SOME CONSIDERATIONS CONCERNING NEWTONIAN CHARTS

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A B S T R A C T

In Part I, spherically symmetric solutions of Rastall's 1971 gravitational field equations for empty space-time are examined. One static solution is found to be just a static spherically symmetric Newtonian metric; i.e., the metric of Rastall's 1968 scalar theory of gravity. However, there are other solutions which satisfy the same boundary conditions at spatial infinity. It is observed that the time-like vector field n^{μ} appearing in the field equations is not uniquely defined when the metric is assumed to be spherically symmetric. Part I concludes with a discussion of the effects of this ambiguity upon the solutions of the field equations.

Part II is a discussion of an alternative procedure for generalizing Rastall's 1968 theory of gravity. The new, *generalized Newtonian metric* is assumed to satisfy the linearized vacuum field equations of General Relativity in the weak-field limit. The quantities from which generalized Newtonian metrics are constructed are then found to exhibit wave-like behavior.

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A C K N O W L E D G M E N T S

I wish to thank Professor P. Rastall for suggesting the topic of this thesis and for his patient supervision of my research.

I N T R O D U C T I O N

A physical theory is said to admit a class of preferred charts (or "co-ordinate systems," the differential geometric definition of which appears in Section I (a)) if the members of the class can be distinguished from charts not in the class by a well-defined physical procedure. For example, the Special Theory of Relativity admits the class of "inertial charts" as preferred charts. It has been suggested (c.f. Einstein, 1956) that the notion of preferred charts leads to philosophical ambiguities.

The General Theory of Relativity, as usually formulated, is without preferred charts. This lack, while perhaps a philosophical strength, often presents technical difficulties and hinders our understanding of many aspects of the theory (for example, the difficulty in defining local gravitational energy stems in large measure from the lack of preferred charts). Some gravitational theorists have attempted to surmount these difficulties by introducing "co-ordinate conditions," a procedure by which most of General Relativity remains intact, but in which preferred charts are introduced. Examples of this are Fock's "harmonic co-ordinates" (Fock, 1959), and the "canonical co-ordinates" of Arnowitt, Deser and Misner (Arnowitt, Deser, Misner, 1962).

However, if one is willing to put aside the philosophical arguments against preferred charts, then it seems that one should look more carefully at theories in which preferred charts enter on a less ad-hoc level than they do in the theories mentioned above. Examples of such theories are the numerous Lorentz covariant theories of gravitation (for a recent example, see Coleman, 1971) and Rastall's scalar theory of gravitation (Rastall, 1968). However, these attempts, to date, have not been as successful as General Relativity in satisfying certain experimental or mathematical criteria.

In this thesis, I shall examine two possible generalizations of Rastall's 1968 scalar theory. One of these generalizations is a generally covariant theory (Rastall, 1971) in which the preferred charts, called *Newtonian charts*, of the scalar theory are in general not present. In Part I, I will examine spherically symmetric solutions of the field equations of the 1971 theory and show that it is conceivable that, in this case, Newtonian charts reappear, in a sense, as preferred charts. To establish this rigorously, however, entails proving an analogue in Rastall's 1971 theory, of Birkhoff's theorem in General Relativity, which states that all spherically symmetric solutions of the vacuum field equations differ from the Schwarzschild solution only by a co-ordinate transformation (Bonner, 1962). In Rastall's theory, the establishment of this theorem depends upon overcoming two problems, neither of which I was able to solve:

(i) what is the nature of the boundary conditions on the field equations?

(ii) do all the mathematical objects appearing in the field equations have unambiguous definitions?

The second possible generalization of the scalar theory involves generalizations of the Newtonian charts. At this writing, these sorts of considerations (Rastall, unpublished notes, 1972) have not attained the status of a full theory of gravitation. However, the "generalized Newtonian charts" should constitute a class of preferred charts in some new theory in a manner roughly analogous to the way Newtonian charts are preferred charts in the scalar theory. In Part II, I will define Generalized Newtonian charts, and show that the metric tensor in a space-time admitting these charts as preferred charts constitutes a rather interesting solution of the linearized vacuum equations of General Relativity.

I. (a) Newtonian Metrics and A Scalar Theory of Gravitation

We assume, as in General Relativity, that space-time, M, is a smooth, pseudo-Riemannian manifold of dimension 4 and metric signature +2. We depart from General Relativity, however, in assuming that space-time is also endowed with preferred charts, called *Newtonian charts:* at each point in the manifold there exists a chart $(u,\chi)^*$ in which the components of the metric have the form

$$
\begin{aligned}
g_{ab} &= e^{-2\phi}\delta_{ab} \\
g_{o\mu} &= g_{\mu o} = -e^{2\phi}\delta_{o\mu}
\end{aligned}
\tag{1}
$$

where ϕ: M → R is smooth. Our convention on the range of indices is that lower-case Latin indices have range 1, 2, 3, while lower-case Greek indices have range 0, 1, 2, 3. The summation convention applies to repeated lower-case indices.

We can promote these considerations to a theory of gravitation by (i) identifying the orbits of test-particles and photons with the geodesics of the manifold (more properly: the geodesics of the metric connection on M); and (ii) by postulating field equations. In particular, we choose (Rastall, 1968)

$$
\nabla^2\phi = \frac{4\pi G_E}{C_E^4}(\varepsilon + \varepsilon_G) \tag{2}
$$

where ∇^2 is the ordinary spatial Laplacian operator, G_E and C_E

---

*(u,χ) is a chart at p in M if and only if u is an open set containing p and χ: u → R^4 is a homeomorphism.

are the classical gravitational constant and speed of light, respectively; ε is the energy-density of non-gravitational matter and fields, and ε_G is the energy-density of the gravitational field. For weak fields ($\varepsilon_G \approx 0$) and slow speeds, equation (2) is just Poisson's equation and ϕC_E^2 is the classical gravitational potential. In vacuum, when $\varepsilon = 0$, it is postulated (Rastall, 1968) that

$$\nabla^2\phi = \tfrac{1}{2}\nabla\phi\cdot\Delta\phi \qquad (3)$$

Newtonian charts differ from the inertial charts of Special Relativity in a very important aspect; namely Newtonian charts, in most cases, define a state of absolute rest. Thus a particle at rest in one Newtonian chart, is at rest in all Newtonian charts whose domains include the location of the particle (Rastall, 1968, Appendix). So when one wishes to calculate the gravitational field produced by a given source, one must specify the motion of the source with respect to the local Newtonian chart. This leads to difficulties in at least one important problem -- the calculation of the paths of planets in the spherically symmetric field of the sun. In particular, the following situation occurs (Rastall, 1968, 1969):

> (i) Assume the sun is at rest in a local Newtonian chart. Then the predictions of the theory are in reasonable agreement with experiment (perhelion advance is 92% of that predicted in General Relativity).
>
> (ii) Assume the sun has some non-zero speed in a local Newtonian chart. Then

> the paths of planets calculated do not agree with observation except for implausibly small speeds.

Since there is no reason for assuming that the sun is at rest in the local Newtonian chart, it is clear that the scalar theory of gravity is incomplete. As a theory of static gravitation, it is acceptable, but in non-static cases one needs a more generaly theory.

In principle, at least, a solution to this difficulty consists in postulating a set of generally covariant field equations which have the property that they reduce to equation (2) in the case of gravitostatics. Rastall proposed such a theory (Rastall, 1971), the field equation of which are

$$Q_{\mu[\nu;\pi]} + n_\mu n_{[\nu} R_{,\pi]} = S_{\mu[\nu\pi]} \tag{4}$$

The symbols appearing here have the following definition:

$[\mu,\nu]$ denotes anti-symmetrization in μ,ν;

$R_{\mu\nu\pi\rho}$ are the components of the Riemann tensor in a given chart.

In particular, if in this chart the metric has components $g_{\mu\nu}$, then

$$R_{\mu\nu\pi\rho} = g_{\mu[\rho,\pi]\nu} - g_{\nu[\rho,\pi]\mu} + g^{\alpha\beta}[\Gamma_{\mu\rho,\alpha}\Gamma_{\nu\pi,\beta} - \Gamma_{\mu\pi,\alpha}\Gamma_{\nu\rho,\beta}]$$

where the $\Gamma_{\mu\nu,\pi} = g_{\pi\rho}\Gamma^\rho_{\mu\nu}$ and $\Gamma^\rho_{\mu\nu}$ are the components of the metric connection; $R_{\mu\nu}$ are components of the Ricci-tensor, $R_{\mu\nu} = g^{\pi\rho}R_{\pi\mu\nu\rho}$; R is the curvature scalar, $R = g^{\mu\nu}R_{\mu\nu}$; n^μ is a time-like vector field which we will discuss in detail below;

$$Q_{\mu\nu} = R_{\mu\nu} + n^{\pi}n^{\rho} R_{\pi\mu\nu\rho};$$

$S_{\mu[\nu\pi]}$ is a tensor giving the distribution of sources. For an ideal fluid, Randall chose

$$S_{\mu[\nu\pi]} = 2T_{\mu[\nu;\pi]} + (\varepsilon_E - p_E)_{,[\nu}g_{\pi]\mu} + 2(\varepsilon_E + p_E)_{,[\nu}n_{\pi]}n_{\mu} \tag{5}$$

where $T_{\mu\nu}$ is the stress-energy-momentum tensor of an ideal fluid, ε_E is the (non-gravitational) energy density, and p_E the pressure density of the fluid. In general, $S_{\mu[\nu\pi]}$ may be constructed from (5) by defining ε_E and p_E by

$$\varepsilon_E = -n^{\mu}n^{\nu}T_{\mu\nu}$$

$$p_E = -\frac{1}{3}(g_{\mu\nu} + n_{\mu}n_{\nu})T^{\mu\nu}$$

Now if we write equation (4) and (5) in a static Newtonian chart (the metric has the form given by equation (1) with ϕ independent of time), and define

$$n^{\mu} = \delta_{\mu o}e^{-\phi} \tag{6}$$

then equation (4) reduces to equation (3). So we seek a geometric definition of the time-like vector field n^{μ} which reduces to equation (6) in the case of a static Newtonian chart. Such a definition is provided by the following consideration:

Consider an orthonormal tetrad $(W_\alpha) = (W_o, W_1, W_2, W_3)$. The *tetrad* or *physical* components of the Riemann tensor with respect to (W_α) are

$$R_{\alpha\beta\gamma\delta} \equiv W^\mu_\alpha \; W^\nu_\beta \; W^\pi_\gamma \; W^\rho_\delta \; R_{\mu\nu\pi\rho} \quad .$$

One can show (Landau and Lifshitz, pp. 305-306) that the Riemann tensor is uniquely determined by a pair of complex 3 x 3 matrices, S and H, defined by

$$S_{ab} = \tfrac{1}{4}[-\hat{R}_{(a)(b)} + \hat{R}_{(a+3)(b+3)} - i(\hat{R}_{(a)(b+3)} + \hat{R}_{(a+3)(b)})]$$

$$H_{ab} = \tfrac{1}{4}[\hat{R}_{(a)(b)} + \hat{R}_{(a+3)(b+3)} + i(-\hat{R}_{(a)(b+3)} + \hat{R}_{(a+3)(b)})] \qquad (7)$$

where Latin indices enclosed by parentheses denote pairs of indices with the convention: 12=(3), 23=(1), 31=(2), 10=(4), 20=(5), 30=(6). If S is non-degenerate and of Petrov type I (i.e., can be diagonalized), then there exists a unique orthonormal tetrad (W_α), called the *Principal tetrad*, such that S is diagonal. Rastall defines the time-like vector field n^μ in equation (4) to be the time-like vector field of the principal tetrad; i.e., $n^\mu \equiv W_o$. Now if we construct the Riemann tensor from the metric (1), then we find that, except for certain trivial functions ϕ, the corresponding matrix S is non-degenerate and of Petrov type I, so we are able to calculate the principal tetrad (W_α) and in particular, we find that

$$n^{\mu} = W_{o} = \delta_{\mu o} e^{-\phi}$$

as desired. In general, n^{μ} will be a function of the metric and its first and second derivatives.

In many cases of physical interest the matrix S and hence n^{μ} may not be uniquely defined. In such cases, one hopes that the physics will not be affected by the choice of any particular mathematically permissible n^{μ}.

I. (b) The Spherically Symmetric Field Equations

Let us consider the case of the metric having the form

$$\begin{aligned} g_{11} &= e^{\alpha} \\ g_{22} &= r^{2} \\ g_{33} &= r^{2}\sin^{2}\theta \\ g_{oo} &= -e^{\gamma} \\ g_{\mu\nu} &= 0 \text{ if } \mu \neq \nu \end{aligned} \tag{8}$$

with respect to so-called "curvature co-ordinates" (r, θ, Φ, t). The functions α and γ depend on r, t only. This is one form of a spherically symmetric metric. We can now write down the connection components $\Gamma^{\pi}_{\mu\nu}$ and find the only non-vanishing ones are:

$$\begin{aligned} \Gamma^{1}_{11} &= \tfrac{1}{2}\alpha' \\ \Gamma^{0}_{11} &= \tfrac{1}{2}\dot{\alpha}\, e^{\alpha-\gamma} \\ \Gamma^{2}_{12} &= \Gamma^{3}_{13} = \frac{1}{r} \end{aligned}$$

$$\Gamma_{10}^{1} = \tfrac{1}{2}\dot{\alpha}$$

$$\Gamma_{10}^{0} = \tfrac{1}{2}\gamma'$$

$$\Gamma_{23}^{3} = \cot\theta$$

$$\Gamma_{22}^{1} = -re^{-\alpha}$$

$$\Gamma_{33}^{r} = -r\sin^2\theta e^{-\alpha}$$

$$\Gamma_{33}^{2} = -\sin\theta\cos\theta$$

$$\Gamma_{00}^{1} = \tfrac{1}{2}\gamma' e^{\gamma-\alpha}$$

$$\Gamma_{00}^{0} = \tfrac{1}{2}\dot{\gamma} \tag{9}$$

where ' denotes $\partial/\partial r$ and · denotes $\partial/\partial t$.

The only non-vanishing components of the Riemann tensor are:

$$\begin{aligned} R_{1212} &= \tfrac{1}{2}r\alpha' \\ R_{1313} &= \sin^2\theta\, R_{1212} \\ R_{2323} &= r^2\sin^2\theta(1-e^{-\alpha}) \\ R_{1220} &= -\tfrac{1}{2}r\dot{\alpha} \\ R_{1330} &= \sin^2\theta\, R_{1220} \\ R_{1010} &= e^{\alpha}A(\alpha,\gamma) + e^{\gamma}B(\alpha,\gamma) \\ R_{2020} &= \tfrac{1}{2}re^{\gamma-\alpha}\gamma' \\ R_{3030} &= \sin^2\theta\, R_{2020} \end{aligned} \tag{10}$$

where

$$A(\alpha,\gamma) = -\tfrac{1}{2}\ddot{\alpha} - \tfrac{1}{4}(\dot{\alpha})^2 + \tfrac{1}{4}\dot{\alpha}\dot{\gamma}$$

$$B(\alpha,\gamma) = \tfrac{1}{2}\gamma'' + \tfrac{1}{4}(\gamma')^2 - \tfrac{1}{4}\alpha'\gamma' \tag{11}$$

The Ricci tensor and the curvature scalar are

$$R_{11} = -\frac{1}{r}\alpha' + e^{\alpha-\gamma}A(\alpha,\gamma)$$

$$R_{22} = \tfrac{1}{2} re^{-\alpha}(\gamma'-\alpha') + e^{-\alpha}-1$$

$$R_{33} = \sin^2\theta\, R_{22}$$

$$R_{00} = -e^{\gamma-\alpha}[B(\alpha,\gamma)+\frac{1}{r}\gamma] - A(\alpha,\gamma)$$

$$R_{01} = R_{10} = -\frac{1}{r}\dot{\alpha} \tag{12}$$

all other $R_{\mu\nu}$ vanish.

$$R = 2[e^{-\alpha}(B(\alpha,\gamma)+\frac{1}{r}(\gamma'-\alpha') + \frac{1}{r^2}) + e^{-\gamma}A(\alpha,\gamma) - \frac{1}{r^2}] \tag{13}$$

It is easy to check that a principal tetrad of the Riemann tensor of equation (10) is (W_α), given by

$$x^\mu \equiv W_1^\mu = \delta_{\mu 1}e^{-\alpha/2}$$

$$y^\mu \equiv W_2^\mu = \frac{1}{r}\delta_{\mu 2}$$

$$z^\mu \equiv W_3^\mu = \frac{1}{r\sin\theta}\delta_{\mu 3}$$

$$n^\mu \equiv W_o^\mu = \delta_{\mu o}\, e^{-\gamma/2} \tag{14}$$

If we now construct the physical components of the Riemann tensor with respect to (W_α) and thence construct the matrix S, we find that S is diagonal but degenerate

$$S = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_2 \end{bmatrix} \tag{15}$$

So the tetrad of equation (4) is not a unique principal tetrad for the system. Nevertheless, for the present we shall assume that the vector field n^μ is given by $n^\mu = \delta_{\mu o} e^{-\gamma/2}$.

Finally, we construct the tensor $Q_{\mu\nu}$ and find

$$\begin{aligned} Q_{11} &= -\frac{1}{r}\alpha' \\ Q_{22} &= -\tfrac{1}{2} r e^{-\alpha}\alpha' + e^{-\alpha} - 1 \\ Q_{33} &= \sin^2\theta\, Q_{22} \\ Q_{00} &= -e^{\gamma-\alpha} B(\alpha,\gamma) - A(\alpha,\gamma) \\ Q_{01} &= Q_{10} = -\frac{1}{r}\dot{\alpha} \end{aligned} \tag{16}$$

all other $Q_{\mu\nu}$ vanish.

Now we can write out explicitly the left-hand sides of the field equation (4). The only components which do not identically vanish are:

$$Q_{1[1;0]} + n_1 n_{[1} R_{,0]} = -\tfrac{1}{4}\dot{\alpha}\left(\frac{2}{r^2} + B(\alpha,\gamma) + e^{\alpha-\gamma} A(\alpha,\gamma)\right) \tag{17}$$

$$\begin{aligned} Q_{0[0;1]} + n_0 n_{[0} R_{,1]} = {} & e^{\gamma-\alpha}\Big[\tfrac{1}{2}B'(\alpha,\gamma) - \tfrac{1}{4}\gamma' B(\alpha,\gamma) - \tfrac{1}{2}\alpha' B(\alpha,\gamma) \\ & + \tfrac{1}{2}\frac{1}{r}\gamma'' - \tfrac{1}{4}\frac{1}{r}(\gamma')^2 - \frac{3}{4}\frac{1}{r}\alpha'\gamma' - \tfrac{1}{2}\frac{1}{r^2}\gamma' - \frac{\alpha''}{r} + \left(\frac{\alpha'}{r}\right)^2 - \frac{2}{r^3}\Big] \\ & + \tfrac{1}{2}A'(\alpha,\gamma) - \frac{3}{4}\gamma' A(\alpha,\gamma) - \tfrac{1}{4}\frac{1}{r}(\dot{\alpha}-\dot{\gamma})\dot{\alpha} + \frac{2}{r^3}e^{\gamma} \end{aligned} \tag{18}$$

$$Q_{2[2;1]} + n_2 n_{[2} R_{,1]} = \tfrac{1}{2}\left[e^{-\alpha}\left(-\tfrac{1}{2} r\alpha'' + \tfrac{1}{2} r(\alpha')^2 - \frac{1}{r}\right) + \frac{1}{r}\right] \tag{19}$$

$$Q_{2[2;0]} + n_2 n_{[2} R_{,0]} = \tfrac{1}{4}\, r e^{-\alpha}(-\dot{\alpha}' + \dot{\alpha}\alpha') \tag{20}$$

In the next section, we will find solutions for the vacuum field equation,

$$Q_{\mu[\nu;\pi]} + n_\mu n_{[\nu}R,_{\pi]} = 0$$

In particular we will solve for α and γ in the following four equations, which are merely equations (17) - (20) with the right-hand sides set to zero (and common, constant, non-zero factors divided out):

$$\dot{\alpha}\left(\frac{2}{r^2} + B(\alpha,\gamma) + e^{\alpha-\gamma}\dot{A}(\alpha,\gamma)\right) = 0 \tag{21}$$

$$e^{-\alpha}\left[\tfrac{1}{2}B'(\alpha,\gamma) - \tfrac{1}{4}\gamma'B(\alpha,\gamma) - \tfrac{1}{2}\alpha'B(\alpha,\gamma) + \tfrac{1}{2}\frac{1}{r}\gamma'' - \tfrac{1}{4}\frac{1}{r}(\gamma')^2 - \frac{3}{4}\frac{1}{r}\alpha'\gamma' - \tfrac{1}{2}\frac{1}{r^2}\gamma' - \frac{1}{r}\alpha'' + \frac{1}{r}(\alpha')^2 - \frac{2}{r^3}\right] + \frac{2}{r^3}$$
$$+ e^{-\gamma}\left[\tfrac{1}{2}A'(\alpha,\gamma) - \frac{3}{4}\gamma'A(\alpha,\gamma) - \tfrac{1}{4}\frac{1}{r}(\dot{\alpha}-\dot{\gamma})\dot{\alpha}\right] = 0 \tag{22}$$

$$e^{-\alpha}\left(-\tfrac{1}{2}r\alpha'' + \tfrac{1}{2}r(\alpha')^2 - \frac{1}{r}\right) + \frac{1}{r} = 0 \tag{23}$$

$$e^{-\alpha}(-\alpha' + \dot{\alpha}\alpha') = 0 \tag{24}$$

I. (c) The Solutions of the Field Equations

Consider equation (24). It is equivalent to

$$\frac{\partial}{\partial r\partial t}(e^{-\alpha}) = 0$$

The general solution of this is well known to be

$$e^{-\alpha} = f(r) + g(t) \tag{25}$$

where f and g are at least c^2 in their respective arguments.

Now we may re-write equation (23) as:

$$\tfrac{1}{2}r(e^{-\alpha})'' - \frac{1}{r}(e^{-\alpha}) + \frac{1}{r} = 0$$

or

$$r^2(e^{-\alpha})'' - 2(e^{-\alpha}) = -2 \tag{26}$$

So for equations (23) and (24) to be consistent, we must have [substitute (25) into (26)]

$$r^2 f''(r) - 2f(r) = -2(1 - g(t)) \tag{27}$$

But this is nonsense unless g(t) is a constant, say $g(t) = g_o$. Equation (27) is an inhomogeneous Cauchy differential equation and the general solution is:

$$f(r) = \frac{c_1}{r} + c_2 r^2 + 1 - g_o \tag{28}$$

So we finally have:

$$e^{-\alpha} = \frac{c_1}{r} + c_2 r^2 + 1 - g_0 + g_0$$

$$= \frac{c_1}{r} + c_2 r^2 + 1 \tag{29}$$

We shall now impose the boundary condition that the metric by asymptotically flat; i.e., that $e^{\gamma} \to 1$, $e^{\alpha} \to 1$ as $r \to \infty$. This implies that we must choose the constant of integration, $c_2 = 0$. Thus, $\quad e^{\alpha} = (1 + \frac{c_1}{r})^{-1}$

or

$$\alpha(r) = -\ln(1 + \frac{c_1}{r}) \tag{30}$$

Now note that $\dot{\alpha} = 0$ is a solution of equation (21), so we are left with equation (22), which becomes:

$$e^{-\alpha}[\tfrac{1}{2}B'(\alpha,\gamma) - \tfrac{1}{4}\gamma' B(\alpha,\gamma) - \tfrac{1}{2}\alpha' B(\alpha,\gamma) + \tfrac{1}{2}\frac{1}{r}\gamma'' - \tfrac{1}{4}\frac{1}{r}(\gamma')^2$$

$$- \frac{3}{4}\,\frac{1}{r}\alpha'\gamma' - \tfrac{1}{2}\frac{1}{r^2}\gamma' - \frac{1}{r}\alpha'' + \frac{1}{r}(\alpha')^2 - \frac{2}{r^3}] + \frac{2}{r^3} = 0$$

This can be further simplified by noticing that the last four terms in the above equation are identically zero by equation (23). Now use equation (11) and simplify to get:

$$\gamma''' - \tfrac{1}{4}(\gamma')^3 + \tfrac{1}{2}\gamma'\gamma'' + (-\frac{3}{2}\alpha' + \frac{2}{r})\gamma'' - (\tfrac{1}{4}\alpha' + \frac{1}{r})(\gamma')^2$$

$$+ (-\tfrac{1}{2}\alpha'' + \tfrac{1}{2}(\alpha')^2 - \frac{3}{2}\alpha' - \frac{2}{r^2})\gamma' = 0 \qquad (31)$$

It would be very difficult to solve (31) directly. But one solution can be found by the following trick, and in the next section I shall prove that this solution (for which γ is a function of r alone) is the only analytic solution given the appropriate boundary conditions.

The metric (8) is only one example of a spherically symmetric metric. By transforming from the "curvature co-ordinates" (r, θ, ψ, t) to "isotropic co-ordinates" $(\rho, \bar{\theta}, \bar{\psi}, \bar{t})$ we obtain another form for the spherically symmetric metric. The transformation from (r, θ, ψ, t) to $(\rho, \bar{\theta}, \bar{\psi}, \bar{t})$ is given by (Landau and Lifschitz, p. 331):

$$r = (1 - \frac{c_1}{4\rho})^2\rho \qquad \psi = \bar{\psi}$$

$$\theta = \bar{\theta} \qquad t = \bar{t} \qquad (32)$$

The metric $\bar{g}_{\mu\nu}$ in isotropic co-ordinates has the form:

$$\begin{aligned} \bar{g}_{11} &= e^{\beta} \\ \bar{g}_{22} &= e^{\beta}\rho^{2} \\ \bar{g}_{33} &= e^{\beta}\rho^{2}\sin^{2}\theta \\ \bar{g}_{00} &= -e^{\delta} \end{aligned} \tag{33}$$

where β and δ are functions of ρ and t only. In particular

$$\beta(\rho) = 4\ \ln(1 - \frac{c_1}{4\rho}) \tag{34}$$

Now notice that if we demand $\beta = -\delta \equiv -2\phi$, ϕ independent of time, then (33) looks like the metric in a spherically symmetric Newtonian chart. In the case we know that the field equations reduce to just

$$\nabla^2\phi = \tfrac{1}{2}\nabla\phi \cdot \nabla\phi$$

Rastall has already found a spherically symmetric solution for this equation (Rastall 1968), and, in the appropriate notation it is just equation (34). So we suspect that

$$\gamma(r) = \delta(\rho(r)) = -\beta(\rho(r)) = -\ 4\ \ln(1 - \frac{c_1}{4\rho(r)}) \tag{35}$$

is a solution of equation (31). Transforming back to curvature co-ordinates, we obtain

$$\gamma(r) = 4\ \ln[\tfrac{1}{2}(1 \pm \sqrt{1 + \frac{c_1}{r}})]$$

and if we demand $e^{\gamma} \to 1$ as $r \to \infty$, we have

$$\gamma(r) = 4\ \ln[\tfrac{1}{2}(1 + \sqrt{1 + \frac{c_1}{r}})] \tag{36}$$

Indeed, when we put (36) and (30) in (31), we see that we have found a solution.

Unfortunately, there may be at least two more solutions of the field equations, namely

$$\alpha = -\ln(1 + \frac{c_1}{r})$$
$$\gamma = \text{constant} \tag{37}$$

and

$$\alpha = 0$$
$$\gamma = \gamma(r) \tag{38}$$

a solution of (31) with $\alpha = 0$, if such a solution exists.

It is obvious that neither solution agrees with experiment (e.g., (38) gives no perhelion advance). Presumably, to obtain a unique solution of the field equations, we would have to impose other boundary conditions. Since asymptotic flatness requires the vanishing at spatial infinity of the partial derivatives of all orders of the metric, these yet-to-be specified boundary conditions would relate in some way to the sources of the gravitational field.

I. (d) Uniqueness

The general uniqueness theorems for solutions of differential equations (Birkhoff and Rota, 1962) are very difficult to apply in equation (31). However, it can be shown by a straightforward procedure that if we demand $\gamma(r) \to 0$ as $r \to \infty$, then the non-trivial solution of (31) near infinity is unique to analyticity, i.e., there is only one analytic non-zero solution near infinity.

In equation (31) set $\alpha = -\ln(1+\frac{c_1}{r})$ and change the independent variable to $x = \frac{c_1}{r}$ to obtain

$$\gamma''' - \tfrac{1}{4}(\gamma')^3 + \tfrac{1}{2}\gamma'\gamma'' + \frac{(11x+8)}{2x(1+x)}\gamma'' + \frac{(9x+8)}{4x(1+x)}(\gamma')^2 + \frac{1}{x(1+x)}\gamma' = 0 \tag{39}$$

This can be written as a second order differential equation by writing

$$f(x) \equiv \gamma'(x) \tag{40}$$

Note that f(x) is analytic at x = 0.

$$f'' + \tfrac{1}{2}(f + \frac{(11x+8)}{x(1+x)})f' + (-\tfrac{1}{4}f^3 + \frac{(9x+8)}{4x(1+x)}f^2 + \frac{1}{x(1+x)}f) = 0 \tag{41}$$

let us examine analytic solutions of (41) in the neighbourhood of x = 0; i.e., set

$$f(x) = \sum_{n=0}^{\infty} a_n X^n \tag{42}$$

Using the result that

$$(\sum_{n=0}^{\infty} a_n X^n)(\sum_{n=0}^{\infty} b_n x^n) = \sum_{n=0}^{\infty} c_n X^n$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

(See Fulks, p. 398) for x in the intersection of the circles of convergence of $\sum_{n=0}^{\infty} a_n X^n$ and $\sum_{n=0}^{\infty} b_n X^n$, we obtain

$$f^2(x) = \sum_{n=0}^{\infty} b_n X^n; \; b_n = \sum_{k=0}^{n} a_k a_{n-k} \tag{43.a}$$

$$f^3(x) = \sum_{n=0}^{\infty} c_n X^n; \; c_n = \sum_{k=0}^{n} a_k b_{n-k} \tag{43.b}$$

$$f'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} X^n \tag{43.c}$$

$$f(x)f'(x) = \sum_{n=0}^{\infty} d_n X^n; \; d_n = \sum_{k=0}^{n} a_k (n-k+1) a_{n-k+1} \tag{43.d}$$

$$f''(x) = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} X^n \tag{43.e}$$

Substitute this into equation (41), simplify, and group like powers of x to get:

$$\sum_{n=0}^{\infty} \{(n+1)(n+2)a_{n+2} + \frac{d_n}{2} + \frac{11}{2} e_n + 4g_n - \frac{c_n}{4} + \frac{9}{4} h_n + 2j_n + k_n\} X^n + (4a_1 + 2b_0 + a_0) \frac{1}{x(1+x)} = 0 \tag{44}$$

where

$$\begin{aligned} e_n &= \sum_{k=0}^{n} r_k (n-k+1) a_{n-k+1} \\ g_n &= \sum_{k=0}^{n} r_k (n-k+2) a_{n-k+2} \\ h_n &= \sum_{k=0}^{n} r_k b_{n-k} \\ j_n &= \sum_{k=0}^{n} r_k b_{n-k+1} \\ k_n &= \sum_{k=0}^{n} r_k a_{n-k+1} \end{aligned} \tag{45}$$

and

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} r_n X^n = 1 - x + \ldots$$

So we see that $\sum_{n=0}^{\infty} a_n X^n$ is a solution if

$$4a_1 + 2b_0 + a_0 = 0 \tag{46}$$

and

$$(n+1)(n+2)a_{n+2} + \frac{d_n}{2} + \frac{11}{2}e_n + 4g_n - \frac{c_n}{4} + \frac{9}{4}h_n + 2j_n + k_n = 0 \tag{47}$$

I claim that (46) and (47) imply that given a_o(i.e.,given $f(0) = \gamma'(0)$) the coefficients a_i, $i > 0$ are unique. Thus there is only one analytic solution, given f(0).

Proof:

From (43.a) and (46), we have

$$a_1 = -a_0(2a_0+1)/4 \tag{48}$$

Thus given a_0, a_1 is unique. Now use equations (43) and (45) in (47) to get

$$[(n+1)(n+2) + 4r_o(n+2)]a_{n+2} = (\text{terms in } a_m,\ m<n+2) \tag{49}$$

i.e., (49) is *linear* in a_{n+2}. This proves the uniqueness of a_n, given a_o.

Does this imply that $\gamma(x)$ is unique near $x = 0$? Suppose $\beta(x)$ is also a solution near $x = 0$ and that $\beta'(0) = \gamma'(0)$. Then $\beta'(x) = \gamma'(x)$ near $x = 0$, or $\beta(x) = \gamma(x) + c(t)$. But if we demand that the metric be asymptotically flat, then we also have

$\beta(0) = \gamma(0) = 0$, which implies $c(t) = 0$. Thus, in terms of r, $\gamma(r)$ given by (36) is the only analytic solution of (31) if the metric is assumed to be asymptotically flat. Note that both α and γ are time-independent.

I. (e) Uniqueness of the Vector Field n^μ

Instead of the orthonormal tetrad defined by equation (14), consider a new tetrad defined by

$$\bar{W}_\alpha = L^\beta{}_\alpha W_\beta \tag{50}$$

where $L^\mu{}_\nu$ is a 4 x 4 real matrix which has the form of a Lorentz transformation in the (r,t)-plane, in particular, at a given point in space-time

$$(L^\alpha{}_\beta) = \begin{bmatrix} \sigma & 0 & 0 & -\beta\sigma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\sigma & 0 & 0 & \sigma \end{bmatrix} \tag{51}$$

where

$$\sigma = (1-\beta^2)^{-\frac{1}{2}}$$

Now form the physical components of $R_{\mu\nu\pi\rho}$ with respect to $(\bar{W}_\alpha)$:

$$\bar{\hat{R}}_{\alpha\beta\gamma\delta} = \bar{W}^\mu_\alpha \bar{W}^\nu_\beta \bar{W}^\pi_\gamma \bar{W}^\rho_\delta R_{\mu\nu\pi\rho}$$

If we now construct the 3 x 3 matrix $\bar{S}$ [equation (7)] from $\bar{\hat{R}}_{\alpha\beta\gamma\delta}$, we find that $\bar{S}=S$, where S is constructed from $\hat{R}_{\alpha\beta\alpha\delta}$ [equation (6)]. Thus all orthonormal tetrads of the form

(50) are also principal tetrads. In particular, our spherically symmetric system admits an infinite number of time-like vector fields of the form

$$(\bar{n}^{\mu}) = (-\beta e^{-\gamma/2},\ 0,\ 0,\ e^{-\gamma/2})\sigma \tag{52}$$

where β is a smooth function of (r,t) and $\sigma = (1-\beta^2)^{-\frac{1}{2}}$

We have shown that for the choice of a particular $\bar{n}^{\mu}$ namely that for which $\beta=0$ in equation (52), there exists a physically acceptable solution of the field equations. If our theory is to be physically acceptable, other choices of $\bar{n}^{\mu}$ must yield no distinct physically meaningful solutions of the field equations. So far we have not been able to prove this; indeed the question is not even well-posed until we say what we mean by "physically acceptable solutions." We shall, however, give a partial answer to the question of whether there are solutions of the field equations that are independent of the choice of β (i.e., in the choice of principal tetrad).

We demand that the field equations (vacuum) hold for the following two choices of the time-like vector field n^{μ}:

$$n^{\mu} = \delta_{\mu o} e^{-\gamma/2} \tag{53}$$

$$\bar{n}^{\mu} = L^{\mu}{}_{\nu} n^{\nu} \tag{54}$$

where $L^{\mu}{}_{\nu}$ is a rotation in the rt-plane. Thus

$$Q_{\mu[\nu;\pi]} + n_{\mu} n_{[\nu} R,_{\pi]} = 0 \tag{55}$$

$$\bar{Q}_{\mu[\nu;\pi]} + \bar{n}_{\mu} \bar{n}_{[\nu} R,_{\pi]} = 0 \tag{56}$$

where

$$\bar{Q}_{\mu\nu} = R_{\mu\nu} + \bar{n}^\rho \bar{n}^\sigma R_{\rho\mu\nu\sigma}$$
$$= R_{\mu\nu} + L^\rho{}_\alpha L^\sigma{}_\beta n^\alpha n^\beta R_{\rho\mu\nu\sigma}$$
$$= R_{\mu\nu} + M^{\rho\sigma} R_{\rho\mu\nu\sigma} \tag{57}$$

and where

$$M^{\rho\sigma} \equiv L^\rho{}_\alpha L^\sigma{}_\beta n^\alpha n^\beta$$

Equation (56) becomes

$$R_{\mu[\nu;\pi]} - (M^{\rho\sigma} R_{\rho\mu\sigma[\nu})_{;\pi]} + M_{\mu[\nu} R_{,\pi]} = 0 \tag{58}$$

where

$$M_{\mu\nu} = g_{\rho\mu} g_{\sigma\nu} M^{\rho\sigma}$$

and we have used $R_{\rho\mu\nu\sigma} = -R_{\rho\mu\sigma\nu}$

But from (55),

$$R_{\mu\ \nu;\pi} = (n^\rho n^\sigma R_{\rho\mu\sigma[\nu})_{;\pi]} - n_\mu n_{[\nu} R_{,\pi]}$$

Using this in (58) and defining

$$K^{\rho\sigma} = n^\rho n^\sigma - M^{\rho\sigma} \tag{59}$$

we obtain

$$(K^{\rho\sigma} R_{\rho\mu\sigma[\nu})_{;\pi]} - K_{\mu[\nu} R_{,\pi]} = 0 \tag{60}$$

This is a set of first order linear equations in the quantities $K^{\rho\sigma}$ (first order non-linear equations in $L^\mu{}_\nu$). If the only solutions of equations (60) were $L^\mu{}_\nu = \delta^\mu{}_\nu$, then it would follow that there are no solutions of the field equations that are independent of the choice of β. In general, equations (60) are quite intractable, so we consider a special case, namely that $L^{\mu}{}_{\nu}$ is an infinitesimal Lorentz transformation, i.e.,

$$\bar{n}^{\mu} = n^{\mu} + \lambda l^{\mu}{}_{\nu} n^{\nu} \tag{61}$$

where

$$l^{\mu\nu} = -l^{\nu\mu} \text{ and } l^{o\mu} = \delta^{1\mu} l(r,t)$$

Then to first order in λ, we have

$$\begin{aligned} K^{\rho\sigma} &= -\lambda(n^{\rho} l^{\sigma}{}_{\beta} + n^{\sigma} l^{\rho}{}_{\beta}) n^{\beta} \\ &= -\lambda(\delta_{\rho o}\delta_{1\sigma} + \delta_{\sigma o}\delta_{1\rho})\varepsilon(r,t) \end{aligned} \tag{62}$$

where

$$\varepsilon(r,t) = e^{-\gamma} l(r,t).$$

Using (62) in (60) and evaluating covariant derivatives, we obtain (to first order in λ):

$$\begin{aligned} &-\lambda\{(R_{o\mu 1\nu} + R_{1\mu o\nu})\varepsilon_{,\pi} - (R_{o\mu 1\pi} + R_{1\mu o\pi})\varepsilon_{,\nu} \\ &+ \varepsilon[R_{o\mu 1\nu,\pi} + R_{1\mu o\nu,\pi} - R_{o\mu 1\pi,\nu} - R_{1\mu o\pi,\nu} \\ &- \Gamma^{\alpha}_{\mu\pi}(R_{o\alpha 1\nu} + R_{1\alpha o\nu}) + \Gamma^{\alpha}_{\mu\nu}(R_{o\alpha 1\pi} + R_{1\alpha o\pi}) \\ &+ (\delta_{1\mu}\delta_{o[\nu} + \delta_{o\mu}\delta_{1[\nu})R_{,\pi]}\, e^{\alpha+\gamma}]\} = 0 \end{aligned} \tag{63}$$

In particular, (63) contains four non-trival equations:

$$R_{1220}\varepsilon_{,1} + \varepsilon(R_{1220,1} - \Gamma^{2}_{12}R_{1220}) = 0 \tag{64}$$

$$R_{1220}\varepsilon_{,0} + \varepsilon(R_{1220,0} + \tfrac{1}{2}\Gamma^{1}_{22}R_{1010}) = 0 \tag{65}$$

$$R_{1010}\varepsilon_{,1} + \varepsilon(R_{1010,1} + (\Gamma^{0}_{01} - \Gamma^{1}_{11})R_{1010} - e^{\alpha+\gamma}R_{,1}) = 0 \tag{66}$$

$$R_{1010}\varepsilon_{,0} + \varepsilon(R_{1010,0} + (\Gamma^{1}_{01} - \Gamma^{0}_{00})R_{1010} - e^{\alpha+\gamma}R_{,0}) = 0 \tag{67}$$

Now $R_{1220} = -\frac{1}{2}r\dot{\alpha}$, and we have shown that in order for (55) to hold we must have $\dot{\alpha}=0$. Thus $R_{1220} = 0$ and (64) becomes trivial. However, (65) becomes

$$\tfrac{1}{2}\varepsilon\,\Gamma^{1}_{22}R_{1010} = 0$$

or

$$-\tfrac{1}{2}re^{-\alpha+\gamma}(\tfrac{1}{2}\gamma'' + \tfrac{1}{4}(\gamma')^2 - \tfrac{1}{4}\alpha'\gamma')\varepsilon = 0$$

or

$$(\gamma'' + \tfrac{1}{2}(\gamma')^2 - \tfrac{1}{2}\alpha'\gamma')\varepsilon = 0$$

If $(\gamma'' + \frac{1}{2}(\gamma')^2 - \frac{1}{2}\alpha'\gamma') \neq 0$, then we must have $\varepsilon=0$. This then gives us the desired result $\bar{n}^{\mu} = n^{\mu}$. It is worth verifying that $(\gamma'' + \frac{1}{2}(\gamma')^2 - \frac{1}{2}\alpha'\gamma') = 0$ is *not* consistent with any solution of the vacuum field equations (21) - (24).

So we assume $\gamma'' + \frac{1}{2}(\gamma')^2 - \frac{1}{2}\alpha'\gamma' = 0$. This implies that the quantity $\beta(\alpha,\gamma)$ appearing in equation (22) is zero. Using this and equation (23) brings (22) into the form:

$$\tfrac{1}{2}re^{-\alpha}(\gamma'' - \tfrac{1}{2}(\gamma')^2 - \tfrac{3}{2}\alpha'\gamma' - \tfrac{1}{r}\gamma') = 0$$

or

$$\tfrac{1}{2}re^{-\alpha}(\gamma'' + \tfrac{1}{2}(\gamma')^2 - \tfrac{1}{2}\alpha'\gamma' - (\gamma')^2 - \alpha'\gamma' - \tfrac{1}{r}\gamma') = 0$$

or

$$\gamma'(\gamma' + \alpha' + \frac{1}{r}) = 0$$

Clearly $(\gamma' + \alpha' + \frac{1}{r}) = 0$ is incompatible with any of the solutions discussed in section I(c). However $\gamma' = 0$ is compatible with the solution (37). But if we reject solutions (37) and (38) on physical grounds, then we have established that

$$(\gamma'' + \tfrac{1}{2}(\gamma')^2 - \tfrac{1}{2}\alpha'\gamma') \neq 0$$

so that we must have $\varepsilon=0$.

We have thus shown that infinitesimal changes of the form (63) in the definition of n^μ [equation (53)] lead to field equations with trivial solutions only. It remains a problem to examine the effect of more general changes in n^μ [equation (54)] on the solution of the resulting field equations.

II. (a) Generalized Newtonian Metrics

We shall now examine space-times that admit a class of preferred charts, called *generalized Newtonian charts*, which include Newtonian charts as a special case.

Let M be a smooth, four-dimensional manifold. A pseudo-Riemannian metric g on M is a 2-covariant, non-degenerate, symmetric, smooth tensor field on M with signature +2. Given a metric g on M, define a new metric $\bar{g}$ by

$$\bar{g} = e^{-2\phi}(g + f\, n \times n) \tag{1}$$

where f and ϕ are smooth real-valued functions on M and n is a smooth covariant vector field on M. It is easy to see that $\bar{g}$ satisfies the definition above of a pseudo-Riemannian metric on M. If (u,χ) is a chart at p in M, then the covariant components of $\bar{g}$, g, n at p with respect to (u,χ) are $\bar{g}_{\alpha\beta}$, $g_{\alpha\beta}$, n_α, while the contravariant components of these objects are, respectively $\bar{g}^{\alpha\beta}$, $g^{\alpha\beta}$, n^α. It is straightforward to show that (Rastall, unpublished notes 1972)

$$\bar{g}_{\alpha\beta} = e^{-2\phi}(g_{\alpha\beta} + f\, n_\alpha n_\beta) \tag{2}$$

$$\bar{g}^{\alpha\beta} = e^{2\phi}(g^{\alpha\beta} - f(1 + f\, j)^{-1} n^\alpha n^\beta) \tag{3}$$

where n^α defined by $n^\alpha = g^{\alpha\beta} n_\beta$ and where $g^{\alpha\beta} n_\alpha n_\beta = j$. Now let us require n to be a unit time-like vector field with respect to g, i.e

$$g^{\alpha\beta} n_\alpha n_\beta = j = -1 \tag{4}$$

Let $\{e_\mu\}$ be a basis of the tangent space to M at p, and let $\{e^\mu\}$ be the dual basis of the cotangent space (i.e., $e^\mu(e_\nu) = \delta_{\mu\nu}$). Then we can choose $\{e_\mu\}$ such that the following are true:

(i) $\{e_\mu\}$ is an orthonormal basis with respect to the metric g, i.e.,

$$g(e_\mu, e_\nu) = \eta_{\mu\nu}$$

where

$$\eta_{ab} = \delta_{ab}$$

$$\eta_{o\mu} = \eta_{\mu o} = -\delta_{\mu o}$$

(ii) $e^o = n$

It follows that

$$\bar{g}(e_\mu, e_\nu) = e^{-2\phi}[g(e_\mu, e_\nu) + f\, e^o(e_\mu)e^o(e_\nu)]$$

$$= e^{-2\phi}[\eta_{\mu\nu} + f\, \delta_{o\mu}\delta_{o\nu}]$$

In particular,

$$\bar{g}(e_\mu, e_a) = e^{-2\phi}\eta_{\mu a};\ a = 1, 2, 3$$

$$\bar{g}(e_\mu, e_o) = e^{-2\phi}(\eta_{\mu o} + f\delta_{\mu o})$$

$$= e^{-2\phi}(1-f)\eta_{\mu o}$$

Let α be in the tangent space of M at p. Define the length of α with respect to the metric g and $\bar{g}$ by

$$L_\alpha = \sqrt{|g(\alpha,\alpha)|}$$

$$\bar{L}_\alpha = \sqrt{|g(\alpha,\alpha)|}$$

Let α be any tangent parallel to $n^*(p)$ and β any tangent orthogonal to $n^*(p)$, where $n^*(p)$ is the contravariant vector field corresponding to n. Then

$$\bar{L}_\alpha = e^{-\phi}(1-f)^{\frac{1}{2}}L_\alpha$$

$$\bar{L}_\beta = e^{-\phi}L_\beta$$

If we choose

$$(1-f) \equiv e^{4\phi} \qquad (5)$$

then

$$\bar{L}_\alpha = e^{\phi}L_\alpha$$

$$\bar{L}_\alpha = e^{-\phi}L_\beta$$

Roughly speaking, $\bar{g}$ "stretches" tangents parallel to n by a factor e^{ϕ} and tangents orthogonal to n by a factor $e^{-\phi}$ in comparison to g.

If we then define j and f appearing in equations (2) and (3) by (4) and (5), then we obtain:

$$\bar{g}_{\alpha\beta} = e^{-2\phi} g_{\alpha\beta} + (e^{-2\phi} - e^{2\phi}) n_\alpha n_\beta$$
$$\bar{g}^{\alpha\beta} = e^{2\phi}\, g^{\alpha\beta} - (e^{-2\phi} - e^{2\phi}) n^\alpha n^\beta \qquad (6)$$

We can now define a generalized Newtonian metric; namely, let g be a flat metric, then $\bar{g}$, defined by equation (1), is a *generalized Newtonian metric* if n is time-like and $(1-f) = e^{4\phi}$. A chart in which $|g_{\alpha\beta}| = \delta_{\alpha\beta}$ in equation (6) is called a *generalized Newtonian chart.* Newtonian charts are a special case of generalized Newtonian charts, defined by the condition

$$n_\mu = \delta_{\mu o}$$

which implies that $\bar{g}_{\alpha\beta}$ reduce to

$$\bar{g}_{ab} = e^{-2\phi} g_{ab}$$
$$\bar{g}_{o\alpha} = -e^{2\phi} \delta_{o\alpha}$$

as in equation (1) of Part I.

II. (b) The Linearized Einstein Equations and Generalized Newtonian Metrics

The eventual goal of these considerations is the construction of a geometric theory of gravitation based upon generalized Newtonian metrics. However, in the remainder of this essay I shall undertake a less ambitious task; namely, I shall show that generalized Newtonian metrics which differ infinitesimally from flat metrics are not uninteresting solutions of the linearized vacuum field equations of General Relativity.

In equation (6), expand $e^{2\phi}$ in a parameter λ:

$$e^{2\phi} = 1 + \lambda\Phi + 0(\lambda^2) \text{ as } \lambda \to o \tag{7}$$

(So we must have $\lambda\phi \equiv 2\Phi$).

Then equations (6) have the form

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} + \lambda h_{\alpha\beta} + 0(\lambda^2)$$

$$\bar{g}^{\alpha\beta} = g^{\alpha\beta} - \lambda h^{\alpha\beta} + 0(\lambda^2) \tag{8}$$

where g is a flat metric in Minkowski co-ordinates, i.e., $g_{\alpha\beta} = \eta_{\alpha\beta}$ and where

$$h_{\alpha\beta} = -(g_{\alpha\beta} + 2n_\alpha n_\beta)\Phi \tag{9}$$

and

$$h^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\delta} h_{\gamma\delta}$$

The components (in a Minkowski chart) $\bar{R}_{\alpha\beta}$ of the Ricci tensor defined by $\bar{g}$ are then

$$\bar{R}_{\alpha\beta} = \tfrac{1}{2}\lambda \left(h_{,\alpha\beta} - h^{\pi}{}_{\alpha,\pi\beta} - h^{\pi}{}_{\beta,\alpha\pi} + \Box h_{\alpha\beta}\right) + 0(\lambda^2) \tag{10}$$

where a comma preceding an index denotes the partial derivative with respect to the co-ordinate labelled by that index and

$$h \equiv h_\alpha{}^\alpha$$

$$\Box \equiv g^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} \tag{11}$$

Also, $\bar{R}$, the curvature scalar is, in the linear approximation:

$$\bar{R} = g^{\alpha\beta}\bar{R}_{\alpha\beta} = \tfrac{1}{2}\lambda\,[\Box h - 2(h^{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta}h)_{,\alpha\beta}] + O(\lambda^2) \qquad (12)$$

As usual, when deaing with the linearized Einstein equations great simplification results if the "Hilbert gauge" is imposed, i.e., if we demand that $h^{\alpha\beta}$ satisfies

$$(h^{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta}h)_{,\alpha} = 0 \qquad (13)$$

In this case

$$\bar{R}_{\alpha\beta} = \tfrac{1}{2}\lambda\Box h_{\alpha\beta} + O(\lambda^2) \qquad (14)$$

$$\bar{R} = \tfrac{1}{2}\lambda\Box h + O(\lambda^2) \qquad (15)$$

Notice that $h_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}\,h = -2n_\alpha n_\beta\Phi$ so the linearized Einstein tensor $\bar{G}_{\alpha\beta} \equiv \bar{R}_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}\bar{R}$ has a very simple form:

$$\bar{G}_{\alpha\beta} = -\lambda\Box(n_\alpha n_\beta\Phi) + O(\lambda^2) \qquad (16)$$

Now we impose the condition that the metric $\bar{g}$ be *Ricci-flat*, i.e., that $\bar{R}_{\alpha\beta} = 0 + O(\lambda^2)$ or equivalently $\bar{G}_{\alpha\beta} = 0 + O(\lambda^2)$ either of which require

$$\Box(n_\alpha n_\beta\Phi) = 0 \qquad (17)$$

If $\bar{R}_{\alpha\beta} = 0 + O(\lambda^2)$, then we also have $\bar{R} = 0 + O(\lambda^2)$, or

$$\Box\Phi = 0 \qquad (18)$$

Thus if we demand that $\bar{g}$ be Ricci-flat, then the function Φ must satisfy the wave-equation. We may also have

$$\Box n_\alpha = 0 \tag{19}$$

if n_α and Φ have the following form:

$$\Phi = \Phi(s_\alpha x^\alpha)$$

$$n_\alpha = n_\alpha(s_\beta x^\beta) \tag{20}$$

where s_α is a constant null vector with respect to the flat metric g. It is easy to check that (20) is a solution of (17) by direct substitution:

$$\Box(n_\alpha n_\beta \Phi) = \Phi(n_\alpha \Box n_\beta + n_\beta \Box n_\alpha) + n_\alpha n_\beta \Box\Phi + 2(\Phi \nabla n_\alpha \cdot \nabla n_\beta$$
$$+ n_\beta \nabla n_\alpha \cdot \nabla\Phi + n_\alpha \nabla n_\beta \cdot \nabla\Phi) \tag{21}$$

where

$$\nabla A \cdot \nabla B \equiv g^{\mu\nu} \frac{\partial A}{\partial x^\mu} \frac{\partial B}{\partial x^\mu} = \sum_{a=1}^{3} \frac{\partial A}{\partial x^a} \frac{\partial B}{\partial x^a} - \frac{\partial A}{\partial x^o} \frac{\partial B}{\partial x^o}$$

Now if Φ and n_α satisfy (20), then clearly (18) and (19) hold, so the first two terms of (21) vanish, and the last term becomes

$$2s_\mu s^\mu(\Phi n'_\alpha n'_\beta + n_\beta n'_\alpha \Phi' + n_\alpha n'_\beta \Phi')$$

where ' denotes differentiation with respect to $(s_\alpha x^\alpha)$. But since s_μ is null, i.e., $s_\mu s^\mu = 0$, the above expression is zero. Notice that Φ and n_α must both be advanced ($s^o > 0$) or both retarted ($s^o < 0$) solutions of the wave-equation.

C O N C L U S I O N

In Part I we examined spherically symmetric solutions of Rastall's 1971 gravitational field equations, which are third order, non-linear,partial differential equations. The imposition of the boundary condition that the metric by asymptotically flat proved to be insufficient to guarantee a unique solution. One solution that is compatible with this boundary condition is equivalent, in appropriate co-ordinates, to a static Newtonian metric. We already know (Rastall, 1968) that such a metric gives experimentally satisfactory results for the case of planetary motion and the deflection of light by the sun. It is possible that by imposing other boundary conditions, or by considering the relation of the field to its sources, one would be able to prove that this metric is the unique spherically symmetric solution of the field equations. In addition, if this were the case, we would have an analogue of "Birkhoff's Theorem" in Rastall's theory.

Another difficulty is that the time-like vector field appearing in the field equations is not unambiguously defined in the case of spherical symmetry. This is clearly an unphysical situation, unless it can be shown that the ambiguity in n^μ does not lead to ambiguity in the solution of the field equations. We found in section I(e) that infinitesimal changes in n^μ do not lead to distinct solutions of the field equations.

In fact, it was shown in that section that the structure of the field equations rules out infinitesimal changes in n^{μ}.

Rastall's 1971 theory is a generalization of his 1968 scalar theory of gravity. Another generalization involves a larger class of metrics, called *generalized Newtonian metrics*. The effect of these metrics is to "stretch" the length of vectors parallel to a certain direction, and "compress" the length of vectors orthogonal to that direction. We have shown that there exist wave-like solutions of the linearized vacuum Einstein equations that have the form of a generalized Newtonian metric.

R E F E R E N C E S

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