MAGNETOHYDRODYNAMICS OF NEUTRON STAR INTERIORS

by

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ABSTRACT

The dynamics of the charged particles in the fluid interior of rotating magnetized neutron stars (pulsars) is investigated. It is shown that a magnetohydrodynamic approach is valid under a wide variety of conditions. The small amplitude waves that can propagate in the "charged" fluid are sound waves (period $\sim 10^{-4}$ sec), inertial waves (≈ 1 sec) and hydromagnetic-inertial waves ($\sim$ months). Generally, the most effective damping mechanism is viscosity. Viscous damping times for hydromagnetic-inertial waves can be as long as hundreds of years.

Several normal sound mode and normal inertial mode calculations are performed. The motion of the charged fluid over the time scales of inertial waves and hydromagnetic-inertial waves is simulated on a computer. Mode-mode coupling, boundary layers, and the acceleration of the crust due to charged fluid motion are also studied.

The most important conclusion in this thesis is that the time the charged fluid as a whole takes to respond to a sudden disturbance such as a change in the angular velocity or acceleration of the crust is the time for a long wavelength hydromagnetic-inertial wave to cross the star. Contrary to an assumption which has been made in models of pulsar post-glitch behaviour, the charged fluid response time is not necessarily small compared to post-glitch relaxation times. In view of this, models of post-glitch behaviour which assume that the charged fluid response time is small compared to post-glitch relaxation times should be re-examined.

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CHAPTER I: INTRODUCTION

Very soon after the discovery of pulsars it was realized that there is probably only one kind of object which can display the observed pulsar characteristics. That object is a rotating neutron star which possesses a large magnetic field.

This thesis is concerned with one particular feature of rotating, magnetized neutron stars: the magnetohydrodynamics of the charged fluid in their interiors. Very little work has hitherto been done in this area. Accordingly, the first part of the thesis is devoted to a careful investigation of various aspects of the charged fluid dynamics and to estimates of their importance. The second part contains detailed numerical calculations of some possible charged fluid flows and modes of oscillation.

This introductory chapter begins with a short review of some theoretically predicted properties of neutron stars, with emphasis on those properties which are most relevant to the charged fluid dynamics.¹ Next, some features of pulsar observations are reviewed, with emphasis on those aspects which one hopes will reveal something about the interior. Previous work on the charged fluid dynamics is critically reviewed. Finally, a number of important questions about the dynamics are posed. In this thesis I will attempt to answer these questions.

¹ For more detailed information on neutron stars and pulsars, see the review articles by ter Haar(1972), Ruderman(1972) and Ginzberg(1971).

Review of Neutron Star Properties

The neutron star structures and properties described in this section are based on the theoretical calculations of many people and not for the most part on direct observational evidence, because while there is a wealth of information contained in pulsar observations, it is not easy to extract from these observations information about neutron star interiors. Nevertheless, a consensus has emerged over the last several years on many aspects of neutron star structure. This view is presented below in general terms. Many points are discussed more thoroughly and quantitatively in Chapter II.

A general overview of neutron star structure is given in Table I.

Table I
Overall Structure Of Neutron Stars

Radius	about 10 km
Composition	mainly neutrons
Mass	about one solar mass
Average density	roughly nuclear (10^{14} gm cm ⁻³)
Temperature	10^6 to 10^8 K
Surface magnetic field	10^{10} to 10^{14} gauss
Rotational period	30 msec to a few seconds
Inner structure	solid core (?) fluid interior solid crust
Surrounded by	magnetosphere

Figure 1 shows a schematic picture of a neutron star. It may help the reader if he refers to Table I and to Figure 1 while reading the present section.

It is convenient for the purposes of this thesis to distinguish three regions in the interior of a neutron star: a solid core, a fluid interior, and a solid crust. These three regions and the region immediately surrounding the star, the magnetosphere, will now be discussed in turn.

Figure 1. A schematic picture of a rotating magnetized neutron star.

The Solid Core

A solid core may or may not exist at the centre of some neutron stars. The uncertainty arises mainly from the unknown nature of the interparticle interaction at the high densities there (about $10^{15} \text{ gm cm}^{-3}$). If present, the solid core is expected to consist of a neutron crystal. Detailed calculations of the crystal properties have been made by Canuto and Chitre (1974). They find that the material moduli of the crystal are about $10^{36} \text{ dyne cm}^{-2}$, which implies a very rigid core. The latest most refined calculations in this area are those of Pandharipande and Smith (1975), who find that solid cores are possible in stars with masses within $.1 M_{\odot}$ of the maximum mass for a stable star (Smith, private communication). For their two different potentials, this maximum mass is 1.8 and $2.3 M_{\odot}$. It is uncertain if neutron stars in this mass range can be formed.

The Fluid Interior

More than 90% of the particles in the fluid interior are degenerate non-relativistic neutrons which form a superfluid of the Bardeen-Cooper-Schrieffer type. The superfluid "rotates" in the sense that there are vortices in whose cores the neutrons are normal. The most widely accepted view is that the superfluid is approximately uniformly rotating, with aligned vortices (Ruderman, 1970; Ruderman and Sutherland, 1974).

The rest of the fluid interior consists of charged particles, primarily of equal numbers of protons and electrons. The protons are degenerate, non-relativistic and probably form a type II superconductor. The electrons are degenerate and relativistic but not superconducting (Baym et al., 1969b).

Because of their strong tendency towards local charge neutrality, it is natural to treat these charged particles as a single fluid - a "charged" fluid. The dynamics of this charged fluid is the subject of this thesis.

The only direct interaction between the superfluid and the charged fluid comes about by the scattering of electrons off those few neutrons which are not in a superfluid state (about 10^{-18} of all neutrons). The small fraction of protons which is not in the superconducting state can also scatter off the normal neutrons. The fluid interior can thus be thought of as being composed of two interpenetrating, very weakly interacting fluids.

Some of the transport coefficients of these fluids depend sensitively on the temperature of the interior. For example, the viscosity calculated by Flowers and Itoh (1975) is $\propto T^{-2}$. Their paper contains the most careful and up-to-date calculations of transport coefficients in neutron stars, so I will use their results in this thesis.

The temperature in the interior may vary from about 10^9 K for neutron stars a few hours old to perhaps as low as 10^5 K for extremely old ones (Tsuruta and Cameron, 1965). Most observed pulsars are so old that their interior temperatures are $\lesssim 10^8$ K. The thermal conductivity of the electrons is so high (about 10^{24} erg/cm-deg-sec at $T = 10^8$ K; Flowers and Itoh, 1975) that the fluid interior is practically isothermal.

Estimates of the interior temperature of a neutron star of a given mass and age are in a very confused state, because the temperature is sensitive to a large number of factors such as magnetic field strength, superfluidity, pion condensation, and so on. Greenstein and McClintock (1974) have recently reviewed the problem of neutron star temperatures, both from a theoretical and an observational point of view.

There is a strong magnetic field in the interior. Just how strong it is and what its structure is are not known. There is fairly general agreement, however, that this field is a remnant of a seed field which was greatly amplified in the collapse which formed the neutron star.

A somewhat simplistic argument illustrates this idea. If a star of 10^{11} cm radius which possesses a poloidal field of 10^2 gauss collapses to form a neutron star of 10^6 cm radius, then according to the argument, conservation of magnetic flux implies the resulting field should have a strength of $10^2 \times (10^{11}/10^6)^2 = 10^{12}$ gauss. Two reasons why this post-collapse value is unreliable are that the choice of initial field strength is somewhat arbitrary and that the amplification of magnetic field by differential rotation during collapse is ignored. Such differential rotation can produce final field strengths which are orders of magnitude larger than those suggested by the flux conservation argument (LeBlanc and Wilson, 1970).

Adding to the uncertainty is the possibility that just after collapse, while the star is completely fluid, large magnetic fields in the interior can be transported to the surface by fluid instabilities, where the fields can decay much more rapidly than they could in the interior. After the star cools sufficiently for a solid crust to form, which may take about an hour (Tsuruta and Cameron, 1965), this process is no longer possible.

After crust formation, the magnetic field in the interior can only decay by resistive diffusion. A magnetic field with length scale λ will decay in a time $\approx 4\pi\sigma\lambda^2/c^2$, where σ is the electrical conductivity. However, the conductivity is so high in the interior (about 10^{29} sec⁻¹ for a temperature $T = 10^8$ K; Baym et al., 1969b) that the decay time for $\lambda \approx 10^6$ cm is much longer than the age of the universe.

An estimate of the interior magnetic field strength in the neutron star forming the Crab pulsar was given by Ostriker and Gunn (1969). Their

magnetic dipole radiation model of pulsar slowing down predicted that the Crab pulsar is $\approx 25\%$ older than it is known to be. They attributed the age discrepancy to the neglect of gravitational quadrupole radiation which is continually emitted because of a small equatorial ellipticity. They estimated that an interior poloidal field of $\approx 10^{15}$ gauss would produce an ellipticity sufficient to eliminate the discrepancy.

However, Gunn and Ostriker assumed that the initial angular speed $\Omega(0)$ of the pulsar was very much greater than its present value $\Omega(t)$. They require this assumption in order to drop terms of order $[\Omega(t)/\Omega(0)]^2$ from their age equation, which then allows them to predict an age for the pulsar which is independent of $\Omega(0)$. If they had used an $\Omega(0)$ which is $2.3\Omega(t)$, rather than a number $\gg \Omega(t)$, there would have been no discrepancy at all. There are several good reasons why $\Omega(0) = 2.3\Omega(t)$ is a reasonable assumption (Ruderman, 1972). It is thus quite unnecessary to invoke an internal field of 10^{15} gauss in order to satisfactorily explain the age discrepancy. Equally important, triaxiality due to the solid crust or to a possible solid core could also produce the required equatorial ellipticity.

Whatever the structure of the interior field and whatever its exact strength, it is almost certainly strong enough to have an important influence on the charged fluid dynamics.

The Crust

The crust is a coulomb lattice composed of neutron rich nuclei bathed in an electron gas. It is between a few hundred meters and several km thick. The bulk and shear moduli of this solid range from 10^{29} to 10^{31} dynes cm^{-2} . The density varies from about 2×10^{14} gm cm^{-3} at the fluid interior-crust

interface to near zero at the surface. At densities higher than about $10^{11} \text{ gm cm}^{-3}$, the solid is expected to be permeated with a neutron superfluid.

The electrical resistivity $\eta = c^2/4\pi\sigma$ due to electron-phonon scattering in the crust falls steadily from about $10^2 \text{ cm}^2 \text{ sec}^{-1}$ near the surface to about $10^{-9} \text{ cm}^2 \text{ sec}^{-1}$ near the crust-fluid boundary, for $T = 10^7 \text{ K}$ (Flowers and Itoh, 1975). The resistivity due to electron-phonon scattering is $\propto T$. The resistivity due to impurity scattering is temperature independent, and is proportional to $\langle(\Delta Z)^2\rangle\chi_i$, where χ_i is the impurity concentration and $\langle(\Delta Z)^2\rangle$ is the mean square charge difference between the impurities and the surrounding nuclei. For $\chi_i \approx 10^{-3}$ and $\langle(\Delta Z)^2\rangle \approx 1$, the impurity scattering resistivity is roughly equal to the electron-phonon resistivity at $T = 10^7 \text{ K}$ throughout the crust.

Taking a simple model in which η varies inversely with distance from the surface at a rate $\propto \text{cm}^3 \text{ sec}^{-1}$, dimensional considerations suggest that the decay time τ for the magnetic field in a crust of thickness λ is $\tau \approx \lambda^3/\alpha$. For $\alpha = 10^{-4} \text{ cm}^3 \text{ sec}^{-1}$ and $\lambda = 1\text{km}$, τ is about 10^{12} years. All pulsars whose ages are known or have been estimated are younger than several million years. A somewhat more realistic model of the dependence of η on distance from the surface will not change the conclusion that the magnetic field in the crust will not decay significantly over the lifetime of the pulsar. Thus, the field can be considered to be effectively "frozen" into the crust.

The Magnetosphere

The magnetosphere is the plasma outside the surface of the star. It owes its existence to the large magnetic field frozen into the highly electrically conducting rotating star. In most models, a portion of the magnetosphere co-rotates rigidly with the crust, and pulses of electromagnetic radiation are emitted in time with the rotation of the crust. The significance of this for studies of the interior is that pulsar timing allows one to determine the angular speed of the crust, from which it may be possible to infer something about the interior.

Review of Pulsar Timing Features

General Timing Features

Pulsars are very steady clocks. However, if the pulse arrival times for a pulsar are recorded for a period of several months, it is observed that on the average the pulsar slows down. As an illustration of timing data, the timing parameters for pulsar NP0532 (the Crab) are given in Table II. The Crab is the fastest pulsar known, and primarily for this reason it is also the only pulsar for which $\ddot{\Omega}$ is measurable.

Table II
Timing Parameters For NP0532

Ω	$= 1.9 \times 10^2$	sec^{-1}
$\dot{\Omega}$	$= -2.4 \times 10^{-9}$	sec^{-2}
$\ddot{\Omega}$	$= 6.9 \times 10^{-20}$	sec^{-3}

The slowing down of a pulsar can be understood in terms of the braking action of the electromagnetic torque acting on the solid surface of the star. The size of the surface magnetic field can be estimated as follows: assume a magnetic field configuration (e.g., a "rotating dipole"). Calculate the Maxwell stress tensor at the surface and, by integration, the electromagnetic torque. Assume a perfectly rigid star and calculate the angular deceleration = torque/(moment of inertia of star). Comparing this number with the observed deceleration allows the surface magnetic field size to be estimated.

Two comments about this procedure are in order. The first is that due to large uncertainties and inconsistencies in the present models of the magnetosphere, the deduced values of the magnetic field are uncertain by a few orders of magnitude. For example, the model of Ostriker and Gunn (1969) implies that surface magnetic fields for most pulsars are a few times 10^{12} gauss, whereas the dense magnetosphere model of Roberts and Sturrock (1972) implies surface magnetic fields about an order of magnitude smaller than this.

The second comment is that neutron stars are not perfectly rigid. The preceding view of a uniformly slowing down star is an oversimplification. It might be expected that there are irregularities in pulsar timings arising from the lack of rigidity of neutron stars.

Timing Irregularities

If the pulse arrival times for a pulsar are recorded for about a year or longer and the results fitted by any smooth function such as a cubic or quartic polynomial, there are sometimes significant differences between the observed and fitted arrival times. These differences, called timing residuals,

are measurable for the Crab pulsar (Drake, 1972), the Vela pulsar PSR0833 (Nelson et al., 1970; Reichley, 1972), and several other pulsars (Manchester et al., 1974; Manchester and Taylor, 1974).

There appear to be two distinct kinds of irregularities. In the Crab, Vela and PSR1508+55 pulsars, the residuals show large "glitches", that is, sudden changes in the rotational period of the star. Glitches are followed by a relaxation phase, lasting a few days for the Crab and about a year for the Vela, in which the sudden change in period relaxes exponentially. These glitches have been interpreted as due to cracking in the solid crust or in a solid core (Ruderman, 1969; Baym and Pines, 1971; Shaham et al., 1973).

Very simple models of the superfluid motion following such glitches are able to account well for the relaxation behaviour immediately following the glitch (Baym et al., 1969c), and this agreement has been interpreted as evidence for an interior which is at least partly superfluid.

This post-glitch theory assumes that both superfluid and charged fluid can be treated as rigidly rotating, with the charged fluid co-rotating with the solid parts of the star, and that the observed relaxation time is the time it takes the superfluid to adjust to the glitch. In support of the assumption that over the relaxation time scale it is a good approximation to treat the charged fluid as if it rigidly co-rotates with the solid parts of the star, Baym et al. (1969c) argue that within roughly the time it takes an Alfvén wave to cross the star, about 100 sec for a 10^{12} gauss magnetic field, the charged fluid can readjust itself to changes in the angular velocities of the solid parts of the star.

The other kind of timing irregularity is present in some pulsars in which large glitches have not yet been seen, as well as in those in which large glitches have been seen. The most complete residual data are for

the Crab in the optical (Boynton et al., 1972; Groth, 1975a,b; Groth, private communication). The residuals are quasi-sinusoidal, with a period between about one and two thirds the length of the data, for any given data set. The quasi-sinusoidal period thus grows longer as more data is collected. A power spectrum analysis of the residuals shows that the power at frequency ω is roughly proportional to ω^{-4} . The amplitude of the residuals also grows with the length of the data, from several 100 μ sec for about half a year of data to a few msec for a few years of data.

Two kinds of explanations for the quasi-sinusoidal behaviour seem possible. The first (Boynton et al., 1972) assumes that some random process is occurring. This assumption is consistent with the growth of the period of the quasi-sinusoidal residuals with the length of the data. Furthermore, if the process is a random walk in the frequency of the crust a power spectrum $\propto \omega^{-4}$ would result, in agreement with observations. Such a frequency walk could be produced by micro-glitches.

The second type of explanation is that the neutron star is oscillating simultaneously in a large number of long lived normal modes. For example, Ruderman (1972) assumed that the superfluid was oscillating mainly in its longest period Tkachenko mode. His theory was able to account for the amplitude and period of the quasi-sinusoidal residuals known at that time. Since then, the amplitude and period have increased as more data has been gathered so the superfluid oscillation model has become untenable.

This example illustrates one basic problem of the second kind of explanation: the longest period normal mode must have a period $\gtrsim$ the length of the data in order for the residuals to appear to be the result of noise. Data collection ceased in April, 1974 (Groth, private communication). A second basic problem is to account for the ω^{-4} trend of the power spectrum in a natural way.

Charged Fluid Dynamics: What Has Been Done And What Needs To Be Done

What Has Been Done

Not very much research has been done on the charged fluid dynamics. One line of research has been the investigation of magnetohydrodynamic instabilities in neutron stars. This work was initiated by Vandakurov (1972) who noted that for the highly degenerate material in neutron stars the pressure depends only very weakly on the temperature, and who speculated that as a result an arbitrarily weak magnetic field could cause convection.

This was investigated further by Tayler (1973b), who considered the possible effects of the very small thermal corrections to the equation of state. He concluded that it is necessary for the magnetic field B to be $\gtrsim 5 \times 10^5 \cdot T$ gauss in order for the star to be convectively unstable.

Tayler's (1973b) work was criticized by Chanmugam (1974), who pointed out that Tayler had tacitly assumed that the growth time τ of the instability was $\gg$ the time τ_N for convected matter to reach nuclear equilibrium. Since nuclear equilibrium is established by the weak interaction, Chanmugam argued that $\tau \ll \tau_N$ is more likely. Assuming this ordering of time scales, he found that a necessary condition for convective instability is $B \gtrsim 10^{16}$ gauss, independently of the temperature.

There are two major factors that have not been taken into account in all this work on convective instability. One is rotation, which generally tends to stabilize fluids (Acheson and Hide, 1973) and, which as Hide (1971) points out, is very important in the dynamics of the fluid core of neutron stars. The other is the highly conducting solid crust through which magnetic lines of force thread. The lines of force tend to be pinned at the solid-fluid interface. This "line-tying" tends to greatly stabilize

a magnetic field configuration (Krall and Trivelpiece, 1973).

Vandakurov (1970) attempted to find some of the normal modes and eigenfrequencies of a completely fluid rotating neutron star with a toroidal magnetic field that vanished at the surface. His calculations were performed for values of a dimensionless parameter which, if the star is rotating at the rate of the Crab pulsar, imply he is considering fields of about 10^{17} gauss. He incorrectly neglected bouyancy forces which are the same order of magnitude as other effects which he included, despite claims to the contrary in his paper. This could be a serious omission for the non-axisymmetric modes he considers. He also incorrectly assumed, in effect, that the neutrons as well as the charged fluid can respond to the $\vec{J} \times \vec{B}$ magnetic body force. As a result, the unperturbed densities which appear in his equations are too large by about a factor ≈ 100 . The detailed calculations of Vandakurov (1970) thus seem to be incorrect, although his conclusion that the toroidal field he considered is unstable may be correct.

Heinzman et al. (1973) attempted to determine the reaction of rotating stars, both magnetized and unmagnetized, to externally applied torques. An essential simplification in their paper is based on their claim that for isochoric rotational motion, in which $\vec{\nabla} \cdot \vec{v} = 0$, the non-linear term $\vec{v} \cdot \vec{\nabla} \vec{v}$ in the fluid momentum equation written in an inertial frame is cancelled by gradient terms. This is equivalent to saying that in a frame of reference which is co-rotating with the mean angular velocity of the star, not only is the centrifugal force conservative (which it is), but so is the Coriolis force (which it is not). This fundamental error restricts the validity of their calculations to the case of neutron stars which to a first approximation do not rotate.

Hide (1971), in probably the most important work in this area to date, pointed out that rotation has an extremely important influence on the dynamics of the charged fluid. As a result, two possible types of waves that can exist in the charged fluid are inertial waves and hydromagnetic-inertial waves. Hide did not investigate these waves in any detail.

What Needs To Be Done

Listed below are some of the important questions about the charged fluid dynamics which need to be answered and the reasons why I consider them important. It is these questions that I attempt to answer in this thesis.

1. What is the range of validity of the fluid description of the charged particles in the interior? Within this regime, to what extent are the usual equations of magnetohydrodynamics valid?

The motivation for these questions is practical because a fluid approach to a plasma problem is usually considerably easier than a kinetic approach. Since the highly degenerate plasma in the interior is in many respects different from non-degenerate laboratory plasmas, it is necessary to determine under what conditions, if any, the usual magnetohydrodynamic equations are applicable to the charged fluid.

2. What types of small amplitude waves can exist in the charged fluid? What can damp them? How much are they damped? How do the answers to these questions depend on the gross parameters of the problem, such as Ω , B , and T ?

Implicit in these questions is the approximation that the charged fluid can be treated as uniform and infinite in extent. Hide (1971) provided a partial answer to the first of these questions.

3. What kinds of mode-mode coupling can occur? Can this weakly non-linear effect lead to heavy damping of a wave which is only lightly damped in the linear approximation?

The answers to these questions are important in determining to what extent the calculated linear damping rates of waves are accurate.

4. What are some normal modes of oscillation of the charged fluid?

This question is a more realistic version of question 2 in that it takes into account the inhomogeneity of the fluid and the effects of boundary conditions.

5. How do charged fluid oscillations affect the crust? What is the relationship, if any, between the charged fluid and the quasi-sinusoidal residuals?

The answer to these questions are important because it is only via the observed crust angular velocity that one can learn anything observationally about the dynamics of the interior.

6. How does the charged fluid respond under a variety of initial conditions and external forces? In particular, how does it respond and how quickly does it respond to changes in the crust angular velocity?

The question of the extent to which the charged fluid can rigidly slow down with the crust is important. Furthermore, there is a published treatment of the problem (Heinzman et al., 1973) which is incorrect. How rigidly the charged fluid acts following a glitch is extremely important to the question of whether or not post-glitch relaxation constitutes strong evidence in favour of a partly superfluid interior.

CHAPTER II: THE MODEL OF THE CHARGED FLUID

In this chapter, the basic assumptions and approximations that will be made about the charged fluid and its interaction with the other parts of the star are outlined. The equations of the charged fluid dynamics are derived, and order of magnitude estimates of the size of various effects are given.

Magnetohydrostatic Equilibrium Of A Rotating Star

A neutron star is formed by collapse. Immediately after formation it is in a very agitated state (LeBlanc and Wilson, 1970). Within roughly the time it takes for a sound wave to cross the star it should be oscillating about hydrostatic equilibrium. These oscillations are damped within seconds by the emission of gravitational radiation and neutrinos.

The star cannot be differentially rotating to any great extent, partly because of the inhibitory effect of viscosity, but more importantly because of the effect of the magnetic field. Any shear in the charged fluid velocity will rapidly "wind up" the magnetic field lines and amplify the magnetic field within an e-folding time $\approx 1/(\text{velocity shear})$. For a large shear, this is approximately the rotational period of the star. After several rotations, the magnetic field will be magnified to such an extent that the magnetic forces are sufficiently large to effectively resist differential rotation. Shortly after differential rotation has effectively stopped, the star should be near magnetohydrostatic equilibrium.

The youngest pulsar known is the Crab, which is 921 years old. Even this young pulsar should be close to rigid rotation in magnetohydrostatic equilibrium, assuming the absence of continual violent excitation. The

proper point of departure for an investigation of the internal motions of neutron stars which are not extremely young is therefore the problem of the magnetohydrostatic equilibrium of a rigidly rotating neutron star.

This problem should best be treated within the framework of a relativistic theory of gravitation. There are at least two problems with any relativistic approach, however. The first is one of principle: which of the several viable relativistic theories of gravitation should be used? The second is a practical problem. For consistency, the same gravitational theory should be used in the magnetohydrostatic calculation as will be used in the calculation of the fluid dynamics. The latter calculation requires a large number of gravitational variables (e.g., six for Einstein's theory) as compared to the one in Newton's theory.

I therefore decided to use Newton's theory of gravitation throughout my calculations. As a result, such equilibrium quantities as the pressure and density will be in error by a maximum of about $\text{pressure}/(\text{density} \times c^2)$, which is $< 1/3$. The upper limit of $1/3$ occurs only at the centre of the most massive neutron stars. For most neutron stars, the error is probably more like 10%. The most important dynamical consequence of the non-relativistic approximation occurs in the Coriolis term in the fluid dynamic equations. The neglect of the relativistic "dragging of inertial frames" (Cohen, 1971) produces an error of about the same size as in the equilibrium equations.

There is one case in which the non-relativistic approximation would produce qualitatively wrong results. Newtonian gravitational theory is not capable of predicting some of the instabilities of massive neutron stars. Such critically massive neutron stars are not considered here.

The Equations of Magnetohydrostatic Equilibrium

The non-relativistic magnetohydrostatic equations are:

$$[\text{II.1}] \quad \vec{0} = -\vec{\nabla} p_0 - \rho_0 \vec{\nabla}(\phi_0 - \frac{1}{2} |\vec{\Omega}_0 \times \vec{r}|^2) - \frac{1}{4\pi} \vec{B}_0 \times (\vec{\nabla} \times \vec{B}_0)$$

$$[\text{II.2}] \quad \nabla^2 \phi_0 = 4\pi G \rho_0$$

$$[\text{II.3}] \quad p_0 = p_0(\rho_0) \quad .$$

In these equations, ρ_0 is the total equilibrium density, p_0 is the total equilibrium pressure, ϕ_0 is the equilibrium gravitational potential, $\vec{B}_0$ is the equilibrium magnetic field, and $\vec{\Omega}_0$ is the angular velocity. The temperature will be denoted T_0 .

Three of the most variable or uncertain parameters in both the equilibrium problem and in the problem of the charged fluid dynamics are Ω_0 , B_0 and T_0 . The angular speeds of known pulsars span roughly two decades, the magnetic field strength is uncertain by at least two orders of magnitude, and the temperature of neutron stars of varying ages probably spans two or three decades. To facilitate making order of magnitude estimates, and to make manifest how physical quantities depend on Ω_0 , B_0 and T_0 , non-dimensional versions of these parameters are introduced in Table III. A non-dimensionalized length scale is also introduced in that table.

Table III
Non-Dimensional Parameters

$$T_7 = T_0(^{\circ}\text{K})/10^7$$

$$B_{11} = B_0(\text{gauss})/10^{11}$$

$$\Omega_2 = \Omega_0(\text{rad/sec})/10^2$$

$$\lambda_6 = \lambda(\text{cm})/10^6$$

The value chosen to non-dimensionalize Ω_0 is such that pulsar angular velocities lie in the range $.01 \lesssim \Omega_2 \lesssim 2$. The surface magnetic fields for most pulsars as deduced from the dense magnetosphere model (Roberts and Sturrock, 1972) correspond to $B_{11} \approx 1$. Values of T_7 in the range $.1 \lesssim T_7 \lesssim 10$ are probably reasonable for neutron stars neither younger than the Crab nor extremely old. A length $\lambda_6 = 1$ corresponds to 10 km, which is a typical radius for a neutron star.

For later reference, the values of a number of microscopic parameters of the charged fluid are collected in Table IV.

Several approximations besides the non-relativistic one are already present in [II.1-3]. In [II.3] the temperature dependence of the pressure has been neglected because of the degeneracy of the stellar material - the Fermi energy ϵ_{Fn} of the neutrons is about 200 Mev, so $k_B T_0 / \epsilon_{Fn} \approx 5 \times 10^{-6} T_7 \ll 1$.

Because of the presence of a magnetic field, the material pressure tensor should be anisotropic rather than isotropic as in [II.1]. However, Ostriker and Gunn (1969) estimated that the effect of the pressure anisotropy on the stellar structure should be very much smaller than the effect of the $\vec{J} \times \vec{B}$ magnetic force, which is itself extremely small as is shown below. Because of the small effect of the magnetic field, fluctuations in the permeability of the stellar material due to the de Haas-Van Alphen effect have been neglected. These fluctuations tend to be thermally smeared out in any case.

In [II.1], the size of the centrifugal term relative to the $\rho_0 \vec{\nabla} \phi_0$ term is $\approx 10^{-4} \Omega_2^2$, which is $\ll 1$. In making this estimate, I have used $\phi_0 \approx GM/R \approx 10^{20} \text{ cm}^2 \text{ sec}^{-2}$, where M is the mass of the star $\approx 10^{33} \text{ g}$ and R is the stellar radius $\approx 10^6 \text{ cm}$. The relative size of the magnetic term is $(B_0^2 / 4\pi\rho_0) / \phi_0 \approx 10^{-14} B_{11}^2$. It is therefore an excellent approximation

Table IV
Some Microscopic Parameters Of The Fluid Region

Name	Symbol	Approximate Value
Electron number density	n_e	$5 \times 10^{36} \text{ cm}^{-3}$
Proton number density	n_p	$5 \times 10^{36} \text{ cm}^{-3}$
Neutron number density	n_n	$5 \times 10^{38} \text{ cm}^{-3}$
Electron Fermi energy	ϵ_{Fe}	110 Mev
Proton Fermi energy	ϵ_{Fp}	5 Mev
Neutron Fermi energy	ϵ_{Fn}	200 Mev
Electron Lorentz factor	γ	220
Electron-electron scattering time	τ_e	$10^{-10} T_7^{-2} \text{ sec}$
Electron mean free path	λ_e	$3 T_7^{-2} \text{ cm}$
Electron plasma frequency	ω_{pe}	$8 \times 10^{21} \text{ sec}^{-1}$
Ion plasma frequency	ω_{pi}	$3 \times 10^{21} \text{ sec}^{-1}$
Electron cyclotron frequency	ω_{ce}	$10^{16} B_{11} \text{ sec}^{-1}$
Ion cyclotron frequency	ω_{ci}	$10^{15} B_{11} \text{ sec}^{-1}$
Electron Debye length	λ_{De}	$3 \times 10^{-13} \text{ cm}$
Charged fluid kinematic shear viscosity	ν_c	$3 \times 10^{10} T_7^{-2} \text{ cm}^2 \text{ sec}^{-1}$
Thermal conductivity	κ	$10^{25} T_7^{-1} \text{ erg/deg-cm-sec}$

to completely neglect the effect of the magnetic field on the stellar structure and to work to only first order in the rotation.

In this approximation, all variables can be expanded as the sum of a spherically symmetric term and a term with an angular dependence of $P_2(\cos\theta)$. A spherical coordinate system (r, θ, ϕ) is used, with $\theta = 0, \pi$ as the rotation axis. The second term arises from the rotation, and is smaller than the first by about $\Omega_0^2 R^2 / \phi_0$. For example, the density may be expanded as

$$[\text{II.4}] \quad \rho_0(r, \theta) = \rho_0^{(0)}(r) + \rho_0^{(2)}(r) P_2(\cos\theta) \quad .$$

Under this expansion scheme, equations [II.1-3] reduce to a set of coupled ordinary differential equations.

The Equation of State

An equation of state must be chosen before these equations can be solved. There are by now a large number of equations of state for neutron stars. All are in substantial agreement below densities of $2 \times 10^{14} \text{ gm cm}^{-3}$, that is, in the crust whose properties are fairly well understood. Above that density, and particularly near the centre of the star, the strong inter-particle interactions are not well understood. As physicists have begun to understand these interactions better, more and more refined equations of state have been devised.

Unfortunately, this refining process has not yet converged - theoretically calculated neutron star structures continue to fluctuate by about the same amount as each new equation of state is proposed. In such a situation it is best to use the simplest equation of state which takes into account the features in which one is interested. The simplest equation of state which takes into account charged particles at densities greater than $2 \times 10^{14} \text{ gm cm}^{-3}$

is the Harrison-Wheeler equation of state (Harrison et al., 1965), so I chose it for the work in this thesis.

The details of this equation of state in the crustal region are not of interest for the purposes of this thesis; however, its behaviour in the fluid region is. The Harrison-Wheeler equation of state assumes that the fluid region is composed of non-interacting degenerate neutrons, protons, and electrons. The relative composition at a given density is fixed by the requirements of charge neutrality and β -equilibrium. It follows from charge neutrality that the average number densities of electrons and protons are equal, and thus their Fermi momenta are as well:

$$[\text{II.5}] \quad m_e c \sinh(t_e/4) = m_p c \sinh(t_p/4) \quad ,$$

where m_e and t_e are the electron mass and dimensionless energy parameter, respectively. Similar notation is used for the protons and the neutrons.

In β -equilibrium the reactions

$$[\text{II.6}] \quad n \rightarrow p + e + \bar{\nu}_e$$

$$[\text{II.7}] \quad p + e \rightarrow n + \nu_e$$

proceed at equal rates. Since the neutrinos escape from the star, β -equilibrium implies that the chemical potential of the neutrons is equal to the sum of the chemical potentials of the electrons and the protons:

$$[\text{II.8}] \quad m_n c^2 \cosh(t_n/4) = m_e c^2 \cosh(t_e/4) + m_p c^2 \cosh(t_p/4) \quad .$$

In [II.8], the rest mass energies are included in the chemical potentials.

The total density is the sum of the partial densities

$$[\text{II.9}] \quad \rho_o = D[(m_e/m_n)^4(\sinh t_e - t_e) + (m_p/m_n)^4(\sinh t_p - t_p) + (\sinh t_n - t_n)] \quad ,$$

which can be written as the sum of the charged fluid density ρ_{co} and the uncharged fluid density ρ_{uo} :

$$[II.10] \quad \rho_o = \rho_{co} + \rho_{uo} .$$

In [II.9], D is the characteristic density:

$$[II.11] \quad D = \pi c \frac{m_e^4}{4h^3} = 5.7 \times 10^{14} \text{ gm cm}^{-3} .$$

The pressure is the sum of the partial pressures:

$$[II.12] \quad p_o = P \left[\left(\frac{m_e}{m_n} \right)^4 (\sinh t_e - 8 \sinh(t_e/2) + 3t_e) \right. \\ \left. + \left(\frac{m_p}{m_n} \right)^4 (\sinh t_p - 8 \sinh(t_p/2) + 3t_p) \right. \\ \left. + (\sinh t_n - 8 \sinh(t_n/2) + 3t_n) \right] ,$$

which may be written as the sum of the charged fluid pressure p_{co} and the uncharged fluid pressure p_{uo} :

$$[II.13] \quad p_o = p_{co} + p_{uo} .$$

In [II.12], P is the characteristic pressure:

$$[II.14] \quad P = \frac{1}{3} D c^2 = 1.7 \times 10^{35} \text{ dynes cm}^{-2} .$$

Given any one of the three energy parameters, e.g. t_p , the other two can be calculated using [II.5] and [II.8]. Then $p_o(t_p)$, $\rho_o(t_p)$, $p_{co}(t_p)$, $\rho_{co}(t_p)$, etc. can be calculated from [II.9] and [II.12]. This procedure was programmed on a computer to produce the equation of state $p_o(\rho_o)$ for use in equilibrium calculations and also to produce two other functions useful in charged fluid dynamical calculations: $\rho_{co}(\rho_o)$ and $s_{co}^2(\rho_o)$. The latter function is defined by

$$[II.15] \quad s_{co}^2 = dp_{co}/d\rho_{co} .$$

s_{co} will be referred to as the charged fluid sound speed; it appears as the propagation speed in the wave equation for sound-like waves in the charged fluid. An analogous quantity for the fluid as a whole is

$$[II.16] \quad s_o^2 = dp_o / d\rho_o .$$

Strictly speaking, it is not proper to refer to s_o as the sound speed of the fluid interior because the neutron fluid and the charged fluid are weakly coupled and therefore do not oscillate together.

The energy parameters in the density range $2 \times 10^{14} \lesssim \rho_o \lesssim 10^{15}$ vary over $20 \lesssim t_e \lesssim 25$, $.2 \lesssim t_p \lesssim .5$ and $1 \lesssim t_n \lesssim 2$. Consequently in this density range the electrons are very relativistic, the protons are non-relativistic, and the neutrons vary from non-relativistic to somewhat relativistic.

The main contribution to ρ_o arises from the neutrons, so [II.9] implies that for $t_n \approx 1$,

$$[II.17] \quad \rho_o \approx \frac{1}{6} t_n^3 D .$$

Similarly the main contribution to the charged fluid mass density arises from the protons, so

$$[II.18] \quad \rho_{co} \approx \frac{1}{6} t_p^3 D ,$$

and thus the fraction of charged fluid by mass over this density range is

$$[II.19] \quad \rho_{co} / \rho_o \approx (t_p / t_n)^3 \approx 10^{-2} .$$

The exact numerical results for ρ_{co} / ρ_o as a function of ρ_o given in Figure 2 bear out the rough estimate [II.19].

The main contribution to the total pressure arises from the neutrons, so for $t_n \approx 1$,

$$[II.20] \quad p_o \approx \frac{1}{160} t_n^5 P .$$

For the charged fluid, the relativistic electrons provide most of the pressure, so after using [II.5] to eliminate t_e in favour of t_p , it follows that

$$[II.21] \quad p_{co} \approx \frac{1}{32} t_p^4 P .$$

It follows from [II.17] and [II.20] that the equation of state is very nearly that of an ideal non-relativistic neutron gas, $p_o \propto \rho_o^{5/3}$, when $t_n \approx 1$. When t_n is closer to two the gas is becoming relativistic, and the exponent should become smaller than 5/3. From [II.18] and [II.21], the charged fluid equation of state $p_{co}(\rho_{co})$ behaves like $p_{co} \approx \rho_{co}^{4/3}$. It also follows that

$$[II.22] \quad s_{co}^2/s_o^2 \approx 4t_p/t_n^2 \approx .5 .$$

These approximate pressure-density relations are born out by the exact numerical calculations. In Figure 3, s_o^2 and s_{co}^2 are plotted as functions of ρ_o . The change in slope of s_o^2 at high densities is due to the changeover from a non-relativistic to a slightly relativistic regime for the neutrons. The s_{co}^2 curve is straighter than the s_o^2 curve because the protons remain non-relativistic over almost the entire density range. Finally, s_{co}^2/s_o^2 is about .55 and .60 throughout, in agreement with [II.22].

A Numerical Example

The coupled differential equations which result from the expansion scheme [II.4] were solved on a computer.¹ Two parameters define a particular model. These parameters are the central density $\rho_o^{(o)}(0)$ and the non-dimensional angular speed Ω_2 .

Spherically symmetric functions such as $\rho_o^{(o)}(r)$ do not depend on Ω_2 in the approximation considered, and functions such as $\rho_o^{(2)}(r)$ scale as Ω_2^2 . Thus, from a particular solution obtained for given values of $\rho_o^{(o)}(0)$ and Ω_2 , the family of solutions corresponding to this central density and any angular velocity may be obtained by scaling.

The expected range of values for $\rho_o^{(o)}(0)$ is not great. Most neutron star models of moderate mass have central densities which lie within a factor of two either way of $10^{15} \text{ gm cm}^{-3}$. In addition, there is a well-known scaling law (§109; Landau and Lifschitz, 1969) which allows one to generate a class of approximate solutions parametrized by $\rho_o^{(o)}(0)$ from a solution with a given $\rho_o^{(o)}(0)$. The scaling law states that $\rho_o^{(o)}(r)R^6$ is a function only of r/R . It is exact in the absence of rotation and when the stellar material is entirely composed of a single non-relativistic degenerate gas. The latter condition was previously shown to be satisfied to a good approximation in the fluid interior.

The value of $\rho_o^{(o)}(0)$ chosen for the numerical calculations was $10^{15} \text{ gm cm}^{-3}$; Ω_2 was taken to be 1.9, the angular speed of the Crab pulsar (see Table II). The results for $\rho_o^{(o)}(r)$ and $\rho_o^{(2)}(r)$ are displayed in Figure 4. The scaling methods just mentioned may be used to generate a large class of approximate solutions from these results.

¹ The routine STIFF from the U.B.C. Computing Centre was used.

The spherically symmetric part of the density drops smoothly from $10^{15} \text{ gm cm}^{-3}$ at the centre to $2 \times 10^{14} \text{ gm cm}^{-3}$ at $r = R_c \approx 8.9 \text{ km}$, the radius of the crust. There are several kinks in the graphs of $\rho_o^{(0)}$ and $\rho_o^{(2)}$ for $\rho_o^{(0)} < 2 \times 10^{14} \text{ gm cm}^{-3}$. These arise from changes in the form of the equation of state in the crust. The density drops to near zero at the stellar radius $r = R \approx 15.8 \text{ km}$. The main features of this particular model are summarized in Table V.

Table V
The Parameters Of A Rotating Neutron Star Model

Central density	$10^{15} \text{ gm cm}^{-3}$
Angular speed	190 sec^{-1}
Mass	$1.71 \times 10^{33} \text{ gm}$ (about $.9 M_\odot$)
Radius	15.8 km
Inner crustal radius	8.9 km
Crust moment of inertia	$4.06 \times 10^{44} \text{ gm cm}^2$

The Magnetic Field Structure

The equations of magnetohydrostatic equilibrium place constraints on the form of the magnetic field. Taking the curl of $1/\rho_o$ times [II.1] and using the dependence of p_o only on ρ_o produces

$$[\text{II.23}] \quad \vec{\nabla} \times \{ [\vec{B}_o \times (\vec{\nabla} \times \vec{B}_o)] / 4\pi\rho_o \} = 0 ,$$

which must be satisfied in addition to the Maxwell equation

$$[\text{II.24}] \quad \vec{\nabla} \cdot \vec{B}_o = 0 .$$

It is an excellent approximation to neglect the dependence of ρ_o on $\vec{B}_o$ in [II.23], and to use the density as determined in a hydrostatic equilibrium calculation, since as previously shown, the magnetic field has only a minor effect on the stellar structure.

There are a very large number of magnetic field structures consistent with [II.23]. One large class is formed by magnetostatic fields which satisfy $\vec{\nabla} \times \vec{B}_0 = \vec{0}$; a more general class is formed by force-free fields, which satisfy $\vec{\nabla} \times \vec{B}_0 \propto \vec{B}_0$. It is not presently possible to decide on the basis of theory alone which of these many solutions are actually realized in neutron star interiors.

It is quite probable that the magnetic field in the interior is magnetohydrodynamically stable, considering the very long time available for it to attain such a stable state. As was indicated in Chapter I, the subject of magnetohydrodynamic stability inside neutron stars is not yet well developed. Further consideration of this question is deferred until Chapter VI.

However, some information about the interior magnetic field structure may be gleaned from the observed slowing down rate of pulsars. The angular velocity of the crust is altered by torques exerted by the magnetic stresses at both its outer and inner surfaces. There is no particular reason to suppose that the torque exerted on the inner surface is very much smaller than that on the outer. It is thus reasonable to assume that the torque on the inner surface is roughly equal to that exterior torque required to produce the observed pulsar slowdown.

From this condition one may obtain as follows an estimate of $B_p B_t$ at the inner surface, where B_p and B_t are the sizes of the poloidal and toroidal parts of $\vec{B}_0$, respectively. The components of magnetic stress at the surface $r = R_c$ which can exert torques are $\approx B_p B_t / 4\pi$, so the force is $\approx B_p B_t R_c^2$, and the torque is $\approx B_p B_t R_c^3$. Equating this torque to $I\dot{\Omega}$, where I is the moment of inertia of the crust, implies that

$$[\text{II.25}] \quad B_p B_t \approx I \dot{\Omega} / R_c^3$$

at the surface of the crust.

It is not necessarily true that $B_p B_t$ has roughly the value [II.25] throughout the fluid interior. However, there is a large class of magnetic fields for which this is the case. For example, if the magnetic field is symmetric about any axis and the angular variations in ρ_0 are negligible, it follows from [II.23] that even though B_p and B_t may separately vary by many orders of magnitude throughout the star, the product $B_p B_t$ is roughly constant.

The values of I and R_c for the particular model given in Table V can be used to obtain a very crude estimate of $B_p B_t$ at the inner crustal surface for any pulsar with known $\dot{\Omega}$. For the Crab (see Table II), the result is $B_p B_t \approx 10^{18} \text{ gauss}^2$.

If one believes that $B_0 \gtrsim 10^{10} \text{ gauss}$, then it follows from [II.25] that for all known pulsars the magnetic field at the inner crustal surface is either almost purely poloidal or purely toroidal.

Charged Fluid Equations Of Motion

In this section, the equations governing the motion of the charged fluid are derived. There are four sets of equations: the mass equation, the fluid momentum equations, the gravitational source equation, and the magnetic induction equations.

Firstly, however, the range of length scales and time scales over which the charged particles may be treated as a fluid are examined. The reason for wanting to treat the charged particles as a fluid is that the charged fluid equations are much simpler than the kinetic equations.

A great deal of information about the system is lost in going to a fluid picture, however. For example, within a purely fluid picture, it is not possible to treat collisionless damping of waves or any velocity space microinstabilities, both of which are often very important in plasmas.

The Range Of Validity Of A Fluid Description

I assume that both the protons and neutrons are superfluid (Baym et al., 1969a), so that the most effective scattering mechanism in establishing local thermodynamic equilibrium is electron-electron scattering. The scattering time τ_e has been estimated by Heinzman and Nitsch (1972):

$$[\text{II.26}] \quad \tau_e = \alpha^{-3/2} \hbar \mu_e / (k_B T_0)^2 \approx 10^{-10} T_7^{-2} \text{ sec} ,$$

where α is the fine structure constant and μ_e is the chemical potential of the electrons, which is about 110 Mev (Ruderman, 1972).

The value given by Flowers and Itoh (1975) for the viscosity η_e of the electron fluid whose relativistic mass density is ρ_e may be used in the approximate kinetic relation

$$[\text{II.27}] \quad \tau_e \approx 3\eta_e / \rho_e v_{Fe}^2 ,$$

to obtain an independent estimate of τ_e . Assuming that the electron Fermi velocity $v_{Fe} \approx c$, which is an excellent approximation, the result given by [II.27] agrees with that of [II.26] to within a factor of two.

Two necessary conditions for the validity of a fluid description of the charged particles are that the time scale τ and the length scale λ of any disturbances satisfy

$$[\text{II.28}] \quad \tau \gg \tau_e \approx 10^{-10} T_7^{-2} \text{ sec}$$

and

$$[\text{II.29}] \quad \lambda \gg \lambda_e = v_{Fe} \tau_e \approx 3 T_7^{-2} \text{ cm} .$$

Here λ_e is the electron mean free path. The most extreme case occurs in very old pulsars where T_7 may get as low as 10^{-2} , in which case $\lambda \gg .3 \text{ km}$ is necessary for a fluid description.

A necessary condition for the validity of a single fluid description of the electrons and the protons, as opposed to a two-fluid description, is that

$$[\text{II.30}] \quad \tau \gg \omega_{pi}^{-1} \approx 3 \times 10^{-22} \text{ sec} ,$$

where ω_{pi} is the ion (proton) plasma frequency, defined by

$$[\text{II.31}] \quad \omega_{pi} = \left(\frac{4\pi n_p e^2}{m_p} \right)^{1/2} \approx 3 \times 10^{21} \text{ sec}^{-1} ,$$

where n_p is the number density of protons, which is $\approx 5 \times 10^{36} \text{ cm}^{-3}$. Note that because the protons are non-relativistic, it is a good approximation to use the proton rest mass in [II.31]. For the relativistic electrons, however, the plasma frequency is

$$[\text{II.32}] \quad \omega_{pe} = \left(\frac{4\pi n_e e^2}{\gamma m_e} \right)^{1/2} \approx 8 \times 10^{21} \text{ sec}^{-1} ,$$

where γ is the electron Lorentz factor, which is $\approx \epsilon_{Fe}/m_e c^2 \approx 220$.

Another necessary condition is that length scales be much longer than electron or proton Debye lengths λ_{De} and λ_{Di} , e.g.,

$$[\text{II.33}] \quad \lambda \gg \lambda_{De} = v_{Fe} \omega_{pe}^{-1} \approx 3 \times 10^{-13} \text{ cm} .$$

A comparison of [II.28] with [II.30] and [II.29] with [II.33] shown that if a fluid description is valid, then quasi-neutrality is an excellent approximation.

The Mass Equation

If the charged fluid is perturbed away from the magnetohydrostatic equilibrium, its density will in general change:

$$[\text{II.34}] \quad \rho_c = \rho_{co} + \rho'_c .$$

In what follows, only small departures from equilibrium are considered, so the density perturbation is relatively small:

$$[\text{II.35}] \quad |\rho'_c / \rho_{co}| \ll 1 .$$

The equation describing changes in ρ_c is

$$[\text{II.36}] \quad \partial \rho'_c / \partial t + \vec{\nabla} \cdot (\rho_c \vec{v}_c) = -\rho'_c / \tau_N ,$$

where $\vec{v}_c$ is the charged fluid velocity. The term $-\rho'_c / \tau_N$ represents the creation or destruction of charged fluid by the weak interaction, which tends to create charged fluid from neutrons if $\rho'_c < 0$, and produces neutrons from charged fluid if $\rho'_c > 0$. As $\tau_N \rightarrow \infty$, the equation expresses the conservation of charged fluid mass. It is important to estimate τ_N because only in the limit of large τ_N can the charged fluid be treated as a distinct fluid with its own equation of state.

If the charged fluid density is perturbed away from equilibrium, the electron and proton Fermi energies change by $\delta \epsilon_{Fe}$ and $\delta \epsilon_{Fp}$, respectively. For relatively small Fermi energy perturbations, the amount of energy E per unit volume which is available for radiation away by neutrinos is

$$[II.37] \quad E \approx \frac{1}{2} \frac{\partial n_e}{\partial \epsilon_{Fe}} (\delta \epsilon_{Fe})^2 + \frac{1}{2} \frac{\partial n_p}{\partial \epsilon_{Fp}} (\delta \epsilon_{Fp})^2 \approx \frac{1}{2} \frac{n_e}{\epsilon_{Fe}} (\delta \epsilon_{Fe})^2.$$

The smallness of $|\delta \epsilon_{Fp}|$ compared to $|\delta \epsilon_{Fe}|$, which results from $\epsilon_{Fp} \ll \epsilon_{Fe}$, was used to drop the second term on the right-hand side of [II.37].

The rate of the single neutron β -decay reaction

$$[II.38] \quad n \rightarrow p + e + \bar{\nu}_e$$

inside the fluid region of neutron stars is very low, because momentum conservation requires the emission of electrons and protons with momenta small compared to ϵ_{Fe} and ϵ_{Fp} , respectively. The probability of holes being available in the electron and proton Fermi seas for these emitted particles is $\approx e^{-\epsilon_{Fn}/k_B T}$, which is $\ll 1$.

Conservation of momentum is easily satisfied in URCA reactions such as

$$[II.39] \quad n + n \rightarrow n + p + e + \bar{\nu}_e.$$

The luminosity L arising from such reactions was calculated by Bahcall and Wolf (1965) for the case in which the available energy is solely thermal. The luminosity for that case is $\propto (k_B T)^8$. However, when the available energy can arise both from thermal effects and from density induced changes $\delta \epsilon_F$ in Fermi energies, the expression for L contains a series of terms in $(k_B T)^8$, $(k_B T)^6 (\delta \epsilon_F)^2$, $(k_B T)^4 (\delta \epsilon_F)^4$, etc (Pethick, private communication). Since the $\delta \epsilon_F$'s considered here are $\ll k_B T$, the most important contribution to L which includes the effects of small density perturbations is $\propto (k_B T)^6 (\delta \epsilon_F)^2$. One may then take Bahcall and Wolf's (1965) expression and replace $(k_B T)^8$ by $(k_B T)^6 (\delta \epsilon_{Fe})^2$ to get a fairly good approximation to the luminosity arising from small charged fluid density perturbations. The result is

$$[II.40] \quad L \approx 10^{23} T_7^6 (\delta \epsilon_{Fe})^2 \text{ Mev cm}^{-3} \text{ year}^{-1},$$

if $\delta\epsilon_{Fe}$ is measured in Mev. In [II.40], a term $\propto (\delta\epsilon_{Fe})^2$ is neglected, because $|\delta\epsilon_{Fe}| \ll |\delta\epsilon_{Fe}|$.

The decay time is thus

$$[II.41] \quad \tau_N = \frac{E}{L} \approx 10^{12} T_7^{-6} \text{ years.}$$

In the calculation of Bahcall and Wolf (1965) it was assumed that the neutrons are not superfluid. If superfluidity is taken into account, the decay time of $10^{12} T_7^{-6}$ years should not be shortened, so [II.40] can be looked upon as a lower limit for τ_N . The effects of charged fluid creation or destruction are thus completely negligible over the time scales of interest, and mass conservation holds:

$$[II.42] \quad \frac{\partial \rho'_c}{\partial t} + \vec{\nabla} \cdot (\rho_c \vec{v}_c) = 0.$$

The Momentum Equations

The equations describing the acceleration of the charged fluid have the form

$$[II.43] \quad \frac{\partial (\rho_c \vec{v}_c)}{\partial t} + \vec{\nabla} \cdot (\rho_c \vec{v}_c \vec{v}_c) = \{\text{sum of all forces/unit volume on the charged fluid}\}.$$

The various forces are given in Table VI. In that table, the terms which support the charged fluid when it is in magnetohydrostatic equilibrium with the uncharged fluid have been subtracted off, although no linearizations have yet been made.

The frame of reference which is used co-rotates with the solid crust at angular velocity $\vec{\Omega}$:

$$[II.44] \quad \vec{\Omega}(t) = \vec{\Omega}_0 + \vec{\Omega}'(t).$$

Table VI
Forces Acting On The Charged Fluid

Centrifugal	$\rho_c \vec{\nabla}(\frac{1}{2} \vec{\Omega} \times \vec{r} ^2) - \rho_{co} \vec{\nabla}(\frac{1}{2} \vec{\Omega}_o \times \vec{r} ^2)$
Coriolis	$- \rho_c 2\vec{\Omega} \times \vec{v}_c$
Poincaré	$- \rho_c \vec{\Omega} \times \vec{r}$
Gravitational	$- (\rho_c \vec{\nabla}\Phi - \rho_{co} \vec{\nabla}\Phi_o)$
Pressure	$- \vec{\nabla} \cdot \vec{P}$
Magnetic	$(1/4\pi)\vec{\nabla} \cdot [\vec{B} \vec{B} - \vec{B}_o \vec{B}_o - \frac{1}{2}(B^2 - B_o^2)\vec{I}]$
Viscous	$- \vec{\nabla} \cdot \vec{\sigma}$
Superfluid drag	$\rho_c (\vec{v}_s - \vec{v}_c)/\tau_D$

The small variations $\vec{\Omega}'$ in $\vec{\Omega}$ are due mainly to electromagnetic torques on the crust which cause its general slowdown. In all calculations in this thesis, the z axis is taken to lie parallel to $\vec{\Omega}_o$.

The various terms listed in Table VI are now discussed. Each will be linearized in the sense that terms which are second order in perturbed quantities will be dropped. This linearization is an excellent approximation for the small amplitude perturbations that are considered in this thesis. A discussion of some of the expected effects of the dropped non-linear terms is given in Chapter III.

1. Centrifugal force

To first order in primed variables, the centrifugal force can be written as $\rho_c \vec{\nabla}(\frac{1}{2}|\vec{\Omega}_o \times \vec{r}|^2) + \rho_{co} \vec{\nabla}(\frac{1}{2}|\vec{\Omega} \times \vec{r}|^2 - \frac{1}{2}|\vec{\Omega}_o \times \vec{r}|^2)$. The second term is conservative and can be balanced by pressure gradients within roughly the time it takes a sound wave to cross the star, about 10^{-4} sec. Over time scales

shorter than this, $\vec{\Omega}$ cannot change appreciably. Thus no matter what the time scale, the second term may be neglected. The first term is discussed later along with the gravitational term.

2. Coriolis force

The Coriolis force is extremely important in the charged fluid dynamics. A fuller discussion of its importance is delayed until Chapter III, which deals with the types of waves which can propagate in the charged fluid.

The linearized form of the Coriolis term is $-2\vec{\Omega}_0 \times \rho_{co} \vec{v}_c$. The non-linear convective term $\vec{v} \cdot (\rho_c \vec{v}_c \vec{v}_c)$ in the fluid momentum equation [II.43] is smaller than the Coriolis term by about a factor $V_c / \Omega_0 \lambda$, where the order of magnitude of $\vec{v}_c$ is denoted V_c , and λ represents the length scale over which $\vec{v}_c$ varies. The condition that the fluid is close to rigid rotation,

$$[II.45] \quad V_c / \Omega_0 R \ll 1,$$

is equivalent to the smallness of the convective term compared to the Coriolis term for long wavelength ($\lambda \approx R$) disturbances.

3. Poincaré force

The Poincaré force¹ is a fictitious force which arises from the acceleration of the coordinate frame attached to the crust. If the crust is precessing the oscillating Poincaré force which results can excite the charged fluid at the precession frequency.

If the crust is not precessing but only gradually slowing down, the resulting Poincaré force is almost constant in time. In such a situation, the fluid has ample time in which to balance this force. Because the force

¹ The force was so named by Malkus (1970), who considered the possibility that it plays an important role in the dynamics of the fluid core of the earth.

is non-conservative, it cannot be balanced by a pressure gradient. However, it can be balanced by the magnetic body forces. This will be shown later in this chapter.

4. Gravitational force

To first order in primed quantities, the gravitational force consists of two terms: $-\rho'_c \vec{\nabla} \phi_0$ and $-\rho_{co} \vec{\nabla} \phi'$, where the gravitational potential perturbation ϕ' is defined by

$$[\text{II.46}] \quad \phi = \phi_0 + \phi'.$$

The first term represents the buoyancy force on an element of charged fluid, and may be conveniently combined with the unbalanced portion of the centrifugal force to produce $-\rho'_c \vec{\nabla} (\phi_0 - \frac{1}{2} |\vec{\Omega}_0 \times \vec{r}|^2)$.

The second term represents the gravitational force on the charged fluid due to perturbations in the gravitational potential. These perturbations arise from changes in ρ_c , whereas ϕ_0 arises from both the charged fluid and superfluid. The second term is thus about $\rho_{co}/\rho_0 \approx 1\%$ the size of the first. It is therefore a fairly good approximation to neglect $-\rho_{co} \vec{\nabla} \phi'$.

5. The pressure term

In the absence of a magnetic field, the perturbation $\vec{P}'_c$ in the charged fluid pressure tensor would be isotropic, and $-\vec{\nabla} \cdot \vec{P}'_c$ could be written as $-\vec{\nabla} p'$. In the presence of a magnetic field, $\vec{P}'_c$ can be anisotropic. It is important to determine whether or not the anisotropy can be neglected.

On the classical level the conditions for a nearly isotropic pressure tensor are (Krall and Trivelpiece, 1973)

$$[\text{II.47}] \quad (a_1/\lambda)^2 \ll 1/(\tau\omega_{ci}) \ll 1,$$

where a_i is the ion (proton) cyclotron radius, ω_{ci} is the ion cyclotron frequency, and λ and τ are macroscopic length and time scales, respectively. ω_{ci} is given by

$$[II.48] \quad \omega_{ci} = eB_o / m_p c \approx 10^{15} B_{11} \text{ sec}^{-1}.$$

Note that a non-relativistic form such as [II.48] for ω_{ci} is a good approximation. For later reference, the electron cyclotron frequency is

$$[II.49] \quad \omega_{ce} = eB_o / \gamma m_e c \approx 10^{16} B_{11} \text{ sec}^{-1}.$$

A comparison of ω_{ci}^{-1} with the electron-electron scattering time τ_e in [II.26] reveals that in the fluid regime where $\tau \gg \tau_e$, the condition $1/(\tau\omega_{ci}) \ll 1$ is obeyed.

By quasi-neutrality, the Fermi momenta of the electrons and protons are equal, so $v_{Fi} \approx v_{Fe} \approx 6 \times 10^9 \text{ cm sec}^{-1}$, and thus

$$[II.50] \quad a_i = v_{Fi} \omega_{ci}^{-1} \approx 6 \times 10^{-6} B_{11}^{-1} \text{ cm}.$$

A comparison of [II.29] and [II.50] shows that when $\lambda \gg \lambda_e$, $(a_i/\lambda)^2 \ll 1$.

The answer to the question of whether or not $(a_i/\lambda)^2$ is $\ll 1/(\tau\omega_{ci})$ depends on the relationship of τ and λ . It is shown in Chapter III that there are three classes of waves which can propagate in the charged fluid: sound waves, inertial waves and hydromagnetic-inertial waves. For each class of wave, there is a different relation between τ and λ . For sound waves, the relation between τ and λ is $\tau \approx \lambda/s_{co}$, so $(a_i/\lambda)^2 \ll 1/(\tau\omega_{ci})$ if $\lambda \gg a_i^2 \omega_{ci} / s_{co} \approx 10^{-5} B_{11}^{-1} \text{ cm}$, which is satisfied in the fluid regime. For inertial waves, $\tau \approx \Omega_o^{-1}$ independently of λ , so if $\lambda \gg (\omega_{ci}/\Omega_o)^{1/2} a_i \approx 200(B_{11}/\Omega_2)^{1/2} \text{ cm}$ then $(a_i/\lambda)^2 \ll 1/(\tau\omega_{ci})$. For hydromagnetic-inertial waves, $\tau \approx 10^{-6} \Omega_2 B_{11}^{-2} \lambda^2$, so $(a_i/\lambda)^2 \ll 1/(\tau\omega_{ci})$ for all wavelengths if $\Omega_2^{-1} B_{11}^3 \gg 4 \times 10^{-2}$.

On the quantum level there is an additional restriction, which is that the electron Landau levels are thermally smeared out:

$$[\text{II.51}] \quad k_B T_o \gg \hbar \omega_{ce} ,$$

which can be written as $T_o B_{11}^{-1} \gg 10^{-2}$.

In what follows, it is assumed that these conditions are met so that the pressure tensor can be treated as isotropic. The pressure perturbation p'_c can arise from both density and temperature perturbations:

$$[\text{II.52}] \quad p'_c = \left(\frac{\partial p_c}{\partial \rho_c} \right)_T \rho'_c + \left(\frac{\partial p_c}{\partial T} \right)_{\rho_c} T' .$$

However, the charged fluid is highly degenerate, and in the limit of extreme degeneracy ($k_B T_o / \epsilon_{Fe} \rightarrow 0$), both factors $(\partial p_c / \partial T)_{\rho_c}$ and T' in the second term of [II.52] separately vanish. In this limit, p'_c becomes independent of T and the heat transfer equation becomes decoupled from the other equations of motion. This point is discussed more quantitatively later. For the moment we shall ignore the small thermal corrections to the charged fluid equation of state $p_c = p_c(\rho_c)$ calculated in the first section of this chapter, so to first order in primed quantities

$$[\text{II.53}] \quad p'_c = s_{co}^2 \rho'_c .$$

Equation [II.53] can be used to eliminate ρ'_c from the momentum equations. Using [II.53] and the magnetohydrostatic equation [II.1] with the negligible magnetic term dropped, the buoyancy term can be written as

$$[\text{II.54}] \quad -\rho'_c \vec{\nabla}(\phi_o - \frac{1}{2} |\vec{\Omega}_o \times \vec{r}|^2) = -p'_c \vec{\nabla} \psi_o ,$$

where the dimensionless quantity ψ_o which depends only on the hydrostatic solution is defined by

$$[\text{II.55}] \quad \psi_o(\rho_o) = \int_{s_{co}}^{\frac{s_o}{2}} \frac{d\rho_o}{\rho_o} .$$

The limits of integration on ψ_o are arbitrary, because ψ_o appears only as a gradient. Collecting the buoyancy and pressure gradient terms, we have

$$[\text{II.56}] \quad -\vec{\nabla} \cdot \vec{p}'_c - \rho'_c \vec{\nabla}(\Phi_o - \frac{1}{2}|\vec{\Omega}_o \times \vec{r}|^2) = -(\vec{\nabla} + \vec{\nabla}\psi_o) p'_c .$$

6. The magnetic body forces

To first order in primed quantities, the magnetic body forces are $-(1/4\pi)\vec{B}_o \times (\vec{\nabla} \times \vec{B}') - (1/4\pi)\vec{B}' \times (\vec{\nabla} \times \vec{B}_o)$, where the magnetic perturbation is defined by

$$[\text{II.57}] \quad \vec{B} = \vec{B}_o + \vec{B}' .$$

As in the magnetohydrostatic equations [II.23] for $\vec{B}_o$, any de Haas-Van Alphen oscillations in the permeability are neglected because of the thermal smearing [II.51] of electron Landau levels.

As previously mentioned, the magnetic body forces can balance the quasi-steady Poincaré force due to crustal slowing down. The condition for such a force balance is

$$[\text{II.58}] \quad -\frac{1}{4\pi}[\vec{B}_o \times (\vec{\nabla} \times \vec{B}') + \vec{B}' \times (\vec{\nabla} \times \vec{B}_o)] - \rho_{co} \dot{\vec{\Omega}} \times \vec{r} - \vec{\nabla} p'_c - p'_c \vec{\nabla}\psi_o = \vec{0} .$$

From [II.58], the relative size of magnetic perturbation required to do this is

$$[\text{II.59}] \quad B'/B_o \approx \rho_{co} \dot{\Omega} R^2 / (B_o^2 / 4\pi) ,$$

which is about $10^{-5} B_{11}^{-2}$ for the Crab. The magnetic field has a very long time in which to adjust to the Poincaré force due to crustal slowing down.

Thus it seems reasonable that given any $\vec{B}_0$ satisfying [II.23], a small $\vec{B}'$ will be present such that [II.58] is satisfied.

However, this is not possible for every magnetohydrostatic $\vec{B}_0$. For example, in the axially symmetric case, the ϕ component of [II.58] is

$$[\text{II.60}] \quad (\vec{B}_0 \cdot \vec{\nabla})(r \sin\theta B'_\phi) = \rho_{co} \dot{\Omega} r^2 \sin^2\theta.$$

The right-hand side of [II.60] is ≤ 0 , so if $\vec{B}_0$ has any closed field lines within the fluid region, [II.60] has no single-valued solution for B'_ϕ .

Magnetic fields $\vec{B}_0$ of this closed field line type and perhaps other types as well cannot balance the quasi-steady Poincaré force by making only small changes $\vec{B}'$ in their structures, and are thus in this sense unstable.

A simple example of a stable $\vec{B}_0$ is a uniform field $\vec{B}_0 = B_0 \hat{z}$, where $\vec{\nabla} B_0 = \vec{0}$. The B'_ϕ required to balance the Poincaré force is displayed in Figure 5. For this calculation, $B_{11} = 1$, $\dot{\Omega}$ is that of the Crab (see Table II) and the total central density is $10^{15} \text{ gm cm}^{-3}$ (the model of Figure 4). This result for B'_ϕ can be easily scaled for other values of B_{11} , $\dot{\Omega}$ and $\rho_o^{(0)}(0)$.

For the remainder of this thesis, only those $\vec{B}_0$'s which are stable in the sense just explained are considered. It is then convenient to absorb the $\vec{B}'$ required to balance the quasi-steady Poincaré force into the definition of $\vec{B}_0$ and to drop the quasi-steady Poincaré force from the equations. The resulting equations are valid only over time scales short compared to the time over which $\dot{\Omega}$ varies. For the Crab this time is roughly $\dot{\Omega}/\ddot{\Omega} \approx$ a thousand years (see Table II). Any Poincaré force due to precession with a period $\ll \dot{\Omega}/\ddot{\Omega}$ should be retained in the equations as a term of the form $-\rho_{co} \dot{\vec{\Omega}} \times \vec{r}$ if it is not small compared to other retained terms.

However, precession will not be considered in this thesis.

7. The Viscous term

The charged fluid experiences viscous effects due to electron-electron scattering. The charged fluid shear viscosity η_c in the absence of a magnetic field has been calculated by Heinzman and Nitsch (1972). Flowers and Itoh (1975) calculated the electron fluid shear viscosity η_e which is related to η_c by $\eta_e/\rho_e = \eta_c/\rho_c$. The two results for η_c are in good agreement.

The charged fluid kinematic shear viscosity ν_c is defined as follows:

$$[\text{II.61}] \quad \nu_c = \eta_c / \rho_c \approx \frac{1}{3} \tau_e c^2 \approx 3 \times 10^{10} T_7^{-2} \text{ cm}^2 \text{ sec}^{-1}.$$

Note that the shear viscosity is $\propto T^{-2}$. On the other hand, the Landau theory of Fermi liquids predicts that the bulk viscosity is $\propto T^2$ (Abrikosov and Khalatnikov, 1958), which vanishes in the extreme degeneracy limit. Accordingly, the bulk viscosity is neglected.

Since ν_c depends on T , which fluctuates slightly due to the fluid flow, ν_c experiences slight fluctuations. The effect of these viscous fluctuations on the flow is negligible if the temperature fluctuations are small. The size of the temperature fluctuations is discussed in Chapter III.

The estimate [II.61] for ν_c is based on the assumption that the magnetic field vanishes. This is a good assumption if the electron cyclotron frequency ω_{ce} is much smaller than the electron-electron collision frequency τ_e^{-1} . From [II.26] and [II.49] it follows that

$$[\text{II.62}] \quad \omega_{ce} \tau_e \approx 10^6 B_{11} T_7^{-2} \gg 1,$$

so it is a poor approximation to neglect the effect of the magnetic field on the viscosity. This was noted by Heinzman et al. (1973).

In the limit $\omega_{ce}\tau_e \rightarrow \infty$, which applies to the charged fluid in view of [II.62], the isotropic viscosity tensor with non-zero components $\approx \eta_c$ for the $\vec{B} = \vec{0}$ case is replaced by an anisotropic tensor. This tensor depends on the direction of $\vec{B}$ but not on its magnitude. Its non-zero components are still $\approx \eta_c$ (Chapman and Cowling, 1939). Thus, the order of magnitude of the viscosity term is equal to $\eta_c v_c / \lambda^2$ in both the $\omega_{ce}\tau_e \rightarrow 0$ limit and the $\omega_{ce}\tau_e \rightarrow \infty$ limit. The ratio of the size of the viscosity term to the size of the Coriolis term is to within a factor of two the Ekman number E:

$$[II.63] \quad E = v_c / \Omega_o \lambda^2 \approx 3 \times 10^{-4} \Omega_2^{-1} T_7^{-2} \lambda_6^{-2},$$

where the dimensionless length scale λ_6 is used (see Table III). Since E is very small for $\lambda_6 \approx 1$ and $T_7 \gtrsim .1$, the viscosity should have only a small effect on large scale charged fluid flow within pulsars which are not extremely old. In such cases where the viscosity has little effect the anisotropy in the viscosity will not have an important effect on the motion.

8. The superfluid drag

Electrons in the charged fluid can scatter off the normal neutrons in the superfluid vortex cores, as can the small fraction of protons which are not superconducting. As a result, there is a drag force which tends to reduce the relative velocity $\vec{v}_s - \vec{v}_c$ between the neutron superfluid and the charged fluid. For small relative velocities, this drag force may be represented by a term $\rho_c (\vec{v}_s - \vec{v}_c) / \tau_D$ in the charged fluid momentum equation.

Feibelman (1971) estimated the relaxation time required to reduce the relative velocity between the electrons and the neutrons, and showed that

the electron drag is larger than the proton drag. Baym et al. (1975) discuss the relationship of this relaxation time with the post-glitch relaxation time of pulsars. For reasonable values of the adjustable parameters, two of which are the temperature and the neutron energy gap size Δ , the theoretically predicted post-glitch relaxation times can be the order of days (as in the Crab) or a year (as in the Vela).

The fluid momentum equation has been discussed term by term, and a number of approximations have been made and justified. Given below are the resulting linearized equations:

$$\begin{aligned}
 \text{[II.64]} \quad \frac{\partial}{\partial t} (\rho_{co} \vec{v}_c) = & -2\vec{\Omega}_0 \times \rho_{co} \vec{v}_c - \vec{\nabla} p'_c - p'_c \vec{\nabla} \psi_0 \\
 & - \rho_{co} \vec{\nabla} \phi' - (1/4\pi) \vec{B}_0 \times (\vec{\nabla} \times \vec{B}') - (1/4\pi) \vec{B}' \times (\vec{\nabla} \times \vec{B}_0) \\
 & - \vec{\nabla} \cdot \vec{\sigma}' + \rho_{co} (\vec{v}_s - \vec{v}_c) / \tau_D .
 \end{aligned}$$

The Gravitational Source Equation

The gravitational potential perturbation satisfies

$$\text{[II.65]} \quad \nabla^2 \phi' = 4\pi G \rho'_c .$$

This equation is coupled to the momentum equation through the term $-\rho_{co} \vec{\nabla} \phi'$, which as previously noted is only about 1% of the size of the buoyancy term for long wavelength motions. In that case, it is a fairly good approximation to neglect the term $-\rho_{co} \vec{\nabla} \phi'$ and not to solve [II.65]. This is done in all the free oscillation calculations in this thesis.

The Magnetic Induction Equations

The magnetic field evolves according to

$$[\text{II.66}] \quad \frac{\partial \vec{B}}{\partial t} = -c \vec{\nabla} \times \vec{E}$$

subject to the constraint

$$[\text{II.67}] \quad \vec{\nabla} \cdot \vec{B} = 0$$

For processes happening so slowly that electromagnetic radiation can be neglected, the current $\vec{J}$ is given by the pre-Maxwell equation

$$[\text{II.68}] \quad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$$

To close these equations, a relationship must be found between the electric field $\vec{E}$ and the other variables in the problem, such as $\vec{B}$, $\vec{J}$, $\vec{v}_c$ and p' , i.e., a generalized Ohm's law appropriate to the charged fluid must be found.

A standard procedure (Krall and Trivelpiece, 1973) used in forming a generalized Ohm's law begins with the momentum equations for the electron and proton fluids. Not all terms in these equations will be written out; those that are omitted are analogous to terms in the one-fluid equations which are small. A term that appears in the two-fluid equations which has no analogue in the one-fluid equations is the electron-proton collision term.

The resulting linearized two-fluid equations are

$$[\text{II.69}] \quad n_{o\alpha} m_{\alpha}^* \frac{\partial \vec{v}_{\alpha}}{\partial t} = -2\vec{\Omega}_o \times n_{o\alpha} m_{\alpha}^* \vec{v}_{\alpha} - \vec{\nabla} p'_{\alpha} - n'_{\alpha} m_{\alpha}^* \vec{\nabla} \phi_o \\ + n_{o\alpha} q_{\alpha} \left(\vec{E}' + \frac{\vec{v}_{\alpha}}{c} \times \vec{B}_o \right) + n_{o\alpha} m_{\alpha}^* \frac{\delta \vec{v}_{\alpha}}{\delta t} \Big|_{\text{coll.}}$$

In [II.69], $\alpha = \{e, p\}$ labels electrons and protons. Subscripts 'o' refer to unperturbed quantities and primes refer to perturbations, as in the

one-fluid equations. The symbols n_α , $\vec{v}_\alpha$, q_α and m_α^* refer to the number density, velocity, charge and effective mass, respectively, of particles of type α . Because the electrons are relativistic and the protons non-relativistic, $m_e^* \approx \gamma m_e$ and $m_p^* \approx m_p$. In terms of these variables the perturbations in $\vec{J}$, in the charge density ρ_q and in the centre of mass velocity are defined by

$$[\text{II.70}] \quad \vec{J}' = \sum_{\alpha} q_{\alpha} n_{\alpha} \vec{v}_{\alpha}$$

$$[\text{II.71}] \quad \rho_q' = \sum_{\alpha} n_{\alpha}' q_{\alpha}$$

$$[\text{II.72}] \quad \vec{v}_c = \frac{1}{\rho_c} \sum_{\alpha} n_{\alpha} m_{\alpha}^* \vec{v}_{\alpha}.$$

Multiplying [II.69] by q_{α}/m_{α}^* and summing over α produces an equation for the time evolution of $\vec{J}'$. This equation becomes considerably simpler if terms of order $m_e^*/m_p^* \approx .1$ are neglected compared to one. It will be shown that this approximation does not produce comparable errors in the generalized Ohm's law. The equation for $\partial \vec{J}' / \partial t$ is

$$[\text{II.73}] \quad \frac{\partial \vec{J}'}{\partial t} = -2\vec{\Omega}_0 \times \vec{J}' + \frac{e}{m_e^*} \vec{\nabla}_{p_e}'^{\text{CM}} - \rho_q' \vec{\nabla} \phi_0 + \frac{e^2 n_e}{m_e^*} (\vec{E}' + \frac{\vec{v}_c}{c} \times \vec{B}_0) - \frac{e}{cm_e^*} \vec{J}' \times \vec{B}_0 + \left. \frac{\delta \vec{J}'}{\delta t} \right|_{\text{coll}}.$$

The term $\vec{\nabla}_{p_e}'^{\text{CM}}$ is the gradient of the perturbation in the electron pressure as viewed in the centre-of-mass frame. Since a small relative velocity between the electron and proton fluids produces an extremely large current which tends to reduce the relative velocity, $\vec{v}_e \approx \vec{v}_p \approx \vec{v}_c$ and thus $\vec{\nabla}_{p_e}'^{\text{CM}}$ equals $\vec{\nabla}_{p_e}'$ to a fair approximation. With the standard approximation that the collision term $(\delta \vec{J}' / \delta t)_{\text{coll}}$ is $\propto \vec{J}'$, [II.73] may be rewritten as the generalized Ohm's law

$$\begin{aligned}
 \text{[II.74]} \quad \vec{E}' = & -\frac{\vec{v}_c}{c} \times \vec{B}_0 + \left\{ \frac{1}{\sigma} + \frac{4\pi}{2\omega_{pe}} \left[\frac{\partial}{\partial t} + (2\vec{\Omega}_0 - \omega_{ce} \hat{B}_0) \times \right] \right\} \vec{J}' \\
 & - \frac{1}{en_e} \vec{\nabla} p_e' + \frac{4\pi\rho_q'}{2\omega_{pe}} \vec{\nabla} \phi_0,
 \end{aligned}$$

where ω_{pe} is the electron plasma frequency, which is $\approx 10^{21} \text{ sec}^{-1}$, $\hat{B}_0$ is a unit vector in the direction of $\vec{B}_0$, and σ is the conductivity in the absence of a magnetic field.

The $-\vec{v}_c \times \vec{B}_0$ term is the induced emf. If it is the only significant term in the Ohm's law, the magnetic field evolves according to the infinite conductivity induction equation [II.85]. There are four terms linear in $\vec{J}'$. One is the usual Ohm term $\vec{J}'/\sigma$. Another is the Hall term $\propto \hat{B}_0 \times \vec{J}'$ which represents the magnetically induced anisotropy in the electrical conduction. Another anisotropic term $\propto \vec{\Omega}_0 \times \vec{J}'$ arises from the rotation of the star. There is a term $\propto \partial \vec{J}'/\partial t$ which represents the electric field set up by very rapidly fluctuating currents. Electric fields are also set up by gradients in the electron pressure and by the action of gravity on fluid regions where the perturbation in the charge density does not vanish.

Because of quasi-neutrality one might expect the term $\propto \rho_q' \vec{\nabla} \phi_0$ to be small. This is indeed the case. Using $\vec{\nabla} \cdot \vec{E}' = 4\pi\rho_q'$ to estimate $\rho_q' \approx E'/4\pi\lambda$, it follows that

$$\text{[II.75]} \quad \frac{|4\pi\rho_q' \vec{\nabla} \phi_0 / \omega_{pe}^2|}{|\vec{E}'|} \approx 10^{-28} \lambda^{-1} \ll 1 \text{ if } \lambda \gg 10^{-28} \text{ cm}.$$

Such short wavelengths are, of course, unphysical. It will be shown in Chapter III that sound waves are the highest frequency waves that the charged fluid can support. Because they have the highest frequency, the pressure perturbation in them is higher for a given V_c than for other kinds of waves. Thus $|\vec{\nabla} p_c'| \lesssim \rho_{co} V_c / \tau$ where the sound wave period $\tau = \lambda / s_{co}$. Since $|p_e'| \lesssim |p_c'|$,

$$[II.76] \quad \frac{|\vec{\nabla} p'_e / en_e|}{|\frac{1}{c} \vec{v}_c \times \vec{B}_o|} \lesssim \frac{\rho_{co} c s_{co}}{en_e \lambda B_o} \approx 10^{-15} B_{11}^{-1} \lambda^{-1},$$

which is $\ll 1$ if $\lambda \gg 10^{-15} B_{11}^{-1}$ cm.

The four terms linear in $\vec{J}'$ stand in the ratio $\frac{1}{\sigma} : \frac{4\pi}{2} \frac{1}{\tau} : \frac{4\pi}{2} \frac{1}{\omega_{pe}} 2\Omega_o :$

$\frac{4\pi}{2} \omega_{ce}$. Using the conductivity σ given by Baym et al. (1969b), and the fact that the shortest period oscillations are sound waves, the ratios are roughly $10^{-31} \tau_7^2 : 10^{-37} \lambda_6^{-1} : 10^{-39} \Omega_2 : 10^{-25} B_{11}$. The Hall term is dominant except for some extreme conditions such as a very low magnetic field or a very high temperature.

To estimate the size of the Hall term one needs an estimate of J' .

From [II.68]

$$[II.77] \quad J' \approx cB'/4\pi\lambda.$$

An assumption about the origin of the perturbations $\vec{B}'$ must be made in order to estimate B' . The assumption I make is that if $\vec{v}_c = \vec{0}$, then $\vec{B}' = \vec{0}$, i.e., any perturbations in $\vec{B}$ originate in the action of charged fluid motion on $\vec{B}_o$. That being the case, it is reasonable to suppose that the induced emf is not very much smaller than the Hall term. Equation [II.66] can then be used to estimate B' in terms of v_c :

$$[II.78] \quad B' \approx v_c B_o \tau / \lambda.$$

The relative size of the Hall term is thus

$$[II.79] \quad \frac{|(4\pi\omega_{ce}/\omega_{pe}^2) \hat{B}_o \times \vec{J}'|}{|\frac{1}{c} \vec{v}_c \times \vec{B}_o|} \approx \frac{\omega_{ce} c^2 \tau}{\omega_{pe}^2 \lambda^2} \approx 10^{-5} B_{11} (\tau/\lambda^2).$$

For sound waves, $\tau \approx \lambda/s_{co}$. For such waves the Hall term is negligible if

$$[II.80] \quad \lambda \gg 10^{-15} B_{11} \text{ cm (sound).}$$

For inertial waves, $\tau \approx \Omega_0^{-1}$ independently of λ . For them the condition is

$$[\text{II.81}] \quad \lambda \gg 10^{-3} (B_{11}/\Omega_2)^{1/2} \text{ cm (inertial)}.$$

For hydromagnetic-inertial waves, $\tau/\lambda^2 \approx 10^{-6} \Omega_2 B_{11}^{-2}$, so over the time scales of these waves, the Hall term is negligible if

$$[\text{II.82}] \quad \Omega_2 B_{11}^{-1} \ll 10^{11} \text{ (hydromagnetic-inertial)}.$$

The conditions [II.80-82] are easily satisfied when the charged particles can be treated as a fluid (see equations [II.28-29]). Under such conditions, the linearized form of the generalized Ohm's law reduces to

$$[\text{II.83}] \quad \vec{E}' + \frac{\vec{v}}{c} \times \vec{B}_0 = \vec{0} \quad ,$$

which gives the linearized induction equation

$$[\text{II.84}] \quad \frac{\partial \vec{B}'}{\partial t} = \vec{v} \times (\vec{v}_c \times \vec{B}_0) \quad ,$$

of which the non-linear form is

$$[\text{II.85}] \quad \frac{\partial \vec{B}}{\partial t} = \vec{v} \times (\vec{v}_c \times \vec{B}) \quad .$$

It follows from [II.84] that a necessary condition for the validity of a linearized treatment is

$$[\text{II.86}] \quad v_c \ll \lambda/\tau \quad .$$

The conclusion from this analysis is that for a wide variety of λ and τ which encompass the entire fluid regime, the magnetic field evolves according to the infinite conductivity induction equation.

The Boundary Conditions

The charged fluid is enclosed within two surfaces which are roughly concentric spheres (or within one if there is no solid core). The transition between crust and fluid is a sudden one (Baym et al., 1971). The transition from the fluid to any solid central core probably also involves a phase change (Pandharipande and Smith, 1975). It is necessary to specify the behaviour of the charged fluid velocity and the magnetic field at these boundaries in order to be able to calculate the fluid motion.

The Velocity Boundary Conditions

Since the charged fluid has a non-zero viscosity, the appropriate velocity boundary conditions at a solid surface Σ are "no-slip" conditions, that is,

$$[\text{II.87}] \quad \vec{v}|_{\Sigma} = \vec{0} \quad .$$

It is assumed in [II.87] that the frame of reference is chosen so that the surface is not moving. That is not the case with the core. For it, the boundary condition is

$$[\text{II.88}] \quad \vec{v}|_{\Sigma} = \Delta\vec{\Omega} \times \vec{r} \quad ,$$

where $\Delta\vec{\Omega}$ is the difference in angular velocity between crust and core.

Although [II.87-88] are the correct velocity boundary conditions, it is not clear that they are an adequate and consistent set of boundary conditions when one formally lets the viscosity tend to zero, which may be a desirable approximation in certain circumstances.

Even in the inviscid limit, the surface is impermeable to the charged fluid, so the component of velocity normal to it is zero:

$$[\text{II.89}] \quad \vec{v} \cdot \vec{n}|_{\Sigma} = 0 ,$$

where $\vec{n}$ is a unit vector normal to Σ . It follows from Maxwell's equations that the tangential component of $\vec{E}$ is continuous at the interface. Just inside the crust $\vec{E}$ vanishes in the infinite conductivity approximation. Thus,

$$[\text{II.90}] \quad \vec{n} \times \vec{E}|_{\Sigma} = 0 .$$

Ohm's law [II.83] and [II.89-90] imply that

$$[\text{II.91}] \quad (\vec{B} \cdot \vec{n})(\vec{n} \times \vec{v})|_{\Sigma} = 0 ,$$

so that

$$[\text{II.92}] \quad \vec{v}|_{\Sigma} = 0 \text{ if } \vec{B} \cdot \vec{n}|_{\Sigma} \neq 0 .$$

Thus, if there is a non-zero component of magnetic field normal to Σ at a point, then the fluid satisfies no-slip boundary conditions there, even in the inviscid limit.

In the derivation of the boundary condition [II.91], it was implicitly assumed that the charged fluid is strongly coupled to the magnetic field. It will be shown in Chapter III that over time scales $\lesssim \Omega_0^{-1}$, the two are only very weakly coupled. Under such conditions, it is a very good approximation to completely neglect the effect of the magnetic field on the fluid flow, and to use the inviscid fluid equations with no magnetic body force term. If such a mathematical approximation is made, only one boundary condition can be imposed on $\vec{v}_c$ at Σ . That condition is the "slip" boundary condition [II.89].

The Magnetic Field Boundary Conditions

The penetration depth δ of variable magnetic fields with frequency ω into the crust is (§45; Landau and Lifshitz, 1966)

$$[\text{II.93}] \quad \delta = c/\sqrt{2\pi\sigma\omega} \quad ,$$

where σ is the crust conductivity. For the lowest frequency waves that can propagate in the charged fluid (see Chapter III), δ is tens of cm, which is considerably smaller than the thickness of the crust. It is thus an excellent approximation to assume that the solid parts of the star are infinitely conducting. Therefore the magnetic field in these regions cannot decay, and since we are assuming these regions are also perfectly rigid, the magnetic field is static there, as seen in the crustal frame.

From Maxwell's equations, $\vec{B} \cdot \vec{n}$ is continuous across a boundary Σ . However, on the solid side of Σ , $\vec{B} \cdot \vec{n}$ is static, and thus $\vec{B} \cdot \vec{n}$ on the fluid side is also. In this sense, the magnetic field lines are "pinned" at the boundaries. However, in the infinite conductivity limit the tangential component of $\vec{B}$, $\vec{n} \times \vec{B}$, is not necessarily continuous across Σ . What actually happens is that if the magnetic field lines in the fluid are oscillating with frequency ω , $\vec{n} \times \vec{B}$ can vary a large amount in the small distance $\delta(\omega)$ given in [II.93].

The Crust And Core Equations

The solid crust and the solid core (if one exists) are very rigid. In this thesis, they will be treated as perfectly rigid. A necessary condition for the approximation to be a good one is that the stress on a solid part of the star is smaller than its material moduli times the maximum strain it can take without fracturing. For example, if the inner crust can take a strain of 10^{-2} , the maximum stress that it could take without fracturing would be about 10^{27} to 10^{29} dynes cm^{-2} .

If the charged fluid is in shearing motion, then there are viscous stresses set up in it. These shear stresses produce torques on the solid parts of the star, and thus perturb their angular velocities. There are also variable electromagnetic stresses on the solid boundaries which can exert torques.

Formulae for the perturbation $\vec{\Omega}'$ in the angular velocity of the crust caused by the perturbations in the magnetic stress are given below. Similar formulae can be derived for a possible solid core. For simplicity, the results are given for the special case of an axisymmetric magnetic field whose axis coincides with the rotational axis. For the same reason, the perturbations in the magnetic stress are linearized. It is a straightforward matter to generalize these formulae to more complicated field geometries and to the non-linear case. Since the effects of rotation on the structure of the star are of order $10^{-4} \Omega_2^2$, the slightly non-spherical shape of boundaries is neglected.

The torques exerted by fluid motions lie along the z axis. The viscous torque on the crust is given by

$$[\text{II.94}] \quad N'_V = 2\pi R_c^3 \int_{\Sigma} \sigma'_{r\phi} \sin^2 \theta \, d\theta \quad ,$$

where $\sigma'_{r\phi}$ is the (r, ϕ) component of the fluid shear stress. The magnetic torque on the crust due to perturbations in the magnetic stresses at the inner crustal surface is

$$\begin{aligned}
 \text{[II.95]} \quad N'_B &= 2\pi R_c^3 \int_{\Sigma} \frac{B_{\text{or}} B'_{\phi}}{4\pi} \sin^2 \theta \, d\theta \\
 &= 2\pi R_c^3 \int_{\Sigma} \frac{B_{\text{or}} [B'_{\phi}]}{8\pi} \sin^2 \theta \, d\theta .
 \end{aligned}$$

In [II.95], the symbol $[B'_{\phi}]$ stands for the jump in B'_{ϕ} at Σ going from the fluid side Σ^- to the solid side Σ^+ . In the infinite conductivity limit $B'_{\phi}|_{\Sigma} \rightarrow \frac{1}{2}(B'_{\phi}|_{\Sigma^-} + B'_{\phi}|_{\Sigma^+}) = \frac{1}{2} B'_{\phi}|_{\Sigma^-} = \frac{1}{2} [B'_{\phi}]$, since $B'_{\phi}|_{\Sigma^+}$ vanishes in this limit.

The total crust angular acceleration is the sum of a term $\dot{\vec{\Omega}}_0$ due to quasi-steady external torques and any precessional motion, and a term $\dot{\vec{\Omega}}'$ due to the shear stresses and the perturbations in the magnetic stresses communicated by the fluid interior:

$$\text{[II.96]} \quad \dot{\vec{\Omega}} = \dot{\vec{\Omega}}_0 + \dot{\vec{\Omega}}' .$$

If the crust is treated as a spherical rigid body with a scalar moment of inertia I , then

$$\text{[II.97]} \quad \dot{\vec{\Omega}}' = (N'_V + N'_B)/I .$$

The power spectrum of the perturbations $\dot{\vec{\Omega}}'(t)$ in the angular acceleration of the crust due to charged fluid flow will be used in Chapters V and VI to determine the periodicities in some computer simulated flows.

Figure 2. Charged fluid mass fraction as a function of total density in the fluid region. The density is in grams.

Figure 3. Logs of squares of sound speeds as a function of total density in the fluid region. The change in slope of the s_0^2 graph above $10^{15} \text{ gm cm}^{-3}$ is due to the neutrons becoming more relativistic.

Figure 4. $\rho_o^{(0)}$ and $\rho_o^{(2)}$ as functions of radius in a rotating neutron star model. $\rho_o^{(0)}$ is the spherically symmetric part of the total mass density. $\rho_o^{(2)}$ is the radial part of the density perturbation due to rotation. The radius is measured in km.

Figure 5. Toroidal magnetic field perturbation required to balance the quasi-steady Poincaré force. The unperturbed magnetic field is assumed to be a uniform one parallel to the rotation axis and to be 10^{11} gauss. The rotation axis is vertical; the equator is horizontal. The toroidal field B'_ϕ is given in a contour plot. Contour plots with similar labeling conventions are used throughout the thesis for the azimuthal parts of $\vec{B}'$ and $\vec{v}_c$. The maximum of the absolute value of the quantity (in this case, $\max|B'_\phi|$, which is 4.93×10^5 gauss) is given on each picture. A value of +10 is assigned to the point where this maximum occurs if the quantity is positive there. Otherwise, -10 is assigned to it.

MAX $B_\phi = 4.93 \times 10^5$ GAUSS

CHAPTER III: TYPES OF CHARGED FLUID WAVES

In the last chapter, the equations of the charged fluid dynamics were derived. The present chapter uses these equations to find the general characteristics of charged fluid waves.

Dispersion Relations

When the length scale λ of a free oscillation in the charged fluid is small compared to the distance R over which the equilibrium solution varies, dispersion relations may be derived relating the frequency ω and the local wavenumber $\vec{k}$ of the oscillation. Even when $\lambda \approx R$, the dispersion relations derived upon neglecting inhomogeneities in the equilibrium solution can serve as useful guides in determining the relation between the length and time scales of the oscillations. In the present section, such inhomogeneities are neglected and the charged fluid is treated as infinite in extent. The motivation for this somewhat crude treatment of the waves in the fluid is that it enables one to determine easily the possible types of waves and their prime characteristics.

It is assumed throughout this thesis that the length scales λ and time scales τ of charged fluid disturbances are such that the conditions derived in the previous chapter for the charged particles to be treated as a fluid are obeyed. These conditions are satisfied for an extremely wide range of λ and τ . In particular, they are easily satisfied in the case $\lambda \approx R$.

To a large extent, the work in this section is an adaptation to neutron stars of the general theory of waves in rotating electrically conducting fluids in the presence of a magnetic field, which has recently been reviewed by Acheson and Hide (1973).

When it is linearized and spatial gradients in ρ_{co} are neglected, the mass conservation equation becomes

$$[III.1] \quad s_{co}^{-2} \frac{\partial P'}{\partial t} + \vec{\nabla} \cdot \vec{v}_c = 0 .$$

The notation $P' = p'_c / \rho_{co}$ has been used in [III.1] for the reduced pressure perturbation.

I shall neglect several terms in the linearized momentum equation. One is the Poincaré term, which is omitted because only free oscillations are of interest here. I shall also neglect $-\rho_{co} \vec{\nabla} \phi'$ and $-p'_c \vec{\nabla} \psi_0$. The gravitational term $-\rho_{co} \vec{\nabla} \phi'$ was shown in the previous chapter to be fairly small. Also, since ϕ' satisfies an elliptic equation, this gravitational term cannot be responsible for any new types of waves.

In the buoyancy term $-p'_c \vec{\nabla} \psi_0$, $\vec{\nabla} \psi_0$ varies roughly linearly with distance to the centre of the star (see Chapter IV). Thus, buoyancy can be properly treated in the dispersion relation approach only if $\lambda \ll R$. However, in this short wavelength limit, $-p'_c \vec{\nabla} \psi_0$ is roughly λ/R times the size of the pressure gradient term $-\vec{\nabla} p'_c$ and is thus negligible. The question of the effects of buoyancy on modes with length scales $\lambda \approx R$ is addressed in Chapters IV through VI for each class of oscillation individually.

The damping terms (viscosity and superfluid drag) are temporarily neglected; they are re-introduced in a later section where they are shown to damp most oscillations only lightly.

The resulting highly simplified momentum equations are

$$[III.2] \quad \frac{\partial \vec{v}_c}{\partial t} = -2\vec{\Omega}_0 \times \vec{v}_c - \vec{\nabla} P' - \frac{1}{4\pi\rho_{co}} \vec{B}_0 \times (\vec{\nabla} \times \vec{B}') .$$

The linearized induction equation is

$$[III.3] \quad \frac{\partial \vec{B}'}{\partial t} = \vec{\nabla} \times (\vec{v}_c \times \vec{B}_0) .$$

If all variables vary in time and space as $e^{i(\vec{k} \cdot \vec{r} - \omega t)}$, then

$$[\text{III.4}] \quad -i\omega s_{co}^{-2} P' + i\vec{k} \cdot \vec{v}_c = 0$$

$$[\text{III.5}] \quad -i\omega \vec{v}_c = -2\vec{\Omega}_o \times \vec{v}_c - i\vec{k}P' - \frac{1}{4\pi\rho_{co}} \vec{B}_o \times (i\vec{k} \times \vec{B}')$$

$$[\text{III.6}] \quad -i\omega \vec{B}' = i\vec{k} \cdot \vec{B}_o \vec{v}_c - i\vec{k} \cdot \vec{v}_c \vec{B}_o$$

Sound Waves

An equation for P' may be derived by taking the scalar product of $\vec{k}$ with [III.5] and using [III.4]:

$$[\text{III.7}] \quad (k^2 - \omega^2 s_{co}^{-2})P' = -i\vec{k} \cdot [-2\vec{\Omega}_o \times \vec{v}_c - \frac{1}{4\pi\rho_{co}} \vec{B}_o \times (i\vec{k} \times \vec{B}')] .$$

Note that in the limit $\Omega_o, B_o \rightarrow 0$, [III.7] reduces to the Fourier analyzed form of a wave equation for P' ; in the absence of rotation and a magnetic field, the only kind of wave which can propagate is a sound wave with dispersion relation

$$[\text{III.8}] \quad \omega = s_{co} k \quad (\Omega_o = B_o = 0) .$$

If the magnetic body force and the Coriolis force are sufficiently small, sound waves with the dispersion relation [III.8] can propagate. The total acceleration, Coriolis acceleration and magnetic acceleration have sizes in the ratio $1 : \frac{2\Omega_o}{\omega} : V_A^2 k^2 \omega^{-2}$, respectively, where V_A is the Alfvén speed:

$$[\text{III.9}] \quad \vec{V}_A = \frac{\vec{B}_o}{\sqrt{4\pi\rho_{co}}} ; \quad V_A \approx 10^4 B_{11} \text{ cm sec}^{-1} .$$

If $\omega \approx s_{co} k$, then the three accelerations are in the ratio $1 : 2\Omega_o/s_{co} k : v_A^2/s_{co}^2 \approx 1 : 10^{-2} \Omega_2 \lambda_6 : 10^{-12} B_{11}^2$, so the conditions necessary for the propagation of sound waves which are not much affected by rotation or the magnetic field are

$$[III.10] \quad \Omega_2 \lambda_6 \ll 10^2 ; \quad B_{11} \ll 10^6 .$$

These conditions are well satisfied unless the magnetic field or the angular velocity are so large that they have a substantial effect on the equilibrium structure of the star. Such cases are not considered in this thesis.

Chapter IV is devoted to a fuller examination of sound waves in the charged fluid.

Inertial Waves And Hydromagnetic-Inertial Waves

If $\omega \ll s_{co} k$, then the left-hand side of [III.7] becomes $k^2 P'$ to a good approximation. Dropping ω^2/s_{co}^2 in comparison to k^2 is equivalent to replacing the wave equation for P' with an elliptic equation in which pressure effects are instantaneously propagated. In such an approximation, equation [III.4] becomes the incompressibility condition

$$[III.11] \quad i\vec{k} \cdot \vec{v}_c = 0$$

and [III.6] simplifies to

$$[III.12] \quad -i\omega \vec{B}' = i\vec{k} \cdot \vec{B}_o \vec{v}_c .$$

Multiplying [III.5] by $i\vec{k}$ and using [III.11-12] produces

$$[III.13] \quad [(-i\omega)^2 - (\vec{v}_A \cdot i\vec{k})^2] i\vec{k} \times \vec{v}_c = (2\vec{\Omega}_o \cdot i\vec{k})(-i\omega)\vec{v}_c .$$

Since the linear operation $(\vec{k} \times)$ has the eigenvalues 0, $\pm ik$, of which the first is excluded by the incompressibility condition, the dispersion relation which follows from [III.13] is

$$[\text{III.14}] \quad \omega^2 \pm \frac{(2\vec{\Omega}_0 \cdot \vec{k})}{k} \omega - (\vec{V}_A \cdot \vec{k})^2 = 0 ,$$

which is equation (4.11a) of Acheson and Hide (1973).

A characteristic dimensionless parameter associated with [III.14] is

$$[\text{III.15}] \quad \xi = \frac{(2\vec{\Omega}_0 \cdot \vec{k})}{k\vec{V}_A \cdot \vec{k}} \approx 10^4 \Omega_2 \lambda_6 B_{11}^{-1} .$$

Note that the estimate of the magnitude of ξ in [III.15] is not accurate if $\vec{k}$ and $\vec{\Omega}_0$ are almost perpendicular. However, the estimate is accurate for almost all directions of $\vec{k}$. In order to save space, statements such as these about the dependence of the size of various quantities and effects on the direction of $\vec{k}$ will be omitted for the remainder of this chapter.

If $\xi \gg 1$, then the roots of [III.14] split into two branches, widely separated in frequency. The high frequency branch is formed by inertial waves:

$$[\text{III.16}] \quad \omega = \pm \frac{2\vec{\Omega}_0 \cdot \vec{k}}{k} \approx 200 \Omega_2 \text{ sec}^{-1} .$$

The lower frequency branch is formed by hydromagnetic-inertial waves:

$$[\text{III.17}] \quad \omega = \pm \frac{(\vec{V}_A \cdot \vec{k})^2 k}{2\vec{\Omega}_0 \cdot \vec{k}} \approx 10^{-6} B_{11}^2 \lambda_6^{-2} \Omega_2^{-1} \text{ sec}^{-1} .$$

Hide (1971) made an important contribution to the understanding of the charged fluid dynamics when he noted that ξ is generally $\gg 1$ for oscillations with length scales $\lambda \approx R$ if the magnetic field is not very much larger than 10^{12} gauss. As a result, two modes of the charged fluid are the inertial modes and the hydromagnetic-inertial modes.

The parameter ξ can be $\ll 1$ if the magnetic field is extremely strong ($B_{11} \gg 10^4$) or if the wavelength is extremely small ($\lambda_6 \ll 10^{-4}$). Under such conditions rotation has only a minor effect on the waves, which are essentially pure hydromagnetic (Alfvén) waves. In this thesis, I am primarily concerned with the less extreme conditions in which $\xi \gg 1$.

Inertial Waves

The periods τ_I of inertial waves are typically half the rotation period P , or slightly longer:

$$[\text{III.18}] \quad \tau_I \gtrsim \frac{1}{2} P.$$

Since the periods of sound waves are $\lesssim R/s_{co} \approx 10^{-4}$ sec, there is no overlap between the frequency range of sound waves and that of inertial waves in neutron stars.

It follows from [III.12] that over the time scales τ_I , the ratio of the magnetic energy of these waves to their kinetic energy is extremely small:

$$[\text{III.19}] \quad \frac{(B')^2/8\pi}{\frac{1}{2} \rho_{co} v_c^2} = \xi^{-2} \approx 10^{-8} \Omega_2^{-2} \lambda_6^{-2} B_{11}^2.$$

The ratio of the magnetic body force to the Coriolis force over time scales τ_I is also equal to ξ^{-2} . Under these circumstances the effect of the magnetic field can be completely neglected.

Hydromagnetic-Inertial Waves

The periods τ_{HI} of these highly dispersive waves can be extremely long:

$$[\text{III.20}] \quad \tau_{HI} \approx B_{11}^{-2} \lambda_6^2 \Omega_2 \text{ months.}$$

If $\xi \gg 1$, there is no overlap between the periods of these waves and the periods of inertial waves. Unlike pure Alfvén waves, the group velocity $\partial\omega/\partial\vec{k}$ of hydromagnetic-inertial waves is not necessarily parallel to $\vec{B}_0$.

The magnetic energy in hydromagnetic-inertial waves is very much larger than the kinetic energy:

$$[\text{III.21}] \quad \frac{(\vec{B}')^2/8\pi}{\frac{1}{2} \rho_{co} v_c^2} = \xi^2 \approx 10^8 B_{11}^{-2} \Omega_2^2 \lambda_6^2 ,$$

in contrast to the inertial waves [III.19] where this ratio is reversed.

Over time scales τ_{HI} , the magnetic body forces are roughly the same size as the Coriolis force:

$$[\text{III.22}] \quad \frac{|[\vec{B}_0 \times (\vec{\nabla} \times \vec{B}')] / 4\pi|}{|2\vec{\Omega}_0 \times \rho_{co} \vec{v}_c|} \approx 1 ,$$

whereas the total force on a fluid element is smaller than either by the factor ξ^2 .

Damping Due To Viscosity And Superfluid Drag

For order of magnitude estimates of viscous damping times, the use of an isotropic viscosity tensor is sufficient. For incompressible flow, isotropic viscosity contributes an extra term $\nu_c \nabla^2 \vec{v}_c$ to [III.2]. The effect of this term on the dispersion relation is to change equation [III.14] to

$$[\text{III.23}] \quad \omega^2 + \left[\pm \frac{2\vec{\Omega}_0 \cdot \vec{k}}{k} + i\nu_c k^2 \right] \omega - (\vec{V}_A \cdot \vec{k})^2 = 0 .$$

For inertial waves, it follows from [III.23] that

$$[\text{III.24}] \quad \omega = \pm \frac{2\vec{\Omega}_0 \cdot \vec{k}}{k} - i\nu_c k^2 .$$

The viscous damping time $\tau_V^{(I)}$ for inertial waves is thus

$$[III.25] \quad \tau_V^{(I)} = 1/\nu_c k^2 \approx 10 \lambda_6^2 T_7^2 \text{ sec.}$$

It is straightforward to show that the viscous damping time for sound waves is also about $1/\nu_c k^2$. For hydromagnetic-inertial waves, the dispersion relation which follows from [III.23] is

$$[III.26] \quad \omega = \frac{(\vec{V}_A \cdot \vec{k})^2}{\pm(2\vec{\Omega}_0 \cdot \vec{k})/k + i\nu_c k^2} \approx \pm \frac{k(\vec{V}_A \cdot \vec{k})^2}{2\vec{\Omega}_0 \cdot \vec{k}} - i\xi^{-2} \nu_c k^2.$$

The viscous damping time $\tau_V^{(HI)}$ for these waves is thus about ξ^2 times $\tau_V^{(I)}$:

$$[III.27] \quad \tau_V^{(HI)} = \xi^2/\nu_c k^2 \approx 100 \Omega_2^2 \lambda_6^4 B_{11}^{-2} T_7^2 \text{ years.}$$

The very long viscous damping time for hydromagnetic-inertial waves when ξ^2 is $\gg 1$ results from the fact that only a small fraction ξ^{-2} of the energy in these waves resides in fluid motion.

Equation [III.23] can also be used to determine the viscous damping times for waves with wavelengths so small that $\xi \ll 1$, i.e. when $k \gg \Omega_0/V_A$. Under such conditions the factor $\pm(2\vec{\Omega}_0 \cdot \vec{k})/k$ in [III.23] may be dropped. The viscous damping time for such Alfvén waves is $\approx 2/\nu_c k^2$, which for $k \gg \Omega_0/V_A$ is $\ll 2V_A^2/\nu_c \Omega_0^2 \approx 10^{-6} B_{11}^2 T_7^2 \Omega_2^{-2}$ sec. Such waves are thus damped in a very short time.

To analyze the effects of superfluid drag on the three basic types of waves, I assume that $\vec{v}_s = 0$ in the rotating frame. These effects may then be determined by formally replacing $\nu_c k^2$ in [III.23-27] by $1/\tau_D$. For example, the inertial wave damping time $\tau_D^{(I)}$ due to superfluid drag is just

$$[III.28] \quad \tau_D^{(I)} = \tau_D.$$

For hydromagnetic-inertial waves the superfluid drag damping time is

$$[\text{III.29}] \quad \tau_D^{(\text{HI})} = \xi^2 \tau_D.$$

Since τ_D can be days to years, $\tau_D^{(\text{HI})}$ can be up to 10^8 years if $\xi = 10^4$.

When $\tau_D \gg 1/v_c k^2$, which is true for a wide range of conditions, superfluid drag is a less important damping mechanism than is viscosity.

Sound waves can be damped by collisionless processes such as Landau damping, but for wavelengths longer than the electron mean free path, collisionless damping is insignificant (Pippard, 1955).

Wave Damping By Heat Conduction

Since the charged fluid is not completely degenerate, the mass density ρ_c depends on the temperature. If the temperature is perturbed by T' , there will be a density perturbation $(\partial \rho_c / \partial T)_{p_c} T'$, which is acted upon by gravity to produce a thermal buoyancy force $-(\partial \rho_c / \partial T)_{p_c} T' \vec{\nabla} \phi_0$. Because temperature perturbations which are created by charged fluid motion can be damped by heat conduction, the thermal buoyancy force can damp waves in the charged fluid.

The appropriate form of the heat flux equation (Acheson and Hide, 1973) when linearized is

$$[\text{III.30}] \quad \alpha \left(\frac{\partial T'}{\partial t} + \vec{v}_c \cdot \vec{\nabla} T_0 \right) = \kappa \nabla^2 T'.$$

α is the heat capacity per unit volume, and κ is the thermal conductivity.

The viscous heating term $\vec{\sigma}' \cdot \vec{\nabla} \vec{v}_c$ does not appear in [III.30] because it

is second order in small quantities. An Ohmic dissipation term of the form

$\vec{j}' \cdot (\vec{E}' + \frac{1}{c} \vec{v}_c \times \vec{B}_0)$ does not appear because of the assumed infinite electrical conductivity.

Because the protons are superfluid, the heat capacity arises almost entirely from the relativistic electrons (§58; Landau and Lifschitz, 1969):

$$[III.31] \quad \alpha = \frac{(3\pi^2)^{2/3}}{3c\hbar} k_B^2 n_e^{2/3} T_o \approx 10^{16} T_7 \text{ erg/deg-cm}^3.$$

The thermal conductivity as calculated by Flowers and Itoh (1975) is not strongly density dependent in the fluid region, and is roughly equal to

$$[III.32] \quad \kappa \approx 10^{25} T_7^{-1} \text{ erg/deg-cm-sec.}$$

The corresponding thermal diffusivity χ is

$$[III.33] \quad \chi = \kappa/\alpha \approx 10^9 T_7^{-2} \text{ cm}^2 \text{ sec}^{-1}.$$

If the gradient in T_o is treated as roughly constant, a Fourier analysis in space and time of [III.30] yields

$$[III.34] \quad T' = \frac{1}{i\omega - \chi k^2} \vec{v}_c \cdot \vec{\nabla} T_o.$$

The temperature gradient in the fluid region can be roughly estimated by equating the temperature gradient heat flux $|\kappa \vec{\nabla} T|$ in the interior with the flux $\sigma_{SB} T_S^4$ of blackbody radiation from the stellar surface at temperature T_S . σ_{SB} is the Stephan-Boltzman constant. The result is

$$[III.35] \quad \left| \frac{\vec{\nabla} T_o}{T_o} \right| \approx \frac{\sigma_{SB} T_S^4}{\kappa T_o} \approx 10^{-13} \text{ cm}^{-1} \text{ for } T_S = 10^6 \text{ K,}$$

which is very much larger than R^{-1} . The mass density of the charged fluid is due primarily to the protons, so the temperature dependence of ρ_c at temperatures low compared to the proton Fermi temperature is

$$[III.36] \quad \rho_c(T_o) \approx \rho_c(T_o = 0) \left[1 + \frac{\pi^2}{8} \left(\frac{k_B T_o}{\epsilon_{Fp}} \right)^2 \right].$$

Thus the thermal buoyancy force is

$$[III.37] \quad - \left(\frac{\partial \rho_c}{\partial T} \right)_{p_c} T \vec{\nabla} \phi_o \approx \rho_{co} \left(\frac{k_B T_o}{\epsilon_{fp}} \right)^2 \frac{\vec{v}_c \cdot \vec{\nabla} T_o}{(i\omega - \chi k^2) T_o} \vec{\nabla} \phi_o.$$

For low frequency waves such that $\omega \ll \chi k^2 \approx 10^{-3} T_7^{-2} \lambda_6^{-2} \text{ sec}^{-1}$, $i\omega$ can be neglected in comparison to χk^2 in [III.37]. For $\omega \gtrsim \chi k^2$, which is probably the case for both sound and inertial waves, $i\omega$ may not be neglected in comparison with χk^2 in estimating the magnitude of the thermal buoyancy force. However, our interest in [III.37] lies in the damping effect that it has. This damping arises only from the χk^2 term in $i\omega - \chi k^2$. Thus if thermal buoyancy damping time $\tau^{(B)}$ is defined by the condition that the thermal buoyancy force with ω set equal to zero be about $\rho_{co} v_c / \tau^{(B)}$ in magnitude, then it follows from [III.37] that

$$[III.38] \quad \tau^{(B)} \approx \left(\frac{\epsilon_{fp}}{k_B T_o} \right)^2 \chi k^2 \frac{T_o}{|\vec{\nabla} T_o|} \frac{R^2}{GM} \approx 10^4 T_7^{-4} \lambda_6^{-2} \text{ sec}.$$

The associated damping time for sound waves and inertial waves is $\tau^{(B)}$, and for hydromagnetic-inertial waves is $\xi^2 \tau^{(B)}$.

Wave damping due to heat conduction seems to be less important than viscous damping if $T_7 \lesssim 1$, but for $T_7 \gg 1$, it could be the dominant damping mechanism.

Resonant Mode-Mode Coupling

It was previously concluded that hydromagnetic-inertial waves in the charged fluid tend to have extremely long damping times. One of the essential assumptions that led to this conclusion was that all waves are of such small amplitude that non-linear effects could be neglected. In this section, some of these non-linear effects are briefly examined.

The question is: can the energy in a hydromagnetic-inertial wave be transferred to other waves at such a rate that it can be dissipated in a time much less than the energy damping time predicted by a linear theory? For the moment, neglect the linear damping effects and consider the simpler question of whether energy can be resonantly transferred from a hydromagnetic-inertial wave with real frequency ω_1 and wavenumber $\vec{k}_1$ to two waves with real frequencies and wavenumbers $\omega_2, \vec{k}_2; \omega_3, \vec{k}_3$. The resonance conditions are (Sagdeev and Galeev, 1969):

$$[\text{III.39}] \quad \omega_1 = \omega_2 \pm \omega_3$$

$$[\text{III.40}] \quad \vec{k}_1 = \vec{k}_2 \pm \vec{k}_3 .$$

Several possibilities are immediately excluded by [III.39-40] and by the dispersion relations [III.8, 16 and 17]. A hydromagnetic-inertial wave cannot beat with itself, because its dispersion relation is non-linear in k . Two hydromagnetic-inertial waves cannot undergo fusion into either a sound wave or an inertial wave because of the frequency mismatch. For the same reason, a hydromagnetic-inertial wave cannot decay into an inertial wave and a sound wave.

The coupling between a hydromagnetic-inertial wave and either two inertial waves or two sound waves should be extremely weak: as previously shown, all but a small fraction ξ^{-2} of the energy in a hydromagnetic-inertial wave resides in the magnetic field, whereas for inertial waves and sound waves almost all the energy resides in the kinetic energy of the wave.

However, even if the coupling were not extremely weak, energy cannot be resonantly transferred from a hydromagnetic-inertial wave to either a pair of inertial waves or sound waves. This may be demonstrated as follows. Because the damping times for these high frequency waves are much shorter

than the periods of hydromagnetic-inertial waves, it suffices to consider the case in which initially almost all the energy is in the low frequency wave. Resonant transfer of energy is possible only if the wave which initially possesses most of the energy has a higher frequency than the other two waves (Sagdeev and Galeev, 1969). However, the frequencies of hydromagnetic-inertial waves are very much lower than the frequencies of either inertial waves or sound waves.

In view of these arguments, it seems unlikely that the energy in a hydromagnetic-inertial wave of period τ_{HI} can be resonantly transferred to sound waves or inertial waves and then dissipated in a time $\ll \tau_{HI}$.

Boundary Layers

Acheson and Hide (1973) review the theory of hydromagnetic boundary layers. In this section, their general results are applied to the problem of boundary layers at the fluid-solid interfaces within neutron stars.

In the charged fluid near the boundary surface Σ with unit normal $\vec{n}$, all physical variables ϕ are split into two parts, a part $\bar{\phi}$ which is roughly constant throughout the boundary layer and a part $\hat{\phi}$ which varies significantly throughout the boundary layer:

$$[III.41] \quad \phi = \bar{\phi} + \hat{\phi} .$$

If the usual boundary layer approximations are made and the curvature of Σ is neglected (Acheson and Hide, 1973), the linearized magnetohydrodynamic equations may be Fourier analyzed in time and space. The time and space variations of all boundary layer quantities are assumed to be of the form

$$[III.42] \quad \hat{\phi} \propto \exp(i\omega t + \gamma z) ,$$

where z measures the distance from Σ along the unit normal $\vec{n}$. The resulting algebraic equations have non-trivial solutions only if a dispersion relation is satisfied:

$$[\text{III.43}] \quad [(i\omega)(i\omega - v_c \gamma^2) - (\vec{V}_A \cdot \vec{n})^2 \gamma^2]^2 - (2\vec{\Omega}_0 \cdot \vec{n})^2 \omega^2 = 0.$$

The boundary layer thickness $\Delta(\omega)$ is defined by

$$[\text{III.44}] \quad \Delta = -1/\text{Re}(\gamma).$$

There are two characteristic frequencies which appear in the dispersion relation [III.43]. One is $2\vec{\Omega}_0 \cdot \vec{n}$ and represents the influence of rotation on the boundary layer structure. The other characteristic frequency is $(\vec{V}_A \cdot \vec{n})^2/v_c$, which is the rate at which viscous diffusion can proceed across the boundary layer along the field lines.

It was shown in Chapter II that the magnetic field at the crustal Σ is probably almost entirely poloidal or almost entirely toroidal. In the poloidal case,

$$[\text{III.45}] \quad (\vec{V}_A \cdot \vec{n})^2/v_c \approx V_A^2/v_c \approx 10^{-2} B_{11}^2 T_7^2 \text{ sec}^{-1}.$$

In the toroidal case, $|\vec{V}_A \cdot \vec{n}|$ is $\ll V_A$, and the characteristic frequency $(\vec{V}_A \cdot \vec{n})^2/v_c$ is $\ll 10^{-2} B_{11}^2 T_7^2 \text{ sec}^{-1}$.

For waves with frequencies ω in the range

$$[\text{III.46}] \quad |2\vec{\Omega}_0 \cdot \vec{n}| > \omega \gg (\vec{V}_A \cdot \vec{n})^2/v_c,$$

the magnetic term in the dispersion relation [III.43] is negligible. The boundary layer thickness for such waves is

$$[\text{III.47}] \quad \Delta = \left(\frac{2v_c}{\omega + |2\vec{\Omega}_0 \cdot \vec{n}|} \right)^{1/2} \approx 10^4 \Omega_2^{-1/2} T_7^{-1} \text{ cm},$$

which is about a hundredth of R_c if $\Omega_2 \approx T_7 \approx 1$. These boundary layers are

the Ekman boundary layers often found at the boundaries of rotating fluids.

In the low frequency range

$$[\text{III.48}] \quad \omega \ll (\vec{V}_A \cdot \vec{n})^2 / \nu_c ,$$

the viscous term in the dispersion relation [III.43] is negligible. The boundary layer thickness for such frequencies is

$$[\text{III.49}] \quad \Delta = \frac{\vec{V}_A \cdot \vec{n}}{\sqrt{2\Omega_0 \cdot \vec{n}\omega}} .$$

It follows from [III.49] that if the charged fluid is oscillating in a hydromagnetic-inertial mode whose length scale is λ and if the magnetic field is predominantly poloidal at Σ , then

$$[\text{III.50}] \quad \Delta \approx \lambda ,$$

i.e. there is effectively no boundary layer. If the magnetic field is predominantly toroidal near Σ , then

$$[\text{III.51}] \quad \Delta \ll \lambda .$$

The presence of a boundary layer in the toroidal case can be understood in terms of a simple physical picture in which the field lines near Σ can easily "slip" by one another parallel to Σ . In the poloidal case, tension in the field lines acts to oppose any such motion and thus inhibits the presence of a boundary layer.

Simplified Charged Fluid Equations

It has been shown in this chapter that for $\xi^2 \gg 1$, there are three basic time scales in the charged fluid dynamics corresponding to the periods of sound waves, inertial waves and hydromagnetic-inertial waves. For each time scale, one or more of the terms in the charged fluid equations are negligible. In this section, a simplified set of charged fluid equations is derived for each of the three time scales by dropping the negligible terms.

There are great advantages to the use of such simplified equations. Not only do they make the mathematics easier, but they also enable one to clearly separate the effects of different time scales. All mode-mode coupling is neglected in such an approach. The remaining three chapters in this thesis are devoted to a more thorough examination of these approximate equations and some of their solutions.

Sound Time Scales

Over the time scales of sound waves, it has been shown that the magnetic body forces are negligible and the Coriolis force is small. A simplified set of equations for these time scales is thus

$$[\text{III.52}] \quad \frac{\partial \rho'_c}{\partial t} + \vec{\nabla} \cdot (\rho_{co} \vec{v}_c) = 0$$

$$[\text{III.53}] \quad \frac{\partial}{\partial t} (\rho_{co} \vec{v}_c) = -\vec{\nabla} p'_c - p'_c \vec{\nabla} \psi_0$$

The viscous term could be added to [III.53] if viscous damping is of interest. In the absence of it, the boundary condition on $\vec{v}_c$ is the slip boundary condition [II.89].

Inertial Time Scales

For time scales longer than those of sound waves, it was shown that $\partial P'/\partial t$ was negligible. However, it was assumed that gradients in ρ_{co} could be neglected, which is not necessarily the case when $\lambda \approx R$. This suggests that the incompressibility condition $\vec{\nabla} \cdot \vec{v}_c = 0$ is not a good approximation over these time scales, but that the anelastic condition (Batchelor, 1953) $\vec{\nabla} \cdot (\rho_{co} \vec{v}_c) = 0$ is.

The anelastic condition is a good approximation if

$$[III.54] \quad |\vec{\nabla} \cdot (\rho_{co} \vec{v}_c)| \ll \rho_{co} V_c / \lambda .$$

From the mass conservation equation it follows that

$$[III.55] \quad |\vec{\nabla} \cdot (\rho_{co} \vec{v}_c)| = |\partial \rho_c' / \partial t| \approx p_c' / s_{co}^2 \tau .$$

From the momentum equation for time scales $\tau \gtrsim \Omega_0^{-1}$, it follows that

$$[III.56] \quad p_c' \approx 2\lambda \Omega_0 \rho_{co} V_c .$$

Together, [III.54] through [III.56] imply that if

$$[III.57] \quad \frac{\lambda^2 \Omega_0^2}{s_{co}^2 \tau} \ll 1 ,$$

then the anelastic condition

$$[III.58] \quad \vec{\nabla} \cdot (\rho_{co} \vec{v}_c) = 0$$

is a good approximation. Condition [III.57] is easily satisfied for all $\tau \gtrsim \Omega_0^{-1}$. The use of [III.58] mathematically "filters" out all sound waves from the solutions to the fluid equations, just as the incompressibility condition $\vec{\nabla} \cdot \vec{v}_c = 0$ does for fluids of nearly constant density.

The magnetic body forces are negligible over the time scales of inertial waves, so the simplified momentum equation is

$$[\text{III.59}] \quad \frac{\partial}{\partial t} (\rho_{co} \vec{v}_c) = -2\vec{\Omega}_o \times \rho_{co} \vec{v}_c - \vec{\nabla} p'_c - p'_c \vec{\nabla} \psi_o.$$

The viscous term may be added if desired, and the slip boundary condition on $\vec{v}_c$ replaced by a no-slip condition. The charged fluid dynamics over the time scale of inertial waves is examined more thoroughly in Chapter V.

Hydromagnetic-Inertial Time Scales

Since [III.57] is satisfied for hydromagnetic-inertial time scales as well as inertial time scales, the anelastic condition

$$[\text{III.60}] \quad \vec{\nabla} \cdot (\rho_{co} \vec{v}_c) = 0$$

is a good approximation on this time scale also.

Over these time scales, it was shown that the $\partial(\rho_{co} \vec{v}_c)/\partial t$ term in the fluid momentum equations is smaller than the Coriolis and magnetic body force terms by a factor ξ^2 . Dropping this relatively small term from the momentum equations results in

$$[\text{III.61}] \quad \vec{0} = -2\vec{\Omega}_o \times \rho_{co} \vec{v}_c - \vec{\nabla} p'_c - p'_c \vec{\nabla} \psi_o - \frac{1}{4\pi} \vec{B}_o \times (\vec{\nabla} \times \vec{B}') - \frac{1}{4\pi} \vec{B}' \times (\vec{\nabla} \times \vec{B}_o).$$

Dropping the $\partial(\rho_{co} \vec{v}_c)/\partial t$ term mathematically filters out inertial waves in the same manner as dropping the $\partial \rho'_c / \partial t$ term in the mass equation filters out sound waves. It was previously shown that resonant coupling of hydro-magnetic-inertial modes and the higher frequency modes is very small so the use of equations [III.60-61], which strictly eliminates such mode coupling, should not lead to qualitatively incorrect results in this regard.

All explicit time evolution over these time scales resides in the magnetic induction equation

$$[\text{III.62}] \quad \frac{\partial \vec{B}'}{\partial t} = \vec{V} \times (\vec{v}_c \times \vec{B}_0) \quad .$$

The appropriate velocity boundary conditions are the no-slip conditions if $\vec{B} \cdot \vec{n}|_{\Sigma} \neq 0$, and slip boundary conditions if $\vec{B} \cdot \vec{n}|_{\Sigma} = 0$. The viscous term may be included in the momentum equation [III.61], but it affects the fluid flow only if $\vec{B} \cdot \vec{n}|_{\Sigma} = 0$.

The charged fluid dynamics over the time scale of hydromagnetic-inertial waves is examined more thoroughly in Chapter VI.

CHAPTER IV: SOUND MODES

In the preceding chapter, it was shown that the highest frequency oscillations of the charged fluid are sound waves. The present chapter explores the properties of these waves in greater detail. One reason for studying these waves in more detail is to determine the effect of the buoyancy force $-p'_c \vec{\nabla} \psi_o$ on them. This force cannot be properly treated in the dispersion relation approach of Chapter III.

The mass conservation equation is

$$[IV.1] \quad s_{co}^{-2} \frac{\partial p'_c}{\partial t} + \vec{\nabla} \cdot (\rho_{co} \vec{v}_c) = 0.$$

The simplified momentum equation is

$$[IV.2] \quad \frac{\partial}{\partial t} (\rho_{co} \vec{v}_c) = -\vec{\nabla} p'_c - p'_c \vec{\nabla} \psi_o.$$

From these two equations follows

$$[IV.3] \quad s_{co}^{-2} \frac{\partial^2 p'_c}{\partial t^2} = \nabla^2 p'_c + \vec{\nabla} \psi_o \cdot \vec{\nabla} p'_c + \nabla^2 \psi_o p'_c.$$

Equation [IV.3] differs from the usual wave equation in that buoyancy terms are present and the sound speed is position dependent. Standing wave solutions of [IV.3] satisfy (cf. equation [III.7])

$$[IV.4] \quad s_{co}^2 \{ \nabla^2 p'_c + \vec{\nabla} \psi_o \cdot \vec{\nabla} p'_c + \nabla^2 \psi_o p'_c \} = -\omega^2 p'_c.$$

In [IV.4], distances are measured in units of the inner crustal radius R_c , and sound speeds in units of $s_{co}(0)$, the sound speed at the centre of the star. The frequency ω is measured in units of $s_{co}(0)/R_c$.

Because the magnetohydrostatic solution departs from spherical symmetry only by a factor $\Omega_o^2 R^2 / \phi_o \approx 10^{-4} \Omega_2^2$, it is a very good approximation to take

ψ_0 and s_{co}^2 to be functions of r only. In this approximation, [IV.4] is separable in spherical polar coordinates. Solutions of the form

$$[IV.5] \quad p'_c(r, \theta, \phi) = f_\ell(r) P_\ell^m(\cos\theta) e^{im\phi}$$

were sought. f_ℓ satisfies the ordinary differential equation

$$[IV.6] \quad s_{co}^2 \left\{ f_\ell'' + \left(\frac{2}{r} + \psi_0' \right) f_\ell' + \left(\psi_0'' + \frac{2\psi_0'}{r} - \frac{\ell(\ell+1)}{r^2} \right) f_\ell \right\} = -\omega_\ell^2 f_\ell$$

where ' means d/dr . In the absence of gravitational effects and variation in the sound speed, [IV.6] becomes the spherical Bessel equation, with solutions $j_\ell(\omega_\ell r)$ and $n_\ell(\omega_\ell r)$. However, the gravitational terms may be neglected only for $r \ll 1$, since ψ_0 is of order 1. The sound speed variation may be neglected only if the fluid region is very thin.

Because of the neglect of viscous effects, slip boundary conditions should be used at the outer boundary $r = 1$ and at any inner boundary $r = a$, representing a solid central core. The r -component of the momentum equation then implies

$$[IV.7] \quad 0 = -\frac{\partial p'_c}{\partial r} - p'_c \frac{d\psi_0}{dr}$$

at the boundaries. In terms of f_ℓ , [IV.7] becomes

$$[IV.8] \quad f_\ell'(a) + f_\ell(a)\psi_0'(a) = 0$$

$$[IV.9] \quad f_\ell'(1) + f_\ell(1)\psi_0'(1) = 0 \quad .$$

If there is no inner core, the appropriate boundary conditions at the centre $r = 0$ are that the pressure perturbation and its derivatives are continuous:

$$[IV.10] \quad f_0'(0) = 0$$

$$[IV.11] \quad f_\ell(0) = 0 \quad \text{if } \ell > 0 \quad .$$

The eigenvalue differential equation for f_ℓ was solved numerically, subject to the appropriate boundary conditions, for various values of ℓ and a .¹ The functions $s_{co}^2(r)$ and $\psi_0(r)$ used in the calculation were based on the magnetohydrostatic equilibrium model of Chapter II. They were determined by least squares fitting² sixth order polynomials to the spherically symmetric parts of $s_{co}^2(r, \theta)$ and $\psi_0(r, \theta)$. This fitting produced

$$[IV.12] \quad s_{co}^2(r) = 1 - .8697 r^2 + .3277 r^4 - .0650 r^6$$

and

$$[IV.13] \quad \psi_0(r) = 1.2168 r^2 - .0129 r^4 + .0593 r^6.$$

The five lowest positive eigenfrequencies ω_ℓ for each of $\ell = 0, 1$ and 2 are given in Table VII, for a variety of core sizes. For every eigenfrequency $\omega_\ell > 0$ there is another one $-\omega_\ell$. Note that all the eigenfrequencies ω_ℓ are real; the buoyancy force $-p'_c \vec{\nabla} \psi_0$ neither damps the waves nor renders them unstable.

The corresponding eigenmodes f_ℓ for some core sizes are plotted in Figures 6, 7 and 8, respectively. The zero frequency eigenmodes in the spherically symmetric case represent the pressure difference between two equilibrium configurations with different central pressures. It can be seen from the figures that the eigenmodes are qualitatively like spherical Bessel functions. Neither the buoyancy force nor the gradients in the sound speed has any qualitative effect on the eigenmodes.

¹ See Appendix A for details of the numerical method used.

² Using the program OLQF from the U.B.C. Computing Centre.

Table VII
Eigenfrequencies Of Some Standing Sound Waves

ℓ	$a = 0.0$	$a = .25$	$a = .50$	$a = .75$
0	0.0	0.0	0.0	0.0
	3.74	3.96	5.07	8.98
	6.56	7.24	9.81	17.8
	9.30	10.6	14.6	26.7
	12.0	14.0	19.4	35.6
1	1.74	1.68	1.44	1.14
	4.94	4.76	5.35	9.05
	7.78	7.73	9.95	17.9
	10.5	10.9	14.7	26.7
	13.3	14.2	19.5	35.6
2	2.64	2.63	2.43	1.97
	5.99	5.88	5.88	9.21
	8.91	8.66	10.2	17.9
	11.7	11.6	14.9	26.8
	14.5	14.7	19.6	35.7

Figure 6. Radial parts of some $\ell = 0$ charged fluid sound modes. In the upper diagram, it is assumed that there is no solid central core. In the lower diagram, a solid central core of .25 times the inner crustal radius is assumed to be present. Similar comments apply to the next two figures. f_0 is plotted against r for the first four modes.

Figure 7. Radial parts of some $\ell = 1$ charged fluid sound modes.

Figure 8. Radial parts of some $\ell = 2$ charged fluid sound modes.

CHAPTER V: INERTIAL MODES AND INERTIAL FLOW

In Chapter III, it was shown that the charged fluid is capable of oscillating at frequencies equal to or somewhat less than twice that of the rotational frequency of the star. The present chapter explores some features of these inertial normal modes, as well as charged fluid flow over these time scales.

Free Inertial Modes Of Oscillation

The equations to be solved are the anelastic condition

$$[\text{V.1}] \quad \vec{\nabla} \cdot (\rho_{co} \vec{v}_c) = 0$$

and the simplified momentum equation

$$[\text{V.2}] \quad \frac{\partial (\rho_{co} \vec{v}_c)}{\partial t} = -2\vec{\Omega}_o \times \rho_{co} \vec{v}_c - \vec{\nabla} p'_c - p'_c \vec{\nabla} \psi_o.$$

When Fourier analyzed in time, equations [V.1-2] can be reduced to an eigenvalue equation for p'_c . To make this reduction, it is convenient to define new variables $\vec{s}$ and $\vec{h}$:

$$[\text{V.3}] \quad \vec{s} = \rho_{co} \vec{v}_c$$

$$[\text{V.4}] \quad \vec{h} = -\vec{\nabla} p'_c - p'_c \vec{\nabla} \psi_o,$$

in terms of which the equations become

$$[\text{V.5}] \quad \vec{\nabla} \cdot \vec{s} = 0$$

$$[\text{V.6}] \quad \frac{\partial \vec{s}}{\partial t} + 2\vec{\Omega}_o \times \vec{s} = \vec{h}.$$

Measuring distances in units of R_c , $\vec{s}$ in units of $2\Omega_0 R_c$ and frequencies in units of $2\Omega_0$, [V.6] becomes, after Fourier analyzing in time,

$$[V.7] \quad -i\omega\vec{s} + \hat{z} \times \vec{s} = \vec{h}.$$

Equation [V.7] can be solved for $\vec{s}$:

$$[V.8] \quad \omega^2(\omega^2 - 1)\vec{s} = -\omega^2\vec{h} + i\omega\hat{z} \times \vec{h} + h_z\hat{z}.$$

Taking the divergence of [V.8] and using the anelastic condition produces

$$[V.9] \quad 0 = -\omega^2\vec{\nabla} \cdot \vec{h} - i\omega\hat{z} \cdot \vec{\nabla} \times \vec{h} + \frac{\partial h_z}{\partial z}.$$

If an azimuthal dependence for p'_c of the form $e^{im\phi}$ is assumed, where cylindrical coordinates (z, x, ϕ) are used, [V.9] becomes

$$[V.10] \quad \nabla^2 p'_c + \vec{\nabla}\psi_0 \cdot \vec{\nabla} p'_c + \left[\nabla^2 \psi_0 + \frac{m\omega^{-1}}{x} \frac{\partial \psi_0}{\partial x} \right] p'_c \\ = \omega^{-2} \left[\frac{\partial^2 p'_c}{\partial z^2} + \frac{\partial \psi_0}{\partial z} \frac{\partial p'_c}{\partial z} + \frac{\partial^2 \psi_0}{\partial z^2} p'_c \right].$$

In the limit $\vec{\nabla}\psi_0 \rightarrow \vec{0}$, [V.10] reduces to an equation used to study the inertial oscillations of a uniform density confined fluid (Greenspan, 1968).

In the present case the buoyancy terms, which are not negligible if $\lambda \approx R_c$, render [V.10] inseparable in both spherical polar and cylindrical coordinates.

Cylindrical Geometry

Equation [V.10] is separable in cylindrical coordinates if, instead of taking ψ_0 to be a function of r only, it is taken to be a function of x only. The resulting situation is that of an external gravitational field everywhere perpendicular to the rotation axis. This idealization will be

made here. In addition, the boundaries will be changed from two concentric spheres to two coaxial cylinders of finite length.

The replacement of a spherical geometry by a cylindrical geometry for reasons of mathematical simplicity is frequently used in the study of normal modes in rotating bodies, both magnetized and unmagnetized (Acheson and Hide, 1973). The resulting cylindrical eigenmodes and eigenfrequencies, while not quantitatively accurate, are usually found to possess many of the qualitative features of the spherical eigenmodes and eigenfrequencies. A cylindrical idealization is in any case more realistic than a dispersion relation approach such as was used in Chapter III because the former takes into account, even if only approximately, the effects of boundary conditions and inhomogeneities in the equilibrium solution, while the latter does not.

In the cylindrical idealization, equation [V.10] becomes

$$[V.11] \quad (1 - \omega^{-2}) \frac{\partial^2 p'_c}{\partial z^2} + \frac{\partial^2 p'_c}{\partial x^2} + \left(\frac{1}{x} + \frac{d\psi_o}{dx} \right) \frac{\partial p'_c}{\partial x} + \left\{ \frac{d^2 \psi_o}{dx^2} + \frac{(1 + m\omega^{-1})}{x} \frac{d\psi_o}{dx} - \frac{m^2}{x^2} \right\} p'_c = 0 .$$

This equation has solutions of the form

$$[V.12] \quad p'_c = Z(z) f(x) e^{im\phi} ,$$

where

$$[V.13] \quad Z(z) = \cos(kz) \text{ or } \sin(kz) .$$

The function $f(x)$ satisfies the ordinary differential equation

$$[V.14] \quad f'' + \left(\frac{1}{x} + \psi'_o \right) f' + \left\{ \psi''_o + \frac{(1 + m\omega^{-1})}{x} \psi'_o + k^2 (\omega^{-2} - 1) - \frac{m^2}{x^2} \right\} f = 0 ,$$

where ' means d/dx .

Since $\psi_0(r)$ is to a very good approximation just proportional to r^2 (see [IV.13]), the $\psi_0(x)$ chosen in the present idealization is

$$[V.15] \quad \psi_0(x) = \frac{A}{2} x^2 ,$$

where A is a positive constant of order 1. For this case, f satisfies

$$[V.16] \quad f'' + \left(\frac{1}{x} + Ax\right)f' - \frac{m^2}{2} f = -\lambda^2 f ,$$

where

$$[V.17] \quad \lambda^2 = k^2(\omega^{-2} - 1) + A(2 + m\omega^{-1}) .$$

To every eigenvalue $-\lambda^2$ there corresponds a countable infinity of normal modes. The eigenfrequencies $\omega(k)$ corresponding to an eigenvalue $-\lambda^2$ are given by

$$[V.18] \quad \omega(k) = \frac{Am + [A^2 m^2 + 4k^2(\lambda^2 - 2A + k^2)]^{1/2}}{2(\lambda^2 - 2A + k^2)}$$

From [V.18], it is possible to learn several things about the eigenfrequency spectrum. For example, in the mathematical limit $A = 0$ of no gravitation, [V.18] implies that $|\omega| < 1$, or, in unnormalized units, $|\omega| < 2\Omega_0$, a result which was given in Chapter III. Another inference is that if λ is real and

$$[V.19] \quad \lambda^2 \geq 2A - k^2 - A^2 m^2 / (4k^2) ,$$

then there is no damping or growth of the inertial waves.

Since viscosity has been neglected, the proper boundary conditions for this problem are slip boundary conditions. It follows from [V.4, 6, & 12] that the vanishing of s_z at the upper and lower surfaces of the cylinders, which are taken at $z = \pm 1$, implies that dZ/dz vanishes there.

The permissible k -values are thus

$$[V.20] \quad k = n \quad \text{for cos,}$$

and

$$[V.21] \quad k = (n + \frac{1}{2})\pi \quad \text{for sin,}$$

where $n = 0, \pm 1, \pm 2, \dots$

The vanishing of s_x at the inner cylinder surface $x = a < 1$ and at the outer cylinder surface $x = 1$ implies

$$[V.22] \quad f'(a) + (Aa - m\omega^{-1}/a) f(a) = 0$$

and

$$[V.23] \quad f'(1) + (A - m\omega^{-1}) f(1) = 0 .$$

If there is no inner cylinder, the proper boundary conditions are that the pressure and its derivatives are continuous on the rotation axis:

$$[V.24] \quad f'(0) = 0 \quad \text{if } m = 0$$

$$[V.25] \quad f(0) = 0 \quad \text{if } m \neq 0 .$$

The boundary conditions for axisymmetric modes are similar to those of the sound modes in Chapter IV, so the differential equation for $f(x)$ was solved numerically in this case in the same way that the sound wave equations were solved.¹ The five eigenvalues $-\lambda^2$ which are smallest in magnitude for this $m = 0$ case are given in Table VIII, for four values of a . A value of $2 \times 1.2168 = 2.4336$ for A was used (see equation [IV.13]).

¹ See Appendix A.

Table VIII
Eigenvalues $-\lambda^2$ For Some Axisymmetric Inertial Modes

$a = 0.0$	$a = .25$	$a = .50$	$a = .75$
-4.86	-4.86	-4.85	-4.84
-20.4	-25.8	-47.7	-169.
-56.0	-80.6	-169.	-654.
-112.	-171.	-372.	-1460
-188.	-298.	-656.	-2590

From Table VIII it can be seen that there is no damping or growth of these modes in the inviscid approximation. The eigenfunctions f for the four eigenvalues of smallest magnitude are plotted in Figure 9.

The eigenfunctions f are qualitatively like linear combinations of Bessel functions $J_0(\lambda x)$ and $N_0(\lambda x)$, which are the $m = 0$ solutions of [V.16] for $A = 0$. It follows that, as with the sound modes, buoyancy has no qualitative effect on axisymmetric inertial modes. While this conclusion is based on the use of a cylindrical geometry, I can see no reason why it should not also hold in a spherical geometry.

The velocity fields corresponding to a given f can be formed by multiplying f by $\cos(kz)$ or $\sin(kz)$ and using [V.8] to find $\vec{s}$. One general feature of the velocity fields in all these axisymmetric modes is that, since s_ϕ and s_x differ by a purely imaginary multiplicative factor when $m = 0$, the meridional and azimuthal flow are out of phase in time by $\pi/2$.

The boundary conditions for the axisymmetric modes are independent of the eigenvalue $-\lambda^2$. However, the boundary conditions for the non-axisymmetric modes do depend on $-\lambda^2$. Furthermore, the dependence of $-\lambda^2$ is in general

non-linear. As a result, the non-axisymmetric mode problem is quite difficult to solve.

There is, however, one class of non-axisymmetric modes for which the problem is tractable, and moreover, which is of some interest. When $k = 0$ (no z -dependence of p'_c), the boundary conditions can be written in a form which is linear in $-\lambda^2$:

$$[V.26] \quad -aAf'(a) - [(aA)^2 + 2A]f(a) = -\lambda^2 f(a)$$

$$[V.27] \quad -Af'(1) - [A^2 + 2A]f(1) = -\lambda^2 f(1)$$

Because these boundary conditions take the same form as the eigenvalue equation [V.16], it is possible to use the methods outlined in Appendix A for the solution of the problem.

It follows from [V.18] that for these $k = 0$ modes,

$$[V.28] \quad \omega = Am/(\lambda^2 - 2A) \quad \text{or} \quad 0$$

One branch corresponds to steady ($\omega = 0$) flow, while the other branch corresponds to oscillatory ($\omega \neq 0$) motion. In the mathematical limit $A \rightarrow 0$, the motion in the latter branch also becomes steady. In this limit (which is not relevant to neutron stars), all buoyancy forces vanish, and the Taylor-Proudman theorem (Acheson and Hide, 1973) holds. That theorem states that steady flow in rotating, non-magnetized constant density fluids is "two-dimensional" in the sense that $\partial \vec{s} / \partial z = 0$. This theorem can be derived by assuming that $\omega = 0$ and $\vec{\nabla} \psi_0 = \vec{0}$, and taking the curl of [V.7].

When $\vec{\nabla} \psi_0 \neq \vec{0}$, the theorem is not necessarily true because the buoyancy force $-p'_c \vec{\nabla} \psi_0$ gives rise to a torque $\propto \vec{\nabla} p'_c \times \vec{\nabla} \psi_0$ which vanishes if and only if p'_c is constant on surfaces of constant ψ_0 . Since $\partial \vec{s} / \partial z$ is $\propto \vec{\nabla} p'_c \times \vec{\nabla} \psi_0$ when $\omega = 0$, $\partial \vec{s} / \partial z$ need not vanish. In particular, it does not vanish for non-axisymmetric modes.

Conversely, non-axisymmetric two-dimensional ($k = 0$) inertial modes need not necessarily be steady, as shown by [V.28]. The physical reason for this is the torque exerted on the fluid by the buoyancy force. Because of the importance of the buoyancy force for the non-axisymmetric oscillatory two-dimensional inertial modes, these modes will be called buoyancy-inertial modes. To my knowledge, these modes have not been previously discussed in the literature.

The eigenfrequencies for some $m = 1, 2$, and 3 buoyancy-inertial modes are given in Table IX. The eigenfrequencies are all real, so these modes are all stable and undamped. Note that all the eigenfrequencies satisfy $|\omega| < 1$, or in ordinary units, $|\omega| < 2\Omega_0$, just as for ordinary inertial waves. However, the highest frequency buoyancy-inertial mode has a period which is more than five times longer than that of the highest frequency axisymmetric inertial mode.

The motion in the buoyancy-inertial modes is such that s_z vanishes, i.e., the motion is everywhere parallel to the equator, since $\partial \vec{s} / \partial z = \vec{0}$ and s_z vanishes at $z = \pm 1$. The $\vec{s}$ field associated with these waves can be calculated from [V.8]. At a given time, it is of the form

$$[V.29] \quad s_x \propto [(f' + Axf) - \omega^{-1} \frac{m}{x} f] \cos(m\phi)$$

$$[V.30] \quad s_\phi \propto [\omega^{-1}(f' + Axf) - \frac{m}{x} f] \sin(m\phi) .$$

These fields for the highest frequency $m = 1, 2$, and 3 modes are plotted in Figures 10, 11, and 12, for $a = 0$ and $a = .25$. The cellular flow pattern arises from the $e^{im\phi}$ form for p'_c ; the number of cells is $2m$.

The buoyancy force can have a qualitatively important effect on $m \neq 0$ modes, in the sense that motion which would be steady in the absence of it is oscillatory in its presence. While this conclusion was arrived at on the basis of a simplified cylindrical geometry, it is true even in the

Table IX
Eigenfrequencies Of Some Buoyancy-Inertial Modes

m	a = 0.0	a = .25	a = .50	a = .75
1	.139	.107	.055	.015
	.047	.032	.015	.004
	.023	.015	.007	.002
	.014	.009	.004	.001
	.009	.006	.003	.001
2	.166	.155	.098	.029
	.066	.056	.029	.008
	.036	.028	.013	.003
	.022	.017	.008	.002
	.015	.011	.005	.001
3	.167	.164	.124	.044
	.075	.070	.041	.011
	.043	.038	.020	.005
	.028	.023	.011	.003
	.020	.016	.007	.002

spherical case. This can be seen most easily in equation [V.10], which applies to any geometry. If the flow is axisymmetric ($m = 0$) or if the buoyancy forces vanish ($\partial\psi_0/\partial x = 0$), then [V.10] is linear in ω^{-2} , as a result of which the two branches of oscillation are the same up to a change in the sign of ω . If the flow is non-axisymmetric ($m \neq 0$) and the buoyancy force perpendicular to the rotation axis does not vanish everywhere ($\partial\psi_0/\partial x \neq 0$), then there is an additional non-zero term $\propto \omega^{-1}$ in [V.10]. As a result of this term, the two branches of oscillation can have quite different eigenmodes and eigenfrequencies.

Inertial Flow

In the first section of this chapter some of the free inertial modes of oscillation of the charged fluid were investigated. In the present section, the charged fluid flow which develops from chosen initial conditions is examined. Since the fluid motion will be followed only over the time scales of inertial waves, I call this flow inertial flow.

In order to filter out sound waves from the solutions to the equations, the anelastic condition

$$[\text{V.31}] \quad \vec{\nabla} \cdot (\rho_c \vec{v}_c) = 0$$

is used. The simplified momentum equation is

$$[\text{V.32}] \quad \partial(\rho_c \vec{v}_c)/\partial t = -\vec{\nabla} p'_c - p'_c \vec{\nabla} \psi_0 - \rho_c \vec{\nabla} \Phi' - \vec{\nabla} \cdot \overleftrightarrow{\sigma}' - 2\vec{\Omega} \times \rho_c \vec{v}_c + \vec{Q}$$

In equation [V.32], $\vec{Q}$ represents any unbalanced body forces which are roughly constant over these time scales. Such forces might occur, for example, if the magnetic body force and the quasi-steady Poincaré force are not balanced

at the initial time. It is convenient to include such a term if one is interested in determining the response of the charged fluid to an applied force.

Taking the divergence of [V.32] and using [V.31] produces an equation for p'_c which is elliptic, rather than hyperbolic, because of the neglect of sound waves:

$$[V.33] \quad \nabla^2 p'_c + \vec{\nabla} \psi_o \cdot \vec{\nabla} p'_c + \nabla^2 \psi_o p'_c = -\vec{\nabla} \cdot [2\vec{\Omega} \times \rho_c \vec{v}_c - \vec{Q} + \rho_c \vec{\nabla} \phi' + \vec{\nabla} \cdot \vec{\sigma}']$$

(cf. equation [III.7]). Equation [V.33], when combined with the momentum equation, ensures that the fluid flow satisfies the anelastic condition for all times $t > 0$ if it is satisfied at $t = 0$. As an alternative to [V.31-32], it is possible to solve [V.32-33] if [V.31] is used as an initial constraint on $\vec{v}_c$.

The gravitational potential perturbation satisfies

$$[V.34] \quad \nabla^2 \phi' = (4\pi G/s_{co}^2) p'_c.$$

Equation [V.34] and the pressure equation [V.33] form a pair of coupled elliptic equations. If the velocity field and $\vec{Q}$ are known at a particular time, [V.33-34] allow the pressure and gravitational potential perturbations at that time to be calculated.

The proper boundary conditions on the velocity are the no-slip conditions [II.87-88], since viscosity is taken into account in [V.32]. For reasons of numerical simplicity, an isotropic viscous tensor was used in place of an anisotropic tensor which depends on the direction of the magnetic field. The use of an isotropic tensor is justifiable because the Ekman number is so small compared to 1 that viscosity, whether isotropic or anisotropic, does not have a very important effect on the fluid flow.

The magnetic induction equation is not included in the equations [V.31-34] which are to be solved because the changes in $\vec{B}$ over the time scale Ω_0^{-1} are so extremely small that time variations in the magnetic body force are negligible, as was shown in Chapter III. However, changes in $\vec{B}$ can cause changes in the magnetic torques on the solid parts of the star (see [II.95]). For the induction equation to be decoupled from the other equations, it is also necessary that these variations in the magnetic torque be small compared to the viscous torque.

From equations [II.78, 94, and 95], the ratio of the variation in the magnetic torque N'_B to the viscous torque N'_V is

$$[V.35] \quad N'_B/N'_V \approx V_A^2 \tau / \nu_c ,$$

for fluid motion with time scale τ . For inertial flow, $\tau \approx \Omega_0^{-1}$, so

$$[V.36] \quad N'_B/N'_V \approx V_A^2 / (\Omega_0 \nu_c) \approx 10^{-5} B_{11}^2 \Omega_2^{-1} T_7^2 .$$

Thus, the variations in the magnetic torque are relatively negligible and the induction equation may be completely dropped unless B is quite large compared to 10^{11} gauss or if the neutron star is so young that its temperature is much higher than 10^7 K. It is assumed in the detailed calculations in this chapter that $N'_B \ll N'_V$, so the induction equation is not solved.

It is necessary to make definite choices for the values of the independent parameters of the problem, such as $\rho_0(0)$, Ω_0 and T_0 , in order to perform detailed calculations such as those reported below. Since no one set of parameters is typical of neutron stars as a class, any particular set may be criticized as being somewhat arbitrary. However, the use of scaling laws such as those mentioned in Chapter II allows one to generate a large class of approximate solutions from any particular solution.

The equilibrium model used in the calculations below was the one calculated in Chapter II - see Table V and Figure 4. The value chosen for the kinematic viscosity was $\nu_c = 2 \times 10^{11} \text{ cm}^2 \text{ sec}^{-1}$, whose associated Ekman number for $\lambda \approx R$ is so small compared to 1 (see [II.63]) that the resulting fluid flow should be almost completely independent of the precise value of ν_c . The temperature corresponding to this value of ν_c is $T_7 \approx .4$.

First Flow Simulation

The charged fluid motion was simulated on a computer by solving very accurate finite difference approximations to the momentum equations and the coupled p'_c , Φ' equations. This was done as follows.¹ At the initial time $t = 0$, a velocity field $\vec{v}_c(t = 0)$ was chosen which satisfies the anelastic condition (or, more precisely, the finite difference approximation to it). The coupled p'_c , Φ' equations were solved for $p'_c(t = 0)$ and $\Phi'(t = 0)$, which were then used in the momentum equations to get $\partial(\rho_c \vec{v}_c)/\partial t(t = 0)$. This time derivative and $\vec{v}_c(t = 0)$ were used to generate an approximation to $\vec{v}_c(t = \Delta t > 0)$, where Δt is a time interval small compared to all characteristic time scales in the problem. The procedure was then repeated at times $t = \Delta t, 2\Delta t$, etc.

Each physical variable was defined on one of four staggered grids which are uniform in spherical coordinates. These grids are displayed in Figure 13. The slight non-sphericities in the solid boundaries were neglected. It can be seen from the figure that the grids are coarsest near the crust. Because a spherical coordinate system becomes singular at the origin, it was numerically necessary to assume that there is a solid central core in order

¹ See Appendix B for further details of the numerical methods used.

to avoid the singularity. A very small core of radius $R_s = 1$ km was chosen. The moment of inertia of this spherical core was determined to be 2.48×10^{40} gm cm² by numerical integration of the model in Table V. Because of computer storage restrictions, only axisymmetric fluid flows could be simulated, in which all quantities such as v_r , v_θ , v_ϕ , etc are independent of azimuthal angle ϕ .

Two of the computer simulations which were performed are reported here. The first simulation was performed in order to see how the fluid responds to a suddenly applied force. This force was taken to be the Poincaré force due to crustal slowing down:

$$[V.37] \quad \vec{Q}(t) = -\rho_c \dot{\vec{\Omega}}_0 \times \mathbf{r} \quad \text{for } t \geq 0.$$

The crustal deceleration $\dot{\vec{\Omega}}_0$ chosen was that observed for the Crab (see Table II). For simplicity, the initial charged fluid velocity was assumed to vanish in the co-rotating frame:

$$[V.38] \quad \vec{v}_c(t = 0) = \vec{0}.$$

Due to the constant Poincaré force in the $+\phi$ direction, v_ϕ initially increased linearly with time, as did the perturbation $\dot{\vec{\Omega}}'$ in the crustal acceleration, which depends only on the ϕ component of $\vec{v}_c$. One might expect that the Coriolis force would cause the meridional flow to grow initially quadratically with time, and become comparable in magnitude with v_ϕ after about one rotation. This does not happen. After one rotation, $|v_{r,\theta}|/|v_\phi|$ is only about 10^{-3} .

The reason for the relatively small meridional flow is that pressure gradients are fairly successful in balancing the Coriolis force in meridional planes. This can be seen by considering a model inviscid, constant density fluid with no gravitational field present. The equations describing the

acceleration of such a fluid by the Poincaré force are

$$[V.39] \quad \partial v_z / \partial t = -\partial P' / \partial z$$

$$[V.40] \quad \partial v_x / \partial t = -\partial P' / \partial x + 2\Omega_o v_\phi$$

$$[V.41] \quad \partial v_\phi / \partial t = -\dot{\Omega}_o x - 2\Omega_o v_x,$$

where P' is the perturbation in the kinematic pressure. The solution to these equations which satisfies the initial conditions [V.38] is

$$[V.42] \quad v_z = v_x = 0$$

$$[V.43] \quad v_\phi = -\dot{\Omega}_o x t$$

$$[V.44] \quad P' = -\dot{\Omega}_o \Omega_o x^2 t.$$

Thus in this model, pressure gradients are completely effective in suppressing meridional flow.

The computer generated solution of [V.32-34] for $\vec{v}_c$ after about two rotations is displayed in Figure 14. The velocity field in that figure is split into a meridional or poloidal part $\vec{v}_p$ and an azimuthal or toroidal part v_ϕ . The meridional flow is represented by a vector plot; the azimuthal flow is represented by a contour plot. For ease of visualization, the vectors $\vec{v}_p$ are attached to a uniform 15 x 15 grid in (z,x) coordinates. It would have been extremely hard to draw and visualize the flow using the grids in spherical coordinates, as can be seen from the high density of these grid points near the core (see Figure 13). It is necessary to interpolate the vectors $\vec{v}_p$ before they be displayed on a uniform (z,x) grid. However, no matter what grid $\vec{v}_p$ is displayed on, interpolation is necessary because v_r and v_θ are computed on two different grids (see Appendix B).

The v_ϕ contour plot shows that the computed solution is essentially rigid rotation in the $+\hat{\phi}$ direction with an extremely large velocity shear near the crust, particularly near the equator. After only two rotations, viscosity has not yet had time to reduce the shear.

Apart from the shear which arises from the presence of no-slip boundaries, the computed solution is practically identical to the solution [V.42-44] of the simple model equations [V.39-41], which do not take boundary conditions into account. This agreement is consistent with the soundness of the numerical method. Because the numerical calculations included buoyancy whereas the simple model [V.39-41] does not, the agreement between the two is also consistent with the conclusion reached earlier in this chapter that the buoyancy force has little qualitative effect on axisymmetric inertial flow.

The undulations which are apparent in the contour lines near the crust in Figure 14 arise because the numerical grid is coarsest at these large radii (see Figure 13) and because coincidentally the shear in v_ϕ is extremely large in the same region. These are the most extreme conditions possible. Thus the size of these artificial wiggles gives some indication of the maximum numerical errors which can possibly arise in the numerical simulations.

The calculation was not carried further than two rotations because it was apparent that nothing of interest was going to happen for a very long time. The question of what happens to the charged fluid if the Poincaré force is "turned on" suddenly is returned to in Chapter VI, where the flow is followed for a time $\approx 10^8$ times longer than it was here.

Second Flow Simulation

A second simulation was performed to see what type of charged fluid flow develops from a simple initial velocity condition. In this simulation, it was assumed that the quasi-steady Poincaré force was balanced by magnetic body forces, as explained in Chapter II:

$$[\text{V.45}] \quad \vec{Q} = \vec{0}.$$

The initial velocity field was chosen to be a simple azimuthal flow which vanishes on the crust and core surfaces:

$$[\text{V.46}] \quad v_{\phi}(r, \theta, t=0) \propto (r - R_s)(r - R_c) \sin\theta,$$

The resulting flow at eight different times is given in Figure 15. It can be seen that because of the Coriolis force, meridional flow develops out of the initially purely azimuthal flow. The maximum of $|\vec{v}_p|$ never exceeds about a third of $\max|v_{\phi}|$, and the oscillations in time of $\max|\vec{v}_p|$ have about the same amplitude as the oscillations of $\max|v_{\phi}|$. Thus, the flow which develops consists of two parts. The first part is a roughly steady flow in the $+\hat{\phi}$ direction. If the calculation had been carried forward for a time $\gg R^2/\nu_c$, viscosity would have slowed this steady flow. The second part of the flow is oscillatory in character, with the amplitude of the meridional and azimuthal oscillations equal to about a half of the magnitude of the steady azimuthal flow.

At .15 of a rotation period after $t = 0$, a large scale meridional flow has developed, consisting of a large swirl centered near $r = .75R_c$, $\theta = 3\pi/8$, plus a small counterswirl near the core. The azimuthal flow is developing a large shear near the crust. At .31 rotations, the counterswirl has grown. By .47 rotations, the original swirl in the meridional flow has disappeared.

What remains is the counterswirl, which is now very large. The shear in the azimuthal flow near the crust has begun to decrease, and by .78 rotations, the flow is much like it was shortly after $t = 0$.

The oscillatory part of the flow thus has a period of roughly .7 of a rotation period. From the large spatial scale of the oscillation, it can be reasonably concluded that the oscillation is either the fundamental mode or another one very close to it in frequency. The flow patterns at about 11 and 16 rotation periods give no indication of the fluid switching to another mode.

To show more clearly the gross features of the flow, the angular accelerations of the crust and the core due to fluid shear stress are plotted in Figure 16. The accelerations of the solid parts of the star have two components, one a steady positive acceleration due to the steady part of the flow in the $+\hat{\phi}$ direction, and the other an oscillatory acceleration due to the oscillatory part of the flow. Figure 17 shows a power spectrum analysis of the crustal acceleration graph.¹ There is a sharp peak in the power spectrum at a little less than .75 rotations, with a much smaller peak at about one rotation. The oscillatory motion thus consists of the fluid oscillating mainly in a normal mode with a period of $\approx .75 P$, and to a smaller degree in a normal mode with period $\approx P$. Both these periods are $\geq .5 P$, which they must be because the modes are inertial.

One feature of the flow pattern which can be seen most predominantly after 16 rotations is some sinuous ripples in both the meridional stream lines and the azimuthal contour lines near the rotation axis above the solid core. This feature can be seen developing at earlier times and is clearly associated with the presence of the core. After only .15 rotations,

¹ In the power spectrum analysis, the routine FOURT from the U.B.C. Computing Centre was used to Fourier analyze the data.

one ripple has formed just above the core, and is connected with the small counterswirl in the meridional flow in the same region. As time goes on, more ripples grow further and further up from the core.

These ripples are not a numerical artifact. There are a number of strong reasons to support this statement. Firstly, these ripples are larger and more pronounced than the ripples near the crust in the first simulation (Figure 14) which definitely are numerical artifacts. Since the latter ripples arise from a combination of the most severe conditions the computer program can handle, any larger ripples such as the ones under consideration should not be artifacts.

Secondly, these ripples do not arise from a grid point to grid point variation in the numerical solution. Even on the scale that the solution has been plotted, the flow is in no sense turbulent; rather, the flow near the rotation axis above the core is a coherent but ripply one. The scale of the ripples is several times the distance between the grid points (see Figure 13).

Thirdly, if the ripples were in any way the result of too coarse a numerical grid, one might reasonably expect that this would result in numerical instability. There is no indication whatsoever of any such instability.

Finally, and most importantly, the wavelength of the ripples is almost precisely equal to the radius of the core, which is in no way simply related to the grid spacing.

Since the ripples are not a numerical artifact, what are they? Consider the graph of the core angular acceleration in Figure 16. Neglecting the steady acceleration, we see that the core is oscillating about the rotation axis. Because the moment of inertia of the core is so much smaller than the crustal moment of inertia, the core angular acceleration is about

a hundred times larger than the crustal angular acceleration. Thus, the core is oscillating at such a rate that it could have a significant effect on the fluid flow.

What is the result of moving a solid body such as the core in an enclosed rotating fluid such as the charged fluid? If the body is moving slowly or is oscillating at a rate much slower than 2Ω , the fluid which lies above the body parallel to the rotation axis moves rigidly with the body and forms a Taylor-Proudman column (Greenspan, 1968). There can be large shears at the interface between the fluid column and the rest of the fluid. If the frequency of oscillation of the solid body is comparable to but not greater than 2Ω , a columnar structure is also formed. However, the column is not translationally invariant along the rotation axis as it is when $\omega \ll 2\Omega$; structure appears in the column which periodically repeats itself along the rotation axis.

The ripply fluid flow above the oscillating solid core is very much like the preceding description of Taylor-Proudman columnar flow for the case in which Ω is comparable to 2Ω . I conclude that the oscillation of the solid core has set up a Taylor-Proudman column, which has ripply structures in it because the core is oscillating at frequencies comparable to 2Ω .

Figure 9. Radial parts of some axisymmetric inertial modes. The upper picture shows the eigenfunctions f which satisfy [V.16] for the four $a = 0$ eigenvalues $-\lambda^2$ which are smallest in size (see Table VIII). The lower picture is similar, except $a = .25$.

Figure 10. Fluid flow in the highest frequency $m = 1$ buoyancy-inertial mode. The flow field $\vec{s}$, which is related to the flow velocity by $\vec{s} = \rho_{co} \vec{v}_c$, is displayed at an instant of time in a vector plot. The rotation axis points out of the paper. The z component of $\vec{s}$ vanishes. What is shown is (s_x, s_ϕ) , which is independent of z . The length of each line segment is proportional to the magnitude of $\vec{s}$, and is oriented along $\vec{s}$. In the upper picture, $a = 0$ (no inner cylinder); in the lower, $a = .25$. Similar comments apply to the next two figures.

Figure 11. Fluid flow in the highest frequency $m = 2$ buoyancy-inertial mode.

Figure 12. Fluid flow in the highest frequency $m = 3$ buoyancy-inertial mode.

Figure 13. Numerical grids used in the inertial flow simulation. The rotation axis is vertical; the equator is horizontal. Both are indicated by heavy lines, as are the solid core and solid crust surfaces. Each physical variable is defined on one of the four overlapping grids. See Appendix B for a detailed discussion of the grids.

GRID LEGEND

○	(i, j)
+	$(i, j+1/2)$
□	$(i+1/2, j)$
◆	$(i+1/2, j+1/2)$

Figure 14. Inertial flow after 1.98 rotations in the first simulation. The velocity field $\vec{v}_c$ is split into a meridional or poloidal part $\vec{v}_p$, which is represented in a vector plot, and an azimuthal part v_ϕ , which is represented in a contour plot. The contour labeling conventions are those of Figure 5.

$$\text{MAX } V_p = 0.85 \times 10^{-6} \text{ cm/sec}$$

$$\text{MAX } V_\phi = 0.12 \times 10^{-3} \text{ cm/sec}$$

Figure 15. Inertial flow at various times in the second simulation.

This figure extends over several pages. Each picture is labeled by the time in rotation periods at which the flow is represented.

$$\text{MAX } V_{\phi} = 0.99 \times 10^{-6} \text{ cm/sec}$$

TIME = 0.00

$$\text{MAX } V_p = 0.27 \times 10^{-6} \text{ cm/sec}$$

$$\text{MAX } V_\phi = 0.91 \times 10^{-6} \text{ cm/sec}$$

MAX $V_p = 0.15 \times 10^{-6}$ cm/sec

TIME = 0.31

MAX $V_\phi = 0.80 \times 10^{-6}$ cm/sec

TIME = 0.31

MAX $V_P = 0.19 \times 10^{-6}$ cm/sec

MAX $V_\phi = 0.97 \times 10^{-6}$ cm/sec

MAX $V_P = 0.20 \times 10^{-6}$ cm/sec

MAX $V_\phi = 0.10 \times 10^{-5}$ cm/sec

$$\text{MAX } V_p = 0.90 \times 10^{-7} \text{ cm/sec}$$

TIME = 11.02

$$\text{MAX } V_\phi = 0.95 \times 10^{-6} \text{ cm/sec}$$

TIME = 11.02

MAX $V_P = 0.20 \times 10^{-6} \text{ cm/sec}$

TIME = 15.89

MAX $V_\phi = 0.77 \times 10^{-6} \text{ cm/sec}$

TIME = 15.89

Figure 16. Core and crust angular acceleration due to shear stress in second inertial flow simulation. Note that the core acceleration is much larger than the crust acceleration, and that each graph consists of a steady component plus a smaller oscillatory component.

CRUST ANGULAR
ACCELERATION
(10^{-12} rad / sec²)

CORE ANGULAR
ACCELERATION
(10^{-10} rad / sec²)

Figure 17. Power spectrum analysis of crust angular acceleration in second inertial flow simulation.

CHAPTER VI: MAGNETOSTROPHIC FLOW

It was shown in Chapter III that the longest period oscillations in the charged fluid are hydromagnetic-inertial waves. This chapter is concerned with the dynamics of the charged fluid over time scales comparable to the periods of these waves. Acheson and Hide (1973) call fluid motion over these time scales magnetostrophic flow.

The simplified equations over these long time scales were derived in Chapter III. They are the induction equation

$$[\text{VI.1}] \quad \frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v}_c \times \vec{B}) ,$$

the Maxwell equation

$$[\text{VI.2}] \quad \vec{\nabla} \cdot \vec{B} = 0 ,$$

the anelastic condition

$$[\text{VI.3}] \quad \vec{\nabla} \cdot (\rho_c \vec{v}_c) = 0 ,$$

and the momentum equation

$$[\text{VI.4}] \quad \vec{0} = -2\vec{\Omega} \times \rho_c \vec{v}_c - \frac{1}{4\pi} [\vec{B} \times (\vec{\nabla} \times \vec{B}) - \vec{B}_0 \times (\vec{\nabla} \times \vec{B}_0)] - \vec{\nabla} p'_c - p'_c \vec{\nabla} \psi_0 .$$

The gravitational term $-\rho_c \vec{\nabla} \phi'$ is not included in [VI.4]. This term was included in the inertial flow simulations of the previous chapter. However, it had little effect on the flow since as I showed in Chapter II, it is smaller than the buoyancy term $-p'_c \vec{\nabla} \psi_0$ by a factor of about $\rho_o / \rho_{co} \approx 100$.

There is no viscous term in [VI.4]. I showed in Chapter II that it is very much smaller than the Coriolis term, unless $T_7 \ll 1$ or $\lambda_6 \ll 1$. Furthermore, over time scales τ_{HI} , viscosity has a negligible effect on the flow in the vicinity of solid boundaries, unless the normal component of $\vec{B}$

at the boundary is very much smaller than its tangential component. Since it would be extremely difficult to take into account viscous effects in the type of computer simulations reported in this chapter, the computations are restricted to conditions in which such effects are negligible.

A superfluid drag term is not present in [VI.4]. In Chapter III, I showed that the damping effect of superfluid drag on hydromagnetic-inertial waves is generally very much less important than the damping effect of viscosity. Since the latter is very small for a wide range of physical conditions (see [III.27]) and has been neglected here, it is consistent to neglect superfluid drag.

The quasi-steady Poincaré term does not appear in equation [VI.4] because I assume that the $\vec{B}_0$ field in [VI.4] is the sum of the magnetohydrostatic field and the small perturbation required to balance $-\rho_c \dot{\vec{\Omega}}_0 \times \vec{r}$. The fluctuations in the crust angular velocity induced by fluid motion are not taken into account in [VI.4]. For the particular fluid flow simulation reported in this chapter, it will be shown that the fluctuating Poincaré force is relatively small. In general it need not be small.

The appropriate boundary conditions on $\vec{v}_c$ are no-slip conditions (see [II.89 and 92]), since I have restricted attention in this chapter to magnetic fields which generally have non-zero normal components near solid boundaries. Thus

$$[\text{VI.5}] \quad \vec{v}_c|_{\Sigma} = \vec{0}.$$

As explained in Chapter II, the boundary condition on $\vec{B}$ in the infinite electrical conductivity limit is that its normal component is static as viewed in a frame of reference co-rotating with the boundary. In terms of the perturbation $\vec{B}'$ in the magnetic field, this boundary condition is

$$[\text{VI.6}] \quad \vec{B}' \cdot \vec{n}|_{\Sigma} = 0 \quad .$$

In the inertial flow simulations it was necessary to assume that there was a solid central core because of numerical difficulties associated with the choice of a spherical coordinate system. Since the calculations in the present chapter were performed in cylindrical coordinates, there is no such necessity here. Accordingly, it was assumed that there is no solid central core. Thus, [VI.5-6] apply at the inner crustal surface.

As explained in Chapter II, the crust is subject to a fluctuating magnetic torque N'_B and to a torque N'_V due to viscous shear stress, both of which arise from the charged fluid motion. The general formula for the relative size of N'_B and N'_V is given by [V.35]. Combining that result with the estimate [III.20] for the time scale τ_{HI} of hydromagnetic-inertial waves produces

$$[\text{VI.7}] \quad N'_B/N'_V \approx 10^3 \Omega_2 \lambda_6^2 T_7^2 \quad .$$

Thus, unless $T_7 \ll 1$ or $\lambda_6 \ll 1$, the fluctuations in the magnetic torque on the crust are larger than the viscous torque on the crust.

The torques on the crust arising from the fluid flow change the angular velocity of the crust. Can the quasi-sinusoidal timing residuals observed in the Crab (Boynton et al., 1972) be due to this effect? It seems unlikely. The period of the lowest frequency hydromagnetic-inertial mode would have to be longer than the length of the data train for the Crab, about five years. It follows from [III.20] that for the period to be five years or longer the mean interior magnetic field strength would have to be $\lesssim 2 \times 10^{10}$ gauss, which is somewhat low compared to the value expected on the basis of the naive flux conservation argument (see Chapter I).

Furthermore, one must find a process for exciting hydromagnetic-inertial waves in such a way that the predicted timing power spectrum $\mathcal{P}(\omega)$ has the observed form $P_0 \omega^{-4}$ with the correct power coefficient p_0 (Boynton et al., 1972).

I have not been able to think of any excitation process which will do this. One can probably fit the observed $\delta(\omega)$ by choosing the amplitudes of the hydromagnetic-inertial modes properly. However, the number of arbitrary parameters in such a procedure is equal to the number of normal modes, whereas a frequency walk model has only one parameter, namely P_0 . Thus, a frequency walk model is far less arbitrary than a hydromagnetic-inertial mode model.

Equations For The Velocity Field

In magnetostrophic flow, the fluid instantaneously adjusts so that the total force on any fluid element vanishes (see [VI.4]). This adjustment is analogous to the instantaneous adjustment of the pressure in incompressible flow in such a way that the incompressibility condition is satisfied. The adjustment times in both cases for realistic flow are of course not zero, but are much smaller than the time scales of interest.

The instantaneous adjustment of the velocity and pressure in magnetostrophic flow is reflected in the mathematical structure of the magnetostrophic equations [VI.1-4]. At a given initial time $t = 0$, p'_c and $\vec{v}_c$ cannot be chosen arbitrarily, but must satisfy the constraint equations [VI.3-4], just as $\vec{B}$ must satisfy [VI.2]. The induction equation [VI.1] can be used to calculate $\vec{B}$ at the next "instant" $t = \Delta t$. Equations [VI.3-4] must then be used to determine $\vec{v}_c(t = \Delta t)$, and so on.

In this dynamical problem, it is necessary to be able to solve [VI.3-4] for the velocity field, given an arbitrary Lorentz force. For that purpose, it is useful to introduce the variables

$$[\text{VI.8}] \quad \vec{u} = 2\Omega\rho_c \vec{v}_c, \quad ,$$

in terms of which [VI.3-4] become

$$[\text{VI.9}] \quad \vec{\nabla} \cdot \vec{u} = 0$$

$$[\text{VI.10}] \quad \vec{0} = -\hat{z} \times \vec{u} - (\vec{\nabla} + \vec{\nabla}\psi_0)p'_c + \vec{Q} \quad ,$$

where $\vec{Q}$ is the body force:

$$[\text{VI.11}] \quad \vec{Q} = -\frac{1}{4\pi}[\vec{B} \times (\vec{\nabla} \times \vec{B}) - \vec{B}_0 \times (\vec{\nabla} \times \vec{B}_0)] \quad .$$

Equations [VI.9-10] must be solved for $\vec{u}$, given $\vec{Q}$.

Because of computer storage restrictions, only axisymmetric fluid flows could be simulated. For the same reason, attention was restricted to flows which possess equatorial symmetry of the following type in cylindrical coordinates (z, x, ϕ) , where $z = 0$ is the equator:

$$[\text{VI.12}] \quad v_z(-z) = -v_z(z)$$

$$[\text{VI.13}] \quad v_x(-z) = v_x(z)$$

$$[\text{VI.14}] \quad v_\phi(-z) = v_\phi(z) \quad .$$

Because of the assumed axisymmetry, the ϕ component of [VI.10] is

$$[\text{VI.15}] \quad u_x = Q_\phi \quad .$$

Equation [VI.15] expresses the balance in the $\hat{\phi}$ direction between the Coriolis force and the magnetic body force. The z component of $\vec{u}$ can be determined from the anelastic condition [VI.9]:

$$[\text{VI.16}] \quad \frac{\partial u_z}{\partial z} = -\frac{1}{x} \frac{\partial}{\partial x} (xu_x) \quad ,$$

and from the vanishing of u_z at $z = 0$ (see [VI.12]).

The pressure perturbation terms can be eliminated from [VI.10] by applying the operator " $(\vec{\nabla} + \vec{\nabla}\psi_0) \times$ " to them. The result is

$$[\text{VI.17}] \quad (\vec{\nabla} + \vec{\nabla}\psi_0) \times (-\hat{z} \times \vec{u}) = -(\vec{\nabla} + \vec{\nabla}\psi_0) \times \vec{Q}.$$

Note that although p'_c does not appear in [VI.17], the effects of buoyancy are still taken into account through the terms involving $\vec{\nabla}\psi_0$. The ϕ component of [VI.17] yields an equation for u_ϕ :

$$[\text{VI.18}] \quad \left(\frac{\partial}{\partial z} + \frac{\partial\psi_0}{\partial z} \right) u_\phi = \left(\frac{\partial}{\partial x} + \frac{\partial\psi_0}{\partial x} \right) Q_z - \left(\frac{\partial}{\partial z} + \frac{\partial\psi_0}{\partial z} \right) Q_x.$$

As an aid in selecting solutions to [VI.18], I require that u_ϕ vanishes when $\vec{Q}$ vanishes. This eliminates the arbitrariness associated with the possible presence of geostrophic flow (Greenspan, 1968). This arbitrariness arises from dropping the $\partial(\rho_c \vec{v}_c)/\partial t$ term in the momentum equation, and is analogous to the arbitrariness in the pressure solution in incompressible hydrodynamic flow. In incompressible flow, an arbitrary constant may be added to the pressure solution.

There is a simple yet instructive example of a solution to [VI.15-18] for the case in which $\vec{B}_0$ is a uniform magnetic field in the $\hat{z}$ direction, and in which the magnetic body force is linearized. For this case, the charged fluid velocity is given by

$$[\text{VI.19}] \quad \rho_{co} \vec{v}_c = - \left(\frac{B_0}{8\pi\Omega} \right) \vec{\nabla} \times \vec{B}',$$

which clearly satisfies the anelastic condition. The flow field [VI.19] has a number of unusual features. The fluid velocity is everywhere parallel to the current $\vec{J}'$. The linearized Coriolis force $2\vec{\Omega}_0 \times \rho_{co} \vec{v}_c$ and the linearized magnetic body force $-\left(\frac{1}{4\pi}\right) \vec{B}_0 \times (\vec{\nabla} \times \vec{B}')$ exactly cancel so that in the linear approximation the pressure perturbation p'_c vanishes everywhere.

These features are possible because $\vec{B}_0$ and $\vec{\Omega}_0$ are parallel, and will probably not be present in the non-aligned case.

Note that for an arbitrary $\vec{B}'$, the charged fluid velocity $\vec{v}_c$ given in [VI.19] does not necessarily satisfy the no-slip boundary conditions [VI.5]. Thus there can be a boundary layer, whose existence seems to be in contradiction with statements made in Chapter III that there are no oscillatory hydromagnetic-inertial boundary layers where $\vec{B} \cdot \vec{n}|_{\Sigma} \neq 0$. However, there is no contradiction because any such boundary layers are transient. An extremely large shear in $\vec{v}_c$ near Σ will cause $\vec{B}'$ to grow in such a way that the shear in $\vec{v}_c$ is rapidly reduced.

The Charged Fluid Response Time Problem

One of the essential simplifying approximations which have been made in two-component models of post-glitch timing behaviour of pulsars (Shaham et al., 1973; Baym et al., 1975) is that the charged fluid co-rotates with the crust following a glitch. In support of this approximation, it has been argued (Baym et al., 1969c) that within the time it takes an Alfvén wave to cross the star, the charged fluid can adjust to the new angular velocity of the crust. Since this crossing time is small compared to the post-glitch relaxation time, it is argued, one may safely neglect the internal motions in the charged fluid and simply assume it co-rotates with the crust.

However, I showed in Chapter III that unless the magnetic field is so large that $B_{11}^{-1} \Omega_2$ is $\ll 10^{-4}$ (i.e. that $\xi \ll 1$ for $\lambda \approx R$), the waves which couple strongly to the magnetic field are hydromagnetic-inertial waves, which behave like Alfvén waves only if $\lambda \ll R$. Such very short wavelength Alfvén waves are generally damped by viscosity in very much less than a second, and so cannot propagate across the star. Thus, the assumption which has been

made in two-component models that the charged fluid as a whole responds to a disturbance in the Alfvén wave crossing time is not well founded, unless B_0 is $\gtrsim 10^{15}$ gauss.

What then is the charged fluid response time? By definition, it is the time for the fluid as a whole to adjust after the magnetic field lines have been sheared. Thus, it is probably the time scale of the longest wavelength oscillations that are strongly coupled to the magnetic field, i.e., roughly the period τ_{HI} of hydromagnetic-inertial modes with $\lambda \approx R$. This period can be the order of a month, for example, if B_{11} is roughly ≈ 1 within the star. The observed post-glitch relaxation time for the Crab is several days (Boynton et al., 1972). Thus, the charged fluid response time need not be small compared to the post-glitch relaxation time.

The magnetostrophic equations can be used to determine the behaviour of the charged fluid in response to changes in the angular velocity of the crust. One possible computer simulation involves a sudden change in the angular velocity of the crust (a glitch). A second possible simulation involves a sudden change in the angular acceleration of the crust. In this chapter, the results of a computer simulation of the second kind are reported. The choice of this simulation over a glitch simulation was based on numerical convenience, and not because of a lack of interest in the details of charged fluid post-glitch flow. However, the charged fluid response time, which is of great interest, should be almost the same in the chosen simulation as in a glitch simulation.

The Magnetic Field Structure

The charged fluid flow which develops following a disturbance depends on the general structure and strength of the magnetic field in the interior. The choice of magnetic field structure and strength in any magnetostrophic calculations must be guided by a number of considerations including pulsar observations, theories of neutron star formation, magnetohydrodynamic stability and simplicity.

One can learn very little about the interior magnetic field from pulsar observations. Its mean value $\langle B_o \rangle$ is probably $\ll 10^{17}$ gauss; if $\langle B_o \rangle$ were $\sim 10^{17}$ gauss, the star would be so distorted that the main energy loss mechanism would be the emission of gravitational radiation, in disagreement with timing observations (Ruderman, 1972). As regards the field structure, I deduced from pulsar slowing down rates that if $B_o \sim 10^{10}$ gauss at the inner crustal surface, the magnetic field there is almost purely poloidal or almost purely toroidal (see [II.25]).

Theories of neutron star formation are of little help in determining the interior field structure and strength. The value of $\langle B_o \rangle$ predicted by the simplistic flux conservation argument mentioned in Chapter I is uncertain by several orders of magnitude. In addition the flux conservation argument makes no prediction of the post-collapse magnetic field structure.

Ruderman and Sutherland (1973) have attempted to remove some of the uncertainty regarding the strength of the pre-collapse magnetic field. They argue that the stellar matter which later collapses to form a neutron star forms a convective core of radius $\approx 4 \times 10^7$ cm and that the magnetic field energy there is equal to the energy of convective motions. This argument leads to a pre-collapse B of about 3×10^9 gauss. If the naive flux conservation argument is then applied, final magnetic field strengths of

about 4×10^{12} gauss are predicted to occur.

As noted in Chapter I, the flux conservation argument completely ignores the amplification of a magnetic field by the shearing action of differential rotation during the collapse forming a rotating neutron star.

However, the most serious problem in applying theories of neutron star formation to the question of the structure and strength of the magnetic field inside neutron stars arises from present ignorance of the magnetohydrodynamics of neutron stars prior to crust formation. Such newly formed stars are probably subject to magnetohydrodynamic instabilities with growth times much less than the crust formation time. As a result, the magnetic field structure and strength in a neutron star a few days old could very well bear little resemblance to what they were at formation. Thus, on the basis of formation arguments alone, one cannot say anything with any degree of certainty about the structure of the magnetic field inside neutron stars.

If the magnetic field structure inside a neutron star is unstable, the magnetic flux might be quickly redistributed in roughly the time τ_{HI} throughout the interior so that the structure becomes stable. If this is not possible, which is the case if the instability arises from the topology of the field lines (Markey and Tayler, 1973), finite amplitude oscillations will occur. The topology of the field lines can be changed in the resistive diffusion time $\approx \lambda^2/\eta$, which for very short wavelength ($\lambda \ll R$) instabilities can be hundreds of years or less depending on the temperature. The ultimate result of these resistive changes will probably be the alteration of the unstable part of the field structure to a stable form. Thus it is reasonable to restrict ones attention at first to magnetic fields which are magnetohydrodynamically stable.

Unfortunately, little work has been done on magnetic field stability inside neutron stars (Vandakurov, 1970, 1972; Tayler, 1973b; Chanmugan, 1974). This work has primarily been concerned with convective instability. Fujimoto and Murai (1972) have considered the buoyant instability of an isolated toroidal magnetic flux tube in a gas layer just outside the surface of a neutron star. Although the concept of magnetic buoyancy may be applied to the question of magnetic stability inside neutron stars, the details of the work of Fujimoto and Murai (1972) on the exterior field are irrelevant to the present discussion.

Convection occurs if an element of fluid which rises adiabatically and which is in (material plus magnetic) pressure balance with its surroundings is lighter than them. The most recent and complete work on adiabatic convection in neutron stars is by Chanmugan (1974), who assumed that the nuclear relaxation time τ_N is much longer than the growth time of magnetic instabilities. Both he and Tayler (1973b) argued that this growth time is an Alfvén wave crossing time. If $B \lesssim 10^{15}$ gauss, the growth time is more likely to be the crossing time τ_{HI} of a hydromagnetic-inertial wave. Comparing the estimate [II.41] for τ_N with the estimate [III.20] for τ_{HI} , we see that indeed $\tau_N \gg \tau_{HI}$, in agreement with Chanmugan's assumption that τ_N is larger than instability growth times. Chanmugan (1974) concluded that B must be $\gtrsim 10^{16}$ gauss for magnetic buoyancy to be able to drive convection inside neutron stars. Thus, convective instabilities are probably unimportant inside neutron stars.

Non-convective magnetic instabilities in non-rotating stars have been considered by Tayler (1973a) and Markey and Tayler (1973). The basic cause of such instabilities is the curvature of the field lines, from which energy can be extracted, causing small perturbations to grow.

Tayler (1973a) considers toroidal magnetic fields and shows that in the absence of rotation they may be unstable to the interchange of flux tubes if there is a non-zero current density on the axis. However, when rotation is present it tends to suppress these instabilities.

Somewhat similar conclusions about the stability of toroidal fields have been reached by Acheson and Hide (1973). They point out that in a rapidly rotating fluid the interchange of two material rings which conserves their angular momenta requires an input of energy greatly in excess of the energy available from an unstable toroidal field structure. More precisely, a toroidal field configuration $B_\phi(x)$ is stable to axisymmetric perturbations if and only if

$$[\text{VI.20}] \quad \frac{4\Omega_0^2}{x} - \frac{d}{dx} (v_A^2/x^2) \geq 0 ,$$

which is equation (4.33) of Acheson and Hide (1973). Thus, if the magneto-hydrostatic toroidal field varies on a length scale λ , where

$$[\text{VI.21}] \quad \lambda \gtrsim 50 B_{11} \Omega_2^{-1} \text{ cm},$$

the structure is stable to small axisymmetric perturbations. However, in a non-axisymmetric disturbance, the toroidal field lines are twisted and the resulting Lorentz force can offset the stabilizing influence of the Coriolis force through a magnetostrophic balance.

Markey and Tayler (1973) discuss the instabilities of poloidal fields in non-rotating stars. Their main conclusion is that if there are any closed field lines within the star, the field is subject to instabilities similar to those in a pinched gas discharge. Closed poloidal field lines occur if the field is much stronger near the centre of the star than elsewhere. In Chapter II, I concluded that an axisymmetric field $\vec{B}$ whose poloidal part $\vec{B} - B_\phi \hat{\phi}$ contains closed field lines cannot balance the

quasi-steady Poincaré force. Thus it seems unlikely that the poloidal part of $\vec{B}$ is very much stronger in the deep interior than near the crust.

At present, the uncertainty about magnetohydrodynamic stability inside rotating neutron stars is great; the only magnetohydrostatic field which is definitely known to be stable is a uniform field parallel to the rotation axis. It is stable because there are no magnetic pressure gradients and because there is no curvature of the field lines.

Finally, simplicity should be a guide in choosing a field structure to use in the magnetostrophic simulation. The simplest magnetohydrostatic field is the uniform one just mentioned. If the field strength is $\ll 10^{17}$ gauss, such a field is compatible with pulsar observations. It trivially satisfies the condition of being either predominantly poloidal or predominantly toroidal near the inner crust. It contains no closed field lines, and as just mentioned, is the only magnetohydrostatic field configuration inside rapidly rotating neutron stars which is known to be stable.

For these reasons, I chose a uniform magnetic field aligned with the rotation axis as the basic field for the magnetostrophic simulation. To this basic field I added the small toroidal field displayed in Figure 5, which is required to balance the quasi-steady Poincaré force.

The actual value of B_0 chosen was 10^{11} gauss. This choice is not restrictive, because a whole class of solutions for other B_0 values can be formed by using the scaling of variables on B_{11} indicated throughout the thesis. The angular speed Ω_0 and uniform deceleration $\dot{\Omega}_0$ were chosen to be those of the Crab (see Table II). The rotating hydrostatic equilibrium model was chosen to be the one calculated in Chapter II.

Flow Simulation

The magnetostrophic flow which develops following a sudden change in the angular acceleration of the crust was simulated on a computer in order to see how quickly the charged fluid responds to the disturbance and to determine some of the flow characteristics. The magnetostrophic equations [VI.1, 15, 16, and 18] were replaced by finite difference equations. All physical variables were defined on a uniform 15×15 grid in the section $z \geq 0$, $x \geq 0$ of the $\phi = 0$ plane, with the region $z^2 + x^2 > R_c^2$ excluded.¹ The simulation ran for 125 days of simulated time.

There are two possible descriptions of the initial disturbance which are equivalent in the sense that the flow which follows is the same in both cases. In the first description, the magnetic field is uniform and the star is not decelerating prior to the initial time $t = 0$. At $t = 0$ a steady external torque is suddenly applied to the crust, i.e., the Poincaré force is "turned on". In the second description, the star is decelerating and the magnetic field $\vec{B}_0$ is nearly uniform prior to $t = 0$. The non-uniformity in $\vec{B}_0$ arises from the relatively small toroidal field required to balance the quasi-steady Poincaré force. This toroidal field was calculated in Chapter II and is displayed in Figure 5. At $t = 0$, the small balancing toroidal field is "turned off", that is, the initial perturbation $\vec{B}'(t = 0)$ in the magnetic field is the negative of the field in Figure 5. This initial perturbation $\vec{B}'(t = 0)$ clearly satisfies the Maxwell equation [VI.2] and the magnetic field boundary condition [VI.6].

The non-linearity in the subsequent flow is of order B'/B_0 , which is $\approx 10^{-5}$ (see [II.59]). If terms of this size had been neglected compared to

¹ See Appendix C for further details of the grid and the numerical methods used.

terms of order 1, which they were not in the actual calculation, the relationship between $\vec{v}_c$ and $\vec{B}'$ would be that given in [VI.19]. Thus to a first approximation, the properties of the flow field $\vec{v}_c$ in [VI.19] hold for the simulated flow. In particular, the pressure perturbation p'_c arises only from the small non-linearities, and thus buoyancy has very little effect on the fluid flow.

Basic Time Scales Of The Flow

In order to show more clearly some of the basic time scales of the flow, the perturbation in the crustal angular acceleration due to viscous stress is displayed in Figure 18. An isotropic viscous tensor was chosen, as in Figure 16, and ν_c was taken to be $2 \times 10^{11} \text{ cm}^2 \text{ sec}^{-1}$, which corresponds to a temperature $T_7 \approx .4$. Any other value of ν_c could have been chosen, or the anisotropy in the viscosity taken into account, or the magnetic torque fluctuations included, but none of these would have changed the time scales.

It will be shown later that the most prominent mode in the flow simulation has a wavelength $\lambda_6 \approx .3$. Using $\Omega_2 = 1.9$ and $T_7 = .4$, equation [VI.7] can be used to estimate that N'_B/N'_V is ≈ 20 for this flow. The total perturbation $\dot{\Omega}'$ in the crust angular acceleration is therefore ≈ 20 times the acceleration shown in Figure 18, i.e., $\dot{\Omega}'$ is $\approx 4 \times 10^{-13} \text{ sec}^{-2}$, which is about 10^{-4} times the size of $\dot{\Omega}_0$. Thus, the fluctuating Poincaré force $-\rho_c \dot{\Omega}' \times \vec{r}$ is relatively negligible, and it was a good approximation not to include it in the fluid momentum equation [VI.4].

The most striking feature in Figure 18 is the oscillations in $\dot{\Omega}'(t)$. There appears to be one predominant period of about a day. On closer inspection, it can be seen that there are two quite different phases to $\dot{\Omega}'(t)$. The first phase, which I call the charged fluid response phase,

lasts a few days after $t = 0$. It is distinguished by a positive time average of $\dot{\Omega}'$. During this phase the charged fluid responds to the crustal slowing down in such a way as to oppose it. The charged fluid response time in this simulation is thus a few days, which is comparable to the observed post-glitch relaxation time for the Crab pulsar (Boynton *et al.*, 1972). The important point is not that the two times are almost equal, but that the charged fluid response time is not very much smaller than the post-glitch relaxation time.

The charged fluid response phase is followed by a phase in which $\dot{\Omega}'$ fluctuates about a mean of zero. In the latter phase the fluid oscillates about a state of co-rotation with the decelerating crust. The oscillations are undamped because viscosity and superfluid drag have been neglected in the fluid equations.

A power spectrum analysis of the crustal acceleration graph in Figure 18 was performed in order to examine more quantitatively the periodicities in the flow.¹ The resulting power spectrum is given in Figure 19, which shows a large number of well defined peaks. These peaks are not the result of fluctuations at a single point in the discrete Fourier analysis, because each is composed of at least several points (there are 2,000 points in the discrete frequency domain). There can be no doubt that the peaks are not the result of numerical "leakage" from the main peak at a period of 1.2 days. If there were a significant amount of such leakage, one would expect it to produce peaks in the frequency region > 25 rad/day, where there are no peaks. In fact, there should be no peaks in this region because waves of that high a frequency cannot propagate on the computational grid, whereas waves whose frequencies are lower than this can.

¹ The program FOURT from the U.B.C. Computing Centre was used in the Fourier analysis. The original 50,000 data points were averaged in blocks of 25 prior to Fourier analysis. The power spectrum was smoothed with a running average of 3 points. I thank G. Fahlman for recommending this procedure to me.

It is thus quite certain that these peaks are not artifacts of the Fourier analysis. The peaks are due to the oscillation of the charged fluid in hydromagnetic-inertial modes with frequencies equal to the frequencies at the peak maxima. Some of the prominent modes have periods of about .30, .60, .96, 1.2, 2.4, 3.8, 4.9, 6.3, 10 and 13 days. These normal mode periods were computed for values of $B_{11} = 1$, $\Omega_2 = 1.9$, and $R_c = 8.9$ km. The periods for other magnetic field strengths, crustal angular velocities and inner crustal radii may be obtained by scaling, because it follows from [III.20] that all normal mode periods are $\propto B_{11}^{-2} \Omega_2 [R_c(\text{km})/8.9]^2$.

Details Of The Flow

Pictures of the charged fluid velocity $\vec{v}_c$ and the perturbation $\vec{B}'$ in the magnetic field at a number of different times are given in Figure 20, which extends over many pages. Each picture is labeled with the time in days at which the situation is depicted. The meridional and azimuthal parts of $\vec{v}_c$ are separately displayed, as are the poloidal and toroidal parts of $\vec{B}'$.

It was previously noted that $\vec{B}'(t = 0)$ is the negative of the toroidal field depicted in Figure 5, so it is unnecessary to provide a separate picture of it in Figure 20. The initial azimuthal flow $v_\phi(t = 0)$ vanishes, so only the meridional component of $\vec{v}_c(t = 0)$ is displayed.

Since the magnetic body force $\vec{Q}$ at $t = 0$ is equal to $-\rho_c \dot{\vec{\Omega}}_0 \times \vec{r}$ by the choice of initial conditions, it follows from [VI.15] that

$$[\text{VI.22}] \quad \vec{v}_x(t = 0) = - \frac{\dot{\vec{\Omega}}_0 \times \vec{r}}{2\Omega_0}.$$

The z component of $\vec{v}_c(t = 0)$ can be determined from [VI.16]. Because ρ_c is

smaller at larger radii, $v_z(t=0)$ is larger near the crust than near the centre. These features of $\vec{v}_c(t=0)$ are displayed in the first picture of Figure 20. Because $\vec{v}_c(t=0)$ does not satisfy the no-slip boundary condition, a thin transient boundary layer is formed at the crust.

After .06 days, an azimuthal flow v_ϕ has begun to develop through the action of the Coriolis force on the meridional flow $\vec{v}_p$. This azimuthal flow is in such a direction as to oppose the crustal slowing down. At this time v_ϕ is still small in comparison to the meridional flow. Because of the large shear near the crust in the initial meridional flow, the developing azimuthal flow has a similar shear.

The magnetic field perturbation $\vec{B}'$ has a non-zero poloidal part $\vec{B}'_p$ at .06 days. It can be seen that the component of $\vec{B}'_p$ normal to the crust vanishes as required by the magnetic field boundary condition [VI.6]. On the other hand, it can be seen from the pictures of $\vec{B}'$ in Figure 20 that the tangential component of $\vec{B}'$ at any point on the crust is not fixed, because the boundary conditions in the limit $\sigma \rightarrow \infty$ do not place any restrictions on $\vec{n} \times \vec{B}'|_\Sigma$.

$\vec{B}'_p$ lies nearly parallel to the unperturbed magnetic field because

$$[\text{VI.23}] \quad \frac{\partial \vec{B}'_p}{\partial t} \approx B_0 \frac{\partial \vec{v}}{\partial z} \approx B_0 \frac{\partial v_z}{\partial z} \hat{z},$$

where the dependence of $v_x(t=0)$ only on x (see [VI.22]) has been used in the last step of [VI.23]. As a result, $|\partial \vec{B}'_x / \partial t|$ is $\ll |\partial \vec{B}'_z / \partial t|$ for some time after $t=0$. Since $B'_z = B'_x = 0$ initially, $|B'_x|$ is $\ll |B'_z|$ during this time.

After a quarter of a day, v_ϕ and B'_ϕ have grown larger. The largest growth in v_ϕ is near the crust at the equator since the unperturbed magnetic field is tangent to the crust there and as a result large shears in the $\hat{x}$ direction cannot be effectively resisted there.

The region in which v_ϕ and B'_ϕ have large shears is larger than it was at .06 days. There is a small region near the rotation axis in which v_ϕ is negative. These features are easily interpreted in terms of the dispersive nature of the hydromagnetic-inertial waves. The sudden onset of the crustal deceleration generates hydromagnetic-inertial waves of various wavelengths near the crust. Because the phase velocity of hydromagnetic-inertial waves is inversely proportional to their wavelength (see [III.17]), the shorter wavelength disturbances travel more quickly than the longer wavelength ones. A short wavelength disturbance is responsible for the small region in which $v_\phi < 0$. By .5 days, somewhat longer wavelength waves have reached this area, and the region has grown. In the meantime, the longest wavelength waves are still near the crust, gradually widening the shear region. By three quarters of a day, the Lorentz forces have grown to such an extent that in the large shear region near the crust at the equator, v_ϕ has become negative, and $\dot{\Omega}'$ is < 0 .

After one day, $\dot{\Omega}'$ is still < 0 and the region of negative v_ϕ near the rotation axis has grown. The lines of constant v_ϕ and B'_ϕ are tending to line up along the direction of the unperturbed magnetic field, so that $|\partial v_\phi / \partial z|$ and thus $|\partial B'_\phi / \partial t|$ are as small as possible.

By 1.25 days, the azimuthal flow near the crust is again in the $+\hat{\phi}$ direction, and $\dot{\Omega}'$ is > 0 . The poloidal and toroidal components of each of $\vec{v}_c$ and $\vec{B}'$ are by now comparable in magnitude. After 1.5 days, the charged fluid is nearing the end of its response phase because the longer wavelength waves have begun to reach the centre.

The situation at the last day of the simulation, day 125, displays characteristics which were present in the first two days of the flow, including the smallness of $|B'_x|$ compared to $|B'_z|$ and the alignment of the contours of equal v_ϕ and B'_ϕ along the direction of $\vec{B}_0$. The length scale

for changes in $\vec{v}_c$ and $\vec{B}'$ is about $R_c/3$, so that the period of the oscillations should be about $(1/3)^2$ times the period for longest wavelength waves. This is the order of days, which is compatible with the calculated predominant period of 1.2 days.

At 125 days, there are indications of waves with wavelengths which are very small compared to R_c . Such small scale structures would not be present after 125 days if viscosity had been taken into account and if T_7 were $\lesssim 1$, because the damping times of such waves can be the order of a few days or less (see [III.27]). Since $\tau_D^{(HI)}$ is an extremely sensitive function of λ , the large scale structures present at 125 days would probably persist for several years.

The results of the computer simulation can be summarized and interpreted as follows. The sudden onset of the crust deceleration shears the magnetic field lines near the crust, particularly near the equator. This generates a variety of hydromagnetic-inertial waves. Because the shorter wavelength waves travel more quickly, they reach the stellar interior after only a fraction of a day, at which time the longer wavelength waves are still near the crust, widening the shear region. After several days, the longer wavelength waves reach the interior. At this time, the charged fluid is oscillating about a state of rigid rotation with the decelerating crust. The oscillations persist for some time thereafter.

Figure 18. Perturbation in crust angular acceleration due to viscous torque. The perturbations in the crust angular acceleration due to variations in the magnetic torque induced by the fluid flow are larger than the viscosity induced perturbations. However, it is not the amplitude of the perturbations but only their character as a function of time that is of interest here. There are two distinct phases apparent in $\dot{\Omega}'(t)$. One is a charged fluid response phase lasting a few days in which the time average of $\dot{\Omega}'(t)$ is positive. This is followed by a second phase, in which the time average of $\dot{\Omega}'(t)$ is zero.

CRUST ANGULAR ACCELERATION (10^{-12} rad/sec²)

Figure 19. Power spectrum analysis of crust angular acceleration data in Figure 18. The zero of the vertical scale is arbitrary. Peaks in the spectrum correspond to frequencies ω of hydro-magnetic-inertial normal modes of the charged fluid. The frequencies ω are $\propto B_{11}^2 \Omega_2^{-1} [R_c(\text{km})/8.9]^{-2}$, and these were computed for $B_{11} = 1$, $\Omega_2 = 1.9$, and $R_c(\text{km})/8.9 = 1$. The lack of structure at frequencies above 25 rad/day is a numerical artifact due to the finite grid size.

Figure 20. Magnetostrophic flow of charged fluid induced by crustal deceleration. In these pictures, the charged fluid velocity $\vec{v}_c$ and the perturbation $\vec{B}'$ in the magnetic field are displayed at various times. Both $\vec{v}_c$ and $\vec{B}'$ are split into poloidal parts $\vec{v}_p$, $\vec{B}'_p$ and toroidal parts v_ϕ , B'_ϕ , which are separately displayed. The conventions for the vector plots and contour plots are as in Chapter V. Time is measured in days. Note that the notation $\text{MAX } B_p$ stands for the maximum value that $|\vec{B}'_p|$ attains at a given time. Similar notation is used for $\vec{v}_p$, v_ϕ and B'_ϕ . The initial velocity field is purely meridional, so $v_\phi(t = 0)$ is not displayed. The initial $\vec{B}'_p$ vanishes, and $B'_\phi(t = 0)$ is equal to the negative of the field in Figure 5.

$$\text{MAX } V_p = 0.28 \times 10^{-4} \text{ cm/sec}$$

TIME = 0.000

$$\text{MAX } V_P = 0.28 \times 10^{-4} \text{ cm/sec}$$

$$\text{MAX } V_\phi = 0.34 \times 10^{-5} \text{ cm/sec}$$

MAX $B_p = 4.38 \times 10^4$ GAUSS

MAX $B_\phi = 4.82 \times 10^5$ GAUSS

MAX $V_p = 0.28 \times 10^{-4}$ cm/sec

TIME = 0.250

MAX $V_\phi = 0.10 \times 10^{-4}$ cm/sec

TIME = 0.250

MAX $B_p = 1.46 \times 10^5$ GAUSS

TIME = 0.250

MAX $B_\phi = 4.56 \times 10^5$ GAUSS

TIME = 0.250

MAX $V_P = 0.29 \times 10^{-4}$ cm/sec

TIME = 0.500

MAX $V_\phi = 0.13 \times 10^{-4}$ cm/sec

TIME = 0.500

MAX $B_p = 1.78 \times 10^5$ GAUSS

TIME = 0.500

MAX $B_\phi = 4.32 \times 10^5$ GAUSS

TIME = 0.500

$$\text{MAX } V_p = 0.29 \times 10^{-4} \text{ cm/sec}$$

$$\text{MAX } V_\phi = 0.15 \times 10^{-4} \text{ cm/sec}$$

MAX $B_p = 1.93 \times 10^5$ GAUSS

MAX $B_\phi = 4.13 \times 10^5$ GAUSS

MAX $B_p = 2.61 \times 10^5$ GAUSS

MAX $B_\phi = 3.94 \times 10^5$ GAUSS

MAX $V_P = 0.31 \times 10^{-4}$ cm/sec

MAX $V_\phi = 0.15 \times 10^{-4}$ cm/sec

MAX $B_p = 3.30 \times 10^5$ GAUSS

MAX $B_\phi = 3.80 \times 10^5$ GAUSS

MAX $V_p = 0.34 \times 10^{-4}$ cm/sec

MAX $V_\phi = 0.15 \times 10^{-4}$ cm/sec

MAX $B_p = 4.00 \times 10^5$ GAUSS

MAX $B_\phi = 3.70 \times 10^5$ GAUSS

MAX $V_P = 0.34 \times 10^{-4}$ cm/sec

MAX $V_\phi = 0.14 \times 10^{-4}$ cm/sec

MAX $B_p = 7.12 \times 10^5$ GAUSS

TIME = 125.0

MAX $B_\phi = 4.55 \times 10^5$ GAUSS

TIME = 125.0

CONCLUDING REMARKS

The conclusions I have reached are presented in the form of answers to the sets of questions which I posed at the end of Chapter I. For ease of reference, the questions are repeated here.

1. What is the range of validity of the fluid description of the charged particles in the interior? Within this regime, to what extent are the usual equations of magnetohydrodynamics valid?

The charged particles may be treated as a fluid if the length scale λ of a disturbance is $\gg 3 T_7^{-2}$ cm (the electron mean free path) and if the time scale τ is $\gg 10^{-10} T_7^{-2}$ sec (the electron mean free time). Under such conditions, quasi-neutrality is an excellent approximation. The magnetic field evolves in time according to the infinite conductivity induction equation. The creation or destruction of charged fluid by the action of the weak interaction on density perturbations is completely negligible, and the usual mass conservation equation holds. Off-diagonal terms in the material pressure tensor due to finite Larmor radius effects are relatively small. Other relatively small terms in the fluid momentum equation are the viscous term and the self-gravitational term. To a very good approximation, the heat flux equation is decoupled from the other equations.

To my knowledge, there has been no previous attempt to answer this first set of questions.

2. What types of small amplitude waves can exist in the charged fluid? What can damp them? How much are they damped? How do the answers to these questions depend on the gross parameters of the problem, such as Ω , B , and T ?

The answers to these questions are given in Table X. The most significant results are the long periods of hydromagnetic-inertial waves and their extremely long damping times. Two additional damping mechanisms not listed in the table are heat conduction and electrical resistivity, which are significant only if the star is so young that T_7 is $\gg 1$.

The possible existence of inertial waves and hydromagnetic-inertial waves inside neutron stars was noted by Hide (1971). However, he did not consider the question of damping or investigate further the magnetohydrodynamics of the charged fluid.

3. What kinds of mode-mode coupling can occur? Can this weakly non-linear effect lead to heavy damping of a wave which is only lightly damped in the linear approximation?

The second of these questions concerns hydromagnetic-inertial waves. If such a wave can resonantly couple to two other waves, one of which is a sound wave or an inertial wave, the possibility should be considered that energy can be transferred to the sound or inertial wave and dissipated in a time short compared to the period of the hydromagnetic-inertial wave. I found that the only such resonant coupling involves a hydromagnetic-inertial wave and either a pair of inertial waves or a pair of sound waves. However, I showed that resonant energy transfer was not possible in this situation.

Table X
Charged Fluid Waves And Their Characteristics

TYPE	PERIOD	VISCOUS DAMPING TIME	SUPERFLUID DRAG DAMPING TIME
SOUND	$10^{-4} \lambda_6 \text{ sec}$	$10 \lambda_6^2 T_7^2 \text{ sec}$	$\tau_1 \text{ days}$
INERTIAL [†]	$\frac{1}{2} \text{ rotation period}$	$10 \lambda_6^2 T_7^2 \text{ sec}$	$\tau_1 \text{ days}$
HYDROMAGNETIC-INERTIAL [†]	$B_{11}^{-2} \lambda_6^2 \Omega_2^2 \text{ months}$	$100 B_{11}^{-2} \Omega_2^2 \lambda_6^4 T_7^2 \text{ years}$	$10^6 B_{11}^{-2} \Omega_2^2 \lambda_6^2 \tau_1 \text{ years}$

NON-DIMENSIONALIZED VARIABLES	NOTATION
WAVELENGTH	$\lambda_6 = \lambda(\text{cm})/10^6$
CRUST ANGULAR SPEED	$\Omega_2 = \Omega(\text{rad/sec})/10^2$
MAGNETIC FIELD	$B_{11} = B(\text{gauss})/10^{11}$
INTERIOR TEMPERATURE	$T_7 = T(^{\circ}\text{K})/10^7$
SUPERFLUID DRAG TIME [*]	$\tau_1 = \tau_D(\text{day})/1$

[†] It is assumed above that $B_{11}^{-1} \Omega_2 \lambda_6 \gg 10^{-4}$.

^{*} τ_D defined by: drag force/unit volume on charged fluid = $\rho_c (\vec{v}_s - \vec{v}_c)/\tau_D$.

4. What are some normal modes of oscillation of the charged fluid?

I calculated some of the sound modes and their eigenfrequencies in the inviscid approximation. Neither the variations in the charged fluid sound speed nor the buoyancy force had any qualitative effect on the modes.

I calculated some inertial eigenmodes and eigenfrequencies in a crude representation of the interior consisting of two co-axial rotating cylinders of finite length with a gravitational field perpendicular to the rotation axis. Viscosity was neglected. All axisymmetric modes were stable and relatively unaffected by the gravitational buoyancy force. I also calculated a number of non-axisymmetric modes which are independent of the distance along the rotation axis. The frequencies of these buoyancy-inertial modes, as I call them, become zero if the gravitational field is turned off.

5. How do charged fluid oscillations affect the crust? What is the relationship, if any, between the charged fluid and the quasi-sinusoidal residuals?

The motion of the charged fluid can change the angular velocity of the crust either through the viscous torque or through variations in the magnetic torque. If the fluid is oscillating at frequencies of inertial waves or higher, the viscous torque is the more effective of the two mechanisms. At hydromagnetic-inertial frequencies, the fluctuations in the magnetic torque are generally more effective.

It appears unlikely that charged fluid oscillations can account for the residuals. A mean interior magnetic field of $\sim 2 \times 10^{10}$ gauss would be required. Furthermore, there seems to be no natural way that hydro-magnetic-inertial modes can produce an ω^{-4} timing residual power spectrum.

6. How does the charged fluid respond under a variety of initial conditions and external forces? In particular, how does it respond and how quickly does it respond to changes in the crust angular velocity?

I simulated the charged fluid flow over two quite different time scales on a computer to see how the fluid responds.

Two simulations were performed in which the fluid motion was followed over the time scales of inertial waves. In one of these simulations, the charged fluid initially has a small azimuthal flow as seen in a reference frame co-rotating with the crust. A meridional flow develops through the action of the Coriolis force, and the fluid begins to oscillate primarily in an inertial mode with a period $\approx .75$ times the rotation period. One interesting feature of the flow is a type of Taylor-Proudman column parallel to the rotation axis above a solid core which oscillates relative to the crust.

I also simulated the fluid motion over the long time scales of hydro-magnetic-inertial waves. It is only over these time scales that the magnetic field and the charged fluid are strongly coupled.

Prior to the initial time $t = 0$, the magnetic field is a uniform 10^{11} gauss field parallel to the rotation axis. The star is not decelerating. At $t = 0$, a steady external torque is applied to the crust so it decelerates steadily. The angular velocity and deceleration chosen were those of the Crab pulsar.

Although the simulation performed is not a glitch simulation, the response time of the charged fluid should be roughly the same in both cases. It takes several days for the longer wavelength ($\lambda \approx R_c/3$) hydromagnetic-inertial waves formed at the crust near the equator to reach the centre of the star. At the end of this time, the charged fluid is oscillating about a state of co-rotation with the decelerating crust. The observed post-glitch

relaxation time for the Crab pulsar is several days. The major conclusion of my thesis is that the charged fluid response time is not necessarily very much shorter than post-glitch relaxation times. In view of this conclusion, simple two-component models of post-glitch behaviour which assume that the charged fluid response time is much shorter than post-glitch relaxation times need to be re-examined.

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APPENDIX A: NUMERICAL METHODS FOR SOUND WAVES

In brief, the differential equation for f_ℓ was solved by approximating it by a finite matrix eigenvalue problem. The interval $[a,1]$ was divided into $n + 1$ equal intervals of length Δr :

$$[A.1] \quad r^{(i)} = a + i\Delta r ,$$

so that $r^{(1)}, \dots, r^{(n)}$ are the interior points, and $r^{(0)}$ and $r^{(n+1)}$ are the boundary points. The function f_ℓ and its first and second derivatives were approximated as follows:

$$[A.2] \quad f_\ell(r^{(i)}) \rightarrow f_\ell^{(i)}$$

$$[A.3] \quad f'_\ell(r^{(i)}) \rightarrow (f_\ell^{(i+1)} - f_\ell^{(i-1)})/2\Delta r$$

$$[A.4] \quad f''_\ell(r^{(i)}) \rightarrow (f_\ell^{(i+1)} - 2f_\ell^{(i)} + f_\ell^{(i-1)})/(\Delta r)^2$$

The differential equation for f_ℓ then becomes a set of n linear equations in $n + 2$ unknowns, the i^{th} equation relating $f_\ell^{(i-1)}$, $f_\ell^{(i)}$ and $f_\ell^{(i+1)}$.

The boundary conditions were handled by solving their difference analogues for $f_\ell^{(0)}$ and $f_\ell^{(n+1)}$, and then substituting these expressions into the first and n^{th} equations. The result is a matrix eigenvalue problem:

$$[A.5] \quad \underline{A} \cdot \underline{F} = -\omega^2 \underline{F} ,$$

where $\underline{A}$ is a non-symmetric, tri-diagonal, $n \times n$ real matrix, and $\underline{F}$ is the column vector $(f_\ell^{(1)}, \dots, f_\ell^{(n)})$.

Equation [A.5] was solved with the use of routines FIGI2 and IMTQL2 of the subroutine package EISPACK.¹ The calculations were performed in

¹ Obtained from the U.B.C. Computing Centre.

double precision on an IBM 370 machine. To test convergence of the answers, the equations were solved with $n = 25, 50$ and 100 . The dependence of the lowest five eigenfrequencies of the $\ell = 1, a = 0$ case on n is displayed in Table X. It can be seen from Table XI that the numerical method is converging well. The final results were all calculated with $n = 100$.

Table XI
Convergence Of Sound Wave Eigenfrequencies

$n = 25$	$n = 50$	$n = 100$
1.750	1.742	1.738
5.018	4.966	4.937
7.888	7.823	7.781
10.65	10.59	10.55
13.33	13.32	13.28

APPENDIX B: NUMERICAL METHODS FOR INERTIAL FLOW

The equations were solved by replacing derivatives by differences, and solving the resulting algebraic equations.

Spatial Mesh Systems

A set of four staggered grids in the upper right-hand quadrant of the $z = 0$ plane was used, similar in many respects to the MAC mesh system of Harlow and Welch (1965). These four grids can be generated by the product of a pair of r grids and a pair of θ grids:

$$[B.1] \quad r_i = R_s + (i - \frac{3}{2})\Delta r$$

$$[B.2] \quad r_{i+1/2} = R_s + (i - 1)\Delta r$$

$$[B.3] \quad \theta_j = (j - \frac{3}{2})\Delta\theta$$

$$[B.4] \quad \theta_{j+1/2} = (j - 1)\Delta\theta \quad ,$$

where $i = \{1, \dots, N_r\}$, $j = \{1, \dots, N_\theta\}$, R_s is the radius of the solid central core, and

$$[B.5] \quad \Delta r = (R_c - R_s)/(N_r - 2)$$

$$[B.6] \quad \Delta\theta = (\pi/2)/(N_\theta - 2) \quad .$$

The four meshes may be labeled by (i, j) , $(i, j + \frac{1}{2})$, $(i + \frac{1}{2}, j)$, and $(i + \frac{1}{2}, j + \frac{1}{2})$. In the calculations reported in this thesis, both N_r and N_θ were taken to be 15. A representation of the four grids is given in Figure 13.

The three components (v_r, v_θ, v_ϕ) of the velocity were placed on the $(i + \frac{1}{2}, j)$, $(i, j + \frac{1}{2})$, and (i, j) meshes, respectively. The magnetic field components were similarly placed. The variables p'_c and ϕ' were placed on the (i, j) mesh. Of the stress tensor components, σ_{rr} , $\sigma_{\theta\theta}$, and $\sigma_{\phi\phi}$ were placed on the (i, j) mesh; $\sigma_{r\phi}$ on the $(i + \frac{1}{2}, j)$ mesh; $\sigma_{\theta\phi}$ on the $(i, j + \frac{1}{2})$ mesh; and $\sigma_{r\theta}$ on the $(i + \frac{1}{2}, j + \frac{1}{2})$ mesh.

This choice of meshes and variable placements was designed so that all derivatives can be replaced by second order accurate central differences. For example, consider the reasons for the placement of p'_c relative to v_r and v_θ . The radial component of the fluid momentum equation contains the terms

$$[B.7] \quad \frac{\partial}{\partial t} (\rho_c v_r) = \dots - \frac{\partial p'_c}{\partial r} \dots$$

With the chosen placement of variables, [B.7] becomes

$$[B.8] \quad \frac{\partial}{\partial t} (\rho_c v_r)_{i+1/2, j} = \dots - [(p'_c)_{i+1, j} - (p'_c)_{i, j}] / \Delta r \dots$$

It can be seen from [B.8] that the placement of $(p'_c)_{i, j}$ between $(v_r)_{i-1/2, j}$ and $(v_r)_{i+1/2, j}$ produces a truncation error of $O((\Delta r)^2)$ in evaluating $\partial p'_c / \partial r$, which is four times smaller than the truncation error $O((2\Delta r)^2)$ which would have arisen if only one grid had been used.

Similarly, the θ component of the fluid momentum equation contains the terms

$$[B.9] \quad \frac{\partial}{\partial t} (\rho_c v_\theta) = \dots - \frac{1}{r} \frac{\partial p'_c}{\partial \theta} \dots,$$

which becomes

$$[B.10] \quad \frac{\partial}{\partial t} (\rho_c v_\theta)_{i, j+1/2} = \dots - [(p'_c)_{i, j+1} - (p'_c)_{i, j}] / (r_i \Delta \theta) \dots$$

It is apparent from [B.10] that the formal truncation error in evaluating $\partial p'_c / \partial \theta$ is four times smaller than it would have been if p'_c and v_θ had been placed on the same grid.

Not only p'_c , v_r and v_θ , but all variables were placed so as to achieve a four-fold decrease in spatial derivative truncation errors compared to that achievable with a single grid with the same N_r , N_θ . In terms of truncation error, the staggered 15×15 grids used were equivalent to the use of a single 30×30 grid, but with no more computer storage space than with a single 15×15 grid.

This choice of staggered grids also allows boundary and symmetry conditions to be easily satisfied. For example, consider the symmetry conditions on the equator $\theta = \pi/2$. These are

$$[B.11] \quad \frac{\partial v_r}{\partial \theta} (\theta = \pi/2) = 0$$

$$[B.12] \quad v_\theta (\theta = \pi/2) = 0$$

$$[B.13] \quad \frac{\partial v_\phi}{\partial \theta} (\theta = \pi/2) = 0 \quad ,$$

whose finite difference versions are simply

$$[B.14] \quad (v_r)_{i+1/2, N_\theta} = (v_r)_{i+1/2, N_\theta-1}$$

$$[B.15] \quad (v_\theta)_{i, N_\theta-1/2} = 0$$

$$[B.16] \quad (v_\phi)_{i, N_\theta} = (v_\phi)_{i, N_\theta-1}$$

Occasionally, variables are required at points where they have not been placed. In that case, averaging of nearby values was used. For example, we define

$$[B.17] \quad (v_r)_{i,j} = \frac{1}{2} [(v_r)_{i+1/2,j} + (v_r)_{i-1/2,j}]$$

The Coupled Pressure And Potential Equations

Given the velocity field and the magnetic field at some time, the two coupled elliptic equations for p'_c and ϕ' were solved in a "self-consistent" manner; that is, a trial solution for ϕ' was assumed, and the equation for p'_c solved. These new values for p'_c were then used to calculate new ϕ' values, and so on, until convergence was reached. At each stage in this iterative process, it is necessary to solve an elliptic equation with given right-hand side. For example, the Poisson equation for ϕ' is

$$[B.18] \quad \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \phi'}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \phi'}{\partial \theta}) = \frac{4\pi G}{s_{co}^2} p'_c,$$

which must be solved for ϕ' assuming that p'_c is known. The finite difference version of [B.18] is

$$[B.19] \quad \left(\frac{r_{i+1/2}}{r_i} \right)^2 (\phi')_{i+1,j} + \left(\frac{r_{i-1/2}}{r_i} \right)^2 (\phi')_{i-1,j} \\ + \left(\frac{\Delta r}{r_i \Delta \theta} \right)^2 \frac{\sin \theta_{j+1/2}}{\sin \theta_j} (\phi')_{i,j+1} + \left(\frac{\Delta r}{r_i \Delta \theta} \right)^2 \frac{\sin \theta_{j-1/2}}{\sin \theta_j} (\phi')_{i,j-1} \\ - \left[\left(\frac{r_{i+1/2}}{r_i} \right)^2 + \left(\frac{r_{i-1/2}}{r_i} \right)^2 + \left(\frac{\Delta r}{r_i \Delta \theta} \right)^2 \left(\frac{\sin \theta_{j+1/2}}{\sin \theta_j} + \frac{\sin \theta_{j-1/2}}{\sin \theta_j} \right) \right] (\phi')_{i,j} \\ = 4\pi G (\Delta r)^2 \frac{(p'_c)_{i,j}}{(s_{co}^2)_{i,j}},$$

which, along with finite difference forms for the ϕ' boundary conditions, constitute a set of linear equations in $N_r N_\theta = 225$ unknowns. The finite difference version of the elliptic equation for p'_c is not given because it is much more complicated than [B.19].

The use of successive overrelaxation (SOR) methods to solve elliptic equations like [B.19] is very time consuming. Since [B.18] has to be solved several times on each time step, the computing time and cost which would arise if SOR methods were used would be completely prohibitive.

I therefore adapted the EVP method of Roache (1972), which had been designed for the Poisson equation in Cartesian coordinates, to the present problem. The EVP method has two great advantages over SOR methods. Firstly, it is not iterative as SOR methods are. Thus, one does not need to worry about whether or not the solution has converged. Secondly, it is about two orders of magnitude faster than SOR methods. The actual calculations of the p'_c , ϕ' solutions were performed in double precision on an IBM 370 computer.

Time Differencing And Time Centering

There are essentially two kinds of time differencing: implicit and explicit. In an implicit method, a set of coupled algebraic equations must be solved in order to calculate variables like $\vec{v}_c$ at the advanced time. In an explicit method, the advanced values of variables are given explicitly in terms of their values at the present and past time steps. Because of its relative simplicity, an explicit method was used in the present calculations.

The particular explicit timing differencing method used was the Leapfrog method (Roache, 1972). This method can be illustrated by the finite difference analogue of [B.7]. If a time grid

$$[B.20] \quad t^k = k\Delta t$$

is used, then [B.7] becomes

$$[B.21] \quad \frac{(\rho_c v_r)^{k+1}_{i+1/2,j} - (\rho_c v_r)^{k-1}_{i+1/2,j}}{2\Delta t} = \dots - \frac{[(p'_c)^k_{i+1,j} - (p'_c)^k_{i,j}]}{\Delta r} \dots ,$$

and so, neglecting ρ'_c compared to ρ_{co} ,

$$[B.22] \quad (v_r)^{k+1}_{i+1/2,j} = (v_r)^{k-1}_{i+1/2,j} \dots - \frac{2\Delta t}{(\rho_{co})_{i+1/2,j} \Delta r} [(p'_c)^k_{i+1,j} - (p'_c)^k_{i,j}]$$

The Leapfrog method is thus a centred second order accurate method.

It was found that the time splitting phenomenon (Roache, 1972) appeared if all steps were leapfrog. This was cured by taking a forward time step about every 40 steps. The viscous terms were retarded by one time step in order to avoid an instability of the Richardson type.

The size of the time step Δt was constrained by several effects: convection, diffusion, and inertial effects. These constraints are roughly

$$[B.23] \quad \Delta t < \Delta r / V_c \approx 10^{-12} \text{ sec} \quad (\text{Courant condition})$$

$$[B.24] \quad \Delta t < (\Delta r)^2 / \nu_c \approx 1 \text{ sec}$$

$$[B.25] \quad \Delta t < 2\pi / \Omega \approx 10^{-2} \text{ sec}$$

The most restrictive of these constraints on the step size arises from the Coriolis term in the fluid equations. The Δt chosen for these calculations was 10^{-3} sec.

APPENDIX C: NUMERICAL METHODS FOR MAGNETOSTROPHIC FLOW

The magnetostrophic equations were replaced by an approximate set of finite difference equations, which were solved on an IBM 370 computer. All variables were defined on a single uniform grid (z_j, x_i) in cylindrical coordinates:

$$[C.1] \quad x_i = (i - 1)\Delta x$$

$$[C.2] \quad z_j = (j - 1)\Delta z,$$

for $i, j = 1, 2, \dots, N$, where

$$[C.3] \quad \Delta x = \Delta z = R_c / (N - 1).$$

For the calculations reported in this thesis, N was 15.

Although the choice of cylindrical coordinates rather than spherical coordinates makes the equations considerably simpler, it complicates the boundary conditions. The representation of the inner surface of the crust in the (z_j, x_i) mesh is one with many sharp edges.

A single cylindrical grid (z_j, x_i) was chosen in preference to a set of staggered cylindrical grids analogous to the staggered spherical grids used in the inertial flow calculations (see Appendix B) because the computer "bookkeeping" involved in representing the essentially spherical crust boundary in a number of staggered cylindrical grids is extremely complicated. In fact, a preliminary version of the computer program which used such staggered grids could not be made to work for this reason, and was abandoned in favour of the present program.

One consequence of the use of a single grid is that the spatial truncation errors are about four times larger than those in the inertial flow calculations. In an effort to determine what influence these truncation errors had on the

computer generated solution, a simulation was performed with $N = 20$ but with the same time step size as in the $N = 15$ calculations. Because the computer cost per time step in the former simulation is about twice that of the latter, the $N = 20$ simulation was terminated after only a few days of simulated time. Within that simulated time interval, the $N = 20$ simulation gave practically the same results as the $N = 15$ one. Thus, neither truncation error nor the irregular representation of the crust boundary is a serious problem in the $N = 15$ calculations.

In this grid, spatial derivatives were replaced by central differences. For example,

$$[C.4] \quad (\vec{\nabla} \times \vec{B}')_{\phi} \rightarrow \frac{[(B'_x)_{i,j+1} - (B'_x)_{i,j-1} - (B'_z)_{i+1,j} + (B'_z)_{i-1,j}]}{2\Delta z}.$$

Also, the spatial differencing of the right-hand side of the induction equation

$$[C.5] \quad \frac{\partial \vec{B}'}{\partial t} = \vec{\nabla} \times (\vec{v}_c \times \vec{B})$$

was designed so that $\vec{\nabla} \cdot \vec{B}$ was numerically locally conserved in time. It is a well-known numerical phenomenon (Roache, 1972) that although the conservation of a quantity such as $\vec{\nabla} \cdot \vec{B}$ is guaranteed by a set of partial differential equations such as [C.5], not all differencing techniques lead to the conservation of the quantity in a numerical simulation.

For example, if central differencing had been applied to the right-hand side of the expanded form

$$[C.6] \quad \frac{\partial \vec{B}'}{\partial t} = \vec{B} \cdot \vec{\nabla} \vec{v}_c - (\vec{v}_c \cdot \vec{\nabla}) \vec{B} - \vec{B} (\vec{\nabla} \cdot \vec{v}_c)$$

of the induction equation, the difference version of $\vec{\nabla} \cdot \vec{B}$ would definitely not have been conserved as the simulation proceeded. Spurious magnetic monopoles would have appeared in the solution.

This problem was avoided by calculating $\vec{v}_c \times \vec{B}$ at all grid points, and then applying central differencing to the form [C.5] of the induction equation. This method is thus numerically flux conserving.

Time derivatives were approximated by Leapfrog time differencing (see Appendix B). Thus, if we define

$$[C.7] \quad \vec{\mathcal{E}} \equiv \vec{v}_c \times \vec{B} ,$$

the difference form of [C.5] is

$$[C.8] \quad (B'_z)_{i,j}^{k+1} = (B'_z)_{i,j}^{k-1} + \frac{2\Delta t}{x_i 2\Delta x} [(x\mathcal{E}_\phi)_{i+1,j}^k - (x\mathcal{E}_\phi)_{i-1,j}^k]$$

$$[C.9] \quad (B'_x)_{i,j}^{k+1} = (B'_x)_{i,j}^{k-1} - \frac{2\Delta t}{2\Delta z} [(\mathcal{E}_\phi)_{i,j+1}^k - (\mathcal{E}_\phi)_{i,j-1}^k]$$

$$[C.10] \quad (B'_\phi)_{i,j}^{k+1} = (B'_\phi)_{i,j}^{k-1} + \frac{2\Delta t}{2\Delta z} [(\mathcal{E}_x)_{i,j+1}^k - (\mathcal{E}_x)_{i,j-1}^k - (\mathcal{E}_z)_{i+1,j}^k + (\mathcal{E}_z)_{i-1,j}^k] .$$

A Von Neumann stability analysis (Roache, 1972) of this differencing method was carried out. It indicates that the method is numerically stable if Δt is chosen to be less than $1/\sqrt{2}$ times the period of a hydromagnetic-inertial wave of wavelength Δz . In the present calculations the numerical factor was taken to be $1/20$, rather than $1/\sqrt{2}$, to be conservative. The stability analysis also indicates that the method has zero numerical damping, a desirable feature that is often associated with Leapfrog time differencing.

A total of 50,000 time steps were taken in the calculation. There was no indication whatsoever of numerical instability even after this very large number of steps, in agreement with Von Neuman stability analysis. I conclude that the numerical method is a sound one.