CONSIDERATIONS REGARDING THE DUALITY ROTATION

by

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ABSTRACT

Maxwell's equations for the vacuum are invariant under the duality rotation; however, the significance of this invariance is not well understood. The purpose of this thesis is to consider the duality rotation in greater detail than has been done previously. The duality invariance of Maxwell's equations is discussed, and it is shown that the only duality invariants bilinear in the electric and magnetic fields are arbitrary linear combinations of the components of the stress-energy-momentum tensor. It is also shown that the most general linear field transformation which leaves Maxwell's vacuum equations invariant is the duality rotation. The usual Lagrangian density for the electromagnetic field does not exhibit duality invariance. It is shown, however, that if one takes the components of the electromagnetic field tensor as field variables, then the most general Lorentz invariant Lagrangian density bilinear in the electomagnetic fields and their first derivatives is determined uniquely by the requirement of duality invariance. The ensuing field equations are identical with the iterated Maxwell equations. It is further shown that in neutrino theory the Pauli transformation of the second kind corresponds to the duality rotation.

CONTENTS

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0. INTRODUCTION

After the introduction of the complex field vector $\underline{F} = \underline{E}+i\underline{B}$ Maxwell's equations for the vacuum[1],

$$
\begin{aligned}
\nabla \times \underline{E} + \partial \underline{B}/\partial t &= 0 \\
\nabla \cdot \underline{E} &= 0 \\
\nabla \times \underline{B} - \partial \underline{E}/\partial t &= 0 \\
\nabla \cdot \underline{B} &= 0 \ ,
\end{aligned}
\tag{0.1}
$$

take the form

$$
\begin{aligned}
\nabla \times \underline{F} - i\partial \underline{F}/\partial t &= 0 \\
\nabla \cdot \underline{F} &= 0 \ .
\end{aligned}
\tag{0.2}
$$

These equations are invariant under the transformation

$$
\begin{aligned}
\underline{F} \rightarrow \underline{F}' &= \underline{F}\, e^{-i\theta} \\
\underline{F}^* \rightarrow \underline{F}^{*\prime} &= \underline{F}^* e^{+i\theta}
\end{aligned}
\tag{0.3}
$$

where $\underline{F}^* = \underline{E} - i\underline{B}$. Misner and Wheeler (1957) have named this transformation the duality rotation. Milner (1927)[2] used the duality rotation to simplify the solution of Maxwell's equations with sources; however, his paper seems to have been forgotten. Recently, interest in this invariance property of Maxwell's vacuum equations has been revived by Witten (1962), Calkin (1965), Schwinger (1969), and Kaempffer (1970). Schwinger has developed a theory of strongly interacting particles by requiring that the duality invariance of Maxwell's equations without sources be maintained in the equations with sources. Kaempffer has considered the quantum field theory associated with duality invariant action principles.

This thesis contains some elementary considerations regarding the duality rotation. The major result of the research outlined here can be summarized in the following three propositions:

1. The only duality invariants bilinear in the electric and magnetic fields are arbitrary linear combinations of the components of the stress-energy-momentum tensor.

2. The most general linear field transformation leaving Maxwell's vacuum equations invariant is, up to a normalization factor, the duality rotation.

3. The most general Lorentz invariant Lagrangian density bilinear in the electromagnetic field tensor and its first derivatives is uniquely determined by the requirement of duality invariance.

1. DUALITY PAIRS AND DUALITY INVARIANTS

In terms of the electric field $\underline{E}$ and the magnetic field $\underline{B}$ the duality rotation (0.3) becomes

$$(1.1)\qquad \begin{aligned} \underline{E} &\longrightarrow \underline{E}' = \underline{E}\cos\theta + \underline{B}\sin\theta \\ \underline{B} &\longrightarrow \underline{B}' = -\underline{E}\sin\theta + \underline{B}\cos\theta \,. \end{aligned}$$

$(\underline{E},\underline{B})$ forms a "duality pair" with "angle of rotation" θ. Another important duality pair exists. From $\underline{E}$ and $\underline{B}$ one can form the Lorentz scalar $\frac{1}{2}(\underline{B}^2 - \underline{E}^2)$ and the Lorentz pseudoscalar[3] $\underline{E}\cdot\underline{B}$. Under the duality rotation they transform as

$$(1.2)\qquad \begin{aligned} \underline{E}'\cdot\underline{B}' &= \underline{E}\cdot\underline{B}\cos 2\theta + \tfrac{1}{2}(\underline{B}^2 - \underline{E}^2)\sin 2\theta \\ \tfrac{1}{2}(\underline{B}'^2 - \underline{E}'^2) &= -\underline{E}\cdot\underline{B}\sin 2\theta + \tfrac{1}{2}(\underline{B}^2 - \underline{E}^2)\cos 2\theta \,; \end{aligned}$$

therefore, they form a duality pair with angle of rotation 2θ. From equation (1.2) it follows that the usual Lagrangian density for the electromagnetic field

$$(1.3)\qquad L = -\tfrac{1}{2}(\underline{B}^2 - \underline{E}^2)$$

is not duality invariant. Misner and Wheeler (1957) have introduced

$$(1.4)\qquad L_{M-W} = (\underline{E}\cdot\underline{B})^2 + \tfrac{1}{4}(\underline{B}^2 - \underline{E}^2)$$

which is duality invariant; however, L_{M-W} is fourth order in the electromagnetic fields.

Besides L_{M-W} there are several other duality invariants.

The Poynting vector $\underline{S} = \underline{E} \times \underline{B}$, the electromagnetic energy density $\mathcal{E} = \frac{1}{2}(\underline{E}^2 + \underline{B}^2)$, and Maxwell's stress tensor[4]

$$\overleftrightarrow{T} \equiv \overleftrightarrow{I} - (\vec{E}\vec{E} + \vec{B}\vec{B})$$

are all duality invariant. It should be noted here that the electromagnetic field potentials $\underline{A}$ and $\underline{\Phi}$,

$$(1.5) \qquad \begin{aligned} \underline{E} &= -\nabla\underline{\Phi} - \partial A/\partial t \\ \underline{B} &= \nabla \times \underline{A}, \end{aligned}$$

transform in a very complicated manner. They are not fundamental quantities, however, since they cannot be defined uniquely. Although classical electrodynamics is simplified by the introduction of potentials, electromagnetic phenomena can be studied, at least in principle, without the introduction of potentials.

The question arises whether any other duality invariants exist. As a partial answer to this question consider an arbitrary function of the fields[5] $f = f(E_\alpha, B_\beta)$. Under an infinitesimal duality transformation E and B transform as

$$(1.6) \qquad \begin{aligned} \underline{E}' &= \underline{E} + \theta\underline{B} \\ \underline{B}' &= \underline{B} - \theta\underline{E} \end{aligned}$$

where $\theta \ll 1$. Then

$$(1.7) \qquad \begin{aligned} \delta f &\equiv f(E'_\alpha, B'_\beta) - f(E_\alpha, B_\beta) \\ &= \theta\frac{\partial f}{\partial E_\alpha} B_\alpha - \theta\frac{\partial f}{\partial B_\beta} E_\beta . \end{aligned}$$

If f is a duality invariant, $\delta f = 0$ and

$$\frac{\partial f}{\partial E_\alpha} B_\alpha = \frac{\partial f}{\partial B_\beta} E_\beta \quad . \tag{1.8}$$

Consider the case where f is bilinear in the electric and magnetic fields; therefore,

$$f = a_{\alpha\beta} E_\alpha E_\beta + b_{\alpha\beta} B_\alpha B_\beta + c_{\alpha\beta} E_\alpha B_\beta \tag{1.9}$$

where $a_{\alpha\beta}$, $b_{\alpha\beta}$, $c_{\alpha\beta}$ are real constants. Without loss of generality we may take $a_{\alpha\beta} = a_{\beta\alpha}$ and $b_{\alpha\beta} = b_{\beta\alpha}$. Equations (1.8) now gives

$$2a_{\alpha\beta} B_\alpha E_\beta + c_{\alpha\beta} B_\alpha B_\beta = 2b_{\alpha\beta} E_\alpha B_\beta + c_{\alpha\beta} E_\alpha E_\beta \quad . \tag{1.10}$$

In equation (1.10) $\underline{E}$ and $\underline{B}$ may be considered to be a set of six arbitrary real numbers. Setting $\underline{E} = 0$ gives $c_{\alpha\beta} + c_{\beta\alpha} = 0$ since $B_\alpha B_\beta$ is symmetric. Equation (1.10) now becomes

$$a_{\alpha\beta} B_\alpha E_\beta = b_{\alpha\beta} E_\alpha B_\beta \tag{1.11}$$

which yields $a_{\alpha\beta} = b_{\alpha\beta}$; therefore, f can be written

$$f = a_{\alpha\beta} (E_\alpha E_\beta + B_\alpha B_\beta) + c_{\alpha\beta} E_\alpha B_\beta \tag{1.12}$$

where $a_{\alpha\beta}$ is symmetric and $c_{\alpha\beta}$ is skew symmetric. Inspection of equation (1.12) reveals that the only duality invariants bilinear in the electric and magnetic fields are arbitrary linear combinations of $\mathcal{E}$ and the components of $\underline{S}$ and $\overset{\rightrightarrows}{T}$. If the electromagnetic stress-energy-momentum tensor T,

$$T = \begin{pmatrix} & \overleftrightarrow{T} & & -S_1 \\ & & & -S_2 \\ & & & -S_3 \\ S_1 & S_2 & S_3 & -\mathcal{E} \end{pmatrix} \quad ,$$

is introduced, the above result can be stated: the only duality invariants bilinear in the electric and magnetic fields are arbitrary linear combinations of the components of T.

In Schwinger's theory of strongly interacting particles Maxwell's vacuum equations are extended to include both electric and magnetic charges by writing

$$\begin{aligned} \nabla \times \underline{E} + \partial \underline{B}/\partial t &= -\underline{J}_m \\ \nabla \cdot \underline{E} &= \rho_e \\ \nabla \times \underline{B} - \partial \underline{E}/\partial t &= \underline{J}_e \\ \nabla \cdot \underline{B} &= \rho_m \quad . \end{aligned} \tag{1.13}$$

ρ_e and ρ_m are the electric and magnetic charge densities; $\underline{J}_e$ and $\underline{J}_m$ are the electric and magnetic current density vectors. It is postulated that under the duality rotation both (ρ_e , ρ_m) and $(\underline{J}_e, \underline{J}_m)$ transform as duality pairs with angle of rotation θ . In this case $(\rho_e \rho_m, \frac{1}{2}[\rho_m^2 - \rho_e^2])$ and $(\underline{J}_e \cdot \underline{J}_m, \frac{1}{2}[\underline{J}_m^2 - \underline{J}_e^2])$ are duality pairs with angle of rotation 2θ . The following are duality invariants: $\rho_e^2 + \rho_m^2$, $\underline{J}_e^2 + \underline{J}_m^2$, $\underline{J}_e \times \underline{J}_m$, and $\underline{J}_e \underline{J}_m + \underline{J}_m \underline{J}_e$.

In conclusion we note for completeness that

$$\begin{aligned} \underline{S} &= i\tfrac{1}{2}\, \underline{F} \times \underline{F}^* \\ \mathcal{E} &= \tfrac{1}{2}\, \underline{F} \cdot \underline{F}^* \\ \overleftrightarrow{T} &= \overleftrightarrow{I} - \tfrac{1}{2}(\vec{F}\vec{F}^* + \vec{F}^*\vec{F}) \end{aligned} \tag{1.14}$$

$$\tfrac{1}{2}\,\underline{F}\cdot\underline{F} = \tfrac{1}{2}(\underline{E}^2 - \underline{B}^2) + i\underline{E}\cdot\underline{B}$$

The transformation properties of the above quantities follow now from equation (0.3) .

2. A COVARIANT FORMULATION OF THE DUALITY ROTATION

Maxwell's equations form the basis of a Lorentz covariant theory. As a result it is convenient to write the duality rotation in a manifestly Lorentz covariant form. In such a form the significance of the term "duality" also becomes apparent.

In the following the Minkowski metric tensor g_{mn} has the diagonal elements -1, -1, -1, +1. The dual of a second rank contravariant tensor A^{mn} is defined as[6]

$$\tilde{A}^{mn} \equiv \tfrac{1}{2}(-g)^{-\frac{1}{2}}\, \mathcal{E}^{mnrs} A_{rs} \tag{2.1}$$

where $A_{mn} = g_{mr} g_{ns} A^{rs}$ is the associated covariant tensor of A^{mn} and $g = \det g_{mn}$. $\mathcal{E}^{mnrs}$ is a completely skew symmetric relative tensor of weight $w = +1$ with $\mathcal{E}^{1234} = +1$. The factor $(-g)^{-\frac{1}{2}}$ ensures that $\tilde{A}^{mn}$ is a pseudotensor[3] rather than a relative tensor. Similarly, the dual of a second rank covariant tensor B_{mn} is defined as

$$\tilde{B}_{mn} \equiv \tfrac{1}{2}(-g)^{\frac{1}{2}}\, \mathcal{E}_{mnrs} B^{rs} \tag{2.2}$$

where $B^{mn} = g^{mr} g^{ns} B_{rs}$ is the associated contravariant tensor of B_{mn} and $g^{mn} = g_{mn}$. $\mathcal{E}_{mnrs}$ is a completely skew symmetric relative tensor of weight $w = -1$ with $\mathcal{E}_{1234} = -1$. With the above conventions

$$g_{mr} g_{ns} \tilde{C}^{rs} = \tilde{C}_{mn} \, . \tag{2.3}$$

It can now be shown that for an arbitrary antisymmetric tensor $D^{mn} = -D^{nm}$ one has $\tilde{\tilde{D}}^{mn} = -D^{mn}$.

With this preparation consider the electromagnetic field tensor

$$(2.4)\qquad F^{mn} = \begin{pmatrix} 0 & B_3 & -B_2 & -E_1 \\ -B_3 & 0 & B_1 & -E_2 \\ B_2 & -B_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{pmatrix} .$$

The dual of F^{mn} is given by

$$(2.5)\qquad \tilde{F}^{mn} = \begin{pmatrix} 0 & E_3 & -E_2 & B_1 \\ -E_3 & 0 & E_1 & B_2 \\ E_2 & -E_1 & 0 & B_3 \\ -B_1 & -B_2 & -B_3 & 0 \end{pmatrix} .$$

F^{mn} and $\tilde{F}^{mn}$ are second rank, contravariant, skew symmetric tensors. Maxwell's vacuum equations (0.1) can now be written

$$(2.6)\qquad \begin{aligned} F^{mn}{}_{,n} &= 0 \\ \tilde{F}^{mn}{}_{,n} &= 0 \end{aligned}$$

where $_{,m} = \partial/\partial x^m$. Setting $G^{mn} = F^{mn} + i\tilde{F}^{mn}$ gives

$$(2.7)\qquad G^{mn}{}_{,n} = 0 .$$

Equation (2.7) is invariant under the transformation

$$(2.8)\qquad G^{mn} \longrightarrow G'^{mn} = G^{mn}\, e^{-i\theta}$$

which can be rewritten as

$$(2.9)\qquad \begin{aligned} F'^{mn} &= F^{mn}\cos\theta + \tilde{F}^{mn}\sin\theta \\ (\tilde{F}^{mn})' &= -F^{mn}\sin\theta + \tilde{F}^{mn}\cos\theta \end{aligned} .$$

Since $\tilde{\tilde{F}}^{mn} = -F^{mn}$, $(\tilde{F}^{mn})' = (\widetilde{F'^{mn}})$. When expanded in terms of $\underline{E}$ and $\underline{B}$ equation (2.9) is exactly the duality rotation (1.1). A problem arises because F^{mn} is a tensor while $\tilde{F}^{mn}$ is a pseudo-tensor. If F'^{mn} is to be a tensor and $\tilde{F}'^{mn}$ is to be a pseudo-tensor, then θ must be a pseudoscalar. With this assumption the duality rotation (2.9) is a Lorentz covariant transformation. All the results of section 1 can now be proved using the covariant notation. One need only observe that

$$\tfrac{1}{4}F^{mn}F_{mn} = \tfrac{1}{2}(\underline{B}^2 - \underline{E}^2)$$

$$\tfrac{1}{4}\tilde{F}^{mn}F_{mn} = \underline{E}\cdot\underline{B} \qquad (2.10)$$

$$T^m{}_n = F^{mr}F_{nr} - \tfrac{1}{4}\delta^m{}_n F^{rs}F_{rs} \quad .$$

Although Maxwell's equations are first order differential equations they are equivalent to the second order system

$$F^{mn}{}_{,nr} = 0$$

$$\tilde{F}^{mn}{}_{,nr} = 0 \qquad (2.11)$$

obtained by differentiating (2.6) by x^r . The integration of equations (2.11) gives

$$F^{mn}{}_{,n} = C^m$$

$$\tilde{F}^{mn}{}_{,n} = D^m \qquad (2.12)$$

where C^m and D^m are vector constants of integration which can be set equal to zero by the principle of isotropy of spacetime. They must be set to zero if the vacuum is to be Lorentz invariant.

3. COORDINATE AND FIELD TRANSFORMATIONS

Any coordinate transformation induces a transformation of the electric and magnetic fields. Under a Lorentz transformation, for example, the electromagnetic fields F^{mn} transform as a second rank contravariant tensor. The duality rotation differs fundamentally from the Lorentz transformation since it does not possess an associated coordinate transformation. The duality rotation is a field transformation rather than a coordinate transformation. Three questions immediately occur:

1. What is the most general group C of coordinate transformations under which Maxwell's vacuum equations are covariant?

2. What is the most general group D of field transformations leaving Maxwell's vacuum equations invariant?

3. For a given field transformation $d \in D$ does there exist a coordinate transformation $c \in C$ which induces d?

It is well known (Appendix A) that the full covariance group of Maxwell's equations is the fifteen parameter group of conformal transformations[7]. In this section a partial answer is obtained to question 2. It is shown that the most general linear field transformation leaving Maxwell's vacuum equations invariant is, up to a normalization factor, the duality rotation. The answer to question 3 is not yet known.

The most general linear electromagnetic field transformation can be written

$$F'^{mn} = A^{mn}{}_{rs} F^{rs} \tag{3.1}$$

where $A^{mn}{}_{rs}$ are real constants. Since $F'^{mn} = -F'^{nm}$, $F^{mn} = -F^{nm}$, and $\widetilde{F}^{mn} = -\widetilde{F}^{nm}$ the constants $A^{mn}{}_{rs}$ satisfy the symmetry condition

$$A^{mn}{}_{rs} + A^{nm}{}_{rs} = A^{mn}{}_{sr} + A^{nm}{}_{sr} \quad . \tag{3.2}$$

The invariance of Maxwell's vacuum equations under transformation (3.1) implies

$$\begin{aligned} A^{mn}{}_{rs} F^{rs}{}_{,n} &= 0 \\ \widetilde{A^{mn}}{}_{rs} F^{rs}{}_{,n} &= 0 \end{aligned} \tag{3.3}$$

where

$$\widetilde{A^{mn}}{}_{rs} = \tfrac{1}{2}(-g)^{-\frac{1}{2}} \mathcal{E}^{mnkl} A_{klrs} \quad . \tag{3.4}$$

In equation (3.3) the coefficients of $F^{rs}{}_{,n}$ cannot be set equal to zero since the $F^{rs}{}_{,n}$ are dependent. The following conditions exist on the $F^{rs}{}_{,n}$:

$$\begin{aligned} \delta^k{}_r \delta^n{}_s F^{rs}{}_{,n} &= 0 \\ \tfrac{1}{2}(-g)^{-\frac{1}{2}} \mathcal{E}^{knab} g_{ar} g_{bs} F^{rs}{}_{,n} &= 0 \\ (\delta^a{}_r \delta^b{}_s + \delta^b{}_r \delta^a{}_s) F^{rs}{}_{,n} &= 0 \quad . \end{aligned} \tag{3.5}$$

Equations (3.3) can now be simplified by Lagrange's method of undetermined multipliers. Each equation of constraint in (3.5) is multiplied by an undetermined real constant. The resulting equations are added to (3.3a) giving

$$(3.6)\qquad \left(A^{mn}{}_{rs} - a^m{}_k \delta^k{}_r \delta^n{}_s - \tfrac{1}{2}(-g)^{-\frac{1}{2}}\, b^m{}_k\, \varepsilon^{knab}\, g_{ar}\, g_{bs} + \Gamma^{mn}{}_{ab}\left[\delta^a{}_r \delta^b{}_s + \delta^b{}_r \delta^a{}_s\right]\right) F^{rs}{}_{,n} = 0$$

where $a^m{}_k$, $b^m{}_k$, and $\Gamma^{mn}{}_{ab}$ are the undetermined multipliers. By choosing the constants so that the coefficients of the dependent $F^{rs}{}_{,n}$ disappear one obtains

$$(3.7)\qquad A^{mn}{}_{rs} = a^m{}_k \delta^k{}_r \delta^n{}_s + \tfrac{1}{2}(-g)^{-\frac{1}{2}}\, b^m{}_k\, \varepsilon^{knab}\, g_{ar}\, g_{bs} + \Gamma^{mn}{}_{ab}\left(\delta^a{}_r \delta^b{}_s + \delta^b{}_r \delta^a{}_s\right).$$

Transformation (3.1) now becomes

$$(3.8)\qquad F'^{mn} = a^m{}_k F^{kn} + b^m{}_k \tilde{F}^{kn} ,$$

the $\Gamma^{mn}{}_{rs}$ not appearing due to the skew symmetry of F^{mn}. Similarly, equations (3.3b) and (3.5) yield

$$(3.9)\qquad \tilde{F}'^{mn} = c^m{}_k F^{kn} + d^m{}_k \tilde{F}^{kn}$$

where $c^m{}_k$ and $d^m{}_k$ are real constants.

Equations (3.8) and (3.9) can be simplified further by use of equation (3.4); however, it is more convenient to work directly with the equations themselves. From the skew symmetry of F'^{mn} and $\tilde{F}'^{mn}$ one can easily show (Appendix B) that the off-diagonal elements of $a^m{}_k$, $b^m{}_k$, $c^m{}_k$, and $d^m{}_k$ are zero so that equations (3.8) and (3.9) read

$$(3.10)\qquad F' = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} F + \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & b_3 & \\ & & & b_4 \end{pmatrix} \tilde{F}$$

$$F' = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & c_4 \end{pmatrix} F + \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{pmatrix} \tilde{F}$$

where a_m , b_m , c_m and d_m are real constants. Expanding these equations in terms of $\underline{E}$ and $\underline{B}$ gives four equations for each component of $\underline{E}'$ and $\underline{B}'$. In Appendix B it is shown that these equations are consistent only if

$$a_m = d_m \equiv a \quad , \quad b_m = -c_m \equiv b \tag{3.11}$$

where a and b are constants. Setting $a=r\cos\theta$, $b=r\sin\theta$, and $a^2 + b^2 = r^2$ gives

$$\begin{aligned} F'^{mn} &= r \cos \theta \, F^{mn} + r \sin \theta \, \tilde{F}^{mn} \\ \tilde{F}'^{mn} &= -r \sin \theta \, F^{mn} + r \cos \theta \, \tilde{F}^{mn} \quad . \end{aligned} \tag{3.12}$$

Equations (3.12) are the duality rotation (2.9) with a normalization factor r .

4. THE CONSTRUCTION OF FIELD THEORIES

A physical system with infinite degrees of freedom is described by a set of field functions $\varphi^A = \varphi^A(x^m)$ and by a set of differential equations for the fields φ^A. In the Lagrangian formulation of field theory all the properties of the system are assumed to be embodied in a Lagrangian density

$$L = L\,(\,\varphi^A;\ \varphi^A{}_{,m};\ \varphi^A{}_{,mn};\ \dots\,) \quad . \tag{4.1}$$

From the action

$$W = \int L d^4x \tag{4.2}$$

one can obtain the field equations

$$\frac{\partial L}{\partial \varphi^A} - \frac{\partial}{\partial x^m}\frac{\partial L}{\partial \varphi^A{}_{,m}} + \frac{\partial^2}{x^m\, x^n}\frac{\partial L}{\partial \varphi^A{}_{,mn}} - \dots = 0 \tag{4.3}$$

by application of Hamilton's principle $\delta W = 0$. If the action W is invariant under a group of transformations any conserved currents which exist may be obtained from Noether's theorems (see Schroeder, 1968). Historically field equations are usually discovered before their corresponding Lagrangian density. Nevertheless, the Lagrangian density is considered to be fundamental. In this section a method is outlined for obtaining a system's Lagrangian density without complete knowledge of the system's field equations.

The form of a Lagrangian density suitable for the description of a physical system is severely restricted if the order and the degree of the system are known. A system is of order k if the field equations contain no derivatives of the fields higher than k th order. A system is of degree m if the fields and the field gradients have maximum total power m in the field equations. The simplest possible case is a system of order and degree 1; however, this condition proves to be too restrictive. It has been found that most physical systems have order 2 and degree 1. Such a system has linear, second order field equations so that the Lagrangian density must be bilinear in the fields and their first derivatives. Only systems of this type will be considered in the following.

If a physical system possesses an intrinsic symmetry, we expect, by Noether's theorems, that the Lagrangian density for the system exhibits this symmetry by being invariant under an associated group of transformations. The universal validity of the principle of special relativity implies, for example, that a physical system's Lagrangian density must be Lorentz invariant. By Noether's first theorem the conservation of linear momentum, angular momentum, and energy follows. The imposition of invariance properties on a Lagrangian density often determines the Lagrangian density up to a multiplicative factor. Two examples will be considered: a phase invariant complex Lorentz scalar field and a gauge invariant Lorentz vector field.

Consider a complex Lorentz scalar field $\varphi = \varphi_1 + i\,\varphi_2$. φ and $\varphi^* = \varphi_1 - i\,\varphi_2$ will be taken as the field variables. If the Lagrangian density is Lorentz invariant and bilinear in the fields and their first derivatives, then it must be a linear combination of the following Lorentz scalars:

$$(4.4) \qquad \begin{aligned} J_1 &= \varphi^2 & J_4 &= \varphi_{,m}\varphi^{,m} \\ J_2 &= \varphi^{*2} & J_5 &= \varphi^*_{,m}\varphi^{*,m} \\ J_3 &= \varphi^*\varphi & J_6 &= \varphi^*_{,m}\varphi^{,m} \end{aligned}$$

If the Lagrangian density is also invariant under the phase transformation

$$(4.5) \qquad \begin{aligned} \varphi &\rightarrow \varphi' = \varphi e^{-i\theta} \\ \varphi^* &\rightarrow \varphi^{*\prime} = \varphi^* e^{+i\theta} \end{aligned} \quad ,$$

then only J_3 and J_6 can appear; therefore, for a phase invariant complex scalar field the Lagrangian density has the form

$$(4.6) \qquad L_S = a\varphi^*\varphi + b\,\varphi^*_{,m}\varphi^{,m}$$

where a and b are constants. The field equations

$$(4.7) \qquad \begin{aligned} b\,\varphi^{,m}{}_{m} - a\varphi &= 0 \\ b\,\varphi^{*,m}{}_{m} - a\varphi^* &= 0 \end{aligned}$$

follow from Hamilton's principle. The phase transformation (4.5) is a one parameter continuous transformation. By Noether's first theorem a conservation law exists. Under the infinitesimal phase transformation

$$(4.8) \qquad \begin{aligned} \varphi' &= \varphi - i\theta\varphi \\ \varphi^{*\prime} &= \varphi^* + i\theta\varphi^* \end{aligned}$$

the Lagrangian density L_S transforms as

$$\delta L_S \equiv L_S' - L_S$$

(4.9)
$$= -i\theta\,(\partial L_S/\partial\varphi)\varphi - i\theta\,(\partial L_S/\partial\varphi_{,m})\varphi_{,m} + i\theta\,(\partial L_S/\partial\varphi^*)\varphi^* + i\theta\,(\partial L_S/\partial\varphi^*_{,m})\varphi^*_{,m}$$

$$= -i\theta\,\frac{\partial}{\partial x^m}\left(\frac{\partial L_S}{\partial\varphi_{,m}}\varphi\right) + i\theta\,\frac{\partial}{\partial x^m}\left(\frac{\partial L_S}{\partial\varphi^*_{,m}}\varphi^*\right)$$

where equation (4.3) has been used for simplification. Since by construction $\delta L_S = 0$, one has that

(4.10) $$\partial j^m/\partial x^m = 0$$

where

(4.11) $$j^m = (\partial L_S/\partial\varphi_{,m})\varphi - (\partial L_S/\partial\varphi^*_{,m})\varphi^* \quad .$$

It is easily shown from equation (4.6) that

(4.12) $$j^m = b\,(\varphi^{*,m}\varphi - \varphi^{,m}\varphi^*) \quad .$$

Inspection of these results shows that the phase invariant complex scalar field is completely equivalent to the complex Klein-Gordon field.

As a final example consider a Lorentz vector field A_m whose Lagrangian density is invariant under the gauge transformation

(4.13) $$A_m \longrightarrow A'_m = A_m + h_{,m}$$

where $h = h(x^m)$ is a scalar function. For linear, second order, Lorentz covariant field equations the Lagrangian density L_V must be of the form

$$L_V = \sum_{m=1}^{4} c_m K_m \tag{4.14}$$

where the c_m are real constants and

$$\begin{aligned} K_1 &= A^m A_m & K_3 &= A^m{}_{,m} A^n{}_{,n} \\ K_2 &= A^{m,n} A_{m,n} & K_4 &= A^{m,n} A_{n,m} \end{aligned} \tag{4.15}$$

are Lorentz scalars. Under transformation (4.13) $L_V \longrightarrow L_V'$ where

$$L_V' = \sum_{m=1}^{4} c_m K_m' \tag{4.16}$$

and

$$\begin{aligned} K_1' &= K_1 + 2A^m h_{,m} + h^{,m} h_{,m} \\ K_2' &= K_2 + 2A^{m,n} h_{,mn} + h^{,mn} h_{,mn} \\ K_3' &= K_3 + 2A^m{}_{,m} h^{,n}{}_n + h^{,m}{}_m h^{,n}{}_n \\ K_4' &= K_4 + 2A^{m,n} h_{,mn} + h^{,mn} h_{,mn} \ . \end{aligned} \tag{4.17}$$

Since the Lorentz scalars K_m are linearly independent, $L_V = L_V'$ implies

$$c_1 = c_3 = 0 \ , \ c_2 = -c_4 \ . \tag{4.18}$$

It is conventional to choose $c_4 = \frac{1}{2}$. This choice gives

$$L_V = \tfrac{1}{2}(A^{m,n} A_{n,m} - A^{m,n} A_{m,n}) \ . \tag{4.19}$$

The resulting field equations are

$$(A_{m,n} - A_{n,m})^{,n} = 0 \ . \tag{4.20}$$

If we set $F^{mn} = A^{m,n} - A^{n,m}$, then equation (4.20) becomes $F^{mn}{}_{,n} = 0$.

Since $F^{mn} = A^{m,n} - A^{n,m}$ implies that $\tilde{F}^{mn}{}_{,n} = 0$ we see that the gauge invariant vector field is equivalent to the electromagnetic field. Note that

$$(4.21)\qquad \begin{aligned} L_V &= -\tfrac{1}{4} F^{mn}F_{mn} \\ &= \tfrac{1}{2}(E^2 - B^2) \quad . \end{aligned}$$

The gauge transformation (4.13) depends analytically on the first derivatives of a scalar function h. A conserved current exists, but the conservation law is satisfied identically.

5. A DUALITY INVARIANT ACTION PRINCIPLE[8]

Schwinger's theory (1969) of strongly interacting particles requires the introduction of magnetic as well as electric charges, and although this procedure symmetrizes Maxwell's equations in a pleasing manner, it introduces a new problem. If magnetic and electric charges and currents are allowed, electromagnetic field potentials may not exist; therefore, a reformulation of the classical theory independent of the existence of potentials is desirable. In this section a development in this direction is attempted.

The usual Lagrangian density for the electromagnetic field

$$L = \tfrac{1}{2}(\underline{E}^2 - \underline{B}^2) \,, \tag{1.3}$$

although Lorentz invariant, is not duality invariant. The question arises whether a Lagrangian density L_D exists which is both Lorentz and duality invariant. This section contains a derivation, using the methods of section 4, of the most general Lorentz and duality invariant Lagrangian density bilinear in the electromagnetic field tensor F^{mn} and its first derivatives $F^{mn}{}_{,k}$. The requirement of duality invariance is sufficient to determine the Lagrangian density L_D up to a multiplicative constant. The components of the field tensor can then be treated as field variables.

If the theory is to be Lorentz covariant, L_D must be formed from the scalars and scalar densities of F^{mn} and its derivatives. For linear field equations containing no derivatives higher than the second, L_D must be bilinear in F^{mn} and $F^{mn}{}_{,k}$; therefore, L_D must be a linear combination of scalars and scalar densities bilinear in F^{mn} and $F^{mn}{}_{,k}$.

For an arbitrary tensor A^{mn} there are fourteen linearly independent Lorentz scalars of the required type (Kaempffer, 1968):

$$
\begin{aligned}
J_1 &= A^{mn}A_{mn} & J_8 &= A^{mn,k}A_{km,n} \\
J_2 &= A^{mn}A_{nm} & J_9 &= A^{mn,k}A_{kn,m} \\
J_3 &= A^m{}_m A^n{}_n & J_{10} &= A^m{}_m{}^{,k}A_{nk}{}^{,n} \\
J_4 &= A^{mn,k}A_{mn,k} & J_{11} &= A^{mn}{}_{,n}A_{mk}{}^{,k} \\
J_5 &= A^{mn,k}A_{nm,k} & J_{12} &= A^{mn}{}_{,n}A_{km}{}^{,k} \\
J_6 &= A^m{}_m{}^{,k}A^n{}_{n,k} & J_{13} &= A^{mn}{}_{,m}A_{kn}{}^{,k} \\
J_7 &= A^{mn,k}A_{mk,n} & J_{14} &= A^m{}_m{}^{,k}A_{kn}{}^{,n} \; .
\end{aligned}
\tag{5.1}
$$

For an arbitrary skew symmetric tensor there are only four linearly independent scalars:

$$
\begin{aligned}
\check{J}_1 &= F^{mn}F_{mn} & \check{J}_2 &= F^{mn,k}F_{mn,k} \\
\check{J}_3 &= F^{mn,k}F_{mk,n} & \check{J}_4 &= F^{mn}{}_{,n}F_{mk}{}^{,k}
\end{aligned}
\tag{5.2}
$$

where $F^{mn} = -F^{nm}$. For an arbitrary tensor A^{mn} there are ten linearly independent scalar densities of the required type (Pellegrini and Plebanski, 1963):

$$K_1 = \varepsilon^{klmn}A_{kl}A_{mn}$$

$$K_2 = \varepsilon^{klmn}g^{rs}A_{kl,m}A_{nr,s}$$

$$K_3 = \varepsilon^{klmn} g^{rs} A_{kl,m} A_{rn,s}$$

$$K_4 = \varepsilon^{klmn} g^{rs} A_{kl,m} A_{rs,n}$$

$$K_5 = \varepsilon^{klmn} g^{rs} A_{kl,r} A_{mn,s}$$

$$K_6 = \varepsilon^{klmn} g^{rs} A_{kr,l} A_{mn,s}$$

(5.3)

$$K_7 = \varepsilon^{klmn} g^{rs} A_{rk,l} A_{mn,s}$$

$$K_8 = \varepsilon^{klmn} g^{rs} A_{kr,l} A_{ms,n}$$

$$K_9 = \varepsilon^{klmn} g^{rs} A_{rk,l} A_{ms,n}$$

$$K_{10} = \varepsilon^{klmn} g^{rs} A_{rk,l} A_{sm,n} \quad .$$

For a skew symmetric tensor F^{mn} the above scalar densities reduce to the following five scalar densities:

$$\check{K}_1 = \varepsilon^{klmn} F_{kl} F_{mn}$$

$$\check{K}_2 = \varepsilon^{klmn} g^{rs} F_{kl,m} F_{nr,s}$$

(5.4)

$$\check{K}_3 = \varepsilon^{klmn} g^{rs} F_{kl,r} F_{mn,s}$$

$$\check{K}_4 = \varepsilon^{klmn} g^{rs} F_{kr,l} F_{mn,s}$$

$$\check{K}_5 = \varepsilon^{klmn} g^{rs} F_{kr,l} F_{ms,n} \quad .$$

The Lagrangian density L_D is therefore of the form

(5.5)
$$L_D = \sum_{m=1}^{4} a_m \check{J}_m + \sum_{m=1}^{5} b_m \check{K}_m$$

where a_m and b_m are real constants.

One can now require that L_D be invariant under the duality rotation (2.9). It is mathematically convenient, however, to require only that L_D be invariant under the transformation

$$F^{mn} \longrightarrow \tilde{F}^{mn} \quad \text{and} \quad F_{mn} \longrightarrow \tilde{F}_{mn} \tag{5.6}$$

obtained by setting $\theta = \frac{1}{2}\pi$ in equations (2.9). As will be seen, the invariance of L_D under this special case of the duality rotation guarantees its invariance under the full duality rotation. Under transformation (5.6) $L_D \longrightarrow L_D'$ where

$$L_D' = \sum_{m=1}^{4} a_m \check{J}_m' + \sum_{m=1}^{5} b_m \check{K}_m' \tag{5.7}$$

and

$$\check{J}_1' = \tilde{F}^{mn}\tilde{F}_{mn}$$

$$\check{J}_2' = \tilde{F}^{mn,k}\tilde{F}_{mn,k}$$

$$\check{J}_3' = \tilde{F}^{mn,k}\tilde{F}_{mk,n}$$

$$\check{J}_4' = \tilde{F}^{mn}{}_{,n}\tilde{F}_{mk}{}^{,k}$$

$$\check{K}_1' = \varepsilon^{klmn}\tilde{F}_{kl}\tilde{F}_{mn} \tag{5.8}$$

$$\check{K}_2' = \varepsilon^{klmn}g^{rs}\tilde{F}_{kl,m}\tilde{F}_{nr,s}$$

$$\check{K}_3' = \varepsilon^{klmn}g^{rs}\tilde{F}_{kl,r}\tilde{F}_{mn,s}$$

$$\check{K}_4' = \varepsilon^{klmn}g^{rs}\tilde{F}_{kr,l}\tilde{F}_{mn,s}$$

$$\check{K}_5' = \varepsilon^{klmn}g^{rs}\tilde{F}_{kr,l}\tilde{F}_{ms,n} .$$

After a short calculation (see Appendix C) one finds

$$\check{J}_1' = -\check{J}_1 \qquad\qquad \check{J}_2' = -\check{J}_2$$

$$\check{J}_3' = -\tfrac{1}{2}\check{J}_2 + \check{J}_4 \qquad \check{J}_4' = -\tfrac{1}{2}\check{J}_2 + \check{J}_3$$

$$\check{K}_1' = -\check{K}_1 \qquad \check{K}_2' = -\check{K}_2$$

(5.9)

$$\check{K}_3' = -\check{K}_3 \qquad \check{K}_4' = -\check{K}_4$$

$$\check{K}_5' = -\tfrac{1}{2}(\check{K}_4 + \check{K}_2) \quad .$$

The requirement $L_D = L_D'$ gives

$$2a_1\check{J}_1 + \tfrac{1}{2}(4a_2 + a_3 + a_4)\check{J}_2 + (a_3 - a_4)\check{J}_3$$

(5.10)

$$+ (a_4 - a_3)\check{J}_4 + 2b_1\check{K}_1 + \tfrac{1}{2}(4b_2 + b_5)\check{K}_2$$

$$+ 2b_3\check{K}_3 + \tfrac{1}{2}(4b_4 + b_5)\check{K}_4 + b_5\check{K}_5 = 0 \quad .$$

Since $\check{J}_m$ and $\check{K}_m$ are linearly independent

$$2a_1 = 0 \qquad 2a_2 + \tfrac{1}{2}(a_3 + a_4) = 0$$

$$a_3 - a_4 = 0 \qquad 2b_1 = 0$$

(5.11)

$$2b_3 = 0 \qquad 2b_2 + \tfrac{1}{2}b_5 = 0$$

$$b_5 = 0 \qquad 2b_4 + \tfrac{1}{2}b_5 = 0 \quad .$$

The solution of equations (5.11) is

$$a_1 = b_1 = b_2 = b_3 = b_4 = b_5 = 0$$

(5.12)

$$a_2 = -\tfrac{1}{2}a_3 = -\tfrac{1}{2}a_4 \quad .$$

By choosing $a_2 = 1$ one can write L_D as

(5.13) $$L_D = F^{mn,k}F_{mn,k} - 2F^{mn,k}F_{mk,n} - 2F^{mn}{}_{,n}F_{mk}{}^{,k} \quad .$$

Only scalars appear in the Lagrangian density L_D, and its Lorentz invariance is thus manifest. Alternatively, if L_D is written in terms of the complex field vectors F and F*,

$$
\begin{aligned}
L_D = \frac{\partial \underline{F}^*}{\partial t}\cdot\frac{\partial \underline{F}}{\partial t} &+ i\left[\frac{\partial \underline{F}^*}{\partial t}\cdot(\nabla\times\underline{F}) - \frac{\partial \underline{F}}{\partial t}\cdot(\nabla\times\underline{F}^*)\right] \\
&+ (\nabla\times\underline{F}^*)\cdot(\nabla\times\underline{F}) - (\nabla\cdot\underline{F}^*)(\nabla\cdot\underline{F}) \quad ,
\end{aligned}
\tag{5.14}
$$

its duality invariance becomes manifest on account of transformation (0.3).

6. THE ITERATED MAXWELL EQUATIONS AND CONSERVED CURRENT

If the components of the electromagnetic field tensor F^{mn} are treated as field variables, one obtains from Hamilton's principle

$$\delta \int L_D(F^{mn}{}_{,k})\, d^4x = 0 \tag{6.1}$$

the field equations

$$F^{mn,k}{}_{k} - 4F^{mk,n}{}_{k} = 0 \quad . \tag{6.2}$$

Since L_D is duality invariant, equations (6.2) imply that

$$\tilde{F}^{mn,k}{}_{k} - 4\tilde{F}^{mk,n}{}_{k} = 0 \quad . \tag{6.3}$$

$F^{mn,k}{}_{k}$ and $\tilde{F}^{mn,k}{}_{k}$ are skew symmetric in (m,n); therefore, equations (6.2) and (6.3) can be rewritten as

$$\begin{aligned} F^{mn,k}{}_{k} - 2(F^{mk,n}{}_{k} - F^{nk,m}{}_{k}) &= 0 \\ \tilde{F}^{mn,k}{}_{k} - 2(\tilde{F}^{mk,n}{}_{k} - \tilde{F}^{nk,m}{}_{k}) &= 0 \quad . \end{aligned} \tag{6.4}$$

In terms of the complex field vector $\underline{F}$ the field equations (6.4) become

$$\partial^2\underline{F}/\partial t^2 - \nabla^2\underline{F} - 2\nabla\times(\nabla\times\underline{F} - i\partial\underline{F}/\partial t) = 0 \quad . \tag{6.5}$$

These equations permit longitudinal modes as solutions. If these modes are eliminated by the imposition of the subsidiary condition

$$\nabla\cdot\underline{F} = 0 \quad , \tag{0.2b}$$

then equation (6.5) can be written as

$$(6.6) \qquad (\nabla\times - i\partial/\partial t)(\nabla\times - i\partial/\partial t)\,\underline{F} = 0$$

which is identical with the iterated Maxwell's vacuum equations. It is clear that Maxwell's vacuum equations imply the duality invariant field equations. Similarly, if equations (6.2) and (6.3) are solved with the subsidiary condition

$$(6.7) \qquad F^{mn,k}{}_{k} = 0$$

Maxwell's equations are obtained in the form (2.12).

In the usual field-theoretical formulation of vacuum electrodynamics the Lagrangian density is

$$(6.8) \qquad L = -1/4\, F^{mn}F_{mn}$$

with the subsidiary condition

$$(6.9) \qquad F_{mn} = A_{m,n} - A_{n,m}$$

where A_m is the field 4-potential. This condition holds if and only if

$$(2.6b) \qquad \tilde{F}^{mn}{}_{,n} = 0 \quad .$$

The subsidiary condition (6.9) is thus equivalent to assuming equation (2.6b). The components of the field potential A_m are treated as the field variables to obtain Maxwell's equations (2.6a). This approach is far from satisfactory since the A_m are not measurable quantities. The field potential is defined only up to a gauge transformation, and appears only as an intermediary quantity in calculations. The approach using the duality invariant Lagrangian density L_D avoids this problem by treating the components of the field tensor F^{mn} as the field variables. It is interesting, nevertheless, to consider the effect of the subsidiary condition (6.9) on the duality invariant formulation.

Since equation (6.9) implies (2.6b) one obtains from equation (6.3)

$$\tilde{F}^{mn,k}{}_{k} = 0 \tag{6.10}$$

and as a result

$$F^{mn,k}{}_{k} = 0 \quad ; \tag{6.7}$$

therefore, condition (6.9) also gives Maxwell's vacuum equations in the form (2.12). The field potential A_m can also be treated as the field variable in the duality invariant action principle (6.1). From equation (5.13) one has

$$L_D = 4A^{k,m}{}_{k}A_{m,n}{}^{n} - 2A^{m,k}{}_{k}A_{m,n}{}^{n} - 2A^{k,m}{}_{k}A_{n,m}{}^{k} \quad . \tag{6.11}$$

Hamilton's principle now yields

$$\frac{\partial L_D}{\partial A^s} - \frac{\partial}{\partial x^m}\frac{\partial L_D}{\partial A^s{}_{,m}} + \frac{\partial^2}{x^m\, x^n}\frac{\partial L_D}{\partial A^s{}_{,mn}} = 0 \tag{6.12}$$

or

$$A^{m,sn}{}_{nm} - A^{s,mn}{}_{mn} = 0 \tag{6.13}$$

or

$$F^{ms,n}{}_{nm} = 0 \quad . \tag{6.14}$$

Equation (6.14) is implied by the duality invariant equation (6.2) since if equation (6.2) is differentiated with respect to x^m one obtains

$$F^{mn,k}{}_{km} - 4F^{mk,n}{}_{km} = 0 \quad . \tag{6.15}$$

The skew symmetry of F^{mn} implies that $F^{mk,n}{}_{km} = 0$ so that (6.14) results from equation (6.15).

The duality rotation is a one parameter continuous transformation; therefore, by Noether's first theorem a conserved current exists. From the infinitesimal duality rotation

$$F'^{mn} = F^{mn} + \theta\,\tilde{F}^{mn} \tag{6.16}$$

one obtains

$$\delta L_D = \theta \frac{\partial L_D}{\partial F^{mn}} \tilde{F}^{mn} + \theta \frac{\partial L_D}{\partial F^{mn}{}_{,k}} \tilde{F}^{mn}{}_{,k} \quad . \tag{6.17}$$

The field equations allow equation (6.17) to be rewritten as

$$\delta L_D = \theta \frac{\partial}{\partial x^k} \left(\frac{\partial L_D}{\partial F^{mn}{}_{,k}} \tilde{F}^{mn} \right) \quad . \tag{6.18}$$

Since the Lagrangian density L_D is duality invariant $\delta L_D = 0$ and

$$\frac{\partial}{\partial x^k} \left(\frac{\partial L_D}{\partial F^{mn}{}_{,k}} \tilde{F}^{mn} \right) = 0 \tag{6.19}$$

or

$$\partial j^k / \partial x^k = 0 \tag{6.20}$$

where $j^k = \frac{\partial L_D}{\partial F^{mn}{}_{,k}} \tilde{F}^{mn}$. A simple calculation shows that

$$(6.21) \qquad \frac{\partial L_D}{\partial F^{mn}{}_{,k}} = 2F_{mn}{}^{,k} + 4F^k{}_{m,n} - 4F_{ml}{}^{,l}\delta^k{}_n$$

so that

$$(6.22) \qquad j^k = 2\tilde{F}^{mn}F_{mn}{}^{,k} + 4\tilde{F}^{mn}F^k{}_{m,n} - 4\tilde{F}^{mk}F_{mn}{}^{,n} \quad .$$

7. THE DUALITY ROTATION IN QUANTUM MECHANICS

Maxwell's equations (0.2) can be written as a Schroedinger equation for momentum-handedness. If this is done, the quantum mechanical significance of the duality rotation becomes apparent, and a duality operator can be introduced. There is then an exact correspondence the duality rotation and the Pauli transformation of the second kind in neutrino theory.

Consider the quantum mechanical momentum operator $\hat{\underline{p}} = -i\,\underline{\nabla}$. Equation (0.2) can be written

$$\begin{pmatrix} 0 & -i\hat{p}_3 & i\hat{p}_2 \\ i\hat{p}_3 & 0 & -i\hat{p}_1 \\ -i\hat{p}_2 & i\hat{p}_1 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = i\frac{\partial}{\partial t}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \tag{7.1}$$

or

$$\hat{H}\,\underline{F} = i\,\partial\underline{F}/\partial t \tag{7.2}$$

where

$$\hat{H} = \hat{S}_\alpha\,\hat{p}_\alpha = \hat{\underline{S}}\cdot\hat{\underline{p}}$$

$$\hat{S}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\hat{S}_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{7.3}$$

$$\hat{S}_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\quad .$$

The 3×3 matrices $\hat{S}_\alpha$ are angular momentum operators of spin $J = 1$. They satisfy the commutation relations

$$[S_\alpha, S_\beta] = i\,\mathcal{E}_{\alpha\beta\gamma} S_\gamma \qquad (7.4)$$
$$S_\alpha S_\alpha = 2I$$

where $\mathcal{E}_{\alpha\beta\gamma}$ is the three dimensional Levi-Civita tensor. For $\underline{F}^*$ the equation corresponding to (7.2) is

$$\hat{H}\,\underline{F}^* = -i\,\partial \underline{F}^*/\partial t \quad . \qquad (7.5)$$

Classically, $\underline{S}\cdot\underline{P}$ is the magnitude of the momentum $\underline{P}$ times the component of the spin angular momentum $\underline{S}$ in the direction of the momentum. To convert to quantum mechanics one uses the well-known recipe of replacing classical quantities by their corresponding quantum mechanical operators; therefore,

$$\underline{S}\cdot\underline{P} \longrightarrow \hat{\underline{S}}\cdot\hat{\underline{P}} \qquad (7.6)$$

$\hat{H}$ is seen to be the momentum-handedness operator, and equations (7.2) and (7.5) are Schroedinger equations in momentum-handedness with $\underline{F}$ and $\underline{F}^*$ as state vectors.

Consider a beam of photons with energy ω and momentum $\underline{k}$. One has

$$\underline{F} = |\varphi\rangle e^{i(k\cdot x - \omega t)} \qquad (7.7)$$
$$\underline{F}^* = |\chi\rangle e^{-i(k\cdot x - \omega t)}$$

where $\omega^2 = k^2$. The substitution of (7.7) into equations (7.2) and (7.5) yields

$$\hat{\underline{S}}\cdot\underline{k}\;\underline{F} = \omega\underline{F} \qquad (7.8)$$
$$\hat{\underline{S}}\cdot\underline{k}\;\underline{F}^* = -\omega\underline{F}^* \quad .$$

Thus F represents a right-handed photon and F* represents a left-handed photon. The duality rotation (0.3) is now seen to be a phase transformation of the quantum mechanical state vectors $\underline{F}$ and $\underline{F}^*$. Note, however, that the phase factor of $\underline{F}^*$ is the complex conjugate of the phase factor of $\underline{F}$. This fact distinguishes the duality rotation from a true phase transformation where all the state vectors are multiplied by the same phase factor.

It is convenient to find a duality operator for the purpose of comparing the duality rotation and the Pauli transformation of the second kind. Let

$$f = \begin{pmatrix} \underline{F} \\ \underline{F}^* \end{pmatrix}$$

(7.9) $$\varsigma^\alpha = \begin{pmatrix} 0 & -\hat{S}_\alpha \\ \hat{S}_\alpha & 0 \end{pmatrix}$$

$$\varsigma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} .$$

f is a 6×1 matrix and the ς^m are 6×6 matrices. Equations (7.2) and (7.5) now read

(7.10) $$\varsigma^m \partial/\partial x^m f = 0.$$

The ς^m satisfy the commutation relations

$$[\varsigma^\alpha, \varsigma^4] = -2\Sigma^\alpha$$

(7.11) $$[\varsigma^\alpha, \varsigma^\beta] = -i\,\varepsilon_{\alpha\beta\gamma}\Sigma^\gamma$$

where $$\Sigma^\alpha = \begin{pmatrix} \hat{S}_\alpha & 0 \\ 0 & \hat{S}_\alpha \end{pmatrix} .$$

The matrix

$$\delta^5 \equiv i \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

has the same properties as the imaginary unit i since

(7.12) $$(\delta^5)^2 = -I, \ (\delta^5)^3 = -\delta^5, \text{ and } (\delta^5)^4 = I \quad .$$

Therefore,

(7.13) $$e^{-\theta\delta^5} = I \cos\theta - \delta^5 \sin\theta \quad .$$

By writing

$$(F) = \begin{pmatrix} \underline{F} \\ 0 \end{pmatrix}$$

$$(F^*) = \begin{pmatrix} 0 \\ \underline{F}^* \end{pmatrix}$$

one obtains

(7.14) $$\begin{aligned} e^{-\theta\delta^5}\,(F) &= e^{-i\theta}\,(F) \\ e^{-\theta\;^5}\,(F^*) &= e^{+i\theta}\,(F^*) \quad . \end{aligned}$$

$e^{-\theta\delta^5}$ is the desired duality operator.

It can now be shown that the duality transformation corresponds to the Pauli transformation of the second kind in neutrino theory. The field equations for the neutrino (see Lurié, 1968; Kaempffer, 1965) are given by

(7.15) $$\gamma^k \partial\Psi/\partial x^k = 0$$

where the γ^k are 4×4 matrices satisfying the anticommutation relations

$$\{\gamma^k, \gamma^l\} = 2g^{kl} \quad . \tag{7.16}$$

$\psi = \begin{pmatrix} R \\ L \end{pmatrix}$ is a 4×1 matrix, R and L being the state functions of the right- and left-handed neutrino respectively. If as above one sets

$$\gamma^5 = i \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$(R) = \begin{pmatrix} R \\ 0 \end{pmatrix} \tag{7.17}$$

$$(L) = \begin{pmatrix} 0 \\ L \end{pmatrix} \quad ,$$

then

$$\begin{aligned} e^{-\theta\gamma^5} (R) &= e^{-i\theta} (R) \\ e^{-\theta\gamma^5} (L) &= e^{+i\theta} (L) \quad . \end{aligned} \tag{7.18}$$

Equations (7.18) are called the Pauli transformation of the second kind. The above transformation clearly corresponds to the duality rotation.

8. CONCLUSION

This thesis demonstrates that the duality invariance of Maxwell's vacuum equations contains greater significance than is generally believed. The considerations outlined here have been mainly classical in nature. Most interesting is the possibility of extending these results by the methods of quantum field theory. Both Schwinger (1969) and Kaempffer (1970) have used the duality invariance of Maxwell's vacuum equations as a stepping stone to a theory of elementary particles. With the evidence now at hand such efforts must be considered to be in the realm of speculation. It is the author's hope, however, that this thesis provides some justification for further work along these lines.

NOTES

1. Natural units ($\hbar = c = 1$) are used throughout the thesis. Electromagnetic quantities are rationalized.

2. The author is indebted to F. Peet for this reference.

3. For the significance of the term "pseudo" see Appendix A.

4. I is always a unit matrix whose dimensionality is determined by context. The components of I are $\delta^A{}_B$ where $\delta^A{}_B$ is the Kronecker symbol.

5. Greek indices run from one to three; latin indices, from one to four. Repeated indices are summed. The indices 1, 2, 3, 4 refer, respectively, to the spatial coordinates x, y, z, and to the time coordinate t.

6. The dual is variously defined in Physics literature. The definition here follows Witten (1962). Robertson and Noonan (1968) omit the factor $(-g)^{-\frac{1}{2}}$. Corson (1953) replaces $(-g)^{-\frac{1}{2}}$ with $(g)^{-\frac{1}{2}}$. For the significance of the factor see Appendix A.

7. The conformal group contains the Lorentz transformations, the translations, the dilations, and the special conformal transformations.

8. This section is based on a paper accepted for publication in the Canadian Journal of Physics (Levman, 1970).

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APPENDIX A: THE CONFORMAL GROUP

Before proceeding to a discussion of the conformal group of transformations C some terminology which is often confused in Physics literature must be clarified (see Schouten, 1951). Consider a group A of coordinate transformations

$$x'^m = x'^m(x^n)$$

(A.1)

$$J = \det(\partial x/\partial x') \neq 0 \quad .$$

A geometrical object with respect to A is a set of functions $G = G^{abc\ldots}_{def\ldots}(x^m)$ transforming under A according to

(A.2) $$G'^{ijk\ldots}_{lmn\ldots} = |J|^T J^W \frac{\partial x'^i}{\partial x^a} \ldots \frac{\partial x^f}{\partial x'^n} \ldots G^{abc\ldots}_{def\ldots}$$

$(i,j,k, \ldots)$ are contravariant indices; $(l,m,n, \ldots)$ are covariant indices. If $T = W = 0$, then G is said to be a tensor with respect to A. If $T = 0$, G is a relative tensor of weight W. For $T = 0$ and $W = 1$ G is a tensor density since $\int G dx$ is a tensor. If $W = 0$, G is an absolute relative tensor of weight T. Lastly, if $W = -T = \pm 1$, G is called a pseudotensor. A pseudotensor transforms like a tensor up to a sign. Consider, as an example, the Levi-Civita symbol ε^{mnrs}. With respect to the general group of non-singular coordinate transformations ε is a relative tensor of weight +1; however, ε may be considered a Lorentz pseuodtensor and a translation tensor.

It is the purpose of this appendix to find the most general group of transformations under which Maxwell's vacuum equations are covariant. If Maxwell's equations

$$F^{mn}{}_{,n} = 0 \tag{2.6}$$
$$\tilde{F}^{mn}{}_{,n} = 0$$

are to be covariant under a group of transformations R, then $F^{mn}{}_{,n}$ and $\tilde{F}^{mn}{}_{,n}$ must be geometrical objects with respect to R. This requirement actually defines what is meant by the covariance of (2.6) with respect to R. Note that Maxwell's equations are Lorentz covariant and that F^{mn} is a Lorentz contravariant tensor; therefore, we require that R contain the Lorentz group and that the transformation property of F^{mn} under R reduce to the tensor transformation under the Lorentz group. Since

$$J(\text{Lorentz}) = \pm 1 \tag{A.3}$$

the above condition requires W = 0 for R so that F^{mn} is at most an R absolute relative tensor.

Consider the scale transformation (dilation)

$$x'^m = s\, x^m \quad . \tag{A.4}$$

Under this transformation charge and current are invariant. Since

$$\text{dimension } (\underline{E}) = \text{charge}/(\text{distance})^2 \tag{A.5}$$
$$\text{dimension } (\underline{B}) = \text{current}/(\text{distance})^2$$

one has

$$\underline{E} \rightarrow \underline{E}' = s^{-2}\, \underline{E} \tag{A.6}$$
$$\underline{B} \rightarrow \underline{B}' = s^{-2}\, \underline{E} \ .$$

Maxwell's vacuum equations (0.1) are clearly invariant under (A.4) and (A.5). The dilation group is therefore a subgroup of R. From equation (A.6) it follows that

$$F^{mn} \longrightarrow F'^{mn} = (s)^{-2} F^{mn} \; ; \tag{A.7}$$

however, from equation (A.2) with $W = 0$

$$F^{mn} \longrightarrow F'^{mn} = (s^{-4})^{T} s^{2} F^{mn} \tag{A.8}$$

since

$$\begin{aligned} J(\text{dilation}) &= s^{-4} \\ \frac{\partial x'^m}{\partial x^n}(\text{dilation}) &= s\,\delta^m{}_n \end{aligned} \quad . \tag{A.9}$$

For equations (A.7) and (A.8) to be consistent $T = 1$. Similarly, $T = 0$ for F_{mn} . Since

$$F_{mn} = g_{mr}\, g_{ns}\, F^{rs} \tag{A.10}$$

g_{mn} must have weight $T = -\frac{1}{2}$. As a result, under the group R g_{mn} transforms as

$$g'_{mn} = |J|^{-\frac{1}{2}} \frac{\partial x^r}{\partial x'^m} \frac{\partial x^s}{\partial x'^n} g_{rs} \quad . \tag{A.11}$$

This equation does not reduce correctly to the Lorentz form unless $g'_{mn} = g_{mn}$. This condition is also necessary if indices are to be raised and lowered in a consistent manner. Hence equation (A.11) becomes

$$g_{mn} = |J|^{-\frac{1}{2}} \frac{\partial x^r}{\partial x'^m} \frac{\partial x^s}{\partial x'^n} g_{rs} \quad . \tag{A.12}$$

The group of transformations C satisfying equation (A.12) is called the conformal group (see Isham; 1970). It will now be shown that Maxwell's vacuum equations (2.6) are covariant under any group of transformations for which F^{mn} is an absolute relative tensor of weight T = 1. From the considerations above, the only physically admissable transformations are those satisfying equation (A.12); therefore, the conformal group C forms the full covariance group of Maxwell's equations.

If F^{mn} is an absolute relative contravariant tensor of weight T = 1, then

$$
\begin{aligned}
F'^{mn}{}_{,n} = {} & (|J|/\partial x'^n)(\partial x'^m/\partial x^r)(\partial x'^n/\partial x^s)\, F^{rs} \\
& + |J|\,(\partial^2 x'^m/\partial x^r \partial x^s)\, F^{rs} \\
& + |J|\,(\partial x^k/\partial x'^n)(\partial x'^m/\partial x^r)(\partial^2 x'^n/\partial x^s \partial x^r)\, F^{rs} \\
& + |J|\,(\partial x'^m/\partial x^r)(\partial x'^n/\partial x^s)\,\partial F^{rs}/\partial x'^n \quad .
\end{aligned}
\tag{A.13}
$$

The second term on the right of equation (A.13) vanishes due to the skew symmetry of F^{mn}. Since

$$-J^{-1}\,\partial J/\partial x^m = (\partial x^k/\partial x'^n)(\partial^2 x'^n/\partial x^m \partial x^k) \tag{A.14}$$

the first and third term subtract to zero. Equation (A.13) now becomes

$$F'^{mn}{}_{,n} = |J|\,(\partial x'^m/\partial x^r)\, F^{rn}{}_{,n} \tag{A.15}$$

so that $F^{mn}{}_{,n}$ is an absolute vector of weight T = 1. One can now conclude that $F^{mn}{}_{,n} = 0$ implies $F'^{mn}{}_{,n} = 0$. Similarly,

$\tilde{F}^{mn}{}_{,n} = 0$ is a covariant equation. Note, however, that $\tilde{F}^{mn}$ transforms as a relative tensor of weight $W = 1$. The factor $(-g)^{-\frac{1}{2}}$ appearing in the definition of $\tilde{F}^{mn}$ is unnecessary for Lorentz covariance, but is important in this context. Note also that the duality rotation (2.9) is conformal covariant if θ is a conformal pseudoscalar.

In summary, the full covariance group of Maxwell's vacuum equations is the conformal group defined by equation (A.12) . Under the conformal group F^{mn} is an absolute relative tensor of weight $T = 1$, $\tilde{F}^{mn}$ is a tensor density, and F_{mn} is a tensor. The duality rotation parameter θ transforms as a pseudoscalar.

If equation (A.12) is solved one finds that the conformal group consists of Lorentz transformations, translations, dilations, and the special conformal transformations

$$x'^m = (x^m + x^n x_n \beta^m)(1 + 2x^n \beta_n + \beta^n \beta_n x^r x_r)^{-1} \tag{A.16}$$

where β^m are constants. It is interesting to note that the conformal group can be defined as the group of transformations for which

$$dx^m dx_m = 0 \tag{A.17}$$

is a covariant equation (see Robertson and Noonan, 1968). This definition of C implies that the conformal group is the group of transformations which leaves the light-cone invariant. Finally, the term "conformal" is used since the group C is also the group of transformations leaving angles invariant.

In conclusion note that Maxwell's equations with charge

$$F^{mn}{}_{,n} = J^m \qquad \text{(A.18)}$$

$$\tilde{F}^{mn}{}_{,n} = 0$$

are also conformal covariant. J^m, the charge current density, must be, by equation (A.18), an absolute relative tensor of weight $T = 1$. This transformation property for J^m is consistent with equation (A.5) since a volume V transforms under dilations according to $V' = s^3 V$.

APPENDIX B: ON EQUATIONS (3.8) AND (3.9)

Since $F'^{mn} = -F'^{nm}$ one obtains from equation (3.8)

(B.1)
$$\begin{aligned} 0 = F'^{11} &= a^1{}_1 F^{11} + a^1{}_2 F^{21} + a^1{}_3 F^{31} + a^1{}_4 F^{41} \\ &\quad + b^1{}_1 \tilde{F}^{11} + b^1{}_2 \tilde{F}^{21} + b^1{}_3 \tilde{F}^{31} + b^1{}_4 \tilde{F}^{41} \\ &= -a^1{}_1 B_3 + a^1{}_3 B_2 + a^1{}_4 E_1 - b^1{}_2 E_3 \\ &\quad + b^1{}_3 E_2 - b^1{}_4 B_1 \quad . \end{aligned}$$

Any two constant fields $\underline{E}$ and $\underline{B}$ satisfy Maxwell's vacuum equations. As a result, for equation (B.1) to be valid for arbitrary constant vectors $\underline{E}$ and $\underline{B}$ we must have

(B.2)
$$a^1{}_2 = a^1{}_3 = a^1{}_4 = b^1{}_2 = b^1{}_3 = b^1{}_4 = 0 \quad .$$

Similarly, all other off-diagonal elements of $a^m{}_k$, $b^m{}_k$, $c^m{}_k$ and $d^m{}_k$ are zero; equation (3.10) results.

The expansion of equation (3.10) in terms of $\underline{E}$, $\underline{B}$, $\underline{E}'$ and $\underline{B}'$ gives

(B.3)
$$\begin{aligned} F'^{41} &= E'_1 = a_4 E_1 - b_4 B_1 \\ -F'^{14} &= E'_1 = a_1 E_1 - b_1 B_1 \\ F'^{23} &= E'_1 = d_2 E_1 + c_2 B_1 \\ -F'^{32} &= E'_1 = d_3 E_1 + c_3 B_1 \quad . \end{aligned}$$

Equations (B.3) are consistent only if

(B.4)
$$a_1 = a_4 = d_2 = d_3 \ , \ b_1 = b_4 = -c_2 = -c_3 \quad .$$

Continuing in the same manner one easily verifies equation (3.11) . With this step the reduction of equation (3.1) to equation (3.12) is completed.

APPENDIX C: ON EQUATIONS (5.9)

In the derivation of equations (5.9) the following identities are useful:

$$\varepsilon^{klmn}\varepsilon_{krst} = -1!\,\delta^{lmn}_{rst}$$

(C.1)
$$\varepsilon^{klmn}\varepsilon_{klrs} = -2!\,\delta^{mn}_{rs}$$

$$g_{rk}g_{sl}g_{tm}g_{un}\,\varepsilon^{klmn} = -g\,\varepsilon_{rstu} \quad .$$

δ^{lmn}_{rst} equals +1 if (l,m,n) is an even permutation of (r,s,t), equals -1 if (l,m,n) is an odd permutation of (r,s,t), and equals 0 in all other cases. δ^{mn}_{rs} is defined similarly. δ^{mn}_{rs} and δ^{lmn}_{rst} can be represented in terms of the Kronecker symbol as

$$\delta^{mn}_{rs} = \det\begin{pmatrix} \delta^m{}_r & \delta^m{}_s \\ \delta^n{}_r & \delta^n{}_s \end{pmatrix}$$

(C.2)

$$\delta^{lmn}_{rst} = \det\begin{pmatrix} \delta^l{}_r & \delta^l{}_s & \delta^l{}_t \\ \delta^m{}_r & \delta^m{}_s & \delta^m{}_t \\ \delta^n{}_r & \delta^n{}_s & \delta^n{}_t \end{pmatrix} \quad .$$

As an example, consider $\check{J}_1' = \tilde{F}^{mn}\tilde{F}_{mn}$. From equations (C.1) and (C.2)

$$\check{J}_1' = \tfrac{1}{2}(-g)^{-\frac{1}{2}}\,\varepsilon^{klmn}F_{mn}\,\tfrac{1}{2}(-g)^{\frac{1}{2}}\,\varepsilon_{klrs}F^{rs}$$

(C.3)
$$= -\tfrac{1}{2}\,\delta^{mn}_{rs}\,F_{mn}F^{rs} \quad .$$

$$= -\check{J}_1$$