RIP CURRENTS ON A CIRCULAR BEACH

by

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ABSTRACT

A mathematical model is developed which extends the theory of rip currents developed by Bowen (1969b) for a straight beach to curved beaches where radii of curvature are large relative to the width of the surf zone. Nine forcing terms are found to cause rip current systems. The terms are functions of the longshore variation in wave height and angle of incidence of the incoming waves at the breakers. The model is applied to the case of a circular beach with conical nearshore bottom topography. A large rip current component is found to exist which is inversely proportional to the radius of curvature of the beach. Another significant rip current component is found to be proportional to the variation in the angle of incidence of the waves at the breakers. This component would cause rip currents on a straight beach where some irregular offshore topography caused some variation in the incident angle of the incoming waves. Another component rip current was found which was essentially the same as the one predicted by Bowen (1969b).

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LIST OF SYMBOLS

A large number of symbols have been used throughout this thesis and they are defined within the text as they are introduced. They are summarized below for the convenience of the reader.

A : constant of proportionality between set-up and wave height (see (2.21)).

B_j where j = a, b, ..., i: constant coefficients of the forcing terms.

C : constant of proportionality for bottom friction (see (3.10)).

D : average value of H_b^+.

d : depth from average water surface $\bar{\eta}$ to bottom; $d = \bar{\eta} + h$.

E : energy density of waves.

$e(\theta)$: θ dependent factor of ϕ, $\phi = \phi_0 e(\theta)$.

g : acceleration of gravity.

H : wave height, twice the amplitude.

H_b^+ : height of waves just before they break.

h : depth from undisturbed water surface to bottom.

I_α : inertial terms in equations of motion.

K : ratio of surface slope to bottom slope, $K = m_\eta / m_b$.

k : a constant equal to $\frac{1}{8} g \gamma^2$, (see (5.17)).

$\tilde{M}_\alpha$: total horizontal momentum per unit surface area; $\tilde{M}_\alpha = \tilde{\mathcal{U}}_\alpha / \rho d$; $\alpha = 1, 2$.

m : rate of change of d with respect to r, $m = -\partial d/\partial r$.

m_η : slope of mean surface, $m_\eta = \partial \bar{\eta}/\partial r$ inside surf zone.

m_b : slope of bottom, $m_b = \partial h/\partial r$.

N : ratio of set-down at the breakers to surf zone width, $N = \bar{\eta}_b / r_{sb}$.

p : amplitude factor of θ dependent component of H_b^+ (see (2.1)).

q : wave number of incoming waves.

r : radial co-ordinate of polar co-ordinate system.

r_b : radial position of breaker line.

r_o : radial position of shoreline for undisturbed water surface.

r_s : radial position of shoreline for disturbed water surface.

r_{sb} : width of surf zone, $r_{sb} = r_s - r_b$.

s : variable defining distance from average shoreline, $s = \bar{r}_s - r$.

s_b : value of s at $r = r_b$.

$\tilde{U}_\alpha$: mean transport velocity, tensor notation.

u, v : mean transport velocity components in the x and y directions respectively.

v_r, v_θ: mean transport velocities in the r and θ directions respectively.

x, y: cartesian co-ordinate axes defined on the bay (see Figure 1).

X, Y: cartesian co-ordinate axes defined on the incoming wave (see Figure 2).

γ : ratio of wave height and depth inside the surf zone, $H = \gamma d$. (see (2.9)).

ϵ : amplitude factor of θ dependent component of d for term a, $\epsilon = \eta_a / \bar{h}_b$.

$\bar{\eta}$: mean surface level, $\bar{\eta} = 0$ for undisturbed surface.

η_1 : amplitude of θ varying component of $\bar{\eta}$, $\eta_1 = ADp$ for refractive variations.

η_a : amplitude of θ varying component of $\bar{\eta}$ from edge wave effects in term a.

θ : angular co-ordinate of polar co-ordinate system (see Figure 1).

λ : ratio of wave height to depth just before waves break, $H_b^+ = \lambda d_b$ (see (2.9)).

σ_0 : longshore scaling distance for refractive variables.

σ_a : longshore scaling distance for edgewave effects in term a.

Υ_α : force per unit mass vector due to divergence of radiation stress tensor (see (3.8) and (3.9)).

$\Phi_{\alpha\beta}$: tensor factor of radiation stress tensor. $\Phi_{\alpha\beta}$ is a function of ϕ only (see (5.18)).

ϕ : angle of incidence of waves at breaker line (see Figure 2).

ψ : transport stream function (see (3.23) and (3.24)).

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CHAPTER 1

INTRODUCTION

On many long sandy ocean beaches signs are posted to warn swimmers of the danger of being swept out by near shore currents to water depths over their heads. These currents flowing away from the beach are commonly known as rip currents. They are associated with variations of the momentum flux of the incoming surface waves and are found at specific locations along a beach at any one time (see Shepard, Emery and Lafond, 1941 for an introduction to the subject of rip currents). In this thesis the term "rip current system" is used to represent the complete flow pattern of near shore currents, including both seaward flowing currents and longshore currents.

This thesis discusses theoretically the effects of bottom topography on rip current systems along a beach and presents a theoretical solution for a circular bay with a conical bottom. The purpose for studying this subject was to gain some physical insight and understanding of the various effects of bottom topography on rip current systems.

In 1941 Shepard, Emery and Lafond wrote a mainly descriptive paper on rip currents, their flow patterns and geological effects on the beach. Various papers have been presented since then attempting physical explanations for the causes of rip currents. But it was Bowen's paper (1969b) which for the first time presented a satisfactory mathematical model of the rip current process on a straight beach. His model was based on the theory of the radiation stress (Longuet-Higgins and Stewart, 1964) of waves proceeding into shallow water. In this thesis we shall expand upon Bowen's model to include bottom topographical effects and beach curvature.

This thesis is in two parts. Part I deals with developing the theoretical model for a general topography and Part II uses this model to solve for rip currents in the particular case of a circular bay with a conical bottom topography.

We chose to test our model on a circular bay because it seemed the next level of complication above a straight beach as modeled by Bowen (1969b). It also provides insight into the effects of beach curvature on rip currents.

In section (2c) we derive formulae for the location of the shoreline and the breaker line (the line describing the locus of points at which the waves break). While this is not a very sophisticated development, nevertheless I could not find similar calculations in any of the literature which I have read.

CHAPTER 2

THE MODEL

2a The Coastline

In Figure 1 is shown a sketch of a sample beach and a definition of the co-ordinate systems. It is assumed that the beach is smooth. By this it is meant that the radius of curvature of the shoreline is much larger than (at least by a factor of five) the width of the surf zone $(r_s - r_b)$ and that the bottom slope is small.

The breaker line is the locus of points at which the incoming waves first break on the beach. The shoreline is the locus of points of the time averaged (over several wave periods) location of the land-sea boundary.

The origin of the co-ordinate system is determined by the average location of the centers of curvature of the beach within a given region. The radial distances r_s and r_b will in general be functions of θ .

2b Wave Height and Refraction

When uniform deep water waves proceed toward a non-linear shore, we expect the waves to be refracted by the bottom topography and thereby to have variable wave heights as functions of their longshore location. The subject of refraction is discussed in the US Hydrographic Office Publication Number 234. Analytical and graphical means are available for estimating, with reasonable accuracy, the refraction patterns of waves on beaches of any shape. However, in this thesis it is sufficient to make an educated guess at the refraction pattern of a given beach based on the discussions in US Hydrographic Office Publication Number 234.

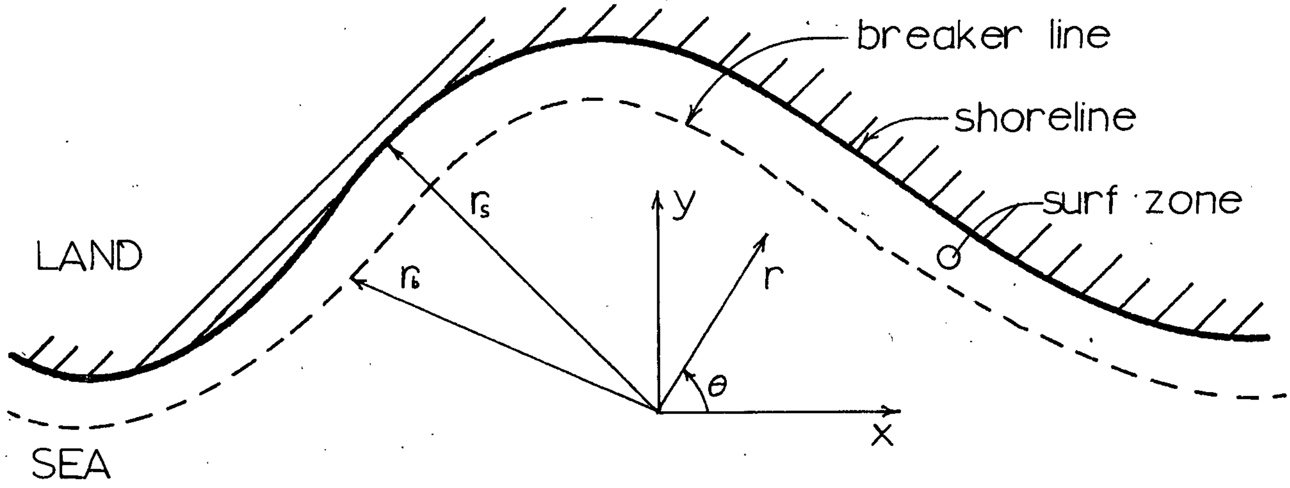

Figure 1

Plan View of Beach

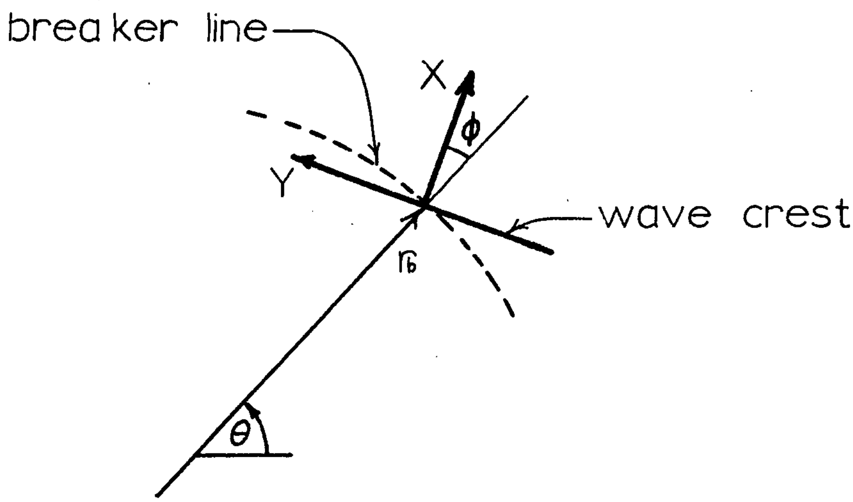

Figure 2

Angle of Refraction

We are interested in determining the wave height at the breaker line Hb^+ as a function of position on the breaker line. We let:

$$Hb^+ = D(1 + pH_1(\theta)) \tag{2.1}$$

where D is the average height along the breaker line, Dp is the amplitude of the varying component of Hb^+ and $H_1(\theta)$ is the θ dependent function of unit amplitude describing the variable part of Hb^+.

In Part II we will consider a circular bay with deep water waves advancing in the positive y co-ordinate direction and we take to represent a reasonable refraction pattern the relation: $H_1 = -\cos 2\theta$ and so:

$$Hb^+ = D(1 - p\cos 2\theta) \tag{2.2}$$

The functions we choose to represent $H_1(\theta)$ and $e(\theta)$ as discussed in the preceding and following paragraphs serve merely as examples of possible incoming waves. If one wished to estimate the rip currents on an actual beach one would first have to measure or otherwise determine the $H_1(\theta)$ and $e(\theta)$ function applicable to that particular beach and wave approach.

We are also interested in the angle of incidence of the wave with the breaker line. To deal with this analytically we define a co-ordinate system as shown in Figure 2. The X axis is in the direction of the phase velocity of the refracted wave at the breaker line. The angle of incidence ϕ is the angle between the X axis and the radius vector rb. The Y axis is parallel to the wave crests. We let:

$$\phi = \phi_0 e(\theta) \tag{2.3}$$

where ϕ_0 is the amplitude and $e(\theta)$ is the variable function of order unity expressing the θ dependence of ϕ. In our circular beach problem we will let:

$$\phi = \phi_0 \cos\theta \tag{2.4}$$

On a large smooth beach we expect the incoming waves to have had sufficient time to adjust to the bottom topography such that ϕ_0 will be at most 15°. Thus we make the approximation:

$$\begin{aligned} \sin\phi &\cong \phi \\ \cos\phi &\cong 1 \end{aligned} \tag{2.5}$$

2c Surf Zone Geometry

In this section we discuss the effects of wave height H on the shoreline. To understand the following discussion, it is necessary to understand the theory of set-down and set-up as described by Longuet-Higgins and Stewart (1964).

Figure 3 shows a cross section normal to a beach. In general the r co-ordinate will not be normal to the beach everywhere, but for the purposes of this discussion we assume this to be the case and this saves us introducing a new co-ordinate system which would, in general, have components normal and parallel to the beach. The bottom slope m_b is assumed constant and the set-up theory indicates that the surface slope m_η is also constant for a given bottom slope and wave train (see equation (A7)). In the following discussion we shall derive relationships between the surf zone measurements r_{bo}, r_{os}, r_{sb}, $\bar{\eta}$ (see Figure 3) and the height at the breaker line of the incoming waves H_b^+.

From Figure 3 we see that:

$$m_b r_{os} = m_\eta r_{sb} - \bar{\eta}_b \tag{2.6}$$

the run-up distance r_{os} is thus:

$$r_{os} = \frac{1}{m_b}\left(m_\eta - \frac{\bar{\eta}_b}{r_{sb}}\right) r_{sb} \tag{2.7}$$

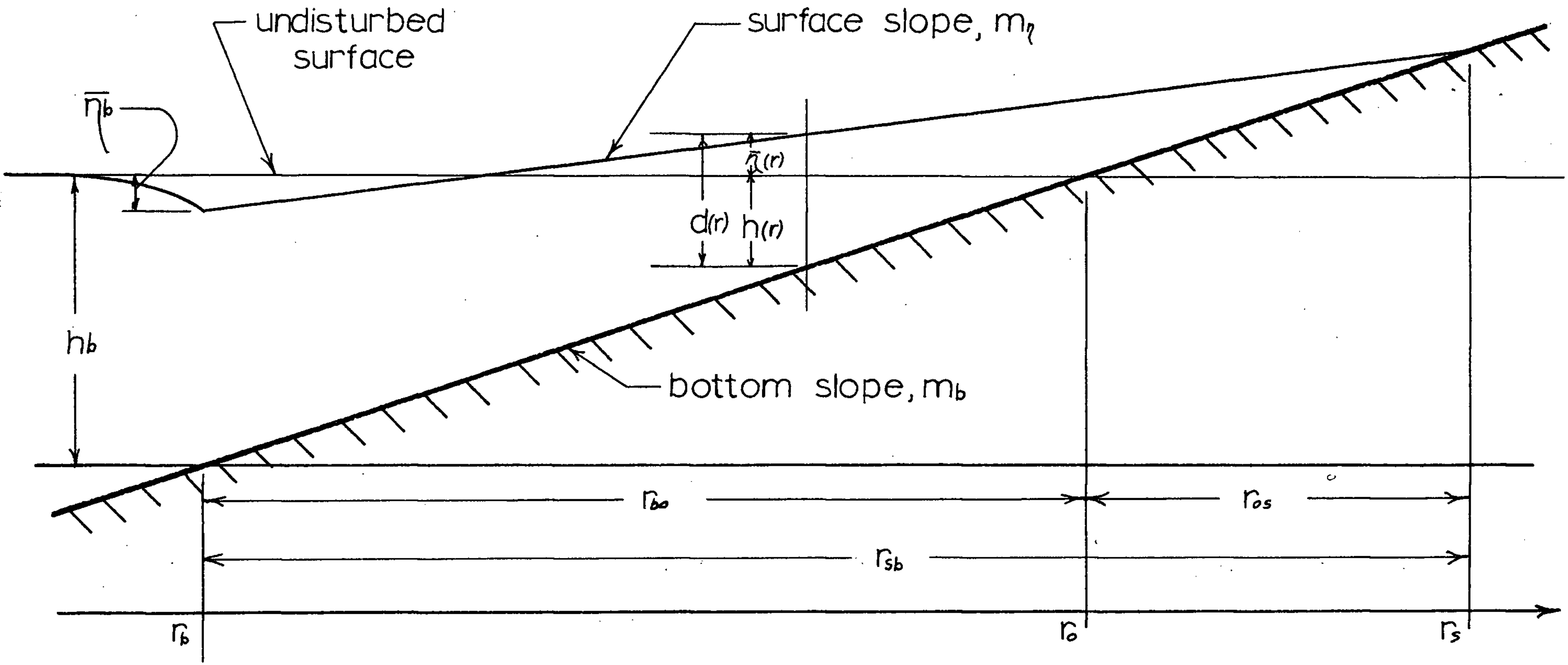

Figure 3

Profile Normal to Beach

Now from (A4) we obtain:

$$\bar{\eta}_b = -\bar{\eta}(r_b) = \frac{1}{8}\frac{H_b^{+2} q}{\sinh 2qh} \tag{2.8}$$

where q is the wave number of the incoming waves.

We denote by H_b^+ and H_b^- respectively the wave height at $r=r_b$ just outside and just inside the surf zone where:

$$H_b^+ = \lambda d_b \quad , \quad H_b^- = \gamma d_b \tag{2.9}$$

or

$$H_b^+/H_b^- = \lambda/\gamma$$

where λ and γ are constants of order unity. Thus from (2.8) and in shallow water we obtain for the set-down at the breaker line:

$$\bar{\eta}_b \cong \frac{1}{16}\frac{H_b^{+2}}{h_b} \tag{2.10}$$

Again from Figure 3 we have $d_b = m\, r_{sb}$ where $m = m_b - m_\eta$, then from (2.9) we obtain:

$$H_b^+ = \lambda\, m\, r_{sb} \tag{2.11}$$

or, for the total width of the surf zone, including run-up, we have:

$$r_{sb} = H_b^+/\lambda m$$

Also we have for the undisturbed depth at the breaker line:

$$h_b = m_b\, r_{bo} \tag{2.12}$$

Thus using (2.12), (2.11) in (2.10), we may write the set-down $\bar{\eta}_b$ as:

$$\bar{\eta}_b = \frac{1}{16}\frac{\lambda^2 m^2}{m_b}\left(\frac{r_{sb}}{r_{bo}}\right) r_{sb} \tag{2.13}$$

or, for short,

$$\bar{\eta}_b = N\, r_{sb} \tag{2.14}$$

where

$$N = \frac{\bar{\eta}_b}{r_{sb}} = \frac{1}{16}\frac{\lambda^2 m^2}{m_b}\frac{r_{sb}}{r_{bo}} \tag{2.15}$$

and

$$N \ll 1$$

Thus (2.7) becomes:

$$r_{os} = \frac{1}{m_b}(m_\eta - N)\, r_{bs} \tag{2.16}$$

From Figure 3 we get for the distance r_{b_0} from breaker line to the undisturbed shoreline:

$$r_{b_0} = r_{sb} - r_{os} = r_{sb} - \frac{1}{m_b}(m_\eta - N)\, r_{sb}$$

or

$$\frac{r_{b_0}}{r_{sb}} = \left(1 - \frac{m_\eta - N}{m_b}\right) = \frac{m+N}{m_b} \tag{2.17}$$

Thus from (2.15) and (2.17) we obtain:

$$N = \frac{1}{16}\,\frac{\lambda^2 m^2}{m_b}\left(\frac{m_b}{m+N}\right) = \frac{1}{16}\,\frac{\lambda^2 m^2}{m+N}$$

or

$$N^2 + mN - \frac{1}{16}\lambda^2 m^2 = 0$$

which yields: $$N = -\frac{m}{2} \pm \frac{m}{2}\left(1 + \frac{\lambda^2}{4}\right)^{\frac{1}{2}}$$

N must be positive and using the binomial expansion we obtain:

$$N = \frac{m\lambda^2}{16}\left(1 - \frac{1}{16} + \cdots\cdots\right) \cong \frac{m\lambda^2}{16} \tag{2.18}$$

Thus N is a constant. From Figure 3 we see that:

$$\bar{\eta} = m_\eta(r - r_b) - \bar{\eta}_b = m_\eta(r_0 - r_b) + m_\eta(r - r_0) - \bar{\eta}_b \tag{2.19}$$

Using (2.17) and (2.14) on (2.19) we obtain:

$$\bar{\eta} = m_\eta\left(\frac{m+N}{m_b}\right) r_{sb} - N r_{sb} + m_\eta(r - r_0)$$

or

$$\bar{\eta} = \frac{m}{m_b}(m_\eta - N)\, r_{sb} + m_\eta(r - r_0) \tag{2.20}$$

From (2.11) and (2.20) we obtain:

$$\bar{\eta} = \left(\frac{m_\eta - N}{\lambda m_b}\right) H_b^+ + m_\eta(r - r_0) = A H_b^+ + m_\eta(r - r_0) \tag{2.21}$$

$$A = (m_\eta - N)/\lambda m_b$$

Thus, according to our present model, the set-up is determined by the breaker height and the slope of the bottom (note that $m_\eta = K m_b$; see Appendix A, equation (A7)). Also, from (2.17) and (2.11) we obtain:

$$r_{bo} = \left(\frac{m+N}{\lambda m_b m} \right) H_b^+ \cong \frac{H_b^+}{\lambda m_b} \qquad (2.22)$$

assuming $\quad \frac{N}{m} = \frac{\lambda^2}{16} << 1$

From (2.16):

$$r_{os} = \left(\frac{m_\eta - N}{\lambda m m_b} \right) H_b^+ \cong \frac{m_\eta}{\lambda m m_b} H_b^+ \qquad (2.23)$$

Thus, we see that the distance of run-up on the beach r_{os} and the position of the breakers r_{bo} are dependent on the bottom slope and the breaker height.

In order to gain some familiarity with the magnitude and accuracy of these equations, we compare calculated results based on these equations with experimental results taken from Bowen, Inman and Simmons (1968). These results are shown in the following table.

				measured			calculated						
EXP	H_b^+ cm	h_b cm	K	r_{sb} cm	η_b cm	η_{max} cm	m_η	r_{ob} cm	r_{os} cm	r_{sb} cm	η_b cm	η_{max} cm	λ
71/3	4.40	4.15	0.27	75	0.17	1.48	.022	52	19	71	0.30	1.56	1.06
51/4	6.60	5.0	0.32	85	0.19	2.07	.026	61	19	80	0.50	1.56	1.32
35/7	7.75	5.9	0.39	110	0.18	3.37	.032	72	28	100	0.54	2.3	1.31
35/15	13.0	9.7	0.37	165	0.43	4.65	.030	120	44	164	0.96	3.6	1.34

where m_b = 0.082

Note that we are able to predict r_{sb} quite accurately. Our prediction of $\bar{\eta}_{max}$ is within 30% of the measured value and $\bar{\eta}_b$ is almost 100% too large.

Bowen, Inman and Simmons (1968) explain this discrepancy by noting that near the break point the wave form is not sinusoidal and therefore, our set-down theory is not reliable at this point, since we assume a sinusoidal wave form.

CHAPTER 3

THE EQUATIONS OF MOTION

In the following discussion the cartesian tensor notation will be used. The equations will be kept general so that they will apply to a general coastline without restriction to a circular bay configuration. However, we shall assume that the radius of curvature of the coastline is always large; that is, at least five times greater than the width of the surf zone.

In deriving the relevant equations of motion we use the equations of conservation of mass and horizontal momentum as given by Phillips (1966), and we assume the time averaged motion to take place only in the horizontal plane. These equations are respectively:

$$\frac{\partial}{\partial t}\left[\rho(\bar{\eta}+h)\right] + \frac{\partial}{\partial x_\alpha}\tilde{M}_\alpha = 0 \tag{3.1}$$

and

$$\frac{\partial}{\partial t}\tilde{M}_\alpha + \frac{\partial}{\partial x_\beta}\left(\tilde{U}_\alpha \tilde{M}_\beta + S_{\alpha\beta}\right) = T_\alpha \tag{3.2}$$

where $T_\alpha = -\rho g d \dfrac{\partial \bar{\eta}}{\partial x_\alpha}$ and $\tilde{M}_\alpha = \rho(\bar{\eta}+h)\tilde{U}_\alpha$

We assume that the short-term time-averaged variables are steady, so we obtain:

$$\frac{\partial}{\partial x_\alpha} M_\alpha = 0 \tag{3.3}$$

and

$$\frac{\partial}{\partial x_\beta}\left(\tilde{U}_\alpha \tilde{M}_\beta + S_{\alpha\beta}\right) = T_\alpha \tag{3.4}$$

By expanding (3.4), using (3.3) and dividing by $\rho d = \rho(\bar{\eta}+h)$ we obtain:

$$\tilde{U}_\beta \frac{\partial}{\partial x_\beta}\tilde{U}_\alpha = -g\frac{\partial}{\partial x_\alpha}\bar{\eta} - \frac{1}{\rho d}\frac{\partial S_{\alpha\beta}}{\partial x_\beta} \tag{3.5}$$

Writing (3.5) in cartesian co-ordinates with $\tilde{U}_1 = u$ and $\tilde{U}_2 = v$ where u and v are the net velocities in the x and y directions respectively, we obtain:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \bar{\eta}}{\partial x} + \tau_x \tag{3.6}$$

and

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial \bar{\eta}}{\partial y} + \tau_y \tag{3.7}$$

where $$\tau_x = -\frac{1}{\rho d}\left(\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y}\right) \tag{3.8}$$

and

$$\tau_y = -\frac{1}{\rho d}\left(\frac{\partial S_{yx}}{\partial x} + \frac{\partial S_{yy}}{\partial y}\right) \tag{3.9}$$

Equations (3.6) and (3.7) were derived for an inviscid fluid. For a viscid fluid we add the extra friction terms R_x, R_y . The friction force should be composed of a bottom friction term and an eddy viscosity term. In this paper, to simplify our equations we use only the bottom friction term and assume that the eddy viscosity is negligible. That this is not the case for the entire region over which we are seeking a solution will be seen later; nevertheless, we shall still obtain useful qualitative results using this assumption.

The bottom friction force acting on a total column of water of unit area and height d with mean velocity $\tilde{U}_\alpha$ we will assume to be a linear function of the mean velocity. Thus the bottom friction force on a column per unit mass of the column is expressed as:

$$R_x = -\frac{cu}{d}, \quad R_y = -\frac{cv}{d}$$

where c is a proportionality constant with units of length over time. In polar co-ordinates we obtain:

$$R_r = -\frac{c\,v_r}{d} \quad , \quad R_\theta = -\frac{c\,v_\theta}{d} \tag{3.10}$$

Thus the complete equations of motion in cartesian co-ordinates are:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \bar{\eta}}{\partial x} + \mathcal{T}_x + R_x \tag{3.11}$$

and $$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial \bar{\eta}}{\partial y} + \mathcal{T}_y + R_y \tag{3.12}$$

In cartesian tensor notation these are:

$$u_\alpha\frac{\partial u_\beta}{\partial x_\alpha} = -g\frac{\partial \bar{\eta}}{\partial x_\beta} + \mathcal{T}_\beta + R_\beta \tag{3.13}$$

and in vector notation

$$(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u} = -g\nabla\bar{\eta} + \underset{\sim}{R} + \underset{\sim}{\mathcal{T}} \tag{3.14}$$

We may eliminate $\bar{\eta}$ by taking the curl of (3.14) which gives:

$$\mathrm{curl}\,(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u} = \mathrm{curl}\,\underset{\sim}{R} + \mathrm{curl}\,\underset{\sim}{\mathcal{T}} \tag{3.15}$$

Transforming (3.15) into polar co-ordinates gives:

$$\frac{\partial I_\theta}{\partial r} - \frac{\partial I_r}{r\partial\theta} + \frac{I_\theta}{r} = \frac{\partial R_\theta}{\partial r} - \frac{\partial R_r}{r\partial\theta} + \frac{R_\theta}{r} + \frac{\partial \mathcal{T}_\theta}{\partial r} - \frac{\partial \mathcal{T}_r}{r\partial\theta} + \frac{\mathcal{T}_\theta}{r} \tag{3.16}$$

where the inertial terms are:

$$I_r = v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} \tag{3.17}$$

$$I_\theta = v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} \tag{3.18}$$

and subscripts r and θ indicate components in the direction of r and θ. Equations (3.17) and (3.18) are nonlinear. In his theoretical paper on rip currents, Bowen (1969b) neglects these nonlinear terms in order to obtain a simple analytic solution.

Arthur (1962) shows that the contribution of these nonlinear terms is to narrow the currents proceeding to deeper water and to widen the currents proceeding into shallow water. Bowen (1969b) solved this nonlinear equation for a linear beach using a computer and the significant difference between his computer solution and his simplified linear analytic solution was this narrowing effect. Therefore, we expect to obtain reasonably good qualitative results by neglecting the nonlinear terms and equation (3.16) becomes:

$$-\,\mathrm{curl}_z\,\underset{\sim}{R} = \mathrm{curl}_z\,\underset{\sim}{\Upsilon} \tag{3.19}$$

or

$$\frac{\partial R_r}{r\partial\theta} - \frac{\partial R_\theta}{\partial r} - \frac{R_\theta}{r} = \frac{\partial \Upsilon_\theta}{\partial r} - \frac{\partial \Upsilon_r}{r\partial\theta} + \frac{\Upsilon_\theta}{r} \tag{3.20}$$

We call $\mathrm{curl}_z\,\underset{\sim}{\Upsilon}$ the forcing function. In Part II, we shall apply equation (3.19) to the case of a circular bay and obtain an analytical solution, however it is important to note that (3.19) is quite general and does not only apply to a circular beach.

From (3.3) we get:

$$\frac{1}{\rho d}\frac{\partial \widetilde{M}_\alpha}{\partial x_\alpha} = 0 \tag{3.21}$$

or using the definition $\widetilde{U}_\alpha = \dfrac{\widetilde{M}_\alpha}{\rho d}$ we obtain:

$$\mathrm{Div}\left(d\,\widetilde{U}_\alpha\right) = 0 \tag{3.22}$$

which is the integrated continuity equation. We introduce a transport stream function ψ (Arthur, 1962) in polar co-ordinates which identically satisfies (3.22). We let:

$$V_r = -\frac{1}{d}\frac{\partial \psi}{r\partial \theta} \tag{3.23}$$

$$V_\theta = \frac{1}{d}\frac{\partial \psi}{\partial r} \tag{3.24}$$

where $d = \bar{\eta} + h$ as previously defined.

CHAPTER 4

THE FRICTION TERMS

It is convenient to speak of two zones: the surf zone $(r_b < r < r_s\,,\ 0 < \theta < \pi)$ and the area outside the surf zone. We shall see in the following discussions that there are specific differences between these two zones.

4a The Surf Zone

Let us examine the individual terms of $\mathrm{curl}_z\, \underset{\sim}{R} = \frac{\partial R_\theta}{\partial r} - \frac{\partial R_r}{r\partial\theta} + \frac{R_\theta}{r}$. We wish to examine the relative sizes of these terms so we introduce the following ordering scheme. Let the primes denote non-dimensional variables of order unity.

$$
\begin{aligned}
v_\theta &= V_b\, v_\theta' & \partial\theta &= \tfrac{\pi}{2}\,\partial\theta' \\
v_r &= V_b\, v_r' & r &= r_0\, r' \\
\partial r &= \overline{r_{sb}}\, \partial r' & \partial d(r) &= \overline{h_b}\, \partial d'(r) \\
d(r) &= \overline{h_b}\, d'(r) & \partial d(\theta) &= \eta_1\, \delta d'(\theta)
\end{aligned}
\qquad (4.1)
$$

where V_b is the maximum expected velocity. It may be of the order of 2 to 4 knots for a linear beach (Shepard and Inman 1950). We introduce two differentials for d representing its changes in the r direction which over the width of the surf zone will be of order $\overline{h_b}$ and its change in the θ direction which will be mostly due to changes in $\bar{\eta}$ in the θ direction and are of order $\eta_1 = AD_p$ (see (2.21) and (2.1)). $\overline{r_{sb}}$ is the average width of the surf zone and $\overline{h_b}$ is the average still water depth at the breaker line.

From (3.10) we obtain:

$$\frac{\partial R_\theta}{\partial r} = -\frac{c}{d}\frac{\partial v_\theta}{\partial r} + \frac{c}{d^2} v_\theta \frac{\partial d}{\partial r} \tag{4.2}$$

Using (4.1) we obtain:

$$\frac{\partial R_\theta}{\partial r} = -\frac{c}{\overline{h_b}}\frac{V_b}{\overline{r_{sb}}}\left(\frac{\partial v_\theta'}{\partial r'} - \frac{v_\theta'}{d'^2}\frac{\partial d'}{\partial r'}\right) \tag{4.3}$$

Using (3.10) again, we obtain:

$$\frac{\partial R_r}{r\partial\theta} = -\frac{c}{d}\frac{\partial v_r}{r\partial\theta} + \frac{c}{d^2} v_r \frac{\partial d}{r\partial\theta} \tag{4.4}$$

and using (4.1) we obtain:

$$\frac{\partial R_r}{r\partial\theta} = -\frac{2cV_b}{\overline{h_b}d' r_o\pi}\left[\frac{\partial v_r'}{\partial\theta'} - \frac{\eta_1}{\overline{h_b}}\frac{v_r'}{d'}\frac{\partial d'}{\partial\theta'}\right] \tag{4.5}$$

The terms in brackets in (4.3) and (4.5) are of order unity. Thus to compare $\partial R_r/r\partial\theta$ to $\partial R_\theta/\partial r$ we take the ratio:

$$\frac{\partial R_r}{r\partial\theta}\Big/\frac{\partial R_\theta}{\partial r} \cong -\frac{2cV_b}{\overline{h_b}r_o\pi}\Big/-\frac{cV_b}{\overline{h_b}\,\overline{r_{sb}}} = \frac{2}{\pi}\frac{\overline{r_{sb}}}{r_o} \tag{4.6}$$

We assume we have a large bay and so $\frac{2}{\pi}\overline{r_{sb}}/r_o \ll 1$.
Thus we can neglect the $\partial R_r/r\partial\theta$ term relative to the $\partial R_\theta/\partial r$ term.

Again using (3.10) we obtain:

$$\frac{R_\theta}{r} = - \frac{c v_\theta}{r d} \tag{4.7}$$

and using (4.1) we get:

$$\frac{R_\theta}{r} = - \frac{c V_b}{r_0 \bar{h}_b} \left(\frac{v_\theta'}{r' d'} \right) \tag{4.8}$$

Comparing $\frac{R_\theta}{r}$ with $\frac{\partial R_\theta}{\partial r}$ we obtain:

$$\frac{R_\theta}{r} \Big/ \frac{\partial R_\theta}{\partial r} = -\frac{c V_b}{r_0 \bar{h}_b} \Big/ -\frac{c V_b}{\bar{h}_b \bar{r}_{sb}} = \frac{\bar{r}_{sb}}{r_0} \ll 1 \tag{4.9}$$

for a large bay. Hence we can neglect the $\frac{R_\theta}{r}$ term relative to $\frac{\partial R_\theta}{\partial r}$. Therefore in a large bay and inside the surf zone region we have:

$$\text{curl}_z \underset{\sim}{R} \cong \frac{\partial R_\theta}{\partial r} \tag{4.10}$$

Using (3.10) and the transport stream function defined by (3.23) and (3.24) we obtain for (4.10):

$$\text{curl}_z \underset{\sim}{R} = - \frac{c}{d^2} \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{2m}{d} \frac{\partial \psi}{\partial r} \right) \tag{4.11}$$

where we define: $\frac{\partial d}{\partial r} = -m$ inside the surf zone. (4.12)

Later we shall use (4.11) in the left hand side of the main equation (3.19) for solving for the current inside the surf zone only.

4b Outside the Surf Zone

In this region we cannot in general neglect the same terms of $\mathrm{curl}_z \underset{\sim}{R}$ as we did in the surf zone because we are no longer dealing with a narrow zone at a large radius r from the origin. We shall use the full expression for $\mathrm{curl}_z \underset{\sim}{R}$ in terms of the transport stream function Ψ. We obtain:

$$\mathrm{curl}_z \underset{\sim}{R} = \frac{\partial R_\theta}{\partial r} - \frac{\partial R_r}{r\partial\theta} + \frac{R_\theta}{r}$$

$$= -\frac{c}{d^2}\left[\frac{\partial^2\Psi}{\partial r^2} - \frac{2}{d}\frac{\partial d}{\partial r}\frac{\partial\Psi}{\partial r} + \frac{1}{r}\frac{\partial\Psi}{\partial r} - \frac{2}{r^2 d}\frac{\partial\Psi}{\partial\theta}\frac{\partial d}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}\right] \tag{4.13}$$

where
$$\frac{\partial R_\theta}{\partial r} = -\frac{c}{d^2}\left(\frac{\partial^2\Psi}{\partial r^2} - \frac{2}{d}\frac{\partial d}{\partial r}\frac{\partial\Psi}{\partial r}\right)$$

$$\frac{\partial R_r}{r\partial\theta} = \frac{c}{d^2 r^2}\left(\frac{\partial^2\Psi}{\partial\theta^2} - \frac{2}{d}\frac{\partial\Psi}{\partial\theta}\frac{\partial d}{\partial\theta}\right) \tag{4.14}$$

$$\frac{R_\theta}{r} = -\frac{c}{r d^2}\frac{\partial\Psi}{\partial r}$$

Equation (4.13) can be simplified. We assume that outside the surf zone the depth d of the mean current is equal to $\bar{\eta} + h_b$. Thus we say that the mean current does not extend below the depth h_b which is the depth of the undisturbed water at the breaker line. Shepard and Inman (1950) found that rip currents seaward of the breakers do not extend to the bottom. Although their results do not entirely support our assumption they do point towards that direction. This assumption should be regarded as a crude simplifying one. Thus we get $\partial d/\partial r \cong 0$.

We shall also neglect the term containing $\partial d/\partial \theta$. To justify this we shall compare the $\frac{2}{r^2 d}\frac{\partial \psi}{\partial \theta}\frac{\partial d}{\partial \theta}$ and $\frac{\partial^2 \psi}{r^2 \partial \theta^2}$ terms using the ordering relationships (4.1):

$$\frac{\partial^2 \psi}{\partial \theta^2} = \frac{\partial}{\partial \theta}(r d v_r) = r v_r \frac{\partial d}{\partial \theta} + r d \frac{\partial v_r}{\partial \theta}$$

$$= \frac{r V_b \bar{h}_b}{\pi/2}\left[\frac{\eta_1}{\bar{h}_b} v_r' \frac{\partial d'}{\partial \theta'} + d' \frac{\partial v_r'}{\partial \theta'}\right] \qquad (4.15)$$

also

$$\frac{2}{d}\frac{\partial \psi}{\partial \theta}\frac{\partial d}{\partial \theta} = -2 r v_r \frac{\partial d}{\partial \theta} = -\frac{2 r V_b \bar{h}_b}{\pi/2}\frac{\eta_1}{\bar{h}_b} v_r' \frac{\partial d'}{\partial \theta'} \qquad (4.16)$$

Thus:

$$\frac{\partial^2 \psi/\partial \theta^2}{\frac{2}{d}\frac{\partial \psi}{\partial \theta}\frac{\partial d}{\partial \theta}} = \left(\frac{r V_b \bar{h}_b}{\pi/2} \Big/ \frac{2 r V_b \bar{h}_b}{\pi/2}\right)\left(\frac{\frac{\eta_1}{\bar{h}_b} v_r' \frac{\partial d'}{\partial \theta'} + d' \frac{\partial v_r'}{\partial \theta'}}{\frac{\eta_1}{\bar{h}_b} v_r' \frac{\partial d'}{\partial \theta'}}\right) \qquad (4.17)$$

$$= \frac{1}{2}\left(1 + \frac{\bar{h}_b}{\eta_1}\frac{d'}{v_r'}\frac{\partial v_r'/\partial \theta'}{\partial d'/\partial \theta'}\right)$$

The primed terms we assume are of order unity. The ratio $\frac{\overline{h_b}}{\eta_1} >> 1$ and so the term containing $\partial d/\partial\theta$ will be insignificant relative to the $\partial^2\psi/\partial\theta^2$ term. We therefore neglect the $\partial d/\partial\theta$ term. Thus equation (4.13) becomes:

$$\operatorname{curl}_z \underset{\sim}{R} \cong -\frac{c}{d^2}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right) \qquad (4.18)$$

To conclude our discussion of the friction terms, we now have two forms for the expression $\operatorname{curl}_z \underset{\sim}{R}$ of equation (3.19). These are as follows:

$$\operatorname{curl}_z \underset{\sim}{R} = -\frac{c}{d^2}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{2m}{d}\frac{\partial\psi}{\partial r}\right) \qquad (4.19)$$

inside the surf zone and

$$\operatorname{curl}_z \underset{\sim}{R} = -\frac{c}{d^2}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right) \qquad (4.20)$$

outside the surf zone. We shall now discuss the radiation stress terms so that we may put equation (3.19) into a workable form and solve for ψ .

CHAPTER 5

THE FORCING TERMS

In this chapter we shall discuss the $curl_z \, \underset{\sim}{T}$ terms in equation (3.19) and as defined in equations (3.8) and (3.9). We call these terms forcing terms because they represent the driving forces, due to the excess momentum flux of the waves, which cause the rip currents.

5a Outside Surf Zone

From equation (3.4) we get:

$$\frac{\partial}{\partial x_\beta}\left(\tilde{\mathcal{U}}_1 \tilde{M}_\beta\right) + \frac{\partial S_{1\beta}}{\partial x_\beta} = T_1 = -\rho g d \frac{\partial \bar{\eta}}{\partial x} \tag{5.1}$$

For a linear beach on which there is no variation in the longshore mass flux (*i.e.* $\frac{\partial \tilde{M}_y}{\partial y} = 0$) we have from (3.3) $\frac{\partial M_x}{\partial x} = 0$. But $\tilde{M}_x$ must be zero at the shore and hence $\tilde{M}_x = 0$, thus $\tilde{\mathcal{U}}_x = \tilde{\mathcal{U}}_1 = 0$ and (5.1) becomes:

$$\frac{\partial S_{xx}}{\partial x} + \rho g d \frac{\partial \bar{\eta}}{\partial x} = 0 \tag{5.2}$$

since $S_{xy} = 0$ for a plane wave traveling in the positive x direction. From (5.2) we obtain the relation:

$$\bar{\eta} = -\frac{1}{8}\frac{H^2 g}{\sinh 2gh} \tag{5.3}$$

where H is the wave height and g is the wave number of the advancing wave (Longuet-Higgins and Stewart (1964)). Bowen (1969b) used this derivation for $\bar{\eta}$ to show that the gradient of the radiation stress is balanced by the induced pressure field and that there are no net forces outside the surf zone that might produce circulation patterns.

However, (5.3) is based on the assumption that $\frac{\partial \tilde{M}_y}{\partial y} = 0$ which is not in general true for a rip current system. The discussion in Appendix A shows that the inertial terms may be neglected in equation (5.1) for a rip current system, thereby making (5.2) and (5.3) reasonably accurate where rip currents are present. Using the y-component of the radiation stress (Longuet-Higgins and Stewart (1964)) where

$$S_{yy} = \frac{1}{8} \rho g H^2 \frac{gh}{\sinh 2gh} \tag{5.4}$$

and equation (5.3) we get:

$$\rho g d \frac{\partial \bar{\eta}}{\partial y} + \frac{\partial S_{yy}}{\partial y} = 0 \tag{5.5}$$

which confirms Bowen's finding as stated above providing (5.3) can be used for the rip current system. For the moment, using the X, Y co-ordinate system shown in Figure 2, equations (5.3) and (5.5) become:

$$\frac{\partial S_{XX}}{\partial X} + \rho g d \frac{\partial \bar{\eta}}{\partial X} = 0 \tag{5.6}$$

$$\rho g d \frac{\partial \bar{\eta}}{\partial Y} + \frac{\partial S_{YY}}{\partial Y} = 0 \tag{5.7}$$

respectively. Note that $S_{YX} = S_{XY} = 0$ (Longuet-Higgins and Stewart, (1964)). Substituting these relations into equations (3.11) and (3.12) where x and y are replaced by X and Y respectively, we obtain:

$$u_X \frac{\partial u_X}{\partial X} + u_Y \frac{\partial u_X}{\partial Y} = R_X \tag{5.8}$$

$$u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = R_y \tag{5.9}$$

or in tensor notation using X_α as a generalized co-ordinate:

$$u_\alpha \frac{\partial u_\beta}{\partial X_\beta} = R_\beta \tag{5.10}$$

Taking the curl of (5.10) we obtain:

$$\mathrm{curl}_z \underset{\sim}{R} = \mathrm{curl}_z \underset{\sim}{I} \tag{5.11}$$

where $\underset{\sim}{I}$ was defined in polar co-ordinates by (3.17) and (3.18). (The $\sim$ under a symbol indicates a vector quantity). However, since we neglect the nonlinear terms $\underset{\sim}{I}$ we obtain:

$$\mathrm{curl}_z \underset{\sim}{R} = 0 \tag{5.12}$$

outside the surf zone where the gradient of the radiation stress balances the pressure gradient.

5b Inside Surf Zone

Bowen (1969) assumed that inside the surf zone the height of the broken wave is directly proportional to the mean water depth d.

$$H = \gamma d \tag{5.13}$$

Also in the surf zone the water is shallow and hence from (5.4):

$$S_{yy} = \frac{1}{16} \rho g H^2 \tag{5.14}$$

Using these relationships Bowen (1969b) showed that the longshore component of the radiation stress is not in equilibrium with the pressure component longshore. Thus we have $\text{curl}_z\, \underset{\sim}{R} \neq 0$ inside the surf zone and there are current generating forces inside the surf zone.

5c Radiation Stress Tensor

As the waves approach the bay from deep water, they are refracted and inclined at an angle ϕ to the normal of the beach at the breaker line (Figure 2). If our beach is large, as we have assumed, then the waves should be almost plane. We shall assume this to be the case at the breaker line and inside the surf zone.

Refering to Figure 2 where the X axis is the axis of advance of these plane waves, we have the radiation stress for the X and Y axis given by Longuet-Higgins and Stewart (1964) as:

$$S_{xx} = \tfrac{3}{2}E \,, \quad S_{yy} = \tfrac{1}{2}E$$

$$S_{xy} = S_{yx} = 0$$

where E is the energy density of the waves

$$E = \frac{1}{8}\rho g H^2 \qquad (5.15)$$

or using (5.13) for H inside the surf zone we obtain:

$$S_{xx} = \frac{3}{2} k\rho d^2 \qquad (5.16)$$

$$S_{yy} = k\rho d^2/2 \qquad (5.17)$$

where $k = \frac{1}{8} g\gamma^2$, a constant.

Transforming $S_{\alpha\beta}$ from the X, Y co-ordinates to the r, θ co-ordinates by a rotation of co-ordinates through the angle ϕ we obtain:

$$S_{\alpha\beta} = k\rho d^2 \begin{bmatrix} \dfrac{3-2\sin^2\phi}{2}, & \dfrac{\sin 2\phi}{2} \\ \dfrac{\sin 2\phi}{2} & , \dfrac{3-2\cos^2\phi}{2} \end{bmatrix} \tag{5.18}$$

or

$$S_{\alpha\beta} = k\rho d^2 \, \Phi_{\alpha\beta} \tag{5.19}$$

inside the surf zone. $\Phi_{\alpha\beta}$ is the tensor shown inside the square brackets in (5.18). Here the indices α and β each take on the values 1 and 2. We assume that d and ϕ are known variables for a given beach with a particular wave system. They can be measured, calculated or assumed. (see Breakers and Surf, HO Pub. 234).

5d Forcing Terms (Inside Surf Zone)

Since we shall be using polar co-ordinates we use the identity:

$$\text{curl}_z \, \underset{\sim}{\Upsilon} = \frac{\partial \Upsilon_\theta}{\partial r} - \frac{\partial \Upsilon_r}{r\partial\theta} + \frac{\Upsilon_\theta}{r} \tag{5.20}$$

where from (3.8) and (3.9) we have:

$$\Upsilon_\alpha = -\frac{1}{\rho d} \frac{\partial S_{\alpha\beta}}{\partial x_\beta} \tag{5.21}$$

in cartesian tensor notation. Prager (1961, p. 37) gives the polar co-ordinate components of the vector $\dfrac{\partial T_{ij}}{\partial x_i}$. Using this relation on (5.21) we obtain:

$$\tau_r = -\frac{1}{\rho d}\left(\frac{\partial S_{rr}}{\partial r} + \frac{\partial S_{\theta r}}{r\partial\theta} + \frac{S_{rr}}{r} - \frac{S_{\theta\theta}}{r}\right) \tag{5.22}$$

and

$$\tau_\theta = -\frac{1}{\rho d}\left(\frac{\partial S_{r\theta}}{\partial r} + \frac{\partial S_{\theta\theta}}{r\partial\theta} + 2\,\frac{S_{r\theta}}{r}\right) \tag{5.23}$$

Now we must determine $S_{\alpha\beta}$ in polar co-ordinates in terms of independent variables which can be somehow measured directly from the surf zone or calculated. These variables are ϕ and H_b^+ . Thus we are now prepared to calculate $curl_z\, \underset{\sim}{\tau}$ in terms of known variables. From (5.23) we obtain:

$$\begin{aligned}
\frac{\partial S_{rr}}{\partial r} &= 2k\rho d\,\Phi_{11}\frac{\partial d}{\partial r}\\
\frac{\partial S_{\theta r}}{r\partial\theta} &= 2k\rho d\,\Phi_{12}\frac{\partial d}{r\partial\theta} + k\rho d^2\frac{\partial\Phi_{12}}{r\partial\theta}\\
\frac{\partial S_{r\theta}}{\partial r} &= 2k\rho d\,\Phi_{12}\frac{\partial d}{\partial r}\\
\frac{\partial S_{\theta\theta}}{r\partial\theta} &= 2k\rho d\,\Phi_{22}\frac{\partial d}{r\partial\theta} + k\rho d^2\frac{\partial\Phi_{22}}{r\partial\theta}
\end{aligned} \tag{5.24}$$

Thus substituting (5.24) into (5.22) and (5.23) we obtain:

$$\tau_r = -2k\Phi_{11}\frac{\partial d}{\partial r} - 2k\Phi_{12}\frac{\partial d}{r\partial\theta} - kd\frac{\partial\Phi_{12}}{r\partial\theta} - \frac{kd}{r}\Phi_{11} + \frac{kd}{r}\Phi_{22} \tag{5.25}$$

$$\tau_\theta = -2k\,\Phi_{12}\frac{\partial d}{\partial r} - 2k\Phi_{22}\frac{\partial d}{r\partial\theta} - 2d\frac{\partial\Phi_{22}}{r\partial\theta} - \frac{2kd}{r}\,\Phi_{12} \tag{5.26}$$

Notice that we have not neglected the $\partial d/\partial\theta$ terms here as we did in discussing the friction terms because they provide significant forcing terms to the overall forcing function $curl_z \underset{\sim}{\tau}$. From (5.25) and (5.26) we obtain:

$$\frac{\partial \tau_r}{r\partial\theta} = -2k\Phi_{11}\frac{\partial^2 d}{r\partial\theta\partial r} - 2k\frac{\partial\Phi_{11}}{r\partial\theta}\frac{\partial d}{\partial r} - 2k\Phi_{12}\frac{\partial^2 d}{r^2\partial\theta^2}$$

$$-2k\frac{\partial\Phi_{12}}{r\partial\theta}\frac{\partial d}{r\partial\theta} - kd\frac{\partial^2\Phi_{12}}{r^2\partial\theta^2} - k\frac{\partial d}{r\partial\theta}\frac{\partial\Phi_{12}}{r\partial\theta} \tag{5.27}$$

$$-\frac{kd}{r^2}\frac{\partial\Phi_{11}}{\partial\theta} - \frac{k\Phi_{11}}{r^2}\frac{\partial d}{\partial\theta} + \frac{k\Phi_{22}}{r^2}\frac{\partial d}{\partial\theta} + \frac{kd}{r^2}\frac{\partial\Phi_{22}}{\partial\theta}$$

and

$$\frac{\partial \tau_\theta}{\partial r} = -2k\Phi_{12}\frac{\partial^2 d}{\partial r^2} + 2k\frac{\Phi_{22}}{r^2}\frac{\partial d}{\partial\theta} + \frac{kd}{r^2}\frac{\partial\Phi_{22}}{\partial\theta}$$

$$-\frac{2k\Phi_{22}}{r}\frac{\partial^2 d}{\partial\theta\partial r} - \frac{k}{r}\frac{\partial d}{\partial r}\frac{\partial\Phi_{22}}{\partial\theta} \tag{5.28}$$

$$+\frac{2kd}{r^2}\Phi_{12} - \frac{2k\Phi_{12}}{r}\frac{\partial d}{\partial r}$$

and

$$\frac{\tau_\theta}{r} = -\frac{2k\Phi_{12}}{r}\frac{\partial d}{\partial r} - \frac{2k\Phi_{22}}{r^2}\frac{\partial d}{\partial\theta}$$

$$-\frac{kd}{r^2}\frac{\partial\Phi_{22}}{\partial\theta} - \frac{2kd}{r^2}\Phi_{12} \tag{5.29}$$

Substituting (5.27), (5.28) and (5.29) into (5.20) and collecting like terms we obtain:

$$
\begin{aligned}
\operatorname{curl}_z \underset{\sim}{T} = {} & \left. \overset{1}{\frac{2k}{r}\left(\bar{\Phi}_{11}-\bar{\Phi}_{22}\right)\frac{\partial^2 d}{\partial\theta\,\partial r}} + \overset{2}{\frac{k}{r^2}\left(\bar{\Phi}_{11}-\bar{\Phi}_{22}\right)\frac{\partial d}{\partial\theta}} \right] \mathrm{I} \\
& \left. \begin{aligned}
& \overset{3}{-\frac{k}{r}\left(\frac{\partial d}{\partial r}+\frac{d}{r}\right)\frac{\partial\bar{\Phi}_{22}}{\partial\theta}} \overset{4}{-2k\,\bar{\Phi}_{12}\frac{\partial^2 d}{\partial r^2}} \\
& \overset{5}{+\frac{k}{r}\left(2\frac{\partial d}{\partial r}+\frac{d}{r}\right)\frac{\partial\bar{\Phi}_{11}}{\partial\theta}} \overset{6}{+\frac{kd}{r^2}\frac{\partial^2\bar{\Phi}_{12}}{\partial\theta^2}} \\
& \overset{7}{-\frac{4k}{r}\bar{\Phi}_{12}\frac{\partial d}{\partial r}}
\end{aligned} \right] \mathrm{II} \\
& \left. \overset{8}{+\frac{2k}{r^2}\bar{\Phi}_{12}\frac{\partial^2 d}{\partial\theta^2}} \overset{9}{+\frac{3k}{r^2}\frac{\partial\bar{\Phi}_{12}}{\partial\theta}\frac{\partial d}{\partial\theta}} \right] \mathrm{III}
\end{aligned}
\tag{5.30}
$$

Notice that equation (5.30) is a general equation regardless of the shape of the bay or its bottom contours. The terms have been numbered from 1 to 9 and grouped into three categories, I, II and III. Category I includes all the terms which would be non-zero if the waves advanced normal to the beach, that is $\phi = 0$. Category II includes all the terms which would be non-zero for a wave system in which ϕ will not be zero everywhere and the mean water depth d is not a function of θ. That is, these terms represent forcing functions due only to the non-zero angle.

of incidence ϕ of the wave system. Category III is the left over terms which are non-zero when Categories I and II terms are both non-zero.

We can calculate the mean water depth d in terms of the breaker wave height $Hb^{+}(\theta)$ as discussed in section 2c. We obtain from equation (2.21) the following:

$$d = \bar{\eta} + h = A\,Hb^{+}(\theta) + m_{\eta}(r - r_0) + m_b(r_0 - r) \tag{5.31}$$

where A is a constant; m_{η} and m_b are the mean slopes of the water surface and beach bottom respectively. Thus we obtain from (5.31):

$$\frac{dd}{dr} = -(m_b - m_{\eta}) = -m \tag{5.32}$$

and

$$\frac{dd}{d\theta} = A\frac{dHb^{+}}{d\theta} + (r_0 - r)\frac{dm}{d\theta} \tag{5.33}$$

Also, inside the surf zone:

$$\frac{dd}{dr} \gg \frac{d}{r} \tag{5.34}$$

and if we assume a circular bay, we have:

$$\frac{\partial m}{\partial \theta} = 0 \tag{5.35}$$

Using (5.35), (5.34) and (5.18) in (5.30) we obtain:

$$
\begin{aligned}
\operatorname{curl}_z \underset{\sim}{\tau} = {} & \mathrm{I} \left[\overset{(a)}{\frac{2k}{r} \cos 2\phi \frac{\partial^2 d}{\partial \theta \partial r}} + \overset{(b)}{\frac{kA}{r^2} \cos 2\phi \frac{\partial H b^+}{\partial \theta}} \right] \\
& \mathrm{II} \left[\overset{(c)}{+\frac{3km}{r} \sin 2\phi \frac{\partial \phi}{\partial \theta}} \overset{(d)}{- k \sin 2\phi \frac{\partial^2 d}{\partial r^2}} \right. \\
& \left. \quad + \frac{kd}{r^2} \left[\overset{(e)}{\cos 2\phi \frac{\partial^2 \phi}{\partial \theta^2}} \overset{(f)}{- 2 \sin 2\phi \left(\frac{\partial \phi}{\partial \theta} \right)^2} \right] \overset{(g)}{+ \frac{2km}{r} \sin 2\phi} \right] \\
& \mathrm{III} \left[\overset{(h)}{+\frac{kA}{r^2} \sin 2\phi \frac{\partial^2 H b^+}{\partial \theta^2}} \overset{(i)}{+ \frac{3kA}{r^2} \cos 2\phi \frac{\partial \phi}{\partial \theta} \frac{\partial H b^+}{\partial \theta}} \right]
\end{aligned}
\tag{5.36}
$$

where $k = \frac{1}{8} g \gamma^2$, a constant.

5e Discussion of Forcing Terms for a Circular Bay

Equations (4.19) and (4.20) are both linear in ψ. Thus we may take each term of $\operatorname{curl}_z \underset{\sim}{\tau}$ and solve for its corresponding value of ψ and add the solutions to obtain the total solution for ψ. We can regard each term in (5.30) or (5.36) as a forcing function for its own rip current system. It is of interest to examine each of these terms and interpret physically what causes them to exist.

We first describe the relative magnitudes of the terms. We do this by ordering each of the terms and comparing them. The exact calculations for (5.36) are given in Appendix B. The only difference between (5.36) and (5.30) is that we assumed $m \neq m(\theta)$ for (5.36) which gives us a circular bay. For a non-circular shoreline (a straight beach is considered circular with $r \rightarrow \infty$) the effect of $m(\theta)$ may be large; however, in this thesis we consider only the next level of complication above a straight beach and that is a circular beach. Thus we order the terms for a circular bay. For a circular bay we assume the independent variables ϕ and Hb^{+} to change significantly over the angle $\theta = \pi/2$ and we use this in our ordering scheme in Appendix B. The results of this ordering process are given below.

The terms are arranged in descending order of magnitude. Beside each pair of terms is the multiplication factor relating the two terms. For example: $\left.\begin{matrix} f \\ b \end{matrix}\right\} -4$ means term $f \cong -4 \times$ term b .

Term	Factor
d	
	$\frac{1}{2} r_0/r_{sb}$
g	
	4
c	
	$2\, r_0/r_{sb}$
a + e	
	−4
f	
	−4
b	
	2
i	
	3/2
h	

(5.37)

We shall discuss each term in descending order of magnitude. Term d, the largest term, exists if $\phi \neq 0$ and the bottom is nonlinear in the r direction. We expect that on most sandy beaches the bottom will be almost linear and hence term d will be small on such a beach. We shall neglect this term in our circular beach problem by assuming the bottom to be linear.

Term g then will be the largest term for a beach with a linear bottom. We see that term g vanishes for a straight coastline $(r \rightarrow \infty)$. For a curved coastline, term g exists only if $\phi \neq 0$ and there exists a bottom slope. Thus we may conclude that term g rip currents on a curved beach are produced by the angle of incidence ϕ of the waves onto the beach. And these currents will be the ones most easily observed since they will be the largest.

Term c depends not only on the existence of ϕ but that ϕ is a variable function of θ. Thus on a straight coastline where it is possible to have ϕ non-zero but constant in the longshore direction term c would be zero and thereby not produce a rip current. In our problem $\phi = \phi(\theta)$ and term c is significant. Along a straight coastline there will still likely be some variation in the bottom topography such as the La Jolla Canyon off the coast of Southern California. This will make ϕ vary in the longshore direction, thereby creating the forcing term c. Term c will be of order of magnitude of term a or greater if $(\phi_0 \Delta\phi)^{\frac{1}{2}}$ is of the order of 3 degrees or more where ϕ_0 is the scaling value of ϕ and $\Delta\phi$ is the expected change in ϕ over the longshore scaling distance σ_0.

Thus we see that term c is very sensitive to changes in ϕ .

Term e depends on the existence of $\phi(\theta)$ being a nonlinear function of θ (ie. $d^2\phi/d\theta^2 \neq 0$). This will usually be the case where the bottom topography is irregular.

Term a is the forcing term Bowen (1969b) found to produce rip currents along a straight beach. It depends on the longshore variation of $m = -dd/dr$. We have discussed this under surf zone geometry and if the r direction is normal to the beach (as it is for a straight or circular beach) then the bottom topography effects will not make m a function of θ . At this point we just let dd/dr be a function of θ and say it is due to edge wave effects. This should be the subject of a future study; however, we shall assume it to be true for now. Quantitatively we let:

$$d = ms(1 + \epsilon H_1(\theta)) \tag{5.38}$$

where ϵ is of order $\eta_a/\overline{h_b} \cong 0.01$.

We see that for a curved beach term g will likely dominate the scene but as the beach becomes more of a straight one, terms c, e and a play a more dominant role. If the beach were perfectly straight without irregular bottom topography, then term a would be the only term available to produce rip currents. This is the case described by Bowen (1969b).

The remaining terms f, b, i and h are of less significance; however, it will be interesting to note in our circular bay solution the character of the flows these terms produce.

CHAPTER 6

SUMMARY OF PART I

We are now prepared to use our mathematical model on a given topographical beach. First, however, it will be useful to summarize our mathematical model.

In deriving it, we made the following assumptions:

a. All averaged motion is in the horizontal plane.

b. Wave reflection is neglected because of the small bottom slope.

c. Eddy viscosity is neglected.

d. Bottom friction is linear.

e. At the breaker line the waves are assumed plane.

f. We assume a large bay such that $r_{sb}/r_0 < 0.2$.

g. The energy density E of the waves is equal to $\frac{1}{8}\rho g H^2$ inside and outside the surf zone.

h. The breaker height H is directly proportional to the mean water depth d inside the surf zone.

i. The rip currents do not extend below $d = h_b$ outside the surf zone.

j. The non-linear inertial terms are neglected.

Apart from these assumptions our model using equations (3.19), (4.19), (4.20), (5.12) and (5.30) is for a general beach with arbitrary bottom topography. For a circular beach we assumed only that $\partial d/\partial r = -m$ was not a function of θ . Thus for a circular beach our model consists of equations: (3.19), (4.19), (4.20), (5.12) and (5.36). For a circular beach we expect term g to produce the most significant rip currents. As the beach straightens out we see that terms a and c will become dominant.

Term a is likely generated by edge wave interaction effects (Bowen 1969b).

The approximated form of $curl_z \underset{\sim}{R}$ (equations (4.19) and (4.20)), does not contain any derivatives with respect to θ. This implies that the θ dependence of ψ will exactly correspond with the θ dependence of each of the forcing terms.

PART II

CIRCULAR BAY SOLUTION

CHAPTER 7

THE PROBLEM

7a The Equations

We shall solve for ψ using the mathematical model derived in Part I for a circular bay. The equations to be solved are as follows: Inside the surf zone we have from equations (3.19), (4.19) and (5.36)

$$\frac{c}{d^2}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{2m}{d}\frac{\partial\psi}{\partial r}\right) = \overset{(a)}{\frac{2k}{r}\cos 2\phi\frac{\partial^2 d}{\partial\theta\partial r}} + \overset{(b)}{\frac{AK}{r^2}\cos 2\phi\frac{\partial H_b^+}{\partial\theta}}$$

$$+\overset{(c)}{\frac{3km}{r}\sin 2\phi\frac{\partial\phi}{\partial\theta}} \overset{(d)}{-k\sin 2\phi\frac{\partial^2 d}{\partial r^2}} + \overset{(e)}{\frac{kd}{r^2}\cos 2\phi\frac{\partial^2\phi}{\partial\theta^2}} - \overset{(f)}{\frac{2kd}{r^2}\sin 2\phi\left(\frac{\partial\phi}{\partial\theta}\right)^2} \tag{7.1}$$

$$+\overset{(g)}{\frac{2km}{r}\sin 2\phi} + \overset{(h)}{\frac{kA}{r^2}\sin 2\phi\frac{\partial^2 H_b^+}{\partial\theta^2}} + \overset{(i)}{\frac{3kA}{r^2}\cos 2\phi\frac{\partial\phi}{\partial\theta}\frac{\partial H_b^+}{\partial\theta}}$$

where $k = \frac{1}{8}g\gamma^2$

The forcing terms in (7.1) are arranged in descending order of magnitude (see (5.37)). Outside the surf zone we have from (4.20) and (5.12):

(a) in polar co-ordinates:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{7.2}$$

(b) in cartesian co-ordinates:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \tag{7.3}$$

We must solve these equations according to the boundary conditions discussed in the following section.

7b The Boundary Conditions

As discussed in Section 1c, the width of the surf zone r_{sb} will change sigificantly due to wave height changes at the breaker line. For example, if the wave height varies by 50% over an interval, the surf zone width will vary by the same percentage. Thus if we were attempting to obtain accurate quantitative results we would have to make the boundaries of the shore and breaker line as accurate as possible. This would involve tedious matching problems at the boundary involving Fourier series. The result would give us stream lines which wiggle with the shoreline, but would not change the overall rip current patterns which we are attempting to determine. Therefore we will take the space average values of the radial components of the shoreline and breaker line, denoted by $\overline{r_s}$ and $\overline{r_b}$ respectively, to represent the boundaries of the surf zone.

We assume that the deep water waves have a phase velocity parallel to the y axis of Figure 4. Since water does not flow across the shoreline $(r_s = \overline{r_s}\ , y > 0)$, the shoreline must be a stream line and we arbitrarily let ψ have the value of zero at the shoreline. We divide the bay into three regions, A, B and C as shown in Figure 4. In regions A and B we use polar co-ordinates and it is convenient to use cartesian co-ordinates in region C.

It is not necessary to assume deep water waves with phase velocity parallel to the y axis; however, since we must assume some sort of wave approach, this one seems the most convenient as it will lead to symmetric rip current distributions. Later we solve for a case where the wave approach is not parallel to the y-axis.

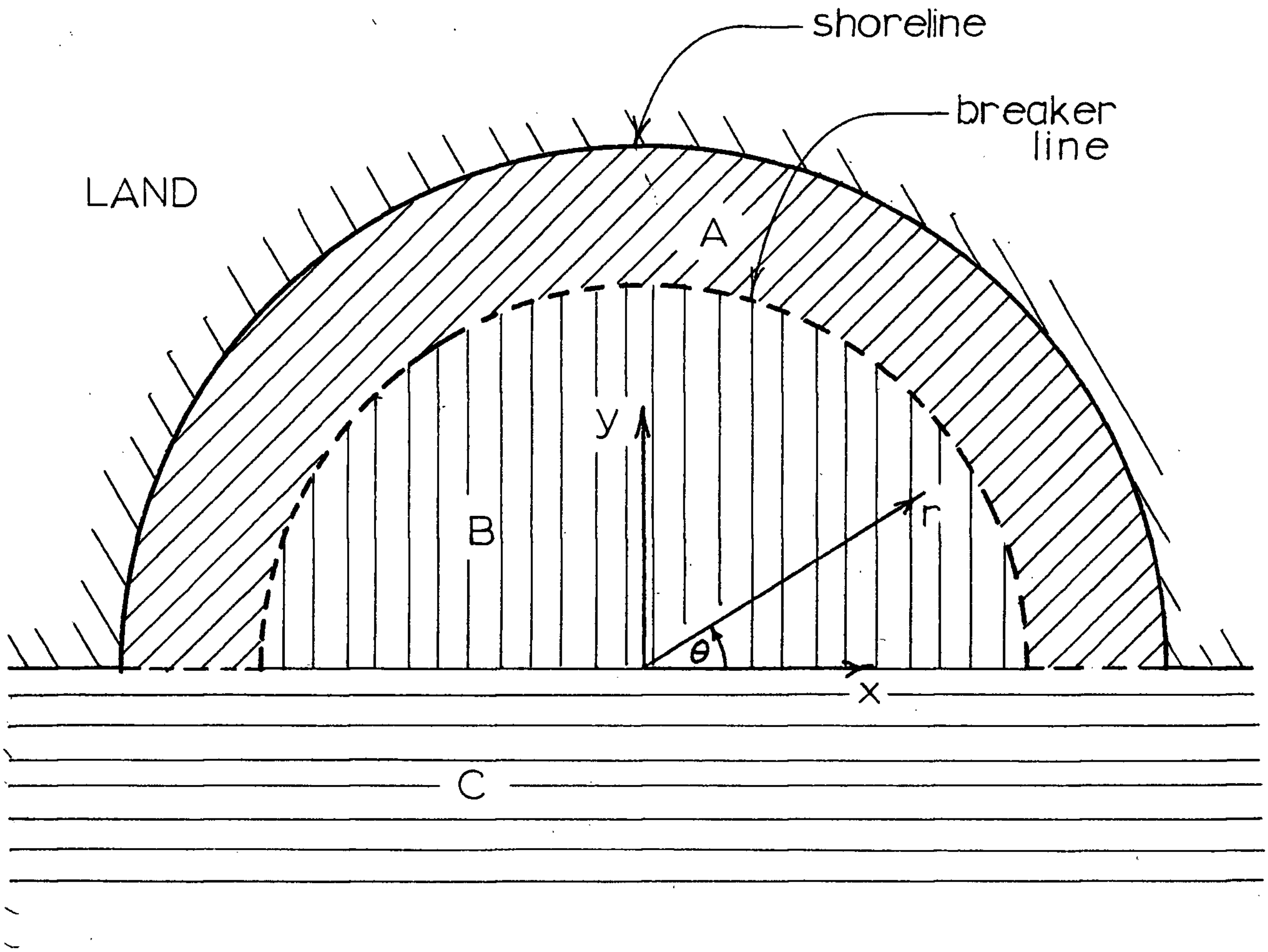

Figure 4

Circular Bay

At the shoreline v_r must be zero. Therefore $\partial\psi/\partial\theta = 0$ at $r = \bar{r}_s$, $0 \leq \theta \leq \pi$. At the shoreline v_θ will be zero due to the bottom friction (see (3.10)). Therefore $\partial\psi/\partial r = 0$ at $r = \bar{r}_s$, $0 \leq \theta \leq \pi$. At the breaker line v_r must be continuous because of mass conservation, hence $\partial\psi/\partial\theta$ is continuous at $r = r_b$, $0 \leq \theta \leq \pi$. Also ψ itself must be continuous at $r = r_b$ since we cannot have $\partial\psi/\partial r \rightarrow \infty$ which would imply $v_\theta \rightarrow \infty$.

Also we know v_θ, in physical reality, must be continuous, otherwise we would have an infinite shear which is not physically possible. However, this continuity is based on there being eddy viscosity present which we have not included in our model. Thus we will not insist that v_θ or $\partial\psi/\partial r$ be continuous at the breaker line or any other boundary; however, it turns out that it will be continuous on the breaker line using only bottom friction in our model.

The boundary conditions are summarized as follows:

a. $\psi = 0$ at $r = \bar{r}_s$, $0 \leq \theta \leq \pi$ (7.4)

b. $\dfrac{\partial\psi}{\partial\theta} = 0$ at $r = \bar{r}_s$, $0 \leq \theta \leq \pi$ (7.5)

c. $\dfrac{\partial\psi}{\partial r} = 0$ at $r = \bar{r}_s$, $0 \leq \theta \leq \pi$ (7.6)

d. $\dfrac{\partial\psi_A}{\partial\theta} = \dfrac{\partial\psi_B}{\partial\theta}$ at $r = \bar{r}_b$, $0 \leq \theta \leq \pi$ (7.7)

where subscripts A and B denote values in Regions A and B respectively.

e. $\psi_A = \psi_B$ at $r = \bar{r}_b$, $0 \leq \theta \leq \pi$ (7.8)

f. If we can allow $\partial\psi/\partial r$ to be continuous everywhere, then we should do so; however, the physical cause of this being so was not built into our model (the cause being eddy viscosity).

CHAPTER 8

THE SOLUTION

8a General

There are nine forcing terms in equation (7.1). Since (7.1) is linear in ψ then we may solve for a ψ for each term of the forcing function and add all the resulting ψ terms and obtain the total ψ due to the total forcing functions of (7.1).

In Section 2b we discussed the effects of the topography on wave height Hb^+ and angle of incidence ϕ at the breaker line. We approximate these effects by assuming the following distributions for Hb^+ and ϕ :

$$Hb^+ = D[1 + p H_1(\theta)] = D[1 - p \cos 2\theta] \tag{8.1}$$

$$\phi = \phi\, e(\theta) = \phi_0 \cos\theta \tag{8.2}$$

The solutions for outside the surf zone are given in the following:

Region C:

Here we use cartesian co-ordinates as shown in Figure 4. We have:

$$\mathrm{curl}_z\, \underset{\sim}{R} = \frac{\partial R_y}{\partial x} - \frac{\partial R_x}{\partial y} = 0$$

where $R_x = -\dfrac{c\,u}{d_b}$, $R_y = -\dfrac{c\,v}{d_b}$, $d_b \doteq \bar{h}_b$

and the transport stream function ψ is defined as:

$$u = -\frac{1}{d_b}\frac{\partial \psi}{\partial y} \quad , \quad v = \frac{1}{d_b}\frac{\partial \psi}{\partial x}$$

Thus we obtain: $$\mathrm{curl}_z \underset{\sim}{R} = -\frac{c}{d_b}\frac{\partial^2 \psi}{\partial x^2} - \frac{c}{d_b}\frac{\partial^2 \psi}{\partial y^2} = 0$$

or $$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0$$

This is Laplace's Equation and its solution is:

$$\psi_c = (A_1 \cos\beta x + B_1 \sin\beta x)(K_1 e^{\beta y} + K_2 e^{-\beta y})$$

and we assume ψ is finite at $y = -\infty$ thus $K_2 = 0$

and we obtain:

$$\psi_c = (A_1 \cos\beta x + B_1 \sin\beta x)\, e^{\beta y} \tag{8.3}$$

Region B:

From (4.20) and (5.12) we obtain:

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{8.4}$$

Solving by separation of variables we let:

$$\psi = R(r)\,\Theta(\theta) \tag{8.5}$$

and (8.4) becomes: $$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0$$

or $$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \mu^2$$

where μ is a constant.

Thus we obtain: $\Theta'' + \mu^2 \Theta = 0$

$$\Theta = A_2 \cos \mu\theta + B_2 \sin \mu\theta \tag{8.6}$$

or

and $$r^2 R'' + rR' - \mu^2 R = 0$$

which is Euler's equation and the solution is:

$$R = K_3 r^{\mu} + K_4 r^{-\mu} \tag{8.7}$$

Thus

$$\psi_B = (A_2 \cos \mu\theta + B_2 \sin \mu\theta)(K_3 r^{\mu} + K_4 r^{-\mu}) \tag{8.8}$$

Since ψ must be finite at $r = 0$, we must have $K_4 = 0$ thus we obtain:

$$\psi_B = (A_2 \cos \mu\theta + B_2 \sin \mu\theta) r^{\mu} \tag{8.9}$$

where K_3 is absorbed in A and B.

We must match this solution with the various term solutions we obtain inside the surf zone. We assume a linear bottom so term d = 0. We now solve for ψ for each term of equation (7.1) in descending order of magnitude. (see Section 5e for physical interpretation of each term).

8b Term g

From equation (7.1) we obtain:

$$\frac{d^2\psi}{dr^2} + \frac{2m}{d}\frac{d\psi}{dr} = \frac{2km}{c r_0} d^2 \sin 2\phi \tag{8.10}$$

where we let r on the right hand side (RHS) be r_o . Since we assume the bay is large the error resulting from this approximation should be small. Now $d = m(r_s - r)$ but we are assuming the shore to be circular as was discussed in Section 7b, therefore we have an approximation:

$$r_s \cong \bar{r}_s \tag{8.11}$$

and

$$d \cong m(\bar{r}_s - r) = ms \tag{8.12}$$

where $s = \bar{r}_s - r$

Transforming the variable from r to s we obtain:

$$\frac{\partial^2 \psi}{\partial s^2} - \frac{2}{s}\frac{\partial \psi}{\partial s} = \frac{2km^3}{c r_o} s^2 \sin 2\phi \tag{8.13}$$

$$= B_g s^2 e(\theta)$$

where $B_g = \frac{4 k m^3 \phi_o}{c r_o}$

and $e(\theta)$ is the θ dependence factor of ϕ as shown in (8.2). We let $\sin 2\phi \cong 2\phi = 2\phi_o e(\theta)$.

The complementary solution is determined from the homogeneous equation:

$$\frac{\partial^2 \psi}{\partial s^2} - \frac{2}{s}\frac{\partial \psi}{\partial s} = 0 \tag{8.14}$$

and is found to be

$$\psi_{(r,\theta)} = (C_1 + C_2 s^3)\,\psi(\theta) \tag{8.15}$$

and the particular solution is:

$$\psi = \left(C_3 s^3 + \frac{B_g}{4} s^4\right) e(\theta)$$

Thus $$\psi_A = e(\theta)\left[C_1 + C_2 s^3 + \frac{B_9}{4} s^4\right] \tag{8.16}$$

We now apply the boundary conditions:

(a) $\psi = 0$ at $r = \bar{r}_s$ or $s = 0$

Thus $C_1 = 0$

and $$\psi_A = e(\theta)\left[C_2 s^2 + \frac{B_9}{4} s^4\right] \tag{8.17}$$

(b) $\frac{\partial \psi}{\partial r} = 0$ at $s = 0$

Thus (8.17) gives:

$$e(\theta)\left[-3C_2 s^2 - B_9 s^3\right] = 0$$

which is identically true at $s = 0$.

(c) $\frac{\partial \psi}{\partial \theta} = 0$ at $s = 0$

which is identically true using (8.17)

(d) $\psi_A(r_b) = \psi_B(r_b)$

or $$e(\theta)\left[C_2 s_b^3 + \frac{B_9}{4} s_b^4\right]$$
$$= (A_1 \cos\mu\theta + B_1 \sin\mu\theta)\, r_b^{\mu}$$ (see (8.9))

Now if $e(\theta)$ were not sinusoidal we would use a Fourier series expansion. To demonstrate this we let:

$$e(\theta) = \sum_n (A_n \cos n\theta + B_n \sin n\theta)$$
$$\equiv \sum_n F_n(\theta) \qquad n = 0, 1, 2, \cdots$$

where $F_n(\theta) = A_n \cos n\theta + B_n \sin n\theta$

Since (8.13) is linear in ψ we say:

$$\psi_A = \sum_n \psi_{An} \tag{8.18}$$ where

$$\psi_{An} = F_n(\theta)\left[C_n s^3 + \frac{B_9}{4} s^4\right]$$

We also let: $\psi_B = \sum_n \psi_{Bn}$ where

$$\psi_{B_n} = (A'_n \cos n\theta + B'_n \sin n\theta) r^n \equiv G_n(\theta) r^n$$

At the boundary $s = sb$ we have: $\psi_{An} = \psi_{Bn}$ which gives us:

$$G_n = \frac{F_n}{rb^n}\left(C_n sb^3 + \frac{B_g}{4} sb^4\right)$$

and $\frac{\partial \psi_{An}}{\partial r} = \frac{\partial \psi_{Bn}}{\partial r}$ which gives us:

$$G_n = -\frac{F_n}{n\, rb^{n-1}}\left(3C_n sb^2 + B_g sb^3\right)$$

Therefore from these two expressions for $G_n(\theta)$ we obtain values for C_n , A'_n and B'_n and our boundary problem is solved. When we discuss term f we use this method of solution for n = 1 and 3.

However, in this problem we let $e(\theta) = \cos\theta$ and hence:

$$B_1 = 0,\ \mu = 1,\ A_1 = \frac{C_2\, sb^3}{rb} + \frac{B_g}{4}\frac{sb^4}{rb} \qquad (8.19)$$

where $sb = \bar{r}_s - \bar{r}_b$.

(e) $\frac{\partial \psi_B}{\partial r}(rb) = \frac{\partial \psi_A}{\partial r}(rb)$

or $\cos\theta\left[-3C_2 sb^2 - B_g sb^3\right] = A_1 \cos\theta$

or $$A_1 = -3C_2 sb^2 - B_g sb^3 \qquad (8.20)$$

Using (8.20) and (8.19) we obtain:

$$C_2 = -\frac{B_g sb}{3}\left[1 - \frac{1}{12}\frac{sb}{rb} + \frac{1}{36}\left(\frac{sb}{rb}\right)^2 - \frac{1}{108}\left(\frac{sb}{rb}\right)^3 + \cdots\right] \qquad (8.21)$$

or $$C_2 \cong -\frac{B_g s_b}{3}\left(1-\frac{1}{12}\frac{s_b}{r_b}\right)$$

for large bays $\frac{s_b}{r_b} \ll 1$

and $$A_1 = -\frac{B_g s_b^3}{12}\left[\frac{s_b}{r_b}-\frac{1}{3}\left(\frac{s_b}{r_b}\right)^2+\frac{1}{9}\left(\frac{s_b}{r_b}\right)^3-\cdots\right] \tag{8.22}$$

or $$A_1 \cong -\frac{B_g s_b^4}{12 r_b} \tag{8.23}$$

Therefore we obtain:

$$\psi_A \cong -\frac{1}{3} B_g\left[\left(1-\frac{s_b}{12 r_b}\right)s_b s^3-\frac{3}{4}s^4\right]\cos\theta \tag{8.24}$$

and $$\psi_B \cong -\frac{B_g s_b^4}{12 r_b}\, r\cos\theta \tag{8.25}$$

Note that by not including eddy viscosity and still saying v_θ is continuous at $r = r_b$, we are forcing a value on C_2 and A_1 for which the model was not specifically designed. If we did not have the boundary condition $\frac{\partial\psi_A}{\partial r}(r_b) = \frac{\partial\psi_B}{\partial r}(r_b)$, then we could chop off ψ_A before it reached its maximum value. If eddy viscosity were included, then the equation (7.1) would be of higher order and would enable us to state with certainty the continuity of v_θ.

However, since we do get a highly satisfactory solution without eddy viscosity, then perhaps it is negligible relative to the bottom friction. This seems reasonable although it should receive more attention in a future study.

(f) $\psi_B = \psi_C$ at $y = 0$

Thus we obtain from (8.3) and (8.25):

$$-\frac{B_g\, s_b^4}{12\, r_b} r = \sum_n \left(A_n \cos\beta_n x + B_n \sin\beta_n x\right) = \sum_n B_n \sin\beta_n x \tag{8.26}$$

since $\psi_B(y=0)$ is an odd function.

Using Fourier series expansion theory we obtain:

$$B_n = \frac{2}{\bar{r}_s} \int_0^{\bar{r}_s} \psi(x,0) \sin \frac{n\pi x}{\bar{r}_s}\, dx \tag{8.27}$$

where $\beta_n = \frac{n\pi}{\bar{r}_s}$

Thus

$$\psi_c = \sum_n B_n \sin\beta_n x\, e^{\beta_n y} \tag{8.28}$$

From (8.28) we obtain:

$$\frac{\partial \psi_c}{\partial y}(y=0) = \sum_n \beta_n B_n \sin\beta_n x \tag{8.29}$$

From (8.25)

$$\frac{\partial \psi_B}{\partial \theta}(\theta = 0, \pi) = 0 \tag{8.30}$$

Thus we cannot match (8.29) and (8.30) at $y = 0$ because (8.29) will not in general be identically zero. Therefore we have a discontinuity in v_r at $y = 0$. In real life this is impossible; however, since we have not included eddy viscosity in our model we obtain this discontinuity.

In summary then, we have for term g of equation (7.1):

$$\psi_A \cong -\frac{1}{3} B_g\left[\left(1 - \frac{1}{12}\frac{s_b}{r_b}\right) s_b s^3 - \frac{3}{4} s^4\right] \cos\theta$$

$$\psi_B \cong -\frac{B_g\, s_b^4}{12\, r_b} r \cos\theta \tag{8.31}$$

$$\psi_c = \sum_n B_n \sin\beta_n x\, e^{\beta_n y}$$

At the boundary between regions B and C we used a Fourier series method to match the solutions. This makes region C periodic in x with period $2\bar{r}_s$. To eliminate this periodicity, we could have used a Fourier integral method by letting:

$$\psi_c(x,y) = \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, e^{|k|y}\, dk$$

which identically satisfies Laplace's equation and we have:

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x,0)\, e^{-ikx}\, dx$$

which satisfies the boundary condition at $y=0$ for any function $\psi(x,0)$. This same method can be used at the boundary between regions A and B where we would let:

$$\psi_B = \int_{-\infty}^{\infty} F(n)\, e^{-in\theta} \left(\frac{r}{r_b}\right)^{|n|} dn$$

and
$$F(n) = \frac{1}{2\pi}\int_{0}^{\pi} \psi(r_b,\theta)\, e^{-in\theta}\, d\theta$$

where we let $\psi(r_b,\theta) = 0$ for $\theta > \pi$ and $\theta < 0$.

We have no need to use this method in this paper for the boundary at $r = \bar{r}_b$ because we let $\phi(\theta)$ and $H_b^+(\theta)$ be simple trigonometric functions.

We find the location of ψ_A maximum as follows:

$$\frac{d\psi_A}{dr} = B_g S_b s^2 \cos\theta \left(1 - \frac{s_b}{12 r_b} - \frac{s}{s_b}\right)$$

therefore the maximum occurs at:

$$\frac{s_b}{12 r_b} + \frac{s}{s_b} = 1$$

or
$$S_{max} = \left(1 - \frac{s_b}{12 r_b}\right) s_b \qquad (8.32)$$

For a large bay $s_b/r_b \ll 1$ and so $S_{max} \cong s_b$.

For $\theta = 0$ we plot ψ vs r as shown in Figure 5.

From (8.32) and (8.31) we get:

$$\psi_{max} \cong \psi_B(r_b) = -\frac{B_g s_b^4}{12} \qquad (8.33)$$

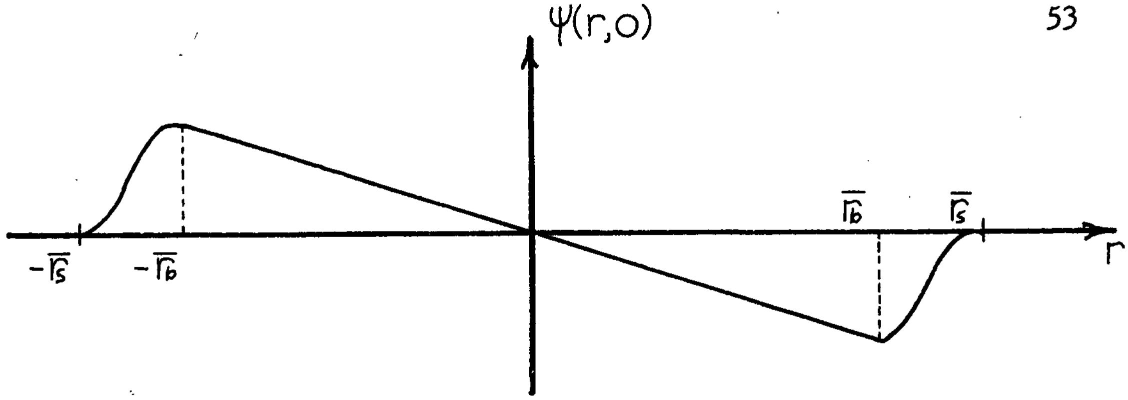

Figure 5

Profile of $\psi_g(x,0)$

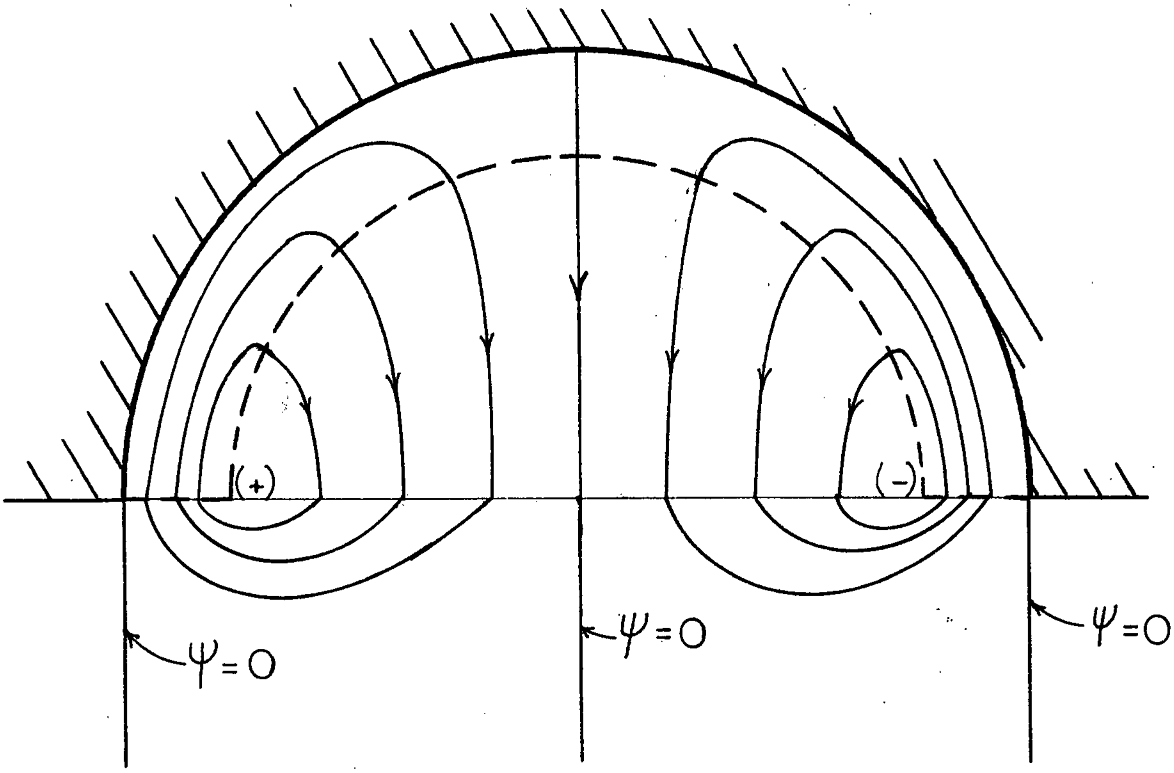

Figure 6

Streamlines, Term g

The entire field of ψ is sketched according to equations (8.31) in Figure 6. We see that there is only one rip current, at $\theta = \pi/2$. The streamlines represent the average paths fluid particles would follow providing the streamline pattern does not change significantly over the time it would take for the particles to complete a round trip along the streamlines. The rip current flows out past the mouth of the bay (where y = o) and then curves around and enters the surf zone with a strong velocity as indicated by the high density of the streamlines. The (+) and (-) symbols in Figure 6 show where the transport stream function ψ is positive or negative respectively. The effect of the non-linear terms, which we neglected (Chapter 3), will be to strengthen the rip current at $\theta = \pi/2$ because the current there is flowing into deeper water, and to decrease the current flowing into the surf zone at $\theta = 0$ and π because the flow there is into shallow water (Arthur 1962).

A sample magnitude of $U_r(r_b, \pi/2)$ is computed as follows:

$$U_r(r_b,\tfrac{\pi}{2}) = -\frac{1}{d}\frac{d\psi}{r d\theta}(r_b,\tfrac{\pi}{2}) = -\frac{1}{12}\frac{B_0 s_b^4}{d_b r_b} = -\frac{g\gamma^2}{24c} m^2 \phi_0 s_b \left(\frac{s_b}{r_b}\right)^2$$

where $d = m s_b$

Let $m = 0.1$ $\quad g = 32\ ft/sec^2$ $\quad s_b = 100\ ft$ $\quad \frac{s_b}{r_b} = 0.1$

$\phi_0 = \frac{\pi}{12} \doteq \frac{1}{4}$ $\quad \gamma = 1.0$ $\quad d_b = 10\ ft$

Then $$U_r(r_b,\tfrac{\pi}{2}) = -\frac{32 \times 1 \times .01 \times 100}{24 \times 4}\frac{1}{c}\left(\frac{s_b}{r_b}\right)^2$$

$$= -\frac{1}{3}\frac{1}{c}\left(\frac{s_b}{r_b}\right)^2 = -\frac{0.0033\ (ft^2/sec^2)}{c}$$

Bowen (1969b) estimates c = 0.2 cm./sec. = 0.0067 ft./sec. which gives:

$$U_r(r_b,\tfrac{\pi}{2}) \cong -\frac{.0033}{.0067} \cong -0.5\ ft/sec$$

From Figure 6 we see that the maximum velocity occurs in the surf zone at $\theta = 0, \pi$. To calculate this velocity we do the following:

$$v_\theta = \frac{1}{d} \frac{\partial \psi_A}{\partial r} = \frac{1}{3} \frac{B_g sb}{d} \left(3s^2 - \frac{3s^3}{sb} \right) \cos\theta$$

$v_{\theta\,max}$ occurs at $s = \frac{2}{3} sb$, $\theta = 0, \pi$

Then $v_\theta\left(\frac{2}{3} sb, 0\right) = \frac{4}{27} \frac{B_g}{d} sb^3 = \frac{8}{9} \frac{k m^2 \Phi_0}{c} sb \left(\frac{sb}{rb}\right)$

and $\dfrac{v_\theta\left(\frac{2}{3} sb, 0\right)}{v_r(rb, \pi/2)} = \frac{8}{3} \frac{rb}{sb} \cong \frac{80}{3}$

If $v_r\left(rb, \frac{\pi}{2}\right) = 0.5 \text{ fps}$

then $v_{\theta_{max}}\left(\frac{2}{3} sb, 0\right) \cong 13 \text{ fps}$

The effects of the nonlinear terms will be to lessen v_θ maximum and increase $v_r\left(rb, \frac{\pi}{2}\right)$.

<u>8c Term c</u>

The equation for ψ inside the surf zone for term c is from (7.1):

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2m}{d} \frac{\partial \psi}{\partial r} = \frac{3km}{c} \frac{d^2}{r} \sin 2\phi \frac{\partial \phi}{\partial \theta} \qquad (8.34)$$

Using the same procedure as for term g we obtain:

$$\frac{\partial^2 \psi}{\partial s^2} - \frac{2}{s} \frac{\partial \psi}{\partial s} = B_c s^2 e \frac{\partial e}{\partial \theta} \qquad (8.35)$$

where $B_c = \dfrac{6 k m^3 \Phi_0^2}{c r_0}$

$s = \bar{r}_s - r$

$e(\theta) = \phi / \phi_0$

The solution is obtained in the same manner as for term g and is:

$$\psi_A = e\frac{\partial e}{\partial \theta}\left(C_1 + C_2 s^3 + \frac{B_c s^4}{4}\right) \tag{8.36}$$

or using $e = \cos\theta$ we get:

$$\psi_A = -\left(C_1 + C_2 s^3 + \frac{B_c}{4} s^4\right)\frac{\sin 2\theta}{2} \tag{8.37}$$

Subjecting (8.37) to the boundary conditions (BCs) with ψ_B, ψ_c given by (8.3) and (8.9) we obtain:

$$\psi_A = \frac{B_c}{2}\left[\frac{sb\, s^3}{3}\left(1 - \frac{sb}{6rb}\right) - \frac{s^4}{4}\right]\sin 2\theta \tag{8.38}$$

$$\psi_B = \frac{B_c}{24}\,\frac{sb^4}{rb^2}\,r^2 \sin 2\theta \tag{8.39}$$

$$S_{max} = sb\left(1 - \frac{sb}{rb}\right) \tag{8.40}$$

$$\psi_c = 0 \tag{8.41}$$

$$\psi_{max} \cong \psi_B(rb) = \frac{B_c}{4} sb^4 \tag{8.42}$$

Again there is a velocity discontinuity at $y = 0$ as was the case for term g . This indicates that eddy viscosity is significant at this location in the model.

A sketch of the rip current system for term c is shown in Figure 7. The dotted lines are estimated flow lines which are expected to exist if we include eddy viscosity in our model. We obtain rip currents where ϕ is maximum or minimum for a sinusoidal ϕ distribution.

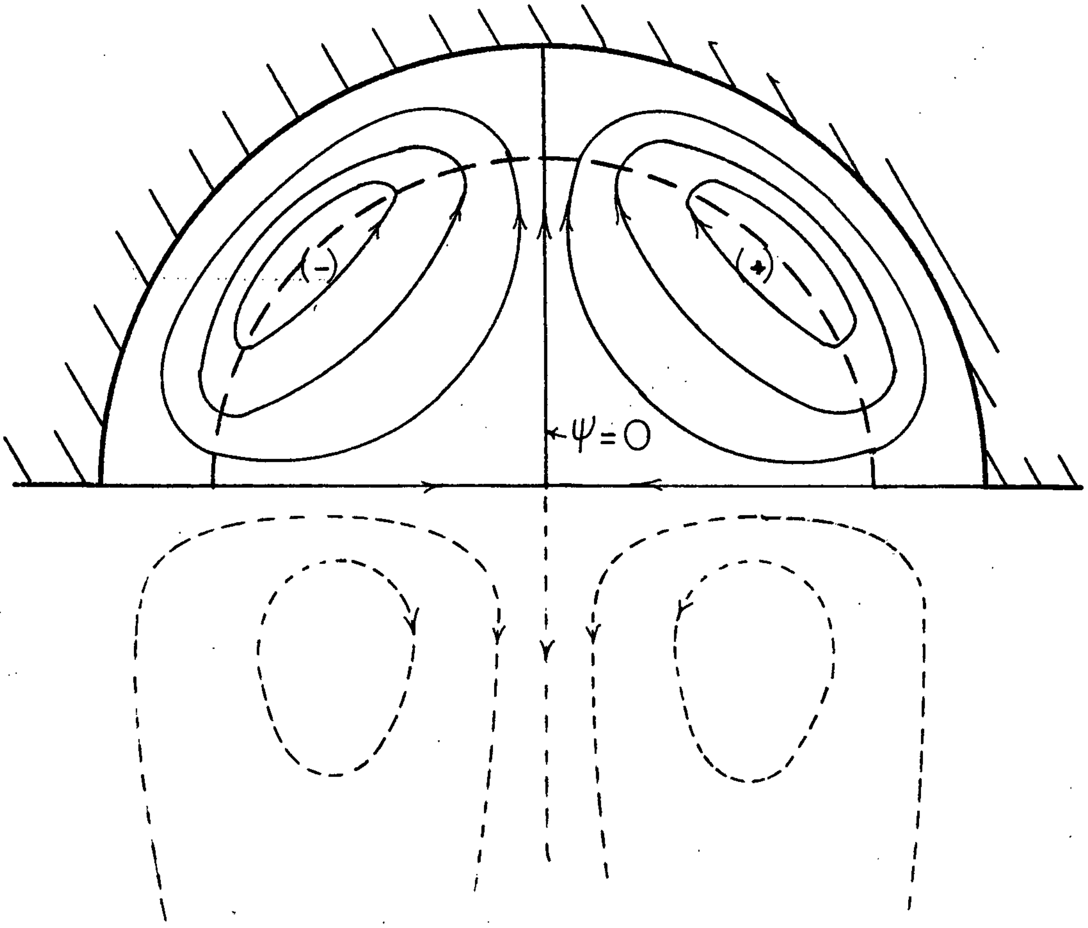

Figure 7

Streamlines, Term c

8d Term e

From (7.1) we obtain:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2m}{d}\frac{\partial\psi}{\partial r} = \frac{kd^3}{cr^2}\cos 2\phi\,\frac{\partial^2\phi}{\partial\theta^2} \tag{8.43}$$

which we alter as we did for terms g and c and obtain:

$$\frac{\partial^2\psi}{\partial s^2} - \frac{2}{s}\frac{\partial\psi}{\partial s} = B_e\, s^3\frac{\partial^2 e}{\partial\theta^2} \tag{8.44}$$

where $$B_e = \frac{k\,m^3\phi_0}{c\,r_0^2}$$

$$s = \bar{r}_s - r$$

Solving we obtain:

$$\psi_A = -\left(C_1 + C_2 s^3 - \frac{B_e}{10}s^6\right)\frac{\partial^2 e}{\partial\theta^2} \tag{8.45}$$

or with $e = \cos\theta$ we obtain:

$$\psi_A = +\left(C_1 + C_2 s^3 - \frac{B_e}{10}s^5\right)\cos\theta \tag{8.46}$$

Subjecting (8.46) to the BCs we obtain:

$$\psi_A = \frac{B_e}{6}\left[\left(1-\frac{2}{15}\frac{s_b}{r_b}\right)s_b^2 s^3 - \frac{3}{5}s^5\right]\cos\theta$$

$$\psi_B = \frac{B_e}{15}\frac{s_b^5}{r_b}\,r\cos\theta \tag{8.47}$$

$$s_{max} = s_b\left(1-\frac{2}{15}\frac{s_b}{r_b}\right)^{\frac{1}{2}}$$

and $$\psi_c = \sum_n B_n \sin\beta_n x\; e^{\beta_n y} \tag{8.48}$$

where $$\beta_n = \frac{n\pi}{\bar{r}_s}$$

$$B_n = \frac{2}{\bar{r}_s}\int_0^{\bar{r}_s}\psi(x,0)\,\sin\frac{n\pi x}{\bar{r}_s}\,dx$$

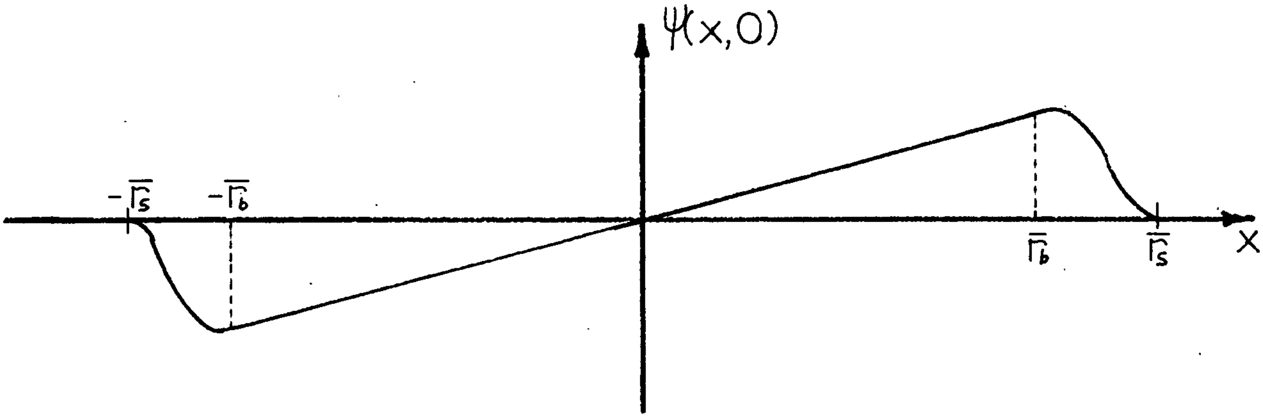

Figure 8

Profile of $\psi_e(x,0)$

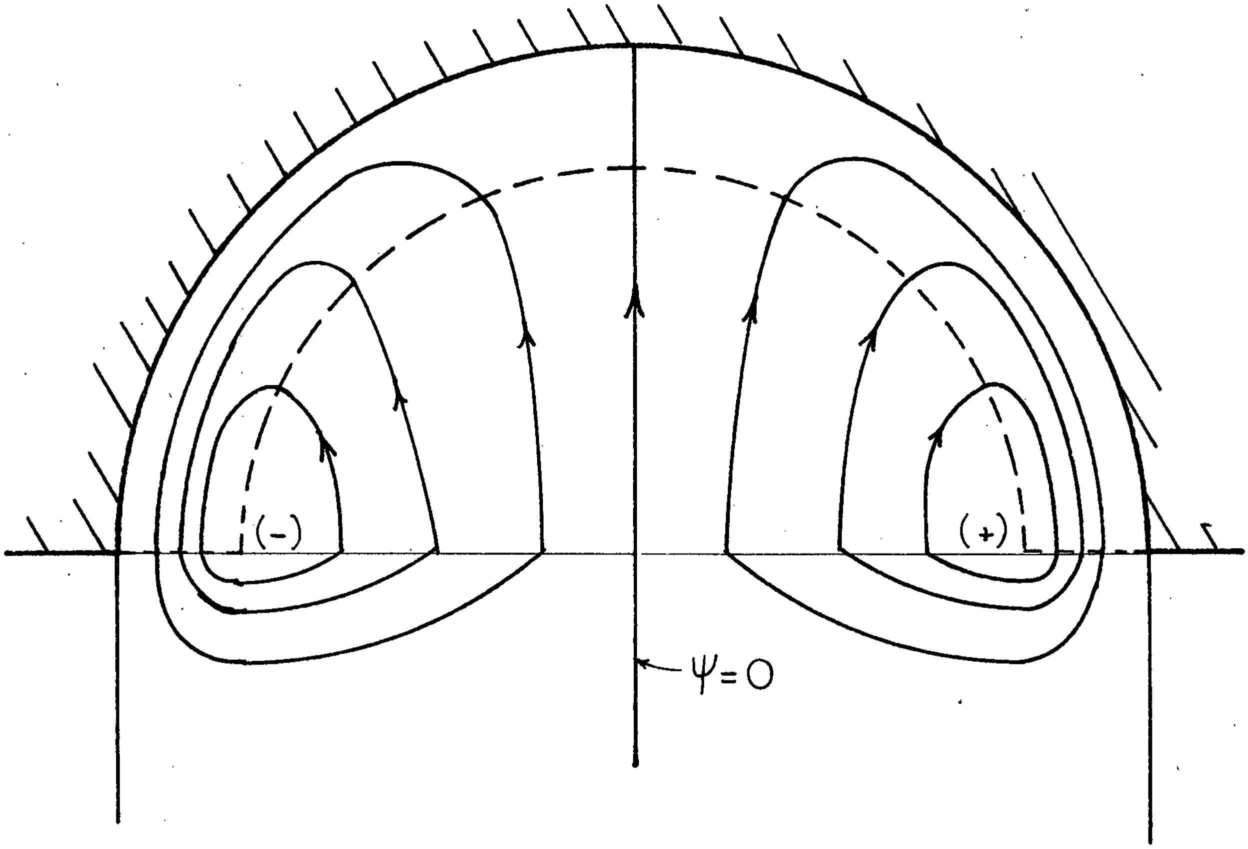

Figure 9

Streamlines, Term e

From ψ_A and ψ_B we obtain the graph of $\psi(x,0)$ as shown in Figure 8.

From (8.47) we obtain:

$$\psi_{max} \cong \psi_B(r_b) = \frac{B_e}{15} s_b^5 \tag{8.49}$$

A sketch of the rip current system for term e is shown in Figure 9. Again we have a discontinuity at $y=0$ because of the absence of eddy viscosity in our model.

8e Term a

From (7.1) we obtain:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2m}{d}\frac{\partial\psi}{\partial r} = \frac{2kd^2}{cr}\cos 2\phi \frac{\partial^2 d}{\partial\theta\partial r} \tag{8.50}$$

or with $s = \bar{r}_s - r$ we obtain:

$$\frac{\partial^2\psi}{\partial s^2} - \frac{2}{s}\frac{\partial\psi}{\partial s} = -B_a s^2 \frac{\partial H_1(\theta)}{\partial\theta} \tag{8.51}$$

where $B_a = \frac{2km\epsilon}{cr_0}$

and $H_1(\theta) = -\cos 2\theta$

Here we let $d = ms(1+\epsilon H_1(\theta))$ (8.52)

where ϵ is of order $\eta_1/\bar{h}_b$. This is similar to the function Bowen(1969b) used, although it cannot be derived from the theory discussed in our section of surf zone geometry. One may circumvent this difficulty by saying $d(r,\theta)$ arises from an edge wave effect. This is speculative, but it gives us a finite value for term a.

The solution to (8.51) is:

$$\psi_A = \frac{B_a}{3}\left[\left(1-\frac{5s_b}{6r_b}\right)s_b s^3 - \frac{3}{4}s^4\right]\sin 2\theta \tag{8.53}$$

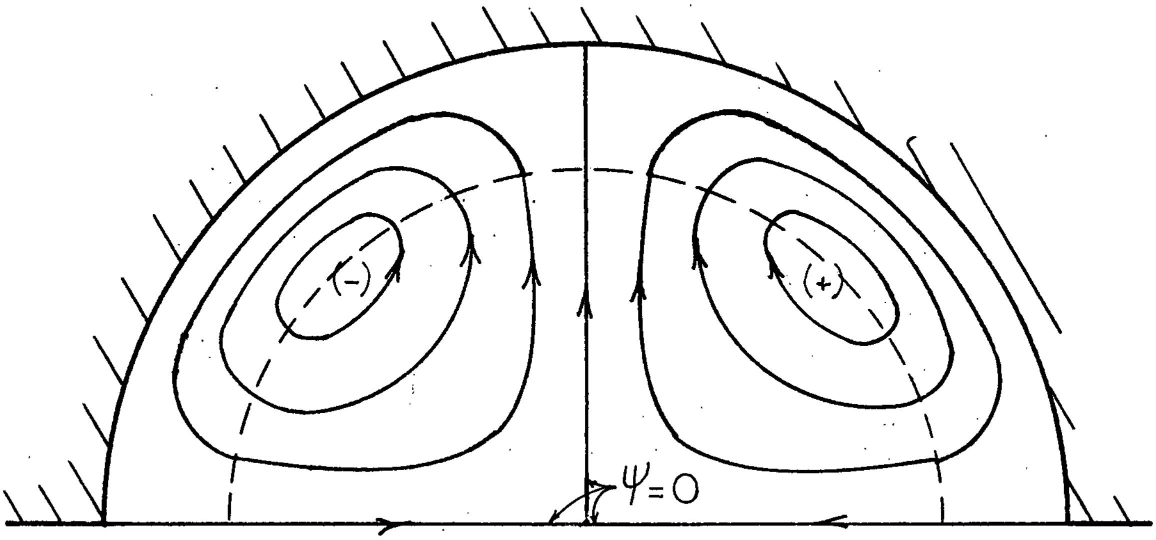

Figure 10

Streamlines, Term a

$$\psi_B = \frac{B_a}{12} \frac{s_b^4}{r_b^2} r^2 \sin 2\theta \tag{8.54}$$

$$\psi_c = 0 \tag{8.55}$$

A sketch of ψ_a is shown in Figure 10. Just as Bowen (1969b) found, we see that the rip current occurs where the waves are the lowest. $(at\ \theta = 0, \pi)$.

It is important to realize that we assumed term a to have a θ dependence the same as H_b^+ . If term a is influenced by phenomenon other than just refraction, then it may very likely have a θ dependence different than that of H_b^+ . If term a varies significantly over a longshore distance of the order of the surf zone width, then we must reconsider our assumptions in obtaining equation (4.10). In particular, we would not be able to neglect the $\partial R_r / r\partial\theta$ term in $curl_z\, \underset{\sim}{R}$.

8f Term f

From (7.1) we obtain:

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2m}{d}\frac{\partial \psi}{\partial r} = -\frac{kd^3}{r^2 c} 2 \sin 2\phi \left(\frac{d\phi}{d\theta}\right)^2 \tag{8.56}$$

Altering this as done for previous terms we obtain:

$$\frac{\partial^2 \psi}{\partial s^2} - \frac{2}{s}\frac{\partial \psi}{\partial s} = -B_f s^3 e \left(\frac{\partial e}{\partial \theta}\right)^2 \tag{8.57}$$

where $$B_f = \frac{4 k m^3 \phi_0^3}{r_0^2 c}$$

We obtain the solution:

$$\psi_A = \left(C_1 + C_2 s^3 - \frac{B_f}{10} s^5\right) e \left(\frac{\partial e}{\partial \theta}\right)^2 \tag{8.58}$$

Letting $e = \cos\theta$, we get:

$$e\left(\frac{de}{d\theta}\right)^2 = \cos\theta \sin^2\theta = \cos\theta - \cos 3\theta \tag{8.59}$$

Since (8.56) is linear in ψ we may treat each of the terms of $e\left(\frac{de}{d\theta}\right)^2$ as separate forcing functions and add their solutions to obtain ψ_f .

We obtain: $$\psi_A = \frac{B_f}{6}\left(s_b^2 s^3 - \frac{3}{5}s^5\right)(\cos\theta - \cos 3\theta) \tag{8.60}$$

$$\psi_B = \frac{B_f}{15}\frac{s_b^5}{r_b} r\left(\cos\theta - \frac{r^2}{r_b^2}\cos 3\theta\right) \tag{8.61}$$

and $$\psi_c = \sum_n A_n \sin\beta_n x \, e^{\beta_n y} \tag{8,62}$$

where $\beta_n = n\pi/\bar{r}_s \; , n = 1,2,3,\cdots$

$$A_n = \frac{2}{\bar{r}_s}\int_0^{\bar{r}_s} \psi(x,0)\sin\beta_n x \, dx$$

and from (8.61) and (8.60) we obtain the sketch of $\psi(x,0)$ shown in Figure 11. A sketch of $\cos\theta - \cos 3\theta$ is shown in Figure 12. We use these sketches to sketch ψ_f as shown in Figure 13.

8g Term b

From (7.1) we obtain: $$\frac{d^2\psi}{dr^2} + \frac{2m}{d}\frac{d\psi}{dr} = \frac{kAd^2}{cr^2}\cos 2\phi \frac{dH_b^+}{d\theta} \tag{8.63}$$

We alter (8.63) to the form: $$\frac{d^2\psi}{ds^2} - \frac{2}{s}\frac{d\psi}{ds} = B_b \, s^2 \frac{dH_1(\theta)}{d\theta} \tag{8.64}$$

where $H_b^+ = D(1 + p\, H_1(\theta))$ and we have discussed in Section 2b the form of $H_1(\theta) = -\cos 2\theta$.

Thus $$\frac{dH_1(\theta)}{d\theta} = 2\sin 2\theta \; .$$

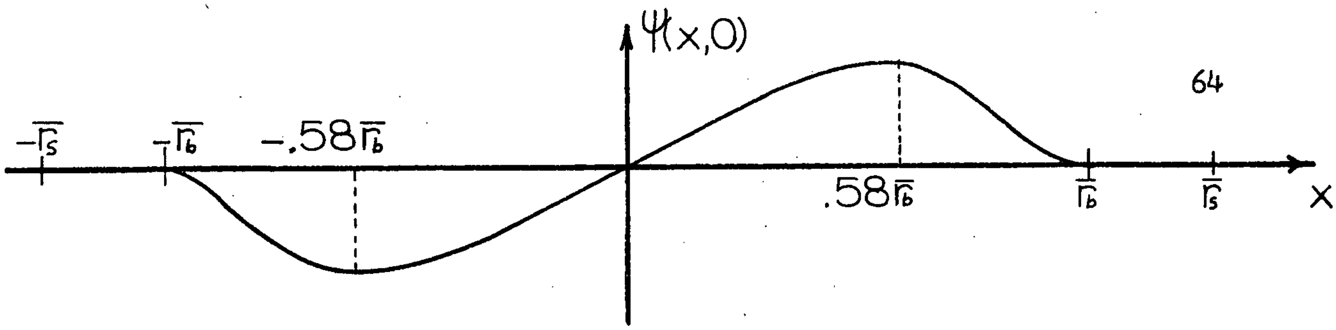

Figure 11

Profile of $\psi_f(x,0)$

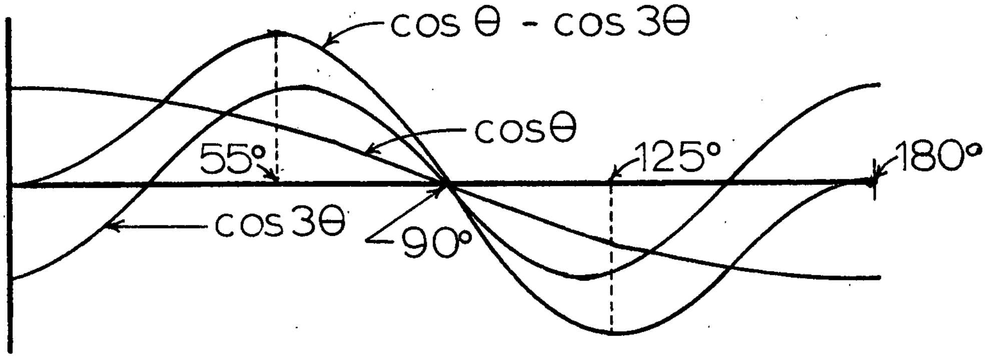

Figure 12

Graph of $\cos\theta - \cos 3\theta$

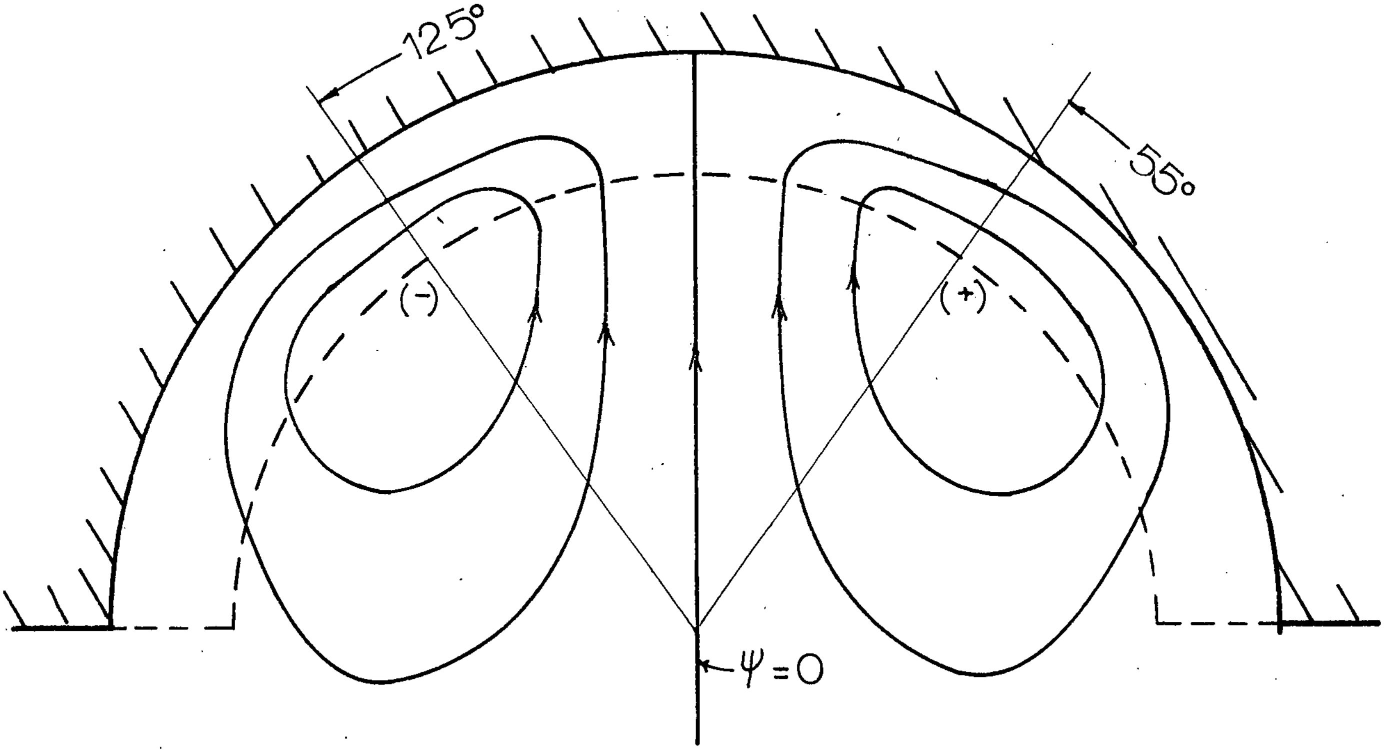

Figure 13

Streamlines, Terms f & i

Hence we obtain the solution:

$$\psi_A = \left(C_1 + C_2 s^3 + \frac{B_b}{4} s^4\right) \frac{dH_1(\theta)}{d\theta} \tag{8.65}$$

$$= \left(C_1 + C_2 s^3 + \frac{B_b}{4} s^4\right) 2\sin 2\theta \tag{8.66}$$

Using the boundary conditions we obtain:

$$\psi_A = -\frac{2}{3} B_b \left[\left(1 - \frac{1}{6}\frac{s_b}{r_b}\right) s_b s^3 - \frac{3}{4} s^4\right] \sin 2\theta \tag{8.67}$$

$$\psi_B = -\frac{1}{6} B_b \frac{s_b^4}{r_b^2} r^2 \sin 2\theta \tag{8.68}$$

and

$$\psi_c = 0 \tag{8.69}$$

ψ_B is sketched in Figure 14.

Notice that the rip current occurs where the wave height is greatest. This is contrary to the result Bowen (1969b) got. However, we obtain Bowen's result for our term a which is similar to his forcing term. Term b is a result of the variation of wave height and the curvature of the beach. It goes to zero value as the beach becomes straight $(r_0 \rightarrow \infty)$.

8h Term i

From (7.1) we obtain:

$$\frac{d^2\psi}{dr^2} + \frac{2m}{d}\frac{d\psi}{dr} = \frac{3kAd^2}{cr^2} \cos 2\phi \frac{d\phi}{d\theta}\frac{dH_b^+}{d\theta} \tag{8.70}$$

We alter this to the form:

$$\frac{d^2\psi}{ds^2} - \frac{2}{s}\frac{d\psi}{ds} = B_i s^2 \frac{de}{d\theta}\frac{dH_1(\theta)}{d\theta}$$

where $B_i = \dfrac{3kAm^2\phi_0 p D}{c r_0^2}$

Thus we obtain:

$$\psi_A = \left(C_1 + C_2 s^3 + \frac{B_i}{4} s^4\right) \frac{de}{d\theta}\frac{dH_1(\theta)}{d\theta} \tag{8.71}$$

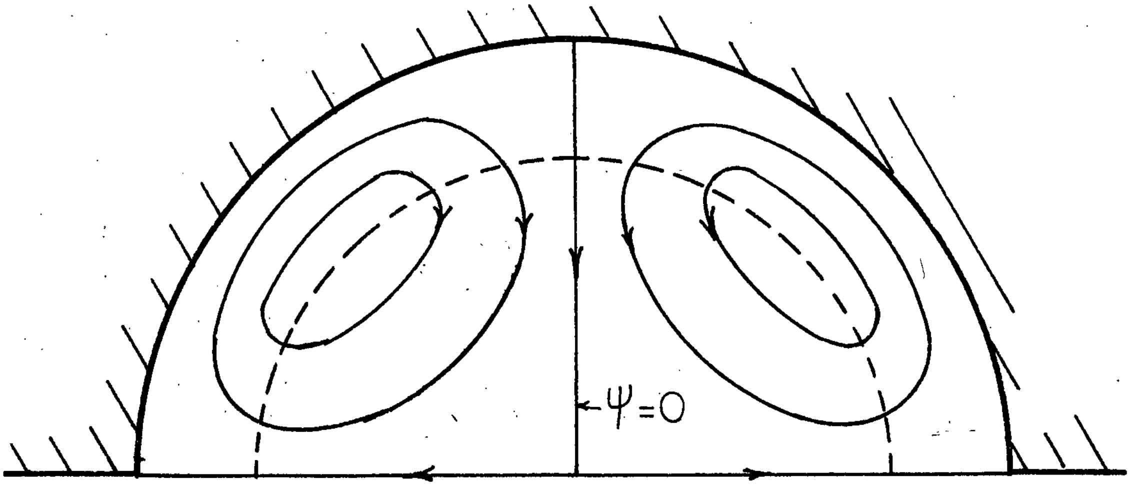

Figure 14

Streamlines, Term b

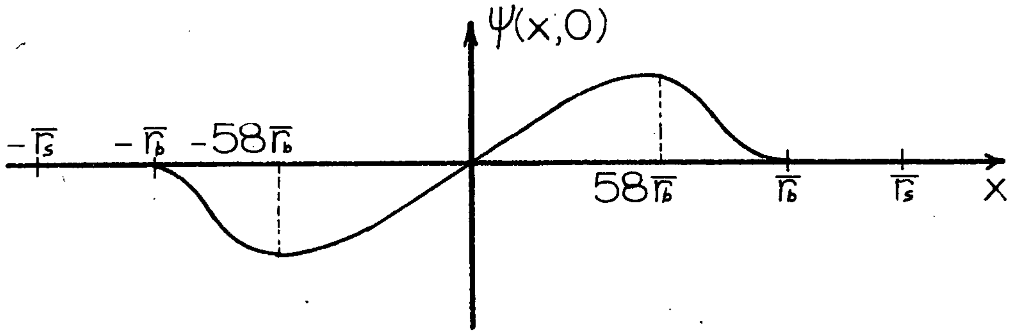

Figure 15

Profile of $\psi_i(x,o)$

or with $e = \cos\theta$ and $H_1(\theta) = -\cos 2\theta$ we get:

$$\psi_A = -\left(C_1 + C_2 s^3 + \frac{B_i}{4} s^4\right)(\cos\theta - \cos 3\theta) \tag{8.72}$$

In a similar manner to the f terms we allow each of the trigonometric terms to represent a forcing function and solve for each. We then add the solutions and obtain:

$$\psi_A \cong \frac{B_i}{3}\left(s_b s^3 - \frac{3}{4} s^4\right)(\cos\theta - \cos 3\theta) \tag{8.73}$$

$$\psi_B = \frac{B_i}{12} \frac{s_b^4}{r_b} r\left(\cos\theta - \frac{r^2}{r_b^2}\cos 3\theta\right) \tag{8.74}$$

and
$$\psi_c = \sum_n A_n \sin\beta_n x \, e^{\beta_n y} \tag{8.75}$$

where $\beta_n = \dfrac{n\pi}{\bar{r}_s}$

and $A_n = \dfrac{2}{\bar{r}_s}\displaystyle\int_0^{\bar{r}_s} \psi(x,0)\sin\beta_n x \, dx$

From (8.73) and (8.74) we obtain the sketch of $\psi(x,0)$ as shown in Figure 15. The sketch of ψ for term i is similar to the sketch for term f in Figure 13.

8i Term h

From (7.1) we obtain:

$$\frac{d^2\psi}{dr^2} + \frac{2m}{d}\frac{d\psi}{dr} = \frac{kAd^2}{cr^2}\sin 2\phi \frac{d^2 H b^4}{d\theta^2} \tag{8.76}$$

which upon rearrangement yields:

$$\frac{d^2\psi}{ds^2} - \frac{2}{s}\frac{d\psi}{ds} = B_h s^2 e(\theta)\frac{d^2 H_1(\theta)}{d\theta^2} \tag{8.77}$$

where $B_h = 2kAm^2\phi_0 p D/c r_0^2$

We obtain the solution:

$$\psi_A = \left(C_1 + C_2 s^3 + \frac{B_h}{4} s^4\right) e(\theta)\frac{d^2 H_1(\theta)}{d\theta^2} \tag{8.78}$$

or substituting for $e(\theta)$ and $H_1(\theta)$:

$$\psi_A = 2\left(C_1 + C_2 s^3 + \frac{Bh}{4} s^4\right)\left(\cos\theta + \cos 3\theta\right)$$

Similarly to term i we obtain:

$$\psi_A = -\frac{2}{3} Bh\left(sb\, s^3 - \frac{3}{4} s^4\right)\left(\cos\theta + \cos 3\theta\right) \tag{8.79}$$

$$\psi_B = -\frac{Bh}{6}\frac{sb^4 r}{r_b}\left(\cos\theta + \frac{r^2}{rb^2}\cos 3\theta\right) \tag{8.80}$$

and
$$\psi_c = \sum_n A_n \sin\beta_n x\, e^{\beta_n y} \tag{8.81}$$

where $\beta_n = n\pi/\bar{r}_s$

and $A_n = \frac{2}{\bar{r}_s}\int_0^{\bar{r}_s} \psi(x,0)\sin\beta_n x\, dx$

From (8.79) and (8.80) we obtain the sketch of $\psi(x,0)$ shown in Figure 16. A sketch of $\cos\theta + \cos 3\theta$ is shown in Figure 17. A sketch of ψ_h is shown in Figure 18.

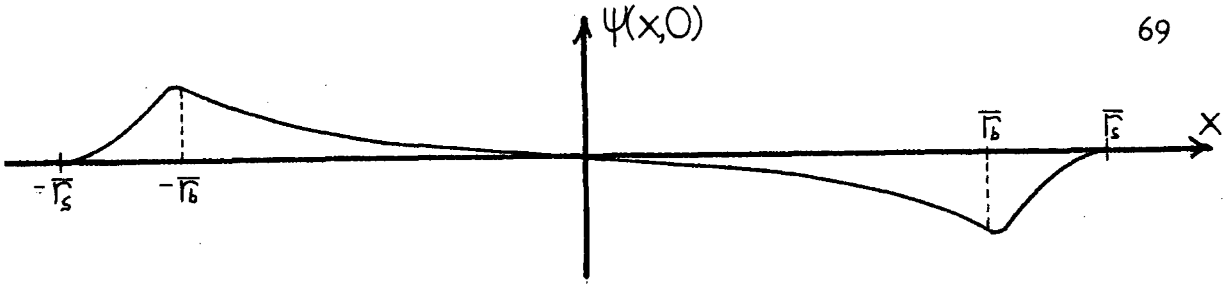

Figure 16

Profile of $\psi_h(x,0)$

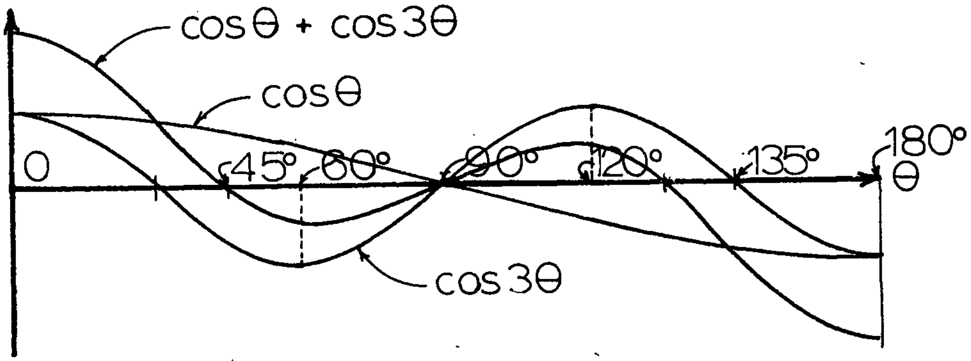

Figure 17

Graph of $\cos\theta + \cos 3\theta$

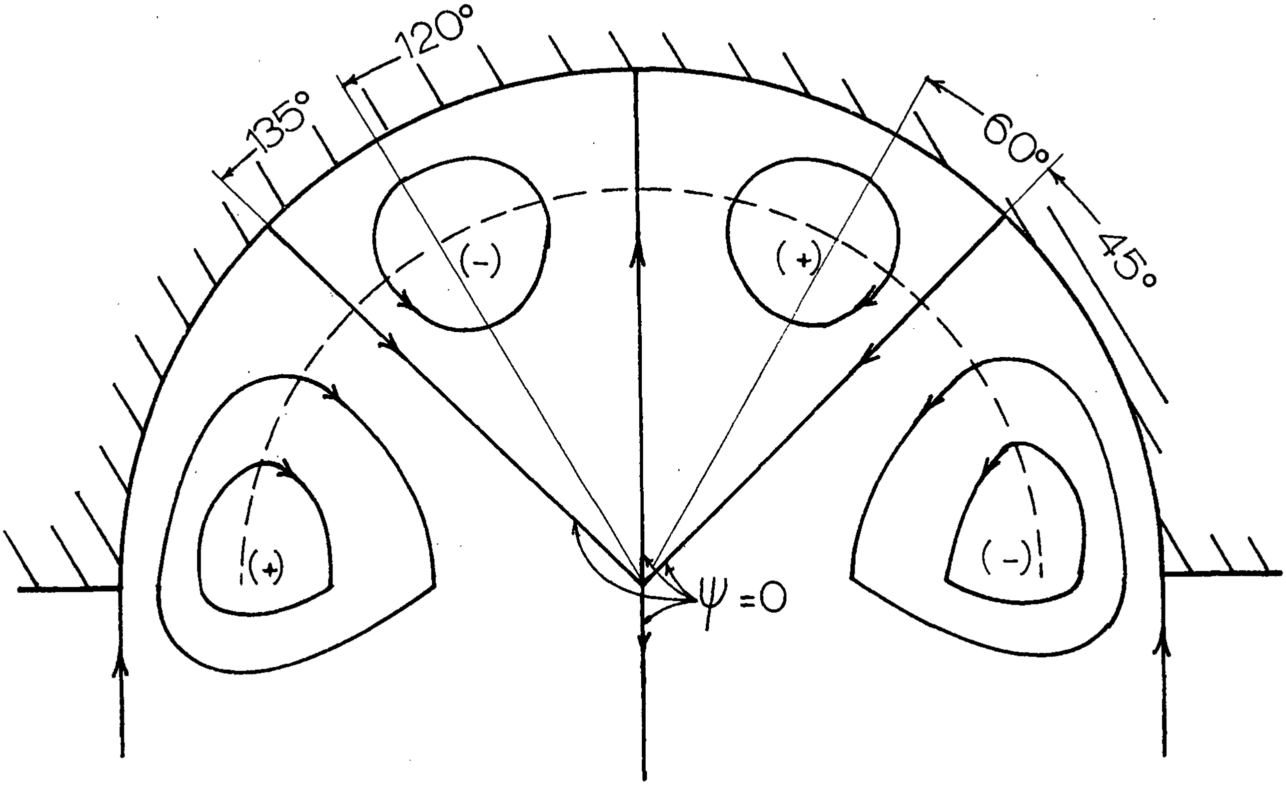

Figure 18

Streamlines, Term h

CHAPTER 9

GENERAL WAVE APPROACH

9a General

In the previous solutions we assumed the deep water wave approach to be in the direction of the radius vector with $\theta = \pi/2$. Let us now examine a more general wave approach in the direction of the radius vector at $\theta = \theta_0$. This does not introduce any new difficulty. We merely determine the functions $Hb^+(\theta)$ and $\phi(\theta)$ for the new wave approach and solve equations (7.1), (7.2) and (7.3) subject to the same boundary conditions.

Figure 19 shows a sketch of a sample refraction pattern for waves approaching with the angle θ_0. The sample bottom topography is shown by contour lines (the dotted lines). The solid lines with direction arrows represent the direction of the wave phase speed and are called orthogonals (see H O pub 234). Where the orthogonals diverge we will have smaller wave heights. From this sketch we assign the following functions to represent the wave characteristics Hb^+ and ϕ:

$$Hb^+ = D(1 + p\, H_1(\theta)) \tag{9.1}$$

$$H_1\theta = \sin \frac{\pi}{2}\frac{\theta}{\theta_0} = \sin \mu\theta$$

where $\mu = \pi/2\theta_0 \;,\; \theta_0 \geq \pi/2$

and

$$\phi = \phi_0\, e(\theta) \tag{9.2}$$

$$e(\theta) = \cos \frac{\pi\theta}{2\theta_0} = \cos\mu\theta,\; \theta_0 \geq \frac{\pi}{2}$$

If $\theta_0 \leq \frac{\pi}{2}$ we would have:

$$H_1(\theta) = \sin \frac{\pi}{2}\left(\frac{180^\circ - \theta}{180^\circ - \theta_0}\right)$$

and

$$e(\theta) = \cos \frac{\pi}{2}\left(\frac{180^\circ - \theta}{180^\circ - \theta_0}\right)$$

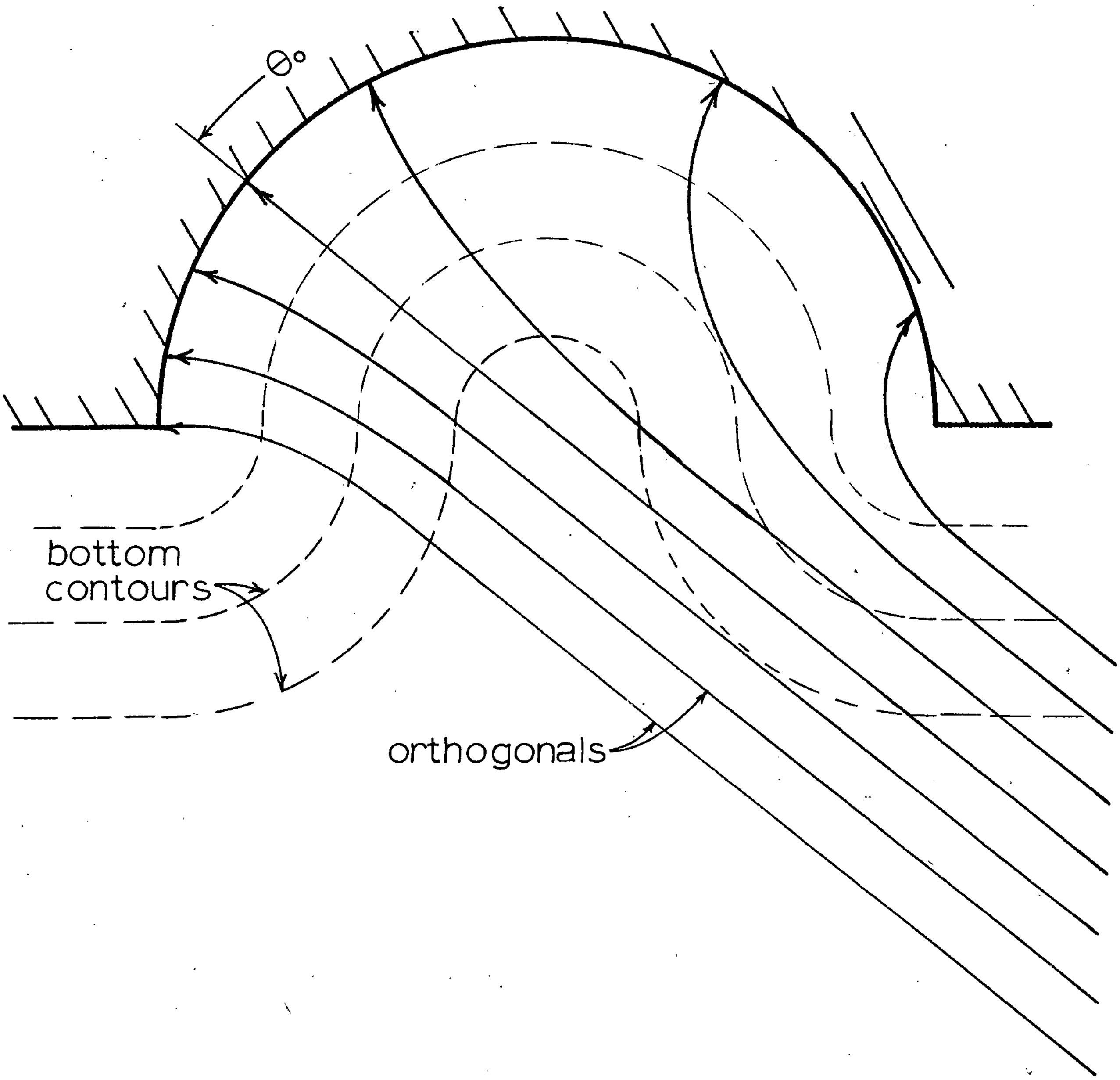

Figure 19

Refraction Pattern, General Approach

Figure 20 is a sketch of these functions. We assume trigonometric functions, however we need not have and for an actual situation we would not expect these functions to be purely trigonometric. If the functions are not trigonometric, then we will have to use Fourier series to solve the boundary conditions. This is unnecessarily complicated for our present discussion (see Section 8b). Using (9.1) and (9.2) in (7.1), (7.2) and (7.3), we arrive at solutions for all the forcing terms. Term g is discussed in the following section.

9b Term g

$$\psi_A = -\frac{1}{3} B_g\left[\left(1-\frac{s_b}{12r_b}\right)s_b s^3 - \frac{3}{4}s^4\right]\cos\mu\theta \tag{9.3}$$

$$\psi_B = -\frac{B_g s_b^4}{12\mu r_b^{\mu}} r^{\mu}\cos\mu\theta \tag{9.4}$$

$$\psi_c = \sum_n A_n \sin\beta_n x\, e^{\beta_n y} \tag{9.5}$$

where $\beta_n = \pi n/\bar{r}_s$

$$A_n = \frac{2}{\bar{r}_s}\int_0^{\bar{r}_s} \psi(x,0)\sin\beta_n x\, dx$$

and where $B_g = \dfrac{4k\, m^3\phi_0}{c\, r_0}$

Note that our solution here is the same as the former case where $\phi_0 = \pi/2$ except that here $\mu = \dfrac{\pi}{2\theta_0}$ instead of unity as for the previous example. This is due to the function $\phi(\theta)$ we chose. However, the result is meaningful providing our $\phi(\theta)$ is fairly realistic. A sketch of the transport stream lines is shown in Figure 21. The remaining rip current systems for each of the other forcing terms can be calculated in a manner similar to that done in CHAPTER 8. Term g, as already calculated, will be the dominant rip current and should be readily observable on a real curved beach.

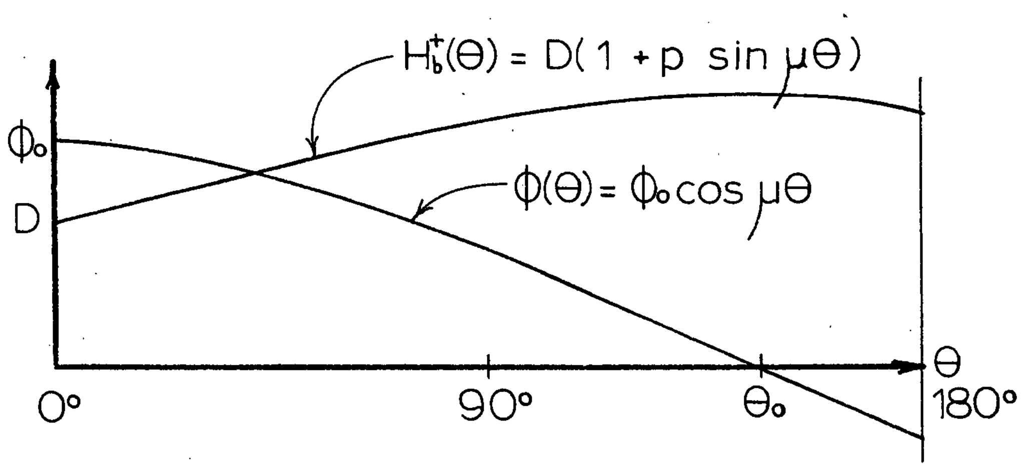

Figure 20

Graph of $\Phi(\theta)$ and $Hb^+(\theta)$
for General Approach

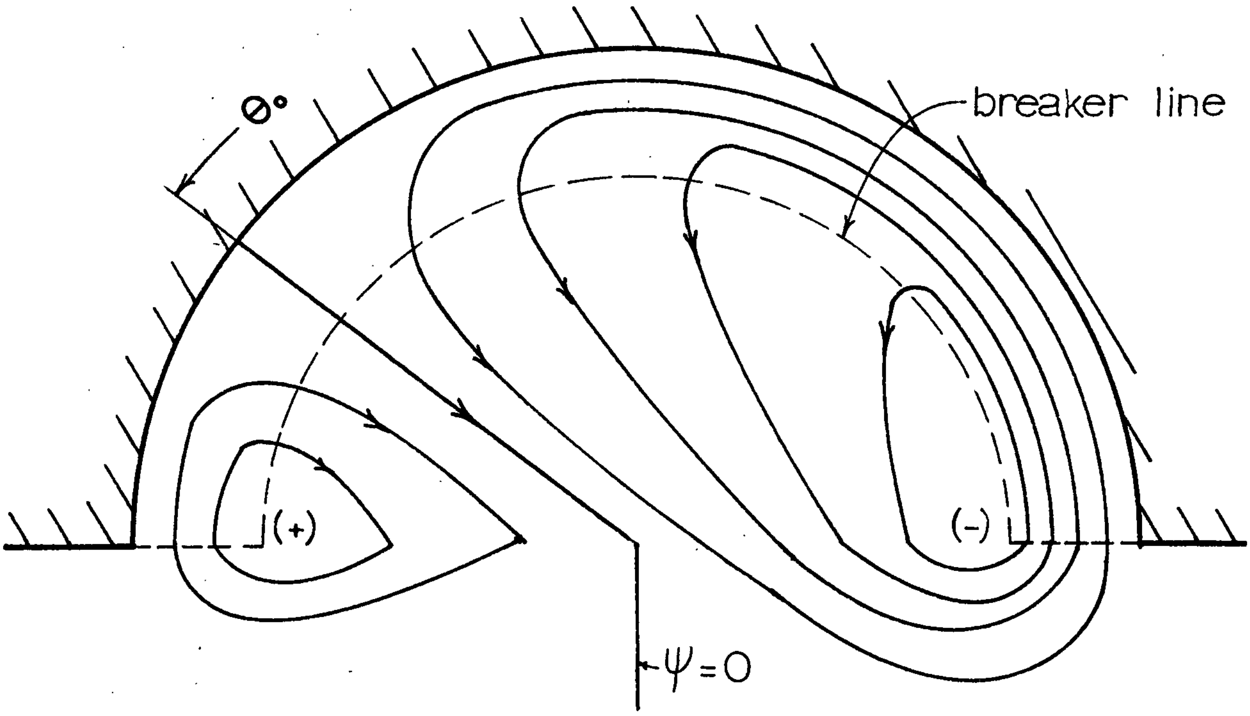

Figure 21

Streamlines,
General Approach, Term g

CHAPTER 10

SUMMARY OF CONCLUSIONS

In this thesis we derived a mathematical model for rip current systems along beaches which takes into account the effects of bottom topography. These effects are: the slope and curvature of the bottom inside the surf zone; the curvature of the shoreline; and the variations in wave height H_b^+ and angle of incidence ϕ of the waves at the breaker line due to wave refraction outside the surf zone.

In this model nine forcing terms caused nine component rip currents whose sum gives the total predicted rip current system. One of these terms (term a) is equivalent to the forcing function Bowen (1969b) used in his straight beach problem. The model was applied to the special case of a circular bay and the rip current patterns which occured were discussed and their streamlines sketched. When one interprets these sketches, it should be remembered that the inertial terms which were neglected in the equations of motion will cause the streamlines to come together where the velocity is towards deeper water and to separate where the velocity flows into shallow water. This effect tends to strengthen the outward flowing rip currents.

It was found that as long as the variations in ϕ and H_b^+ were caused by the circular bay topography only, the dominant flow pattern in the circular bay would be that due to term g (Figure 6). Term c was the next lower term on the magnitude scale (See (5.37)). Figure 22 shows a sketch of the sum of the transport stream functions ψ_g and ψ_c along

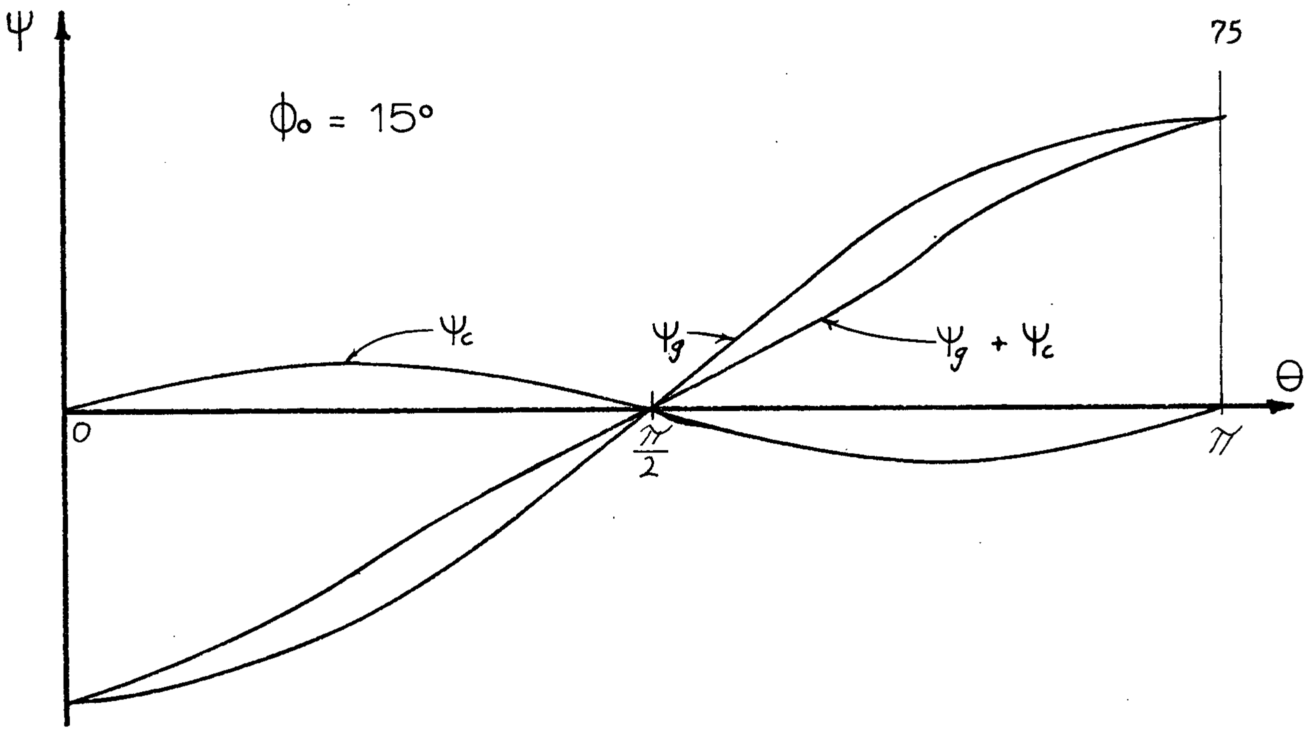

Figure 22

Profile of $\Psi_g + \Psi_c$
at the breaker line

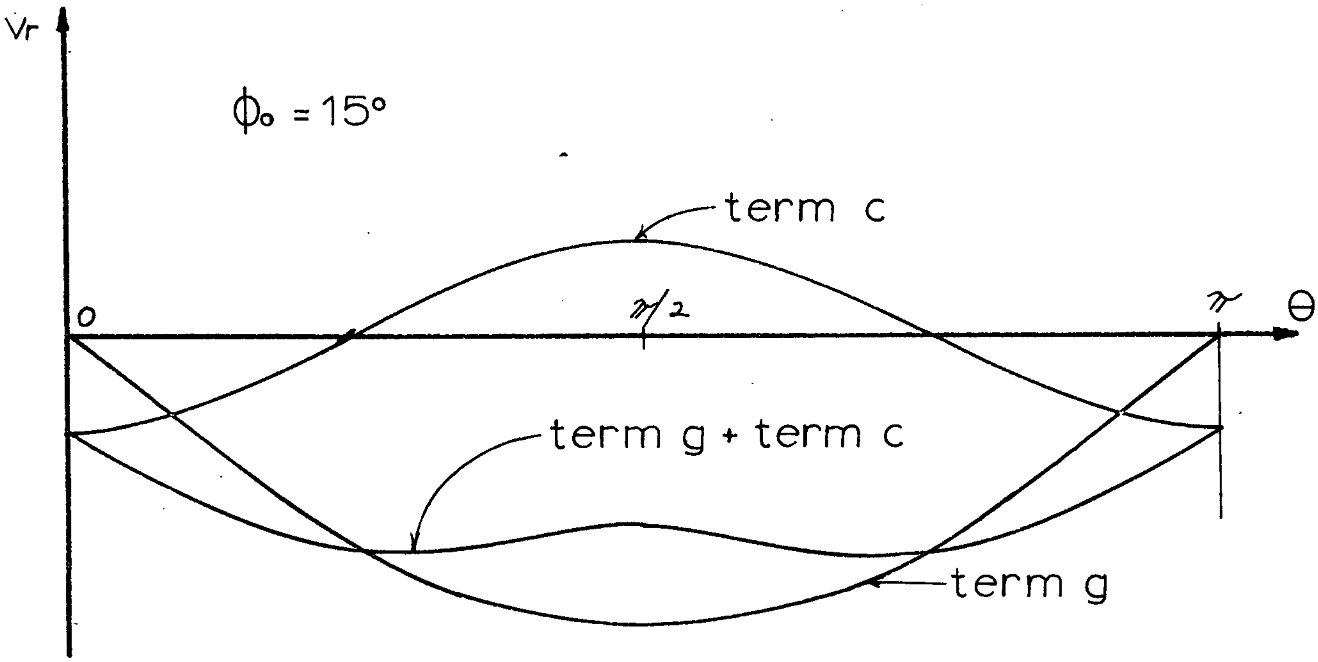

Figure 23

Profile of V_r at the breaker line
due to terms g and c for $\phi_0 = 15°$.

the breaker line for $\phi_o = 15^\circ$. The differences between the curves for ψ_g and $\psi_g + \psi_c$ are seen to be small. The remaining stream functions due to terms e, a, f, b, i and h will be insignificant since they are at least an order of magnitude smaller than term c (see (5.37)). Thus we see that the sketch of ψ_g in Figure 6 represents a first order approximation of the total flow pattern that one would expect to see in a circular bay as we have described it. A sketch of the sum of the velocities normal to the breaker line at the breaker line due to terms g and c is shown in Figure 23. We see that the tendency is to establish a uniform seaward velocity along the breaker line. However, one must be careful when adding these velocities because the effect of the non-linear inertial terms, which we have neglected (Chapter 3), will be to weaken the current due to term c between $\frac{\pi}{4} < \theta < \frac{3}{4}\pi$ and to strengthen the currents due to term g in the same region thereby tending to cause a maximum combined seaward current at $\theta = \pi/2$ which is not apparent from Figure 23. The currents due to the remaining forcing terms are at least an order of magnitude smaller than those due to terms g and c and can be neglected in the first order current approximation.

When discussing this model one must recall that the equation we used to represent $curl_z \underset{\sim}{R}$ inside the surf zone (4.10) assumes that the distance σ_o of significant longshore variations is much larger than the width of the surf zone. This enables us to neglect the $\partial R_r / r\partial\theta$ term in $curl_z \underset{\sim}{R}$. This assumption is valid for refractive effects in large circular bays with conical bottom topography; however, this may not be a valid assumption if there were irregular topographic features present to give small fluctuations to the refractive variables (H_b^+ and ϕ). Edge wave disturbances may also give significant fluctuations over longshore distances σ_a which are comparable to the surf zone width. In our circular bay problem we limit ourselves to the case where $\sigma_o, \sigma_a \gg r_{sb}$, thereby making (4.10) valid.

We find that forcing term g, which is proportional to $\frac{\sin\phi}{r}$, will be the most significant forcing term providing that the radius of curvature r_o of the beach and the longshore distance σ_o over which ϕ and H_b^+ change significantly are of the same order of magnitude (See (B8)). This is the case in our circular bay problem. Because of the $\sin\phi$ dependence of term g we obtain ψ_g along the breaker line to be largest where ϕ is largest. This necessitates the longshore current being largest where ϕ is largest. When the radius of curvature of the beach r_o is much larger than the longshore distance σ_o we have the beach tending towards a straight beach solution and terms c and a will become more significant than term g (See (B8) and (B7)). In this case the refraction effects would be caused by irregularities in the local bottom topography such as the La Jolla Canyon near the Scripps Institution of Oceanography in California (see Shepard and Inman, 1950).

It is interesting to note that term a, which is the forcing function Bowen (1969b) used for a straight beach, is smaller than term c providing ϕ exists and $(\phi_o \Delta\phi)^{\frac{1}{2}} \geqslant \frac{\sigma_o}{\sigma_a} \times 3^o$ where ϕ_o is the scale of ϕ and $\Delta\phi$ is the change in ϕ over the longshore distance σ_o , and σ_a is the significant longshore distance for changes in d in term a (See (B1)). This suggests that term c, as well as term a, may be responsible for the rip currents along the straight beach opposite La Jolla Canyon near San Diego, California due to any small refraction caused by the offshore topography. Term c is in fact very sensitive to any small amounts of refraction and must be considered as a potential cause for rip currents even on a so-called straight beach where some small degree of refraction will almost always be present. The following paragraphs state some recommendations for future studies.

It is suggested that this linear analytical model of a circular beach could be extended to more realistically shaped beaches by fitting the shoreline with circular arcs, concave or convex, and solving for each circular arc region in a similar manner to our circular beach solution and matching each arc solution to its adjacent arc solution.

It would be interesting to use a numerical computer method to solve this circular bay problem including nonlinear inertial terms and eddy viscosity as did Bowen (1969b) for his straight beach model.

A computer program could also be made, using our general mathematical model, to predict rip current systems for general beaches with irregular bottom topography. A part of this program could predict refraction patterns for given wave approaches and wave heights and determine the location of the breaker line and compute $H_b^+(\theta)$ and $\phi(\theta)$.

It should be worth studying the effects of edge waves on the mean surface height $\bar{\eta}$, thus giving us more insight in discussing term a.

Experimental work should be done to verify our mathematical model and to examine the effects of a nonlinear bottom slope as predicted by term d. Experiments on straight beaches could be conducted to examine the relative sizes of currents produced by forcing terms c and a.

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APPENDIX A

DISCUSSION OF SET DOWN

The momentum equation (3.4) gives us:

$$\frac{\partial}{\partial x_\beta}\left(\tilde{u}_1 \tilde{M}_\beta\right) + \frac{\partial S_{1\beta}}{\partial x_\beta} = -\rho g d \frac{\partial \bar{\eta}}{\partial x} \tag{A1}$$

If we have a straight beach and the waves are advancing normal to the beach, we have:

$$\tilde{u}_1 = 0 \tag{A2}$$

as discussed in (5.1) and we obtain:

$$\frac{\partial S_{1\beta}}{\partial x_\beta} = -\rho g d \frac{\partial \bar{\eta}}{\partial x} \tag{A3}$$

from which Longuet-Higgins and Stewart (1964) got:

$$\bar{\eta} = -\frac{1}{8} \frac{H^2 g}{\sinh 2gh} \tag{A4}$$

as described in (5.3) of this paper for the region outside the surf zone for $S_{12} = 0$. Inside the surf zone we assume:

$$H = \gamma(\bar{\eta} + h) = \gamma d \tag{A5}$$

and

$$\left.\begin{aligned} S_{xx} &= 3/2\ E = \tfrac{3}{16}\rho g H^2 \\ S_{xy} &= 0 \end{aligned}\right\} \tag{A6}$$

as did Bowen (1969b). That is we assume that the radiation stress is the same function of wave energy inside the surf zone as outside. Using (A5), (A6) and (A3) we obtain:

$$\frac{\partial \bar{\eta}}{\partial x} = -K\frac{\partial h}{\partial x} = m_\eta = K m_b \tag{A7}$$

where

$$K = \left(1 + \frac{8}{3\gamma^2}\right)^{-1}$$

and m_η is the absolute value of the surface slope.

m_b is the absolute value of the bottom slope.

A problem arises when we have a rip current system where we know $\tilde{u}_1 \neq 0$. Does (A3) still apply and give us (A4) and (A7) which Bowen (1969b) used in his rip current analysis?

If (A3) is still applicable for our rip current system, we must have

$$\frac{\partial}{\partial x_\beta}(\tilde{u}_1 \tilde{M}_\beta) \ll \frac{\partial S_{1\beta}}{\partial x_\beta} \tag{A8}$$

Now

$$\frac{\partial}{\partial x_\beta}(\tilde{u}_1 \tilde{M}_\beta) = \tilde{M}_\beta \frac{\partial}{\partial x_\beta}\tilde{u}_1 + \tilde{u}_1 \frac{\partial}{\partial x_\beta}\tilde{M}_\beta \tag{A9}$$

and using (3.3) we obtain:

$$\frac{\partial}{\partial x_\beta}(\tilde{u}_1 \tilde{M}_\beta) = \tilde{M}_\beta \frac{\partial}{\partial x_\beta}\tilde{u}_1 \tag{A10}$$

or

$$= \rho d\left(\tilde{u}_1 \frac{\partial \tilde{u}_1}{\partial x} + \tilde{u}_2 \frac{\partial \tilde{u}_1}{\partial y}\right) \tag{A11}$$

by the definition of $\tilde{M}_\beta$ in (3.2).

Near the surf zone in a large bay, we expect that velocity changes in the onshore direction will be much larger than in the longshore direction. Thus we neglect the $\partial \tilde{u}_1/\partial y$ term of (A11) and obtain:

$$\frac{\partial}{\partial x_\beta}(\tilde{u}_1 \tilde{M}_\beta) \cong \rho d \tilde{u}_1 \frac{\partial \tilde{u}_1}{\partial x} \tag{A12}$$

Using (A6) and (A5) we obtain:

$$\frac{\partial S_{1\beta}}{\partial x_\beta} \cong \frac{\partial S_{11}}{\partial x} = \frac{3}{8}\rho g \gamma^2 d \frac{\partial d}{\partial x} \tag{A13}$$

Thus by forming a ratio we obtain from (A12) and (A13):

$$\frac{\partial S_{1\beta}/\partial x_\beta}{\partial \tilde{u}_1 \tilde{M}_\beta/\partial x_\beta} \cong \frac{3}{8} g \gamma^2 \frac{\partial d/\partial x}{\tilde{u}_1 \partial \tilde{u}_1/\partial x} \tag{A14}$$

Rip current measurements were made by Shepard and Inman (1950) and to estimate the magnitude of (A14) we use some of their sample results:

Let $\gamma = 1$ $\qquad d_b = 10$ ft.

$\tilde{u}_1 = 2$ kn. $= 3.4$ ft./sec.

$g = 32$ ft./sec.^{2}

Thus (A14) becomes with $\partial d/\partial x \cong \frac{10}{r_{sb}}$, $\frac{\partial \tilde{u}_1}{\partial x} \cong \frac{3.4}{r_{sb}}$;

$$\frac{\partial S_{\beta\beta}/\partial x_\beta}{\partial(\tilde{u}_1 M_\beta)/\partial x_\beta} \cong \frac{3}{8} \times 32 \times 1^2 \times \frac{10}{3.4^2} \cong 10 \qquad \text{(A15)}$$

or in general equals $12\, d_b/\tilde{u}_1^2$ in units of feet and seconds. Hence it is expected that $\partial(\tilde{u}_1 M_\beta)/\partial x_\beta \ll \partial S_{\beta\beta}/\partial x_\beta$ by an order of magnitude and therefore (A4) and (A7) may be used with reasonable accuracy when rip currents exist.

APPENDIX B

ORDERING OF FORCING TERMS

We shall discuss the relative magnitudes of the forcing terms given in equation (5.36). From (2.21) and (2.1) we see that ADp is the amplitude of the θ varying component of $\bar{\eta}$ and also of d if we let m be independent of θ as it is for a circular bay (see (5.33)). We introduce the ordering scheme:

$$
\begin{aligned}
&\eta_1 = ADp && \text{for refractive variables.}\\
&\eta_1 = \eta_a && \text{for term a only.}\\
&\phi = \phi_0 \phi'(\theta) && \\
&\delta\phi = \Delta\phi\, \delta\phi' && \text{in general, and } \Delta\phi = \phi_0 \text{ for a circular bay.}\\
&Hb^+ = D[1 + pH_1(\theta)] && \\
&\delta\theta = \frac{\pi}{2}\delta\theta' && \text{for the refraction variables } Hb^+ \text{ and } \phi \text{ in a circular bay.}\\
&r\delta\theta = \delta\sigma = \sigma_0 \delta\sigma' && \text{for refraction variables on a general shoreline.}\\
&\sigma_0 = \pi/2 && \text{for a circular bay.}\\
&r\delta\theta = \delta\sigma = \sigma_a \delta\sigma' && \text{for the longshore variable of term a only.}\\
&r = r_0 r' && \\
&\delta r = r_{sb}\delta r' && \\
&d = \overline{hb}\, d'(r,\theta) && \\
&\delta d(r) = \overline{hb}\, \delta d'(r) && \\
&\delta d(r,\theta) = \eta_1 \delta d'(r,\theta) &&
\end{aligned}
\tag{B1}
$$

where the primed symbols are variables of order unity. The differentials $\partial d'(r)$ and $\partial d'(r,\theta)$ are introduced to allow for the different rates of change of d in the r and θ component directions as we did in (4.1). We introduce two longshore scales σ_o and σ_a to allow us to differentiate between the longshore variation due to refraction and any possible longshore variation in term a due to edge waves. We also let $\eta_1 = \eta_a$ for term a. This is because η_a may not be caused by refraction but by edge waves and therefore may not equal $\eta_1 = ADp$.

<u>Terms d and g</u>

$$\text{Terms } \frac{d}{g} = -\frac{k \sin 2\phi \, \partial^2 d/\partial r^2}{\frac{2km}{r} \sin 2\phi} \tag{B2}$$

Using the ordering scheme we obtain:

$$\text{Terms } \frac{d}{g} = -\frac{1}{2m} \cdot \frac{\bar{h}_b}{r_{sb}} \frac{r_o}{r_{sb}} \left(r' \frac{\partial^2 d'}{\partial r'^2} \right) \tag{B3}$$

$$\bar{h}_b / r_{sb} \cong m \tag{B4}$$

therefore $$\text{Terms } \frac{d}{g} \cong \frac{1}{2} \frac{r_o}{r_{sb}} \left(r' \partial^2 d' / \partial r'^2 \right) \tag{B5}$$

If we have a large bay, $\frac{1}{2} \frac{r_o}{r_{sb}} \gg 1$ and if $\partial^2 d' / \partial r'^2$ is of order unity we see that term d is much larger than term g. However, in a beach with a linear bottom $\partial^2 d' / \partial r'^2 = 0$ and term d is thus zero. However, this ratio does show us that if the bottom were not linear then term d could be of great influence.

Terms g and a

$$\text{Terms } \frac{g}{a} = \frac{\frac{2km}{r}\sin 2\phi}{\frac{2k}{r}\cos 2\phi \frac{\partial^2 d}{\partial\theta\partial r}} = 2\phi_0\left(\frac{\overline{hb}}{\eta_a}\right)\left(\frac{\sigma_a}{r_0}\right)\left(\frac{\phi'}{r'\frac{\partial^2 d'}{\partial\sigma'\partial r'}}\right) \tag{B6}$$

where we let $m \cong \overline{hb}/r_{sb}$

If $\phi_0 \cong \pi/12$ then $\text{terms } \frac{g}{a} \cong \frac{1}{2}\left(\frac{\overline{hb}}{\eta_a}\right)\left(\frac{\sigma_a}{r_0}\right)$

Now if we say $\overline{hb}/\eta_a \cong 100$ then

$$\text{Terms } \frac{g}{a} \cong 50\,\sigma_a/r_0 \tag{B7}$$

In the absence of short wave length edge wave modulation of the incoming swell, we can say that the longshore length scale will be determined by the beach curvature and then we can say roughly that $r_0 \cong \sigma_a$ and so term g = 50 term a. As the bay gets large $(r_0 \rightarrow \infty)$ then term a will be more significant than term g, providing edge waves are generated and η_a is of finite value.

Terms g and c

$$\text{Terms } \frac{g}{c} = \frac{\frac{2km}{r}\sin 2\phi}{\frac{3km}{r}\sin 2\phi\frac{\partial\phi}{\partial\theta}} = \frac{2}{3}\frac{1}{\Delta\phi}\left(\frac{\sigma_0}{r_0}\right)\left(\frac{1}{r'\frac{\partial\phi'}{\partial\theta'}}\right)$$

For a circular bay we have $\Delta\phi = \phi_0 \cong \pi/12$ and $\sigma_0 = \pi/2$ and so we obtain:

$$\text{Terms } \frac{g}{c} = \frac{2}{3}\frac{1}{\Delta\phi}\left(\frac{\sigma_0}{r_0}\right) \cong 4 \tag{B8}$$

Terms c and e

$$\text{Terms } \frac{c}{e} = \frac{\frac{3km}{r}\sin 2\phi\,\partial\phi/\partial\theta}{\frac{kd}{r^2}\cos 2\phi\,\frac{\partial^2\phi}{\partial\theta^2}} = 6\phi_0\frac{\sigma_0}{r_{sb}}\frac{r'\phi'}{d'}\frac{\partial\phi'/\partial\sigma'}{\partial^2\phi'/\partial\sigma'^2} \tag{B9}$$

and for a circular bay $\phi_0 \cong \frac{\pi}{12}$, $\frac{\sigma_0}{r_{sb}} = \frac{\pi}{2}\frac{r_0}{r_{sb}}$ we get:

$$\text{Terms } \frac{c}{e} \cong 2\frac{r_0}{r_{sb}}\left(\frac{r'\phi'}{d'}\frac{\partial\phi'/\partial\sigma'}{\partial^2\phi'/\partial\sigma'^2}\right) \tag{B10}$$

Thus, if the primed terms are of order unity and we expect them to be so, we see that

$$\text{Term } c \cong \frac{2r_0}{r_{sb}}\text{Term } e \tag{B11}$$

<u>Terms c and a</u>

$$\text{Terms } \frac{c}{a} = \frac{\frac{3km}{r}\sin 2\phi\,\partial\phi/\partial\theta}{\frac{2k}{r}\cos 2\phi\,\partial^2 d/\partial\theta\partial r} = 3m\phi\,\frac{\partial\phi/\partial\sigma}{\partial^2 d/\partial\sigma\partial r}$$

where $\partial\sigma = r\,\partial\theta$ the longshore arc length. We say ϕ and d change by $\Delta\phi$ and η_a over the distances σ_0 and σ_a respectively, then: $\frac{\partial\phi}{\partial\sigma} = \frac{\Delta\phi}{\sigma_0}\frac{\partial\phi'}{\partial\sigma'}$,

and
$$\frac{\partial^2 d}{\partial r\partial\sigma} = -\frac{\eta_a}{r_{sb}\sigma_a}\frac{\partial^2 d'}{\partial\sigma'\partial r'}$$

Thus $\text{Terms } \frac{c}{a} = 3\left(\frac{\sigma_a}{\sigma_0}\right)\left(\frac{\overline{h_b}}{\eta_a}\right)\Delta\phi\,\phi_0\left(\phi'\partial\phi'/\partial\sigma'\right)/\left(\partial d'/\partial r'\partial\sigma'\right)$ (B12)

If $\Delta\phi = \phi_0 \cong \pi/12$ and $\sigma_a = \sigma_0$ then $\text{Terms } \frac{c}{a} \cong \frac{1}{4}\,\overline{h_b}/\eta_a$ thus $\text{Term } c \gg \text{Term } a$ for our circular bay problem. Also, $\text{Term } C \cong \text{Term } a$ when $(\Delta\phi\,\phi_0)^{\frac{1}{2}} \cong 3^\circ$ assuming $\overline{h_b}/\eta_a \cong 100$ and $\sigma_a = \sigma_0$.

<u>Terms a and e</u>

$$\text{Terms } \frac{a}{e} = \frac{\frac{2k}{r}\cos 2\phi\frac{\partial^2 d}{\partial\theta\partial r}}{\frac{kd}{r^2}\cos 2\phi\frac{\partial^2\phi}{\partial\theta^2}} = \frac{2}{\Delta\phi}\left(\frac{\eta_a}{\overline{h_b}}\right)\left(\frac{\sigma_0}{r_{sb}}\right)\left(\frac{\sigma_0}{\sigma_a}\right)\left(\frac{r'}{d'}\frac{\partial^2 d'/\partial\sigma'\partial r'}{\partial^2\phi'/\partial\sigma'^2}\right)$$

If we assume the primed terms to be of order unity and assume a circular bay where $\Delta\phi = \phi_0 \cong \pi/12$ and $\sigma_0 = r_0\pi/2$

we get:

$$\text{Term a} \cong 12 \frac{\eta_a}{\overline{hb}} \frac{r_o}{r_{sb}} \frac{\sigma_o}{\sigma_a} \text{Term e} \tag{B13}$$

and if $\eta_a \cong \frac{\overline{hb}}{100}$ and $5 \leqslant \frac{r_o}{r_{sb}} \leqslant 10$ and $\frac{\sigma_o}{\sigma_a} = 1$

then we obtain:

$$0.6 \text{ Term e} \lesssim \text{Term a} \leqslant 1.2 \text{ Term e} \tag{B14}$$

Thus term a is of the order of magnitude of term e if $\partial^2 d' / \partial \sigma' \partial r'$ is of order unity. However, in our problem of the circular bay, we assumed $m = -\partial d/\partial r$ to be a constant and so $\partial^2 d' / \partial \sigma' \partial r' = 0$. Bowen (1969b) admitted a θ dependence to m and thereby obtained a rip current system; however, according to our present theory of set-up, we do not obtain a variable m. Perhaps if one were to consider the effects of edge waves and not just bottom topography effects, one would get m to be a function of θ. If this was found to be the case, then it is also possible that σ_a, the longshore scaling parameter for term a, would not equal σ_o, the longshore scaling factor for the refraction variables Hb^+ and ϕ. If $\sigma_o/\sigma_a > 1$ then term a would be more significant by a factor of σ_o/σ_a. This factor is also relevant for comparing terms c and a (see (B12)).

<u>Terms e and f</u>

$$\text{Terms} \frac{e}{f} = -\frac{\cos 2\phi \, \partial^2\phi/\partial\theta^2}{2\sin 2\phi \, (\partial\phi/\partial\theta)^2} = -\frac{1}{4\phi_o \Delta\phi}\left(\frac{\partial^2\phi'/\partial\sigma'^2}{\phi' \, (\partial\phi'/\partial\sigma')^2}\right)$$

We expect the primed term to be of order unity, so for a circular bay:

$$\text{Terms} \frac{e}{f} \cong -\frac{1}{4\phi_o^2} \cong -4 \tag{B15}$$

Terms f and b

$$Terms\ \frac{f}{b} = \frac{-\frac{kd}{r^2}\, 2\sin 2\phi \left(\frac{d\phi}{d\theta}\right)^2}{\frac{kA}{r^2}\cos 2\phi\, \frac{dHb^+}{d\theta}} = -4\,\phi_o\,\Delta\phi^2 \left(\frac{\overline{hb}}{\eta_1}\right)\left(\frac{r_o}{\sigma_o}\right)\left(\frac{r'd'\phi'(d\phi'/d\sigma')^2}{dH'/d\sigma'}\right)$$

We expect the primed terms to be of order unity and $\overline{hb}/\eta_1 \cong 100$ thus for a circular bay:

$$Terms\ \frac{f}{b} \cong -\frac{1}{24}\,\frac{\overline{hb}}{\eta_1} \cong -4 \qquad \text{(B16)}$$

Terms i and b

$$Terms\ \frac{i}{b} = \frac{\frac{3kA}{r^2}\cos 2\phi\, \frac{d\phi}{d\theta}\, \frac{dHb^+}{d\theta}}{\frac{kA}{r^2}\cos 2\phi\, \frac{dHb^+}{d\theta}} = 3\,\frac{r_o}{\sigma_o}\,\Delta\phi\left(\frac{d\phi'}{d\sigma'}\right)$$

or $Terms\ \frac{b}{i} \cong 2$ for a circular bay. (B17)

Terms i and h

$$Terms\ \frac{i}{h} = \frac{\frac{3kA}{r^2}\cos 2\phi\, \frac{d\phi}{d\theta}\, \frac{dHb^+}{d\theta}}{\frac{kA}{r^2}\sin 2\phi\, \frac{d^2Hb^+}{d\theta^2}} = \frac{3}{2}\,\frac{\Delta\phi}{\phi_o}\left(\frac{1}{\phi'}\,\frac{\frac{d\phi'}{d\sigma'}\frac{dH'}{d\sigma'}}{d^2H'/d\sigma'^2}\right)$$

We expect the primed terms to be of order unity and $\Delta\phi = \phi_o$ for a circular bay, therefore:

$$Terms\ \frac{i}{h} \cong \frac{3}{2} \qquad \text{(B18)}$$

Thus we obtain a magnitude structure as follows:

Term	Circular Bay Factor	General Factor	
d, g	$\frac{1}{2} r_0/r_{sb}$	$\frac{1}{2}\frac{r_0}{r_{sb}}$	
g, c	4	$\frac{2}{3}\frac{1}{\Delta\phi}\frac{\sigma_0}{r_0}$	
c, e	$2\, r_0/r_{sb}$	$6\phi_0\frac{\sigma_0}{r_{sb}}$	
e, a	1	$\frac{\Delta\phi}{2}\frac{\overline{hb}}{\eta_a}\frac{r_{sb}}{\sigma_0}\frac{\sigma_a}{\sigma_0}$	(B19)
e, f	-4	$-\frac{1}{4}\frac{1}{\phi_0\Delta\phi}$	
f, b	-4	$-4\phi_0\Delta\phi^2\frac{\overline{hb}}{\eta_1}\frac{r_0}{\sigma_0}$	
b, i	2	$\frac{1}{3}\frac{\sigma_0}{r_0}\frac{1}{\Delta\phi}$	
i, h	$3/2$	$\frac{3}{2}\frac{\Delta\phi}{\phi_0}$	