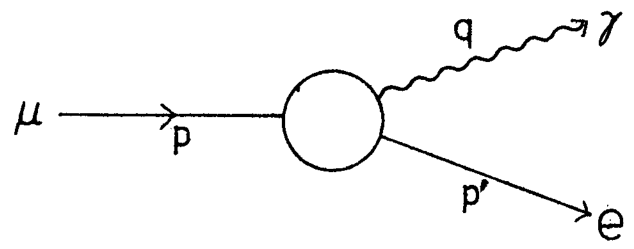

Fig. 11: The radiative decay $\mu \to e\gamma$

$$j_\nu \sim \bar{u}_e \, J_\nu \, u_\mu \, , \tag{5.1.2}$$

and current conservation

$$\partial \cdot j = \partial_\nu j^\nu = 0 \, , \tag{5.1.3}$$

or, in momentum space

$$q \cdot j = 0 \, , \tag{5.1.4}$$

requires that J_ν have the form

$$J_\nu = q_\nu (a + a'\gamma_5) + q \wedge \gamma_\nu (b + b'\gamma_5) \tag{5.1.5}$$

since a real photon satisfies $q^2 = 0$ and

$$\epsilon \cdot q = \epsilon^\nu q_\nu = 0 = \tfrac{1}{2}(\epsilon q + q\epsilon) \, , \tag{5.1.6}$$

where the vector products described in App. B have been employed. The result (5.1.6) makes the first term of (5.1.5) unnecessary. Thus the transition amplitude has the form

$$A \sim \epsilon^\nu_{(\alpha)}(q) \, \bar{u}_e(p') \, q \wedge \gamma_\nu \, (a + b\gamma_5) \, u_\mu(p) \, . \tag{5.1.7}$$

In what follows the electron's mass will be ignored because $M_\mu >> m_e$.

The basis spinors satisfy

$$p \, u_\mu(p) = M_\mu \, u_\mu(p) \tag{5.1.8a}$$

$$p' \, u_e(p') = m_e \, u_e(p') \, , \tag{5.1.8b}$$

and as

$$\gamma_5 \; (q \wedge \gamma_\nu) = (q \wedge \gamma_\nu) \; \gamma_5 \tag{5.1.9}$$

we need only look for the general form, with or without a γ_5,

$$\epsilon^\nu \; \bar{u}_e(p') \; q \wedge \gamma_\nu \; u_\mu(p) = \bar{u}_e q \wedge \epsilon u_\mu = \bar{u}_e q \epsilon u_\mu \; . \tag{5.1.10}$$

This has a number of equivalent forms, using $q = p-p'$ and (5.1.8)

$$\begin{aligned} \bar{u}_e q \epsilon u_\mu &= \bar{u}_e p \epsilon u_\mu - m_e \bar{u}_e \epsilon u_\mu \\ &= -\bar{u}_e \epsilon q u_\mu = \bar{u}_e \epsilon p' u_\mu - M_\mu \bar{u}_e \epsilon u_\mu \; , \end{aligned} \tag{5.1.11}$$

and as

$$p \epsilon = -\epsilon p + 2p \cdot \epsilon \tag{5.1.12}$$

we have

$$\begin{aligned} \bar{u}_e q \epsilon u_\mu &= (2p \cdot \epsilon) \; \bar{u}_e u_\mu + \cdots \\ &= (2p' \cdot \epsilon) \; \bar{u}_e u_\mu + \cdots \; , \end{aligned} \tag{5.1.13}$$

where the dropped terms, such as the $\bar{u}_e \epsilon u_\mu$ terms of (5.1.11), have either the wrong structure for a radiative transition or are negligible (if a factor m_e appears).

To summarize, all terms in the transition amplitudes to follow except those that contain one of the equivalent forms

$$\begin{aligned} &\bar{u}_e q \epsilon (a+b\gamma_5) u_\mu \sim \bar{u}_e p \epsilon (a+b\gamma_5) u_\mu \\ &\sim \bar{u}_e \epsilon p' (a+b\gamma_5) u_\mu \sim (2p \cdot \epsilon) \bar{u}_e (a+b\gamma_5) u_\mu \sim (2p' \cdot \epsilon) \bar{u}_e (a+b\gamma_5) u_\mu \end{aligned} \tag{5.1.14}$$

are to be ignored as having the wrong structure for an electromagnetic transition. This short list will be a considerable aid in simplifying the complex diagrams to follow. It is especially gratifying to discover that no divergent term has one of the wanted forms.

V.2 THE DIAGRAMS AND AMPLITUDES FOR THE DECAY $\mu \to e\,\gamma$

It is appropriate now to consider and list the five Feynman diagrams that are responsible for the radiative decays of muons as permitted by the SU(2)×U(1) gauge theory outlined to this point. The propagators and vertices of the previous chapter provide for an unambiguous translation from these diagrams to their mathematical representation.

The W propagator is rather complex and it will help to first redefine it, given in (4.3.19), by factoring out the i:

$$
\begin{aligned}
-i\Delta_{\mu\nu}(k) &\equiv -i\{\eta_{\mu\nu}/(k^2-M_W^2) - k_\mu k_\nu/[M_W^2(k^2-M_W^2)] \\
&\qquad + k_\mu k_\nu/[M_W^2(k^2-\xi M_W^2)]\} \\
&\equiv -i[\Delta_1(k) + \Delta_2(k) + \Delta_3(k)]_{\mu\nu} \qquad (5.2.1)
\end{aligned}
$$

The amplitude for Fig. 12 (or Diag. W) is, from the Feynman rules of Chap. IV,

$$
\begin{aligned}
A(W) = \Sigma_i \int \frac{d^4k}{(2\pi)^4}\, \bar{u}_e(p')(U_{i1} ig\gamma_\mu \Sigma_-)\left(\frac{1/2\ i}{(p-k)-m_i}\right)(U_{i2} ig\gamma_\nu \Sigma_-)u_\mu(p) \\
\times\ (-i)\Delta^{\mu\rho}(k-q)\ (-i)\Delta^{\sigma\nu}(k)\ (-ie\epsilon^\lambda \Gamma_{\lambda\sigma\rho})\ , \qquad (5.2.2)
\end{aligned}
$$

where

$$
\Gamma_{\lambda\sigma\rho} \equiv (-q-k)_\rho \eta_{\lambda\sigma} + [k-(-k+q)]_\lambda \eta_{\sigma\rho} + [(-k+q)-(-q)]_\sigma \eta_{\lambda\rho} \qquad (5.2.3)
$$

arises from the W-W-A vertex, Eq. (4.4.27), where i=1= electron, i=2= muon have been used, and where the other vertices and propagators are as given in Sec. IV.3 and IV.4. The momentum and other symbols employed in A(W) are given in

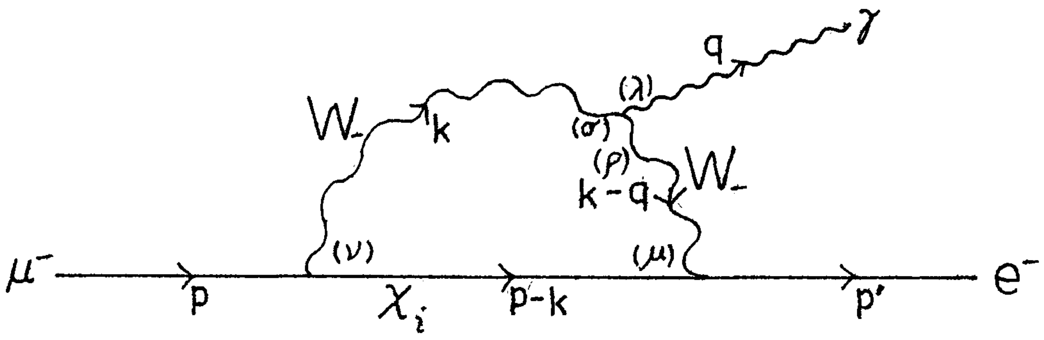

Fig. 12: Feynman diagram W in the decay $\mu \to e\gamma$

---

the appropriate place in Fig. 12. The integration is over the W_- loop: as the 4-momentum k is undetermined by energy-momentum conservation (which guarantees only $p = p' + q$) all values of the W_- momentum make a contribution (Bjorken and Drell, 1964, App. B).

The amplitude for Fig. 13 (Diag. S) is

$$A(S) = \Sigma_i \int \frac{d^4k}{(2\pi)^4} \bar{u}_e(p') \left(\frac{-igU}{M_W}_{i1} (m_e \Sigma_- - m_i \Sigma_+) \right) \left(\frac{{}^1\!/\!{}_2 i}{(p-k)-m_i} \right)$$

$$\times \left(\frac{-igU}{M_W}_{i2} (M_\mu \Sigma_+ - m_i \Sigma_-) \right) u_\mu(p) \left(\frac{i}{(k-q)^2 - \xi M_W^2} \right) \left(\frac{i}{k^2 - \xi M_W^2} \right)$$

$$\times\ ie\ [k + (k-q)]_\lambda\ \epsilon^\lambda\ , \tag{5.2.4}$$

where the vertices (4.4.15), (4.4.16) and (4.4.34) have been used, as has the mass $M_S^2 = \xi M_W^2$ of (2.6.10) for the scalar field propagator, given in (4.3.9).

The amplitude for Fig. 14 (Diag. WS) is

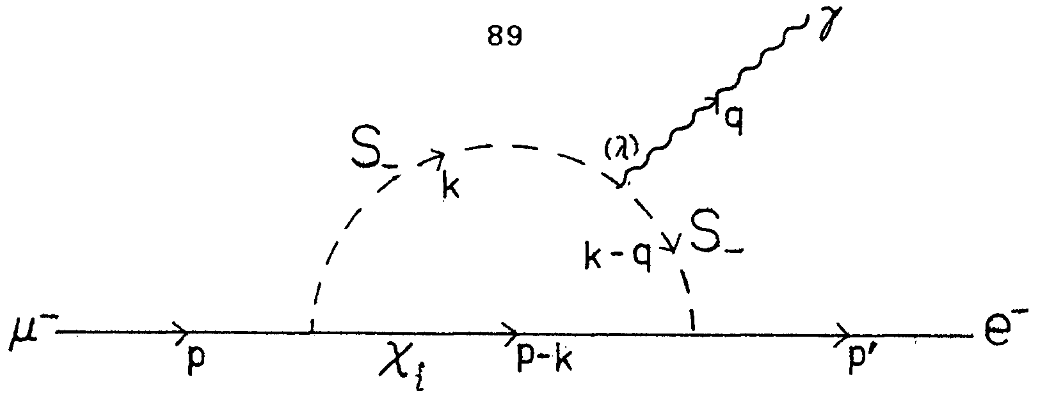

Fig. 13: Feynman diagram S in the decay $\mu \to e\gamma$

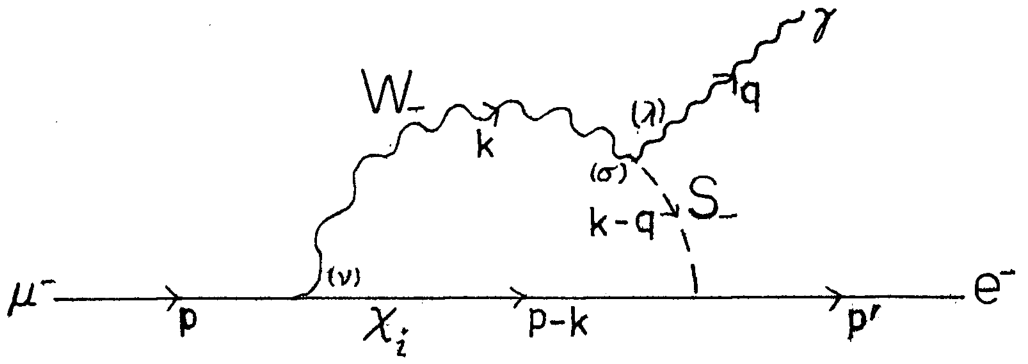

Fig. 14: Feynman diagram WS in the decay $\mu \to e\gamma$

$$A(WS) = \sum_i \int \frac{d^4k}{(2\pi)^4}\ \bar{u}_e(p')\left(\frac{-igU_{i1}}{M_W}(m_e\Sigma_- - m_i\Sigma_+)\right)\left(\frac{1/2\ i}{(p-k)-m_i}\right)$$

$$\times\ (U_{i2}ig\gamma_\nu\Sigma_-)u_\mu(p)\left(\frac{i}{(k-q)^2-\xi M_W^2}\right)(-i)\Delta^{\nu\sigma}(k)$$

$$\times\ (-ieM_W\eta_{\sigma\lambda})\ \epsilon^\lambda\ , \tag{5.2.5}$$

where the one new vertex, Eq. (4.4.35), has been employed.

The amplitude for Fig. 15 (Diag. SW) is, similarly,

$$A(SW) = \sum_i \int \frac{d^4k}{(2\pi)^4}\ \bar{u}_e(p')(U_{i1}ig\gamma_\mu\Sigma_-)\left(\frac{1/2\ i}{(p-k)-m_i}\right) \tag{5.2.6}$$

$$\times\left(\frac{-igU_{i2}}{M_W}(M_\mu\Sigma_+ - m_i\Sigma_-)\right)u_\mu(p)\left(\frac{i}{k^2-\xi M_W^2}\right)(-i)\Delta^{\sigma\mu}(k-q)$$

$$\times\ (-ieM_W\ \eta_{\sigma\lambda})\ \epsilon^\lambda\ .$$

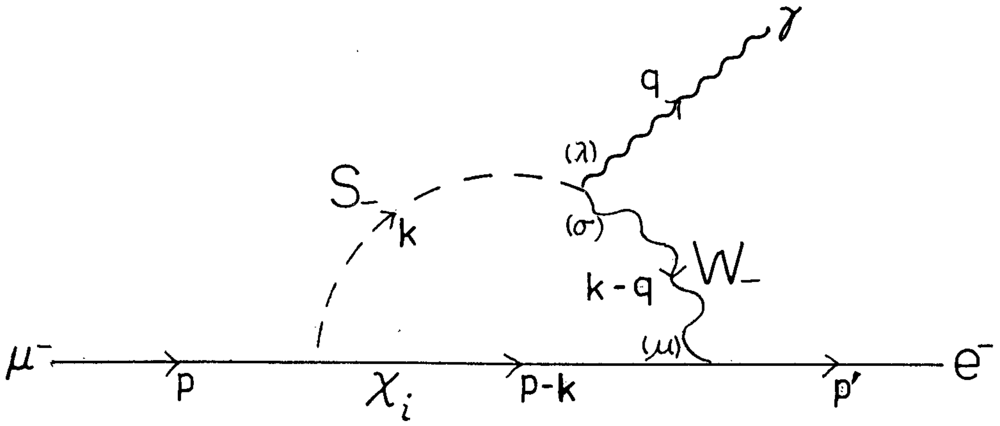

Fig. 15: Feynman diagram SW in the decay $\mu \to e\gamma$

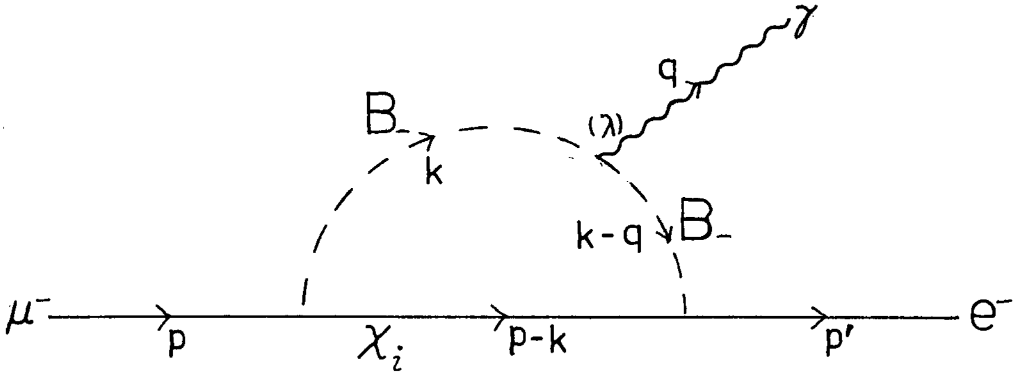

Fig. 16: Feynman diagram B in the decay $\mu \to e\gamma$

Finally, the decay mediated by the physical Higgs boson (B) is, from Fig. 16 (Diag. B),

$$A(B) = \sum_i \int \frac{d^4k}{(2\pi)^4}\, \bar{u}_e(p') \left(\frac{igU_{i1}}{M_W} (m_e \Sigma_- \tan\theta + m_i \Sigma_+ \cot\theta) \right) \left(\frac{{}^1\!/_2\, i}{(p-k) - m_i} \right)$$

$$\times \left(\frac{igU_{i2}}{M_W} (M_\mu \Sigma_+ \tan\theta + m_i \Sigma_- \cot\theta) \right) u_\mu(p) \left(\frac{i}{(k-q)^2 - M_B^2} \right) \left(\frac{i}{k^2 - M_B^2} \right)$$

$$\times\ ie\ [k + (k-q)]_\lambda\ \epsilon^\lambda\ , \qquad (5.2.7)$$

where the vertices of (4.4.19), (4.4.20) and (4.4.33) have been used.

V.3 EVALUATION OF THE AMPLITUDES FOR $\mu \to e \gamma$

The five integrals of the previous section, in their present form, pose a rather formidable mathematical problem. It is fortunately possible to considerably reduce their complexity by the appropriate use of approximations, such as $m_i, m_e \ll M_\mu \ll M_W, M_B$, and by keeping only the lowest possible order of the neutrino masses.

The neutrino propagator $(p-m_i)^{-1}$ is approximated by

$$1/(p-m) = (p+m)/(p^2-m^2) = [(p+m)/p^2]\,(1 - m^2/p^2)^{-1} \qquad (5.3.1)$$

$$= p/p^2 + m/p^2 + m^2p/p^4 + m^3/p^4 + \cdots$$

and only the first non-vanishing term will be kept.

From A(W) (Eq. (5.2.2)) we have a factor (using $p-k \to p$)

$$\Sigma_i U_{i1}U_{i2}\gamma_\mu\Sigma_-(p/p^2 + m_i/p^2 + m_i^2p/p^4 + m_i^3/p^4 + \cdots)\gamma_\nu\Sigma_- . \qquad (5.3.2)$$

The first term vanishes because U is an orthogonal matrix:

$$\Sigma_i\ U_{i1}U_{i2} = \delta_{12} = 0\ ; \qquad (5.3.3)$$

the second vanishes because

$$\Sigma_- m_i\gamma_\nu\Sigma_- = m_i\gamma_\nu\Sigma_+\Sigma_- = 0\ . \qquad (5.3.4)$$

The next term -- the first non-vanishing one -- reduces the factor (5.3.2) to, approximately,

$$\Sigma_i U_{i1}U_{i2}m_i^2\Sigma_+\gamma_\mu p\gamma_\nu\ /\ p^4\ . \qquad (5.3.5)$$

Similarly, the factor

$$\Sigma_i U_{i1}U_{i2}(m_e\Sigma_- - m_i\Sigma_+)[(p+m_i)/p^2 + m_i^2p/p^4 + \cdots](M_\mu\Sigma_+ - m_i\Sigma_-) \qquad (5.3.6)$$

in A(S) from (5.2.4) becomes (setting $m_e = 0$: there would be no m_em_i term anyway)

$$\sum_i U_{i1} U_{i2} m_i^2 \Sigma_+ (p - M_\mu)/p^2 \ ; \tag{5.3.7}$$

the factor

$$\sum_i U_{i1} U_{i2} (m_e \Sigma_- - m_i \Sigma_+) [(p+m_i)/p^2 + m_i^2 p/p^4 + \cdots] \Sigma_+ \gamma_\nu \tag{5.3.8}$$

in A(WS) from (5.2.5) becomes

$$-\sum_i U_{i1} U_{i2} m_i^2 \Sigma_+ \gamma_\nu \ / \ p^2 \ ; \tag{5.3.9}$$

the factor

$$\sum_i U_{i1} U_{i2} \Sigma_+ \gamma_\mu [(p+m_i)/p^2 + m_i^2 p/p^4 + \cdots] (M_\mu \Sigma_+ - m_i \Sigma_-) \tag{5.3.10}$$

in A(SW) from (5.2.6) becomes

$$\sum_i U_{i1} U_{i2} m_i^2 \Sigma_+ \gamma_\mu (-1/p^2 + M_\mu p/p^4) \ ; \tag{5.3.11}$$

and, finally, the factor

$$\sum_i U_{i1} U_{i2} (m_e \Sigma_- \tan\theta + m_i \Sigma_+ \cot\theta) [(p+m_i)/p^2 + m_i^2 p/p^4 + \cdots]$$
$$\times (M_\mu \Sigma_+ \tan\theta + m_i \Sigma_- \cot\theta) \tag{5.3.12}$$

in A(B) from (5.2.7) becomes

$$\sum_i U_{i1} U_{i2} m_i^2 \Sigma_+ \ (M_\mu / p^2 + p \cot^2\theta / p^2) \ . \tag{5.3.13}$$

The amplitudes are now

$$A(W) = -i^6 \frac{g^2 e}{(2\pi)^4} \tfrac{1}{2} \sum_i U_{i1} U_{i2} m_i^2 \int d^4k \bar{u}_e \Sigma_+ \gamma_\mu \frac{(p-k)}{(p-k)^4} \gamma_\nu u_\mu$$
$$\times \Delta^{\mu\rho}(k-q) \Delta^{\nu\sigma}(k) \ \epsilon^\lambda \Gamma_{\lambda\sigma\rho} \tag{5.3.14}$$

$$A(S) = +i^6 \frac{g^2 e}{(2\pi)^4} \tfrac{1}{2} \sum_i \frac{U_{i1} U_{i2} m_i^2}{M_W^2} \int d^4k \bar{u}_e \Sigma_+ \frac{(p-k-M_\mu)}{(p-k)^2} u_\mu$$
$$\times \frac{2k \cdot \epsilon}{(k^2 - \xi M_W^2)[(k-q)^2 - \xi M_W^2]} \tag{5.3.15}$$

$$A(WS) = +i^6 \frac{g^2 e}{(2\pi)^4} \tfrac{1}{2} \sum_i U_{i1} U_{i2} m_i^2 \int d^4k \bar{u}_e \Sigma_+ \gamma_\nu u_\mu$$
$$\times \frac{\Delta^{\nu\sigma}(k)}{(p-k)^2 [(k-q)^2 - \xi M_W^2]} \ \epsilon_\sigma \tag{5.3.16}$$

$$A(SW) = -i^6 \frac{g^2 e}{(2\pi)^4} \tfrac{1}{2}\Sigma_i U_{i1} U_{i2} m_i^2 \int d^4k \bar{u}_e \Sigma_+ \gamma_\nu \left(\frac{-1}{(p-k)^2} + \frac{M_\mu (p-k)}{(p-k)^4}\right) u_\mu$$

$$\times \frac{\Delta^{\nu\sigma}(k-q)}{(k^2-\xi M_W^2)} \epsilon_\sigma \qquad (5.3.17)$$

$$A(B) = +i^6 \frac{g^2 e}{(2\pi)^4} \tfrac{1}{2}\Sigma_i \frac{U_{i1} U_{i2}}{M_W^2} m_i^2 \int d^4k \bar{u}_e \Sigma_+ \left(\frac{M_\mu}{(p-k)^2} + \frac{(p-k)\cot^2\theta}{(p-k)^2}\right) u_\mu$$

$$\times \frac{2\, k\cdot\epsilon}{(k^2-M_B^2)[(k-q)^2-M_B^2]} \;. \qquad (5.3.18)$$

Further reduction of the integrands, and the actual integrations, are much too lengthy to display here, and so have been relegated to App. E. With the notations

$$A \equiv \left(\Sigma_i \frac{g^2 e}{2(2\pi)^4} U_{i1} U_{i2} m_i^2\right)\left(\frac{i\pi^2}{M_W^4} M_\mu\right)(\bar{u}_e \Sigma_+ q\epsilon u_\mu) \;, \qquad (5.3.19)$$

$$f(\xi) \equiv (\ln \xi)/(\xi-1) \qquad (5.3.20a)$$

$$g(\xi) \equiv (\xi \ln \xi)/(\xi-1)^2 - 1/(\xi-1) \;, \qquad (5.3.20b)$$

following Cheng and Li (1984, Sec. 13.3), the final results for the amplitudes are

$$A(W) = A[1/4 - f(\xi)/12 + g(\xi)/2] \qquad (5.3.21a)$$

$$A(S) = A(-5/12\xi) \qquad (5.3.21b)$$

$$A(WS) = \tfrac{1}{4} A[5/6\xi - f(\xi) + g(\xi)/3] \qquad (5.3.21c)$$

$$A(SW) = \tfrac{1}{4} A[5/6\xi + 4f(\xi)/3 - 7g(\xi)/3] \qquad (5.3.21d)$$

$$A(B) = \tfrac{1}{2} A\, (M_W^2/M_B^2)(1 + \tfrac{1}{6}\cot^2\theta) \qquad (5.3.21e)$$

with the sum

$$A(\mu\to e\gamma) = \tfrac{1}{2} A[\tfrac{1}{2} + (M_W^2/M_B^2)(1 + \tfrac{1}{6}\cot^2\theta)] \;, \qquad (5.3.22)$$

which is independent of the gauge parameter ξ as it must be. This result, with the omission of the last term which is not present in the case of purely Dirac neutrinos, is almost identical to Cheng and Li's result (1984, Sec. 13.3) (ours contains the spinor amplitude that Cheng and Li seem to have inadvertantly dropped). Thus, with the exception of the terms that depend on the physical, charged Higgs boson B, the decay $\mu \rightarrow e\gamma$ is indifferent to the neutrino type -- factors of $(g/\sqrt{2})^2$ in the coupling of the Dirac neutrinos to the charged leptons and the bosons become a factor $\frac{1}{2}$ in the Majorana neutrino propagator. But with the charged Higgs particles and the potential for $\cot\theta$ to be large (or more accurately, for $m_i \cot\theta$ to be independent of neutrino mass) the radiative decay rate of the muon has the potential of becoming significantly enhanced in the case of the vector Higgs field. This is discussed in Chap. VII.

V.4 THE DECAY RATE FOR $\mu \rightarrow e\,\gamma$

Standard techniques can now be employed to calculate the transition rate for the decay $\mu \rightarrow e\gamma$ from the invariant amplitude $A(\mu \rightarrow e\gamma)$ of the previous section (Bjorken and Drell, 1964, App. B; Ityzkson and Zuber, 1980, App. A3).

The transition rate is

$$\Gamma(\mu \rightarrow e\gamma) = \int |A(\mu \rightarrow e\gamma)|^2 \frac{d^3q}{(2\pi)^3 2q_0} \frac{m_e}{p_0'} \frac{d^3p'}{(2\pi)^3} (2\pi)^4 \delta^4(p-q-p') \quad (5.4.1)$$

which has the meaning of the probability of decay per unit time per particle, or $\Gamma = 1/\tau$, where τ is the particle's (the muon's) mean lifetime for this decay mode. With

$$C \equiv \left| \frac{g^2 e}{2(2\pi)^4} (\Sigma_i U_{i1} U_{i2} m_i^2) \left(\pi^2 \frac{M_\mu}{M_W^4} \right) \frac{1}{2} [\tfrac{1}{2} + (M_W^2/M_B^2)(1 + \tfrac{1}{6}\cot^2\theta)] \right| \tag{5.4.2}$$

we have, from (5.3.22)

$$|A(\mu\to e\gamma)|^2 = C^2 \; |\bar{u}_e(p')\Sigma_+ q\epsilon u_\mu(p)|^2 \; . \tag{5.4.3}$$

One must now average over the initial muon spin and sum over the final electron and photon spins (designated by $\bar{\Sigma}_{\text{spins}}$). One thereby obtains

$$\bar{\Sigma}_{\text{spins}} |A(\mu\to e\gamma)|^2 = \tfrac{1}{2} \; C^2 \; \Sigma_{\text{spins}} |\bar{u}_e \Sigma_+ q\epsilon u_\mu|^2 \tag{5.4.4a}$$

and with (Bjorken and Drell, 1964, Chap. 7)

$$\Sigma_\alpha \; \epsilon^\mu_{(\alpha)} \; \epsilon^\nu_{(\alpha)} = -\eta^{\mu\nu} \tag{5.4.4b}$$

we have

$$\begin{aligned} &\Sigma_{\text{spins}} |\bar{u}_e \Sigma_+ q\epsilon u_\mu|^2 \\ &\quad = -\eta^{\mu\nu} \; 4S[\Sigma_+ q\gamma_\mu \tfrac{1}{2}(1+p/M_\mu)\gamma_\nu q\Sigma_- \tfrac{1}{2}(1+p'/m_e)] \\ &\quad = 2 \; (p\cdot q) \; (p'\cdot q) \; / \; (m_e M_\mu) \; , \end{aligned} \tag{5.4.5}$$

which is evaluated in App. E and where (App. B) $4S[\cdots]=\mathrm{Tr}[\cdots]$ (in the notation of the matrix representation of the Dirac algebra). We now have

$$\Gamma(\mu\to e\gamma) = \frac{C^2 \; m_e}{m_e M_\mu (2\pi)^2} \int \frac{d^3q}{2q_0} \frac{d^3p'}{p'_0} \; (p\cdot q) \; (p'\cdot q) \; \delta^4(p-q-p') \; . \tag{5.4.6}$$

Since

$$q^2 = (p-p')^2 = 0 = M_\mu^2 - 2p\cdot p' + m_e^2 \approx M_\mu^2 - 2p\cdot p' \tag{5.4.7}$$

we have

$$p\cdot q = p\cdot(p-p') \approx \tfrac{1}{2} \; M_\mu^2 \tag{5.4.8a}$$

$$p'\cdot q = p'\cdot(p-p') \approx \tfrac{1}{2} \; M_\mu^2 \; , \tag{5.4.8b}$$

so that

$$\Gamma(\mu\to e\gamma) = \frac{C^2 M_\mu^3}{4(2\pi)^2} \int \frac{d^3q}{2q_0} \frac{d^3p'}{p_0'} \delta^4(p-q-p') . \qquad (5.4.9)$$

As the integral has the value π (App. E) we have

$$\Gamma(\mu\to e\gamma) = C^2 M_\mu^3 / 16\pi . \qquad (5.4.10)$$

To compare with the usual rate for muon decay (Cheng and Li, 1984, Sec. 13.3)

$$\Gamma(\mu\to e\nu\nu) = G_F^2 M_\mu^5 / 192\pi^3 , \qquad (5.4.11)$$

where (from (2.2.20))

$$G_F / \surd 2 = g^2 / (8M_W^2) , \qquad (5.4.12)$$

we have

$$\frac{\Gamma(\mu\to e\gamma)}{\Gamma(\mu\to e\nu\nu)} = \frac{12 \pi^2 C^2}{G_F^2 M_\mu^2} \qquad (5.4.13)$$

$$= (3\alpha/8\pi)(\textstyle\sum_i U_{i1} U_{i2} m_i^2/M_W^2)^2 [\frac{1}{2} + (M_W/M_B)^2 (1 + \frac{1}{6}\cot^2\theta)]^2$$

where $\alpha \equiv e^2/4\pi \approx 1/137$ is the fine structure constant.

One can gather from (5.4.13) that the $\mu \to e\gamma$ decay mode appears to be suppressed (and is, in the case of Dirac neutrinos, where $1/M_B \equiv 0$) by the (probably) extremely small factor $(m_i/M_W)^4 \sim 10^{-40}$, for $m_i \sim 10$eV (assuming the U_{11}, U_{12} values are not negligible -- they are quite unknown). But if $M_B \sim M_W$ this is not the case, since $(\cot\theta)^4$ is quite large. This point, and its implications, will be discussed in detail in Chap. VII.

VI. THE DECAY $\mu \rightarrow 3e$

VI.1 THE DIAGRAMS FOR THE DECAY $\mu \rightarrow 3e$

It so happened that the diagrams of the previous chapter for the radiative decay of the muon were each of the same order of magnitude in the first non-vanishing approximation, a fact required by the gauge invariance of the theory. A possible exception was the decay sequence $\mu \rightarrow B\chi \rightarrow \gamma e$ (not dependent on the arbitrary gauge parameter) because of the values of the theory's parameters. In this chapter the decay mode $\mu \rightarrow 3e$ will be considered. This decay mode, not being a radiative transition, will result in a much greater variety of possible decay sequences, with a correspondingly greater variety in their relative contributions to the invariant amplitude. The primary task, therefore, is to isolate those diagrams that yield the first non-vanishing approximation to the decay mode $\mu \rightarrow 3e$.

When diagrams similar to those used in the previous chapter are considered one observes that as the photon is "off-shell" (or virtual, or has a 4-momentum q that satisfies $q^2 > 0$) its role can also be played by the neutral Z particle, or even by neutral Higgs particles. Thus in Fig. 17, for example, one has twelve separate diagrams: either a W or an unphysical, internal S (one of each possible case, or four choices), and for each such choice a γ or Z or Φ_0 (representing neutral Higgs

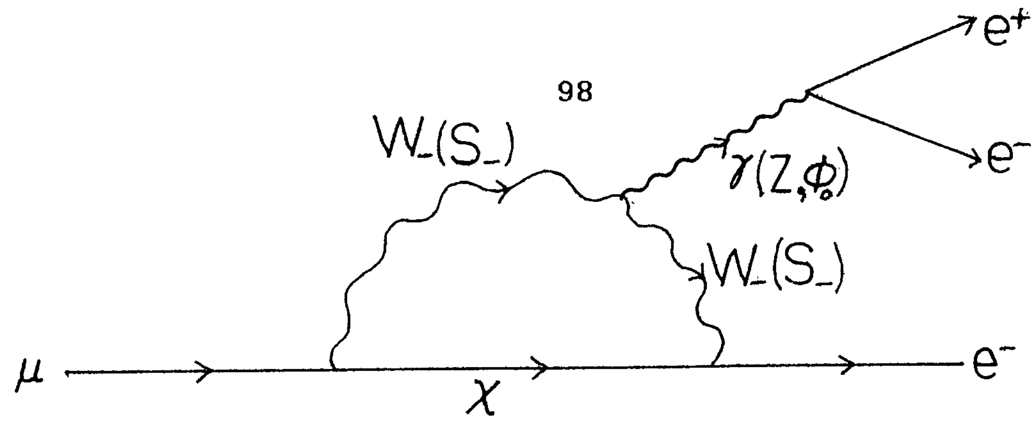

Fig. 17: Possible sequences for the decay mode $\mu \to 3e$

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fields) could give rise to the e^+-e^- pair.

The neutral Higgs coupling to the electron is proportional to m_e (as in Eq. (2.2.7) and (2.2.14)), and they are in all cases much, much smaller than γ-e couplings, because, from (2.2.14) and (2.2.17a),

$$\beta_e \sim m_e/v \sim g\ (m_e/M_W) \sim e\ (m_e/M_W) \qquad (6.1.1)$$

that is, of order $(m_e/M_W) \sim 10^{-5}$ times smaller, even without taking into account the (probably) heavy mass of the neutral Higgs particle.

Further, at the energies involved here ($\sim M_\mu$), the photon propagator ($\sim 1/q^2 \sim 1/M_\mu^2$) is very much greater in magnitude than that of the Z-particle (propagator $\sim 1/M_Z^2$). Thus one should also ignore the possible replacement of a virtual photon with a virtual Z.

The detailed calculation of the four Feynman diagrams of Fig. 17 will be done in the next section (and App. F), but we quote here their combined order of magnitude:

$$A(W,S) \sim g^2 e^2\ (\Sigma_i U_{i1} U_{i2} m_i^2)/(M_W^2 q^2)\ . \qquad (6.1.2)$$

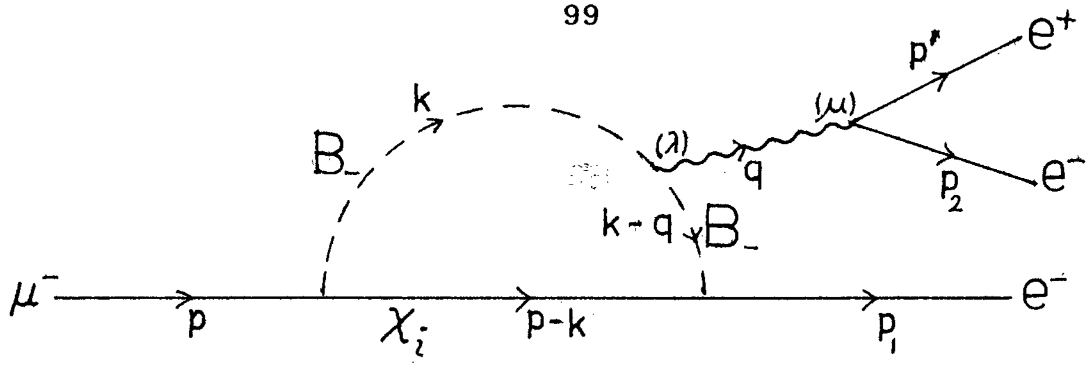

Fig. 18: Contribution of $B_\pm$ to the decay $\mu \to 3e$

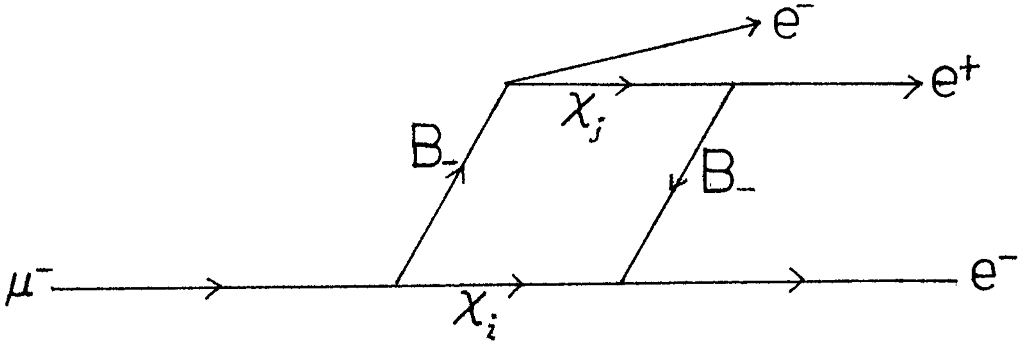

Fig. 19: A 2χ Feynman diagram for the decay $\mu \to 3e$

The physical Higgs particle B (Fig. 18) will be shown to make the contribution

$$A(B) \sim g^2 e^2 \ (\textstyle\sum_i U_{i1} U_{i2} m_i^2 \cot^2 \theta) \ M_\mu^2/(M_W^2 q^2 M_B^2) \ , \qquad (6.1.3)$$

which is (possibly) of the same order of magnitude of (6.1.2) because of the $\cot^2\theta$ factor.

There remain two further diagrams to consider. First, in Fig 19, is a diagram that involves two internal Majorana neutrinos. Its amplitude is, roughly,

$$A(2B,2\chi) \sim g^4 \ (\textstyle\sum_i U_{i1} U_{i2} m_i^2 \cot^2 \theta)^2/(M_W^4 M_B^2) \ , \qquad (6.1.4)$$

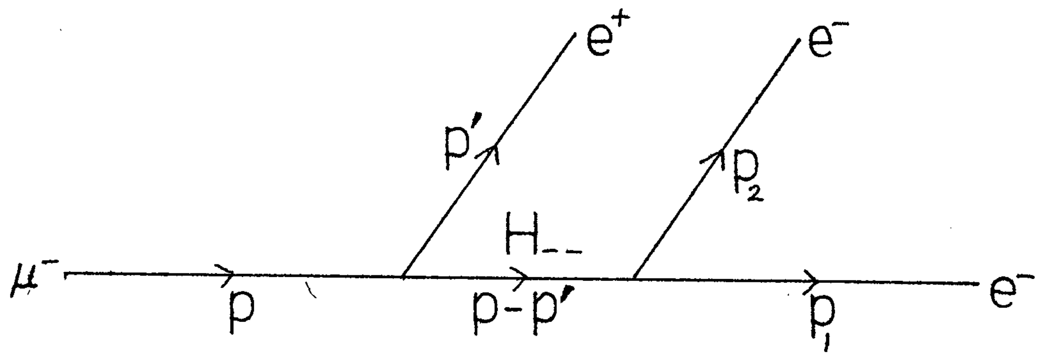

Fig. 20: Contribution of $H_{\pm\pm}$ to the decay $\mu \to 3e$

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which, compared to A(B) is (as $q^2 \sim M_\mu^2$)

$$A(2B,2\chi)/A(B) \sim (\textstyle\sum_i U_{i1} U_{i2} m_i^2 \cot^2\theta)/M_W^2 \ll 1 \qquad (6.1.5)$$

because a generous value for $m_i \cot\theta$ is of the order of a lepton mass.

Finally, there is the diagram of Fig. 20 which involves the doubly charged Higgs particle, a contribution that follows from the interaction term (4.1.6) which is neutrino independent. This isn't even a loop diagram and is therefore readily written down. Its amplitude is, roughly, assuming $M_H \gg M_\mu$,

$$A(H_{--}) \sim \beta_{11}\beta_{12}/M_H^2 \qquad (6.1.6)$$

so that

$$\frac{A(H)}{A(B)} \sim \frac{\beta_{11}\beta_{12}}{M_H^2} \; \frac{M_W^2 M_B^2}{g^2 e^2 (m_i \cot\theta)^2} \; . \qquad (6.1.7)$$

From (4.1.8) follows

$$m_i \cot\theta \sim (v_H \beta_{ij})\,(v/v_H) = v\beta_{ij} \qquad (6.1.8)$$

and as $g^2 v^2 \sim M_W^2$, we have

$$A(H)/A(B) \sim M_B^2/(e^2 M_H^2) \; , \qquad (6.1.9)$$

which could have any value because the Higgs boson masses are unknown.

Thus we see that any one of three diagrams -- Figs. 17,18,20 -- could make the largest contribution to the decay $\mu \to 3e$. Because the masses m_i, M_B, M_H and the parameter cotθ are unknown there is no way of deciding which makes the greatest contribution: there is even the possibility that they are all of the same order of magnitude. Unfortunately, there are divergences involved in the integrals associated with these diagrams which have to be carefully handled, a task to which we now turn our attention.

VI.2 EVALUATION OF THE AMPLITUDES FOR $\mu \to 3e$

We begin the detailed evaluation of the amplitudes for the decay $\mu \to 3e$ by considering the role played by the physical Higgs boson B. The reason for this is that divergent integrals soon appear that have to be carefully manipulated, integrals that appear as well in the amplitudes in which the unphysical Higgs boson S plays an important role in keeping the overall result gauge invariant.

In this section there is truly an overwhelming advantage to be gained by selecting the Feynman-t'Hooft gauge ($\xi = 1$), so such a choice will not be resisted, in contrast with the previous section where it was possible to keep ξ arbitrary. In this section it is not possible to identify the relevant, finite contributions directly, and propagator terms like $k^\mu k^\nu/(k^2-M^2)$ considerably exacerbate the divergence difficulties. The choice $\xi = 1$ eliminates such terms.

Most of the propagators and vertices for the decay sequence of Fig. 18 were used earlier, in (5.2.7), for a contribution to the decay $\mu \to e\gamma$. Here, however, only the terms with $\cot\theta$ will be retained, reflecting the fact that $\cot\theta \gg 1$, and that such terms make by far the largest contribution. In this case the amplitude for Fig. 18 is

$$A(B) = \Sigma_i \int \frac{d^4k}{(2\pi)^4} \bar{u}_e(p_1)\left(\frac{igU_{i1}m_i\cot\theta\Sigma_+}{M_W}\right)\left(\frac{1/2\ i}{(p-k)-m_i}\right)\frac{igU_{i2}m_i\cot\theta\Sigma_- u_\mu(p)}{M_W}$$
$$\times\left(\frac{i}{(k-q)^2-M_B^2}\right)\left(\frac{i}{k^2-M_B^2}\right) ie(2k-q)_\lambda\left(\frac{-i\eta^{\lambda\mu}}{q^2}\right)\bar{u}_e(p_2)(ie\gamma_\mu)v_e(p')$$
$$-\ (p_1 \leftrightarrow p_2) \qquad (6.2.1)$$

where the gauge parameter $\xi = 1$ has been used in the photon propagator (4.3.18) and where the last term indicates that an antisymmetrization over the two final identical electrons must be performed. This expression can be rewritten

$$A(B) = \frac{-i^8g^2e^2}{2(2\pi)^4M_W^2}\ \Sigma_i U_{i1}U_{i2}m_i^2\cot^2\theta \int d^4k\ \frac{\bar{u}_e\Sigma_+(p-k)u_\mu}{q^2(p-k)^2} \qquad (6.2.2)$$
$$\times\ \frac{\bar{u}_e(2k-q)v_e}{(k^2-M_B^2)[(k-q)^2-M_B^2]}\ -\ (p_1 \leftrightarrow p_2)\ .$$

Call the factor in front of the integral sign C. When an integration is performed (App. F) after dimensional regularization ($d^4k \to d^nk$, $n<4$) has been imposed on the integral, one obtains

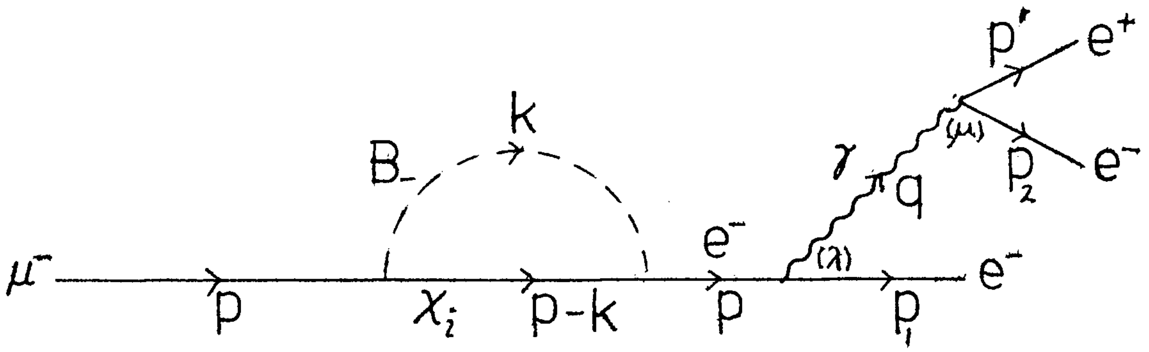

Fig. 21: Feynman diagram needed to renormalize Fig. 18

$$A(B) = C(-\bar{u}_e\Sigma_+\gamma_\lambda u_\mu)(\bar{u}_e\gamma^\lambda v_e) \int\frac{d\alpha d^nK\, K^2\, (1-\alpha)}{q^2[K^2-(1-\alpha)M_B^2]^3} \qquad (6.2.3)$$

$$- \frac{iC\pi^2}{6q^2M_B^2} M_\mu(\bar{u}_e\Sigma_+ u_\mu)(\bar{u}_e p v_e) \quad - (p_1 \leftrightarrow p_2)\ .$$

The integral, for $n = 4$, is not finite. This decay sequence, or diagram, has to be renormalized, or augmented by the diagram of Fig. 21 in which the virtual photon emerges from the emerging electron line. A virtual photon emerging from the entering muon line is of order (m_e/M_μ) relative to Fig. 21 (App. F), and is therefore to be ignored.

The amplitude for Fig. 21 is

$$A(\text{renorm}) = \Sigma_i\int\frac{d^4k}{(2\pi)^4}\bar{u}_e(p_1)ie\gamma_\lambda\left(\frac{i}{p-m_e}\right)\frac{igU}{M_W}{}_{i1}m_i\cot\theta\Sigma_+\left(\frac{{}^1\!/_2\ i}{(p-k)-m_i}\right)$$

$$\times\ \frac{igU}{M_W}{}_{i2}m_i\cot\theta\Sigma_-\ u_\mu(p)\left(\frac{i}{k^2-M_B^2}\right)\left(\frac{-i\eta^{\lambda\mu}}{q^2}\right)\bar{u}_e(p_2)ie\gamma_\mu v_e(p')$$

$$- (p_1 \leftrightarrow p_2) \qquad (6.2.4)$$

with only the highest order terms kept, as in (6.2.1), which becomes, with C defined as in (6.2.2), and after dimensional regularization has been imposed,

$$A(\text{renorm}) = C(\bar{u}_e\Sigma_+\gamma_\lambda u_\mu)(\bar{u}_e\gamma^\lambda v_e) \int\frac{d\alpha d^nK\, (1-\alpha)}{q^2[K^2-(1-\alpha)M_B^2]^2} - (p_1 \leftrightarrow p_2) \qquad (6.2.5)$$

which, too, is divergent for $n = 4$. We now employ special formulae from dimensional regularization (Nash, 1978, Appendix) to evaluate (6.2.3) as

$$A(B) = \frac{-C}{q^2}\,\bar{u}_e \Sigma_+ \gamma_\lambda u_\mu\;\bar{u}_e \gamma^\lambda v_e \int (1-\alpha)\,d\alpha\; \frac{i\;\pi^{n/2}}{\Gamma(3)}\,\frac{2\Gamma(3-1-n/2)}{B^{3-1-n/2}}$$

$$-\;\frac{Ci\pi^2}{6q^2 M_B^2}\,M_\mu\,\bar{u}_e \Sigma_+ u_\mu\;\bar{u}_e p v_e \;-\; (p_1 \leftrightarrow p_2) \qquad (6.2.6)$$

where $B \equiv -(1-\alpha)M_B^2$, Γ is the gamma function and (6.2.5) as

$$A(\text{renorm}) = \frac{C}{q^2}\,\bar{u}_e \Sigma_+ \gamma_\lambda u_\mu\;\bar{u}_e \gamma^\lambda v_e \int (1-\alpha)\,d\alpha\; \frac{i\pi^{n/2}\,\Gamma(2-n/2)}{\Gamma(2)\;B^{2-n/2}} \qquad (6.2.7)$$

$$-\;(p_1 \leftrightarrow p_2)\;,$$

which exactly cancels the first term of (6.2.6). Thus the amplitude for $\mu \to 3e$ via B is

$$A(B) = \frac{-iC\pi^2}{6q^2 M_B^2}\,M_\mu\,\bar{u}_e \Sigma_+ u_\mu\;\bar{u}_e p v_e \;-\; (p_1 \leftrightarrow p_2)\;. \qquad (6.2.8)$$

The amplitude for Fig. 22, with internal W's, is, similar to its $\mu\to e\gamma$ diagram described in (5.2.2),

$$A(W) = \Sigma_i \int \frac{d^4k}{(2\pi)^4}\,\bar{u}_e(p_1)\,U_{i1}\,ig\gamma_\mu \Sigma_- \left(\frac{\tfrac{1}{2}\,i}{(p-k)-m_i}\right) U_{i2}\,ig\gamma_\nu \Sigma_-\, u_\mu(p)$$

$$\times \left(\frac{-i\eta^{\mu\rho}}{(k-q)^2-M_W^2}\right)\left(\frac{-i\eta^{\nu\sigma}}{k^2-M_W^2}\right)\left(\frac{-i\eta^{\lambda\alpha}}{q^2}\right)(-ie\Gamma_{\lambda\sigma\rho})\;\bar{u}_e(p_2)\,ie\gamma_\alpha v_e(p')$$

$$-\;(p_1 \leftrightarrow p_2) \qquad (6.2.9)$$

in the $\xi = 1$ gauge, where

$$\Gamma_{\lambda\sigma\rho} = -(q+k)_\rho\,\eta_{\lambda\sigma} + (2k-q)_\lambda\,\eta_{\sigma\rho} - (k-2q)_\rho\,\eta_{\lambda\sigma}\;. \qquad (6.2.10)$$

This is easily reduced, using (5.3.5), to

$$A(W) = C' \int \frac{d^4k}{q^2}\,\bar{u}_e \Sigma_+ \gamma^\rho \frac{(p-k)}{(p-k)^4}\,\gamma^\sigma u_\mu\;\frac{\Gamma_{\lambda\sigma\rho}\,\bar{u}_e \gamma^\lambda v_e}{[(k-q)^2-M_W^2](k^2-M_W^2)} \;-\; (p_1 \leftrightarrow p_2) \qquad (6.2.11)$$

where

$$C' \equiv i^8 \tfrac{1}{2} g^2 e^2 \Sigma_i U_{i1} U_{i2} m_i^2 \;/\; (2\pi)^4\;. \qquad (6.2.12)$$

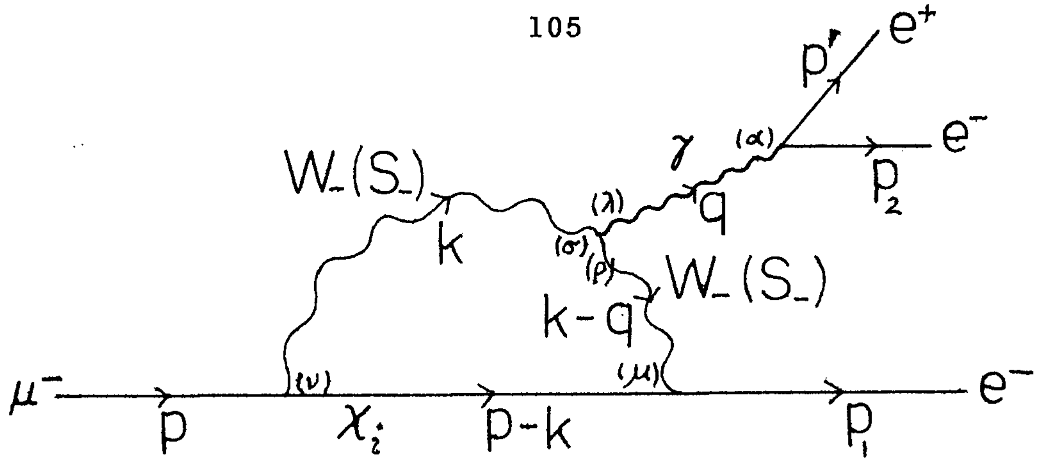

Fig. 22: Contribution of W,S to the decay $\mu \to 3e$

This integral is evaluated in App. F. The result is, using only the leading k^2 terms in the numerator,

$$A(W) = \frac{-i\pi^2 C'}{M_W^2\, q^2}\, 3\bar{u}_e \Sigma_+ \gamma_\lambda u_\mu \; \bar{u}_e \gamma^\lambda v_e \; - (p_1 \leftrightarrow p_2) \; . \qquad (6.2.13)$$

The amplitude for Fig. 22 with the second W replaced by an S is (setting $m_e = 0$)

$$A(WS) = \Sigma_i \int \frac{d^4k}{(2\pi)^4}\, \bar{u}_e \left(\frac{-igU_{i1}}{M_W}(-m_i)\Sigma_+\right)\left(\frac{1/2\; i}{(p-k)-m_i}\right) U_{i2}\, ig\gamma_\nu \Sigma_- \, u_\mu$$
$$\times \left(\frac{i}{(k-q)^2 - M_W^2}\right)\left(\frac{-i\eta^{\nu\sigma}}{k^2 - M_W^2}\right)(-ieM_W \eta_{\sigma\lambda})\left(\frac{-i\eta^{\lambda\alpha}}{q^2}\right)\bar{u}_e ie\gamma_\alpha v_e \; - (p_1 \leftrightarrow p_2) \qquad (6.2.14)$$

which is evaluated (App. F) to be

$$A(WS) = \frac{-i\pi^2 C'}{q^2\, M_W^2}\, \bar{u}_e \Sigma_+ \gamma_\lambda u_\mu \; \bar{u}_e \gamma^\lambda v_e \; - (p_1 \leftrightarrow p_2) \; . \qquad (6.2.15)$$

The amplitude for Fig 22 with the first W replaced by an S is

$$A(SW) = \Sigma_i \int \frac{d^4k}{(2\pi)^4}\, \bar{u}_e U_{i1}\, ig\gamma_\mu \Sigma_- \left(\frac{1/2\; i}{(p-k)-m_i}\right)\left(\frac{-igU_{i2}}{M_W}(M_\mu \Sigma_+ - m_i \Sigma_-)\right) u_\mu$$
$$\times \left(\frac{-i\eta^{\mu\sigma}}{(k-q)^2 - M_W^2}\right)\left(\frac{i}{k^2 - M_W^2}\right)(-ieM_W \eta_{\sigma\lambda})\left(\frac{-i\eta^{\lambda\alpha}}{q^2}\right)\bar{u}_e ie\gamma_\alpha v_e \; - (p_1 \leftrightarrow p_2) \qquad (6.2.16)$$

which is similarly evaluated (App. F) to be

$$A(SW) = \frac{-i\pi^2 C'}{q^2\ M_W^2}\ \bar{u}_e \Sigma_+ \gamma_\lambda u_\mu\ \bar{u}_e \gamma^\lambda v_e - (p_1 \leftrightarrow p_2)\ . \qquad (6.2.17)$$

The amplitude for Fig. 22 with both W's replaced by S's is

$$A(S) = \Sigma_i \int \frac{d^4k}{(2\pi)^4}\ \bar{u}_e \left(\frac{-igU}{M_W}{}_{i1}(-m_i)\Sigma_+\right)\left(\frac{1/2\ i}{(p-k)-m_i}\right)\frac{igU}{M_W}{}_{i2}(M_\mu \Sigma_+ - m_i \Sigma_-)\ u_\mu$$
$$\times \left(\frac{i}{(k-q)^2-M_W^2}\right)\left(\frac{i}{k^2-M_W^2}\right) ie(2k-q)_\lambda \left(\frac{-i\eta^{\lambda\alpha}}{q^2}\right)\bar{u}_e ie\gamma_\alpha v_e - (p_1 \leftrightarrow p_2) \qquad (6.2.18)$$

This integral diverges and must be handled exactly as was done for the similar Higgs B particle, which is to say, it must be considered in addition to a diagram similar to Fig 21. The only difference is in the factors multiplying the integral, so the result is

$$A(S) = \left(\frac{i\pi^2 C' M_\mu}{6q^2 M_W^4}\right)\bar{u}_e \Sigma_+ u_\mu\ \bar{u}_e p v_e - (p_1 \leftrightarrow p_2) \qquad (6.2.19)$$

which is of order (M_μ^2/M_W^2) less than (6.2.13), (6.2.15) and (6.2.17) and so is ignored.

Thus the final result for the $\mu \to 3e$ decay mediated by the W with its co-requisite gauge particles, the S's, designated collectively as A(W), is

$$A(W) = \left(\frac{-i\pi^2 C'}{q^2\ M_W^2}\right) 5\bar{u}_e \Sigma_+ \gamma_\lambda u_\mu\ \bar{u}_e \gamma^\lambda v_e - (p_1 \leftrightarrow p_2)\ . \qquad (6.2.20)$$

Finally, the amplitude for Fig. 20 is (from (4.1.6) and C.7))

$$A(H) = (i\sqrt{2}\beta_{11})(i\sqrt{2}\beta_{12})\ \frac{\bar{v}_\mu(p)v_e(p')}{(p-p')^2-M_H^2}\ [\bar{u}_e(p_1)v_e(p_2)-\bar{u}_e(p_2)v_e(p_1)] \qquad (6.2.21)$$

and as $(p-p')^2 \sim M_\mu^2 << M_H^2$ (presumably), we have

$$A(H) \approx \frac{2\beta_{11}\beta_{12}}{M_H^2}\ \bar{v}_\mu(p)v_e(p')\ [\bar{u}_e(p_1)v_e(p_2)-\bar{u}_e(p_2)v_e(p_1)]. \qquad (6.2.22)$$

As mentioned, we have no way of knowing which of the three amplitudes (6.2.8), (6.2.20) or (6.2.22) makes the greatest contribution to the decay $\mu \to 3e$, because each contains unknown masses. We now consider the $\mu \to 3e$ decay rates as implied by these amplitudes.

VI.3 THE DECAY RATE FOR $\mu \to 3e$

With the final form for the $\mu \to 3e$ decay amplitudes at hand standard techniques, such as those applied in the previous chapter, will now be used to obtain the transition rate. As we are mainly concerned with establishing the approximate μ lifetime for this decay we will only evaluate them for each of the three decay modes separately, and not consider the interference effects. The reason for this is that each amplitude has a different unknown parameter in it (such as a mass or Yukawa coupling constant or the parameter θ).

The spinor amplitudes to be evaluated are, first, from A(W) in (6.2.20),

$$\begin{aligned} S_W^2 \equiv \overline{\sum}_{\text{spins}} | & \bar{u}_e(p_1)\Sigma_+\gamma_\lambda u_\mu(p)\ \bar{u}_e(p_2)\gamma^\lambda v_e(p')\ /\ q_2^2 \\ & - \bar{u}_e(p_2)\Sigma_+\gamma_\lambda u_\mu(p)\ \bar{u}_e(p_1)\gamma^\lambda v_e(p')\ /\ q_1^2 |^2 \end{aligned} \tag{6.3.1}$$

where the overbar in the summation sign once again refers to an average over the initial muon spin (a factor $\frac{1}{2}$, essentially) and where, using $p = p_1+p_2+p'$,

$$q_{1,2} \equiv p' + p_{1,2} = p - p_{2,1} \quad , \tag{6.3.2}$$

from A(B) in (6.2.8),

$$S_B^2 \equiv \bar{\sum}_{spins} |\bar{u}_e(p_1)\Sigma_+ u_\mu(p)\ \bar{u}_e(p_2) p v_e(p') \ /\ q_2^2 \tag{6.3.3}$$

$$- \bar{u}_e(p_2)\Sigma_+ u_\mu(p)\ \bar{u}_e(p_1) p v_e(p') \ /\ q_1^2|^2 \quad ,$$

and finally, from A(H) in (6.2.22),

$$S_H^2 \equiv \bar{\sum}_{spins} \bar{v}_\mu(p) v_e(p')\ [\bar{u}_e(p_1)v_e(p_2) - \bar{u}_e(p_2)v_e(p_1)]|^2 . \tag{6.3.4}$$

The results of these rather lengthy calculations are (App. F), using $M_\mu \equiv M$, $m_e \equiv m$ in this section,

$$S_W^2 = (1/2Mm^3)[(m^2 p\cdot p_1 + p\cdot p_2 p_1\cdot p' + p\cdot p' p_1\cdot p_2)/(p-p_1)^4 \tag{6.3.5}$$

$$+ (m^2 p\cdot p_2 + p\cdot p_1 p_2\cdot p' + p\cdot p' p_1\cdot p_2)/(p-p_2)^4$$

$$+ (m^2 p\cdot p_1 + m^2 p\cdot p_2 - m^2 p\cdot p' + 2p\cdot p' p_1\cdot p_2)/(p-p_1)^2(p-p_2)^2] \ ,$$

$$S_B^2 = (1/4Mm^3)\{p\cdot p_1(2p\cdot p' p\cdot p_2 - M^2 p'\cdot p_2 - M^2 m^2)/(p-p_1)^4 \tag{6.3.6}$$

$$+ p\cdot p_2(2p\cdot p' p\cdot p_1 - M^2 p'\cdot p_1 - M^2 m^2)/(p-p_2)^4$$

$$- [4p\cdot p_1 p\cdot p_2 p\cdot p' - M^2(p\cdot p_2 p'\cdot p_1 + p\cdot p_1 p'\cdot p_2 + p_1\cdot p_2 p\cdot p')$$

$$+ M^2 m^2(p\cdot p' - p\cdot p_1 - p\cdot p_2)]/2(p-p_1)^2(p-p_2)^2\}$$

and,

$$S_H^2 = 2(1+p\cdot p'/mM)(p_1\cdot p_2/m^2 - 1) \tag{6.3.7}$$

In the rest frame of the muon where $p = M\gamma_0$, we have

$$p\cdot p_i = ME_i \quad (i=1,2) \tag{6.3.8}$$

$$p\cdot p' = p\cdot(p-p_1-p_2) = M^2 - M(E_1+E_2) \ ,$$

and $p_1\cdot p_2$ follows from

$$p'^2 = (p-p_1-p_2)^2 = m^2 = M^2 + 2m^2 - 2M(E_1+E_2) + 2p_1\cdot p_2 \ . \tag{6.3.9}$$

If we now define

$$E_i \equiv \tfrac{1}{2}Mx_i \quad (i=1,2) \tag{6.3.10}$$

and set $m = 0$ wherever it occurs in the numerator only, the spinor amplitudes squared are

$$S_W^2 \approx \frac{1}{8Mm^3}\left(\frac{(2-x_1-x_2)(x_1+x_2-1) + x_2(1-x_2)}{(1-x_1+m^2/M^2)^2}\right. \\ + \frac{(2-x_1-x_2)(x_1+x_2-1) + x_1(1-x_1)}{(1-x_2+m^2/M^2)^2} \\ \left. + \frac{2(2-x_1-x_2)(x_1+x_2-1)}{(1-x_1+m^2/M^2)(1-x_2+m^2/M^2)}\right), \tag{6.3.11}$$

$$S_B^2 \approx \frac{M}{16m^3}\left(\frac{x_1x_2(2-x_1-x_2) + x_1(x_1-1)}{(1-x_1+m^2/M^2)^2}\right. \\ + \frac{x_1x_2(2-x_1-x_2) + x_2(x_2-1)}{(1-x_2+m^2/M^2)^2} \\ \left. + \frac{(2-x_1-x_2)(2x_1x_2-x_1-x_2+1) + x_1(x_1-1) + x_2(x_2-1)}{(1-x_1+m^2/M^2)\ (1-x_2+m^2/M^2)}\right) \tag{6.3.12}$$

and

$$S_H^2 \approx (M^3/2m^3)(2-x_1-x_2)(x_1+x_2-1)\ . \tag{6.3.13}$$

The transition rate, as in the previous chapter, is

$$\Gamma(\mu\to 3e) = \tfrac{1}{2}\int |A(\mu\to 3e)|^2\ (2\pi)^4\delta^4(p-p_1-p_2-p') \\ \times \frac{m}{E'}\frac{d^3p'}{(2\pi)^3}\frac{m}{E_1}\frac{d^3p_1}{(2\pi)^3}\frac{m}{E_2}\frac{d^3p_2}{(2\pi)^3}, \tag{6.3.14}$$

with the required statistical factor ($\frac{1}{2}$) because of the two identical electrons in the final state. A first integration over p' (that is, over the e^+ states) results in (App. F)

$$\Gamma = m^3\int\frac{|A|^2}{(2\pi)^5E_1E_2}\ d^3p_1\ d^3p_2\ \delta[(p-p_1-p_2)^2-m^2]\tfrac{1}{2}[1+\epsilon(p-p_1-p_2)] \tag{6.3.15}$$

and because $|A|^2$ depends only on E_1 and E_2 we obtain (App. F)

$$\Gamma = [m^3M^2/4(2\pi)^3]\ \int|A|^2\ dx_1\ dx_2 \tag{6.3.16}$$

which is exact.

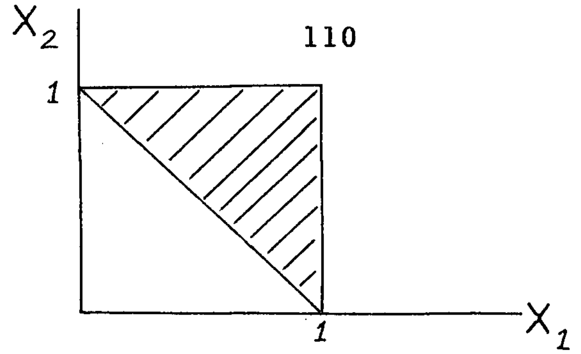

Fig. 23: Approximate integration region in the x_1-x_2 plane

Fig. 24: Maximum and minimum electron energies

Now, the x-values vary from 0 to 1, approximately, as indicated in Fig. 23, but it is important to be more precise so as to avoid (6.3.11) and (6.3.12) diverging in the $m \to 0$ limit. One electron could be at rest (Fig. 24a) so the minimum x-value from (6.3.10) is

$$x_{min} = 2E_{min}/M = 2m/M \ , \tag{6.3.17}$$

and the maximum x-value is (Fig. 24b)

$$x_{max} = 1 - 3m^2/M^2 \tag{6.3.18}$$

which follows from elementary energy-momentum conservation principles. From $(p')^2 = (p-p_1-p_2)^2$ follows

$$M^2(x_1+x_2) = M^2 + m^2 + 2p_1 \cdot p_2 \tag{6.3.19}$$

which gives the exact x_1-x_2 integration region, and

$$(x_1+x_2)_{max} = 2 - 2m/M \ , \tag{6.3.20}$$

which follows from Fig. 24a with the positron at rest, and

$$(x_1+x_2)_{min} = 1 + 3m^2/M^2 \ , \tag{6.3.21}$$

because the minimum value of $p_1 \cdot p_2$ is m^2, when the electrons are at rest relative to each other. We must now integrate (6.3.11) over the appropriate x_1-x_2 region.

The first two terms of (6.3.11) will give the same integral, namely

$$\int_{\epsilon_1}^{1-\epsilon_2} dx_1 \int_{1+\epsilon_2-x_1}^{1-\epsilon_2} dx_2 \ \frac{(2-x_1-x_2)(x_1+x_2-1)+x_2(1-x_2)}{(1-x_1+m^2/M^2)^2} \tag{6.3.22}$$

where $\epsilon_1 \equiv 2m/M$ and $\epsilon_2 \equiv 3m^2/M^2$. The integral is, with $x \equiv x_1$,

$$\int_{\epsilon_1}^{1-\epsilon_2} dx \ \frac{(-2x^3/3 + x^2)}{(1-x+m^2/M^2)^2} \tag{6.3.23}$$

where the ϵ terms from the x_2 integration can be ignored at this stage. This integral is

$$-\frac{2}{3}\left(\frac{1}{2}(1-x+m^2/M^2)^2 - 3(1+m^2/M^2)(1-x+m^2/M^2) + \frac{(1+m^2/M^2)^3}{(1-x+m^2/M^2)} + 3(1+m^2/M^2)\ln(1-x+m^2/M^2)\right)$$
$$+\left(-(1-x+m^2/M^2) + \frac{(1+m^2/M^2)^2}{(1-x+m^2/M^2)} + 2(1+m^2/M^2)\ln(1-x+m^2/M^2)\right)\Bigg|_{\epsilon_1}^{1-\epsilon_2} \tag{6.3.24}$$

in which the dominant terms are the $(1-x+m^2/M^2)$ factors in the denominator evaluated at the upper limit. The value of this integral is, approximately,

$$-\tfrac{2}{3}[1/(4m^2/M^2)] + [1/(4m^2/M^2)] = M^2/12m^2 \ . \tag{6.3.25}$$

The second integral from (6.3.11) is, approximately,

$$\int_{\epsilon_1}^{1-\epsilon_2} dx_1 \ \int_{1-x_1}^{1} dx_2 \ \frac{2(2-x_1-x_2)(x_1+x_2-1)}{(1-x_1+m^2/M^2)(1-x_2+m^2/M^2)} \tag{6.3.26}$$

and its highest order term is a logarithmic term like those in

(6.3.24), which, compared to (6.3.25), is negligible.

Thus the final value for the W-mediated decay rate is

$$\Gamma_W(\mu\to 3e) = \frac{m^3\ M^2}{2^2(2\pi)^3}\left|\frac{5\pi^2 C'}{M_W^2}\right|^2\left(\frac{1}{8Mm^3}\right)\left(\frac{2M^2}{12m^2}\right) \tag{6.3.27}$$

$$= \frac{5^2}{3(2^{12})(2\pi)^7}\ g^4 e^4\ (\textstyle\sum_i U_{i1}U_{i2}m_i^2)^2\left(\frac{M^3}{m^2 M_W^4}\right)$$

$$= \frac{5^2}{3(2^7)(2\pi)^7}\ G_F^2\ e^4\frac{M^3}{m^2}\ (\textstyle\sum_i U_{i1}U_{i2}m_i^2)^2$$

where, again, $G_F/\surd 2 = g^2/(8M_W^2)$. In comparison with the $\mu\to e\nu\nu$ transition rate (5.4.11) we have

$$\frac{\Gamma_W(\mu\to 3e)}{\Gamma(\mu\to e\nu\nu)} = \frac{5^2}{2^4(2\pi)^4}\ \frac{e^4}{m^2M^2}\ (\textstyle\sum_i U_{i1}U_{i2}m_i^2)^2\ . \tag{6.3.28}$$

For the B-mediated decay sequence we have for the first term of (6.3.12)

$$\int dx_1\,dx_2\ \frac{x_1x_2(2-x_1-x_2)\ +x_1(x_1-1)}{(1-x_1+m^2/M^2)^2} = \int_{\epsilon_1}^{1-\epsilon_2}\frac{x^4\ dx}{6(1-x+m^2/M^2)^2} \tag{6.3.29}$$

$$\approx M^2\ /\ 24m^2\ ,$$

obtained in a fashion similar to the integral (6.3.22). The second term will give the same result, while the third term is negligible by comparison, as it was in the previous case, and for the same reason.

The final form for this decay rate is then

$$\Gamma_B(\mu\to 3e) = \frac{m^3\ M^2}{2^2(2\pi)^3}\left|\frac{C\pi^2 M}{6M_B^2}\right|^2\left(\frac{M}{16m^3}\right)\left(\frac{2M^2}{24m^2}\right) \tag{6.3.30}$$

$$= \frac{g^4\ e^4\ M^7}{3^3 2^{12}(2\pi)^7 m^2 M_W^4 M_B^4}\ (\textstyle\sum_i U_{i1}U_{i2}m_i^2\cot^2\theta)^2$$

$$= \frac{e^4\ M^7}{3^3 2^7 (2\pi)^7 m^2 M_B^4}\ G_F^2\ (\textstyle\sum_i U_{i1}U_{i2}m_i^2\cot^2\theta)^2$$

and in comparison to the $\mu\to e\nu\nu$ rate

$$\frac{\Gamma_B(\mu\to 3e)}{\Gamma(\mu\to e\nu\nu)} = \frac{e^4\ M^2}{3^2 2^4 (2\pi)^4 m^2 M_B^4}\ (\textstyle\sum_i U_{i1}U_{i2}m_i^2\cot^2\theta)^2 \tag{6.3.31}$$

and the W-mediated rate

$$\frac{\Gamma_B(\mu\to 3e)}{\Gamma_W(\mu\to 3e)} = \frac{(M\cot\theta)^4}{(3^2)M_B^4} \tag{6.3.32}$$

which may indicate that the rates are comparable.

For the doubly charged Higgs particle we have the integral, from (6.3.13),

$$\int dx_1\,dx_2\ \ (2-x_1-x_2)(x_1+x_2-1) = 1/12 \tag{6.3.33}$$

and the decay rate becomes

$$\Gamma_H(\mu\to 3e) = \frac{(\beta_{11}\beta_{12})^2\ M^5}{24\ (2\pi)^3\ M_H^4}\ , \tag{6.3.34}$$

which, in comparison to the $\mu\to e\nu\nu$ rate, is

$$\frac{\Gamma_H(\mu\to 3e)}{\Gamma(\mu\to e\nu\nu)} = \frac{(\beta_{11}\beta_{12})^2}{G_F^2\ M_H^4}\ . \tag{6.3.35}$$

Having no neutrino mass or θ-parameter in it, there seems to be no point in comparing Γ_H with Γ_W or Γ_B at this time.

VI.4 THE CROSS-SECTIONS FOR $\mu^+ + X \rightarrow e^+ + X$

An experiment searching for the decay $\mu \rightarrow 3e$ could conceivably be complicated by other processes and reactions in which muons disappear with the production of electrons, processes that, like the $\mu \rightarrow 3e$ decay, violate electron and muon number conservation. We note here two possibilities in which an incoming μ^+ beam is subject to the possibility, through physical, but virtual, charged Higgs particles, that a μ^+ is converted to an e^+. It is important to establish whether such reaction rates would be large enough to interfere with a $\mu \rightarrow 3e$ experiment.

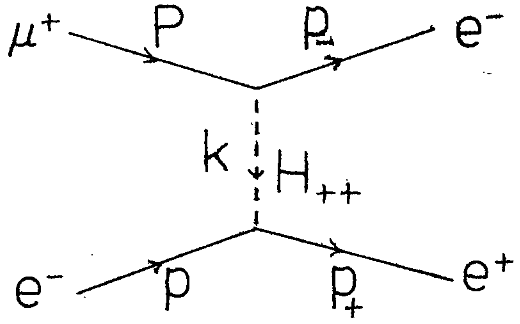

Fig. 25: Feynman diagram for $\mu^+ + e^- \rightarrow e^+ + e^-$

In Fig. 25 the doubly charged Higgs field could play this role. The amplitude for the Feynman diagram of Fig. 25 is

$$A(\mu^+, e^- \rightarrow e^+, e^-) = \frac{(i\sqrt{2}\beta_{11})(i\sqrt{2}\beta_{12})}{k^2 - M_H^2}\, \bar{v}(p)\Sigma_- v(p_+)\ \bar{v}_\mu(P)\Sigma_- v(p_-) \tag{6.4.1}$$

$$\approx (2\beta_{11}\beta_{12}/M_H^2)\ \bar{v}(p)\Sigma_- v(p_+)\ \bar{v}_\mu(P)\Sigma_- v(p_-)\ ,$$

since $M_H^2 \gg k^2 \sim M_\mu^2$. This amplitude follows from the interaction term (4.1.6):

$$\mathcal{L}(\ell_i, \ell_j, H_{\pm\pm}) = \sqrt{2}\beta_{ij}\ (\bar{\ell}_i^c \Sigma_- H_{++} \ell_j + \bar{\ell}_j \Sigma_+ H_{--} \ell_i^c) \tag{6.4.2}$$

and ψ^c, from (C.7), where the first term applies to the lower vertex in Fig. 25, and the second to the upper vertex. This is summed and averaged over spins to become

$$\bar{\Sigma}_{\text{spins}} |A|^2 \approx [(\beta_{11}\beta_{12})^2/4M_H^4]\ (1+E_+/m_e)(1+E_-/m_e)\ . \tag{6.4.3}$$

There would be no need for the muon beam to have a high kinetic energy so that $E_+ \approx E_- \approx \frac{1}{2}M_\mu$. The cross-section ($\sigma$) is evaluated using standard techniques (Bjorken and Drell, 1964, App. B) to be, assuming $|\vec{v}_\mu| \sim 1$,

$$\sigma \sim [(\beta_{11}\beta_{12})^2/16\pi]\ (M_\mu^2/M_H^4)\ (\hbar^2/c^2)\ , \tag{6.4.4}$$

where the last factor was put in for unit reasons. If as before, one generously assumes

$$|\beta_{ij}| \sim |\beta_i| \sim g\ (M_\ell/M_W)\ , \tag{6.4.5}$$

where M_ℓ is a lepton mass ($\sim\sqrt{m_e M_\mu}$), one finds

$$\begin{aligned} \sigma &\sim (g^4/16\pi)\ (m_e^2 M_\mu^4/M_W^8)\ (M_W/M_H)^4\ (\hbar^2/c^2) \\ &\sim 10^{-56}\ (M_W/M_H)^4\ \text{cm}^2\ , \end{aligned} \tag{6.4.6}$$

with a mean "scattering length" in matter of

$$\bar{\ell} \equiv 1/n\sigma \sim 10^{18}\ \text{lt yr}\ , \tag{6.4.7}$$

for an electron density $n \sim 10^{20}$ cm^{-3}, $M_H \sim M_W$, which shows just how negligible such a reaction is.

With similar assumptions the quark induced process shown in Fig. 26 would have the very rough amplitude (after a rather crude loop integration)

$$A(\mu^+, q \to e^+, q) \sim (gm_e/M_W)(gM_\mu/M_W)(gM_q/M_W)^2(1/M_H^2) \tag{6.4.8}$$

from (4.5.8) and (4.5.10) where M_q is a quark mass, which, when compared to (6.4.1), is

$$A(\mu^+, q \to e^+, q) \,/\, A(\mu^+, e^- \to e^+, e^-) \sim g^2 \, (M_q^2/M_W^2) \ll 1 \,, \qquad (6.4.9)$$

so that such a process would be even more negligible.

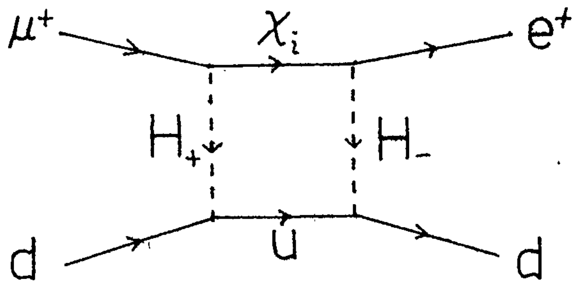

Fig. 26: Feynman diagram for $\mu^+ + d \to e^+ + d$

We may conclude, then, that although the decay $\mu \to 3e$ is, at most, very rare, other muon lepton number non-conserving processes, such as the direct conversion of a μ into an e, are probably very much rarer by comparison, and do not mimic processes by which one would attempt to measure the decay rate $\mu \to 3e$.

VII. DISCUSSION AND CONCLUSIONS

It is appropriate, now, to consider the quantitative predictions that follow from the decay rates calculated in the previous two chapters. Unfortunately most of the parameters that went into the model's construction are unknown. But some, however, cancel, and others can perhaps be estimated on the basis of previous experimental work or on the basis of others' work on limitations from astrophysical considerations.

From (2.3.12) one has

$$\rho \equiv M_W^2/(M_Z^2\cos^2\theta_W) = (v^2+2v_H^2)/(v^2+4v_H^2) \qquad (7.1)$$
$$\approx 1 - 2v_H^2/v^2 = 1 - \tan^2\theta ,$$

and from (2.3.13) (and references mentioned there) one can consider as experimentally established that $\rho \approx 1$ and $v_H \ll v$. Current experiments are consistent with $\rho = 1$, and an estimate of $\rho = .985\pm.015$ (Marshak, Riazuddin and Mohapatra, 1981) leads to

$$\cot^2\theta \gtrsim 10^2 . \qquad (7.2)$$

An independent estimate based on astrophysical evolution (Gelmini, Nussinov and Roncadelli, 1982; Dugan *et al.*, 1985; Glashow and Manohar, 1985) gives $v_H \sim 100$ keV (which is very rough) so that with

$$v = (1/\surd 2G_F)^{1/2} \approx 250 \text{ GeV} \qquad (7.3)$$

(Cheng and Li, 1984, p. 353), we have

$$\cot\theta = v/\surd 2v_H \sim 10^6 , \qquad (7.4)$$

which, although a crude estimate, is a more stringent bound than (7.2). Of course, $v_H = 0$ is not ruled out.

As pointed out earlier, in (4.1.8), the product $m_i \cot\theta$ is actually independent of v_H, and therefore of the neutrino masses, however small:

$$m_i \cot\theta \sim |v_H \beta_{ii}| (v/v_H) = |v\beta_{ii}| , \qquad (7.5)$$

because from (3.3.2b) we have

$$m_i = |2\Sigma_{jk} U_{ij} U_{ik} \beta_{jk} v_H| \sim |\beta_{ii} v_H| \qquad (7.6)$$

if, in the first approximation $U_{ij} \approx \delta_{ij}$, as seems likely (see the references before (7.3)). An assumption we will make now, although there really is no evidence for it, is that the Yukawa coupling constants for the iso-spinor Higgs field and those for the iso-vector Higgs field are of the same order of magnitude. In such a case

$$|v\beta_{ii}| \sim v\beta \sim M_\ell , \qquad (7.7)$$

where M_ℓ is a lepton mass, say $M_\ell \sim (M_\mu m_e)^{1/2}$. These rough estimates will now be applied to the decay rates calculated earlier. It is surprising that (7.4) and (7.7) turn out to be of the same order of magnitude.

The basic radiative decay rate for the muon is given in (5.4.10). Ignoring, for now, the physical, charged Higgs boson B, the rate as a fraction of the usual decay rate is, from (5.4.13),

$$\frac{\Gamma(\mu \to W \to e\gamma)}{\Gamma(\mu \to e\nu\nu)} = \frac{3\alpha}{32\pi} (\Sigma_i U_{i1} U_{i2}\, m_i^2/M_W^2)^2 \qquad (7.8)$$

in agreement with Cheng and Li (1984, p. 427), and with a neutrino mass in the 10 eV range one has

$$\frac{\Gamma(\mu\to W\to e\gamma)}{\Gamma(\mu\to e\nu\nu)} \sim 6\times10^{-44}\ [\Sigma_i U_{i1}U_{i2}(m_i/10\mathrm{eV})^2]^2 \ , \qquad (7.9)$$

an extremely small fraction that would be expected to be even further reduced by the components of the orthogonal matrix ($|U_{ij}| < 1$, for all i,j), the off-diagonal elements of which might be reasonably expected to be quite small. In the case of two families, for example, the matrix U could be written

$$U = \begin{matrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{matrix} \qquad (7.10)$$

so the factor in (7.9) becomes

$$\Sigma_i U_{i1}U_{i2}m_i^2 = \cos\theta\ \sin\theta\ (m_2^2-m_1^2) \ , \qquad (7.11)$$

in agreement with other work (e.g., Marshak, Riazuddin and Mohapatra, 1981). Recent estimates do seem to indicate that θ is small (Gelmini, Nussinov and Roncadelli, 1982; Dugan *et al.*, 1985). Thus this decay, first calculated by Cheng and Li (1980b), is hopelessly beyond any expectation of experimental verification, a conclusion indifferent to the species of neutrino employed, whether of the Dirac type employed by Cheng and Li or the Majorana type considered here.

Let us now consider this decay using the result of the Higgs particle B. Its contribution is the highest order term from (5.4.13), the one with $\cot\theta$:

$$\frac{\Gamma(\mu\to B\to e\gamma)}{\Gamma(\mu\to e\nu\nu)} = \frac{\alpha}{96\pi}\ [\Sigma_i U_{i1}U_{i2}(m_i/M_B)^2\cot^2\theta]^2 \ . \qquad (7.12)$$

From (7.4) we immediately see that this ratio is of the order of 10^{24} times the pessimistic result of Cheng and Li, provided that $M_B \sim M_W$. From (7.7) one obtains

$$\frac{\Gamma(\mu\to B\to e\gamma)}{\Gamma(\mu\to e\nu\nu)} \sim 10^{-21}\ (M_W/M_B)^4\ (\Sigma_i U_{i1} U_{i2})^2\ , \tag{7.13}$$

which is conceivably some 22 orders of magnitude greater than (7.9), and not all that much different from (7.12) which was an estimate based on totally different assumptions. In any case both are still about 10 to 12 orders of magnitude below current detectability, which is (Cooper, 1978; Kinnison *et al.*, 1982)

$$\Gamma(\mu\to e\gamma)/\Gamma(\mu\to e\nu\nu) < 10^{-10}\ , \tag{7.14}$$

with

$$\Gamma(\mu\to e\gamma\gamma)/\Gamma(\mu\to e\nu\nu) < 8\times10^{-9} \tag{7.15}$$

(Azuelos, 1983). Current hopes are that the ratio in (7.14) can be reduced to about 10^{-12} in the relatively near future (Weinberg, 1984; Bowen, 1985).

For the $\mu \to 3e$ decay we have the three illustrative rates calculated in Chap. VI. For the decay via the W we have, from (6.3.28),

$$\frac{\Gamma(\mu\to W\to 3e)}{\Gamma(\mu\to e\nu\nu)} = \frac{5^2\ \alpha^2}{2^2(2\pi)^2 m^2 M^2}\ (\Sigma_i U_{i1} U_{i2} m_i^2)^2 \tag{7.16}$$

$$\sim 3\times10^{-29}\ [\Sigma_i U_{i1} U_{i2} (m_i/10\mathrm{eV})^2]^2\ ,$$

where m and M are the electron and muon masses, respectively. This rate should be compared with the $\mu \to e\gamma$ (via W) rate of (7.8):

$$\frac{\Gamma(\mu\to W\to 3e)}{\Gamma(\mu\to W\to e\gamma)} = \frac{5^2(2^2)\alpha}{3(2\pi)}\ \frac{M_W}{m^2 M^2}\ \sim 10^{15}\ . \tag{7.17}$$

Thus the W-mediated decay rate for the decay $\mu \to 3e$ looks very much more promising than the radiative decay, but is still very much ($\sim 10^{20}$) beyond the limit of experimental detectability which is (Korenchenko *et al.*, 1976; Bolton *et al.*, 1984)

$$\Gamma(\mu\to 3e)/\Gamma(\mu\to e\nu\nu) < 10^{-10} , \tag{7.18}$$

which rate, too, is expected to improve by two more orders of magnitude in the near future (Bertl *et al.*, 1984; Eichler, 1984).

For the $\mu \to 3e$ decay mediated by the Higgs boson B we have, from (6.3.31),

$$\frac{\Gamma(\mu\to B\to 3e)}{\Gamma(\mu\to e\nu\nu)} = \frac{\alpha^2}{3^2(2^2)(2\pi)^2}\,\frac{M^2}{m^2 M_B^4}\,(\Sigma_i U_{i1}U_{i2}m_i^2\cot^2\theta)^2 \tag{7.19}$$

$$\sim 10^{-18}\,(M_W/M_B)^4\,(\Sigma_i U_{i1}U_{i2})^2$$

using the approximation (7.7), or

$$\sim 10^{-19}\,(M_W/M_B)^4\,[\Sigma_i U_{i1}U_{i2}(m_i/10\mathrm{eV})^2]^2 \tag{7.20}$$

using the approximation (7.4). Thus this decay mode seems to be about 10 or so orders of magnitude below current detectability. As a comparison with the B-mediated decay mode we have

$$\Gamma(\mu\to B\to 3e)/\Gamma(\mu\to B\to e\gamma) = (2\alpha/3\pi)\,(M/m)^2 \approx 60 , \tag{7.21}$$

not nearly as dramatic a comparison as (7.17).

As mentioned earlier, these rates would be expected to be even further reduced by the factors such as U_{12}. Nor can one expect them to be much raised by relatively small values for M_B: being electrically charged, such bosons should be readily detectable at current accelerator energies, and a lower limit of 15 GeV or so (Adeva *et al.*, 1982) has already been established.

For the $\mu\to 3e$ decay mediated by the doubly charged Higgs boson H we have, from (6.3.35),

$$\frac{\Gamma(\mu\to H\to 3e)}{\Gamma(\mu\to e\nu\nu)} = \frac{(\beta_{11}\beta_{12})^2}{G_F^2\ M_H^4} \quad . \tag{7.22}$$

With the approximation (7.7) we have

$$\frac{\Gamma(\mu\to H\to 3e)}{\Gamma(\mu\to e\nu\nu)} \sim 10^{10} e^4\ (M_\ell/M_W)^4\ (M_p/M_H)^4 \tag{7.23}$$

$$\sim 10^{-16}$$

with the generous assumptions $M_\ell \sim (m_e M_\mu)^{1/2}$, $\beta_{11} \sim \beta_{12} \sim \beta \sim e(M_\ell/M_W)$, and $M_H \sim M_W$. For the off-diagonal β_{12} we might presume that $\beta_{12} \sim 10^{-2}\beta_{11}$ or smaller (Glashow and Manohar, 1985), in which case (7.23) falls to 10^{-20}, the same rough level of the other $\mu\to 3e$ transition rates.

Thus, overall it seems that the $\mu \to e\gamma$, $\mu \to 3e$ decay modes are 10 or more orders of magnitude below current detectability, although this is a considerable improvement over the 30 or so orders of magnitude that follows without the iso-vector Higgs fields.

It now remains to be seen whether the model described in this thesis has anything to contribute to other processes that are within the reach of current experimental capacity.

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APPENDIX A

NOTATION AND CONVENTIONS

Four-Vectors and Metric

Basis 4-vectors for a reference frame are generally denoted (γ_μ) $(\mu = 0,1,2,3)$, where γ_0 is a unit timelike 4-vector (the 4-velocity of an observer attached to the reference frame) and where γ_i $(i = 1,2,3)$ are unit spacelike 4-vectors:

$$\gamma_\mu \cdot \gamma_\nu \equiv \eta_{\mu\nu} \equiv \begin{cases} 1, & \mu=\nu=0 \\ -1, & \mu=\nu=1,2,3 \\ 0, & \mu\neq\nu \end{cases} \tag{A.1}$$

A general 4-vector such as the 4-momentum p of a particle of mass m can then be written

$$p = p^\mu \gamma_\mu = p_\mu \gamma^\mu \tag{A.2}$$

(where the usual summation convention is employed) with

$$\gamma_\mu \cdot \gamma^\nu \equiv \delta_\mu^\nu \tag{A.3}$$

and

$$p^2 = p\cdot p = p_\mu p^\mu = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = m^2. \tag{A.4}$$

The differential operator ∂ with components $\partial_\mu \equiv \partial/\partial x^\mu$ is defined by

$$\partial \equiv \gamma^\mu \partial_\mu = \gamma_\mu \partial^\mu . \tag{A.5}$$

The reader should note that the (γ_μ) are not matrices. No matrix representation of the Dirac algebra is used here because the (γ_μ) are given certain algebraic properties (App. B) that makes unnessary any use of matrices.

Units

Units are employed here such that

$$\hbar \equiv c \equiv 1 , \qquad (A.6)$$

and, as is also common in field theory, the proton charge $e>0$ is measured in rationalized units such that the fine structure constant is

$$\alpha \equiv e^2/(4\pi\hbar c) = e^2/(4\pi) \approx 1/137 . \qquad (A.7)$$

Conjugates

The notation (*) is employed for the complex conjugates of complex numbers and the hermitian conjugates of Fock-space creation and annihilation operators. The notation (†) is reserved for a special definition of algebraic importance (App. B).

APPENDIX B

THE GEOMETRIC ALGEBRA: CONVENTIONS AND APPLICATIONS

Matrix representations of the Dirac and Pauli algebras can be avoided totally if certain algebraic properties are postulated for 4-vectors in Minkowskian space-time and 3-vectors in ordinary Euclidean space, the iso-space of the SU(2) gauge transformations. This algebra -- a Clifford algebra -- is based on the definition, given two vectors A and B ,

$$\begin{aligned} A\ B &= \tfrac{1}{2}(AB + BA) + \tfrac{1}{2}(AB - BA) \qquad (B.1) \\ &\equiv A\cdot B + A\wedge B \\ &= B\cdot A - B\wedge A\ , \end{aligned}$$

which gives the three basic products (AB, $A\cdot B$, $A\wedge B$) of the algebra, AB and $A\wedge B$ being defined as well to be associative. The "Dirac algebra" is that Clifford algebra based on 4-dimensional space-time, while that using ordinary 3-vectors is called the "Pauli algebra". Accounts of these algebras and their association with spinors can be found in Hamilton (1984a,b) and references contained therein. This summary will do no more than list the essentials of the subject.

The Dirac and Pauli Algebras

The Dirac algebra, assuming that real numbers only are being used, is a 16-dimensional vector space with a general "multivector" M having the form

$$\begin{aligned} M = \alpha\ &(\text{scalar}) + V\ (\text{4-vector}) + B\ (\text{bivector}) \qquad (B.2) \\ &+ \gamma_5 U\ (\text{trivector or pseudovector}) + \gamma_5 \beta\ (\text{pseudoscalar})\ , \end{aligned}$$

where α, β are (real) scalars, V, U are 4-vectors, $B = \frac{1}{2}B^{\mu\nu}\gamma_\mu \wedge \gamma_\nu$ is a bivector and where

$$\gamma_5 \equiv \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_0\gamma_1\gamma_2\gamma_3 \tag{B.3}$$

which satisfies

$$\gamma_\mu \gamma_5 = -\gamma_5 \gamma_\mu , \quad \gamma_5\gamma_5 = -1 , \tag{B.4}$$

where the basis 4-vectors are denoted (γ_μ; μ=0,1,2,3) as explained in App. A.

The Pauli algebra, again assuming real coefficients and a basis (σ_i) for 3-dimensional space, is an 8-dimensional vector space with a general multivector m having the form

$$m = \alpha \text{ (scalar)} + \vec{u} \text{ (3-vector)} + i\vec{v} \text{ (bivector or pseudovector)} + i\beta \text{ (pseudoscalar)} , \tag{B.5}$$

where

$$i \equiv \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = \sigma_1\sigma_2\sigma_3 \tag{B.6}$$

satisfies

$$i\, \sigma_i = \sigma_i\, i , \quad i^2 = -1 , \tag{B.7}$$

which is why the symbol i is used here.

The Pauli algebra is a sub-algebra of the Dirac algebra, which is established by the identification

$$\sigma_i \equiv \gamma_i \wedge \gamma_0 = \gamma_i\gamma_0 . \tag{B.8}$$

In this thesis, however, each algebra is used separately and independently (the Dirac algebra for space-time, the Pauli algebra for the SU(2) 3-vectors), so that (B.8) is really of no further interest here.

Pauli Spinors

The Pauli algebra admits but one set of projection operators (as any other would not commute with it) $\frac{1}{2}(1 \pm \sigma_3)$, σ_3 being the conventional unit 3-vector to use here. A suitable spinor basis would then be $\chi_\pm$, where

$$\tfrac{1}{2}(1 \pm \sigma_3)\ \chi_\pm \equiv \chi_\pm \ . \tag{B.9}$$

One defines an operation (†) by $m^\dagger \equiv m$, but with all vector products reversed, so that $\sigma_i^\dagger = \sigma_i$, $(m_1 m_2)^\dagger = m_2^\dagger\ m_1^\dagger$, whence $i^\dagger = -i$ and one defines the dual spinors $\chi_\pm^\dagger$ by

$$\chi_r^\dagger\ \chi_s \equiv \delta_{rs} \ . \tag{B.10}$$

With the definition

$$\sigma_1\ \chi_+ \equiv \chi_- \tag{B.11}$$

to fix an otherwise undetermined phase, one has

$$\sigma_1\ \chi_- = \chi_+ \ , \quad \sigma_2\ \chi_\pm = \pm\, i\chi_\mp \tag{B.12}$$

and

$$\begin{aligned} \chi_+\ \chi_+^\dagger &= \tfrac{1}{2}(1 + \sigma_3) \ , \quad & \chi_+\ \chi_-^\dagger &= \tfrac{1}{2}(\sigma_1 + i\sigma_2) \\ \chi_-\ \chi_-^\dagger &= \tfrac{1}{2}(1 - \sigma_3) \ , \quad & \chi_-\ \chi_+^\dagger &= \tfrac{1}{2}(\sigma_1 - i\sigma_2) \ . \end{aligned} \tag{B.13}$$

The basic theorem for spinor amplitude calculations is, given Pauli spinors Φ, Ψ and a multivector m,

$$\Phi^\dagger\ m\ \Psi = 2\ S[m\ \Psi\ \Phi^\dagger] \ , \tag{B.14}$$

where $S[\cdots]$ means the complex scalar part (the first and last terms of (B.5)) of whatever multivector is enclosed in the square brackets. From (B.14) follows

$$\Psi\ \Phi^\dagger = \tfrac{1}{2}(\Phi^\dagger\ \Psi) + \tfrac{1}{2}(\Phi^\dagger\ \sigma_i\ \Psi)\ \sigma_i \ , \tag{B.15}$$

which displays the important structural connection between spinors and vectors.

<u>The Gauge Fixing Terms of Section II.6</u>

The gauge fixing terms of Sec. II.6, Eq. (2.6.1), (2.6.2), were written in terms of the Pauli algebra. Here they will be evaluated in more detail.

Given the Higgs spinor field

$$\Phi = \phi_+ \, \chi_+ + \phi_0 \, \chi_- \, , \tag{B.16a}$$

with

$$\langle\Phi\rangle = (v/\sqrt{2}) \, \chi_- \tag{B.16b}$$

$$\Phi^\dagger = \phi_- \, \chi_+^\dagger + \phi_o^* \, \chi_-^\dagger \tag{B.16c}$$

$$\langle\Phi^\dagger\rangle = (v/\sqrt{2}) \, \chi_-^\dagger \, , \tag{B.16d}$$

where (*) denotes the complex and hermitian conjugates of the fields (such as $\phi_-^* = \phi_+$), we have

$$S[\langle\Phi\rangle \, \Phi^\dagger - \Phi \, \langle\Phi^\dagger\rangle] \tag{B.17}$$

$$= (v/\sqrt{2}) \, S[\phi_- \, \chi_-\chi_+^\dagger + \phi_o^* \, \chi_-\chi_-^\dagger - \phi_+ \, \chi_+\chi_-^\dagger - \phi_o \, \chi_-\chi_-^\dagger]$$

$$= -\tfrac{1}{2}(v/\sqrt{2}) \, (\phi_o - \phi_o^*) \, ,$$

where the scalar parts were obtained from (B.13). If $V[\cdots]$ stands for the (real plus pseudo-) vector part of (B.5) we also have

$$V[\langle\Phi\rangle \, \Phi^\dagger - \Phi \, \langle\Phi^\dagger\rangle] = \tfrac{1}{2}(v/\sqrt{2}) \tag{B.18}$$

$$\times[(\phi_o - \phi_o^*) \, \sigma_3 + \phi_- \, (\sigma_1 - i\sigma_2) + \phi_+ \, (\sigma_1 + i\sigma_2)] \, ,$$

again employing (B.13). Both of these results were needed in (2.6.8) and (2.6.9).

The Higgs vector field $\vec{H}$ is, from (2.6.4a),

$$\vec{H} = H_i \sigma_i = (1/\sqrt{2}) H_0 (\sigma_1 - i\sigma_2) + (1/\sqrt{2}) H_{++} (\sigma_1 + i\sigma_2) + H_+ \sigma_3 \tag{B.19}$$

with

$$\langle\vec{H}\rangle = \tfrac{1}{2} \, v_H \, (\sigma_1 - i\sigma_2) \tag{B.20}$$

$$\vec{H}^\dagger = (1/\sqrt{2}) H_o^* (\sigma_1 + i\sigma_2) + (1/\sqrt{2}) H_{--} (\sigma_1 - i\sigma_2) + H_- \sigma_3 \tag{B.21}$$

$$\langle \vec{H}^{+} \rangle = \tfrac{1}{2}\, v_H\, (\sigma_1 + i\sigma_2) \; . \tag{B.22}$$

If we use the identities

$$(\sigma_1 \pm i\sigma_2)^2 = 0 \tag{B.23a}$$

$$(\sigma_1 \pm i\sigma_2)(\sigma_1 \mp i\sigma_2) = 2\,(1 \pm \sigma_3) \; , \tag{B.23b}$$

we obtain

$$S[\langle \vec{H} \rangle \vec{H}^{+} - \vec{H} \langle \vec{H}^{+} \rangle] = -(1/\surd 2) v_H (H_o - H_o^{*}) \tag{B.24}$$

and

$$V[\langle \vec{H} \rangle \vec{H}^{+} - \vec{H} \langle \vec{H}^{+} \rangle] = \tfrac{1}{2}\, v_H \tag{B.25}$$
$$\times\; [\surd 2 (H_o - H_o^{*})\sigma_3 + H_-(\sigma_1 - i\sigma_2) - H_+(\sigma_1 + i\sigma_2)] \; .$$

These, too, were needed in the evaluation of (2.6.8) and (2.6.9).

<u>Dirac Spinors</u>

A basis of the 4-dimensional space of Dirac spinors ($u_\pm(p)$, $v_\pm(p)$) is defined by (Hamilton, 1984b)

$$\tfrac{1}{2}(1 + p/m)\; \tfrac{1}{2}[1 \pm (-i)\gamma_5 Sp/m]\; u_\pm = u_\pm \tag{B.26a}$$

$$\tfrac{1}{2}(1 - p/m)\; \tfrac{1}{2}[1 \pm (-i)\gamma_5 Sp/m]\; v_\mp = v_\mp \; , \tag{B.26b}$$

where p is the 4-momentum of a particle of mass m and S ($S^2 = -1$, $S \cdot p = 0$) is a unit spacelike 4-vector that points in the spin direction in the rest frame of the particle.

If M of (B.2) is a general multivector of the Dirac algebra (M=S+V+B+T+P) one defines

$$\tilde{M} \equiv M, \text{ with all products reversed} \tag{B.27}$$
$$= S + V - B - T + P \; .$$

A spinor equation $\Phi = \alpha M \Psi$ for arbitrary spinors Φ, Ψ, a (complex) scalar α, and assuming M has real coefficients -- the basis vectors (γ_μ) are defined to be real -- has as an "adjoint"

$$\Phi = \alpha M\Psi \quad \leftrightarrow \quad \bar{\Phi} = \alpha^{*}\bar{\Psi}\tilde{M} \; , \tag{B.28}$$

with the basis spinors satisfying

$$\bar{u}_r u_s = \delta_{rs} = -\bar{v}_r v_s \tag{B.29a}$$

$$\bar{u}_r v_s = 0 \quad = \bar{v}_r u_s \; . \tag{B.29b}$$

One further defines

$$\bar{M} \equiv M, \text{ with all vectors reversed} \tag{B.30}$$

$$= S - V + B - T + P \; ,$$

and the complex conjugate of $\Phi = \alpha M\Psi$ is taken to be

$$\Phi^{*} \equiv \alpha^{*} \bar{M} \Psi^{*} \tag{B.31}$$

(again assuming M to have real coefficients). (In (B.31) $\bar{M}$ is used rather than M because in the latter case contradictions would ensue. An alternative would be to use M, but with $\gamma_{\mu}^{*} \equiv -\gamma_{\mu}$: I simply prefer (B.31).)

Evaluation of Spinor Amplitudes

The basic theorem for the evaluation of spinor amplitudes is

$$\bar{\Phi} \, M \, \Psi = 4 \; S[M \, \Psi \, \bar{\Phi}] \; , \tag{B.32}$$

where Φ, Ψ are any two Dirac spinors, M is any multivector of the Dirac algebra and where S[$\cdots$] refers to the scalar part, in the sense of (B.2), of the general multivector inside the square brackets. In more conventional matrix terminology 4S[$\cdots$] is equivalent to Trace[$\cdots$].

As an example of these methods consider the amplitude squared that must be evaluated in the $\mu \to e\gamma$ transition, the expression from (5.4.5):

$$\Sigma_{spins} |\bar{u}_e(p')\Sigma_+ q\epsilon u_\mu(p)|^2 \tag{B.33}$$
$$= \Sigma_{spins}[\bar{u}_e\Sigma_+ q\epsilon u_\mu \bar{u}_\mu (\Sigma_+ q\epsilon)^* u_e]$$
$$= \Sigma_{spins}[\bar{u}_e\Sigma_+ q\epsilon u_\mu \bar{u}_\mu \epsilon q \Sigma_- u_e]$$
$$= \Sigma_{spins} 4S[\Sigma_+ q\epsilon u_\mu \bar{u}_\mu \epsilon q \Sigma_- u_e \bar{u}_e] \ .$$

From (B.26a) we have

$$\Sigma_{spins}(u\bar{u}) = \tfrac{1}{2}(1 + p/m) \ , \tag{B.34}$$

and (B.33) becomes

$$(\Sigma_{spins}\epsilon^\mu_{(\alpha)}\epsilon^\nu_{(\alpha)}) \ 4S[\Sigma_+ q\gamma_\mu \tfrac{1}{2}(1+p/M_\mu)\gamma_\nu q\Sigma_- \tfrac{1}{2}(1+p'/m_e)] \ , \tag{B.35}$$

a calculation that will be completed in App. E. Further relations needed are

$$\Sigma_\pm \gamma_\mu = \gamma_\mu \Sigma_\mp \tag{B.36a}$$
$$\gamma_\nu p \gamma^\nu = \gamma_\nu(p\cdot\gamma^\nu + p\wedge\gamma^\nu) \tag{B.36b}$$
$$= \gamma_\nu p^\nu + \gamma_\nu\cdot(p\wedge\gamma^\nu) + \gamma_\nu\wedge(p\wedge\gamma^\nu)$$
$$= -2\ p \ ,$$

where, if M is a multivector,

$$\gamma_\mu M = \gamma_\mu\cdot M + \gamma_\mu \wedge M \tag{B.37a}$$

and, if A, B_i are 4-vectors,

$$A\cdot(B_1\wedge B_2\wedge\cdots\wedge B_n) = \Sigma_i(-1)^{(i+1)}(A\cdot B_i)(B_1\wedge\cdots\wedge B_n) \tag{B.37b}$$

where B_i is missing from the last factor.

The foregoing is far from the list of valuable relations that would be needed in the general case but is adequate for the needs of this thesis. The brief example above illustrates how readily the matrix-free Dirac algebra can be exploited to evaluate the ubiquitous "traces" of relativistic quantum mechanics.

A Theorem for Spinor Amplitudes

An identity that is most useful in establishing the C and T invariance properties of Lagrangians is as follows. Let ψ, ϕ be quantum fermion fields and M any element of the Dirac algebra, not assumed here to have real coefficients. Then

$$\bar{\psi} M \phi = - \bar{\phi}^{*} \gamma_5 \tilde{M} \gamma_5 \psi^{*} \tag{B.38}$$

$$= \begin{cases} \bar{\phi}^{*} M \psi^{*}, & \text{if } M = S,T,P \\ -\bar{\phi}^{*} M \psi^{*}, & \text{if } M = V,B \end{cases}$$

where the negative sign arises from the anticommutivity of the fermion fields (the signs are reversed for non-quantum fields). The proof of (B.38) is tedious but straightforward: one need only write, for example,

$$\psi = \alpha_r u_r + \beta_r v_r \quad (r = \pm) , \tag{B.39}$$

where the α_r, β_r are anticommuting Fock-space operators and u_r, v_r basis spinors, and then establish (B.38) for each of the five different types of multivector elements.

APPENDIX C

MANIFESTLY COVARIANT C, P AND T TRANSFORMATIONS

It is conventional (if not universal) to employ specific matrix representations of the Dirac algebra to construct the important C, P and T operators of quantum field theory. But this is not necessary: the Dirac algebra described in the previous appendix can be used to construct appropriate transformations that are not only free of any matrix representation of the algebra, but that are manifestly covariant as well. This appendix will display such operators and, as an example, apply them to quantum electrodynamics. Section II.7 contains their application to more general gauge theories.

The parity or P transformation to be used here is quite conventional (Bjorken and Drell, 1965, Sec. 15.11). The appropriate transformation for a spinor field ψ is

$$\begin{aligned} \psi^P(t,\vec{r}) &\equiv P\ \psi(t,\vec{r})\ P^{-1} \\ &\equiv \gamma_0\ \psi(t,-\vec{r}) \ , \end{aligned} \tag{C.1}$$

where P is a unitary Fock-space operator made up of the creation and annihilation operators of the spinor field ψ. The electromagnetic vector field A satisfies

$$\begin{aligned} A^P(t,\vec{r}) &= P\ A(t,\vec{r})\ P^{-1} \\ &= \gamma_0\ A(t,-\vec{r})\ \gamma_0 \ , \end{aligned} \tag{C.2}$$

and the electromagnetic interaction is seen to be P-invariant in the usual way (Table II).

The charge conjugation or C transformation used throughout this thesis is simpler than the conventional one (Bjorken and Drell, 1965, Sec. 15.12). It is

$$\psi^c(x) \equiv C\,\psi(x)\,C^{-1} \qquad (C.3)$$
$$\equiv \psi^*(x) \; ,$$

where C is a linear, unitary Fock-space operator and where ψ^* is defined in (B.31). The electromagnetic vector field A satisfies

$$A^c = C\,A\,C^{-1} = -A \; , \qquad (C.4)$$

and the Dirac equation is thereby made C-invariant:

$$C(i\partial - m - eA)C^{-1}C\psi C^{-1} = 0 \qquad (C.5)$$
$$= (i\partial - m - eA^c)\psi^c$$
$$= (i\partial - m + eA)\psi^*$$
$$= [(i\partial - m - eA)\psi]^*$$
$$= (i\partial - m - eA)\psi \; ,$$

where care must be taken in that (B.31) reverses the sign of vectors under the (*) operation. Table II lists the invariance properties of the QED Lagrangian under the C transformation.

When applied to a spinor field

$$\psi(x) \sim \int d^3p\,(e^{-ip\cdot x}a_r u_r + e^{ip\cdot x}b_r^* v_r) \qquad (C.6)$$

with basis spinors u_r, v_r satisfying (B.26), we have $(u_r)^* = v_r$, from (B.26) and (B.31) (with an arbitrary phase set equal to one), and so

$$\psi^c(x) = \psi^*(x) \sim \int d^3p\,(e^{ip\cdot x}a_r^* v_r + e^{-ip\cdot x}b_r u_r) \qquad (C.7)$$

which was needed in Chap. VI.

The CPT transformation to be adopted is ($\theta \equiv$ cpt, $\Theta \equiv$ TPC)

$$\psi^{cpt}(x) \equiv \psi^{\Theta}(x) \equiv \Theta\, \psi(x)\, \Theta^{-1} \qquad (C.8)$$
$$\equiv \psi^{c}(-x) \quad = \psi^{*}(-x) \ ,$$

which requires the time reversal (or reversal of motion) or T transformation to be

$$\psi^{t}(t,\vec{r}) \equiv T\, \psi(t,\vec{r})\, T^{-1} \qquad (C.9)$$
$$\equiv \gamma_0\ \psi(-t,\vec{r}) \ ,$$

in marked contrast to the usual T and CPT transformations (Bjorken and Drell, 1965, Sec. 15.13). As usual the operators T and Θ are antilinear. That this property is crucial will be seen when the full invariance of the spinor-vector electromagnetic interaction is demonstrated (Table II). The electromagnetic vector potential A transforms as

$$A^{\Theta}(x) = \Theta\, A(x)\, \Theta^{-1} \qquad (C.10)$$
$$= -A(-x) \ .$$

Table II contains a detailed list of the transformation properties of the various components of the fermion-photon Lagrangian, except for the kinetic terms for the photon field which are rather trivial. One should note that the important identity (B.38) has been used wherever ψ^* appears, and note the importance of the antilinearity of T and Θ in the $(i\bar{\psi}\partial\psi)$ part of the fermion Lagrangian. Only real coefficients have been ignored.

Some other useful manipulations that are freely employed are $x \to -x$, $t \to -t$, $\vec{r} \to -\vec{r}$ in the action integral $\int d^4x\ \mathcal{L}(x)$, which are permitted by the action integral's invariance, and the properties of the 4-vector γ_0 and of γ_5 discussed in App.

B. Also permitted is $\mathcal{L} \to \mathcal{L} + \partial_\nu(\cdots)^\nu$.

The C, P and T properties of the electromagnetic interaction are considered in detail because of the assumption made in Sec. II.7 that the vector gauge fields transform exactly as does the electromagnetic vector potential.

TABLE II: C, P AND T INVARIANCE OF ELECTROMAGNETISM

INVARIANCE	LAGRANGIAN TERM	
	$j^\nu=\bar{\psi}\gamma^\nu\psi$	$i\bar{\psi}\partial\psi$
P	$Pj^\nu P^{-1}$	$Pi\bar{\psi}\partial\psi P^{-1}$
	$= \bar{\psi}\gamma_o\gamma^\nu\gamma_o\psi(-\vec{r})$	$= i\bar{\psi}\gamma_o\gamma^\nu\gamma_o\partial_\nu\psi(-\vec{r})$
	$= (j^o,-j^i)(-\vec{r})$	$(\vec{r} \rightarrow -\vec{r})$
		$\rightarrow i\bar{\psi}\partial\psi$
C	$Cj^\nu C^{-1}$	$Ci\bar{\psi}\partial\psi C^{-1}$
	$= \bar{\psi}^*\gamma^\nu\psi^*$	$= i\bar{\psi}^*\gamma^\nu\partial_\nu\psi^*$
	$= -\bar{\psi}\gamma_5\gamma^\nu\gamma_5\psi$	$= -i\partial_\nu\bar{\psi}\gamma_5\gamma^\nu\gamma_5\psi$
	$= -j^\nu$	$= i\bar{\psi}\gamma^\nu\partial_\nu\psi + \partial_\nu(\cdots)$
		$\rightarrow i\bar{\psi}\partial\psi$
T	$Tj^\nu T^{-1}$	$Ti\bar{\psi}\partial\psi T^{-1}$
	$= \bar{\psi}\gamma_o\gamma^\nu\gamma_o\psi(-t)$	$= -i\bar{\psi}^t\partial\psi^t$
	$= (j^o,-j^i)(-t)$	$= -i\bar{\psi}\gamma_o\gamma^\nu\gamma_o\partial_\nu\psi(-t)$
		$\rightarrow i\bar{\psi}\partial\psi$ (as $t \rightarrow -t$)
$TPC \equiv \Theta$	$\Theta j^\nu\Theta^{-1}$	$\Theta i\bar{\psi}\partial\psi\Theta^{-1}$
	$= \bar{\psi}^*\gamma^\nu\psi^*(-x)$	$= -i\bar{\psi}^\theta\partial\psi^\theta$
	$= -\bar{\psi}\gamma_5\gamma^\nu\gamma_5\psi(-x)$	$= -i\bar{\psi}^*\gamma^\nu\partial_\nu\psi^*(-x)$
	$= -\ j^\nu(-x)$	$= i(\partial_\nu\bar{\psi}\gamma_5\gamma^\nu\gamma_5\psi)(-x)$
		$= -i(\bar{\psi}\gamma^\nu\partial_\nu\psi)(-x) + \partial_\nu(\cdots)$
		$\rightarrow i\bar{\psi}\partial\psi$ (as $x \rightarrow -x$)

TABLE II (cont'd.)

INVARIANCE	LAGRANGIAN TERM	
	$\underline{\bar{\psi}\psi}$	$\underline{\bar{\psi}A\psi}$
P	$P\bar{\psi}\psi P^{-1}$	$P\bar{\psi}A\psi P^{-1}$
	$= \bar{\psi}\gamma_0\gamma_0\psi(-\vec{r})$	$= \bar{\psi}\gamma_0\gamma_0 A\gamma_0\gamma_0\psi(-\vec{r})$
	$(\vec{r} \rightarrow -\vec{r})$	$(\vec{r} \rightarrow -\vec{r})$
	$\rightarrow \bar{\psi}\psi$	$\rightarrow \bar{\psi}A\psi$
C	$C\bar{\psi}\psi C^{-1}$	$C\bar{\psi}A\psi C^{-1}$
	$= \bar{\psi}^*\psi^*$	$= \bar{\psi}^*(-A)\psi^*$
	$= -\bar{\psi}\gamma_5\gamma_5\psi$	$= \bar{\psi}\gamma_5 A\gamma_5\psi$
	$= \bar{\psi}\psi$	$= \bar{\psi}A\psi$
T	$T\bar{\psi}\psi T^{-1}$	$T\bar{\psi}A\psi T^{-1}$
	$= \bar{\psi}\gamma_0\gamma_0\psi(-t)$	$= \bar{\psi}\gamma_0\gamma_0 A\gamma_0\gamma_0\psi(-t)$
	$(t \rightarrow -t)$	$(t \rightarrow -t)$
	$\rightarrow \bar{\psi}\psi$	$\rightarrow \bar{\psi}A\psi$
$TPC \equiv \Theta$	$\Theta\bar{\psi}\psi\Theta^{-1}$	$\Theta\bar{\psi}A\psi\Theta^{-1}$
	$= \bar{\psi}^*\psi^*(-x)$	$= \bar{\psi}^*(-A)\psi^*(-x)$
	$= -(\bar{\psi}\gamma_5\gamma_5\psi)(-x)$	$= (\bar{\psi}\gamma_5 A\gamma_5\psi)(-x)$
	$(x \rightarrow -x)$	$(x \rightarrow -x)$
	$\rightarrow \bar{\psi}\psi$	$\rightarrow \bar{\psi}A\psi$

APPENDIX D

ON INVARIANT SPACE-TIME INTEGRALS

Most of the integrals that resulted from the Feynman diagrams of Chap. V, VI had the form

$$\int d^4p\ f(p^2) \quad \text{or} \quad \int d^4p\ p_\mu p_\nu\ f(p^2)\ , \tag{D.1}$$

which are not at all trivial to evaluate. Thirty years of experience with quantum electrodynamics has, however, resulted in standard methods of computing such integrals when they converge, and methods of handling them, such as dimensional regularization, when they do not. This appendix will list the results that are needed in this thesis.

First, it is not difficult to see that all such integrals must have the form of the first of (D.1) because

$$\int d^4p\ f(p^2)\ p_\mu = 0 \tag{D.2}$$

and

$$\int d^4p\ f(p^2)\ p_\mu p_\nu = \tfrac{1}{4}\ \eta_{\mu\nu} \int d^4p\ p^2\ f(p^2) \tag{D.3}$$

(Bjorken and Drell, 1964, Chap.8; Akhiezer and Berestetskii, 1965, Sec. 47.1).

Next, one notes that all such integrals that are relevant have poles. One involving a fermion propagator would be

$$\int d^4p\ \frac{f(p^2)}{(p\!\!\!/-m+i\epsilon)} = \int d^4p\ \frac{f(p^2)\ (p\!\!\!/+m-i\epsilon)}{p^2 - (m-i\epsilon)^2} \tag{D.4}$$

where the $i\epsilon$ prescription is very briefly explained in Sec. IV.3, which shows how the poles at $p_0 = \pm(m-i\epsilon)$ are to be avoided (Fig. 27). As the poles can be avoided by a rotation in the complex p_0-plane (Fig. 27) the integral can be

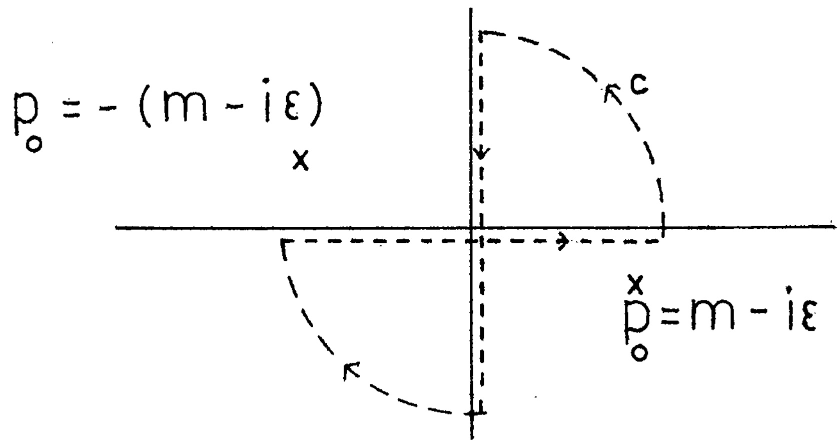


Fig. 27: The complex p_0-plane

transformed into one with a Euclidean metric. From Cauchy's theorem, since no poles are enclosed within the contour c (Fig. 27),

$$\int_c f(p_0)\, dp_0 = 0 = \left(\int_{-\infty}^{\infty} + \int_{\infty i}^{-\infty i}\right) f(p_0)\, dp_0 \ , \qquad (D.5)$$

where the assumption that the integrands vanish sufficiently rapidly as $|p_0| \to \infty$ has been made. With a change of variable, then,

$$p_0 \equiv i p_0' \ , \qquad (D.6)$$

we have

$$\int_{-\infty}^{\infty} f(p_0)\, dp_0 = i \int_{-\infty}^{\infty} f(i p_0')\, dp_0' \ . \qquad (D.7)$$

Since

$$p^2 = p_0^2 - |\vec{p}|^2 = -({p_0'}^2 + |\vec{p}|^2) \equiv -q^2 \ , \qquad (D.8)$$

one can use the parametrization

$$p_1 = q \sin\alpha \sin\theta \cos\Phi \tag{D.9}$$

$$p_2 = q \sin\alpha \sin\theta \sin\Phi$$

$$p_3 = q \sin\alpha \cos\theta$$

$$p_0' = q \cos\alpha ,$$

$(0 \leq \Phi \leq 2\pi,\ 0 \leq \theta, \alpha \leq \pi)$ where (D.8) is satisfied, and

$$d^4 p' \equiv dp_1 dp_2 dp_3 dp_0' = q^3 \sin^2\alpha \sin\theta \, dq \, d\alpha \, d\theta \, d\Phi , \tag{D.10}$$

so on performing the angular integrations we have

$$d^4 p' = 2\pi^2 q^3 \, dq = \pi^2 \, q^2 \, d(q^2) . \tag{D.11}$$

Thus we have, finally,

$$\begin{aligned} \int d^4 p \, f(p^2) &= i \int d^4 p' \, f(-q^2) \\ &= i\pi^2 \int_0^\infty q^2 \, dq^2 \, f(-q^2) \\ &\equiv i\pi^2 \int_0^\infty z \, f(-z) \, dz . \end{aligned} \tag{D.12}$$

We are now in a position to evaluate the three (finite) types of integrals appearing in Chap. V, VI and the next two appendices:

$$\int d^4 p \frac{1}{(p^2+A)^3} = \frac{i \, \pi^2}{2 \, A} \tag{D.13}$$

$$\int d^4 p \frac{1}{(p^2+A)^4} = \frac{i \, \pi^2}{6 \, A^2} \tag{D.14}$$

$$\int d^4 p \frac{p^2}{(p^2+A)^4} = \frac{i \, \pi^2}{3 \, A} \tag{D.15}$$

which agree with the more general results of Nash (1978).

The method of Feynman parameters (Bjorken and Drell, 1964, Chap. 8) is used to reduce integrals with several propagators into the form (D.12):

$$\frac{1}{a_1 a_2 \cdots a_n} = (n-1)! \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_n \frac{\delta[\sum\alpha_n - 1]}{[a_1\alpha_1 + \cdots + a_n\alpha_n]^n} \tag{D.16}$$

The special cases that are needed here are

$$\frac{1}{a_1 a_2} = \int_0^1 d\alpha \int_0^1 d\beta \, \frac{\delta(\alpha+\beta-1)}{(a_1\alpha+a_2\beta)^2} \tag{D.17}$$

$$= \int_0^1 d\alpha \, \frac{1}{[a_1\alpha+a_2(1-\alpha)]^2}$$

$$\frac{1}{a_1 a_2 a_3} = 2\int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma \, \frac{\delta(\alpha+\beta+\gamma-1)}{(a_1\alpha+a_2\beta+a_3\gamma)^3} \tag{D.18}$$

$$= 2\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \, \frac{1}{[a_1\alpha+a_2\beta+a_3(1-\alpha-\beta)]^3}$$

and

$$\frac{1}{a_1^2 a_2 a_3} = -\frac{\partial}{\partial a_1}\left(\frac{1}{a_1 a_2 a_3}\right) \tag{D.19}$$

$$= 6\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \, \frac{\alpha}{[a_1\alpha+a_2\beta+a_3(1-\alpha-\beta)]^4} \; .$$

This parametrization technique, along with (D.12), is all that is required to explicitly evaluate the integrals in this thesis that result from the Feynman diagrams of Chap.V, VI.

APPENDIX E

DETAILS OF THE CALCULATION $\mu \to e\,\gamma$

This appendix contains the details of the calculations of Sec. V.3, V.4 that were omitted there because of their length and essentially technical nature.

The Integrals (5.3.14-18)

Apart from a common factor of $[i^6(2\pi)^{-4}g^2 e\,\frac{1}{2}\Sigma_i U_{i1}U_{i2}m_i^2]$ the integrals to be evaluated are

$$I(W) = -\int d^4k\; \bar{u}_e \Sigma_+ \gamma_\mu \frac{(p-k)}{(p-k)^4}\, \gamma_\nu u_\mu\; \Delta^{\mu\rho}(k-q)\Delta^{\nu\sigma}(k)\; \epsilon^\lambda \Gamma_{\lambda\sigma\rho} \tag{E.1}$$

$$I(S) = \int \frac{d^4k}{M_W^2}\; \bar{u}_e \Sigma_+ \frac{(p-k-M_\mu)}{(p-k)^2} u_\mu\; \frac{2k\cdot\epsilon}{(k^2-\xi M_W^2)[(k-q)^2-\xi M_W^2]} \tag{E.2}$$

$$I(WS) = \int d^4k\; \bar{u}_e\; \frac{\Sigma_+\gamma_\nu}{(p-k)^2} u_\mu\; \frac{\Delta^{\nu\sigma}(k)\quad \epsilon_\sigma}{[(k-q)^2-\xi M_W^2]} \tag{E.3}$$

$$I(SW) = -\int d^4k\; \bar{u}_e \Sigma_+ \gamma_\nu \left(\frac{-1}{(p-k)^2} + \frac{M_\mu (p-k)}{(p-k)^4} \right) u_\mu\; \frac{\Delta^{\nu\sigma}(k-q)\quad \epsilon_\sigma}{(k^2-\xi M_W^2)} \tag{E.4}$$

$$I(B) = \int \frac{d^4k}{M_W^2}\; \bar{u}_e \Sigma_+ \left(\frac{M_\mu}{(p-k)^2} + \frac{(p-k)\cot^2\theta}{(p-k)^2} \right) u_\mu\; \frac{2k\cdot\epsilon}{(k^2-M_B^2)[(k-q)^2-M_B^2]} \tag{E.5}$$

where, from (5.2.1),

$$\Delta(k) = \Delta_1(k) + \Delta_2(k) + \Delta_3(k) \tag{E.6}$$

$$\Delta_1^{\mu\nu}(k) = \eta^{\mu\nu} \;/\; (k^2-M_W^2)$$

$$\Delta_2^{\mu\nu}(k) = -k^\mu k^\nu \;/\; [M_W^2(k^2-M_W^2)]$$

$$\Delta_3^{\mu\nu}(k) = k^\mu k^\nu \;/\; [M_W^2(k^2-\xi M_W^2)]$$

and, from (5.2.3),

$$\Gamma_{\lambda\sigma\rho} = (-q-k)_\rho \eta_{\lambda\sigma} + (2k-q)_\lambda \eta_{\sigma\rho} + (-k+2q)_\sigma \eta_{\lambda\rho}\;. \tag{E.7}$$

Because $q^2=0$, $q\cdot\epsilon=0$, one can show that

$$(k-q)^\rho k^\sigma \epsilon^\lambda \Gamma_{\lambda\sigma\rho} = 0 = (k-q)^\sigma k^\rho \epsilon^\lambda \Gamma_{\lambda\sigma\rho}\;, \tag{E.8}$$

which means that the only non-vanishing factors in (E.1) are

$$\Delta_1^{\mu\rho}(k-q)\Delta_1^{\nu\sigma}(k)\ ,\quad \Delta_1^{\mu\rho}(k-q)\Delta_{2,3}^{\nu\sigma}(k)\ ,\quad \Delta_{2,3}^{\mu\rho}(k-q)\Delta_1^{\nu\sigma}(k)\ . \tag{E.9}$$

For simplicity we define (i=2,3)

$$\Delta_i^{\mu\nu}(k) \equiv a_i k^\mu k^\nu \,/\, [M_W^2(k^2-b_i M_W^2)] \tag{E10a}$$

with

$$a_2 = -1,\ a_3 = +1,\ b_2 = +1,\ b_3 = \xi, \tag{E.10b}$$

so only three integrals will have to be worked out to evaluate (E.1). The first part of (E.1) is, from (D.19),

$$-\int d^4k\ \bar{u}_e \Sigma_+ \gamma^\rho (p-k)\gamma^\sigma u_\mu \ \frac{[-\epsilon_\sigma(q+k)_\rho + 2k\cdot\epsilon\eta_{\sigma\rho} + \epsilon_\rho(2q-k)_\sigma]}{(p-k)^4(k^2-M_W^2)[(k-q)^2-M_W^2]} \tag{E.11}$$

$$= -\int d^4k\ 6\int \alpha d\alpha d\beta \frac{[\cdots]}{\{\alpha(p-k)^2 + \beta[(k-q)^2-M_W^2] + (1-\alpha-\beta)(k^2-M_W^2)\}^4}$$

$$= -\int d^4k\ 6\int \alpha d\alpha d\beta \frac{[\cdots]}{[(k-V)^2+B]^4}$$

where (D.19) has been used, with

$$V \equiv \alpha p + \beta q \tag{E.12a}$$

$$B \equiv -V^2 + \alpha p^2 - (1-\alpha)M_W^2 \approx -(1-\alpha)M_W^2\ . \tag{E.12b}$$

We define the new integration variable $K \equiv k-V$ and obtain

$$-\int d^4K\ 6\int \alpha d\alpha d\beta\ \bar{u}_e \Sigma_+ \gamma^\rho (p-K-V)\gamma^\sigma u_\mu \tag{E.13}$$

$$\times\ [-\epsilon_\sigma(q+K+V)_\rho + \eta_{\sigma\rho} 2(K+V)\cdot\epsilon + \epsilon_\rho(2q-K-V)_\sigma] \,/\, (K^2+B)^4\ .$$

The numerator in (E.13) has factors $K_\mu K_\nu \to \frac{1}{4}\eta_{\mu\nu}K^2$ (from (D.3)), K_μ (which vanish, from (D.2)), and a constant. Each of the K^2 factors gives the spinor amplitude $\bar{u}_e\Sigma_+\epsilon u_\mu$, which, from (5.1.14), has the wrong structure and so the K^2 terms are thus ignored. Integral (E.13) then becomes

$$-\int d^4K 6\int \alpha d\alpha d\beta \bar{u}_e \Sigma_+ \frac{\gamma^\rho(p-V)\gamma^\sigma}{(K^2+B)^4} u_\mu [-\epsilon_\sigma(q+V)_\rho + 2V\cdot\epsilon\eta_{\sigma\rho} + \epsilon_\rho(2q-V)_\sigma] \tag{E.14}$$

$$= -\frac{i\pi^2}{M_W^4} \int \alpha d\alpha d\beta\ \bar{u}_e \Sigma_+ \frac{\gamma^\rho(p-V)\gamma^\sigma}{(1-\alpha)^2}\ u_\mu [-\epsilon_\sigma(q+V)_\rho + 2V\cdot\epsilon\eta_{\sigma\rho} + \epsilon_\rho(2q-V)_\sigma]$$

from (D.14) and (E.12b). The first term is (scalar)$\cdot\bar{u}_e\Sigma_+\epsilon u_\mu$ (wrong structure, so ignore), the second becomes

$$\left(-\frac{i\pi^2}{M_W^4}\right)\left(-4p\cdot\epsilon M_\mu\right)\bar{u}_e\Sigma_+u_\mu\int_0^1\alpha d\alpha\int_0^{1-\alpha}d\beta\ \alpha^2\ \frac{(1-\alpha-\beta)}{(1-\alpha)^2} \tag{E.15}$$

$$= \frac{2}{3}\ i\pi^2\ \frac{p\cdot\epsilon}{M_W^4}\ M_\mu\ \bar{u}_e\Sigma_+u_\mu\ ,$$

and the third becomes

$$\left(-\frac{i\pi^2}{M_W^4}\right)M_\mu\ \bar{u}_e\Sigma_+\epsilon p'u_\mu\ \int_0^1\alpha d\alpha\int_0^{1-\alpha}d\beta\ \frac{[2(1-\alpha)-\beta]}{(1-\alpha)^2} \tag{E.16}$$

$$= -\frac{3}{4}i\pi^2\ \bar{u}_e\Sigma_+\epsilon p'u_\mu\ M_\mu/M_W^4\ .$$

The second part of (E.1) uses the second factor in (E.9):

$$-\int d^4k\ \bar{u}_e\Sigma_+\gamma_\mu\frac{(p-k)}{(p-k)^4}\gamma_\nu u_\mu\frac{\eta^{\mu\rho}}{[(k-q)^2-M_W^2]}\ a_i\frac{k^\nu k^\sigma}{M_W^2(k^2-b_iM_W^2)} \tag{E.17}$$

$$\times\ [-\epsilon_\sigma(q+k)_\rho\ +\ 2k\cdot\epsilon\eta_{\sigma\rho}\ +\ \epsilon_\rho(2q-k)_\sigma]$$

$$= \frac{-6a_i}{M_W^2}\int\alpha d\alpha d\beta\int d^4k\bar{u}_e\Sigma_+\frac{[-k\cdot\epsilon(q+k)+2k\cdot\epsilon k+\epsilon(2q\cdot k-k^2)](p-k)k}{\{\alpha(p-k)^2+\beta[(k-q)^2-M_W^2]+(1-\alpha-\beta)(k^2-b_iM_W^2)\}^4}u_\mu$$

$$= \frac{-6a_i}{M_W^2}\int\alpha d\alpha d\beta\int d^4K\ \bar{u}_e\Sigma_+[-(K+V)\cdot\epsilon(q+K+V)\ +\ 2(K+V)\cdot\epsilon(K+V)$$

$$+\ \epsilon(2q\cdot(K+V)-(K+V)^2)]\ \frac{(p-K-V)(K+V)u_\mu}{(K^2+B)^4}$$

with the substitutions (E.12) and $k \to K+V$. In the numerator all K^4 terms give the wrong spinor amplitude $\bar{u}_e\Sigma_+\epsilon u_\mu$, so are dropped; the K^3 and K terms vanish on integration; the constant term gives an overall $1/M_W^6$ factor which is ignored because the K^2 factor is the dominant factor ($\sim 1/M_W^4$). Thus (E.17) becomes

$$\frac{-6a_i}{M_W^2}\int\alpha d\alpha d\beta\int d^4K\ \bar{u}_e\Sigma_+\frac{[14\text{ terms of order }K^2]}{(K^2+B)^4}u_\mu \tag{E.18}$$

$$= \frac{-6a_i}{M_W^2}\left(\frac{i\pi^2}{-3M_W^2}\right)M_\mu\bar{u}_e\Sigma_+q\epsilon u_\mu\ \int_0^1\alpha d\alpha\int_0^{1-\alpha}d\beta\frac{(-\alpha^2+\alpha/2-\alpha\beta-5\beta/4+1/2)}{[b_i(1-\alpha)\ +\ \beta(1-b_i)]}$$

where (D.15) was used. This integral alone has the values

$$\begin{cases} 1/48\ , & b_2 = 1 \\ \frac{7}{24}\frac{1}{(\xi-1)}\left(1 - \frac{\xi\ln\xi}{\xi-1}\right) + \frac{1}{6}\frac{\ln\xi}{\xi-1}\ , & b_3 = \xi\ . \end{cases} \quad \begin{matrix}(E.19a)\\(E.19b)\end{matrix}$$

The third, and final, part of (E.1) used the third factor in (E.9):

$$-\int d^4k\ \bar{u}_e\Sigma_+\gamma_\mu\frac{(p-k)}{(p-k)^4}\gamma_\nu u_\mu\ a_i\frac{(k-q)^\mu(k-q)^\rho\ \eta^{\nu\sigma}}{M_W^2[(k-q)^2-b_iM_W^2](k^2-M_W^2)} \quad (E.20)$$

$$\times\ [-\epsilon_\sigma(q+k)_\rho + 2k\cdot\epsilon\eta_{\sigma\rho} + \epsilon_\rho(2q-k)_\sigma]$$

$$= \frac{-6a_i}{M_W^2}\int\alpha d\alpha d\beta\int\frac{d^4K}{(K^2+B)^4}\ \bar{u}_e\Sigma_+[-(K+V-q)\cdot(q+K+V)(K+V-q)(p-K-V)\epsilon$$

$$+\ 2\epsilon\cdot(K+V)(K+V-q)(p-K-V)(K+V-q)$$

$$+\ \epsilon\cdot(K+V-q)(K+V-q)(p-K-V)(2q-K-V)]u_\mu$$

where, again, only the K^2 factors in the numerator make the most important contribution. We obtain

$$\frac{-6a_i}{M_W^2}\left(\frac{i\pi^2}{-3M_W^2}\right)M_\mu\bar{u}_e\Sigma_+q\epsilon u_\mu\ \int_0^1\alpha d\alpha\int_0^{1-\alpha}d\beta\frac{(-\alpha^2+\alpha/2-\alpha\beta+\beta/4)}{[(1-\alpha)+\beta(b_i-1)]} \quad (E.21)$$

and the integral alone has the value

$$\begin{cases} -5/48\ , & b_2 = 1 \\ \frac{-1}{12}\frac{\ln\xi}{\xi-1} - \frac{1}{24}\frac{1}{\xi-1}\left(1 - \frac{\ln\xi}{\xi-1}\right)\ , & b_3 = \xi\ . \end{cases} \quad \begin{matrix}(E.22a)\\(E.22b)\end{matrix}$$

When (E.16), (E.19) and (E.22) are added we obtain the final result for

$$I(W) = \left(\frac{-i\pi^2 M_\mu}{M_W^4}\right)\bar{u}_e\Sigma_+q\epsilon u_\mu\ [1/4 - f(\xi)/12 + g(\xi)/2] \quad (E.23)$$

where

$$f(\xi) \equiv (\ln\xi)\ /\ (\xi-1) \quad (E.24)$$

$$g(\xi) \equiv (\xi\ln\xi)/(\xi-1)^2 - 1/(\xi-1)\ ,$$

as quoted in (5.3.21a).

The integral I(S) of (E.2) is, compared with I(W) immediately preceding, much simpler. We have, from (D.18),

$$I(S) = 2\int d\alpha d\beta \int d^4k \; \frac{\bar{u}_e \Sigma_+ (p-k-M_\mu) u_\mu \; 2k\cdot\epsilon}{\{\alpha(p-k)^2+\beta[(k-q)^2-\xi M_W^2]+(1-\alpha-\beta)(k^2-\xi M_W^2)\}^3}$$

$$= 4\int d\alpha d\beta \int d^4K \; \bar{u}_e \Sigma_+ (p-K-V-M_\mu) u_\mu \; \frac{(K+V)\cdot\epsilon}{(K^2+B)^3} \qquad (E.25)$$

where the substitution $k \to K+V$ is again made, where V is the same as in (E.12a) and B the same as in (E.12b), except $M_W^2 \to \xi M_W^2$. The K^2 term in the numerator gives the wrong spinor amplitude $\bar{u}_e \Sigma_+ \epsilon u_\mu$, and so is ignored. Thus (E.25) becomes, using (D.13),

$$I(S) = \left(\frac{4i\pi^2}{-2\xi M_W}\right) M_\mu \; \bar{u}_e \Sigma_+ u_\mu \; p\cdot\epsilon \; \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \; \frac{\alpha(-\alpha-\beta)}{(1-\alpha)} \qquad (E.26)$$

and the integral alone has the value (-5/12). This gave the result (5.3.21b) as quoted.

Integral (E.3) is

$$I(WS) = \int d^4k \; \frac{\bar{u}_e \Sigma_+ \gamma_\nu u_\mu \; \epsilon_\sigma}{(p-k)^2[(k-q)^2-\xi M_W^2]} \left(\frac{\eta^{\nu\sigma}}{k^2-M_W^2} + \Sigma_i a_i \frac{k^\nu k^\sigma}{M_W^2(k^2-b_i M_W^2)} \right) \qquad (E.27)$$

and we see at once that the first term (using $\eta^{\nu\sigma}$) leads to the wrong spinor structure $\bar{u}_e \Sigma_+ \epsilon u_\mu$ and is to be ignored. The remainder is

$$I(WS) = \Sigma_i \frac{a_i}{M_W^2} \; 2\int d\alpha d\beta \int d^4k \; \frac{\bar{u}_e \Sigma_+ k u_\mu \; k\cdot\epsilon}{\{\alpha(p-k)^2+\beta[(k-q)^2-\xi M_W^2]+(1-\alpha-\beta)(k^2-b_i M_W^2)\}^3} \qquad (E.28)$$

$$= \Sigma_i \frac{a_i}{M_W^2} \; 2\int d\alpha d\beta \int d^4K \; \bar{u}_e \Sigma_+ (K+V) u_\mu \; \frac{(K+V)\cdot\epsilon}{(K^2+B)^3}$$

where, here, $B = -M_W^2[b_i(1-\alpha)+\beta(\xi-b_i)]$. The K^2 term gives the wrong structure and we have, keeping only the rest,

$$I(WS) = \Sigma_i \frac{a_i}{M_W^2} \left(-\frac{i\pi^2}{2M_W^2}\right) 2p\cdot\epsilon \; \bar{u}_e \Sigma_+ u_\mu M_\mu \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \; \frac{\alpha(\alpha+\beta)}{b_i(1-\alpha)+\beta(\xi-b_i)} \qquad (E.29)$$

and the integral alone has the values

$$\left\{\begin{array}{ll} \frac{1}{6}\left(\frac{1}{\xi-1} + (2\xi-3)\frac{\ln\xi}{(\xi-1)^2}\right) , & b_2 = 1 \\ \frac{5}{12}\frac{1}{\xi} , & b_3 = \xi , \end{array}\right. \tag{E.30}$$

which, with (E.24), gives the result quoted in (5.3.21c).

Integral (E.4) is

$$I(SW) = -\int d^4k\ \bar{u}_e \Sigma_+ \gamma_\nu \left(\frac{-1}{(p-k)^2} + \frac{M_\mu (p-k)}{(p-k)^4}\right) u_\mu \quad \frac{\epsilon_\sigma}{(k^2-\xi M_W^2)} \tag{E.31}$$

$$\times \left(\frac{\eta^{\nu\sigma}}{(k-q)^2-M_W^2} + \Sigma_i a_i \frac{(k-q)^\nu (k-q)^\sigma}{M_W^2[(k-q)^2-b_i M_W^2]}\right)$$

so, expanded, is four integrals. The product of the first of each of the two sums gives the wrong spinor structure, and as $pu_\mu = M_\mu u_\mu$, all that remains of the $\eta^{\nu\sigma}$ term is

$$M_\mu \int d^4k\ \bar{u}_e \frac{\Sigma_+ \epsilon k}{(p-k)^4} u_\mu \left(\frac{1}{k^2-\xi M_W^2}\right)\left(\frac{1}{(k-q)^2-M_W^2}\right) \tag{E.32}$$

$$= 6M_\mu \int \alpha d\alpha d\beta \int d^4K\ \bar{u}_e \frac{\Sigma_+ \epsilon (K+V)}{(K^2+B)^4} u_\mu$$

with the usual V and $B = -M_W^2[\xi(1-\alpha)+\beta(1-\xi)]$. The term linear in K vanishes and we have, deleting terms with $(\bar{u}_e \epsilon p u_\mu)$ (wrong structure),

$$6M_\mu\left(\frac{i\pi^2}{6M_W^4}\right) (-\bar{u}_e \Sigma_+ \epsilon p' u_\mu) \int_0^1 \alpha d\alpha \int_0^{1-\alpha} d\beta \ \frac{\beta}{\xi(1-\alpha)+\beta(1-\xi)} \tag{E.33}$$

with the integral having the value

$$\frac{1}{2}\left(\frac{1}{\xi-1} - \frac{\ln\xi}{(\xi-1)^2}\right) . \tag{E.34}$$

Back in (E.31) the first term in the first sum, and the second term in the second sum give the wrong structure, and what remains is

$$\sum_i \frac{a_i M_\mu}{M_W^2} \int d^4k\ \bar{u}_e \frac{\Sigma_+(k-q)k}{(p-k)^4(k^2-\xi M_W^2)} u_\mu \frac{k\cdot\epsilon}{[(k-q)^2-b_i M_W^2]} \tag{E.35}$$

$$= 6\sum_i \frac{a_i M_\mu}{M_W^2} \int \alpha d\alpha d\beta \int d^4K\ \bar{u}_e \Sigma_+ \frac{(K+V-q)(K+V)}{(K^2+B)^4} u_\mu\ (K+V)\cdot\epsilon$$

where B here is $-M_W^2[\xi(1-\alpha)+\beta(b_i-\xi)]$, which becomes, as only K^2 terms are needed,

$$6\sum_i \frac{a_i M_\mu}{M_W^2} \left(\frac{i\pi^2}{-3M_W^2}\right) \tfrac{1}{2}\int_0^1 \alpha d\alpha \int_0^{1-\alpha} d\beta\ \frac{(-1+3\alpha)}{\xi(1-\alpha)+\beta(b_i-\xi)} \tag{E.36}$$

and the integral alone is

$$\begin{cases} \dfrac{1}{2}\dfrac{\ln\xi}{\xi-1}\ , & b_2 = 1 \\[2ex] 1/(2\xi)\ , & b_3 = \xi\ . \end{cases} \tag{E.37}$$

The sum of (E.34) and (E.37) gives the value of I(SW) quoted earlier in (5.3.21d).

The final integral, (E.5), is

$$I(B) = 2\int d\alpha d\beta \int d^4k\ \frac{\bar{u}_e \Sigma_+[M_\mu+(p-k)\cot^2\theta]u_\mu\ \ 2k\cdot\epsilon}{M_W^2\{\alpha(p-k)^2+\beta[(k-q)^2-M_B^2]+(1-\alpha-\beta)(k^2-M_B^2)\}^3} \tag{E.38}$$

$$= 2\int d\alpha d\beta \int d^4K\ \bar{u}_e \Sigma_+ \frac{[M_\mu+(p-K-V)\cot^2\theta]}{M_W^2(K^2+B)^3} u_\mu\ \ 2(K+V)\cdot\epsilon$$

where $B = -(1-\alpha)M_B^2$, which is virtually the same as (E.25). Its final value is

$$I(B) = \frac{i\pi^2}{M_W^2 M_B^2} M_\mu (1 + \tfrac{1}{6}\cot^2\theta)\ \tfrac{1}{2}\bar{u}_e \Sigma_+ q\epsilon u_\mu \tag{E.39}$$

as quoted in (5.3.21e).

<u>The Spinor Amplitude (5.4.5)</u>

The evaluation of the spinor amplitude

$$\sum_{spins} |\bar{u}_e \Sigma_+ q\epsilon u_\mu|^2 \tag{E.40}$$

is neither new nor particularly difficult, but will be evaluated using the novel techniques described in App. B. We use (5.4.4b) and (B.35) to obtain

$$\Sigma_{\text{spins}} |\bar{u}_e \Sigma_+ q\epsilon u_\mu|^2 = \Sigma_\alpha \epsilon^\mu_{(\alpha)} \epsilon^\nu_{(\alpha)} 4S[\Sigma_+ q\gamma_\mu \tfrac{1}{2}(1+p/M_\mu)\gamma_\nu q\Sigma_- \tfrac{1}{2}(1+p'/m_e)]$$

$$= -\eta^{\mu\nu} S[\Sigma_+ q\gamma_\mu p\gamma_\nu qp'] \;/\; M_\mu m_e = 2S[\tfrac{1}{2}(1-i\gamma_5)qpqp'] \;/\; M_\mu m_e$$

$$= S[qpqp'] \;/\; M_\mu m_e = 2\; q\cdot p\; q\cdot p' \;/\; M_\mu m_e \tag{E.41}$$

since $q^2 = 0$, and $\Sigma_+\Sigma_- = 0$, $\Sigma_+\Sigma_+ = \Sigma_+$, and

$$qpqp' = qp(q\cdot p' + q\wedge p') = q[pq\cdot p' + p\cdot(q\wedge p') + p\wedge(q\wedge p')]$$

$$= (q\cdot p + q\wedge p)q\cdot p' + q\cdot[p\cdot(q\wedge p')] + q\cdot(p\wedge q\wedge p') \tag{E.42}$$

of which only the scalar part is needed, obtained from (B.37b).

<u>The Integral (5.4.9)</u>

The phase space integral

$$\int \frac{d^3q}{2q_0} \frac{d^3p'}{p'_0} \delta^4(p-q-p') \tag{E.43}$$

will be evaluated using a technique described in Kaempffer (1965, App. 6). First it is rewritten as

$$\int d^4q \;\; \delta(q^2)\; \tfrac{1}{2}[1+\epsilon(q)] \; \frac{d^3p'}{p'_0} \; \delta^4(p-q-p') \tag{E.44}$$

where

$$\epsilon(q) \equiv \begin{cases} +1, & q_0>0 \\ -1, & q_0<0 \end{cases}, \tag{E.45}$$

which becomes, because the q-integration can be performed,

$$\int \frac{d^3p'}{p'_0} \; \delta[(p-p')^2] \; \tfrac{1}{2}[1+\epsilon(p-p')] \;. \tag{E.46}$$

Now, since

$$(p-p')^2 = p^2 + p'^2 - 2p\cdot p' = M_\mu^2 + m_e^2 - 2M_\mu p'_0 \tag{E.47}$$

which vanishes at $p'_0 \equiv E'$, so that as

$$\delta[(p-p')^2]\; \tfrac{1}{2}[1+\epsilon(p-p')] = \left(\left. \frac{d(p-p')^2}{dp'_0} \right|_{p'=E'} \right)^{-1} \delta(p'_0 - E') \tag{E.48}$$

$$= \frac{\delta(p'_0 - E')}{2M_\mu}$$

and

$$d^3p' = 4\pi|\vec{p}'|^2 d|\vec{p}'| = 4\pi|\vec{p}'|p_0' dp_0'$$

(E.49)

we have

$$4\pi\int|\vec{p}'|\frac{p_0' dp_0'}{p_0'}\ \frac{\delta(p_0'-E')}{2M_\mu} = (2\pi/M_\mu)\ (E'^2-m_e^2)^{1/2} \tag{E.50}$$

$$= (2\pi/M_\mu)\ [(M_\mu^2 - m_e^2)/(2M_\mu)] \approx \pi$$

since $M_\mu >> m_e$. This was the value claimed earlier, in (5.4.9).

APPENDIX F

DETAILS OF THE CALCULATION $\mu \to 3e$

This appendix contains the details of the calculations needed in Chap. VI for the $\mu \to 3e$ Feynman diagrams, transition amplitude and decay rate.

The Integral (6.2.2)

We now evaluate the integral

$$I(B) = \int d^4k \, \frac{\bar{u}_e\Sigma_+(p-k)u_\mu \; \bar{u}_e(2k-q)v_e}{(p-k)^2[(k-q)^2-M_B^2](k^2-M_B^2)} \tag{F.1}$$

$$= 2\int d\alpha d\beta \int d^nk \, \frac{\bar{u}_e\Sigma_+(p-k)u_\mu \; \bar{u}_e(2k-q)v_e}{\{\alpha(p-k)^2+\beta[(k-q)^2-M_B^2]+(1-\alpha-\beta)(k^2-M_B^2)\}^3}$$

$$= 2\int d\alpha d\beta \int d^nK \, \frac{\bar{u}_e\Sigma_+(p-K-V)u_\mu}{(K^2+B)^3} \, \bar{u}_e(2K+2V-q)v_e$$

where by dimensional regularization one imposes $n<4$ to allow the shift in the integration variable, and where $V = \alpha p+\beta q$, $B \approx -(1-\alpha)M_B^2$. The K^2 or diverging part as $n \to 4$ is, from (D.3),

$$-(\bar{u}_e\Sigma_+\gamma_\nu u_\mu)(\bar{u}_e\gamma^\nu v_e) \int d\alpha d\beta \int d^nK \, \frac{K^2}{(K^2+B)^3} \tag{F.2}$$

as given in (6.2.3). The finite part is, from (D.13),

$$\left(\frac{i\pi^2}{-2M_B^2}\right) 4\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \; \bar{u}_e\Sigma_+\frac{(p-\alpha p-\beta q)}{(1-\alpha)}u_\mu \; \bar{u}_e \alpha p v_e \tag{F.3}$$

because, as $q \equiv p_2+p'$,

$$\bar{u}_e(p_2)qv_e(p')=\bar{u}_e(p_2)(p_2+p')v_e(p')=\bar{u}_e(p_2)(m_e-m_e)v_e(p')=0. \tag{F.4}$$

Integral (F.3) becomes

$$4[i\pi^2/(-2M_B^2)]M_\mu(\bar{u}_e\Sigma_+u_\mu)(\bar{u}_e p v_e)\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \alpha(1-\alpha-\beta)/(1-\alpha) \tag{F.5}$$

and the integral alone has the value 1/12, all of which lead to the result quoted in (6.2.3).

The Integral (6.2.4)

With the neutrino propagator expanded as in (5.3.1) the integral (6.2.4) becomes

$$A(\text{renorm}) = \frac{C}{p^2 q^2} \int d^4k \, \frac{\bar{u}_e \gamma_\lambda p \Sigma_+ (p-k) u_\mu}{(p-k)^2 (k^2 - M_B^2)} \, \bar{u}_e \gamma^\lambda v_e \tag{F.6}$$

$$= \frac{C}{M_\mu^2 q^2} \int d\alpha \int d^n k \, \frac{\bar{u}_e \gamma_\lambda p \Sigma_+ (p-k) u_\mu}{[\alpha(p-k)^2 + (1-\alpha)(k^2 - M_B^2)]^2} \, \bar{u}_e \gamma^\lambda v_e$$

$$= \frac{C}{M_\mu^2 q^2} \int d\alpha \int d^n K \, \frac{\bar{u}_e \gamma_\lambda p \Sigma_+ (p-K-\alpha p) u_\mu}{[K^2 - (1-\alpha) M_B^2]^2} \, \bar{u}_e \gamma^\lambda v_e$$

$$= \frac{C}{M_\mu^2 q^2} M_\mu^2 \, (\bar{u}_e \Sigma_+ \gamma_\lambda u_\mu)(\bar{u}_e \gamma^\lambda v_e) \int d\alpha \int d^n K \, \frac{(1-\alpha)}{(K^2 + B)^2}$$

as claimed in (6.2.5), since $pu_\mu = M_\mu u_\mu$. If, in Fig. 21, the virtual photon were to emerge from the incoming muon line the diagram would have the amplitude (using the symbols of Fig. 21)

$$A'(\text{renorm}) = \sum_i \int \frac{d^4k}{(2\pi)^4} \bar{u}_e(p_1) \left(\frac{igU_{i1}}{M_W} m_i \cot\theta \Sigma_+ \right) \left(\frac{1/2 \, i}{(p_1 - k) - m_i} \right) \tag{F.7}$$

$$\times \left(\frac{igU_{i2}}{M_W} m_i \cot\theta \Sigma_- \right) \left(\frac{i}{p_1 - M_\mu} \right) (ie\gamma_\lambda) u_\mu(p) \left(\frac{i}{k^2 - M_B^2} \right) \left(\frac{-i\eta^{\lambda\mu}}{q^2} \right)$$

$$\times \bar{u}_e(p_2) ie\gamma_\mu v_e(p')$$

$$= C \int d^4k \, \frac{\bar{u}_e \Sigma_+ (p_1 - k)(p_1 + M_\mu) \gamma_\lambda u_\mu}{q^2 (p_1 - k)^2 (k^2 - M_B^2)(p_1^2 - M_\mu^2)} \, \bar{u}_e \gamma^\lambda v_e \quad .$$

The denominator will require $k \to K + \alpha p_1$, but as the linear K term in the numerator vanishes, and $p_1 u_e = m_e u_e$, the whole integral will be of order $m_e/M_\mu \ll 1$ relative to (F.6) and is therefore to be ignored as was claimed in the remarks just preceding Eq. (6.2.4).

The Integral (6.2.9)

The following integral is needed to establish the contribution of the W particle (and, because of gauge invariance, the S Higgs particle) to the $\mu \to 3e$ transition amplitude:

$$I(W) = \int d^4k \; \bar{u}_e \Sigma_+ \gamma^\rho (p-k) \gamma^\sigma u_\mu \frac{\Gamma_{\lambda\sigma\rho}}{(p-k)^4 [(k-q)^2 - M_W^2](k^2 - M_W^2)} \tag{F.8}$$

where $\Gamma_{\lambda\sigma\rho}$ is given in (E.7). We have

$$\begin{aligned} I(W) = 6\int \alpha d\alpha d\beta \int d^4k \; \bar{u}_e \Sigma_+ [&-(q+k)(p-k)\gamma_\lambda + \gamma_\sigma (p-k)\gamma^\sigma (2k-q)_\lambda \\ &+ \frac{\gamma_\lambda (p-k)(2q-k)] u_\mu}{\{\alpha(p-k)^2 + \beta[(k-q)^2 - M_W^2] + (1-\alpha-\beta)(k^2 - M_W^2)\}^4} \end{aligned} \tag{F.9}$$

$$= 6\int \alpha d\alpha d\beta \int \frac{d^4K}{(K^2+B)^4} \; \bar{u}_e \Sigma_+ (K^2 \gamma_\lambda - 2\gamma_\sigma K \gamma^\sigma K_\lambda + \gamma_\lambda K^2) u_\mu$$

where only the highest order terms -- the K^2 terms -- have been kept in the numerator, and where $B \approx -(1-\alpha)M_W^2$. This becomes, from (D.15),

$$I(W) = [6i\pi^2/(-3M_W^2)] \; 3\bar{u}_e \Sigma_+ \gamma_\lambda u_\mu \int_0^1 \alpha d\alpha \int_0^{1-\alpha} d\beta [1/(1-\alpha)] \; . \tag{F.10}$$

The integral has the value 1/2 which gives

$$I(W) = (-i\pi^2/M_W^2) \; 3\bar{u}_e \Sigma_+ \gamma_\lambda u_\mu \; , \tag{F.11}$$

as recorded in (6.2.13).

<u>The Integral (6.2.14)</u>

Integral (6.2.14) becomes, when the neutrino propagator is reduced,

$$\begin{aligned} A(WS) &= C' \int d^4k \; \frac{\bar{u}_e \Sigma_+ \gamma_\lambda u_\mu \;\; \bar{u}_e \gamma^\lambda v_e}{q^2 (p-k)^2 [(k-q)^2 - M_W^2](k^2 - M_W^2)} \\ &= C' \; 2\int d\alpha d\beta \int \frac{d^4K}{q^2 (K^2+B)^3} \; \bar{u}_e \Sigma_+ \gamma_\lambda u_\mu \; \bar{u}_e \gamma^\lambda v_e \\ &= C' \left(\frac{i\pi^2}{-2q^2 M_W^2} \right) 2 \int \frac{d\alpha d\beta}{(1-\alpha)} \; \bar{u}_e \Sigma_+ \gamma_\lambda u_\mu \; \bar{u}_e \gamma^\lambda v_e \\ &= (-i\pi^2 C'/q^2 M_W^2) \; \bar{u}_e \Sigma_+ \gamma_\lambda u_\mu \; \bar{u}_e \gamma^\lambda v_e \end{aligned} \tag{F.12}$$

as given in (6.2.15).

The Integral (6.2.16)

Similarly, integral (6.2.16) reduces to

$$-C'\int d^4k\ \bar{u}_e\Sigma_+\gamma_\lambda\left(\frac{-1}{(p-k)^2}+\frac{M_\mu(p-k)}{(p-k)^4}\right)u_\mu\ \frac{\bar{u}_e\gamma^\lambda v_e}{q^2[(k-q)^2-M_W^2](k^2-M_W^2)}. \quad (F.13)$$

The second of these terms is a factor M_μ^2/M_W^2 times the first, and so is negligible, while the first is identical to (F.12).

The Spinor Amplitude (6.3.1)

The lengthy spinor amplitude squared

$$\bar{\Sigma}_{spins}|[\bar{u}_e(p_1)\Sigma_+\gamma_\nu u_\mu(p)\ \bar{u}_e(p_2)\gamma^\nu v_e(p')]/(p-p_1)^2\ -(p_1\leftrightarrow p_2)|^2 \quad (F.14)$$

required in the $\mu \to 3e$ transition rate can be evaluated in two parts. The first requires the evaluation

$$\bar{\Sigma}_{spins}\bar{u}_e(p_1)\Sigma_+\gamma_\nu u_\mu(p)\ \bar{u}_e(p_2)\gamma^\nu v_e(p')$$
$$\times\ \bar{v}_e(p')\gamma^\mu u_e(p_2)\ \bar{u}_\mu(p)\gamma_\mu\Sigma_- u_e(p_1)$$
$$=(\tfrac{1}{2})4S[\Sigma_+\gamma_\nu\tfrac{1}{2}(1+p/M)\gamma_\mu\Sigma_-\tfrac{1}{2}(1+p_1/m)] \quad (F.15)$$
$$\times\ 4S[\gamma^\nu\tfrac{1}{2}(-1+p'/m)\gamma^\mu\tfrac{1}{2}(1+p_2/m)]$$
$$=(1/2mM)\ S[\Sigma_+\gamma_\nu p\gamma_\mu p_1]\ S[-\gamma^\nu\gamma^\mu+\gamma^\nu p'\gamma^\mu p_2/m^2]$$
$$=(1/4m^3M)\ (p_\nu p_{1\mu}+p_\mu p_{1\nu}-\eta_{\mu\nu}p\cdot p_1)[p'^\nu p_2^\mu+p'^\mu p_2^\nu-\eta^{\mu\nu}(m^2+p'\cdot p_2)]$$
$$=(1/2m^3M)\ (p\cdot p'\,p_1\cdot p_2\ +\ p\cdot p_2\,p'\cdot p_1\ +\ m^2\,p\cdot p_1)$$

where $m \equiv m_e$, $M \equiv M_\mu$.

By interchanging p_1 and p_2 we have two terms of the expansion. The cross-term from (F.14) is twice the real part of

$$-\bar{\Sigma}_{spins}\bar{u}_e(p_1)\Sigma_+\gamma_\nu u_\mu(p)\ \bar{u}_e(p_2)\gamma^\nu v_e(p')$$
$$\times\ \bar{v}_e(p')\gamma^\mu u_e(p_1)\ \bar{u}_\mu(p)\gamma_\mu\Sigma_- u_e(p_2) \quad (F.16)$$

divided by $(p-p_1)^2(p-p_2)^2$. This can be evaluated by a rearrangement:

$$-\bar{\Sigma}_{spins}\bar{u}_e(p_1)\Sigma_+\gamma_\nu u_\mu(p)\ \bar{u}_\mu(p)\gamma_\mu\Sigma_- u_e(p_2)$$
$$\times\ \bar{u}_e(p_2)\gamma^\nu v_e(p')\ \bar{v}_e(p')\gamma^\mu u_e(p_1) \tag{F.17}$$
$$= (-\tfrac{1}{2})4S[\Sigma_+\gamma_\nu\tfrac{1}{2}(1+p/M)\gamma_\mu\Sigma_-\tfrac{1}{2}(1+p_2/m)\gamma^\nu\tfrac{1}{2}(-1+p'/m)\gamma^\mu\tfrac{1}{2}(1+p_1/m)]$$
$$= (-1/8M)S[\Sigma_+\gamma_\nu p\gamma_\mu(1+p_2/m)\gamma^\nu(-1+p'/m)\gamma^\mu(1+p_1/m)]$$
$$= (1/4m^3M)(m^2 p\cdot p_1 + m^2 p\cdot p_2 - m^2 p\cdot p' + 2p\cdot p' p_1\cdot p_2)$$

after a lengthy but straightforward manipulation. Twice this expression, divided by $(p-p_1)^2(p-p_2)^2$, plus (F.15) (including p_1 and p_2 interchanged) give the spinor amplitude squared (6.3.5).

<u>The Spinor Amplitude (6.3.3)</u>

The spinor amplitude squared

$$\bar{\Sigma}_{spins}|\bar{u}_e(p_1)\Sigma_+ u_\mu(p)\ \bar{u}_e(p_2)pv_e(p')/(p-p_1)^2 - (p_1\leftrightarrow p_2)|^2 \tag{F.18}$$

is evaluated similarly. The first term "squared" is (again, $m_e\equiv m$, $M_\mu\equiv M$)

$$\bar{\Sigma}_{spins}|\bar{u}_e(p_1)\Sigma_+ u_\mu(p)\ \bar{u}_e(p_2)pv_e(p')|^2 \tag{F.19}$$
$$= \tfrac{1}{2}\Sigma_{spins}[\bar{u}_e(p_1)\Sigma_+ u_\mu(p)\ \bar{u}_\mu(p)\Sigma_- u_e(p_1)$$
$$\times\ \bar{u}_e(p_2)pv_e(p')\ \bar{v}_e(p')pu_e(p_2)]$$
$$= \tfrac{1}{2}\ 4S[\Sigma_+\tfrac{1}{2}(1+p/M)\Sigma_-\tfrac{1}{2}(1+p_1/m)]\ 4S[p\tfrac{1}{2}(-1+p'/m)p\tfrac{1}{2}(1+p_2/m)]$$
$$= \tfrac{1}{2}S[\tfrac{1}{2}(1-i\gamma_5)pp_1/mM]\ S[-p^2+pp'pp_2/m^2]$$
$$= [(p\cdot p_1)/(4m^3M)]\ (2p\cdot p' p\cdot p_2 - m^2M^2 - M^2p'\cdot p_2)\ .$$

The second term "squared" merely requires the interchange of p_1 and p_2. The cross-term is the negative of twice the real part of

$$\bar{\Sigma}_{spins}[\bar{u}_e(p_1)\Sigma_+ u_\mu(p)\ \bar{u}_e(p_2) p v_e(p')][\bar{v}_e(p') p u_e(p_1)\bar{u}_\mu(p)\Sigma_- u_e(p_2)]$$

$$= (\tfrac{1}{2})4S[\Sigma_+\tfrac{1}{2}(1+p/M)\Sigma_-\tfrac{1}{2}(1+p_2/m)p\tfrac{1}{2}(-1+p'/m)p\tfrac{1}{2}(1+p_1/m)] \qquad (F.20)$$

$$= [4p\cdot p_1 p\cdot p_2 p\cdot p' - M^2(p\cdot p_2 p'\cdot p_1 + p\cdot p_1 p'\cdot p_2 + p_1\cdot p_2 p\cdot p')$$
$$- m^2M^2(p\cdot p' - p\cdot p_1 - p\cdot p_2)] \ /\ (16m^3M) \ ,$$

after a lengthy, but straightforward, reduction.

The Spinor Amplitude (6.3.4)

The spinor amplitude squared

$$\bar{\Sigma}_{spins}|\bar{v}_\mu(p)v_e(p')\ [\bar{u}_e(p_1)v_e(p_2) - \bar{u}_e(p_2)v_e(p_1)]|^2 \qquad (F.21)$$

we evaluate in parts. The first factor is readily found to be

$$\tfrac{1}{2}\Sigma_{spins}|\bar{v}_\mu(p)v_e(p')|^2 = \tfrac{1}{2}4S[\tfrac{1}{2}(-1+p'/m)\tfrac{1}{2}(-1+p/M)] \qquad (F.22)$$
$$= \tfrac{1}{2}(1+p\cdot p'/mM) \ ,$$

while the second factor

$$\Sigma_{rs}|\bar{u}_r(p_1)v_s(p_2) - \bar{u}_s(p_2)v_r(p_1)|^2 \ , \qquad (F.23)$$

where r,s = ±, seems unlike other amplitudes normally encountered in quantum field theories because the cross-term is not readily reducible to a "trace" over projection operators, and must therefore be obtained more indirectly.

Consider a spinor basis

$$\tfrac{1}{2}(1+\gamma_0)\tfrac{1}{2}[1\pm(-i)\gamma_5\gamma_3\gamma_0]\Phi_\pm = \Phi_\pm \qquad (F.24)$$
$$\tfrac{1}{2}(1-\gamma_0)\tfrac{1}{2}[1\pm(-i)\gamma_5\gamma_3\gamma_0]\chi_\mp = \chi_\mp$$

with

$$\gamma_5\Phi_\pm \equiv \alpha_\pm\ \chi_\mp \qquad (F.25)$$

and $\alpha_+ \equiv 1$ to define a phase. Then we may take $|\vec{p}_2| \equiv 0$ and

$$u_r(p_2) = \Phi_r \ , \qquad v_r(p_2) = \chi_r \ , \qquad (F.26)$$
$$u_r(p_1) \equiv L\Phi_r \ , \qquad v_r(p_1) \equiv L\chi_r \ ,$$

where L is the Lorentz boost operator (Hamilton, 1984b)

$$L = (1 + p_1 \gamma_0/m) \ / \ [2(1 + \epsilon_1/m)]^{1/2} \ , \tag{F.27}$$

and

$$L^{-1} = (1 + \gamma_0 p_1/m) \ / \ [2(1 + \epsilon_1/m)]^{1/2} \ , \tag{F.28}$$

with $\epsilon \equiv p \cdot \gamma_0$.

From (F.24) and (F.25) follows

$$\gamma_3 \chi_{\pm} = \pm(i/\alpha_{\mp})\phi_{\mp} \ , \tag{F.29}$$

and we also have (to within an irrelevant phase)

$$\gamma_1 \chi_{\pm} \propto \phi_{\pm} \ , \qquad \gamma_2 \chi_{\pm} \propto \mp\phi_{\pm} \ . \tag{F.30}$$

The amplitude squared (F.23) now becomes, with $p_1 \equiv p$, $L = L(p)$,

$$\begin{aligned}
&\Sigma_{rs} |\bar{u}_r(p_1) v_s(p_2) - \bar{u}_s(p_2) v_r(p_1)|^2 && \text{(F.31)} \\
&\quad = \Sigma_{rs} |\bar{\phi}_r L^{-1} \chi_s - \bar{\phi}_s L \chi_r|^2 \\
&\quad = [2m^2(1+\epsilon/m)]^{-1} \ \Sigma_{rs} |\bar{\phi}_r \gamma_0 p \chi_s - \bar{\phi}_s p \gamma_0 \chi_r|^2 \\
&\quad = [2m^2(1+\epsilon/m)]^{-1} \ \Sigma_{rs} |\bar{\phi}_r p^i \gamma_i \chi_s + \bar{\phi}_s p^i \gamma_i \chi_r|^2 \\
&\quad = [2(1+\epsilon/m)]^{-1} \ [8(p^1)^2 + 8(p^2)^2 + 2(p^3)^2 |1-\alpha_-|^2]/m^2 \ .
\end{aligned}$$

Thus to achieve a covariant result we must make the selection $\alpha_- \equiv -1$ in (F.25). The amplitude used in Sec. VI.3 is thus

$$\begin{aligned}
&\Sigma_{spins} |\bar{u}(p_1) v(p_2) - \bar{u}(p_2) v(p_1)|^2 && \text{(F.32)} \\
&\quad = 4(|\vec{p}|^2/m^2) \ / \ (1+\epsilon/m) = 4(\epsilon/m - 1) \\
&\quad = 4(p_1 \cdot p_2/m^2 - 1) \ ,
\end{aligned}$$

on reverting back to the original covariant notation.

The Phase Space Integral (6.3.14)

The transition rate of (6.3.14),

$$\Gamma = \tfrac{1}{2}\int |A|^2 (2\pi)^4 \delta^4(p-p_1-p_2-p') \frac{m}{E'}\frac{d^3p'}{(2\pi)^3}\frac{m}{E_1}\frac{d^3p_1}{(2\pi)^3}\frac{m}{E_2}\frac{d^3p_2}{(2\pi)^3}, \tag{F.33}$$

is first integrated over the positron states (p') because the amplitude squared $|A|^2$ depends only on E_1 and E_2. We use the same technique as in App. E and obtain

$$\Gamma = \frac{m^3}{(2\pi)^5}\int |A|^2 \frac{d^3p_1 d^3p_2}{E_1 \quad E_2}\delta^4(p-p_1-p_2-p')\delta(p'^2-m^2)\tfrac{1}{2}[1+\epsilon(p')]\ d^4p' \tag{F.34}$$

$$= \frac{m^3}{(2\pi)^5}\int |A|^2\ \delta[(p-p_1-p_2)^2-m^2]\ \tfrac{1}{2}[1+\epsilon(p-p_1-p_2)]\ \frac{d^3p_1}{E_1}\ \frac{d^3p_2}{E_2}$$

which is (6.3.15). We now integrate over all angles. One replaces d^3p_1 with $4\pi|\vec{p}_1|^2 d|\vec{p}_1| = 4\pi|\vec{p}_1|E_1 dE_1$ and defines

$$\hat{p}_1 \cdot \hat{p}_2 \equiv \cos\theta_{12} \equiv c\ , \tag{F.35}$$

so that

$$f(c) \equiv (p-p_1-p_2)^2-m^2 = M^2+m^2-2ME_1-2ME_2+2E_1E_2-2|\vec{p}_1||\vec{p}_2|c \tag{F.36}$$

and defines

$$f(c_0) \equiv 0. \tag{F.37}$$

Since

$$|f'(c_0)| = 2\ |\vec{p}_1|\ |\vec{p}_2| \tag{F.38}$$

we have

$$\Gamma = \frac{4\pi\ m^3}{(2\pi)^5}\int |A|^2\ |\vec{p}_1|\ dE_1\ \frac{\delta(c-c_0)}{|f'(c_0)|}\ 2\pi|\vec{p}_2|\ dE_2\ dc \tag{F.39}$$

$$= [m^3/(2\pi)^3]\int |A|^2\ dE_1\ dE_2$$

$$= [m^3M^2/4(2\pi)^3]\int |A(x_1,x_2)|^2\ dx_1\ dx_2\ ,$$

where $E_i \equiv \tfrac{1}{2}Mx_i$ from (6.3.10) has been used. Thus we have derived the exact result (6.3.16).

APPENDIX G

LIST OF VERTICES FOR FEYNMAN DIAGRAMS

The vertices computed in Sec. V.4 that were subsequently used to construct the Feynman diagrams for muon decay in Chap. V, VI are listed here for the convenience of the reader. Questions of sign or momentum direction are to be resolved by referring to the original derivation.

The notation used here is

ℓ(or ℓ_i) = lepton (i=1=electron, i=2=muon, etc., of mass M_i);

x(or x_i) = Majorana neutrino (of mass m_i);

A = photon;

Z = massive, neutral electro-weak gauge boson;

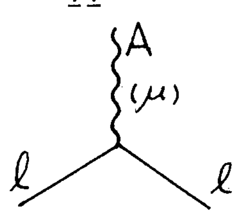
$W_\pm$ = massive, charged electro-weak gauge boson;

$S_\pm$ = unphysical, charged Higgs boson;

$B_\pm$ = physical, charged Higgs boson;

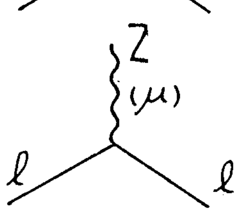
$H_{\pm\pm}$ = physical, doubly-charged Higgs boson.

ℓ-ℓ-A



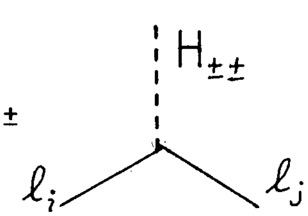
$ie\gamma_\mu$ [Eq.(4.4.9)]

ℓ-ℓ-Z

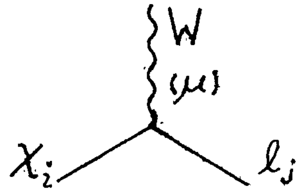


$ig\gamma_\mu(\sin^2\theta_W - \frac{1}{2}\Sigma_-)$ [Eq.(4.4.11)]

ℓ-ℓ-$H_{\pm\pm}$

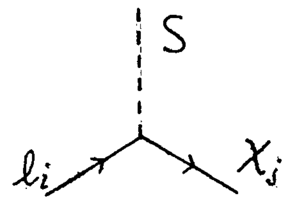


$i\sqrt{2}\beta_{ij}\Sigma_\pm$ [Eq.(4.4.22)] ℓ-x-W

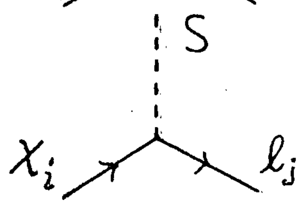


$igU_{ij}\gamma_\mu\Sigma_-$ [Eq.(4.4.13)]

ℓ-x-S

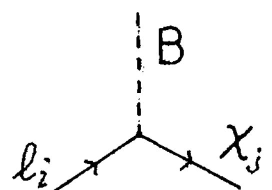


$-i(g/M_W)U_{ji}(M_i\Sigma_+ - m_j\Sigma_-)$ [Eq.(4.4.15)]



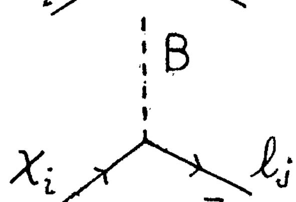
$-i(g/M_W)U_{ij}(M_j\Sigma_- - m_i\Sigma_+)$ [Eq.(4.4.16)]

ℓ-x-B



$i(g/M_W)U_{ji}(M_i\tan\theta\Sigma_+ + m_j\cot\theta\Sigma_-)$

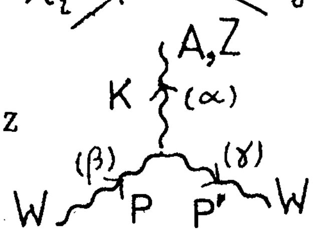
[Eq.(4.4.19)]



$i(g/M_W)U_{ij}(M_j\tan\theta\Sigma_- + m_i\cot\theta\Sigma_+)$

[Eq.(4.4.20)]

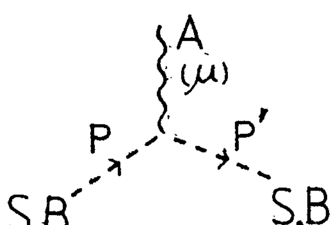
W-W-A,Z



$ig\cos\theta_W[(P'_\alpha+P_\alpha)\eta_{\beta\gamma} + (K_\beta-P'_\beta)\eta_{\alpha\gamma}$
$+ (-P_\gamma-K_\gamma)\eta_{\alpha\beta}]$ [Eq.(4.4.28)]

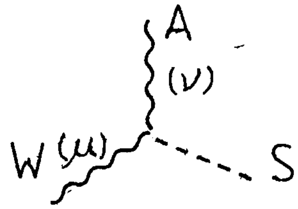
S-S-A

B-B-A



$ie(P'_\mu+P_\mu)$ [Eq.(4.4.33,34)]

W-A-S



$-ieM_W\eta_{\mu\nu}$ [Eq.(4.4.36)]

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