A METHOD TO MEASURE LINE SHAPES
AND RELATIVE TRANSITION
PROBABILITIES
by
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ABSTRACT

An accurate method to measure line shapes and relative transition probabilities between transitions with a common lower state has been developed. Argon was used as a test gas to show that this method is more accurate than other current procedures.

The experimental observations agree well with the theory developed in this thesis. It is shown that the ratio of the Lorentzian to Doppler half-widths of the line can be measured to a precision of 10% when the Doppler width is as much as one hundred times larger than the Lorentzian half-width. The relative transition probabilities can be measured to within 10% for weak lines and to 1% for strongly absorbing lines. The line shapes measured are consistent with those predicted by Griem (2) and the relative transition probabilities agree well with the values given by Wiese (5).

This absorption experiment uses a glow discharge for the source and absorber. The absorption tube is placed between Nicol prisms and a longitudinal magnetic field is applied to Zeeman split the absorption lines. The transmission as a function of field depends on both the transition
probabilities and the line shape of the absorption lines. The high resolving power required to measure the line shapes is obtained from the Zeeman splitting of the absorption lines rather than a spectrometer. The method is an inexpensive, accurate way to measure line shapes and relative transition probabilities especially suited for strongly absorbing lines.
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CHAPTER 1

INTRODUCTION

Accurate values of transition probabilities are important in order to obtain accurate information from quantitative spectroscopy. In astrophysics the transition probabilities are needed in order to deduce the conditions responsible for the emission and/or absorption spectra observed. Quantities such as elemental abundance, degree of ionization, temperatures and electron densities can be determined from detailed examination of the spectral line shapes and intensities. In the laboratory, plasma spectroscopy uses these same probabilities in a similar manner to determine the properties of the plasmas being observed there.

Ever since the original work of Einstein (6) experimental and theoretical work has been carried out to determine accurate values for these transition probabilities. Unfortunately, in general, there has not been good agreement between various workers who have measured the same quantities.
Due to this lack of agreement it was decided to develop yet another method to measure relative transition probabilities.

This new method involves the propagation of linearly polarized line radiation along a magnetic field which is embedded in a plasma. The dispersion in the vicinity of the optical resonance along with the effect of the magnetic field is shown in Fig. 1-1. (All figures are at the end of thesis.) The frequency of the unshifted source line is centered on frequency $V_0$ and the right- and left-hand circularly polarized components of the absorption line (subscripts + and - respectively) are separated due to the Zeeman effect. Referring to Fig. 1-1 let us consider linearly polarized light of frequency $V'$ propagating through the plasma with a magnetic field strength of $H_a$. Since the left- and right-hand components of the linearly polarized source have different indices of refraction there will be a phase shift between the two polarizations resulting in a rotation of the plane of polarization. The difference in the absorption coefficients ($k$) at frequency $V'$ changes the polarization from linear to elliptical. As the magnetic field increases the splitting of the absorption line will increase, resulting in a change in the absorption and rotation at frequency $V'$. 
In the present work the transmission of the source radiation is measured as a function of magnetic field to determine the absorption coefficient \( k \) (cm\(^{-1}\)) and the index of refraction \( n \) as a function of frequency. It is shown that \((n-1)\) and \(k\) are proportional to the transition probability and that the frequency dependence gives information about the state lifetimes and therefore absolute transition probabilities.

The Lorentzian component of the line profile is produced by interruptions of the oscillator. Thus an extrapolation of the Lorentzian component to zero pressure could give an estimate of the lifetimes of the states involved. These levels must have a common lower state to give the relative transition probabilities but the Lorentzian component of the line shape can be determined for any line.

The present work measures the relative transition probabilities for a number of argon lines and discusses the feasibility of measuring absolute lifetimes by this method.

The first application of the method was to neon by Seka and Curzon (8). Their work indicated that the agreement between experiment and theory was sufficient to give relative transition probabilities to 10%-20% and that there was insufficient Lorentzian component in the line shape to be measured. The criterion they used to ignore the
Lorentzian component was to introduce the Lorentzian into the calculation of the absorption coefficient and note that there was little effect on the calculated curves.

The present work introduces the Lorentzian line shape into the calculation of both the absorption coefficient and the index of refraction along with variations in the population of the energy levels with magnetic field. The effect of the Lorentzian component is found to be greatest in the index of refraction which determines the rotation of the plane of polarization. These additions improve the 'fit' between experiment and theory to the point where they agree within experimental error and as a result the relative transition probabilities are determined to an accuracy of 1% for strongly absorbing lines. In addition the new treatment determines the ratio of the Lorentzian to Doppler components within 10%.

Thus the method as applied in this thesis gives the relative transition probabilities to an accuracy much better than the 'best' previous estimates by Wiese (5) and it also gives the small Lorentzian component of a basically Doppler line shape (Lorentzian = 7% Doppler) to a precision of about 10%. Further it must be stressed that the detailed line shape (Lorentzian half-width ~10^{-4} cm^{-1}) is measured by using a monochromator which need only isolate the line which is being measured (10 cm^{-1} instrument width).
CHAPTER 2

THEORY

2.1 INTRODUCTION

This chapter deals with the propagation of a monochromatic circularly polarized wave along a constant magnetic field B embedded in a plasma (Fig. 2-1).

The vector notation used throughout this work will be of the form $E_i$ which is defined as the component of the three dimensional vector $E$ in the i-th direction. The subscripts take on the values 1, 2, and 3 which refer to the three orthogonal axes of a right handed cartesian co-ordinate system. Derivatives with respect to position are written in the form $E_{i,j} = \frac{\partial E_i}{\partial x_j}$ and with respect to time are written as $E_i = \frac{\partial E_i}{\partial t}$. $\xi_{i,j,k}$ is the alternating unit tensor so Curl can be written as

$$\xi_{i,j,k} E_{k,j} = \nabla \times E$$

The first step in treating the propagation of electromagnetic radiation is to determine what the characteristic modes of propagation are for our particular situation. Section 2 of this chapter does this and then
proceeds to express the propagation parameters for these modes in terms of the polarizability of the plasma. Once the propagation parameters are known then the amount of light which will be transmitted through the experimental plasma can be calculated. Since we have calculated these parameters in terms of the polarizability we then proceed to calculate the polarizability in terms of atomic parameters. In section 2-3 we calculate the contribution made to the polarizability by a classical harmonic oscillator, and then assume that in the quantum mechanical case the polarizability is multiplied by the corresponding oscillator strength $f$. The polarizability of the classical oscillator was calculated for an isolated oscillator at rest; the next two sections generalise to the experimental situation. Section 2.4 is devoted to the Doppler shift due to thermal motion of the atoms; section 2.5 takes into account the Zeeman splitting of the absorption lines produced by the applied magnetic field. The polarizability due to the free charges in the plasma is calculated in section 2.6 and compared to the effects of the bound charges in order to show that the free charges can be ignored. At this point we have expressed the propagation parameters for the characteristic modes of propagation in terms of the atomic parameters by means of the polarizability. Section 2.7
uses Einstein's theory of radiation to derive an expression for the relative transition probabilities in terms of atomic parameters which were measured experimentally.

The expressions for the propagation parameters are then used in section 2.8 to calculate how the light transmitted by the plasma depends on the applied magnetic field and the atomic parameters.

During the derivation of these results it is formally assumed that the frequency dependence of the absorption coefficient can be approximated by a convolution of a Lorentzian and Doppler line shape, i.e., a Voigt profile. Section 2.9 discusses the form of these two line shapes and then continues to give some of the phenomena which would produce these shapes in the experimental situation, together with formulae to calculate the amount of each type of broadening.

The chapter is concluded with a few comments in section 2.10.

2.2 WAVES IN A PLASMA - Why use circularly polarized light?

Consider a monochromatic electromagnetic wave of frequency $\nu$ propagating in the $\hat{x}_3$ direction along a magnetic field embedded in a plasma (Fig. 2-1).

The radiation will interact with the atoms to a significant extent only when the frequency of the radiation
is close to the frequency of an allowed atomic transition. Using standard notation we will define the component of the angular momentum in the direction of the field $\mathbf{B}$ to be equal to $m\hbar$, where $\hbar$ is Planck's constant divided by $2\pi$. Photons have two possible directions of spin corresponding to $m=+1$ and $-1$. The photons with $m=+1$ are right hand circularly polarized and the others with $m=-1$ are left-handed.

The polarization is said to be right- or left-handed according to whether the electric vector at a fixed point rotates in a clockwise or counterclockwise direction for an observer looking toward the source. The only atomic transitions which can absorb the photons moving along the field lines are those with $\Delta m(\text{atom})$ equal to $+1$ or $-1$. Since angular momentum must be conserved, $\Delta m(\text{atom}) + \Delta m(\text{photon})=0$ and therefore the right-handed wave can only interact with transitions with $\Delta m=+1$ and the left-handed wave can only interact with transition with $\Delta m=-1$. Thus the right- and left-hand waves interact with different transitions with the result that they are not coupled by the atoms. The magnetic field affects the atoms in such a way that the frequencies of the transitions with $\Delta m=+1$ are changed in an opposite sense to those with $\Delta m=-1$. Therefore the right- and left-hand waves of the same
frequency will "see" atoms with different polarizabilities but the two waves remain uncoupled. It will be shown (Eq. 2-19) that the motion of charged particles can not couple the waves. The fact that these two polarizations are uncoupled will be used to allow us to treat independently the two polarization components of the incident wave.

2.2-1 PROPAGATION OF CIRCULARLY POLARIZED LIGHT

The electric vectors associated with the circular polarizations are written in complex notation as

\[ E_\pm = \text{Re}(E_1 \pm iE_2) \]  \hspace{1cm} (2-1)

where \( E_+ \) and \( E_- \) are respectively right- and left-handed waves. That is, \( E_+ \) rotates at a fixed point in a right-hand sense for an observer looking along the propagation vector toward the source and \( E_- \) rotates in a left-hand sense.

Assuming that the space charge density is zero and that the velocities are not relativistic then Maxwell's equations give

\[ E_{k;jj} = \mu_0 (j_k + \ddot{D}_k) \]  \hspace{1cm} (2-2)
where $E_k$ is the electric field intensity, $J_k$ is the current density, and $D_k$ is the electric displacement.

Assuming that all non-linear effects can be ignored, the propagation of the wave at frequency $\nu$ will induce responses of the medium at the same frequency, that is, the scattering is coherent. The responses of the medium at frequency $\nu$ can be found by taking the Fourier component at $\nu$. Thus the time dependence for $E, J, D, X, \ldots$ will be of the form $\exp(i2\pi \nu t)$.

Since the ions or atoms have a large mass compared to the electrons ($1:10^{-5}$) it is assumed that their velocities are unaffected by the imposed field and that as a result all the induced currents are due to the motion of the electrons. The plasma will be treated as a dielectric medium so that

$$D_k = \varepsilon_0 E_k + P_k$$  \hspace{1cm} (2-3)

$$J_k = 0$$

where $\varepsilon_0$ is the permittivity of free space and $P_k$ is the polarization or dipole moment per unit volume. Substituting this expression for $D_k$ into (2-2), we obtain

$$E_{k,\nu} = \varepsilon_0 \frac{\partial E_k}{\partial \nu} + \frac{\partial P_k}{\partial \nu}$$  \hspace{1cm} (2-4)
which, in terms of the characteristic modes defined in 2-1 can be written as

\[ E_{\pm,ij} = \varepsilon_0 \bar{E}_{\pm} + \bar{P}_{\pm} \]  

(2-5)

where \( P_{\pm} = \text{Re}(P_1 \pm iP_2) \)  

(2-6)

It should be noted that the term \( P_k \) in 2-4 is not proportional to \( E_k \) since the polarization in the \( x_1 \) direction depends on the field \( E_1 \) and \( E_2 \). However, the term \( P_{\pm} \) in 2-5 is proportional to \( E_{\pm} \) since \( E_+ \) and \( E_- \) are uncoupled. This will be shown in more detail when the microscopic motions are considered later. The result of this uncoupling is that when (2-5) is rewritten as

\[ E_{\pm,ij} = \mu_0 \varepsilon_{\pm} \bar{E}_{\pm} \]  

(2-7)

then the permittivity is a vector rather than a tensor and is given by

\[ \varepsilon_{\pm} = \varepsilon_0 + \frac{P_{\pm}}{E_{\pm}} = \varepsilon_0 (\varepsilon'_{\pm} + i\varepsilon''_{\pm}) \]  

(2-8)

which define \( \varepsilon'_\pm \) and \( \varepsilon''_\pm \).

Equating the real and imaginary parts of 2-8 gives the relation between permittivity and the microscopic variable \( \frac{P_{\pm}}{\varepsilon_0 E_{\pm}} \) namely
\[ \varepsilon'_\pm = 1 + \text{Re} \left( \frac{P_\pm}{\varepsilon_0 E_\pm} \right) \quad (2-9) \]

\[ \varepsilon''_\pm = \text{Im} \left( \frac{P_\pm}{\varepsilon_0 E_\pm} \right) \quad (2-10) \]

### 2.2.2 PERMITTIVITY AND PROPAGATION PARAMETERS OF
CHARACTERISTIC MODES

Since \( E_\pm \) are the characteristic modes they can be expressed in the form of a standard equation for a monochromatic attenuated wave, i.e.

\[ E_\pm = E_{e\pm} \exp \left( -i2\pi \left[ (n_\pm - iK_\pm) \frac{X_0}{c} - t + \mathcal{J}_\pm \right] \right) \quad (2-11) \]

\( n_\pm \) is the index of refraction, \( c \) the speed of light, \( K_\pm \) is related to the absorption coefficient and \( \mathcal{J}_\pm \) is the phase factor (\( n_\pm \) and \( K_\pm \) are real). Substituting these expressions for \( E_\pm \) into the wave equations (2-7) enables us to express the propagation parameters \( n \) and \( K \) in terms of the permittivity \( \varepsilon \)

\[ (n_\pm - K_\pm)^2 = \mu_0 \varepsilon_0 (\varepsilon'_\pm + i\varepsilon''_\pm) c^2 \quad (2-12) \]

Noting that \( \mu_0 \varepsilon_0 = \frac{1}{c^2} \) and equating the real and imaginary parts of 2-12 gives
\[ n_+^2 - K_+^2 = \varepsilon'_+ \]
\[ -2n_+K_+ = \varepsilon''_+ \]

The connection between the constants in the wave equation and the permittivity is now completed by solving this pair of equations for \( n_\pm \) and \( K_\pm \), i.e.

\[ n_\pm^2 = \frac{\varepsilon'_\pm}{2} + \frac{1}{2} \sqrt{\varepsilon'^2_{\pm} + \varepsilon''^2_{\pm}} \]  \( (2-13) \)

\[ K_\pm = -\frac{\varepsilon'_\pm}{2} + \frac{1}{2} \sqrt{\varepsilon'^2_{\pm} + \varepsilon''^2_{\pm}} \]  \( (2-14) \)

\( K_\pm \) and \( n_\pm \) can be related to \( \varepsilon_\pm \) by the direct application of \( 2-13, 14 \) but noting that for the experimental plasma \(|\varepsilon'_\pm - 1| \ll 10^{-5} \) and \(|\varepsilon''_\pm| \ll 10^{-5} \) so that \( \varepsilon' \approx 1 \) and \( \varepsilon' \gg |\varepsilon''| \) \( (2-13 \text{ and } 2-14) \) can be simplified to

\[ n_\pm = \sqrt{\varepsilon'_\pm} \]
\[ K_\pm = \frac{1}{2} |\varepsilon''_\pm| \]

The relation between \( \varepsilon_\pm \) and \( \frac{P_\pm}{\varepsilon_o E_\pm} \) is then obtained through \( 2-9 \text{ and } 2-10 \) i.e.

\[ n_\pm = 1 + \frac{1}{2} \text{Re} \left[ \frac{P_\pm}{\varepsilon_o E_\pm} \right] \]  \( (2-15) \)
\[ K_\pm = \frac{1}{2} \text{Im} \left[ \frac{P_\pm}{\varepsilon_o E_\pm} \right] \]  \( (2-16) \)
Thus we have an expression for the propagation parameters \( n \) and \( K \) in terms of the polarizability \( \frac{P_\pm}{\varepsilon_0 E_\pm} \). The problem now is to find the functional dependence of the polarizability on wavelength. This will be accomplished by deriving an expression for the polarizability in terms of properties of the atoms and the conditions in the discharge. First we will deal with an isolated harmonic oscillator.

2.3 POLARIZATION DUE TO ATOMIC PARAMETERS

The polarizability contributed by each spectral line is proportional to that which a classical oscillator would contribute and the proportionality constant 'f' is called the 'oscillator strength' (Ref. 3., Chp. 4-9).

First the problem of the classical oscillator will be treated. Consider an elastically bound electron of mass \( m \), charge \( q \), natural frequency \( \omega \), and half-life \( \tau \) in a magnetic field \( B \). The equation of motion for such a particle when perturbed by an electromagnetic field is

\[
\dot{x}_j + \frac{1}{\tau} x_j + \omega^2 x_j = \frac{q}{m} E_j + \xi_{ijk} B_k \dot{x}_j
\]  

(2-17)

where \( \xi_{ijk} \) is the alternating unit tensor.
The dipole moment of this single oscillator is \( qX_j \) so the polarization from it will be \( p_\pm \) given by

\[
p_\pm = q (x_\pm + i x_\mp) = qx_\pm
\] (2-18)

Note that \( p_\pm \) refers to one oscillator while \( P_\pm \) refers to total medium.

Writing 2-17, the equation of motion, in terms of \( x_\pm \) gives the following equation

\[
\ddot{x}_\pm + \frac{1}{j} \dot{x}_\pm + \omega_0^2 x_\pm = \frac{q}{m} (E_\pm + i B_3 \dot{x}_\pm)
\] (2-19)

We have made the assumption that the response of the medium to the electric field is linear in order to derive (2-9,10) i.e., the relation between permittivity and the macroscopic variable \( \frac{P_\pm}{\varepsilon_\circ E_\pm} \). This same assumption is imposed on the classical oscillator by taking the fourier component of equation (2-19) at frequency \( \nu \).

\[
\therefore i \dot{x}_\pm = \frac{\ddot{x}_\pm}{2 \pi \nu}
\] (2-20)

Thus we have assumed that all non-linear or second order effects can be neglected compared to the coherent effects. Using (2-20) the equation of motion at frequency \( \nu \) can be written as
where \( \omega_b \) is the electron cyclotron frequency \( qB/m \),

The solution of this pair of differential equations for \( \chi_{\pm} \) is

\[
\chi_{\pm} = \frac{q}{m} \frac{E_{\pm}}{\omega_0^2 - \omega(\omega \pm \omega_b) + i \omega/\tau}
\]  \hspace{1cm} (2-22)

Eliminating \( \chi_{\pm} \) from (2-17) by use of (2-22) enables us to express \( \frac{p_{\pm}}{\varepsilon_0 E_{\pm}} \) in terms of the atomic parameters, namely

\[
\frac{p_{\pm}}{\varepsilon_0 E_{\pm}} = \frac{q^2}{\varepsilon_0 m} \frac{1}{\omega_0^2 - \omega(\omega \pm \omega_b) + i \omega/\tau}
\]  \hspace{1cm} (2-23)

This then is the polarization due to a classical oscillator; to obtain the correct quantum mechanical expression this must be multiplied by the oscillator strength \( f \) to give

\[
\frac{p_{\pm}}{\varepsilon_0 E_{\pm}} = \frac{q^2f}{\varepsilon_0 m} \frac{1}{\omega_0^2 - \omega(\omega \pm \omega_b) + i \omega/\tau}
\]  \hspace{1cm} (2-24)

The total polarization due to the bound electron is found by summing over all possible spectrum lines, being careful to allow for the motion of the atoms relative to the source. Finally it should be noted that equation (2-24) is valid only for transitions with a normal Zeeman effect. The anomalous Zeeman effect is treated in section 2.5.
2.4 EFFECT OF THERMAL MOTION ON POLARIZABILITY

It must be stressed that the frequency $\omega$ used in (2-24) is the frequency of the radiation in the rest frame of the atom. Although the motion of the ions in response to the field can be ignored (due to the atomic mass) the thermal motion of the atoms must be considered. The oscillators of the previous section are assumed to be moving at the thermal speeds of the atoms in motion. We assume that the atoms are in thermal equilibrium so the velocity distribution is Maxwellian. Hence, the number of atoms with $\dot{x}_3$ between $v$ and $v+dv$ is $dN$ where

$$dN = \frac{N}{\sqrt{\pi}} \exp(-y^2) \, dy$$  \hspace{1cm} (2-25)

$$y = \sqrt{\beta} \quad ; \quad \beta = \frac{M}{2k_B T}$$  \hspace{1cm} (2-26)

where $N$ is the number density of atoms in the ground state of the transition being considered, $M$ is the mass per atom, $T$ is the temperature in °K, and $k_B$ is Boltzmann's constant.

Rewriting equation (2-24) in a slightly different form

$$\frac{P_x}{\varepsilon_0 E_z^2} = \frac{fq^2}{\varepsilon_0} \frac{1}{(\omega_0 - \omega)(\omega_0 + \omega) + \omega \omega_0 + i \omega/T}$$  \hspace{1cm} (2-27)
but under experimental conditions we have \[ |\frac{\omega - \omega_0}{\omega}| \leq 10^{-5}, \]

thus

\[ \frac{p_x}{\varepsilon_0 E_+} \approx \frac{f q^2}{\varepsilon_0 m \omega_0} \frac{1}{2(\omega_0 - \omega + i \frac{\omega_0}{2}) + \frac{i}{\tau}} \]  \hspace{1cm} (2-28)

which can be written in a dimensionless form

\[ \frac{p_x}{\varepsilon_0 E_+} = \frac{f q^2 \ln 2}{\varepsilon_0 m D \omega_0} \frac{1}{i a + W_\pm} \]  \hspace{1cm} (2-29)

where

\[ a = \frac{\sqrt{\ln 2}}{D} \frac{1}{\tau} \]  \hspace{1cm} (2-30)

\[ W_\pm = -2 \frac{\sqrt{\ln 2}}{D} (\omega - \omega_0 \pm \omega_0 / 2) \]  \hspace{1cm} (2-31)

and \( D \) is the full width at half maximum of the Doppler broadened spectral line and is given in units of angular frequency by

\[ D = \frac{2 \omega_0}{c} \sqrt{\frac{\ln 2}{\beta}} \]  \hspace{1cm} (2-32)

where \( \omega_0 \) is the frequency of the center of the spectral line; \( c \) is the speed of light; \( \omega \) is the frequency at which the polarizability is evaluated.

Typical values for the Doppler half-width for argon lines at room temperature are \( \sim 0.01 \text{A} = 0.02 \text{cm}^{-1} = 20 \text{mk} \).
Note that polarizability is defined in terms of the angular frequency $\omega$ which is seen in the frame at rest with respect to the atom. The frequency seen in the frame of reference of the source is given by $\omega'$. If the atom is moving with $\dot{x}_3 = V$ then the Doppler shifted frequency is

$$\omega = \omega' \left(1 - \frac{V}{c}\right) \quad (2-33)$$

substituting $2-33$ into $2-31$ we find

$$W_\pm = -2\left(\omega'[1 - \frac{V}{c}] - (\omega_o \pm \frac{\omega_b}{2}) \frac{\sqrt{\ln 2}}{D}\right)$$

$$W_\pm = W'_\pm + \frac{2\sqrt{\omega_0 \ln 2}}{D}$$

$$W_\pm = W'_\pm + y$$

where

$$W'_\pm = -2\frac{\sqrt{\ln 2}}{D} \left(\omega' - \omega_o \pm \frac{\omega_b}{2}\right)$$

$$y = V\sqrt{\beta}$$

Thus the polarizability per atom with $\dot{x}_3 = V$ is

$$\frac{p_\pm}{\varepsilon_o E_\pm} = \frac{f q^2 \sqrt{\ln 2}}{\varepsilon_o m D \omega} \frac{1}{y + W'_\pm + ia} \quad (2-34)$$

The total polarizability $\frac{P_\pm}{\varepsilon_o E_\pm}$ is found by integrating over all atoms

$$\frac{P_\pm}{\varepsilon_o E_\pm} = \int \frac{P_\pm}{\varepsilon_o E_\pm} dN$$
or substituting (2-25) and (2-29) into this integral we obtain

\[
\frac{P_\pm}{\varepsilon_0 E_\pm} = k_0 \frac{\lambda}{\pi^2} \frac{1}{\pi} \int_{-\infty}^{\infty} \exp(-y^2) \frac{dy}{y + W_\pm + ia}
\]

(2-35)

where

\[
k_0 = \frac{N f q^2 \sqrt{\ln 2}}{\varepsilon_0 mc D} \]

(2-36)

Equation (2-35) can be written in the form

\[
\frac{P_\pm}{\varepsilon_0 E_\pm} = \frac{i \lambda}{2 \pi} k_0 \Phi(Z_\pm)
\]

(2-37)

where \( \Phi(Z) \) is the complex error function

\[
\Phi(Z) = \exp(-Z^2) \left[ 1 + \frac{2i}{\sqrt{\pi}} \int_0^Z \exp(+t^2) dt \right]
\]

(2-38)

by using Appendix I (Theorem 1.3) which shows

\[
\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\exp(-y^2)}{y + W_\pm + ia} \, dy = i \Phi(Z_\pm)
\]

where

\[ Z_\pm = W_\pm + ia \]
Substituting the expression for polarizability from (2-37) into (2-15, 16) gives

\[ n_\pm = 1 - \frac{\lambda k_0 \text{Im}(\Phi(Z_\pm))}{4\pi} \]  

(2-39)

\[ K_\pm = \frac{\lambda k_0 \text{Re}(\Phi(Z_\pm))}{4\pi} \]  

(2-40)

Thus we have an expression for the propagation parameters in terms of the complex error function \( \Phi \) and \( k_0 \), which is given in equation (2-36) as

\[ k_0 = \frac{Nfq^2\sqrt{\ln 2} \sqrt{\pi}}{\varepsilon_0 mc D} \]

If \( k_0 \) and \( a \) are set then the propagation parameters are completely determined which give us the expected transmission for our experimental conditions. Alternately we can treat \( k_0 \) and \( a \) as parameters which are varied in order to give the best possible agreement between experiment and theory.

Before this is done there remains one further correction. When the effect of the magnetic field was calculated the fact that a bound electron will have its angular momentum quantized was not taken into account correctly (classical Zeeman effect). This quantization will be treated in the next section. The section following that will show that the polarizability due to free charges can be ignored near optical resonances because of the much larger polarizability
from the bound electrons. Hence equations (2-39), (2-40) give the final form for $n_\pm$ and $K_\pm$.

2.5 ANOMALOUS ZEEMAN EFFECT

In the treatment of the semi-classical oscillator the fact that the angular momentum of an electron in a bound state will be quantized was ignored. If we impose the condition that the angular momentum in the direction of the magnetic field is quantized then the shift due to the magnetic field is not $\frac{\omega_B}{2}$ as indicated in equation (2-31) but rather, as in reference (3),

$$\frac{\omega_B}{2} = 2\pi \mu_\beta (M_2 g_2 - M_i g_i) B$$

where $M_i$ is the magnetic quantum number in state $i$
$g_i$ is the Lande g-factor for the state $i$
$\mu_\beta$ is the Bohr Magneton

$$= 1.3996 \times 10^6 \text{ sec}^{-1} \text{ gauss}^{-1} = \frac{q \hbar}{4\pi m}$$

$q$ is the charge of the electron with mass $m$ and $\hbar$ is Planck's constant. (see Fig. 2-5).

Thus the expressions derived for the polarizability are valid if equation (2-31) is replaced by

$$W_\alpha'' = \frac{2 \sqrt{\ln 2}}{D} \left[ \omega' - \omega_\alpha \pm \mu_\beta 2\pi B(M_2 g_2 - M_i g_i) \right]$$
rewriting equations (2-39), (2-40) we have

\[ n_{\pm} = 1 - \frac{\lambda k_e}{4\pi} \text{Im} \left[ \Phi(W''_{\pm} + ia) \right] \]
\[ K_{\pm} = \frac{\lambda k_e}{4\pi} \text{Re} \left[ \Phi(W''_{\pm} + ia) \right] \]

Thus we have the values for \( n_{\pm} \) and \( K_{\pm} \) if the Zeeman splitting consists of one component with each polarization. Most Zeeman lines consist of many components and each one must be taken into account. This can be done by summing over all the Zeeman components.

\[ n_{\pm} - 1 = \sum_{i=1}^{U} (n_{\pm i} - 1) \beta_i \]
\[ K_{\pm} = \sum_{i=1}^{U} K_{\pm i} \beta_i \]

(2-41)

where \( U \) is the total number of components with each polarization, \( \beta_i \) is the relative intensity of the \( i \)-th component, \( K_{\pm i} \), \( n_{\pm i} \) are the propagation parameters for the \( i \)-th component as defined in equation (2-11).

The relative intensity \( \beta_i \) is derived in reference (3) as

\[ \beta_i = C(J \pm M)(J \pm M+1) \text{ if } J \rightarrow J ; M \rightarrow M \pm 1 \]
\[ \beta_i = C(J \pm M)(J \pm M+2) \text{ if } J \rightarrow J+1 ; M \rightarrow M \pm 1 \]
\[ \beta_i = C(J \pm M)(J \pm M-1) \text{ if } J \rightarrow J-1 ; M \rightarrow M \pm 1 \]

where \( J \) is the total angular momentum and \( M \) is the component
in the direction of the field. The constant $C$ is used to normalise the intensities so that

$$\sum_{i=1}^{u} \beta_i = 1$$

2.6 FREE ELECTRONS

Secondly the free electrons must be considered. Since they are free the resonant frequency will be zero. Writing (2-23) for this case we get

$$\frac{p_\pm}{\varepsilon_0 E_\pm} = \frac{q^2}{\xi m} \frac{1}{\omega (\omega \pm \omega_b - i/\tau)} = \frac{\omega_p^2}{N_e \omega} \frac{1}{\omega \pm \omega_b - i/\tau}$$

(2-42)

where $\omega_p = \sqrt{N q^2 / m \varepsilon_0}$ is the plasma frequency, $N_e$ is the electron density and $f = 1$ (2-24) with the assumption that the radiation is near resonance and that $\omega > \omega_b$ we have

$$\frac{p_\pm}{\varepsilon_0 E_\pm} \approx \frac{-\omega_p^2}{\omega^2 + 1/\tau^2} \left(1 + i \frac{1}{\tau \omega}\right)$$

For the experimental plasma with $N_e \sim 10^{12}$, $\lambda \sim 6000\,\text{Å}$ and assuming that $1/\omega \tau \ll 1$ (collision frequency small compared to $\omega$) then
This is compared with the experimental values for the bound electrons of $10^{-5}$. Thus the effect of the free electrons on the polarizability in the vicinity of a spectral absorption line can be ignored. Therefore by investigating the values of the index of refraction and absorption coefficient close to spectral lines we will be able to determine the atomic parameters ($f$ and $a$) since they dominate the polarizability. One of the atomic parameters which can be determined is the oscillator strength ($f$) of section 2.3 but the parameter which is of interest is the transition probability. The relation between the relative oscillator strengths and relative transition probabilities for two transitions with a common ground state is developed next.

2.7 THE EINSTEIN THEORY OF RADIATION

Consider the atomic system shown in the term diagram below.

Following the treatment of Mitchell and Zemansky (13, p. 93) we define the probability coefficients:
\[ B_{1 \rightarrow 2} I_\nu = \text{the probability per second that the atom in state 1 exposed to isotropic radiation of frequency between } \nu \text{ and } \nu + d\nu \text{ and intensity } I_\nu, \text{ will absorb a quantum } h\nu \text{ and pass to the state 2.} \]

\[ A_{2 \rightarrow 1} = \text{the probability per second that the atom in state 2 will spontaneously emit, in a random direction, a quantum } h\nu \text{ and pass to the state 1.} \]

\[ B_{2 \rightarrow 1} I_\nu = \text{probability per second that the atom will undergo the transition from 2 to 1 when it is exposed to isotropic radiation of frequency between } \nu \text{ and } \nu + d\nu \text{ intensity } I_\nu, \text{ emitting thereby a quantum in the same direction as the stimulating quantum.} \]

\[ k_\nu = \text{the absorption coefficient and is the probability per unit length of absorber that a photon of energy } h\nu \text{ to } h(\nu + d\nu) \text{ will be absorbed.} \]

\[ \tau = \text{the mean life of an isolated atom in state 2.} \]

\[ g_1, g_2 = \text{the statistical weights of state 1 and 2 respectively.} \]

From the definition of \( \tau \) and \( A_{2 \rightarrow 1} \), it is clear that

\[ A_{2 \rightarrow 1} = \frac{1}{\tau} \quad (2-44) \]
By considering the thermodynamic equilibrium between the radiation and the atoms, Einstein showed that

\[
\frac{A_{2\rightarrow 1}}{B_{1\rightarrow 2}} = \frac{2h\nu^3 g_1}{c^2 g_2}
\]  \hspace{1cm} (2-45)

\[
\frac{B_{2\rightarrow 1}}{B_{1\rightarrow 2}} = \frac{g_1}{g_2}
\]  \hspace{1cm} (2-46)

where \( c \) is the velocity of light.

Consider a parallel beam of light of frequency between \( \nu \) and \( \nu + d\nu \) and intensity \( I_\nu \) travelling along \( x_1 \) through a layer of atoms bounded by \( x_1 \) and \( x_1 + dx_1 \). If \( dN_{1\nu} \) is the number density of atoms in state 1 capable of absorbing \( I_\nu \) and \( dN_{2\nu} \) is the number density of atoms in state 2 capable of stimulated emission in the frequency range \( \nu \) to \( \nu + d\nu \) and neglecting the effect of spontaneous emission since it occurs in a random direction we can write the decrease in energy of the beam as

\[
d(I_\nu d\nu) = I_\nu \frac{h\nu}{4\pi} dx (B_{2\rightarrow 1} dN_{2\nu} - B_{1\rightarrow 2} dN_{1\nu})
\]  \hspace{1cm} (2.47)

where \( I_\nu /4\pi \) is the intensity of the equivalent isotropic radiation for which \( B_{2\rightarrow 1} \) and \( B_{1\rightarrow 2} \) are defined. This can be rewritten as
\[
\frac{1}{I_v} \frac{dI_v}{dx} = -\frac{h \nu}{4\pi} (B_{1\rightarrow 2} dN_{1\nu} - B_{2\rightarrow 1} dN_{1\nu})
\] (2-48)

The left-hand side of this equation is the probability per unit length of absorption which is just \(-k_i d\nu\).

Integration over the entire line, neglecting the small variation in \(\nu\) gives

\[
\int k_i d\nu = \frac{h \nu}{4\pi} (B_{1\rightarrow 2} N_1 - B_{2\rightarrow 1} N_2)
\] (2.49)

which by use of equation (2-44), (-45) and (-46)

\[
\int k_i d\nu = \lambda g \frac{g_i g_j}{g_i g_j} \left[ 1 - \frac{g_i N_1}{g_j N_2} \right]
\] (2-50)

Let us now proceed to derive an expression for the integral in (2-50) in terms of the measured parameter \(k_i\).

The intensity of radiation due to \(E_\pm\) is proportional to \((\text{Re}E_\pm)^2\) which from thmI-2, Appendix I, is equal to

\[
\frac{1}{2} \text{Re}(E_\pm^* E_\pm)
\]

Substituting the expression for \(E_\pm\) from (2-11) gives

\[
I_v \propto E_{i\pm}^2 \exp(-4\pi K_{i\pm} x_3 / \lambda)
\]

thus the intensity as a function of \(x_3\) is given by

\[
I = I_o \exp(-4\pi K_i x_3 / \lambda)
\] (2-51)
where \( B \) is set equal to zero so \( K_+ = K_- = K \),

therefore the absorption coefficient is given by

\[
k_\nu = \frac{4\pi K\nu}{\lambda}
\]  

(2-52)

or using (2-40) gives

\[
k_\nu = k_0 \text{Re} \Phi(Z_\nu)
\]  

(2-53)

since \( Z_+ = Z_- = Z_\nu \) for \( B = 0 \)

if \( B \neq 0 \) then \( k_\nu \rightarrow k_\pm \) with the corresponding changes,

\( Z_\nu \rightarrow Z_\pm \)

therefore

\[
\int_0^\infty k_\pm d\nu = k_0 \int_0^\infty \text{Re} \Phi(Z_\pm) d\nu
\]  

(2-54)

or using Theorem II-4, Appendix II

\[
\int_0^\infty k_\pm d\nu = \frac{k_0 D}{2} \sqrt{\frac{\pi}{\ln 2}}
\]  

(2-55)

comparing (2-50) and (2-55) shows that

\[
k_{o_2} = \frac{2}{D_2} \frac{\lambda^2 g_2 N_i}{8\pi g_1 \pi} \left[ 1 - \frac{g_{g_2 N_i}}{g_2 N_i} \right] \sqrt{\frac{\ln 2}{\pi}}
\]  

(2-56)
where \( k_{o2} \) refers to \( k_0 \) for transition from state 1 to state 2.

\[
A'_{2\rightarrow1} \equiv A_{2\rightarrow1} \left[ 1 - g_i N_i \right] \frac{g_j N_j}{g_i N_i}
\]  

(2-57)

(see section 2.8.5)

If there are two transitions with a common lower state then if we call the second upper state 3 and take the ratio of the two absorption coefficients (Eq. 2-56)

\[
\frac{k_{o2}}{k_{o3}} = \frac{\lambda_3^2 D_3 A'_{2\rightarrow1} g_2}{\lambda_3^3 D_3 A'_{3\rightarrow1} g_3} = \frac{\lambda_2^3 g_2 A'_{2\rightarrow1}}{\lambda_3^3 g_3 A'_{3\rightarrow1}}
\]  

(2-58)

Thus if we are able to determine the values \( k_{o2} \) and \( k_{o3} \) for two atomic transitions with a common ground state then if \( g_1 N_2 \ll g_2 N_1 \) we know the relative transition probabilities for these lines (see section 3.2.7).

2.8 EXPERIMENTAL METHODS

The experimental plasma is a glow discharge as shown in Fig. 2-1 and 3-1. The light enters the plasma at \( X_3 = 0 \) linearly polarized in the \( X_1 \) direction and after traversing a distance \( L \) in the \( X_3 \) direction passes through an analyzing nicol prism which has been rotated through an angle \( \Omega \) (see Fig. 2-1) with respect to the \( X_1 \) direction. For this case (2-11) can be written as
\[ E_\pm = E_\nu \exp \left( i2\pi \nu (t - \left[ \frac{n_- - iK_+}{c} \right] \ell) \right) \quad (2-59) \]

since for this case \( E_\pm = E_\nu \)

\( \mathcal{J} = 0 \)

From theorem I-1 in Appendix I the amplitude of the electric vector passing through the analysing nicol at the angle \( \Omega \) is given in terms of \( E_+ \) and \( E_- \) as

\[ E_\Omega = \frac{1}{2} (E_+ \exp[-i\Omega] + E_- \exp[i\Omega]) \quad (2-60) \]

The observed intensity is proportional to the time average of the amplitude squared of the electric vector, i.e.

\[ I_\Omega = \mathcal{E}_\Omega^2 \]

or using theorem I-2 in Appendix I this can be written as

\[ I_\Omega = \frac{1}{2} \text{Re}(E_\Omega E_\Omega^*) \quad (2-61) \]

where \( E_\Omega^* \) is the complex conjugate of \( E_\Omega \).

Eliminating \( E_\Omega \) using equation (2-60) gives as the observed intensity \( I_\Omega \)

\[ I_\Omega = \frac{E_\nu^2}{8} \text{Re} \left( E_+ E_+^* + E_- E_-^* + E_+ E_-^* \exp(-i2\Omega) \right. \]

\[ \left. + E_- E_+^* \exp(i2\Omega) \right) \quad (2-62) \]
which can be simplified by using equation (2-59) for $E_\pm$ since
\[
\cos(\tau) = \frac{\exp(i\tau) + \exp(-i\tau)}{2}
\]
to give
\[
I_\pm = \frac{E_\pm^2}{8} \left( \exp(-k_+l) + \exp(-k_-l) + 2\cos(\Theta - 2\phi) \exp(-l[k_+ + k_-]) \right) \tag{2-63}
\]
where
\[
k_\pm = \frac{4\pi K_\pm}{\lambda} \tag{2-64}
\]
\[
\Theta = \frac{4\pi}{\lambda} l (n_+ - n_-) \tag{2-65}
\]
Now $k_\pm$ and $\Theta$ can be expressed in terms of $k_0$ and
by using equations (2-39) and (2-40), i.e.,
\[
k_\pm l = k_0 l \text{Re}[\phi(Z_\pm)] \tag{2-66}
\]
\[
\Theta = k_0 l \frac{1}{2} \left( \text{Im}[\phi(Z_-)] - \text{Im}[\phi(Z_+)] \right) \tag{2-67}
\]
The experiment consists of observing the transmitted intensity (Eq. 2-63) as a function of magnetic field
and then comparing the observed variations with those predicted by theory. The values of $k_0$ (Eq. 2-36) and $'a'$
(Eq. 2-30) were used as parameters which were varied in
order to obtain a 'best fit.' Through equation (2-58) $k_0$
gives us the relative transition probabilities and $'a'$ gives
gives us the line shape (in particular the Lorentz line width provided the Doppler width is known).

It is now evident that if the polarizing nicol N\(_1\) (see Fig. 3-1) were omitted and the source were not polarized for any other reason then the total intensity observed I would be given by

\[
I = I_{\Omega} + I_{\Omega+2\pi} = \frac{E_{\nu}^2}{2} \left[ \exp(-k_+l) + \exp(-k_-l) \right]
\]

which is independent of the phase change (Fig. 2-2).

By using the polarizers we are able to obtain information on both the absorption coefficient and the index of refraction since we now measure the amplitudes and the relative phase of the two transmitted waves E\(_\pm\).

Let us now look at this equation for the transmitted intensity more closely. The maximum intensity is obtained when \( \Omega = \frac{\Theta}{2} \) namely

\[
I_{\text{max}} = \frac{E_{\nu}^2}{8} \left[ \exp(-k_+l) + \exp(-k_-l) + 2\exp\left(\frac{k_+ + k_-}{2}l\right) \right]
\]

and the minimum when \( \Omega = \frac{\Theta \pm \pi}{2} \) and is given by the expression

\[
I_{\text{min}} = \frac{E_{\nu}^2}{8} \left( \exp(-k_+l) + \exp(-k_-l) - 2\exp\left(\frac{k_+ + k_-}{2}l\right) \right)
\]
Thus the plane polarized light from the source is transmitted as elliptically polarized light with the major axis rotated by $\Theta/2$, which depends only on the difference in the values of $n_+$ and $n_-$. The relative amplitude of the major and minor axis is determined by the values of $k_+$ and $k_-$. If $k_+ = k_-$ then $I(\text{max}) = \frac{E_0^2}{2}$, $I(\text{min}) = 0$ and the transmitted radiation is plane polarized. The rotation of the axis of the polarization of the transmitted light is referred to as Faraday rotation (Fig. 2-2).

It must be stressed that the expression given in (2-63) is valid only in the neighbourhood of a spectral line and it assumes that the source is monochromatic at frequency $\nu$. In our experimental set-up two identical plasmas were used for the source and absorber. The only difference between the two is that the absorber was in a magnetic field while the source was kept at zero field. Since the source then only emits light in the neighbourhood of spectral lines (2-63) is valid and the only requirement of the monochromator is that it allow the entire line from the source to pass but exclude the neighbouring lines. The total intensity observed is then given by integrating the intensity from (2-63) over the entire line

$$T'_0 = \int_{\nu} E_0^2 \left[ \exp(-k_+ \nu) + \exp(-k_- \nu) + 2 \cos(\Theta - 2\Omega) \exp(-l \left( \frac{k_+ + k_-}{2} \right)) \right] d\nu$$

(2-71)
The term $E^2_\nu$ is the source intensity at frequency $\nu$ and must be determined experimentally. The form of this term will be dealt with in section 2.7.1.

Fig. 2-3 is a graph obtained from the experimental apparatus. This shows the transmission observed when the two nicols are in position ($T'_o$ in equation (2-71)) and also that observed if the polarizing nicol is not in position, i.e.,

$$T'_o = \int \frac{E^2_\nu}{4} \left[ e^{-k+l} + e^{-k-l} \right] d\nu \quad (2-72)$$

The relative amounts of information in these two curves will be discussed in greater detail in section 5.3.

In the experiment the percentage transmission is actually measured so we must divide the transmission with the absorber on by that with it off. Thus we get with the polarizing nicol $N_1$ in

$$T_o = \int E^2_\nu \left[ e^{-k+l} + e^{-k-l} + 2\cos(\Theta - 2\Omega) e^{-(k_+ + k_2)l} \right] d\nu \quad (2-73)$$

and with the nicol out

$$T_T = \int E^2_\nu \left( e^{-k+l} + e^{-k-l} \right) d\nu \quad (2-74)$$
2.8-1 SOURCE PROFILE

In order to evaluate the relative transmission, \( T_0 \) and \( T_r \) (Eq. (2-73) (-74)), the source profile \( E_\nu \) must be known. Since the source and absorber are identical plasmas the transmission parameters \( k_\pm \) will be identical for both. The source is kept free of any magnetic field so we have \( k_+ = k_- = k_\nu \). This section derives an expression for the source intensity in terms of \( k_\nu \). Since the source and absorber have the same propagation parameters there is only one set of constants which can be varied in order to obtain a theoretical fit to the experimental results. Let us consider a cylinder of plasma as shown below.

The probability that a photon of frequency \( \nu \) is emitted from a slab of thickness \( dx \) in a direction parallel to the axis of the source (\( P_\nu \)) is equal to \( bN_\nu A \) where \( b \) is a constant, \( N_\nu \) is the number density of atoms capable of emitting a photon of frequency and \( A \) is Einstein's A coefficient. The probability that a photon of frequency \( \nu \) will be absorbed per unit length is \( k_\nu \) which is equal to \( b'N'_\nu B \) where \( N'_\nu \) is the number density of atoms capable of absorbing the photon and \( B \) is Einstein's B coefficient.
The probability that a photon of frequency $\nu$ will be emitted in $dx$ and will leave the plasma in the $X_3$ direction is

$$P_{\nu} e^{-k_{\nu}x} \quad (2-82)$$

Thus the total intensity leaving the plasma is

$$\int_0^L P_{\nu} e^{-k_{\nu}x} \, dx \quad (2-83)$$

which is equal to

$$I_{\nu} = I_c \frac{P_{\nu}(1 - e^{-k_{\nu}L})}{k_{\nu}} = \frac{E_{\nu}^2}{2} \quad (2-84)$$

where $I_c$ is a constant.

A & B are constants that are determined by the interaction of the atom with the radiation field and have the same values whether there is thermal equilibrium or not. We cannot claim thermal equilibrium for the glow discharge but we can assume that the number densities of the various states have reached a constant value so that $P_{\nu}/k_{\nu}$ is a ratio which is independent of frequency.

Since this experiment measures relative transmission only, an arbitrary constant in the source strength will cancel so it need not be considered. In the calculation $I_c$ was chosen arbitrarily so that
2.8.2 NON-CONSTANT MAGNETIC FIELD

In the derivation of equation (2-31) it was implicitly assumed that \( n \) and \( K \) were constant over the length of the absorber. If they are not then the equation for the transmitted intensity given in equation (2-63) can be corrected if the following substitutions are made for (2-68) and (2-67).

\[
\frac{E_{\nu}^2}{2} = 1 - e^{-k_{\nu}L} \tag{2-85}
\]

\[
k_{\pm}L = k_o \int_0^L \text{Re}[\Phi(Z_{\pm})] \, dx \tag{2-87}
\]

\[
\Theta = k_o \int_0^L \left( \text{Im}[\Phi(Z_+)] - \text{Im}[\Phi(Z_-)] \right) \, dx \tag{2-88}
\]

Recall from equation (2-38) that

\[
\Phi(Z_\pm) = \exp(-Z_\pm^2) \left[ 1 + i \frac{2}{\sqrt{\pi}} \int_0^{Z_\pm} \exp(t^2) \, dt \right]
\]

where \( Z_\pm = W_\pm + i\alpha \). If we are considering the transmission of the radiation in a magnetic field \( H(x) \) at the normalized frequency \( \nu \) and the Zeeman splitting \( \mathcal{Y}H(x) \), \( \mathcal{Y} = \) constant) then we can write for the absorber

\[
W_\pm = \nu \pm \mathcal{Y}H(x) \tag{2-89}
\]
Thus the value of $\phi(Z_{\pm})$ as a function of field (and hence of $X$) is known. In order to carry out the integrations in (2-87) and (2-88) the variation of the magnetic field along the optical path was determined experimentally and the corrected values given by (2-87) and (2-88) were used to calculate the transmitted intensity. The maximum change caused by this procedure was approximately equal to the standard deviation of the experimental points on the $T$ vs. $H$ curves. It is estimated that the maximum error after the correction is carried out, is one-tenth of experimental error from other sources and as such this particular error has been ignored in the treatment of errors.

2.9 LINE SHAPE CONSIDERATIONS

Now that the formal derivations have been completed it is profitable to return to some key equations and discuss the significance of the distribution of spectral intensity which was assumed.

Using equations (2-16) and (2-64) we see that

$$k_{\pm}l \propto \text{Im} \left[ \frac{P_{\pm}}{\mathcal{E}_0 \mathcal{E}_{\pm}} \right]$$

Taking the case when the magnetic field is equal to zero we can write

$$k_+ = k_- = k_{\nu}, \quad n_+ = n_- = n_{\nu}$$
Let us first consider the spectral distribution of intensity of the light emitted from an optically thin source, i.e. take the limit of equation (2-85) as the source thickness goes to zero

$$\lim_{L\to 0} \frac{E^2_v}{2} = \lim_{L\to 0} (1 - e^{k_v L}) = k_v L$$

For the spectral distribution of an absorption line we take the limit of equation (2-63) to give

$$\lim_{L\to 0} \frac{I_v}{E^2_v} \propto 1 - k_v L + \ldots$$

Thus when we refer to line shapes in this work we are dealing with the spectral distribution of the absorption coefficient $k_v$.

Let us first assume that the thermal motion of the atoms can be ignored in order that we can use equation (2-24) for the polarizability to give us

$$k_v l \propto \text{Im} \frac{P_v}{\varepsilon_0 E_v} \text{Im} \left[ \frac{1}{(\omega_0^2 - \omega^2 + i \omega / \tau)} \right]$$

(2-91)

Noting that over any given absorption line $\nu \sim \nu_0$ we may write

$$k_v \propto \frac{\Delta \omega / 2}{(\omega_0 - \omega)^2 + (\Delta \omega / 2)^2}$$

(2-92)

where $\Delta \omega = 1 / \tau$ is referred to as the Lorentzian half-width (see Fig. 2-6).
In the final expression for polarizability (2-37) the Lorentzian component is used to define the Voigt 'a' factor, i.e.,

\[ a = \frac{\sqrt{\ln 2}}{D} \cdot \frac{1}{\tau} = \frac{\sqrt{\ln 2}}{D} \cdot \frac{\Delta \omega}{\tau} \]  

(2-93)

Thus if the effect of thermal motion can be ignored then \( D \ll \Delta \omega \) and \( a \to \infty \). This then leads to a Lorentzian line profile described by equation (2-92).

If the opposite case is taken then the thermal motion dominates the line shape and equation (2-66) can be simplified for this case to

\[ Z = W + ia \to W \]

\[ W^* = W^* = W \to W = \frac{2\sqrt{\ln 2}}{D} (\omega - \omega_0) \]

so that

\[ \lim_{a \to 0} \text{Re} \phi(Z) = e^{-W^2} \]  

(2-94)

(From equation (2-38))

Thus we have for this case

\[ k \propto e^{-W^2} \]  

(2-95)
This dependence of the spectral distribution is called Doppler Broadening (see Fig. 2-6). Note that the value of $k_v$ will drop to half its maximum value when $|\omega - \omega_0| = \frac{D}{2}$. Thus D is called the Doppler half-width or more precisely the full width at half maximum.

Thus the derivation of the polarizability and the observed transmission has assumed that the spectral distribution of the absorption coefficient can be approximated by a convoluted Doppler and Lorentzian line shape. The temperature of the gas that will determine the value of the Doppler half-width and their ratio (Voigt 'a') is left as a parameter to be determined when the experimental and theoretical results are compared. The phenomena which give rise to the Lorentzian component will be discussed in section 2.9-2.

2.9-1 DOPPLER LINE SHAPE

The Doppler line shape is produced by an ensemble of emitters moving with a Maxwellian velocity distribution (section 2.4). This treatment should be valid since there is negligible heating of the atoms above room temperature and the temperature is constant throughout the experiment. The mean free path of argon at 320°K at 2 mm Hg pressures approximately $10^{-3}$ cm so collisional narrowing can be
ignored (1). A temperature of 300°K was used (section 3.2-7) to calculate the Doppler half-width and the Voigt 'a' was determined experimentally.

From equations (2-32) and (2-26) the Doppler half-width \( D \) is given by

\[
D = \frac{2\nu_0}{c} \sqrt{2\ln 2 \frac{k_BT}{M}} \text{ sec}^{-1}
\]

(2-96)

For experimental conditions \( \lambda = 8115 \text{ Å} \) and

\[
D \sim 24 \times 10^{-3} \text{ cm}^{-1}
\]

2.9-2 LORENTZIAN LINE SHAPE - NATURAL

A stochastic process which affects the atom by shortening the duration for which it radiates will have the effect of giving the line a Lorentzian component (1). If an atom emitted an infinitely long sinusoidal wave the spectral line produced would be a Dirac \( \delta \)-function. The most obvious limit to the duration of the wave train is the fact that an atom will emit radiation in a finite length of time. If the half-life (\( \tau \)) of the excited state is the finite length of this train then a Lorentzian line profile of half-width \( \Delta \nu_N \) will result where

\[
\Delta \nu_N = \frac{1}{2 \pi \tau}
\]
From equation (2-44) we see that the lifetime of state 2 \( (\mathcal{T}_2) \) is related to the transition probability from state 2 to state 1 \( (A_{2 \rightarrow 1}) \) by

\[
A_{2 \rightarrow 1} = \frac{1}{\mathcal{T}_2}
\]

if the transition from state 2 to state 1 is the only transition possible. In general the lifetime of an atom in state \( n \) is equal to

\[
\frac{1}{\mathcal{T}_n} = \sum_{n'} A_{n \rightarrow n'}
\]

where \( n' \) is summed over all possible states. If the upper and lower states both have finite lifetimes then the natural width of the line \( \Delta \nu_{n_2 \rightarrow 1} \) is given by

\[
\Delta \nu_{n_2 \rightarrow 1} = \frac{1}{2\pi \mathcal{T}_2} + \frac{1}{2\pi \mathcal{T}_1} = \frac{1}{2\pi} \left( \sum_{n'} A_{2 \rightarrow n'} + \sum_{n''} A_{1 \rightarrow n''} \right)
\]

For the transitions treated in the present experiment

\[
\Delta \nu_n \sim 2 \times 10^{-4} \text{ cm}^{-1}
\]

**COLLISIONAL BROADENING - RESONANCE BROADENING**

If a particle collides with the radiating atom this will also modify its lifetime, producing a Lorentzian line profile. If the radiating and perturbing atoms are of the same kind then the expected broadening is given by Griem (2) and
\[ \Delta \nu_R = 3 \left( \frac{g_r}{g_p} \right)^{1/2} \frac{e^2 f_{pr}}{8 \pi^2 \varepsilon_0 m c} \lambda N \]

where \( N \) is the number density of the perturbers, \( e \) is the electronic charge, \( m \) is the electronic mass, \( c \) is the speed of light, \( g_p \) and \( g_r \) are the statistical weights of the state of the perturber and radiator, respectively, \( \lambda \) is the wavelength of the transition between the perturber and radiator states and \( f_{pr} \) is the absorption oscillator strength for that same transition.

The glow discharge, which we use in this work, is a slightly ionized, low density plasma and as such the majority of the argon atoms are in their neutral ground state. Thus the majority of the perturbers which will collide with the radiating atom will be these unexcited neutral atoms. The transitions which are treated in this work are between energy levels which are not connected to the ground state by an allowed transition which implies that \( f_{pr} \) is equal to zero for collisions with neutrals. Therefore resonance broadening can be ignored.

**COLLISIONAL BROADENING - VAN DER WAALS**

Following Griem (2) let us calculate the impact parameter \( \rho_{min} \), which gives rise to practically complete
destruction of the correlation between the states of the system before and after a Van Der Waals collision.

\[ \rho_{\text{min}} \approx \left( \frac{9 \hbar R^2}{16 m^3 \nu E^2} \right)^{\nu_5} \]

\[ \bar{R}^2 = \frac{3}{a^2} \left[ \frac{1}{2 E_\infty} - E_\infty \right] \left[ \frac{5 Z^2 E_\infty}{E_\infty - E_\infty} + 1 - 3 l_\infty (l_\infty + 1) \right] \]

where \( N \) is the number density of the perturbing atoms, \( m \) is the mass of an electron, \( e \) is the electronic charge, \( \nu \) is the relative velocity of the radiator and perturber, \( E_p \) is excitation energy of the first excited resonance state of the perturber which is connected to the ground state by an allowed optical transition and \( \bar{R}^2 \) is the square of the co-ordinate vector of the radiating electron with respect to its nucleus in atomic units, \( a_0 \) is the first Bohr radius, \( E_H \) and \( E_\infty \) are the ionization potentials of hydrogen and the radiating atom respectively, \( E_\infty \) is the excitation potential of the upper state of the line, \( l_\infty \) is its orbital quantum and \( Z \) is the charge of the radiator with the radiating electron removed. Under experimental conditions,

\[ N \sim 7 \times 10^{22} \text{ m}^{-3} ; E_H = 13.53 \text{ ev} ; E_p = 1.8 \times 10^{-16} \text{ joule} = 11.3 \text{ ev} \]

\[ \bar{\nu} \sim 400 \text{ m sec}^{-1} ; E_\infty - E_\infty \sim 3 \text{ ev} \]

\[ \rho_{\text{min}} \sim 6 \times 10^{-10} \text{ m} \quad \bar{R}^2 \sim 40 \]
The collision frequency corresponding to \( \rho_{\text{min}} \) is given by

\[ \rho_{\text{min}}^2 \sqrt{N} \sim 10^{-3} \text{ cm}^{-1} \]

which gives rise to a Lorentzian component equal to

\[ \Delta \nu_{\nu} = \rho_{\text{min}}^2 \sqrt{N} \approx N \sqrt{\frac{9 \hbar R}{16m E_p^2}} \]

(2-99)

Along with the broadening of the line there is a shift given by

\[ \text{SHIFT} = -\frac{\Delta \nu_{\nu}}{3} \]

(2-100)

In order to obtain the above estimates of the line width and shift due to Van der Waals' interactions the impact approximation was used to treat adiabatic collisions. The impact approximation assumes that

\[ N \ll (\pi \rho_{\text{min}}^3)^{-1} \]

Under experimental conditions \( N \sim 10^{22} \text{ m}^{-3} \), \( \rho_{\text{min}} \sim 10^{-10} \text{ m} \) thus the perturber density is low compared to the interaction volume.

In order for the collision to be adiabatic the perturber must pass through the interaction volume without appreciable change of energy, i.e.,
\[
\frac{\hbar v}{\rho_{\text{min}}} \ll E_p
\]

or experimentally \(10^{-23} \ll 10^{-16}\)

In order that the collision parameter \(\rho_{\text{min}}\) can have any physical significance it must be larger than the sum of the "radius" of the broadened state \(a_0\sqrt{R_x^2}\) and the perturbing ground state. Experimentally we have

\[
a_0\sqrt{R_x^2} \sim 4 \times 10^{-10} \text{ m} < \rho_{\text{min}}
\]

The expression for \(R_x^2\) as given in equation (2-84) is derived for a one-electron system and is a good approximation whenever the Coulomb approximation is valid. (An excited state in argon is approximated very well by a one-electron system.) Note that the value \(\rho_{\text{min}}\) is the value of the impact parameter which will destroy the correlation between the states of the system before and after the collision and that collisions with impact parameters greater than this have been ignored. Thus we can expect that the broadening predicted by this treatment will give a minimum value.

If we consider two upper states \(\alpha\) and \(\alpha'\) then the ratio of the broadening of the two states is given by equation (2-85)
\[ \frac{\Delta \nu_{\nu_\alpha}}{\Delta \nu_{\nu_{\alpha'}}} = \left( \frac{R_{\nu_\alpha}^2}{R_{\nu_{\alpha'}}^2} \right)^{2/5} \]  

(2-101)

Experimentally = .65 for \( I_{\infty} = 2 \) and \( I_{\infty'} = 1 \)

Even if the magnitude of the broadening is in error the ratio of the broadening of two different levels is expected to be given by the above formula since the validity of the ratio depends solely on the facts that argon can be treated as a one-electron system and the broadening is of the Van der Waals type.

2.10 COMMENTS

The assumptions made to obtain the expression for the observed transmission (equations (2-73) and (2-74)) must be stressed. The atoms must have a Maxwellian velocity distribution and the spectral line shape must be approximated by a Voigt profile; that is, it must be a convolution of a Doppler and a Lorentzian line shape. The Lorentzian component relative to the Doppler component of the line shape gives the value of the Voigt 'a' which is used as a parameter to produce the 'best fit' between experiment and theory.
It must also be stressed that, since the two discharges are made as similar as possible, the same propagation parameters can be used to calculate the source function as well as the transmitted intensity. Once the propagation parameters have been determined for a given source then this known source can be used to investigate other absorbers to determine the propagation parameters and atomic parameters in them. Since the relative transition probabilities are intrinsic atomic constants these should not change with different discharges but the Lorentzian component will change since it is determined by the conditions surrounding the atom.
3.0 INTRODUCTION

Chapter 2 has developed the theory necessary to analyse the results of the experiment. The first section in this chapter (3.1) outlines the apparatus used while following sections give the method of taking data and the acceptance criterion. This chapter then discusses some of the modifications which had to be introduced in order to take account of the nonideal nature of the experiment.

3.1 EXPERIMENTAL APPARATUS

A block diagram of the experimental arrangement is shown in Fig. 3.1. The source and absorber are identical glow discharge tubes (Fig. 3.2). The light from the source is linearly polarized by Nicol prism N₁ and then traverses the absorption tube before passing through the analysing Nicol N₂ to the monochromator. The light intensity passed by the monochromator is detected by a photomultiplier and a digital phase sensitive detector (DPSD).
3.1-1 TEST GAS

The choice of argon for the test gas was made for a number of reasons. Since it is a noble gas it is easily handled and the production of the glow discharges used was relatively simple. It is a gas which is used widely in the development stages of the plasma devices used in this laboratory and indeed transition probabilities have been measured here by Jacobson (7). In addition there have been previous reports of measured transition probabilities (Ref. 7-11).

3.1-2 SOURCE AND ABSORBER

The source and absorber (Fig. 3-2) were made as dimensionally similar as possible. In order to reduce the impurities to an absolute minimum both tubes were connected to the same high vacuum system at the same time. A pressure of 1x10^-8 torr. was maintained for six hours while the tubes were held at 400° C before being simultaneously filled to 2 torr. with research grade argon (Airco). When the tubes were first used it was found that the light output had a higher than expected variance but after being left on for a number of hours the light output became more constant. To avoid the varying source strength the tubes were left on for at least 48 hours before the
experiment was carried out. It was found that the transmission curves varied with the current in the absorber. In order to eliminate errors from this, the current was kept at $1\pm 0.02$ ma. The transmission did not depend on the current in the source as long as it was kept between 1 and 10 ma. No evidence of striations was observed in the source or the absorber even when viewed side on with a photomultiplier.

The discharge tubes were operated from a 1500 volt regulated power supply (regulation and ripple $\leq 5\%$). The current was controlled by a series pentode regulator (regulation and ripple $\leq 1\%$).

3.1-3 POLARIZERS

Nicol prisms were used as polarizer and analyser because they give excellent polarization and have good transmission qualities. The analyser $N_2$ (Fig. 3-1) was permanently aligned with the monochromator so that the light output was a maximum. This was done because the monochromator polarizes the light itself. The polarizer $N_1$ was positioned accurately in the crossed position with respect to $N_2$ by observing the minimum of the transmitted intensity. The mounting of $N_1$ had machined slots
by which it was rotated accurately to the parallel position. This prism itself could be removed and replaced without changing the position of the mount.

As an additional check that the polarizers were parallel the magnetic field was run through zero to verify that the experimental curves were symmetric about zero field. From equation (2-63) it can be seen that the transmitted intensity will be symmetric if and only if $\mathcal{O} = 0$, that is, if the two polarizers are parallel.

3.1-4 MONOCHROMATOR

A Spex grating monochromator ($f/10$, $10 \text{ Å}^\circ/\text{mm in first order}$) was used to isolate individual spectral lines. The slit widths varied from line to line but they were typically about $400 \mu$. This maximum was imposed because the portion of the entrance plane, which is illuminated by the source, is limited by the aperture stops (Fig. 3-1).

3.1-5 DIVERGENCE AND APERTURES

The size of the apertures used were determined experimentally by starting with large diameters and reducing the size of the apertures until the observed relative transmission was independent of this size (Fig. 3-3). This
procedure was carried out to eliminate any radial variation in transmission due to varying number density or any other cause. The apertures used were 2mm in diameter which gives a divergence of about .7°.

3.1-6 MAGNET

The magnet used was constructed in the laboratory according to the specifications shown in Figure 3-4. The maximum field produced was about 2400 gauss from a coil current of approximately 35 amps. The current was measured using a one milliohm shunt across which a digital voltmeter was connected. It was experimentally verified that the shunt was temperature compensated to 50°C, which is well above the operating temperatures (≈30°C). The current was obtained from two power supplies in parallel as shown in Fig. 3-5. The ripple on the current passing through the magnet was thus kept below .1%. The voltmeter was calibrated in terms of the magnetic field in two ways. An absolute calibration was established by the use of an N.M.R. probe which gave the field to within .05% and for relative field values below the range of the N.M.R. probe (1200 gauss) a Hall probe gaussmeter (Bell "240" Incremental Gaussmeter #41228, probe #4274) was used. In the range between 1200 and 2400 gauss where both probes
could be used they gave readings which were consistent to within .5% and the field was linear with respect to current to within the same accuracy, from 0 to 2400 gauss. The variation of field with length along the axis and transverse to the axis was measured using the Hall probe. The percentage variation along the axis was found to be symmetric about the center of the coil and independent of the coil current. The variation across the radius of the tube was found to be less than .5% over the length occupied by the absorber (see Fig. 3-13).

3.1-7 DETECTION SYSTEM

In order to measure a desired signal in the presence of unwanted background noise the detection system must be able to distinguish between these two kinds of input. In the experimental set-up the desired signal is the light which the source emits and which is transmitted through the absorber (Fig. 3-1). The unwanted noise comes from two separate places. The detection system will contribute some noise (photomultiplier dark current) but the major contribution of noise is from light which the absorber itself emits. Since the source and absorber were identical plasmas they will emit identical spectra except for the small Zeeman splitting in the absorber. The monochromator was used to isolate individual spectral lines but due to
their similarity, the spectra of the source and absorber could not be distinguished from each other.

In order to separate the source intensity from the absorber "noise" it was decided to impose a square wave modulation on the source and use a DPSD (Digital Phase Sensitive Detector) to obtain a signal. The DPSD will be treated in three separate subsections. The theory is presented first followed by a section which gives a statistical treatment of the errors of a DPSD. The treatment concludes with operational details on the system used in this experiment.

DPSD - THEORY

The signal which is observed from the photomultiplier (PM) (Fig. 3-10) will be made up from three sources; signal (S), absorber noise (N) and photomultiplier dark current (DC). If the source is constant in time the output is shown schematically in trace A of Fig. 3-6. If the source is now modulated with a square wave as in trace B then the PM output will be as shown in trace C. A gating pulse is now generated (trace D) such that a series of equal length pulses are produced one for each 'on' and 'off' section of the reference (trace B). This gating pulse (trace D) is used to gate the PM output to produce
a series of bursts of signal separated by an off period (trace E). Note that this series of pulses are all of exactly the same length and that each one is completely within one 'off' or 'on' section of the reference. The last step is to direct the bursts of pulses to two separate outputs. One takes all the bursts produced while the reference is 'on' and the other takes all those produced while the reference is 'off.' The 'on' output will have S+N+DC while the 'off' output will have only N+DC. Thus the signals(S) is just the difference between the two outputs. Let us now deal with some statistics.

**DPSD - STATISTICS**

This section will deal with various sources of error introduced by the DPSD used in this experiment. First let us deal with gating times.

Figure 3-6 shows that the gate #1 (trace D) passes a series of pulses of equal length, one for each 'on' and 'off' period of the reference pulse (trace B). Since the same physical element gates both the 'on' and the 'off' channels the mean length of the pulses will be the same but the variance or jitter of the gating times need not be correlated. If \( t_{on} \) is the gating 'on' time and \( t_{off} \) is the following 'off' time then
\[ t_{\text{on}} = t \pm \sigma_t \]
\[ t_{\text{off}} = t \pm \sigma_t \]

where \( \sigma_t^2 \) is the variance of the gating times (Poisson, distribution assumed). The percentage difference in the gating time for these successive pulses is thus given by

\[ \frac{t_{\text{on}} - t_{\text{off}}}{t} = \sqrt{\frac{\sigma_t^2 + \sigma_t^2}{t}} \]

If now \( N_G \) sets of successive 'on' and 'off' times are considered the percentage error introduced because of jitter in the gating times is

\[ \frac{N_G (t_{\text{on}} - t_{\text{off}})}{N_G t} = \sqrt{\frac{2}{N_G}} \frac{\sigma_t}{t} \]

The minimum time of counting for any experimental points is 100 sec. Therefore at 1 KHZ, \( N_G \sim 10^5 \).

Thus if the mean of the gating time is constant then the error introduced due to the jitter in the length will be lower than the variance of one pulse by a factor of the order of \( 10^{-3} \). The experimental test to prove that the gating times are equal is to replace the PM with a signal generator and observe the resulting values on both counters. The signal generator produces a constant frequency \( f \) so
the counts recorded in the two channels will be given by

\[ f N_G t_{on} \quad \text{and} \quad f N_G t_{off} \]

and the difference in the two channels is

\[ \frac{f N_G t_{on} - f N_G t_{off}}{f N_G t_{on}} \quad \text{(experimentally} \quad 10^{-8}) \]

The reason for this extreme accuracy is that the mean time \( \bar{t} \) need only be constant over successive 'on' and 'off' times which experimentally requires only approximately 1 msec. Thus in order to produce equal 'on' and 'off' times the gate #1 (Fig. 3-6) need only be constant over this same time scale. The experimental procedure requires that the ratio of the transmitted signal with the absorber on to that with the absorber off be taken. The counts recorded in the two channels are given by \( C_1 \) and \( C_2 \) respectively, where \( T' \) is the total gating time

\[ C_1 = (N+DC+S)T' \pm \sqrt{(N+DC+S)T'} \]
\[ C_2 = (N+DC)T' \pm \sqrt{(N+DC)T'} \]

thus the transmitted signal is \( S \)

\[ C_1 - C_2 = S T' \pm \sqrt{2(N+DC+S)T'} \]

If the absorber is now turned off the transmitted signal is \( S' \) and \( N' \) is the noise count
\[
C_1' - C_2' = S'T' = \sqrt{2(N+DC) + S'} \frac{1}{T'}
\]

thus the ratio of the transmitted signals T is given by

\[
T = \frac{C_1 - C_2}{C_1' - C_2'} = \frac{S}{S'} \pm \Delta T
\]

where

\[
\frac{\Delta T}{T} = \left[ \frac{(2(N+DC) + S)^2}{S^2} + \frac{(2(N' + DC) + S')^2}{S'^2} \right]^{\frac{1}{2}}
\]

The percentage error in the transmission was maintained at approximately 1% throughout the experiment by adjusting the total gating time (100 sec \(\ll T' \ll 1000 \text{ sec}\)).

**DPSD - Operation**

A block diagram of the system used is given in Fig. 3-7. The square wave modulation is imposed on the source by the chopping wheel (Fig. 3-8) and the reference is taken off the wheel as shown in that figure. The wheel is driven by a synchronous motor in order to maintain a constant chopping speed of 990 Hz. This frequency was chosen to minimize the effect of the 60 cycle noise in the room. Let us now go through the block diagram (Fig. 3-7)
explaining the different components and giving the critical specifications.

In order to integrate the two outputs for a long time (up to 30 min.) it was decided to use a digital system. In this system the individual photon pulses were amplified to give pulses which could be counted in standard counting circuits. The photon pulses were amplified at the PM. to give an output pulse as shown in Fig. 3-9. The PM. used (EMI 9558B) had a pulse width of 42ns. so the pulse circuitry used to treat the pulses had to be made such that it was capable of handling pulses at rates up to 20 MHz. All the circuitry used was designed to handle pulses at this rate in order that the system would be free from pile up effects (i.e., two pulses arrive so close together that the counters only see them as one pulse).

A tunnel diode current discriminator (Fig. 3-10) was used to produce a standard shape for all photon pulses. The discrimination level was set by experimentally reducing it and noting the signal to noise produced for a constant input. The discrimination level was set to give the maximum signal to noise. The optimum level was found to be independent of signal strength.
Gate #1 (Fig. 3-7) is the gate which determines the length of time that the 'on' and 'off' channels are open. Gate #2 is designed to send the bursts gated by #1 to the two outputs. The timing for gate #2 is not critical provided that this gate allows the entire burst to go to the correct output. The reason for this particular arrangement is so the same physical element will alternately gate both the on and the off times. Provided that the gating time of #1 changes slowly compared to the reference frequency (1kc.) then the sum of the 'on time' will be equal to the sum of the 'off time.' This was checked experimentally by introducing a small leak to the PM light seal and varying the intensity of the light to give counting rates over the range found during the experiment. It was found that the two channels gave readings equal to each other to within the expected variance. This was checked periodically throughout the experiment and in all cases the results were consistent with the hypothesis of equal gating times.

In order to check that there was no pileup effects a small signal was fed into the system and counted at minimum noise level. A light leak was then introduced to increase the background counting rate to well above any rates encountered during the experiment and the signal
was measured again. The resulting signal was found to be independent of the size of the light leak. Since the signal counts were the same none had been lost due to pileup and therefore we can ignore the effects of pileup for this experiment.

3.2 MEASUREMENT PROCEDURE

All the elements of the optical path shown on Fig. 3-1 except the monochromator and chopping wheel were mounted on a heavy optical bench. A He-Ne CW laser was positioned as shown to define the optical axis. The aperture stops and the monochromator were then placed in position with respect to the laser beam. It was found that the most reproducible results were obtained when the source and the absorber were positioned so that the optical path was down the center of the tube rather than positioning them for maximum light intensity. The maximum light from the source is obtained when the 'hot area' (Fig. 3-2) is on the optical path but since this spot moves with time the source is not as constant as when that region is off the axis. The absorber tends to transmit light down the glass walls and unless the tube is aligned to keep the walls off the axis unreliable results will result.
The source was turned on and the absorber was left off in order to adjust the phase of the reference with respect to the modulation of the source. The phase was adjusted so that the increase in counting rate was in one counter only (Fig. 3-8). That is, all the counts due to the source intensity were routed to one counter while the counting rate of the other counter remained unchanged whether the source was on or off.

Once the phase was adjusted the bias of the two channels was checked by turning the absorber on and placing a black cloth between the absorber and the chopping wheel (see Fig. 3-1) to insure that there was no signal passing through the wheel. The counters then read the same, for any counting period, within the expected variance between two Poisson processes with the same mean. It was found that the two channels were the same to within $1:10^8$ or better.

In order to minimise the effect of the magnetic field on the electronics, source, photomultiplier and discriminator they all were covered with $\mu$-metal. With the source on and the absorber off the magnetic field was increased to the maximum and the counting rate remained constant throughout.
The actual measurement was carried out by first removing the polarizing Nicol \( N_1 \) (Fig. 3-1) and measuring the signal with the source on and the absorber off, for zero field. The absorber was then turned on and the signal was remeasured. The ratio of these two signals was then the desired relative transmission \( T_r \). This process was then repeated for a series of equally space magnetic fields. \( N_1 \) was then replaced and the procedure was repeated to give \( T_0 \). The only runs that were considered as reliable were the ones during which the magnetic field could be increased to a maximum and back to zero in increments and the transmissions measured were completely reproducible.

It was found that the reliable runs done on different days, even with new optical alignments between runs, agreed within experimental error. The actual data used was the average of all the reliable runs that were made on a given line.

3.3 OPERATING CONDITIONS

Typical dark current counts encountered in this experiment were of the order of 500 Hz. while the absorber noise varied from line to line with a typical value that varied from 1 kHz. for the weak lines to as much as 10 kHz. for the stronger lines. The source strength was usually the same order as the absorber strengths. The times for integration were varied from point to point to give a
constant standard deviation of the transmission of about 1% for all data points. The sum of the standard deviations for the individual lines are given in the table with the results.

The DPSD was checked over a range of 50 Hz. to 10MHz. and over integration times of 1 sec. to 120 min. Pile-up of pulses limited the upper frequency limit and PM. dark current limited the lower end. The signal measured increased linearly with time for as long as the source remained constant and the reason the limit of two hours was imposed was that the longest time used in the experiment was 30 min. There does not appear to be any upper limit on the integration time due to the DPSD.

3.4 REPRODUCIBILITY

A minimum of two independent runs taken on different days with different optical alignments were averaged for any given line. It was found that if the integration times were increased the reproducibility of the results would improve until the error per point was about 1% or less and if the integration time were increased further the reproducibility did not improve beyond this level. The integration time was adjusted to give a maximum accuracy (\(\sim 1\%\)) for each individual line.
The accuracy attained by increasing the integration time was limited by variations in source strength. These variations had two effects on the reading taken. First the variations with periods much less than the measuring period did not affect the mean but did increase the variance of the individual readings. Secondly there were variations in the mean of the source strength which occurred at random with a period of the order of one hour or more. The mean would shift by approximately 1 to 5% in a step function and then maintain the new mean. Due to these jumps in level it was not possible to extend the integrating time per reading longer than about 5 min. It must be stressed that when the transmission was measured before and after the 'jump' the result was constant. That is, the percentage transmission was constant with respect to a change in the source strength.

3.5 'BEST FIT'

The experimentally measured transmission are not exact values but rather are estimates of the true value with an experimental variance \( \sigma_e^2 \). The magnitude of the variance is determined from the reproducibility of the data. A second variance \( \sigma_c^2 \) is the mean square deviation between the experimentally measured mean and the
theoretically calculated value, i.e.

\[ \sigma^2_c = \frac{1}{J} \sum_{i=1}^{J} (T_{\sigma_i} - T_{c_i})^2 \]

where \( T_{\sigma_i} \) is the experimentally measured mean of the transmission at a magnetic field designated by \( i \)

\( T_{c_i} \) is the calculated transmission for the same field

\( J \) is the total number of data points taken

If the theoretical curve is the true transmission curve then

\[ T_{c_i} = \langle T_{\sigma_i} \rangle \]

where \( \langle T_{\sigma_i} \rangle \) means expectation value, which means that \( \sigma^2_c \) is an estimate of the true experimental variance.

Thus if theory agrees with experiment the values of \( \sigma^2_c \) and \( \sigma^2_E \) are both estimates of the same variance. To test the equality of variances we use Snedecor's \( \bar{F} \) distribution with the degrees of freedom for both \( \sigma^2_c \) and \( \sigma^2_E \) chosen equal to \( J \). \( \bar{F} \) is defined by

\[ \bar{F} = \frac{\sigma^2_c}{\sigma^2_E} \]

or rewriting \( \sigma^2_c \) we have
\[ J = \frac{F}{\sigma^2} \]

where

\[ F = \sum_{i=1}^{J} (T_{e_i} - T_{e_i})^2 \]

Thus for any given curve the values of the parameters (\( k_0 l \), \( a \) etc.) are adjusted to give a minimum value to \( F \) (least squares) which gives a minimum value to \( F \). If \( F < F_5 \) (\( F_5 \) is the value of \( F \) which one would expect to exceed by chance alone 5% of the time) then it is assumed that the difference between experiment and theory is due to experimental variances and a 'fit' is assumed.

3.6 UNIQUENESS OF FIT

It was found that the effects of the various parameters were nearly independent. As an example the effects of changing \( k_0 l \) and \( a \) are shown in Figs. 3-11 and 3-12. Thus it is seen that increasing \( k_0 l \) moves the maxima and minima of the curves \( T_o \) and \( T_T \) to higher field with very little change in the depth of the minimum while \( a \) determines the depth of the minima. During the fitting procedure it was found that the effects
of all the parameters could be separated and a unique set obtained within the errors quoted for them.

In order to remove any bias of the analyst in the choice of the parameters the fitting procedure was completed before the results from the literature were considered.

3.7 ERROR LIMITS

The error limits for a parameter are determined by changing the set of parameters and noting the effect on $F$. The error limits for any given parameter are set at the amount the parameter must be changed to give a value of $F$ equal to twice $F_{\text{min}}$. These limits must be considered maximum possible errors (see section 3.2-4). Why this is the case can be seen from the graph showing the effect of changing $a$. Note that the major effect is to change the depth of the minima so points near the minima have the strongest effects on $F$. The criterion for a 'fit' is the value of $F$ which weights all points equally. Thus the difference near the minima must become much larger than experimental errors for these points in order to increase $F$ to twice $F_{\text{min}}$. 
The precision to which 'a' and 'k₀₁' can be measured is a function of the magnitude of these parameters. For example, if this experiment were repeated with a lower pressure in the absorber than the value of 'a' would decrease. Theoretical calculations indicate that for these smaller values of 'a' the accuracy obtained would increase. The theory also predicts that the uncertainty in the value of 'a' is a minimum for large values of k₀₁.

In general the most accurate values of all the parameters used to fit the theoretical curves to the experimental results are obtained for strongly absorbing lines (k₀₁ ≫ 10). The reason for this can be seen from Fig. 2-3, 4 which shows six lines with different k₀₁s. The larger the value of k₀₁ the more absorption at higher fields and the more maxima and minima observed in T₀. The portion of these curves which changes most rapidly with changing parameters is in the region of the extremes, thus increases in k₀₁ increase the sensitivity of the method.
leaf 73 omitted in page numbering
3.8 STIMULATED EMISSION

The effect of stimulated emission on this experiment can be seen in equation (2-57). Stimulated emission from an upper state 2 to a lower state 1 can be ignored compared to the absorption from the lower state 1.

As discussed in section 2.7 stimulated emission involves the de-excitation of an atom from an upper state 2 to a lower state 1 in response to a passing photon while the transition probability which this experiment is attempting to measure involves the absorption of the passing photon by an atom in state 1 which raises the energy of this atom to state 2. If the population of the upper state 2 is much less then the population of the lower state then effects of the stimulated emission can be ignored. This can be seen from equation (2-57) since then

\[ N_2 \ll N_1 \]

and

\[ \frac{A'_{2-1}}{A_{2-1}} = 1 - \frac{g_1 N_2}{g_2 N_1} \approx 1 \]

From the measured values of \( k \) (using equation (2-36)) and the transition probabilities from Wiese (5) the population density of the lower state \( N_1 \) is \( \sim 10^2 \text{ cm}^{-3} \). From the observed intensity, photomultiplier efficiency, experimental geometry and again the transition probabilities of
Wiese the population density of the upper state $N_2$ is estimated as $\sim 10^6 \text{ cm}^{-3}$. Thus $1 - \frac{A'_{2-1}}{A_{2-1}} \approx 10^{-4}$.

3.9 MAGNETIC FIELD EFFECTS ON THE DISCHARGE

In order to obtain a consistent fit between the experimental and theoretical results if was necessary to make $k_0 l$ a function of $B$ (magnetic field). It was found that the dependence of $k_0 l$ on magnetic field was of the form $k_0 l = k_0' l (1 + \alpha_1 |B| + \alpha_2 B^2 + \alpha_3 |B|^3)$

$$\alpha_1 \approx 1.5 \times 10^{-4}, \ B (\text{Gauss}), \ \alpha_2 = 0 = \alpha_3$$

A similar dependence was imposed on $\alpha$, but the 'best' value for $\alpha$ is zero. The same dependence of $k_0 l$ was found for all lines with a common lower state. A possible explanation is that the number density in the ground state ($^3P_2$) is a function of the magnetic field. The value of $\alpha_1$ was determined experimentally using the results of the strongest line ($\lambda=8115 \text{Å}$) The same value was then used for all the rest of the lines with the same lower level to obtain a consistent set of parameters.

3.10 ADDITIONAL PARAMETERS

The temperature of the atoms in the tube is assumed to be the same as the inside of the discharge tube. In
order to estimate the temperature the total power dissipated in the discharge was calculated from the voltage drop across the tube and the current passing through. Assuming that this power was conducted through the walls of the tube, we know the temperature differential across the glass tube. From this differential (\(\sim 20^\circ K\)) and knowing the temperature of the outside of the tube (\(\sim 290^\circ K\)) we are able to estimate the maximum possible temperature of the inside walls (\(\sim 310^\circ K\)). This estimate is an upper limit since we have assumed that all the power is dissipated through the walls and have ignored the power dissipated by the electrodes. The actual value of the temperature was also treated as a variable in fitting the theory to the measured transmission curves. Varying the temperature did improve the value of \(F_{\text{min}}\) but for variations of temperature between room temperature and the maximum estimate as given above there were insignificant effects on the best values for \(a\) and the relative values of \(k_e^l\). Thus the temperature assumed (300° K), as long as it is within reasonable limits, will have little effect on the atomic parameters measured.

In deriving the source strength (Eq. (2-85) we have assumed that there is a uniform plasma in the source tube (Fig. 3-2). The argon atoms which collide with end
windows will tend to be de-excited thus producing a thin layer of atoms which are all in the lower state. Since they are in the ground state they will not be able to emit any radiation but they will still be capable of absorbing radiation. The emitted radiation will then not be as given in equation (2-85) but rather

\[
\frac{E_{\nu}^2}{2} = \frac{E_{\nu}^2}{2} e^{-k_{\nu}'l'} = (1 - e^{-k_{\nu}L}) e^{-k_{\nu}'l'}
\]

where \( l' \) is the thickness of this nonradiating layer of gas. It is assumed that the thermal motion and number density of this thin layer is the same as in the plasma which is emitting \( E_{\nu} \), so that the absorption coefficient will be the same, i.e., \( k_{\nu}' = k_{\nu} \). The parameter \( l' \) was varied in order to produce agreement between theory and experiment and in all cases the 'best' value was zero. Therefore the layer of nonemitting gas can be ignored.
CHAPTER 4

RESULTS

4.1 RESULTS - RELATIVE TRANSITION PROBABILITIES

The results obtained for the relative transition probabilities are given in the table below. They are compared with those of Wiese (5) because his values were obtained from a survey of the literature. Wiese claims the uncertainty in his absolute values does not exceed 20% and the uncertainty in his relative values does not exceed 10%.

Note that for all lines the relative transition probabilities agree with Wiese to within the accuracy of 10% which he claims. The value of Snedecor's $F$ is in all cases less than the value for a 95% confidence limit. Thus the observed variances between experiment and theory are within the limits predicted by observed experimental uncertainties.
<table>
<thead>
<tr>
<th>WAVELENGTH A°</th>
<th>k.</th>
<th>A(rel) A( rel)</th>
<th>A( rel)</th>
<th>No. of Data Points</th>
<th>σ_c⁺</th>
<th>σ_E</th>
<th>g⁺</th>
<th>g⁺⁺</th>
</tr>
</thead>
<tbody>
<tr>
<td>8115</td>
<td>29.0 ± 1%</td>
<td>1.00</td>
<td>1.00</td>
<td>66</td>
<td>0.006</td>
<td>0.006</td>
<td>0.6</td>
<td>1.5</td>
</tr>
<tr>
<td>8014</td>
<td>5.25 ± 5%</td>
<td>0.262 ± 2%</td>
<td>0.264</td>
<td>42</td>
<td>0.006</td>
<td>0.007</td>
<td>1.0</td>
<td>1.7</td>
</tr>
<tr>
<td>7635</td>
<td>12.52 ± 2%</td>
<td>0.725 ± 1%</td>
<td>0.750</td>
<td>62</td>
<td>0.002</td>
<td>0.002</td>
<td>0.5</td>
<td>1.5</td>
</tr>
<tr>
<td>7147</td>
<td>0.160 ± 10%</td>
<td>0.0188 ± 10%</td>
<td>0.0179</td>
<td>10</td>
<td>0.001</td>
<td>0.001</td>
<td>0.06</td>
<td>3.0</td>
</tr>
<tr>
<td>7067</td>
<td>1.52 ± 3%</td>
<td>0.111 ± 3%</td>
<td>0.108</td>
<td>32</td>
<td>0.001</td>
<td>0.003</td>
<td>0.13</td>
<td>1.8</td>
</tr>
<tr>
<td>6965</td>
<td>1.52 ± 5%</td>
<td>0.193 ± 3%</td>
<td>0.184</td>
<td>32</td>
<td>0.001</td>
<td>0.002</td>
<td>0.20</td>
<td>1.8</td>
</tr>
</tbody>
</table>


+See section 3.5.
4.2 RESULTS - LORENTZIAN LINE WIDTH

The results obtained from this experiment for the Lorentzian component of the line shape are presented in Table 2.

From the value of the Voigt 'a' (Eq. (2-30)) and the Doppler half-width D (Eq. (2-96)) the experimental Lorentzian half-width is calculated and entered in column three. The natural half-width is then calculated using equation (2-97) and Wiese's values (5) for the transition probabilities or lifetimes. Subtracting the natural width from the observed width we are left with the experimentally measured value of the Van der Waals broadening. Since the source and the absorber are at the same pressure and temperature there will be no shift of the source and absorption lines with respect to each other.

Column six gives the estimate of a lower limit of the Van der Waals broadening as given in equation (2-99). The experimental values are above this lower limit for all the lines measured.

If the broadening is proportional to the polarizability of the upper state (1), as is expected for Van der Waals broadening, then the ratio of the broadening for two levels with a common ground state is given by equation (2-101) (Table 3).
Table 2

LORENTZIAN HALF-WIDTHS

<table>
<thead>
<tr>
<th>WAVELENGTH</th>
<th>VOIGT (a) ((10^{-3}\text{ cm}^{-1}))</th>
<th>(\Delta \nu_L) ((10^{-3}\text{ cm}^{-1})) (\pm 20%)</th>
<th>(\Delta \nu_N^*) ((10^{-3}\text{ cm}^{-1}))</th>
<th>(\Delta \nu_L - \Delta \nu_N) ((10^{-3}\text{ cm}^{-1}))</th>
<th>(\Delta \nu_{\text{Theory}}^+) ((10^{-3}\text{ cm}^{-1}))</th>
<th>UPPER LEVEL</th>
</tr>
</thead>
<tbody>
<tr>
<td>8115</td>
<td>.072 ±.01</td>
<td>2.1 ±.3</td>
<td>.196 ±.3</td>
<td>1.9 ±.3</td>
<td>.66 ±.3</td>
<td>3(D)_3</td>
</tr>
<tr>
<td>8014</td>
<td>.065 ±.01</td>
<td>1.9 ±.3</td>
<td>.184 ±.3</td>
<td>1.8 ±.3</td>
<td>.67 ±.3</td>
<td>3(D)_2</td>
</tr>
<tr>
<td>7635</td>
<td>.072 ±.01</td>
<td>2.2 ±.3</td>
<td>.202 ±.3</td>
<td>2.0 ±.3</td>
<td>.70 ±.3</td>
<td>2(D)_2</td>
</tr>
<tr>
<td>7147</td>
<td>.08 ±.05</td>
<td>2.6 ±1.5</td>
<td>.191 ±1.5</td>
<td>2.5 ±1.5</td>
<td>1.0 ±1.5</td>
<td>1(p)_1</td>
</tr>
<tr>
<td>7067</td>
<td>.10 ±.02</td>
<td>3.3 ±.7</td>
<td>.204 ±.7</td>
<td>3.1 ±.7</td>
<td>1.0 ±.7</td>
<td>3(p)_2</td>
</tr>
<tr>
<td>6965</td>
<td>.10 ±.02</td>
<td>3.4 ±.7</td>
<td>.212 ±.7</td>
<td>3.2 ±.7</td>
<td>1.0 ±.7</td>
<td>3(p)_1</td>
</tr>
</tbody>
</table>

*Natural line width using Eq. 2-83 and Wiese's values of A and \(\mathcal{J}\)  

+Van der Waals Broadening (2), Eq. 2-85.
Table 3
RELATIVE LORENTZIAN HALF-WIDTHS

<table>
<thead>
<tr>
<th></th>
<th>AVERAGE LORENTZIAN</th>
<th>$\Delta \nu_{l=2}$</th>
<th>RATIO $\frac{\Delta \nu_{l=2}}{\Delta \nu_{l=3}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l=1$</td>
<td>$3.0 \pm 0.6$</td>
<td>$1.9 \pm 0.2$</td>
<td>$0.63 \pm 0.20$</td>
</tr>
<tr>
<td>$l=2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>THEORY</td>
<td>$1.0$</td>
<td>$0.68$</td>
<td>$0.68^*$</td>
</tr>
</tbody>
</table>

*Equation 2-86.*
Thus the observed Lorentzian components are above the lower limit calculated by Griem (2) and the ratio of the Lorentzian half-widths for lines with a common ground state agree with that ratio expected if the broadening is proportional to the polarizability of the upper state (1), that is, Van der Waals broadening.
5.1 WALL STABILIZED ARCS AND SHOCK TUBES

Both these methods involve emission experiments (9,10) and depend on local thermodynamic equilibrium (LTE) for their values to be valid. A high dispersion instrument must be used in order to find the integrated line intensity and the line shape must be determined accurately so the intensity in the wings can be calculated.

The present method does not depend on LTE and indeed the glow discharge is not in LTE. All that our method requires is that the population levels be in dynamic equilibrium. Since source and absorber are identical discharges they are both determined by the same parameters \( (k_o, a) \) and self-absorption is easily dealt with (section 2.8-2). If a series of pressures are used in the absorber once the source conditions have been determined then the limit in the value of the Lorentzian component of the line as pressure goes to zero will give the absolute value of the lifetimes (Eq. 2-97) without having
to make any absolute intensity measurements. The extrapolation to zero pressure is only valid if de-excitation by collisions with the walls is negligible compared to the natural lifetime. Since natural lifetimes are of the order of $10^{-8}$ sec. and the atoms have an average velocity of $4 \times 10^4$ cm. sec.$^{-1}$ wall collisions can be ignored provided the characteristic length in the tubes $\gg 4 \times 10^{-4}$ cm.

5.2 HOOK METHOD

The Roschdestwensky 'hook' method (11) uses the rate of change of the index of refraction as a function of wavelength ($\frac{dn}{d\lambda}$) in order to measure relative transition probabilities. The 'hooks' occur at wavelengths when the derivative is equal to zero and thus they are separated by a few line widths. In order to obtain a relationship between the separation of the hooks and the transition probability it is usually assumed that the hooks occur far enough from the line center that the line shape can be assumed to be completely Lorentzian. Thus this method requires the use of a high dispersion instrument to measure the separation between two hooks separated by a few line widths. This single measurement is then used to determine the transition probability. There is no indication from this method what the line shape is and an assumption about
the shape must be made to determine the transition probability. If the absorber is not of uniform optical thickness with respect to radius then the method will give unreliable results. The method used in the present work uses the entire transmission vs. magnetic field curve to determine $k_0$ and $'a'$. The independence of these parameters depends on the fact that each parameter affects different parts of the transmission curve in different ways. If only two points on the index of refraction curve are treated then $k_0$ and $'a'$ cannot be evaluated independently.

5.3 ZEEMAN SCANNING

Zeeman scanning (12) has been used to determine line shapes. Normally when Zeeman scanning, only one circular polarization is used but the measured transmission will still be of the form given in equation (2-68) if the source and absorber are not shifted with respect to each other. In order to compare Zeeman scanning to the present method all we need do is compare the relative amount of information, in the two measured curves $T_0$ and $T_T$ (equations (2-73), (2-74)) since the present method uses both curves to determine $k_0$ and $'a'$.

The transmission measured with the Nicols in position ($T_0$) is much more sensitive to changes of $k_0$ and
'a' then the normal Zeeman scanning \((T_T)\) (polarizing Nicol removed). This can be seen by calculating the two quantities

\[
F_o = \left| \frac{\sum \limits_{i=1}^{N} (T_{z_i} - T_{t_i})^2}{N(k_o - k_o)} \right|
\]

\[
F_T = \left| \frac{\sum \limits_{i=1}^{M} (T_{z_i} - T_{t_i})^2}{M(k_o - k_o)} \right|
\]

where \(T_{t_i}\) is the transmission measured with \(k_o = k_{o_1}\)

\(T_{z_i}\) is the transmission measured with \(k_o = k_{o_2}\)

\(F_o\) and \(F_T\) are obtained by summing over the curves, \(T_o\) and \(T_T\) respectively.

A typical value for the ratio of these two changes for \((k_{o_1} - k_{o_2})\) comparable to the experimental uncertainty is

\[
\frac{F_o}{F_T} \sim 10
\]

with

\[
F_o \sim \sigma_E
\]

where \(\sigma_E\) is the standard deviation of the experimental points.

If the absorption line center is shifted with respect to the center of the emission line from the source then the transmission curves will be modified. If the magnitude of the shift is used as a parameter to fit theory
to experiment the present method will give an estimate of the shift but from the symmetry relations of equation (2-73) it can be seen that the transmission curves are independent of the direction of the shift. These symmetry relations essentially show that if an emission line is moved from one side of a symmetric absorption line to the other there is no change in the transmission. The direction of the shift can be determined by the Zeeman scanning technique if only one of the circularly polarized components are used. With only one polarization the absorption line is no longer symmetric about its position at zero field; thus the direction is given for the shift.

The Zeeman scanning techniques used to date have been limited to normal Zeeman splitting in order to simplify the computations. However the method could be extended to 'anomalous' splittings so this restriction to normal splittings is not an intrinsic disadvantage of the Zeeman scanning technique.

It was found that it was not possible to obtain a mutually independent set of parameters if the Zeeman scanning curve alone were used (transmission curve with polarizing Nicol removed). That is, we were able to obtain more than one set of values of $k_0$, $a$ and $\alpha_1$ which would fit $T_T$ alone but when both $T_0$ and $T_T$ were used there was only one set of parameters possible.
Thus the method we have used is much more sensitive than the standard Zeeman scanning technique; gives better, independent values for the relative transition probabilities and Lorentzian line widths. In addition the present method gives accurate results even when the absorption line width is comparable to emission line width.
CHAPTER 6

CONCLUDING DISCUSSION

The use of the dispersion in the vicinity of absorption lines has been established as a powerful spectroscopic tool. This method was used to determine the relative transition probabilities between transitions with a common lower level. Although it gives the most accurate values for strongly absorbing lines it also gives values as accurate as any other method for the weaker lines. In addition to the relative transition probabilities this method gives information about the line shape.

Using the present method we can determine the line shape more accurately and with less technical difficulties than with any other method. For example in the present work the line shape of the $8115 \, \AA$ line was measured (Doppler width $0.024 \, \text{cm}^{-1}$; Lorentzian width $0.002 \, \text{cm}^{-1}$). The measurement of the Doppler width required a simple temperature measurement and the Lorentzian component was determined by the depths of the maxima and minima in the transmission curve (Fig. 2-3). The measurement of this Lorentzian
component with a conventional method would require the use of a high dispersion instrument such as a Fabry-Perot interferometer, with its accompanying loss of signal strength (14), and difficulties in alignment with infrared radiation. A Fabry-Perot interferometer with a high finesse would require a plate separation of approximately 60 cm to give equivalent resolution.

Calculations indicate that the accuracy of the present method increases as the percentage of the Lorentzian component decreases provided the strength of the line \( (k_o l) \) remains constant. For low pressures in the absorber (~.1torr) the Lorentzian component is determined by the natural line width (~\(2 \times 10^{-3} \text{cm}^{-1}\)) which can be measured accurately (±10%) provided the line is strong (\(k_o l \geq 20\)).

There are other advantages of the present method. No expensive high dispersion device is required because the transmission of the entire source line is measured. The monochromator need only exclude all other lines rather than resolve the line being investigated. The apparatus does not require critical alignment and the data is easily obtained. Since the resolving power of this method depends on the Zeeman effect in the absorber any part of the electromagnetic spectrum can be investigated provided there are polarizers and detectors available. As the Zeeman splitting
increases the absorption of both circular polarizations will decrease leaving the rotation of the plane of polarization as the major effect of the plasma. Thus the wings of the index of refraction curve are investigated at a magnetic field where the transmitted intensity is a maximum.

The relative transition probabilities which were measured by this method agree with the best estimates available in the literature (5) and the values of the Lorentzian components are consistent with broadening of the Van der Waals type (2).

The variance between the experimental and theoretical transmission curves is within limits set by experimental uncertainties. Furthermore the set of parameters which are adjusted to give a 'fit' \( (k_0 \ell, a, \text{ etc.}) \) are determined independently of each other. For example, when this method was applied by Seka (8) with \( a' = 0 \), the values of the relative transition probabilities obtained were the same as those obtained by fitting the experimental curve with the correct value of \( a' \). This is because the experimental curves were fitted so the maxima and minima (Fig. 2-3) occurred at the correct magnetic fields. The only effect of using an incorrect \( a' \) was that the depth of the minimum predicted by theory did not agree with the observed value.
Now that the advantages of the method have been established, it is possible to use it to measure transition probabilities and line shapes to accuracies which would be very difficult, if not impossible, to measure in any other manner. If the pressure in the absorber is changed while the source is left constant then each of the absorber pressures will yield a new measurement of the relative transition probabilities which should be independent of the pressure used. Plotting the variation of the Lorentzian component with pressure will give two parameters. The extrapolation of the plot to zero pressure will give the absolute value of the state lifetimes and the slope of the plot will give information on the broadening mechanisms involved (12).

This method can thus be applied to many problems where accurate transition probabilities or line shapes are required. It gives the most accurate results when the lines being considered are strongly absorbed and as such it is well suited to study resonance transitions.
ANOMALOUS DISPERSION

FIG. 1-1
PARTIAL TERM DIAGRAM FOR NEUTRAL ARGON

FIG.1-2
EXPERIMENTAL GEOMETRY

FIG. 2-1
Locus of $I_\omega$ (Equ. 2-63)

POLARIZATION ELLIPSE

FIG. 2-2
EXPERIMENT AND THEORY (8115)

$\lambda = 8115 \text{ Å}$

$k_0 l = 29.0$

$T_T$ (without Nicol)

$T_0$ (with Nicols)

Transmission

Magnetic Field (KG)

FIG. 2-3
EXPERIMENT AND THEORY (7067)

\[ \lambda = 7067 \text{Å} \]

\[ k_0 l = 1.5 \]

+++ Experiment

--- Theory

**FIG. 2-4a**
RESULTS (8015Å, 7147Å)

λ = 7147 Å
k₀l = 0.16

λ = 8015 Å
k₀l = 5.25

--- Theory
++ Observed

Fig. 2-4b
RESULTS (6965Å, 7635Å)

\[ \lambda = 6965 \text{ Å} \]
\[ k_0l = 1.52 \]

\[ \lambda = 7635 \text{ Å} \]
\[ k_0l = 12.5 \]

\[ T_T \]
\[ T_o \]

--- Theory
++ Observed

FIG. 2-4c
\( \mu_B = \text{Bohr Magneton} \)
\( B = \text{Magnetic Field} \)
\( g = \text{Lande' g-factor} \)

ZEEMAN SPLITTING TERM DIAGRAM
FIG. 2-5
Abs. Coeff.

\[ \text{Index of Refraction} \]

\[ \Delta \omega_L \]

\[ \omega_0 \]

\[ k \text{ (Doppler)} \]

\[ k \text{ (Lorentzian)} \]

\[ (n - 1) \text{ (Lorentzian)} \]

\[ (n - 1) \text{ (Doppler)} \]

LINE PROFILES

FIG. 2-6
All Dimensions in cm. Constructed of Pyrex Glass

DISCHARGE TUBES
FIG. 3-2

\[ \lambda = 8115\text{Å} \]
\[ B = 0 \text{ gauss} \]

BEAM WIDTH
FIG. 3-3
Coil Run Immersed in Flowing Cold Water

Constructed of 3.2 mm Brass

Coil Windings

Circulation Holes

3.8 cm

20 cm

12.7 cm. Dia.

Solenoid

Fig. 3-4
SORENSEN NOBATRONS
DCR150-15A
DCR80-18
(DC Power Supplies)

DIGITAL VOLTMETER

1 mΩ SHUNT

Solenoid

DIGITAL VOLTMETER

1 mΩ SHUNT

Solenoid

SOLENOID POWER SUPPLY

FIG. 3-5
WAVEFORMS FOR THE DIGITAL PHASE SENSITIVE DETECTOR

FIG. 3-6
See Fig. 3-10

Pulse Shape Given in Fig. 3-9

PM Amplifier Discriminator

Gate #1

Gate #2

Output ‘on’

Output ‘off’

Letters Refer to Waveforms in FIG.3-6

Reference

See FIG.3-8

BLOCK DIAGRAM OF THE DPSD

FIG.3-7
Reference Pick-up Moved to Adjust Phase

33 Equally Spaced Holes

Reference Light Beam

Experimental Optical Axis

Rotated at 1800 rpm by a .04 hp. synchronous motor

CHOPPING WHEEL

FIG.3-8
Type 549 Storage Scope
Type 1S1 Sampling Unit
Risetime $\leq 0.35$ nsec.

SINGLE PHOTON PULSE AFTER THE DISCRIMINATOR

FIG. 3-9

EMI 9558B

Tunnel diode current discriminator with amplification
(output is shown in Fig. 3-9)

current gain $\approx 10$
input $Z = 5 \Omega$
output $Z = 5 \Omega$
risetime $\approx 3$ nsec

PHOTO TUBE AND DISCRIMINATOR

FIG. 3-10
CONTOH FIELD (KG)

FIG. 3-11

\[ \lambda = 8115 \text{ Å} \]

- \( k_0l = 29.0 \)
- \( ++k_0l = 30.5 \)

Transmission

Magnetic Field (KG)

FIG. 3-11
Magnetic Field (KG)

FIG. 3-12

CHANGES IN 'a'

\[
\begin{align*}
\lambda &= 8115 \text{A} \\
K_0 l &= 29.0 \\
- a &= 0.072 \\
++ a &= 0.065
\end{align*}
\]
UNIFORMITY OF THE MAGNETIC FIELD

FIG. 3-13
FIG.A-1

FIG.A-2
BIBLIOGRAPHY


APPENDIX I

THEOREMS

THEOREM I-1 — ROTATION OF CO-ORDINATES

If a cartesian co-ordinate system is rotated about the $x_3$ axis by an angle $\Omega$ then the new circularly polarized components $E'_{\pm}$ can be expressed in terms of the old circularly polarized components $E_{\pm}$ by

$$E'_{\pm} = E_{\pm} e^{\mp i\Omega}$$  \hspace{1cm} (A-1)

PROOF: Referring to Figure A-1 the definition of circular polarization gives

$$E'_{\pm} = E_1' \mp i E_2'$$  \hspace{1cm} (A-2)

From the figure we see that

$$E_1' = E_1 \cos \Omega + E_2 \sin \Omega$$  \hspace{1cm} (A-3)

$$E_2' = -E_1 \sin \Omega + E_2 \cos \Omega$$  \hspace{1cm} (A-4)

Substituting A-3 and A-4 into A-2 gives
\[ E' = (E, \pm i E_2) \cos \Omega \pm i (E, \pm i E_2) \sin \Omega \quad \text{A-5} \]

but
\[ E_\pm = E, \pm i E_2 \]

therefore
\[ E'_\pm = E_\pm \left( \cos \Omega \pm i \sin \Omega \right) \]

or
\[ E'_\pm = E_\pm e^\pm i \Omega \]

THEOREM I-2

The time average of the product of the real parts of two complex functions \( f \) and \( F \) which vary sinusoidally in time with frequency \( \omega \) and \( \omega' \) respectively, is given by

\[ \frac{\text{Re}(f) \, \text{Re}(F)}{2} = \frac{1}{2} \text{Re}(f \, F^*) \delta(\omega - \omega') + \frac{1}{2} \text{Re}(f \, F) \delta(\omega + \omega') \]

PROOF: It is assumed that \( f \) and \( F \) have a sinusoidal time dependence so we may write
\[ f = f_0 \, e^{i(\omega t + \alpha)} \]
\[ F = F_0 \, e^{i(\omega' t + \alpha')} \]

where \( \alpha \) and \( \alpha' \) are arbitrary phase factors. Thus we have

\[ \text{Re}(f) \, \text{Re}(F) = f_0 \, F_0 \, \cos(\omega t + \alpha) \cos(\omega' t + \alpha') \]
which can be rewritten as
\[
\text{Re}(f) \text{Re}(F) = \frac{f_0 F_0}{4} \left\{ e^{i[(\omega + \omega')t + (\alpha + \alpha')] - e^{i[(\omega - \omega')t + (\alpha - \alpha')]} + e^{-i[(\omega - \omega')t + (\alpha - \alpha')]} - e^{-i[(\omega + \omega')t + (\alpha + \alpha')]} \right\}
\]

taking the time average over the lowest common period gives
\[
\text{Re}(f) \text{Re}(F) = \frac{1}{2} \text{Re}(f F^*) \delta(\omega - \omega') + \frac{1}{2} \text{Re}(f F) \delta(\omega - \omega')
\]

Note if \( \omega = \omega' \) and \( f_0 = F_0 \) we have
\[
\text{Re}(f) \text{Re}(f) = \frac{1}{2} \text{Re}(f F^*) = \frac{f_0^2}{2} \cos(\alpha - \alpha')
\]

**THEOREM 1-3**

\[
I = -\frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{e^{-y^2}}{y + \omega_0 + \alpha i} \, dy' = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{e^{-y^2}}{\omega_0^2 \alpha i - y} \, dy \quad (y = -y')
\]
The real part of this expression gives the 'Voigt line shape' of absorption coefficient \( k_v \). There are many computer routines which have been written to evaluate this real part but for the present work it required both the real and imaginary parts of the expression. In order to evaluate \( I \) it is derived in terms of the complex error function \( \Phi \) which is given by

\[
\Phi(z) = e^{-z^2} \left\{ 1 + \frac{z}{\pi} \int_0^\infty e^{t^2} dt \right\}
\]

\( z \equiv \nu + i \alpha \)

The relation between the standard Voigt function and the complex error function is

\[
I = \Phi(z)
\]

where \( \Phi \) if the complex error function

**PROOF:** Multiplying the numerator and denominator of \( I \) by \( -i \)

\[
I = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{i(y - z)} dy
\]

where we define \( z \equiv \nu + i \alpha \)

From Laplace transforms we recall that

\[
\frac{1}{\alpha} = \int_0^\infty e^{-\alpha x} dx \quad \therefore \quad \frac{1}{i(y - z)} = \int_0^\infty e^{-i(y - z)x} dx
\]
Substituting this into \( I \) gives
\[
I = \frac{1}{\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \ e^{-y^2 - x(y-z)x}
\]
Collecting the terms and integrating over \( y \) noting that
\[
\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}
\]
gives
\[
I = \frac{1}{\pi} \int_{0}^{\infty} dx \ e^{-x^2/4} \int_{-\infty}^{\infty} dy \ e^{-(y-i\pi x)^2} \ e^{-i\pi x}
\]
\[
= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} dx \ e^{-x^2/4 + i\pi x}
\]
Completing the square and integrating over the last variable gives
\[
I = \frac{1}{\sqrt{\pi}} e^{-\pi^2} \int_{0}^{\infty} e^{-\left(\frac{x}{\sqrt{2}} - i\pi x\right)^2} dx
\]
\[
= \frac{2}{\sqrt{\pi}} e^{-\pi^2} \int_{-i\pi}^{i\pi} e^{-z^2} dz
\]
where \( z = u + iv \)
Referring to Fig. A-2 the integration along path \( C \) must equal the sum along \( C_2 \), \( C_3 \) and \( C_4 \) since there are no poles within the contour.
\[
\int_{C_4} = \lim_{u \to \infty} \int_{u}^{u-i\pi} e^{-u^2} - 2iuv + v^2 dv \to 0
\]
\[
\int_{C_3} = \int_{0}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}
\]
\[
\int_{C_2} = \int_{i\pi}^{0} e^{-z^2} dz
\]
Substitute $t = iZ$, \( \therefore \mathcal{Z} = -it \)

\[
\mathcal{Z} = -i \int_{-\infty}^{\infty} e^{+t^2} \, dt = i \int_{0}^{\infty} e^{+t^2} \, dt
\]

\[
\int_{-\infty}^{\infty} e^{-z^2} \, dz = \frac{\sqrt{\pi}}{2} + i \int_{0}^{\infty} e^{+t^2} \, dt
\]

\[
= \frac{\sqrt{\pi}}{2} \left\{ 1 + \frac{2 \pi i}{\sqrt{\pi}} \int_{0}^{\infty} e^{+t^2} \, dt \right\}
\]

Substitute into A-7

\[
I = e^{-z^2} \left\{ 1 + \frac{2 \pi i}{\sqrt{\pi}} \int_{0}^{\infty} e^{+t^2} \, dt \right\} = \Phi(z)
\]

where

\[
\Phi(z) \equiv e^{-z^2} \left\{ 1 + \frac{2 \pi i}{\sqrt{\pi}} \int_{0}^{\infty} e^{+t^2} \, dt \right\}
\]

complex error function.

also

\[
-i I = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-y^2}}{y + w + i \alpha} \, dy = -i \Phi(z) + \Phi(z^*)
\]
APPENDIX II

EVALUATION OF THE COMPLEX ERROR FUNCTION

\[
\Phi(z) = e^{-z^2} \left\{ 1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{t^2} dt \right\}
\]

\[z = \omega + ia\]

The evaluation was carried out numerically with an accuracy of approximately five significant figures in order to insure that the possible error in the calculation be small compared to the experimental uncertainty.

Theorem II-1 shows that

\[
\int_0^z e^{t^2} dt = \int_0^\omega e^{x^2} dx - e^{-z^2} \int_0^\omega e^{-y^2} \sin 2\omega y dy
\]

\[+ i \int_0^\omega e^{-z^2} e^{-y^2} \cos 2\omega y dy\]

Putting this into the expression for \(\Phi(z)\) in A-30 we obtain

\[
\Phi(z) = e^{-z^2} \left\{ 1 - \frac{2e^{-z^2}}{\sqrt{\pi}} \int_0^\omega e^{-y^2} \cos 2\omega y dy + \frac{2i}{\sqrt{\pi}} \left[ \int_0^\omega e^{x^2} dx - e^{-z^2} \int_0^\omega e^{-y^2} \sin 2\omega y dy \right] \right\}
\]

Separating the real and imaginary parts we get

\[
\text{Re}\{\Phi(z)\} = A(\omega) \cos(2\omega a) + B(\omega) \sin(2\omega a)
\]

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\[ \text{Im}\{\Phi(Z)\} = A(w) \cos(2\omega a) - A(w) \sin(2\omega a) \]  

where

\[ A(w) = e^{-\omega^2} - \frac{2e^{\omega^2}}{\sqrt{\pi}} \int_0^\infty e^{-y^2} \cos{2\omega y} \, dy \]  

\[ B(w) = \frac{2e^{\omega^2}}{\sqrt{\pi}} \left\{ F(w) - \int_0^\infty e^{-y^2} \sin{2\omega y} \, dy \right\} \]  

\[ F(w) = e^{-\omega^2} \int_0^\omega e^{t^2} \, dt \quad \text{(Dawson's Integral)} \]  

Thus the problem has been reduced to evaluating

\[ \int_0^\omega e^{-y^2} \cos(2\omega y) \, dy, \quad \int_0^\omega e^{-y^2} \sin(2\omega y) \, dy, \text{ and } F(w) \]  

The range of the variables which must be covered is

\[ -\infty < w < +\infty \quad 0 \leq a \leq 1.0 \]  

For \( w < 3.9 \) and \( a \leq 3 \) we evaluate A-38 directly.

**DAWSON'S INTEGRAL \( F(w) \):**

By noting that \( F(-W) = -F(W) \) we need only deal with \( 0 \leq W \leq \infty \). The method used for \( W 5 \) is given by Hummer (THE VOIGT FUNCTION: An eight-significant figure table and Generating Procedure, University of Colorado NBS).
It involves a Chebyshev expansion and the Clenshaw Algorithm. For \( W > 5 \) the series expansion

\[
F(w) = \frac{1}{2w} + \frac{1}{2w^3} + \frac{1 \cdot 3}{2^3 w^5} + \frac{1 \cdot 3 \cdot 5}{2^4 w^7}
\]

was used as given by ERDELY A., OBERHETTINGER F., TRICOMI, I.G. '53 (Higher Trnascedental Functions, Vol. II, McGraw-Hill Book Co. Inc., New York). The above calculation was done in the subroutine DAWSON \((W, Y, RL, RLN)\).

Since we are dealing with small 'a' in the experimental situation, we can expand the term \( e^{-y^2} \) to give according to Theorem \( \Pi-2.0 \).

\[
\int_0^\infty e^{-y^2} \cos(2wy) \, dy = \sum_{n=0}^{\infty} J_n
\]

where

\[
J_n = \frac{2n-1}{2w} J_{n-1} + (-1)^n \left\{ \frac{a^{2n} \sin 2wQ}{2w} + \frac{n a^{2n-1} \cos 2wQ}{2w^2} \right\}
\]

and

\[
J_0 = \frac{\sin 2wQ}{2w}
\]

Note that this series will converge quickly if \( W \) is large but for small \( W \) we must expand \( \cos(2Wy) \) to give according to Theorem \( \Pi-2.2 \).
\[
\int_0^a e^{-y^2} \cos 2\omega y \, dy = \sum_{n=0}^{\infty} J_n
\]

where

\[
J_n = -\left\{ \frac{\omega^2}{n} J_{n-1} + \frac{(-1)^n a^{2n-1}}{2 (2n)!} (2\omega)^{2n} e^{-q^2} \right\}
\]

and

\[
J_0 = \int_0^a e^{-y^2} \, dy
\]

Note that this series converges quickly for \( \omega \) small but A-40 is better for larger \( \omega \). Similar expansions can be obtained for the last integral Theorem II-2.1

\[
\int_0^a e^{-y^2} \sin 2\omega y \, dy = \sum_{n=0}^{\infty} I_n
\]

\[
I_0 = 1 - \frac{\cos 2\omega a}{2\omega a}
\]

\[
I_n = \frac{\omega^{n-1}}{2\omega^2} I_{n-1} + \frac{(-1)^n}{n!} \left\{ \frac{\omega^{2n-1}}{2\omega^2} \sin 2\omega a - \frac{a^{2n}}{2\omega} \cos 2\omega a \right\}
\]
or using Theorem II-2.3 (\( 2\omega a \leq 1 \))

\[
I_0 = \omega \left[ 1 - e^{q^2} \right]
\]

\[
I_n = -\left\{ \frac{\omega^2}{2^{n+1}} I_{n-1} + \frac{(-1)^n \omega}{2 (2n+1)!} a^{2n} e^{q^2} \right\}
\]
Therefore A-38 is evaluated for \( W < 3.9 \) and \( a < 1.0 \). For \( W > 3.9 \) or \( a > 3 \) then a series expansion exists of the form (Handbook of Mathematical Functions by Abramowitz and Stegun, p 328)

\[
\phi(z) = -i \frac{z}{\sqrt{\pi}} \sum_{j=1}^{3} \frac{A_j}{z - B_j} \pm 2 \times 10^{-6}
\]

\[
A_1 = 0.4613135 \quad B_1 = 0.1901635 \\
A_2 = 0.09999216 \quad B_2 = 1.7844927 \\
A_3 = 0.002883894 \quad B_3 = 5.5253437
\]

For \( W > 6 \) or \( a > 6 \) then another series of the form

\[
\phi(z) = -i \frac{z}{\sqrt{\pi}} \sum_{j=1}^{3} \frac{A_j}{z - B_j} \pm 10^{-6}
\]

\[
A_1 = 0.5124242 \quad B_1 = 0.2752551 \\
A_2 = 0.05176536 \quad B_2 = 2.724745
\]

This is done in subroutine approx \((W, Y, RL, RLN)\).

**THEOREM II-1**

When evaluating the complex error function we are confronted with the integral in the complex plane

\[
Q = \int_{0}^{\infty} e^{zt} dt \quad z = w + i \alpha
\]

This integral is calculated by breaking it up into three integrals in the real plane, i.e.
\[ \int_{0}^{\infty} e^{t^2} dt = \int_{0}^{\infty} e^{x^2} dx - \int_{0}^{\infty} e^{-y^3} \sin \omega y dy + i \int_{0}^{\infty} e^{-y^3} \cos \omega y dy \]

PROOF: Since \( Q \) is entire above the real axis the sum of the contour integral shown in Figure A-3 must give \( \oint \), i.e.,

\[ Q = \int_{0}^{\infty} e^{x^2} dx + \int_{0}^{\infty} e^{(w+iy)^2} d(iy) \]

\[ Q = \int_{0}^{\infty} e^{x^2} dx + i \int_{0}^{\infty} e^{w^2+2ixwy-y^2} dy \]

or writing the exponential in its real and imaginary parts gives

\[ Q = \int_{0}^{\infty} e^{x^2} dx - \int_{0}^{\infty} e^{w^2-y^2} \sin \omega y dy + i \int_{0}^{\infty} e^{w^2-y^2} \cos \omega y dy \]

or

\[ Q = \int_{0}^{\infty} e^{x^2} dx - e^{w^2} \int_{0}^{\infty} e^{-y^2} \sin \omega y dy + i e^{w^2} \int_{0}^{\infty} e^{-y^2} \cos \omega y dy \]

The following series of theorems number \( \text{II}-2.0 \) to \( \text{II}-2.3 \) are to prove the algorithms used to calculate the complex error function over different ranges of the argument.

THEOREM II-2.0

Used for \( 2\omega a > 1 \)
\[ I \equiv \int_0^a e^{-y^2} \cos z w y \, dy = \sum_{n=0}^{\infty} I_n \]

where \[ I_0 = \frac{\sin z w a}{2w} \]

\[ I_n = \frac{2n-1}{2w} I_{n-1} + (-1)^n \left\{ \frac{a^{2n} \sin z w a}{2w} + \frac{n a^{2n-1} \cos z w a}{2w^2} \right\} \]

**PROOF:**

\[ e^{-y^2} = \sum_{n=0}^{\infty} \frac{(-y^2)^n}{n!} \]

\[ \therefore I = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^a y^{2n} \cos z w y \, dy = \sum_{n=0}^{\infty} I_n \]

Integrating by parts gives us that

\[ n! I_n = \int_0^a y^{2n} \cos z w y \, dy = \frac{a^{2n} \sin z w a}{2w} - \frac{2n}{2w} \int_0^a y^{2n-1} \sin z w y \, dy \]

and by parts again

\[ n! I_n = \frac{2n}{2w} \frac{a^{2n} \sin z w a}{2w} + \frac{n}{2w^2} a^{2n-1} \cos z w a - \frac{2n-1}{2w^2} \int_0^a y^{2n-1} \cos z w y \, dy \]

\[ \therefore I_n = (-1)^n \left\{ \frac{a^{2n} \sin z w a}{2w} + \frac{n a^{2n-1} \cos z w a}{2w^2} \right\} + \frac{(2n-1)}{2w^2} (-1)^{n-1} \int_0^a y^{2n-1} \cos z w y \, dy \]

\[ I_n = \frac{2n-1}{2w^2} I_{n-1} + (-1)^n \left\{ \frac{a^{2n} \sin z w a}{2w} + \frac{n a^{2n-1} \cos z w a}{2w^2} \right\} \]
\[ I_0 = \int_0^a \cos zwy \, dy = \frac{\sin 2wa}{2w} \quad \text{Q.E.D.} \]

**Theorem II-2.** \((2wa \geq 1)\)

In a way analogous to (II-2.0) it can be shown that

\[ J \equiv \int_0^a e^{-y^2} \sin zwy \, dy = \sum_{n=0}^{\infty} J_n \]

where \( J_0 = \frac{1 - \cos 2za}{2za} \)

\[ J_n = \frac{2^{n-1}}{2w^2} J_{n-1} + (-1)^n \sum_{n=0}^{\infty} \left\{ \frac{a^{2n} \cos 2za + \frac{a^{2n-1}}{2w^2} \sin 2za}{n!} \right\} \]

**Theorem II-2.2**

Used for \(2Wa \leq 1\)

\[ J \equiv \int_0^a e^{-y^2} \cos 2wy \, dy = \sum_{m=0}^{\infty} J_m \]

where \( J_0 = \int_0^a e^{-y^2} \, dy \)

\[ J_m = \left\{ \frac{w^2}{m} J_{m-1} + (-1)^m \frac{a^{2m-1}(2w)^{2m}}{2(2m)!} e^{-a^2} \right\} \]

**Proof:** We must first consider the series expansion of

\[ \cos 2wy = \sum_{m=0}^{\infty} \frac{(2wy)^{2m}}{2m!} (-1)^m \]
\[ J = \int_0^a e^{-y^2} \cos 2w y \, dy = \sum_{m=0}^{\infty} \frac{(-1)^m (2w)^{2m}}{(2m)!} \int_0^a y^{2m} e^{-y^2} \, dy \quad \text{(1)} \]

Let us now develop an algorithm to evaluate the integral.

Consider \( \beta (\alpha, a) = \int_0^a e^{-\alpha y^2} \, dy \)

\[ \frac{d^m \beta}{d \alpha^m} = (-1)^m \int_0^a y^{2m} e^{-\alpha y^2} \, dy \]

\[ \frac{d^m \beta}{d \alpha^m} \bigg|_{\alpha=1} = (-1)^m \int_0^a y^{2m} e^{-y^2} \, dy = \beta_m \]

\[ \int_0^a y^{2m} e^{-y^2} \, dy = (-1)^m \beta_m \]

Integrating by parts gives

\[ \beta_m = -\frac{1}{2} \left\{ (2m-1) \beta_{m-1} + (-1)^m a^{2m-1} e^{-a^2} \right\} \]

Substituting this into (1) gives

\[ J = \sum_{m=0}^{\infty} \frac{(2w)^{2m}}{(2m)!} \beta_m \]

\[ J = \sum_{m=0}^{\infty} \frac{(2w)^{2m}}{(2m)!} \left[ -\frac{1}{2} \left\{ (2m-1) \beta_{m-1} + (-1)^m a^{2m-1} e^{-a^2} \right\} \right] \]

\[ J_m = \frac{(2w)^{2(m-1)}}{2(m-1)!} \beta_{m-1} \left\{ -\frac{2w^2 (2m-1)}{(2m-1)(2m)} \right\} + \frac{(2w)^m a^{2m-1} (-1)^m e^{-a^2}}{(2m)!} \frac{1}{2} \]

\[ J_m = \frac{(2w)^{2(m-1)}}{2(m-1)!} \beta_{m-1} \left\{ -\frac{2w^2 (2m-1)}{(2m-1)(2m)} \right\} + \frac{(2w)^m a^{2m-1} (-1)^m e^{-a^2}}{(2m)!} \frac{1}{2} \]
\[ J_m = - \left\{ \frac{w^2}{m} J_{m-1} + \frac{(-1)^m a^{2m-1} (2w)^m e^{-a^2}}{2 (2m)!} \right\} \]

\[ J_0 = \int_0^a e^{-y^2} \, dy \]

**THEOREM II-2.3**

for \( 2w \leq 1 \)

\[ I = \sum_{m=0}^\infty I_m \]

\[ I_0 = w \left[ 1 - e^{-a^2} \right] \]

\[ I_m = \frac{-2w^2}{2m+1} I_{m-1} - \frac{(-1)^m (2m)^{2m+1}}{2 (2m+1)!} a^{2m} e^{-a^2} \]

**PROOF**

Now \( \sin(2wy) = \sum_{m=0}^\infty \frac{(-1)^m (2wy)^{2m+1}}{(2m+1)!} \)

\[ I = \sum_{m=0}^\infty \frac{(-1)^m (2w)^{2m+1}}{2 (2m+1)!} \int_0^a e^{-y^2} 2y y^{2m} \, dy \]

\[ = \sum_{m=0}^\infty \frac{(-1)^m (2w)^{2m+1}}{2 (2m+1)!} \int_0^{a^2} e^{-t} t^m \, dt \quad (t = y^2) \]

\[ \therefore I_m = \frac{(-1)^m (2w)^{2m+1}}{2 (2m+1)!} \int_0^{a^2} e^{-t} t^m \, dt \]
Integrating by parts twice gives

\[ I_m = -\frac{\lambda \omega^2}{2m+1} I_{m-1} - \frac{(-1)^m (2m)^{2m+1} a^{2m} e^{-a^2}}{(2m+1)!} \]

\[ I_o = \int_0^a e^{-\gamma^2} (\omega y) \, dy = \omega \{1 - e^{-a^2}\} \]

**THEOREM II-3**

For various reasons it is desired that the limiting values of the integrals derived in Theorem I-3 should be known for \( \text{Im}(Z) = a \to \infty \). These limiting values are,

\[ \int_0^\infty e^{-t^2} \cos(\omega t) \, dt = \frac{\sqrt{\pi}}{\omega} e^{\omega^2} \quad \text{A-20} \]

\[ \int_0^\infty e^{-t^2} \sin(\omega t) \, dt = F(\omega) \quad \text{A-21} \]

where \( F \) is Dawson's integral \( F(\omega) = e^{-\omega^2} \int_0^\omega e^{y^2} \, dy \)

**PROOF:** In order to carry out the integrals given in A-20 and A-21 note that they form the real and imaginary parts of the entire function I where
\[ I = \int_{0}^{\infty} e^{-t^2 + 2i\omega t} \, dt \]

i.e.

\[ \text{Re}(I) = \int_{0}^{\infty} e^{-t^2} \cos(\omega t) \, dt \]

\[ \text{Im}(I) = \int_{0}^{\infty} e^{-t^2} \sin(\omega t) \, dt \]

Referring to the contours in Fig. A-4 we have

\[ I = e^{-\omega^2} \int_{0}^{\infty} e^{(\omega + it)^2} \, dt \]

\[ I = -i e^{-\omega^2} \oint_{C_1} e^{z^2} \, dz \]  \hspace{1cm} A-22

where \( Z_1 = \omega + it \) so \( dZ_1 = idt \)

Since \( I \) is entire within the contour in Fig. A-4 we have

\[ - \int_{C_1} e^{z^2} \, dz = \int_{C_2} e^{z^2} \, dz + \int_{C_3} e^{z^2} \, dz + \int_{C_4} e^{z^2} \, dz \]

Considering contour \( C_2 \) we see that if \( Z = \infty + iy \) then

\[ \lim_{y \to \infty} \left| \int_{C_2} \right| < e^{-y^2} \rightarrow 0 \]

therefore using A-22 + A-23 we have
\[ I = i e^{-w^2} \left\{ \int_0^w e^x \, dx + \int_{-\infty}^0 e^{-y^2} \, dy \right\} \]

since \( \mathbb{Z}_4 = \mathbb{C} \)
\[ d\mathbb{Z}_4 = dx \]
\[ \mathbb{Z}_3 = \mathbb{C} \]
\[ d\mathbb{Z}_3 = idy \]

therefore
\[ I = i e^{-w^2} \int_0^w e^x \, dx + e^{-w^2} \int_{-\infty}^0 e^{-y^2} \, dy \]
\[ = i F(w) + e^{-w^2} \frac{\sqrt{n\pi}}{2} \]

which gives us
\[ \text{Re}\{ I \} = e^{-w^2} \frac{\sqrt{n\pi}}{2} \]
\[ \text{Im}\{ I \} = F(w) \]

**THEOREM II-4**
\[ \int_0^\infty \text{Re}\{ \phi(yz) \} \, dz = \frac{\pi}{2} \sqrt{\frac{n}{\ln^2}} \]

**PROOF:** From the definition of \( \phi(yz) \) given in Appendix II we see that
\[ \text{Re} \left\{ \Phi(Z_\pm) \right\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} a \, dy}{(y - \omega)^2 + a^2} \]

\[ W = 2 \left( \omega_0 - \omega = \omega_0 \right) \frac{\sqrt{\ln 2}}{D} \]

therefore

\[ \int_{0}^{\infty} \text{Re} \left\{ \Phi(Z_\pm) \right\} \, d\omega = I = \int_{0}^{\infty} \frac{d\omega}{\pi} \int_{0}^{\infty} \frac{e^{-y^2} a \, dy}{(y - \omega)^2 + a^2} \]

making the substitution

\[ x = y - \omega \]
\[ d\omega = \frac{2\sqrt{\ln 2}}{D} \, dx \]

gives

\[ I = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \int_{-\infty}^{\infty} \frac{e^{-x^2} a \, dx \, dy}{x^2 + a^2} \]
\[ = \frac{D}{2\sqrt{\ln 2}} \int_{-\infty}^{\infty} e^{-y^2} \, dy = \frac{D}{2} \sqrt{\frac{\pi}{\ln 2}} \]

Note that the lower limit of integration has been set to \(-\infty\) rather than \(2 \left( \omega_0 + \frac{\omega_0}{a} \right) \frac{\ln 2}{D}\). This is valid since \(\omega_0 \gg \omega_b\) i.e. there is not absorption at zero frequency, due to the transition in question.

\[ \therefore \int_{0}^{\infty} \text{Re} \left\{ \Phi(Z_\pm) \right\} \, d\omega = \frac{D}{2} \sqrt{\frac{\pi}{\ln 2}} \]
APPENDIX III

COMPUTER PROGRAM

The complex error function was evaluated as in Appendix II. The subroutine for calculating this function was shown to be accurate for $-\infty < W < \infty$, $0 \leq a \leq 1$ by comparing the results in this range with the values given in the table in reference 4.

The required integrals given in equations (2-73) and (2-74) were carried out using a Simpson's rule numerical integration routine. The integration was carried out, then the number of integration steps was doubled and the sum was repeated. This doubling of the number of integration steps was continued until two successive answers differed by less than .1%. This routine integrated the line out to a value of $W = 8.5$ (Eq. 2-31). Beyond this limit Simpson's rule was used with increasing width of integrating interval out to $W = 42.0$. Beyond this upper limit an analytical continuation was used.