CROSS-SECTIONS FOR THE GRAVITATIONAL SCATTERING OF MASSLESS PARTICLES

by

FREDERICK GORDON PEET

B.A., University of Saskatchewan, 1963
M.A., Columbia University, 1966

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN THE DEPARTMENT

OF

PHYSICS

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

October, 1970

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree tha permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Physics

The University of British Columbia
Vancouver 8, Canada

Date October 16, 1970

ABSTRACT

The purpose of this thesis is to examine the gravitational scattering of massless scalar particles, photons, four component neutrinos, and two component neutrinos by one another. A modification of the quantum theory of the weak gravitational field developed by Gupta is used as a basis for the considerations. Cross-sections are given for the gravitational scattering of scalar particles by: scalar particles, photons, four component neutrinos, and two component neutrinos; of photons by: photons, four component neutrinos, and two component neutrinos; of four component neutrinos by four component neutrinos, and of two component neutrinos by two component neutrinos. The cross-section given by Barker et al and by Boccaletti et al for the scattering of photons by photons is confirmed. The cross-section for the scattering of massive scalar particles by massive scalar particles quoted by DeWitt and the cross-section for the scattering of photons by massive scalar particles given by Boccaletti et al are found to be in error and are corrected.

CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

This thesis topic was suggested by Professor F. A. Kaempffer. It would have been difficult to find a more suitable topic for me. For this, I thank him. In addition, his comments and criticisms guided the thesis to its completion.

The remarks of Dr. L. de Sobrino were very much appreciated.

I thank B. G. Keefer for reading the thesis in draft form.

The National Research Council of Canada provided generous financial support in the form of a Postgraduate Scholarship.

INTRODUCTION

During the past decade several articles, for which references will be given shortly, have appeared dealing with the quantum mechanical treatment of the gravitational interaction between particles. In most of these the interacting particles have had a non-zero rest mass. Because of the all pervasive nature of the gravitational field, particles of zero rest mass also interact gravitationally. In this thesis the gravitational interaction of particles of zero rest mass--photons, neutrinos, and massless scalar particles, is studied, using a modification of the quantum theory of the weak gravitational field developed by Gupta(1952).

This latter theory allows one to treat the gravitational interaction in much the same manner as the electrodynamic interaction is treated in quantum electrodynamics. The particle corresponding to the photon is the graviton. Ten types of gravitons are possible. In a free gravitational field eight of these can be eliminated by means of a subsidiary condition just as longitudinal and time-like photons are eliminated from the free electromagnetic field by the Lorentz condition. When sources are present the eight gravitons cannot be eliminated, but instead mediate the interaction. The interaction Lagrangian for a given field and the gravitational field may be found in the following manner. Essentially, one starts from the Lorentz covariant Lagrangian for the field and replaces all partial derivatives with covariant derivatives. The covariant derivative of a spinor has been given by Fock(1929). Alternatively, one may use the method of the compensating field(Utiyama 1956) which yields the same result. The Lagrangian thus obtained transforms as a scalar density under general coordinate transformations. One expands this Lagrangian in terms of the gravitational coupling constant and extracts the interaction Lagrangian. One may now apply the S-matrix formalism(Bogoliubov and Shirkov 1959) and work out matrix elements and cross-sections just as is done in quantum electrodynamics. This is the procedure which is followed here.

Though there are no experimental observations to either confirm or refute the results of the calculations performed to date, there are some problems in the quantum region in which the gravitational field does not play an insignificant role. For example, the gravitational contribution to the cross-section for the scattering of photons by photons dominates the electrodynamic contribution, for very low and very high frequencies. This result follows upon comparison of the gravitational cross-section to be given here and the electrodynamic cross-section given by Akhiezer and Berestetskii(1965). Another example is the following. If one sets the mass to zero in the Dirac equation one obtains the equation for a four component neutrino(Muirhead 1965, Lurié 1968). Altogether there are four types of particles--two neutrinos and two antineutrinos. These can be put into correspondence with the electron and muon neutrinos and their antineutrinos(Appendix G). If one wants to discuss just the electron neutrino, then one imposes a subsidiary condition. This has the effect of reducing the Dirac four component spinor to a two component spinor. These mathematical entities are treated in the books by Corson(1953), Roman(1960), and Aharoni(1965). Before the discovery of the muon neutrino, Kobzarev and Okun(1963) pointed out that the gravitational interaction could be used to detect the then so-called anomalous(muon) neutrinos. Here, cross-sections are given for collisions involving two component neutrinos and for collisions involving unpolarized beams of four component neutrinos.

The history of the subject runs as follows. The weak gravitational field was quantized, as mentioned earlier, by Gupta in 1952. The formalism was used by him in the same year in considerations on the gravitational self-energies of the photon and the electron. Corinaldesi(1956), using Gupta's formalism, considered the two-body problem. Vladimirov(1964) treated the gravitational annihilation of fermions. Barker et al(1966) worked out the matrix elements for the scattering of massive particles of various spins and in 1967 gave the gravitational cross-section for photon-photon scattering. Boccaletti et al(1969) considered this problem again and arrived at the result of Barker et al. Kuchowicz(1969) reviewed neutrino dynamics in quantum and non-quantum theories of gravitation. In the present work cross-sections are given for collisions(due to the gravitational interaction) between scalar particles and: scalars, photons, four component neutrinos and two component neutrinos; between photons and: four component neutrinos and two component neutrinos; between four component neutrinos and four component neutrinos; and, between two component neutrinos and two component neutrinos.

In Chapter 1 the relevant notions of the classical (Einstein 1918, 1956) and quantum(Gupta 1952) theories of the weak gravitational field are given. The scalar field, photon field, four component neutrino field and two component neutrino field are described and the gravitational interaction Lagrangian for each is given in Chapters 2, 3, 4, and 5, respectively. The matrix elements for the various collisions are given in Chapter 6 and in Chapter 7 the cross-sections are calculated. The thesis is concluded in Chapter 8 with a discussion of the results.

Natural units ($\hbar=c=1$) and an imaginary time coordinate are used. All Greek indices run from 1-4 and are summed if repeated, unless otherwise stated. The Latin indices i,j,k, l,m, and n run from 1-3 and are summed if repeated, unless otherwise stated. The Latin indices p,q,r,s, and t run from 1-4 and are summed if repeated. The Latin indices b and c run from 1-4, unless otherwise stated, and are never summed if repeated. The symbols $|$, $*$, $\dagger$, and τ signify respectively, partial differentiation, complex conjugation, Hermitian conjugation and transposition. The scalar product of 4-vectors is denoted by (V,W) or just VW, and of 3-vectors by $\vec{p}\cdot\vec{q}$.

1. The Classical and Quantum Descriptions of the Weak Gravitational Field

a) Classical Theory

According to Einstein(1956) the gravitational field variables, $g_{\mu\nu}$, are the coefficients which appear in the definition of the scalar product, (V,W), of two vectors in space-time, V and W, whose components are v^μ and w^ν:

$$(1.1)\quad (V,W) = g_{\mu\nu}\, v^\mu\, w^\nu \,.$$

The scalar product is assumed to be symmetric so that

$$(1.2)\quad g_{\mu\nu} = g_{\nu\mu} \,.$$

If one can find a coordinate system such that

$$(1.3)\quad g_{\mu\nu} = \delta_{\mu\nu} + \varkappa h_{\mu\nu} \,, \quad \left|\varkappa h_{\mu\nu}\right| \ll 1,$$

one says that the gravitational field is weak. In (1.3) $\delta_{\mu\nu}$ is the Kronecker delta and $\varkappa$ is a constant whose value will be given shortly. The $h_{\mu\nu}$ could be used as field variables, but the field equations have a simpler form if one introduces new variables $\gamma_{\mu\nu}$ defined by

$$(1.4)\quad \gamma_{\mu\nu} = h_{\mu\nu} - (h\delta_{\mu\nu})/2 \,, \quad h=\mathrm{Trace}(h_{\mu\nu}).$$

According to Einstein(1918) one may impose on the $\gamma_{\mu\nu}$ the subsidiary condition

$$(1.5)\quad \gamma_{\mu\nu|\nu} = 0.$$

The Lagrangian for a weak gravitational field is(Gupta 1952)

(1.6) $\mathcal{L}_G = -(1/4)\left[\gamma_{\mu\nu|\tau}\,\gamma_{\mu\nu|\tau} - (\gamma_{|\tau}\,\gamma_{|\tau})/2\right]$, $\gamma = \mathrm{Trace}(\gamma_{\mu\nu})$.

The Lagrangian density which describes the interaction between the gravitational field and any one of the fields to be considered, whose energy-momentum tensor is $T_{\mu\nu}$, is, as will be shown in Chapters 2, 3, 4, and 5,

(1.7) $\mathcal{L}_{int} = -(\varkappa/2)\left[\gamma_{\mu\nu} - (\gamma\delta_{\mu\nu})/2\right]T_{\mu\nu}$.

The field equations which follow from the Lagrangian density

(1.8) $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{int}$

are those given by Einstein for a weak gravitational field with a source whose energy-momentum tensor is $T_{\mu\nu}$:

(1.9) $\Box^2\,\gamma_{\mu\nu} = \varkappa T_{\mu\nu}$.

An examination of the solutions of these equations for two choices of $T_{\mu\nu}$ provides the link with Newton's law of gravitation and provides an introduction to the quantum treatment.

If one assumes a static distribution of mass with density μ, the only non-zero component of $T_{\mu\nu}$ is

(1.10) $T_{44} = \mu$.

If the $\gamma_{\mu\nu}$ are assumed to be time independent and 0 at infinity, then it follows from (1.9) that all the $\gamma_{\mu\nu}$ are 0 except for γ_{44}, which is,

(1.11) $\gamma_{44} = -(\varkappa/4\pi)\int(\mu/r)\,d\tau$.

The equation of motion of a test particle is(Einstein 1956)

(1.12) $d^2\vec{r}/dt^2 = -(\kappa/4)\ \mathrm{grad}(\gamma_{44})$.

This agrees with Newton's law of motion if one takes

(1.13) $\kappa^2 = 16\pi G$.

In c.g.s. units the relationship is

(1.14) $\kappa^2 = 16\pi G/c^4$.

If one sets $T_{\mu\nu}$ to zero in (1.9), the wave equations for a free gravitational field are obtained:

(1.15) $\Box^2 \gamma_{\mu\nu} = 0.$

Solutions of (1.15) are

(1.16) $\gamma_{\mu\nu} = (1/\sqrt{V}) \sum_k (1/\sqrt{2\omega_k})(a_{\mu\nu}(\vec{k})\ e^{ikx} + \bar{a}_{\mu\nu}(\vec{k})\ e^{-ikx}).$

The $a_{\mu\nu}$ satisfy the following reality properties:

(1.17) $a_{ij}{}^* = \bar{a}_{ij}\ ,\quad a_{i4}{}^* = -\bar{a}_{i4}\ ,\quad a_{44}{}^* = \bar{a}_{44}$.

If it is assumed that the wave propagates along the x^3 axis, the subsidiary condition (1.5) yields

(1.18) $a_{\mu 3} + i a_{\mu 4} = 0,\ \mu=1,4$

(1.19) $\bar{a}_{\mu 3} + i \bar{a}_{\mu 4} = 0,\ \mu=1,4$.

The Hamiltonian density in such a wave is

(1.20) $\mathcal{H} = (1/2)\ \dot{\gamma}_{\mu\nu}\ \dot{\gamma}_{\mu\nu} - (1/4)\ \dot{\gamma}\ \dot{\gamma} - \mathcal{L}$

The Hamiltonian, H, is

(1.21) $H = \int \mathcal{H} d^3x = \sum \omega_k (|(a_{11} - a_{22})/2|^2 + |a_{12}|^2)$

where the conditions (1.18) and (1.19) have been used. If one makes the substitutions

$$(1.22)\quad a'_{11} = (a_{11} - a_{22})/2\ ,\quad a'_{22} = (a_{11} + a_{22})/2,$$

H becomes

$$(1.23)\quad H = \Sigma\, \omega_k \left(\left|a'_{11}\right|^2 + \left|a_{12}\right|^2\right).$$

Thus, in a free gravitational wave the energy depends only on two independent modes. The subsidiary condition (1.5) prevents the other modes from contributing to the energy.

On the other hand, particular solutions of the inhomogeneous equations (1.9) automatically satisfy (1.5) as a consequence of the vanishing divergence of $T_{\mu\nu}$.

b) Quantum Theory

The momenta $\pi_{\mu\nu}$, conjugate to the $\gamma_{\mu\nu}$, which follow from the Lagrangian (1.6) are

$$(1.24a) \quad \pi_{\mu\nu} = \partial\mathcal{L}/\partial\dot{\gamma}_{\mu\nu} = \dot{\gamma}_{\mu\nu} \,, \quad \mu\neq\nu$$

$$(1.24b) \quad \pi_{bb} = \partial\mathcal{L}/\partial\dot{\gamma}_{bb} = \dot{\gamma}_{bb}/2 - \dot{\gamma}/4 \quad .$$

The $\gamma_{\mu\nu}$ and $\pi_{\mu\nu}$ are defined to be operators which satisfy the commutation relations(at equal times)

$$(1.25a) \quad \left[\gamma_{\mu\nu}(\vec{r}),\pi_{\alpha\beta}(\vec{r}\,')\right] = \left[\gamma_{\mu\nu}(\vec{r}),\dot{\gamma}_{\alpha\beta}(\vec{r}\,')\right] =$$

$$i(\delta_{\mu\alpha}\,\delta_{\nu\beta} + \delta_{\mu\beta}\,\delta_{\nu\alpha})\delta(\vec{r}-\vec{r}\,'), \quad \mu\neq\nu, \ \alpha\neq\beta,$$

$$(1.25b) \quad \left[\gamma_{bb}(\vec{r}),\pi_{bb}(\vec{r}\,')\right] = i\delta(\vec{r}-\vec{r}\,').$$

All other basic commutators are zero. From (1.24b), (1.25b), and the condition

$$(1.26) \quad \left[\gamma_{bb},\pi_{cc}\right] = 0, \ b\neq c,$$

it follows that

$$(1.27) \quad \left[\gamma_{bb},\dot{\gamma}_{bb}\right] = i\delta(\vec{r}-\vec{r}\,')$$

and that

$$(1.28) \quad \left[\gamma_{bb}(\vec{r}),\dot{\gamma}_{cc}(\vec{r}\,')\right] = -i\delta(\vec{r}-\vec{r}\,'), \ b\neq c.$$

One may study the spectrum of the operators $\gamma_{\mu\nu}$ by decomposing the $\gamma_{\mu\nu}$ according to (1.16). The relations (1.25a), (1.27), and (1.28) lead to

$$(1.29a) \quad \left[a_{\mu\nu},\bar{a}_{\alpha\beta}\right] = (\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}), \ \mu\neq\nu, \ \alpha\neq\beta,$$

(1.29b) $[a_{bb}, \bar{a}_{bb}] = 1$

(1.29c) $[a_{bb}, \bar{a}_{cc}] = -1, \quad b \neq c.$

All other basic commutators are zero.

The reality conditions (1.17) now read

(1.30) $a_{ij}^{\dagger} = \bar{a}_{ij}\ , \quad a_{i4}^{\dagger} = -\bar{a}_{i4}\ , \quad a_{44}^{\dagger} = \bar{a}_{44}\ .$

One would like to, in the customary fashion, use a representation in which the operators

(1.31) $N_{\mu\nu} = \bar{a}_{\mu\nu} a_{\mu\nu}$ (no summation implied)

are diagonal. This is not possible because, by (1.29c), N_{bb} and N_{cc} do not commute. This is a direct consequence of the appearance of γ in the Lagrangian (1.6). In the treatment given by Gupta(1952) this problem does not appear explicitly. There, Gupta treated γ as an independent variable and imposed a subsidiary condition

(1.32) $\gamma = \mathrm{Trace}(\gamma_{\mu\nu}).$

Here, a different method of circumventing the problem is followed. One first of all writes the Lagrangian (1.6) in the form

(1.33) $\mathcal{L}_G = -(1/4)(\mathcal{L}_1 + \mathcal{L}_2)$

where

(1.34) $\mathcal{L}_1 = \gamma_{\underline{\mu\nu}|\tau}\ \gamma_{\underline{\mu\nu}|\tau}$

and

(1.35) $\mathcal{L}_2 = \sum_1^4 \gamma_{bb|\tau}\,\gamma_{bb|\tau} - (1/2)\gamma_{|\tau}\,\gamma_{|\tau}$.

The bar under $\mu\nu$ in $\mathcal{L}_1$ signifies summation over the off-diagonal terms only. If one now introduces new variables $\gamma'_{\mu\nu}$ defined by

(1.36a) $\gamma'_{\mu\nu} = \gamma_{\mu\nu}, \quad \mu\neq\nu$

(1.36b) $$\begin{bmatrix}\gamma'_{11}\\ \gamma'_{22}\\ \gamma'_{33}\\ \gamma'_{44}\end{bmatrix} = (1/\sqrt{2})\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2} & 0 & 0\\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 1/2 & 1/2 & -1/2 & -1/2\\ 1/2 & 1/2 & 1/2 & 1/2\end{bmatrix}\begin{bmatrix}\gamma_{11}\\ \gamma_{22}\\ \gamma_{33}\\ \gamma_{44}\end{bmatrix}$$

then $\mathcal{L}_2$ reduces to

(1.37) $\mathcal{L}_2 = 2\left(\sum_1^3 \gamma'_{bb|\tau}\;\gamma'_{bb|\tau} - \gamma'_{44|\tau}\;\gamma'_{44|\tau}\right)$.

In (1.37) the quadratic form $\mathcal{L}_2$ has been reduced to a sum and difference of squares, in contradistinction to (1.35) which contains the term $\gamma_{|\tau}\gamma_{|\tau}$. A commutation relation like (1.29c) will not arise now. The conjugate momenta are

(1.38a) $\pi'_{\mu\nu} = \dot{\gamma}'_{\mu\nu}$, $(\mu\nu)\neq(44)$

(1.38b) $\pi'_{44} = -\dot{\gamma}'_{44}$.

The commutation relations for the $\gamma'_{\mu\nu}$ and $\pi'_{\mu\nu}$, which are equivalent to the relations (1.25a,b) for the $\gamma_{\mu\nu}$ and $\pi_{\mu\nu}$, are

(1.39a) $\left[\gamma'_{\mu\nu}(\vec{r}),\pi'_{\alpha\beta}(\vec{r}\,')\right] = i(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha})\delta(\vec{r}-\vec{r}\,'),$
$\mu\neq\nu,\ \alpha\neq\beta,$

(1.39b) $\left[\gamma'_{bb}(\vec{r}),\pi'_{bb}(\vec{r}\,')\right] = i\delta(\vec{r}-\vec{r}\,')$.

The $\gamma'_{\mu\nu}$ are expanded as

(1.40) $\gamma'_{\mu\nu} = (1/\sqrt{V}) \sum_k (1/\sqrt{2\omega_k}) \, (c_{\mu\nu}(\vec{k}) e^{ikx} + \bar{c}_{\mu\nu}(\vec{k}) e^{-ikx}).$

The commutation relations (1.39a-b) together with (1.38a-b) imply

(1.41a) $[c_{\mu\nu}, \bar{c}_{\alpha\beta}] = (\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}), \quad \mu \neq \nu, \ \alpha \neq \beta,$

(1.41b) $[c_{bb}, \bar{c}_{bb}] = 1, \quad b \neq 4$

(1.41c) $[c_{44}, \bar{c}_{44}] = -1.$

All other basic commutators are zero. A commutation relation like (1.29c) does not appear now.

From the expansions (1.16) and (1.40) and the transformation (1.36b) it follows that the $c_{\mu\nu}$ and $\bar{c}_{\mu\nu}$ are related to the $a_{\mu\nu}$ and $\bar{a}_{\mu\nu}$ by

(1.42a) $c_{\mu\nu} = a_{\mu\nu}, \quad \mu \neq \nu,$

(1.42b) $$\begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \\ c_{44} \end{bmatrix} = (1/\sqrt{2}) \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{22} \\ a_{33} \\ a_{44} \end{bmatrix}$$

and the above two equations with $c_{\mu\nu}$ and $a_{\mu\nu}$ replaced with $\bar{c}_{\mu\nu}$ and $\bar{a}_{\mu\nu}$. The symbols $c_{\mu\nu}$ and $\bar{c}_{\mu\nu}$ were used in (1.40) instead of the symbols $a'_{\mu\nu}$ and $\bar{a}'_{\mu\nu}$, which are more natural in view of (1.36a,b), in order to abbreviate the notation. It follows from the conjugation properties (1.30) for the $a_{\mu\nu}$ and the transformation (1.42a,b) that the conjugation properties of the $c_{\mu\nu}$ are

(1.43) $c_{ij}^{\dagger} = \bar{c}_{ij}, \quad c_{i4}^{\dagger} = -\bar{c}_{i4}, \quad c_{44}^{\dagger} = \bar{c}_{44}.$

The operators $N_{mn}(\vec{k})$ defined by

(1.44a) $N_{ij}(\vec{k}) = \bar{c}_{ij}(\vec{k})c_{ij}(\vec{k})$ (no summation implied)

(1.44b) $N_{i0}(\vec{k}) = c_{i0}(\vec{k})\bar{c}_{i0}(\vec{k})$

(1.44c) $N_{44}(\vec{k}) = c_{44}(\vec{k})\bar{c}_{44}(\vec{k})$

where

(1.45) $c_{i0} = c_{0i} = -ic_{i4}$, $\bar{c}_{i0} = \bar{c}_{0i} = -i\bar{c}_{i4}$

form a complete set of commuting operators. From the commutation relations and conjugation properties of the $c_{\mu\nu}$ and $\bar{c}_{\mu\nu}$ it follows that the eigenvalues, $n_{mn}(\vec{k})$, of the $N_{mn}(\vec{k})$ are the non-negative integers. For given $\vec{k}$ and given (mn) the eigenvectors, $|n_{mn}(\vec{k})\rangle$, of $N_{mn}(\vec{k})$ are just the eigenvectors of the harmonic oscillator. If $E_{mn}(\vec{k})$ denotes the vector space spanned by these eigenvectors, the space E of all possible dynamical states can be written

(1.46) $E = \prod_{m,n,k}{}' \otimes E_{mn}(\vec{k})$

where $\prod' \otimes$ denotes the tensor product with the proviso that only one of the pair E_{mn} and E_{nm} is to be included since N_{mn} is symmetric in m and n. An arbitrary vector $|\;\rangle$ in E can be written as a linear combination of tensor products of vectors in the $E_{mn}(\vec{k})$. The operators c_{ij}, c_{i0}, c_{44} and their conjugates act in the following manner where, for brevity, the argument $\vec{k}$ has been suppressed, and only the relevant n_{mn} has been written in $|\;\rangle$:

(1.47) $c_{ij}|n_{ij}\rangle = \sqrt{n_{ij}}\,|n_{ij}-1\rangle$

$$\bar{c}_{i0}|n_{i0}\rangle = \sqrt{n_{i0}}\,|n_{i0}-1\rangle$$

$$\bar{c}_{44}|n_{44}\rangle = \sqrt{n_{44}}\,|n_{44}-1\rangle$$

$$\bar{c}_{ij}|n_{ij}\rangle = \sqrt{n_{ij}+1}\,|n_{ij}+1\rangle$$

$$c_{i0}\,|n_{i0}\rangle = \sqrt{n_{i0}+1}\,|n_{i0}+1\rangle$$

$$c_{44}\,|n_{44}\rangle = \sqrt{n_{44}+1}\,|n_{44}+1\rangle$$

One says that $\bar{c}_{44}$, the $\bar{c}_{i0}$, and the c_{ij} destroy gravitons of momentum $\vec{k}$ and polarizations (44), (i0) and (ij) respectively. Similarly, c_{44}, the c_{i0} and the $\bar{c}_{ij}$ create gravitons of momentum $\vec{k}$ and polarizations (44), (i0) and (ij).

At this point however, a new problem arises. The quantum mechanical form of the subsidiary condition (1.5) is(Gupta 1952)

(1.48) $\gamma^{-}_{\mu\nu|\nu}|\,\rangle = 0,$

where $\gamma^{-}_{\mu\nu}$ is the negative frequency part of $\gamma_{\mu\nu}$. This condition, together with the inverse of (1.42b), yields

(1.49a) $(c_{13} + ic_{14})|\,\rangle = 0$

(1.49b) $(c_{23} + ic_{24})|\,\rangle = 0$

(1.49c) $(c_{22} -(c_{33} - c_{44})/\sqrt{2} + ic_{34})|\,\rangle = 0$

(1.49d) $(ic_{43} + c_{22}+(c_{33} - c_{44})/\sqrt{2})|\,\rangle = 0.$

The last two imply

(1.50a) $(c_{22} + ic_{34})|\,\rangle = 0$

$$(1.50b)\quad (c_{33} - c_{44})| > = 0.$$

That part of an arbitrary vector which describes the (33) and (44) polarizations can be written as

$$(1.51)\quad |33,44\rangle = \Sigma\,\Sigma\, A(n_{33},n_{44})|n_{33},n_{44}\rangle$$

It follows from (1.50b) that

$$(1.52)\quad \Sigma\,\Sigma\, A(n_{33},n_{44})\left[\sqrt{n_{33}}\,|n_{33}-1,n_{44}\rangle - \sqrt{n_{44}+1}\,|n_{33},n_{44}+1\rangle\right] = 0.$$

It can be shown(Akhiezer and Berestetskii,1965,page 168) that a relation of this form implies that $|33,44\rangle$ cannot be normalized. Similar conclusions can be drawn from (1.49a), (1.49b) and (1.50a).

A solution to this problem is to introduce the indefinite metric formalism. One assumes that the operators γ_{14}, γ_{24}, γ_{34} and χ_{44}, which is defined by

$$(1.53)\quad \chi_{44} = i\gamma'_{44}\ ,$$

are Hermitian. Then c_{14}, c_{24}, c_{34}, and ϕ_{44} are the Hermitian conjugates of $\bar{c}_{14}$, $\bar{c}_{24}$, $\bar{c}_{34}$, and $\bar{\phi}_{44}$ where ϕ_{44} and $\bar{\phi}_{44}$ are defined by

$$(1.54)\quad \phi_{44} = ic_{44}\ ,\quad \bar{\phi}_{44} = i\bar{c}_{44}.$$

The commutation rule for ϕ_{44} and $\bar{\phi}_{44}$ is, from (1.41c),

$$(1.55)\quad \left[\phi_{44},\bar{\phi}_{44}\right] = 1.$$

The eigenvalues, $n_{\mu\nu}$, of the operators

$$(1.56a)\quad N_{\mu\nu}(\vec{k}) = \bar{c}_{\mu\nu}(\vec{k})c_{\mu\nu}(\vec{k})\ ,\quad (\mu\nu)\neq(44)$$

and

$$(1.56b) \quad N_{44}(\vec{k}) = \bar{\phi}_{44}(\vec{k})\phi_{44}(\vec{k})$$

are the non-negative integers because of the commutation rules (1.41a), (1.41b) and (1.55). The eigenvectors of each $N_{\mu\nu}(\vec{k})$ are just the harmonic oscillator eigenvectors. The space E of dynamical states is the space spanned by the tensor products of the eigenvectors of the $N_{\mu\nu}(\vec{k})$.

One now defines a unitary and Hermitian operator η by its matrix elements

$$(1.57) \quad \langle n'|\eta|n\rangle = (-1)^{n_{14}+n_{24}+n_{34}+n_{44}} \delta_{nn'}$$

where $|n\rangle$ and $|n'\rangle$ are basis vectors of E formed from the tensor product of eigenvectors of the $N_{\mu\nu}(\vec{k})$. A generalized scalar product and a generalized expectation value are defined by

$$(1.58) \quad \langle\Psi|\phi\rangle_G = \langle\Psi|\eta|\phi\rangle$$

and

$$(1.59) \quad \langle A\rangle_G = \langle |\eta A| \rangle$$

One now interprets all expectation values as generalized expectation values. If an Hermitian operator commutes with η it has a real generalized expectation value and if it anti-commutes with η it has a pure imaginary generalized expectation value. From (1.57) it follows that c_{14}, c_{24}, c_{34}, and ϕ_{44} together with their conjugates anti-commute with η; hence, γ'_{14}, γ'_{24}, γ'_{34} and χ_{44} have pure imaginary generalized expectation values. All the other $c_{\mu\nu}$ commute with η and hence all the other $\gamma'_{\mu\nu}$ have real generalized expectation values. Therefore, the correct reality properties for the $\langle \gamma'_{\mu\nu} \rangle_G$ are obtained.

The operators $c_{\mu\nu}$ act on the eigenvectors of $N_{\mu\nu}(k)$ in the following fashion:

$$
\begin{aligned}
(1.60) \quad & c_{ij}|n_{ij}\rangle = \sqrt{n_{ij}}\,|n_{ij}-1\rangle \\
& c_{i4}|n_{i4}\rangle = -i\sqrt{n_{i4}}\,|n_{i4}-1\rangle \\
& \not{c}_{44}|n_{44}\rangle = -i\sqrt{n_{44}}\,|n_{44}-1\rangle \\
& \bar{c}_{ij}|n_{ij}\rangle = \sqrt{n_{ij}+1}\,|n_{ij}+1\rangle \\
& \bar{c}_{i4}|n_{i4}\rangle = i\sqrt{n_{i4}+1}\,|n_{i4}+1\rangle \\
& \bar{\not{c}}_{44}|n_{44}\rangle = i\sqrt{n_{44}+1}\,|n_{44}+1\rangle
\end{aligned}
$$

The subsidiary condition (1.50b) now reads

$$(1.61) \quad (c_{33} + i\not{c}_{44})|\,\rangle = 0.$$

If one expresses, as in (1.51), that part of an arbitrary vector which describes the (33) and (44) polarizations, one obtains from (1.61)

$$(1.62) \quad \Sigma\,\Sigma\, A(n_{33},n_{44})\left[\sqrt{n_{33}}\,|n_{33}-1,n_{44}\rangle + \sqrt{n_{44}}\,|n_{33},n_{44}-1\rangle\right] = 0.$$

It can be shown (Akhiezer and Berestetskii, 1965) that this condition implies that $|33,44\rangle$ is normalizable with non-negative norm. Similar conclusions follow from the subsidiary conditions (1.49a,b) and (1.50a). Thus the problem of infinite norms has been removed.

The condition (1.61) together with its conjugate

$$(1.63) \quad \langle\,|(\bar{c}_{33} - i\bar{\not{c}}_{44}) = \langle\,|\eta(\bar{c}_{33} + i\bar{\not{c}}_{44}) = 0$$

leads to

(1.64a) $\langle N_{33}\rangle_G + \langle N_{44}\rangle_G = 0.$

The subsidiary conditions (1.49a), (1.49b) and (1.50a) and their conjugates lead to

(1.64b) $\langle N_{13}\rangle_G + \langle N_{14}\rangle_G = 0$

(1.64c) $\langle N_{23}\rangle_G + \langle N_{24}\rangle_G = 0$

(1.64d) $\langle N_{22}\rangle_G + \langle N_{34}\rangle_G = 0.$

Equations (1.64a-d) are useful in the computation of the dynamical variables of the free field. For example, the Hamiltonian which follows from the Lagrangian (1.33) is

(1.65) $$H = (1/2) \sum_k \omega_k \left[N_{\underline{\mu\nu}}(\vec{k}) + 2 \sum_1^4 N_{bb}(\vec{k}) \right].$$

Hence, because of (1.64a-d)

(1.66) $$\langle H\rangle_G = \sum_k \omega_k \left[\langle N_{12}(\vec{k})\rangle_G + \langle N_{11}(\vec{k})\rangle_G \right]$$

This is just the quantum mechanical form of (1.23) if one takes into consideration (1.42a,b). The subsidiary condition allows only the (11) and (12) gravitons to contribute to the Hamiltonian.

From (1.60) it follows that ten types of gravitons are possible, though in the free field, only the (11) and (12) gravitons contribute to the dynamical variables as exemplified in (1.66). In Gupta's treatment eleven types of gravitons are possible. This is because he treats γ as an independent variable. Contributions from γ to the free field variables are removed by imposition of the additional subsidiary condition (1.32).

The covariant commutation relations for the fields $\gamma'_{\mu\nu}$ are, from (1.41a,b) and (1.55),

$$(1.67a)\quad \left[\gamma'_{\mu\nu}(x),\gamma'_{\alpha\beta}(y)\right] = i(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha})D(x-y), \mu\neq\nu, \alpha\neq\beta$$

$$(1.67b)\quad \left[\gamma'_{bb}(x),\gamma'_{bb}(y)\right] = iD(x-y), \quad b\neq 4$$

$$(1.67c)\quad \left[\gamma'_{44}(x),\gamma'_{44}(y)\right] = -\left[\gamma_{44}(x),\gamma_{44}(y)\right] = -iD(x-y)$$

where

$$(1.68)\quad D(x-y) = (1/(2\pi)^3)\int(1/\omega_k)\,\sin(k(x-y))\,d^3k$$

If one uses the commutation relations (1.41a-c) for the $c_{\mu\nu}$ and the transformation (1.42a,b) relating the $c_{\mu\nu}$ and the $a_{\mu\nu}$, one obtains for the commutation relations for the $a_{\mu\nu}$ the relations (1.29a-c). The covariant commutation relations for the fields $\gamma_{\mu\nu}$ which follow from (1.29a-c) are

$$(1.69a)\quad \left[\gamma_{\mu\nu}(x),\gamma_{\alpha\beta}(y)\right] = i(\delta_{\mu\alpha}\delta_{\beta\nu} + \delta_{\mu\beta}\delta_{\nu\alpha})D(x-y), \mu\neq\nu, \alpha\neq\beta$$

$$(1.69b)\quad \left[\gamma_{bb}(x),\gamma_{bb}(y)\right] = iD(x-y)$$

$$(1.69c)\quad \left[\gamma_{bb}(x),\gamma_{cc}(y)\right] = -iD(x-y)$$

Also useful are

$$(1.70a)\quad \left[\gamma_{bb}(x),\ \gamma(y)\right] = -2iD(x-y)$$

$$(1.70b)\quad \left[\gamma(x),\gamma(y)\right] = -8iD(x-y)$$

The covariant relations for the $h_{\mu\nu}$ are

$$(1.71)\quad \left[h_{\mu\nu}(x),h_{\alpha\beta}(y)\right] = \left[\gamma_{\mu\nu}(x)-\delta_{\mu\nu}\gamma(x)/2,\ \gamma_{\alpha\beta}(y)-\delta_{\alpha\beta}\frac{\gamma(y)}{2}\right] =$$

$$i(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta})D(x-y),$$

an expression which is good for all $(\mu\nu)$ and all $(\alpha\beta)$.

The vacuum state, $|0\rangle$, can be defined as the state for which

$$(1.72)\quad c_{\mu\nu}|0\rangle = 0,\ (\mu\nu)\neq(44);\ \phi_{44}|0\rangle = ic_{44}|0\rangle = c_{44}|0\rangle = 0.$$

The vacuum expectation value of $P(h_{\mu\nu}(x)\ h_{\alpha\beta}(y))$, where P is the time ordering operator, is required for later use. By following the same procedure as was followed for the commutators, that is, transforming back from the $\gamma'_{\mu\nu}$ to the $\gamma_{\mu\nu}$ to the $h_{\mu\nu}$, one obtains

$$(1.73)\quad \langle 0|\eta P(h_{\mu\nu}(x)\ h_{\alpha\beta}(y))|0\rangle = \langle 0|P(h_{\mu\nu}(x)\ h_{\alpha\beta}(y))|0\rangle =$$

$$-i(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta})D_F(x-y)$$

where

$$(1.74)\quad D_F(x-y) = \lim_{\varepsilon\to 0}\ (1/(2\pi)^4)\int(1/(k^2-i\varepsilon))\ e^{ik(x-y)}\ d^4k\ .$$

This result is good for all $(\mu\nu)$ and all $(\alpha\beta)$. It agrees with the expression given by Gupta(1952).

As an example of the formalism constructed, the change in energy of two non-quantized mass points, situated at $\vec{x}_1$ and $\vec{x}_2$, due to their gravitational interaction is computed. A suitable interaction Hamiltonian density is, since it leads to the correct field equations (1.9),

$$(1.75)\quad \mathcal{H}_{int} = (\kappa/2)\ T_{44}\ h_{44} = (\kappa/2)\ \mu\ h_{44}$$

where

$$(1.76) \quad \mu = \sum_{n=1}^{2} m_n \, \delta(\vec{x}-\vec{x}_n)$$

and

$$(1.77) \quad h_{44} = \gamma_{44} - \gamma/2 = -\gamma'_{22} - \gamma'_{33}/\sqrt{2} + \gamma'_{44}/\sqrt{2} - \gamma/2$$

$$= -\gamma'_{22} - \gamma'_{33}/\sqrt{2} + i\chi_{44}/\sqrt{2}$$

Since the Hamiltonian (1.75) affects only the (22), (33) and (44) gravitons it is sufficient to label a state by $|n_{22},n_{33},n_{44}\rangle$. The initial and final states contain no gravitons. The non-zero matrix elements are

$$(1.78) \quad \langle 000|H|100\rangle = \langle 001|H|000\rangle^* = A$$

$$\langle 000|H|010\rangle = \langle 010|H|000\rangle^* = A/\sqrt{2}$$

$$\langle 000|H|001\rangle = -\langle 100|H|000\rangle^* = A/\sqrt{2} \ ,$$

where

$$(1.79) \quad H = \int \mathcal{H}_{int} \, d^3x$$

and

$$(1.80) \quad A = -(\kappa/2)(\mu/\sqrt{V})(1/\sqrt{2\omega_k}) \sum_{n=1}^{2} m_n \, e^{i\vec{k}\cdot\vec{x}_n}$$

The first order correction to the energy is zero since there are no non-zero matrix elements of the form $\langle 0|H|0\rangle$. The second order correction is

$$(1.81) \quad U = \sum_k \langle 000|H|n_{22}(\vec{k})n_{33}(\vec{k})n_{44}(\vec{k})\rangle\langle n_{22}(\vec{k})n_{33}(\vec{k})n_{44}(\vec{k})| H|000\rangle/(-\omega_k)$$

$$= (-\kappa^2/8V) \sum_k \sum_n \sum_{\substack{n' \\ n\neq n'}} m_n m_{n'} e^{i\vec{k}\cdot(\vec{x}_n-\vec{x}_{n'})} (1+1/2-1/2)/(\omega_k^2)$$

$$= -(\kappa^2/4V)m_1 m_2 \sum_k e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}/(\omega_k^2)$$

$$= -\kappa^2 m_1 m_2/(16\pi r), \quad r=|\vec{x}_1-\vec{x}_2|.$$

From (1.13) one has

(1.82) $U = -Gm_1 m_2/r$

which is just the Newtonian potential energy.

Attention is now turned to a more detailed treatment of interactions.

2. The Description of the Scalar Field and Its Interaction with the Gravitational Field

In a flat space-time the Lagrangian density for the massless and Hermitian scalar field ϕ, when referred to rectilinear coordinates, is(Corson 1953)

$$(2.1) \qquad \mathcal{L} = -(\delta^{\mu\nu}\, \phi_{|\mu}\, \phi_{|\nu})/2 \ .$$

The field equation which follows from the Lagrangian (2.1) is

$$(2.2) \qquad \Box^2 \phi = 0$$

for which a solution is

$$(2.3) \qquad \phi = (1/\sqrt{V}) \sum_p (1/\sqrt{2\omega_p})(a(\vec{p})e^{ipx} + a^\dagger(\vec{p})e^{-ipx}) \ .$$

When one makes the transition to quantum field theory $a(\vec{p})$ is interpreted as an operator which destroys a scalar particle of momentum p and $a^\dagger(\vec{p})$ is interpreted as an operator which creates a scalar particle of momentum p. The canonical energy-momentum tensor which follows from the Lagrangian (2.1) is

$$(2.4) \qquad T_{\mu\nu} = -\phi_{|\mu}\, \phi_{|\nu} + \delta_{\mu\nu}\, (\phi^{|\alpha}\, \phi_{|\alpha})/2 \ , \quad \phi^{|\alpha} = g^{\alpha\beta}\, \phi_{|\beta} \ .$$

In general relativity one seeks, for a tensor field, a Lagrangian density which

1) transforms as a scalar density under general coordinate transformations, and
2) reduces to the known flat space-time Lagrangian when no gravitational field is present and rectilinear coordinates are used.

Such a Lagrangian density can be obtained from the flat space-time form by

1) replacing the partial derivative by the covariant derivative,
2) replacing the metric $\delta_{\mu\nu}$ by the metric $g_{\mu\nu}$, and
3) introducing a factor $\sqrt{g}$ where g is the determinant of the metric tensor.

For a scalar field the covariant derivative is the same as the partial derivative. Hence, the appropriate generalization of (2.1) is

$$\mathcal{L}' = -\sqrt{g}\, g^{\mu\nu}\, \phi_{|\mu}\, \phi_{|\nu}/2. \tag{2.5}$$

When the gravitational field is weak

$$g^{\mu\nu} = \delta^{\mu\nu} - \varkappa h^{\mu\nu} \tag{2.6}$$

and

$$\sqrt{g} = 1 + \varkappa\delta^{\alpha\beta}h_{\alpha\beta}/2 \tag{2.7}$$

so that

$$\begin{aligned} \mathcal{L}' &= -(1 + \varkappa\delta^{\alpha\beta}h_{\alpha\beta}/2)(\delta^{\mu\nu} - \varkappa h^{\mu\nu})\, \phi_{|\mu}\, \phi_{|\nu}/2 \\ &= -(\delta^{\mu\nu}\, \phi_{|\mu}\, \phi_{|\nu})/2 - (\varkappa/2)h_{\alpha\beta}\, T^{\alpha\beta} \end{aligned} \tag{2.8}$$

where $T^{\alpha\beta}$ is the energy-momentum tensor defined by (2.4). Therefore, the desired quantity, the interaction Lagrangian, is

$$\mathcal{L}_{int} = -(\varkappa/2)h_{\alpha\beta}T_{\alpha\beta} \tag{2.9}$$

where the distinction between upper and lower indices has been dropped since $\delta_{\mu\nu}$ is the Kronecker delta.

3. The Description of the Photon Field and Its Interaction with the Gravitational Field

In a flat space-time the Lagrangian density for the photon field A_μ, when referred to rectilinear coordinates, is(Corson 1953)

$$(3.1)\quad \mathcal{L} = -(1/4)\delta^{\alpha\mu}\,\delta^{\beta\nu}\,F_{\alpha\beta}\,F_{\mu\nu}$$

where

$$(3.2)\quad F_{\mu\nu} = A_{\nu|\mu} - A_{\mu|\nu} \quad (\text{and } A_{\mu|\mu} = 0)\ .$$

The field equations which follow from the Lagrangian (3.1) and the relations (3.2) are

$$(3.3)\quad \Box^2 A_\mu = 0$$

for which solutions are

$$(3.4)\quad A_\mu = (1/\sqrt{V})\sum_p (1/\sqrt{2\omega_p})(a_\tau(\vec{p})e^\tau{}_\mu\, e^{ipx} + a^\dagger{}_\tau(\vec{p})e^{\tau *}_\mu\, e^{-ipx}).$$

where the $e^\tau{}_\mu$ are polarization vectors. When one makes the transition to quantum theory, $a_\tau(\vec{p})$ destroys a photon of momentum $\vec{p}$ and polarization e^τ and $a^\dagger{}_\tau(\vec{p})$ creates a photon of momentum $\vec{p}$ and polarization e^τ.

The energy-momentum tensor which follows from the Lagrangian (3.1) is, after symmetrization,

$$(3.5)\quad T_{\mu\nu} = -F_\mu{}^\alpha F_{\nu\alpha} + (1/4)\delta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\ .$$

In order to find a Lagrangian for the photon field in the realm of general relativity, one applies the rules laid down in Chapter 2. The result is

$$(3.6)\quad \mathcal{L}' = -(1/4)\sqrt{g}\,g^{\alpha\mu}\,g^{\beta\nu}\,F_{\alpha\beta}\,F_{\mu\nu}$$

where

$$(3.7) \quad F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu|\mu} - A_{\mu|\nu} \,.$$

The semicolon denotes covariant differentiation.

When the gravitational field is weak one obtains using (2.6) and (2.7)

$$(3.8) \quad \mathcal{L}' = -(1/4)(1 + \varkappa h_{\gamma\tau}\, \delta^{\gamma\tau}/2)(\delta^{\alpha\mu} - \varkappa h^{\alpha\mu})(\delta^{\beta\nu} - \varkappa h^{\beta\nu}) \times F_{\alpha\beta}F_{\mu\nu}$$

$$= -(1/4)\delta^{\alpha\mu}\, \delta^{\beta\nu}\, F_{\alpha\beta}F_{\mu\nu} - (1/2)\varkappa h_{\alpha\beta}T^{\alpha\beta}$$

where $T^{\alpha\beta}$ is the energy-momentum tensor defined by (3.5). Therefore, the desired quantity, the interaction Lagrangian, is

$$(3.9) \quad \mathcal{L}_{int} = -(1/2)\varkappa h_{\alpha\beta}\, T_{\alpha\beta}$$

where the distinction between upper and lower indices has been dropped since $\delta_{\mu\nu}$ is the Kronecker delta.

4. The Description of the Four Component Neutrino Field and Its Interaction with the Gravitational Field

In a flat space-time the Lagrangian for the massless four-spinor field ϕ-the four component neutrino field, when referred to rectilinear coordinates, is(Corson 1953)

$$(4.1)\quad \mathcal{L} = (1/2)(\bar{\Phi}_{|p}\, \gamma^p\, \phi - \bar{\Phi}\, \gamma^p\, \phi_{|p})$$

where

$$(4.2)\quad \bar{\Phi} = \phi^\dagger\, \gamma^4$$

and where the γ matrices are 4×4 matrices which satisfy

$$(4.3)\quad \gamma^p\gamma^q + \gamma^q\gamma^p = 2\delta^{pq}\ .$$

Calculations are easier if one uses the representation

$$(4.4a)\quad \gamma^p = \begin{bmatrix} 0 & -i\sigma^p \\ i\sigma^p & 0 \end{bmatrix} \quad , \quad \gamma^4 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

where

$$(4.4b)\quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ,\ \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} ,\ \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ,\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The only additional properties of the γ matrices which are required are the following trace theorems which follow from the equation (4.3)

$$(4.5)\quad \mathrm{Tr}(\gamma^p\gamma^q) = 4\delta^{pq}$$

$$\mathrm{Tr}(\gamma^p\gamma^q\gamma^r\gamma^s) = 4(\delta^{pq}\delta^{rs} - \delta^{pr}\delta^{qs} + \delta^{ps}\delta^{qr}).$$

Under an infinitesimal Lorentz transformation with parameters ω_{pq}, ϕ and $\overline{\phi}$ transform as follows

(4.6a) $\phi \to \phi' = (1 + (1/2)\omega_{pq}M^{pq})\phi$

(4.6b) $\overline{\phi} \to \overline{\phi}' = \overline{\phi}(1 + (1/2)\omega_{pq}\overline{M}^{pq})$

where

(4.6c) $M^{pq} = (1/2)\gamma^p\gamma^q$

(4.6d) $\overline{M}^{pq} = (1/2)\gamma^q\gamma^p$.

These expressions are required later when gravitation is introduced.

The field equations which follow from the Lagrangian (4.1) are

(4.7) $\gamma^p\phi_{|p} = 0, \quad \overline{\phi}_{|p}\gamma^p = 0$

for which solutions are

(4.8) $\phi_+ = u(p)e^{i(\vec{p}\cdot\vec{x} - |E|t)}, \quad \phi_- = v(p)e^{i(\vec{p}\cdot\vec{x} + |E|t)}$

where u and v satisfy

(4.9) $(\vec{\gamma}\cdot\vec{p} + i|E|\gamma^4)u = 0, \quad (\vec{\gamma}\cdot\vec{p} - i|E|\gamma^4)v = 0.$

Solutions for u and v are

(4.10) $u_1(\vec{n}) = \frac{1}{\sqrt{2}}\begin{bmatrix}\eta_1\\ \eta_1\end{bmatrix}, \quad u_2(\vec{n}) = \frac{1}{\sqrt{2}}\begin{bmatrix}\eta_2\\ -\eta_2\end{bmatrix},$

$v_1(\vec{n}) = \frac{1}{\sqrt{2}}\begin{bmatrix}\eta_1\\ -\eta_1\end{bmatrix}, \quad v_2(\vec{n}) = \frac{1}{\sqrt{2}}\begin{bmatrix}\eta_2\\ \eta_2\end{bmatrix}$

where

(4.11) $\vec{n} = \vec{p}/|E|$

and where

(4.12)
$$\eta_1 = \begin{cases} \eta_1^+ = (1/\sqrt{2(1+n_z)}) \begin{bmatrix} 1 + n_z \\ n_x + in_y \end{bmatrix} & \text{if } n_z \geq 0 \\ \eta_1^- = (1/\sqrt{2(1+|n_z|)}) \begin{bmatrix} n_x - in_y \\ 1 + |n_z| \end{bmatrix} & \text{if } n_z < 0 \end{cases}$$

$$\eta_2 = \begin{cases} \eta_2^+ = (1/\sqrt{2(1+n_z)}) \begin{bmatrix} -n_x + in_y \\ 1 + n_z \end{bmatrix} & \text{if } n_z \geq 0 \\ \eta_2^- = (1/\sqrt{2(1+|n_z|)}) \begin{bmatrix} 1 + |n_z| \\ -n_x - in_y \end{bmatrix} & \text{if } n_z < 0 \end{cases}$$

The components η_1 and η_2 satisfy

(4.13) $\sigma\cdot\vec{n}\eta_1 = \eta_1$, $\sigma\cdot\vec{n}\eta_2 = -\eta_2$

and have the normalization

(4.14) $\eta_1^\dagger\ \eta_1 = \eta_2^\dagger\ \eta_2 = 1.$

With this normalization for η_1 and η_2, u_r and v_r have the normalization

(4.15) $u_r^\dagger\ u_r = v_r^\dagger\ v_r = 1$

The solutions u_r and v_r can be characterized according to the following scheme(Lurié 1968)

(4.16a) $hu_1 = u_1$, $hu_2 = -u_2$, $hv_1 = v_1$, $hv_2 = -v_2$

(4.16b) $\gamma^5 u_1 = -u_1$, $\gamma^5 u_2 = u_2$, $\gamma^5 v_1 = v_1$, $\gamma^5 v_2 = -v_2$

where h, the helicity operator, and γ^5 are

(4.16c) $h = \begin{bmatrix} \vec{\sigma}\cdot\vec{n} & 0 \\ 0 & \vec{\sigma}\cdot\vec{n} \end{bmatrix}$ $\qquad \gamma^5 = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$

The solutions u_r and v_r have the following additional properties which are required later

(4.17a) $v_1(-\vec{n}) = u_2(\vec{n})$

(4.17b) $v_2(-\vec{n}) = u_1(\vec{n})$

(4.17c) $\sum_{r=1}^{2} u_r(\vec{n})\,\bar{u}_r(\vec{n}) = \sum_{r=1}^{2} v_r(-\vec{n})\,\bar{v}_r(-\vec{n}) = -i(\gamma n)/2$

where

(4.18) $n = (\vec{n}, i)$.

An arbitrary solution of the field equation (4.7) can be expanded as

(4.19) $\phi = (1/\sqrt{V}) \sum_p \sum_r (a_r(\vec{p})u_r(\vec{p})e^{ipx} + b^\dagger{}_r(\vec{p})u_r(\vec{p})e^{-ipx})$.

When one makes the transition to quantum field theory the a_r, b_r, $a^\dagger{}_r$, $b^\dagger{}_r$ are interpreted as operators such that the a_r destroy neutrinos, the b_r destroy antineutrinos, the $a^\dagger{}_r$ create neutrinos and the $b^\dagger{}_r$ create antineutrinos.

The symmetrized energy-momentum tensor is

(4.20) $T_{pq} = (1/4)(\bar{\phi}_{|q}\gamma_p\phi - \bar{\phi}\gamma_p\phi_{|q} + \bar{\phi}_{|p}\gamma_q\phi - \bar{\phi}\gamma_q\phi_{|p})$

The system described by the Lagrangian (4.1) is invariant under the charge conjugation operation and under the space inversion operation in contrast to the system to be described in Chapter 5.

The generalization of the Lagrangian (4.1) to general relativity is not as simple as it was in the previous two cases where tensor fields were considered. This is because ψ transforms according to a representation of the Lorentz group which cannot be extended to the group of all linear transformations, which general coordinate transformations are when considered as changes of the natural basis (to be defined shortly) at each point of the space-time manifold (Weyl 1929). Thus a definition of a covariant derivative of ψ requires special attention. Fock(1929) arrived at a suitable definition by considering the transformation properties of the vector determined by ψ. His method will be applied to the two component spinor in Chapter 5. In this chapter the method of Utiyama(1956) will be used which is similar to that of Weyl(1931). No matter which method is used, the vierbein formalism is required and so this is now described.

At each point of the manifold the set of vectors $\{e_\alpha\}$ which are tangent to the coordinate lines at the given point span a vector space called the tangent space. The set of vectors $\{e_\alpha\}$ is called the natural basis. It is not in general orthonormal at every point but if it is, the space is flat. Contravariant vectors and more generally tensors can be defined with respect to this basis and its tensor products. By the Gram-Schmidt process one can introduce an orthonormal basis $\{\bar{e}_p\}$ at each point which is variously called a vierbein basis, a tetrad or an orthogonal quadruple. Thus, one can write

$$(4.21a)\quad \bar{e}_p = f^{\alpha}{}_p e_{\alpha}$$

and

$$(4.21b)\quad e_{\alpha} = F^{p}{}_{\alpha} \bar{e}_p$$

where

$$(4.22)\quad f^{\alpha}{}_p F^{p}{}_{\beta} = \delta^{\alpha}_{\beta}, \quad f^{\alpha}{}_p F^{q}{}_{\alpha} = \delta^{q}_{p}$$

From the definition (1.1) of the metric tensor one obtains

$$(4.23)\quad g_{\alpha\beta} = F^{p}{}_{\alpha} F^{q}{}_{\beta} \delta_{pq} , \quad \delta_{pq} = f^{\alpha}{}_p f^{\beta}{}_q g_{\alpha\beta}$$

Given a contravariant vector A, its components $\{A^{\alpha}\}$ in the natural basis are related to its components $\{\bar{A}^{p}\}$ in the vierbein basis by

$$(4.24)\quad \bar{A}^{p} = F^{p}{}_{\alpha} A^{\alpha} .$$

For a covariant vector one has

$$(4.25)\quad \bar{A}_p = f^{\alpha}{}_p A_{\alpha}$$

and in particular if

$$(4.26)\quad A_{\alpha} = \partial/\partial x^{\alpha}$$

one may call

$$(4.27)\quad d_p = f^{\alpha}{}_p A_{\alpha}$$

the vierbein partial derivative. For a curved manifold there does not exist a set of coordinates y^p which would allow one to write

$$(4.28)\quad d_p = \partial/\partial y^p .$$

If at each point of the manifold a vierbein frame is defined, so that one has a set of vierbein frames, one can obtain a new set by applying a Lorentz transformation to the original set. The new set of vierbein frames is just as good a set of basis frames as the old set, even when the Lorentz transformation at each point of the manifold depends on the point. Under such a reorientation of frames the components of a vector transform according to a Lorentz transformation. However, the quantity

$$(4.29)\quad d_p\bar{A}^q = f^\alpha{}_p\, \partial\bar{A}^q/\partial x^\alpha$$

which may be called the vierbein partial derivative of the vector A does not transform as a mixed tensor of rank two under coordinate dependent Lorentz transformations. One can, however define a new derivative D_p such that $D_p\bar{A}^q$ does transform as a second rank tensor under coordinate dependent Lorentz transformations. This is done by adding the term(Utiyama 1956)

$$(4.30)\quad -f^\alpha{}_p\, B_\alpha{}^{rs}\, M_{rs}/2$$

to d_p to obtain D_p. The M_{rs} are the matrices which represent the infinitesimal Lorentz transformations for the vector A, and the $B_\alpha{}^{rs}$ are given by

$$(4.31)\quad B_\alpha{}^{rs} = \delta^{sp}\, f^\pi{}_p(F^r{}_{\pi|\alpha} - F^r{}_\beta \Gamma^\beta_{\pi\alpha})$$

$$= -\delta^{sp}\,\delta^{rq}\,F^t{}_\alpha\,\gamma_{pqt},\ (\gamma_{pqt}=\delta_{qs}\, f^\alpha{}_{p;\beta}\, F^s{}_\alpha\, f^\beta{}_t)$$

where the $\Gamma^\beta_{\pi\alpha}$ are the Christoffel symbols and the γ_{pqt} are the Ricci coefficients of rotation. The new quantity

(4.32) $D_p \bar{A}^q = f^{\alpha}{}_p \, (\partial \bar{A}^q/\partial x^{\alpha} - (1/2)B_{\alpha}{}^{rs}(M_{rs})^q_t \, \bar{A}^t)$

is related to the usual covariant derivative by

(4.33) $A^{\alpha}{}_{;\beta} = f^{\alpha}{}_q \, F^p{}_{\beta} \, (D_p \bar{A}^q).$

Hence, $D_p \bar{A}^q$ can be called the vierbein covariant derivative. In summary, $D_p \bar{A}^q$ is a quantity which transforms as a second rank mixed tensor under coordinate dependent Lorentz transformations, and is an invariant under general coordinate transformations.

Utiyama has shown that if ϕ is a spinor, then under coordinate dependent Lorentz transformations the quantity $D_p\phi$ defined by

(4.34) $D_p\phi = f^{\alpha}{}_p(\partial\phi/\partial x^{\alpha} - (1/2)B_{\alpha}{}^{rs} \, M_{rs} \, \phi),$

where the M_{rs} are the matrices which represent the infinitesimal Lorentz transformations of ϕ, transforms as a covariant vector and as a spinor simultaneously.

The passage to general relativity can now be made for a spinor field. One seeks a Lagrangian which

1) transforms as a scalar density under general coordinate transformations,
2) is invariant under coordinate dependent reorientations of the vierbein frames, and
3) reduces to the flat space-time Lagrangian when no gravitational field is present and rectilinear coordinates are used.

Such a Lagrangian density can be obtained from the flat space-time form by

1) replacing the partial derivative by the vierbein covariant derivative $D_p\phi$,
2) replacing the metric $\delta_{\mu\nu}$ by the metric $g_{\mu\nu}$, and
3) introducing a factor $\sqrt{g}$ where g is the determinant of the metric tensor.

When the above rules are applied to the Lagrangian (4.1) one obtains

$$(4.35) \quad \mathcal{L}' = \sqrt{g}((d_p\bar{\Phi})\gamma^p\phi - \bar{\Phi}\gamma^p(d_p\phi))/2$$
$$-\sqrt{g}\bar{\Phi}\, f^\alpha{}_p\, B_\alpha{}^{rs}(M_{rs}\,\gamma^p - \gamma^p\, M_{rs})\phi/2 \;.$$

When the gravitational field is weak one writes

$$(4.36) \quad F^p{}_\alpha = \delta^p{}_\alpha + \eta^p{}_\alpha \quad , \quad f^\alpha{}_p = \delta^\alpha{}_p + \phi^\alpha{}_p$$

but the orthogonality conditions (4.22) imply

$$(4.37) \quad \phi^\alpha{}_p = -\eta^\alpha{}_p$$

and the conditions (4.23) imply

$$(4.38) \quad \eta_{pq} + \eta_{qp} = \varkappa h_{pq} \;.$$

The system is invariant under change of vierbein frame so one may choose

$$(4.39) \quad \eta_{pq} = \varkappa h_{pq}/2.$$

The term $B_\alpha{}^{rs}$ becomes

$$(4.40) \quad B_\alpha{}^{pq} = \varkappa\delta^{p\beta}\,\delta^{q\sigma}(h_{\alpha\sigma|\beta} - h_{\alpha\beta|\sigma})$$

It is shown in Appendix A that when (4.40) and (4.6c,d) are substituted into the Lagrangian (4.35) the second term on the right in equation (4.35) vanishes as pointed out by Vladimirov(1964). Therefore,

$$(4.41) \quad \mathcal{L}' = (1/2)(1 + \varkappa h_{\alpha\beta}\, \delta^{\alpha\beta}/2)\{(\delta^{\alpha}{}_{p} - \eta^{\alpha}{}_{p})\bar{\Phi}_{|\alpha}\, \gamma^{p}\, \phi - \bar{\Phi}\, \gamma^{p}(\delta^{\alpha}{}_{p} - \eta^{\alpha}{}_{p})\phi_{|\alpha}\}$$

$$= (1/2)(\bar{\Phi}_{|p}\, \gamma^{p}\, \phi - \bar{\Phi}\, \gamma^{p}\, \phi_{|p}) - (1/2)\eta^{\alpha}{}_{p}(\bar{\Phi}_{|\alpha}\, \gamma^{p}\, \phi - \bar{\Phi}\, \gamma^{p}\, \phi_{|\alpha}) + (\varkappa/4)h_{\alpha\beta}\, \delta^{\alpha\beta}\, (\bar{\Phi}_{|p}\, \gamma^{p}\, \phi - \bar{\Phi}\, \gamma^{p}\, \phi_{|p})$$

The third term vanishes when the field equations are satisfied. Therefore, one obtains

$$(4.42) \quad \mathcal{L}_{int} = -(1/4)\varkappa h^{\alpha}{}_{p}(\bar{\Phi}_{|\alpha}\, \gamma^{p}\, \phi - \bar{\Phi}\, \gamma^{p}\, \phi_{|\alpha})$$

$$= -(1/2)\varkappa h_{\alpha\beta}\, T_{\alpha\beta}$$

where the symmetry of $h_{\alpha\beta}$ has been used to write the last line and where the distinction between upper and lower indices has been dropped since $\delta_{\mu\nu}$ is the Kronecker delta.

5. The Description of the Two Component Neutrino Field and Its Interaction with the Gravitational Field

In a flat space-time the Lagrangian density for the massless two component spinor field Ψ - the two component neutrino field, when referred to rectilinear coordinates, is(Corson 1953)

$$(5.1)\quad \mathcal{L} = (i/2)(\Psi^{\dagger}\ \sigma^{p}\ \Psi_{|p} - \Psi^{\dagger}{}_{|p}\ \sigma^{p}\ \Psi)$$

where the σ^p are the fundamental spin-tensors. A suitable representation is

$$(5.2)\quad \sigma^1 = (\sigma^{1\dot{A}B}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma^4 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

These spin-tensors have contravariant spinor indices. The corresponding spin-tensors with covariant spinor indices are

$$(5.3)\quad \bar{\sigma}^1 = (\sigma^1{}_{\dot{A}B}) = (\varepsilon_{\dot{A}\dot{C}}\ \varepsilon_{BD}\ \sigma^{1\dot{C}D}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\bar{\sigma}^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \bar{\sigma}^3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ \bar{\sigma}^4 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

The σ and $\bar{\sigma}$ satisfy

$$(5.4a)\quad \bar{\sigma}^{p\,T}\ \sigma^{q} + \bar{\sigma}^{q\,T}\ \sigma^{p} = -2\delta^{pq}$$

$$(5.4b)\quad \sigma^{p}\ \bar{\sigma}^{q\,T} + \sigma^{q}\ \bar{\sigma}^{p\,T} = -2\delta^{pq}$$

from which follow the trace theorems

$$(5.5a) \quad \mathrm{Tr}(\bar{\sigma}^{p\,T} \sigma^q) = \mathrm{Tr}(\sigma^p \bar{\sigma}^{q\,T}) = -2\delta^{pq}$$

$$(5.5b) \quad \mathrm{Tr}(\bar{\sigma}^{p\,T} \sigma^q \bar{\sigma}^{r\,T} \sigma^s) = 2(\delta^{pq}\delta^{rs} - \delta^{pr}\delta^{qs} + \delta^{ps}\delta^{qr})$$

Under an infinitesimal Lorentz transformation with parameters ω_{pq}

$$(5.6a) \quad \Psi \to \Psi' = (I + (1/2)\omega_{pq}M^{pq})\Psi$$

$$(5.6b) \quad \Psi^\dagger \to \Psi'^\dagger = \Psi^\dagger(I + (1/2)\omega_{pq}\bar{M}^{pq})$$

where

$$(5.6c) \quad M^{pq} = -(1/2)\bar{\sigma}^{p\,T} \sigma^q$$

$$(5.6d) \quad \bar{M}^{pq} = -(1/2)\sigma^q \bar{\sigma}^{p\,T}.$$

The field equations which follow from the Lagrangian (5.1) are

$$(5.7) \quad \sigma^p \Psi_{|p} = 0 , \quad \Psi^\dagger_{|p} \sigma^p = 0$$

for which solutions are

$$(5.8) \quad \Psi_+ = u(p)e^{i(\vec{p}\cdot\vec{x} - |E|t)} , \quad \Psi_- = v(p)e^{i(\vec{p}\cdot\vec{x} + |E|t)}$$

where u and v satisfy

$$(5.9) \quad (\vec{\sigma}\cdot\vec{p} - |E|)u = 0 , \quad (\vec{\sigma}\cdot\vec{p} + |E|)v = 0 .$$

Solutions for u and v are

$$(5.10) \quad u(\vec{n}) = \eta_1 , \quad v(\vec{n}) = \eta_2$$

where $\vec{n}$ is given by (4.11) and where η_1 and η_2 are given by (4.12). They are eigenvectors of the helicity operator $\vec{\sigma}\cdot\vec{n}$

(5.11) $\vec{\sigma}\cdot\vec{n}\eta_1 = \eta_1$, $\vec{\sigma}\cdot\vec{n}\eta_2 = -\eta_2$.

The solutions u and v have the following additional properties which are required later

(5.12a) $u(\vec{n})u^{\dagger}(\vec{n}) = -(n\bar{\sigma}^{T})/2$

(5.12b) $v(\vec{n})v^{\dagger}(\vec{n}) = -(n\sigma)/2$

(5.13) $v(-\vec{n}) = u(\vec{n})$

where n is defined by (4.18). An arbitrary solution of the field equation (5.7) can be expanded as

(5.14) $\Psi = (1/\sqrt{V}) \sum_{p} (a(\vec{p})u(\vec{p})e^{ipx} + b^{\dagger}(\vec{p})u(\vec{p})e^{-ipx})$.

When the transition to quantum field theory is made, $a(\vec{p})$ destroys antineutrinos, $a^{\dagger}(\vec{p})$ creates antineutrinos, $b(\vec{p})$ destroys neutrinos and $b^{\dagger}(\vec{p})$ creates neutrinos. Here, the usual convention is followed--neutrinos have negative helicity and antineutrinos have positive helicity.

The symmetrized energy-momentum tensor is

(5.15) $T_{\mu\nu} = (i/4)(\Psi^{\dagger}\sigma_{\mu}\Psi_{|\nu} + \Psi^{\dagger}\sigma_{\nu}\Psi_{|\mu} - \Psi^{\dagger}_{|\nu}\sigma_{\mu}\Psi - \Psi^{\dagger}_{|\mu}\sigma_{\nu}\Psi)$.

The system described by the Lagrangian (5.1) is not invariant under the space reflection and charge conjugation operations but it is invariant under the combined space reflection and charge conjugation operations.

The Lagrangian for the two component neutrino in general relativity, which satisfies the criteria laid down in Chapter 4, can be found by using Utiyama's prescription where the matrices representing the infinitesimal Lorentz transformations are given by (5.6c,d). However, instead of using this method, Fock's method(1929) of generalizing the Dirac equation will be used. Fock's method runs as follows.

From $\Psi^\dagger$, σ^p and Ψ one can form a contravariant (under Lorentz transformations) vector A with vierbein components

$$(5.16) \quad A^p = (\Psi^\dagger \sigma^p \Psi)/2 \,.$$

This vector is defined at each point of the space-time manifold. If one displaces the vector A an amount dx^μ by parallel displacement, the change in the vierbein components is

$$(5.17) \quad \delta(A^p) = -\delta^{pq} \gamma_{qrs} A^r F^s{}_\mu \, dx^\mu \,.$$

Following Fock, one writes, from (5.16)

$$(5.18) \quad \delta(A^p) = (1/2) \, (\delta\Psi^\dagger)\sigma^p\Psi + \Psi^\dagger\sigma^p(\delta\Psi)$$

and assumes that under parallel displacement

$$(5.19) \quad \delta\Psi = -C_s \, \Psi \, F^s{}_\mu \, dx^\mu \quad , \quad \delta\Psi^\dagger = -\Psi^\dagger \, C_s^\dagger \, F^s{}_\mu \, dx^\mu$$

where the C_s are to be determined. By substituting (5.19) into (5.18) and equating the result to (5.17), one obtains

$$(5.20) \quad \Psi^\dagger(C_s^\dagger\sigma^p + \sigma^p C_s)\Psi \, F^s{}_\mu \, dx^\mu = \delta^{pq} \gamma_{qrs} \Psi^\dagger\sigma^r\Psi \, F^s{}_\mu \, dx^\mu$$

or

$$(5.21) \quad C_s^\dagger \, \sigma^p + \sigma^p \, C_s = \delta^{pq} \, \gamma_{qrs} \, \sigma^r$$

for which solutions are

$$(5.22a) \quad C_s = -(1/4)(\bar{\sigma}^{p^T} \sigma^q)\gamma_{pqs}$$

$$(5.22b) \quad C_s^\dagger = -(1/4)(\sigma^q \bar{\sigma}^{p^T})\gamma_{pqs}$$

One now defines the vierbein covariant derivative of the spinor Ψ to be

$$(5.23) \quad D_p\Psi = f^\alpha{}_p \Psi_{|\alpha} - C_p\Psi .$$

This is a satisfactory definition because under coordinate dependent Lorentz transformations it can be shown that $D_p\Psi$ transforms as a vector and a spinor simultaneously. The definition (5.23) is the same as that obtained from Utiyama's procedure.

In general relativity, a Lagrangian which is suitable for describing the two component neutrino can be obtained from the Lagrangian (5.1) by applying the rules laid down in Chapter 4. Thus one replaces $\Psi_{|p}$ and $\Psi^\dagger_{|p}$ with $D_p\Psi$ and and $D_p\Psi^\dagger$ to obtain

$$(5.25) \quad \mathcal{L}' = \sqrt{g}(i/2)\left\{\Psi^\dagger \sigma^p(D_p\Psi) - (D_p\Psi^\dagger)\sigma^p \Psi\right\}$$

$$= \sqrt{g}(i/2)\left\{\Psi^\dagger \sigma^p(d_p\Psi) - (d_p\Psi^\dagger)\sigma^p \Psi\right\} -$$

$$\sqrt{g}(i/2)\Psi^\dagger(C_p^\dagger \sigma^p - \sigma^p C_p)\Psi$$

In the weak field approximation the second term can be shown to vanish by exactly the same procedure as given in Appendix A for the four component neutrino. The first term becomes, with the help of (4.36) and (4.39),

$$(5.26)\quad \mathcal{L}' = (i/2)(\Psi^{\dagger}\,\sigma^{p}\,\Psi_{|p} - \Psi^{\dagger}_{|p}\,\sigma^{p}\,\Psi) + (i/4)\varkappa h^{\alpha}{}_{\alpha}(\Psi^{\dagger}\,\sigma^{p}\,\Psi_{|p} - \Psi^{\dagger}_{|p}\,\sigma^{p}\,\Psi) - (i/4)\varkappa h^{\alpha}{}_{p}(\Psi^{\dagger}\,\sigma^{p}\,\Psi_{|\alpha} - \Psi^{\dagger}_{|\alpha}\,\sigma^{p}\,\Psi)\,.$$

The second term vanishes when the field equations are satisfied so that

$$(5.27)\quad \mathcal{L}_{int} = -(1/2)\varkappa h_{\alpha\beta}\,T_{\alpha\beta}$$

where $T_{\alpha\beta}$ is the energy-momentum tensor defined by (5.15). The symmetry of $h_{\alpha\beta}$ has been used to obtain (5.27) from (5.26) and the distinction between upper and lower indices has been dropped since $\delta_{\mu\nu}$ is the Kronecker delta.

6. The Matrix Elements for the Collisions

In the interaction picture

$$(6.1)\quad i\frac{\partial}{\partial t}|\phi(t)\rangle = H_I(t)\,|\phi(t)\rangle .$$

One lets

$$(6.2)\quad |i\rangle = |\phi(-\infty)\rangle$$

and looks for a solution of (6.1) such that

$$(6.3)\quad |\phi(+\infty)\rangle = S\,|i\rangle$$

Such a solution is(Muirhead 1965)

$$(6.4)\quad S = \sum_{n=0}^{\infty} (-i)^n/n! \int\cdots\int dx_1 \quad dx_n\, P\left[\mathcal{L}_{int}(x_1)\cdots\mathcal{L}_{int}(x_n)\right]$$

$$= \sum_{n=0}^{\infty} S_n$$

where P is the chronological product. One is interested in the matrix element $\langle f|S|i\rangle$ where $|f\rangle$ is some particular final state.

The collisions under consideration are those which take place through the creation and annihilation of a single graviton. The lowest order contribution to such events is contained in S_2 which is

$$(6.5)\quad S_2 = (-1/2)\iint dx\, dy\, P\left[\mathcal{L}_{int}(x)\;\mathcal{L}_{int}(y)\right] .$$

The quantity $P[\quad]$ can be decomposed by means of Wick's Theorem. The term in the decomposition which contributes to the aforementioned collisions is

$$(6.6)\quad I = (\kappa^2/4)\; h_{\mu\nu}(x)\; h_{\alpha\beta}(y)\; N\left[T_{\mu\nu}(x)\; T_{\alpha\beta}(y)\right]$$

where $h_{\mu\nu}(x)\, h_{\alpha\beta}(y)$ is the graviton propagator, given by (1.73), N is the normal product and $T_{\mu\nu}$ is the energy-momentum tensor for the two fields which describe the particles being scattered--if the scattering of a scalar particle and a photon is under consideration then $T_{\mu\nu}$ is the sum of (2.4) and (3.5). The quantity I, given by (6.6), will be referred to as the integrand of the scattering matrix.

One is now in a position to calculate the matrix elements for the scattering problems listed in the Introduction. Only the highlights of the calculations are given in this Chapter. This is because any one calculation can serve as an example, and in Appendix F the detailed calculation for the scattering of two massive scalar particles is given. The latter calculation can serve as the example for the results presented in this Chapter.

Briefly, however, to work out a matrix element one substitutes the appropriate $T_{\mu\nu}$ into the integrand I and expands the product $T_{\mu\nu}(x)T_{\alpha\beta}(y)$, thus obtaining an expansion of the integrand I. The Feynman graphs then follow from the expansion of I. The matrix element, s, is computed by performing the integrations indicated in (6.5) and evaluating the scalar product $\langle f|S_2|i\rangle$.

In the following pages the individual scattering problems are considered. For each, the relevant energy-momentum tensor, the expansion of the integrand I, the Feynman graphs and the matrix elements are given. The positive and negative frequency parts of the fields, that is, those parts which contain e^{-ikx} and e^{+ikx} respectively, are denoted by affixing a + or − to the symbols for the fields. The initial momenta of the two interacting particles are denoted by p and q and the final momenta are denoted by p' and q'. The center-of-momentum frame is used so that

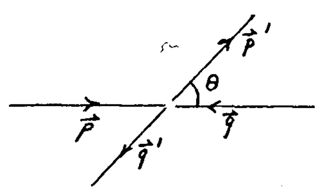
$$\text{(6.7a)} \quad \vec{p} = -\vec{q} \ , \quad \vec{p}\,' = -\vec{q}\,' \ .$$

The particle energies are all equal and this common energy is denoted by p_o. The quantities $\vec{k}^2$ and $\vec{s}^2$ are defined by

$$\text{(6.7b)} \quad \vec{k}^2 = |\vec{p}-\vec{p}\,'|^2 = 4\ p_o^2 \sin^2(\theta/2)$$

$$\vec{s}^2 = |\vec{p}+\vec{p}\,'|^2 = 4\ p_o^2 \cos^2(\theta/2)$$

where θ is the scattering angle:



In the Feynman graphs a dashed line represents a graviton and a solid line represents a scalar particle, photon or neutrino. Where necessary, scalar, photon, and neutrino lines are distinguished by juxtaposing an η, γ or ν respectively. Finally, : : denotes normal ordering.

a) The Matrix Elements for Scalar-Scalar Scattering

The energy-momentum tensor for two massless, Hermitian scalar fields $\overset{1}{\phi}$ and $\overset{2}{\phi}$ is

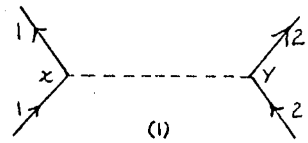
$$(6.8) \quad T_{\mu\nu} = -\sum_{i=1}^{2}\sum_{j=1}^{2} :\!\left(\overset{i}{\phi}_{|\mu}\,\overset{j}{\phi}_{|\nu} - \delta_{\mu\nu}\,\overset{i}{\phi}_{|\tau}\,\overset{j}{\phi}_{|\tau}/2\right)\!: .$$

When this is substituted into I, defined by (6.6), one obtains, after algebraic manipulation

$$\begin{aligned}
(6.9) \quad I &= (\varkappa^2/4)\ \underbrace{h_{\mu\nu}(x)h_{\alpha\beta}(y)}\ N\left[T_{\mu\nu}(x)\ T_{\alpha\beta}(y)\right] \\
&= (2\varkappa^2)\ \underbrace{h_{\mu\nu}(x)\ h_{\alpha\beta}(y)}\times \\
&\Big\{\overset{1}{\phi}{}^{+}_{|\mu}(x)\ \overset{2}{\phi}{}^{+}_{|\nu}(x)\ \overset{1}{\phi}{}^{-}_{|\alpha}(y)\ \overset{2}{\phi}{}^{-}_{|\beta}(y) \\
&-(\delta_{\alpha\beta}/2)\ \overset{1}{\phi}{}^{+}_{|\mu}(x)\ \overset{2}{\phi}{}^{+}_{|\nu}(x)\ \overset{1}{\phi}{}^{-}_{|\tau}(y)\ \overset{2}{\phi}{}^{-}_{|\tau}(y) \\
&-(\delta_{\mu\nu}/2)\ \overset{1}{\phi}{}^{+}_{|\tau}(x)\ \overset{2}{\phi}{}^{+}_{|\tau}(x)\ \overset{1}{\phi}{}^{-}_{|\alpha}(y)\ \overset{2}{\phi}{}^{-}_{|\beta}(y) \\
&+(\delta_{\mu\nu}\ \delta_{\alpha\beta}/4)\ \overset{1}{\phi}{}^{+}_{|\tau}(x)\ \overset{2}{\phi}{}^{+}_{|\tau}(x)\ \overset{1}{\phi}{}^{-}_{|\eta}(y)\ \overset{2}{\phi}{}^{-}_{|\eta}(y) \\
&+\overset{1}{\phi}{}^{+}_{|\mu}(x)\ \overset{2}{\phi}{}^{+}_{|\alpha}(y)\ \overset{1}{\phi}{}^{-}_{|\nu}(x)\ \overset{2}{\phi}{}^{-}_{|\beta}(y) \\
&-(\delta_{\alpha\beta}/2)\ \overset{1}{\phi}{}^{+}_{|\mu}(x)\ \overset{2}{\phi}{}^{+}_{|\tau}(y)\ \overset{1}{\phi}{}^{-}_{|\nu}(x)\ \overset{2}{\phi}{}^{-}_{|\tau}(y) \\
&-(\delta_{\mu\nu}/2)\ \overset{1}{\phi}{}^{+}_{|\tau}(x)\ \overset{2}{\phi}{}^{+}_{|\alpha}(y)\ \overset{1}{\phi}{}^{-}_{|\tau}(x)\ \overset{2}{\phi}{}^{-}_{|\beta}(y) \\
&+(\delta_{\mu\nu}\ \delta_{\alpha\beta}/4)\ \overset{1}{\phi}{}^{+}_{|\eta}(x)\ \overset{2}{\phi}{}^{+}_{|\tau}(y)\ \overset{1}{\phi}{}^{-}_{|\eta}(x)\ \overset{2}{\phi}{}^{-}_{|\tau}(y) \\
&+\overset{1}{\phi}{}^{+}_{|\mu}(x)\ \overset{2}{\phi}{}^{+}_{|\alpha}(y)\ \overset{1}{\phi}{}^{-}_{|\beta}(y)\ \overset{2}{\phi}{}^{-}_{|\nu}(x) \\
&-(\delta_{\alpha\beta}/2)\ \overset{1}{\phi}{}^{+}_{|\mu}(x)\ \overset{2}{\phi}{}^{+}_{|\tau}(y)\ \overset{1}{\phi}{}^{-}_{|\tau}(y)\ \overset{2}{\phi}{}^{-}_{|\nu}(x) \\
&-(\delta_{\mu\nu}/2)\ \overset{1}{\phi}{}^{+}_{|\tau}(x)\ \overset{2}{\phi}{}^{+}_{|\alpha}(y)\ \overset{1}{\phi}{}^{-}_{|\beta}(y)\ \overset{2}{\phi}{}^{-}_{|\tau}(x) \\
&+(\delta_{\mu\nu}\ \delta_{\alpha\beta}/4)\ \overset{1}{\phi}{}^{+}_{|\eta}(x)\ \overset{2}{\phi}{}^{+}_{|\tau}(y)\ \overset{1}{\phi}{}^{-}_{|\tau}(y)\ \overset{2}{\phi}{}^{-}_{|\eta}(x)\Big\} .
\end{aligned}$$

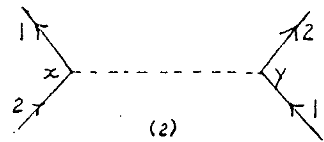
Upon inspection of (6.9) one sees that:

1) the fifth to ninth terms contribute to a process whose graph is



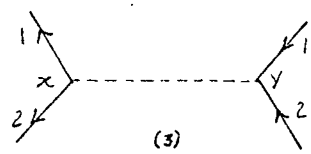
(1)

2) the last four terms contribute to a process whose graph is



(2)

3) the first four terms contribute to a process whose graph is



(3)

The corresponding matrix elements follow from (6.9). They are, in the center-of-momentum frame,

(6.10a) $s_1 = i(\varkappa^2/V^2)(2\pi)^4\ \delta(p+q-p'-q')\left\{\cos^2(\theta/2)/\sin^2(\theta/2)\right\}/2$

(6.10b) $s_2 = i(\varkappa^2/V^2)(2\pi)^4\ \delta(p+q-p'-q')\left\{\sin^2(\theta/2)/\cos^2(\theta/2)\right\}/2$

(6.10c) $s_3 = i(\varkappa^2/V^2)(2\pi)^4\ \delta(p+q-p'-q')\left\{\sin^2\theta\right\}/8$

where p and q denote the initial momenta of the two particles and p' and q' denote the final momenta.

b) The Matrix Elements for Scalar-Photon Scattering

The energy-momentum tensor for the combined scalar field ϕ and photon field A_μ is

$$(6.11)\quad T_{\mu\nu} = :-\phi_{|\mu}\phi_{|\nu} + \delta_{\mu\nu}\,\phi_{|\tau}\phi_{|\tau}/2 - F_{\mu\alpha}F_{\nu\alpha} + \delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta}/4 :$$

When this is substituted into the integrand I, one obtains

$$(6.12)\quad I = (\varkappa^2/4)\; h_{\mu\nu}(x)\; h_{\alpha\beta}(y)\; N\left[T_{\mu\nu}(x)\; T_{\alpha\beta}(y)\right]$$

$$= (2\varkappa^2)\; h_{\mu\nu}(x)\; h_{\alpha\beta}(y)\times$$

$$\{\phi^+_{|\mu}\, A^+_{\gamma|\alpha}\, \phi^-_{|\nu}\, A^-_{\gamma|\beta} - \phi^+_{|\mu}\, A^+_{\gamma|\alpha}\, \phi^-_{|\nu}\, A^-_{\beta|\gamma}$$

$$-\phi^+_{|\mu}\, A^+_{\alpha|\gamma}\, \phi^-_{|\nu}\, A^-_{\gamma|\beta} + \phi^+_{|\mu}\, A^+_{\alpha|\gamma}\, \phi^-_{|\nu}\, A^-_{\beta|\gamma}$$

$$-(\delta_{\alpha\beta}/2)\phi^+_{|\mu}\, A^+_{\gamma|\tau}\, \phi^-_{|\nu}\, A^-_{\gamma|\tau} + (\delta_{\alpha\beta}/2)\phi^+_{|\mu}\, A^+_{\gamma|\tau}\, \phi^-_{|\nu}\, A^-_{\tau|\gamma}$$

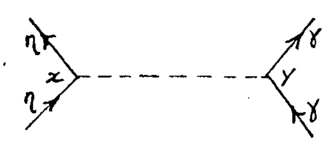
$$-(\delta_{\mu\nu}/2)\phi^+_{|\tau}\, A^+_{\gamma|\alpha}\, \phi^-_{|\tau}\, A^-_{\gamma|\beta} + (\delta_{\mu\nu}/2)\phi^+_{|\tau}\, A^+_{\gamma|\alpha}\, \phi^-_{|\tau}\, A^-_{\beta|\gamma}$$

$$+(\delta_{\mu\nu}/2)\phi^+_{|\tau}\, A^+_{\alpha|\gamma}\, \phi^-_{|\tau}\, A^-_{\gamma|\beta} - (\delta_{\mu\nu}/2)\phi^+_{|\tau}\, A^+_{\alpha|\gamma}\, \phi^-_{|\tau}\, A^-_{\beta|\gamma}$$

$$+(\delta_{\mu\nu}\,\delta_{\alpha\beta}/4)\phi^+_{|\tau}\, A^+_{\gamma|\eta}\, \phi^-_{|\tau}\, A^-_{\gamma|\eta} - (\delta_{\mu\nu}\,\delta_{\alpha\beta}/4)\phi^+_{|\tau}\, A^+_{\gamma|\eta}\, \phi^-_{|\tau}\, A^-_{\eta|\gamma}\}$$

In (6.12) ϕ is a function of x and A_μ is a function of y.

All the terms in (6.12) contribute to a process whose graph is



The corresponding matrix element is, in the center-of-momentum frame and in the Coulomb gauge,

$$(6.13)\quad s = i(\varkappa^2/V^2)(2\pi)^4\;\delta(p+q-p'-q')\;(1/\vec{k}^2)\times$$

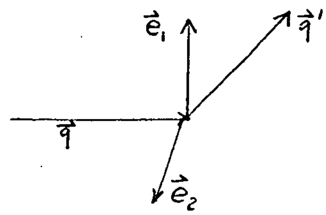
$$\left\{\vec{e}\cdot\vec{e}\,'^{*}(p_o{}^2 + \vec{p}\cdot\vec{p}\,') - (\vec{p}\cdot\vec{e}\,'^{*})(\vec{e}\cdot\vec{p}\,')\right\}.$$

where p and p' denote the initial and final momenta of the scalar particle, q and q' denote the initial and final momenta of the photon, and $\vec{e}$ and $\vec{e}\,'$ denote the initial and final polarization vectors of the photon.

The equation (6.13) has a simpler form if expressed in a different basis. If $\vec{e}_1$ is a unit vector perpendicular to the plane determined by $\vec{q}$ and $\vec{q}\,'$, then

$$(6.14)\quad \vec{e}_2 = \vec{q}\times\vec{e}_1/p_o$$

is a second unit vector which together with $\vec{e}_1$ can be used as a basis for the space spanned by the polarization vectors.



Thus, $\vec{e}$ can be expressed as

$$(6.15)\quad \vec{e} = a_1\vec{e}_1 + a_2\vec{e}_2 \ .$$

In a basis $\vec{e}_+$, $\vec{e}_-$ defined by

$$(6.16)\quad \vec{e}_+ = (\vec{e}_1 + i\vec{e}_2)/\sqrt{2}\ , \quad \vec{e}_- = (\vec{e}_1 - i\vec{e}_2)/\sqrt{2}$$

one has

$$(6.17)\quad \vec{e} = a_+\vec{e}_+ + a_-\vec{e}_-$$

where

$$(6.18)\quad a_+ = (a_1 - ia_2)/\sqrt{2}\ , \ a_- = (a_1 + ia_2)/\sqrt{2} \ .$$

Similarly one can write, for the scattered photon,

$$(6.19)\quad \vec{e}\,' = a_1'\vec{e}_1\,' + a_2'\vec{e}_2\,'$$

where

(6.20) $\vec{e}_1' = \vec{e}_1$, $\vec{e}_2' = \vec{q}' \times \vec{e}_1'/p_o$.

If one introduces

(6.21) $\vec{e}_+' = (\vec{e}_1' + i\vec{e}_2')/\sqrt{2}$, $\vec{e}_-' = (\vec{e}_1' - i\vec{e}_2')/\sqrt{2}$

$a_+' = (a_1' - ia_2')/\sqrt{2}$, $a_-' = (a_1' + ia_2')/\sqrt{2}$

then $\vec{e}'$ becomes

(6.22) $\vec{e}' = a_+'\vec{e}_+' + a_-'\vec{e}_-'$.

In the basis $\{\vec{e}_+, \vec{e}_-, \vec{e}_+', \vec{e}_-'\}$ the matrix element (6.13) is

(6.23) $s = i(\varkappa^2/V^2)(2\pi)^4\ \delta(p+q-p'-q')\ (a_+a_+'^* + a_-a_-'^*)\times$

$\{\cos^2(\theta/2)/\sin^2(\theta/2)\}/2$.

c) The Matrix Element for Scalar-Four Component Neutrino Scattering

The energy-momentum tensor for the combined scalar field $\emptyset$ and four component neutrino field ϕ is

$$(6.21)\quad T_{\mu\nu}(x) = :-\emptyset_{|\mu}\emptyset_{|\nu} + \delta_{\mu\nu}\, \emptyset_{|\tau}\emptyset_{|\tau}/2 + (\overline{\phi}_{|\mu}\, \gamma_\nu\, \phi - \overline{\phi}\, \gamma_\nu\, \phi_{|\mu})/2:$$

When this is substituted into the integrand I, one obtains

$$(6.22)\quad I = (\varkappa^2/4)\, h_{\mu\nu}(x)\, h_{\alpha\beta}(y)\, N\left[T_{\mu\nu}(x)\, T_{\alpha\beta}(y)\right]$$

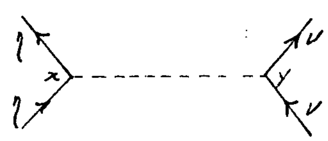
$$= (\varkappa^2/2)\, h_{\mu\nu}(x)\, h_{\alpha\beta}(y) \times$$

$$\Big\{\emptyset^+_{|\mu}\, \overline{\phi}^+\, \gamma_\alpha\, \phi^-_{|\beta}\, \emptyset^-_{|\nu} - \emptyset^+_{|\mu}\, \overline{\phi}^+_{|\alpha}\, \gamma_\beta\, \phi^-\, \emptyset^-_{|\nu}$$

$$-(\delta_{\mu\nu}/2)\emptyset^+_{|\tau}\, \overline{\phi}^+\, \gamma_\alpha\, \phi^-_{|\beta}\, \emptyset^-_{|\tau} + (\delta_{\mu\nu}/2)\emptyset^+_{|\tau}\, \overline{\phi}^+_{|\alpha}\, \gamma_\beta\, \phi^-\, \emptyset^-_{|\tau}\Big\}$$

In (6.22) $\emptyset$ is a function of x and ϕ is a function of y.

All the terms in (6.22) contribute to a process whose graph is



The corresponding matrix element is, in the center-of-momentum frame,

$$(6.23)\quad S = i(\varkappa^2/V^2)(2\pi)^4\, \delta(p+q-p'-q')(1/2)(1/\vec{k}^2)(\vec{p}\cdot\vec{p}' + 3p_0^{\,2})\bar{u}_{r'}(p')\, \gamma_4\, u_r(p)$$

where q and q' denote the initial and final momenta of the scalar particle, p and p' denote the initial and final momenta of the neutrino, and u_r and $u_{r'}$ are the initial and final helicity states of the neutrino.

d) The Matrix Element for Scalar-Two Component Neutrino Scattering

The energy-momentum tensor for the combined scalar field ϕ and two component neutrino field Ψ is

$$(6.24) \quad T_{\mu\nu}(x) = :-\phi_{|\mu}\phi_{|\nu} + \delta_{\mu\nu}\, \phi_{|\tau}\phi_{|\tau}/2 + i(\Psi^{\dagger}_{|\mu}\, \sigma_{\nu}\, \Psi - \Psi^{\dagger}\, \sigma_{\nu}\, \Psi_{|\mu})/2 :$$

When this is substituted into the integrand I, one obtains

$$(6.25) \quad I = (\kappa^2/4)\, \underbrace{h_{\mu\nu}(x)\ h_{\alpha\beta}}_{}(y)\ N\left[T_{\mu\nu}(x)\ T_{\alpha\beta}(y)\right]$$

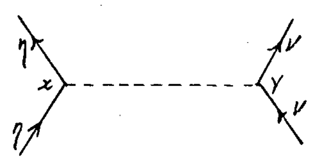
$$= i(\kappa^2/2)\, \underbrace{h_{\mu\nu}(x)\ h_{\alpha\beta}}_{}(y) \times$$

$$N\Big\{\phi^{+}_{|\mu}\, \phi^{-}_{|\nu}\, \Psi^{\dagger -}\, \sigma_{\alpha}\, \Psi^{+}_{|\beta} - \phi^{+}_{|\mu}\, \phi^{-}_{|\nu}\, \Psi^{\dagger -}_{|\alpha}\, \sigma_{\beta}\, \Psi^{+}$$

$$-(\delta_{\mu\nu}/2)\phi^{+}_{|\tau}\, \phi^{-}_{|\tau}\, \Psi^{\dagger -}\, \sigma_{\alpha}\, \Psi^{+}_{|\beta} + (\delta_{\mu\nu}/2)\phi^{+}_{|\tau}\, \phi^{-}_{|\tau}\, \Psi^{\dagger -}_{|\alpha}\, \sigma_{\beta}\, \Psi^{+}\Big\}$$

In (6.25) ϕ is a function of x and Ψ is a function of y.

All the terms in (6.25) contribute to a process whose graph is



The corresponding matrix element is, in the center-of-momentum frame,

$$(6.26) \quad s = -i(\kappa^2/V^2)(2\pi)^4\ \delta(p+q-p'-q')(1/2)(1/\vec{k}^2)(\vec{p}\cdot\vec{p}' + 3p_0^{\,2})\ u^{\dagger}(p)\ u(p')$$

where q and q' denote the initial and final momenta of the scalar particle, and p and p' denote the initial and final momenta of the neutrino.

e) The Matrix Element for Photon-Four Component Neutrino Scattering

The energy-momentum tensor for the combined photon field A_μ and four component neutrino field ϕ is

$$(6.27)\quad T_{\mu\nu}(x) = :-F_{\mu\alpha}F_{\nu\alpha} + \delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta} + (\bar{\phi}_{|\mu}\,\gamma_\nu\,\phi - \bar{\phi}\,\gamma_\nu\,\phi_{|\mu})/2:$$

When this is substituted into I, one obtains

$$(6.28)\quad I = (\kappa^2/4)\; h_{\mu\nu}(x)\; h_{\alpha\beta}(y)\; N\left[T_{\mu\nu}(x)\; T_{\alpha\beta}(y)\right]$$

$$= (\kappa^2/2)\; h_{\mu\nu}(x)\; h_{\alpha\beta}(y) \times$$

$$N\Big\{\bar{\phi}^+\,\gamma_\mu\,\phi^-_{|\nu}\,A^+_{\tau|\alpha}\,A^-_{\tau|\beta} - \bar{\phi}^+\,\gamma_\mu\,\phi^-_{|\nu}\,A^+_{\tau|\alpha}\,A^-_{\beta|\tau}$$

$$-\bar{\phi}^+\,\gamma_\mu\,\phi^-_{|\nu}\,A^+_{\alpha|\tau}\,A^-_{\tau|\beta} + \bar{\phi}^+\,\gamma_\mu\,\phi^-_{|\nu}\,A^+_{\alpha|\tau}\,A^-_{\beta|\tau}$$

$$-(\delta_{\alpha\beta}/2)\bar{\phi}^+\,\gamma_\mu\,\phi^-_{|\nu}\,A^+_{\tau|\eta}\,A^-_{\tau|\eta} + (\delta_{\alpha\beta}/2)\bar{\phi}^+\,\gamma_\mu\,\phi^-_{|\nu}\,A^+_{\tau|\eta}\,A^-_{\eta|\tau}$$

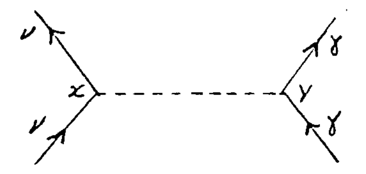
$$-\bar{\phi}^+_{|\mu}\,\gamma_\nu\,\phi^-\,A^+_{\tau|\alpha}\,A^-_{\tau|\beta} + \bar{\phi}^+_{|\mu}\,\gamma_\nu\,\phi^-\,A^+_{\tau|\alpha}\,A^-_{\beta|\tau}$$

$$+\bar{\phi}^+_{|\mu}\,\gamma_\nu\,\phi^-\,A^+_{\alpha|\tau}\,A^-_{\tau|\beta} - \bar{\phi}^+_{|\mu}\,\gamma_\nu\,\phi^-\,A^+_{\alpha|\tau}\,A^-_{\beta|\tau}$$

$$+(\delta_{\alpha\beta}/2)\bar{\phi}^+_{|\mu}\,\gamma_\nu\,\phi^-\,A^+_{\tau|\eta}\,A^-_{\tau|\eta} - (\delta_{\alpha\beta}/2)\bar{\phi}^+_{|\mu}\,\gamma_\nu\,\phi^-\,A^+_{\tau|\eta}\,A^-_{\eta|\tau}\Big\}$$

In (6.28) ϕ is a function of x and A_μ is a function of y.

All the terms in (6.28) contribute to a process whose graph is



The corresponding matrix element is, in the center-of-momentum frame and in the Coulomb gauge,

$$(6.29) \quad s = -(\varkappa^2/V^2)(2\pi)^4\delta(p+q-p'-q')(1/8p_o)(1/\vec{k}^2)\ e_j\ e_i'^*\times$$
$$\bar{u}_{r'}(p')\Big\{((\gamma q') + (\gamma q))\Big[\delta_{ij}(-\vec{p}\cdot\vec{p}' - 3p_o^{\,2}) +$$
$$q_i q_j'\Big] + (\gamma_j q_i + \gamma_i q_j')(4p_o^{\,2})\Big\}u_r(p)$$

where q and q' denote the initial and final momenta of the photon, $\vec{e}$ and $\vec{e}'$ denote the initial and final polarization vectors of the photon, p and p' denote the initial and final momenta of the neutrino, and u_r and $u_{r'}$ denote the initial and final helicity states of the neutrino.

f) The Matrix Element for Photon-Two Component Neutrino Scattering

The energy-momentum tensor for the combined photon field A_μ and two component neutrino field Ψ is

$$(6.30)\quad T_{\mu\nu}(x) = : -F_{\mu\alpha}F_{\nu\alpha} + \delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta} + i(\Psi^\dagger_{|\mu}\sigma_\nu\Psi - \Psi^\dagger\sigma_\nu\Psi_{|\mu})/2 :$$

When this is substituted into I, one obtains

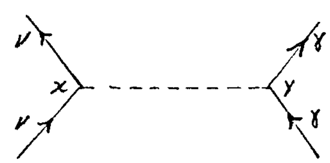
$$(6.31)\quad I = (\varkappa^2/4)\, h_{\mu\nu}(x)\, h_{\alpha\beta}(y)\, N\left[T_{\mu\nu}(x)\, T_{\alpha\beta}(y)\right]$$

$$= i(\varkappa^2/2)\, h_{\mu\nu}(x)\, h_{\alpha\beta}(y) \times$$

$$\begin{aligned} N\Big\{ & \Psi^{\dagger-}\sigma_\mu\Psi^+_{|\nu}A^+_{\tau|\alpha}A^-_{\tau|\beta} - \Psi^{\dagger-}\sigma_\mu\Psi^+_{|\nu}A^+_{\tau|\alpha}A^-_{\beta|\tau} \\ & -\Psi^{\dagger-}\sigma_\mu\Psi^+_{|\nu}A^+_{\alpha|\tau}A^-_{\tau|\beta} + \Psi^{\dagger-}\sigma_\mu\Psi^+_{|\nu}A^+_{\alpha|\tau}A^-_{\beta|\tau} \\ & -(\delta_{\alpha\beta}/2)\Psi^{\dagger-}\sigma_\mu\Psi^+_{|\nu}A^+_{\tau|\eta}A^-_{\tau|\eta} + (\delta_{\alpha\beta}/2)\Psi^{\dagger-}\sigma_\mu\Psi^+_{|\nu}A^+_{\tau|\eta}A^-_{\eta|\tau} \\ & -\Psi^{\dagger-}_{|\mu}\sigma_\nu\Psi^+A^+_{\tau|\alpha}A^-_{\tau|\beta} + \Psi^{\dagger-}_{|\mu}\sigma_\nu\Psi^+A^+_{\tau|\alpha}A^-_{\beta|\tau} \\ & +\Psi^{\dagger-}_{|\mu}\sigma_\nu\Psi^+A^+_{\alpha|\tau}A^-_{\tau|\beta} - \Psi^{\dagger-}_{|\mu}\sigma_\nu\Psi^+A^+_{\alpha|\tau}A^-_{\beta|\tau} \\ & +(\delta_{\alpha\beta}/2)\Psi^{\dagger-}_{|\mu}\sigma_\nu\Psi^+A^+_{\tau|\eta}A^-_{\tau|\eta} - (\delta_{\alpha\beta}/2)\Psi^{\dagger-}_{|\mu}\sigma_\nu\Psi^+A^+_{\tau|\eta}A^-_{\eta|\tau}\Big\} \end{aligned}$$

In (6.31) Ψ is a function of x and A_μ is a function of y.

All the terms in (6.31) contribute to a process whose graph is



The corresponding matrix element is, in the center-of-momentum frame and in the Coulomb gauge,

$$(6.32) \quad s = -i(\varkappa^2/V^2)(2\pi)^4\, \delta(p+q-p'-q')(1/8p_o)(1/\vec{k}^2)\, e_j\, e_i'^{*} \times$$
$$u^{\dagger}(p)\Big\{((\sigma q') + (\sigma q))\Big[\delta_{ij}(-\vec{p}\cdot\vec{p}' - 3p_o^2) +$$
$$q_i q_j'\Big] + (\sigma_j q_i + \sigma_i q_j')(4p_o^2)\Big\}\, u(p')$$

where q and q' denote the initial and final momenta of the photon, $\vec{e}$ and $\vec{e}'$ denote the initial and final polarization vectors of the photon and p and p' denote the initial and final momenta of the neutrino.

g) The Matrix Elements for Four Component Neutrino-Four Component Neutrino Scattering

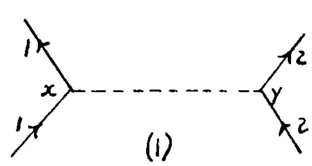
The energy-momentum tensor for two, four component neutrino fields, $\overset{1}{\phi}$ and $\overset{2}{\phi}$, is

$$(6.33)\quad T_{\mu\nu}(x) = (1/2) \sum_{i=1}^{2} \sum_{j=1}^{2} :(\overset{i}{\bar{\phi}}_{|\mu}\, \gamma_\nu\, \overset{j}{\phi} - \overset{i}{\bar{\phi}}\, \gamma_\nu\, \overset{j}{\phi}_{|\mu}):$$

When this is substituted into I, one obtains

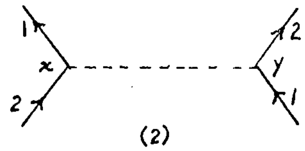
$$
\begin{aligned}
(6.34)\quad I &= (\kappa^2/4)\, \underbrace{h_{\mu\nu}(x)\, h_{\alpha\beta}(y)}\, N\left[T_{\mu\nu}(x)\, T_{\alpha\beta}(y)\right] \\
&= (\kappa^2/2)\, \underbrace{h_{\mu\nu}(x)\, h_{\alpha\beta}(y)} \times \\
& N\Big\{\overset{1}{\bar{\phi}}{}^{+}(x)\, \gamma_\mu\, \overset{1}{\phi}{}^{-}_{|\nu}(x)\, \overset{2}{\bar{\phi}}{}^{+}(y)\, \gamma_\alpha\, \overset{2}{\phi}{}^{-}_{|\beta}(y) \\
& -\overset{1}{\bar{\phi}}{}^{+}(x)\, \gamma_\mu\, \overset{1}{\phi}{}^{-}_{|\nu}(x)\, \overset{2}{\bar{\phi}}{}^{+}_{|\beta}(y)\, \gamma_\alpha\, \overset{2}{\phi}{}^{-}(y) \\
& -\overset{1}{\bar{\phi}}{}^{+}_{|\nu}(x)\, \gamma_\mu\, \overset{1}{\phi}{}^{-}(x)\, \overset{2}{\bar{\phi}}{}^{+}(y)\, \gamma_\alpha\, \overset{2}{\phi}{}^{-}_{|\beta}(y) \\
& +\overset{1}{\bar{\phi}}{}^{+}_{|\nu}(x)\, \gamma_\mu\, \overset{1}{\phi}{}^{-}(x)\, \overset{2}{\bar{\phi}}{}^{+}_{|\beta}(y)\, \gamma_\alpha\, \overset{2}{\phi}{}^{-}(y) \\
& +\overset{1}{\bar{\phi}}{}^{+}(x)\, \gamma_\mu\, \overset{2}{\phi}{}^{-}_{|\nu}(x)\, \overset{2}{\bar{\phi}}{}^{+}(y)\, \gamma_\alpha\, \overset{1}{\phi}{}^{-}_{|\beta}(y) \\
& -\overset{1}{\bar{\phi}}{}^{+}(x)\, \gamma_\mu\, \overset{2}{\phi}{}^{-}_{|\nu}(x)\, \overset{2}{\bar{\phi}}{}^{+}_{|\beta}(y)\, \gamma_\alpha\, \overset{1}{\phi}{}^{-}(y) \\
& -\overset{1}{\bar{\phi}}{}^{+}_{|\nu}(x)\, \gamma_\mu\, \overset{2}{\phi}{}^{-}(x)\, \overset{2}{\bar{\phi}}{}^{+}(y)\, \gamma_\alpha\, \overset{1}{\phi}{}^{-}_{|\beta}(y) \\
& +\overset{1}{\bar{\phi}}{}^{+}_{|\nu}(x)\, \gamma_\mu\, \overset{2}{\phi}{}^{-}(x)\, \overset{2}{\bar{\phi}}{}^{+}_{|\beta}(y)\, \gamma_\alpha\, \overset{1}{\phi}{}^{-}(y)\Big\}
\end{aligned}
$$

Upon inspection of (6.34) one sees that the first four terms contribute to a process whose graph is



(1)

and that the last four terms contribute to a process whose graph is



(2)

The corresponding matrix elements are, in the center-of-momentum frame,

$$(6.35a)\quad s_1 = i(\varkappa^2/V^2)(2\pi)^4\ \delta(p+q-p'-q')(1/8\vec{k}^2)\Big\{(\vec{p}\cdot\vec{p}' + 3p_o^2)\bar{u}_{r'}(p')\ \gamma_\mu\ u_r(p)\ \bar{u}_{s'}(q')\ \gamma_\mu\ u_s(q) + (8p_o^2)\bar{u}_{r'}(p')\ \gamma_4\ u_r(p)\ \bar{u}_{s'}(q')\ \gamma_4\ u_s(q)\Big\}$$

$$(6.35b)\quad s_2 = -i(\varkappa^2/V^2)(2\pi)^4\ \delta(p+q-p'-q')(1/8\vec{s}^2)\Big\{(-\vec{p}\cdot\vec{p}' + 3p_o^2)\bar{u}_{r'}(p')\ \gamma_\mu\ u_s(q)\ \bar{u}_{s'}(q')\ \gamma_\mu\ u_r(p) + (8p_o^2)\bar{u}_{r'}(p')\ \gamma_4\ u_s(q)\ \bar{u}_{s'}(q')\ \gamma_4 u_r(p)\Big\}$$

where p, p' and u_r, $u_{r'}$ denote the initial and final momenta and helicity states of one neutrino and q, q' and u_s, $u_{s'}$ denote the corresponding quantities for the second neutrino.

h) The Matrix Elements for Two Component Neutrino-Two Component Neutrino Scattering

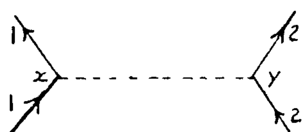
The energy-momentum tensor for two, two component neutrino fields $\overset{1}{\Psi}$ and $\overset{2}{\Psi}$, is

$$(6.36)\quad T_{\mu\nu}(x) = (i/2) \sum_{i=1}^{2} \sum_{j=2}^{2} :(\overset{i}{\Psi}{}^{\dagger}_{|\mu}\, \sigma_\nu\, \overset{j}{\Psi} - \overset{i}{\Psi}{}^{\dagger}\, \sigma_\nu\, \overset{j}{\Psi}_{|\mu}):$$

When this is substituted into I, one obtains

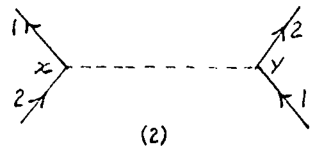
$$\begin{aligned}
(6.37)\quad I &= (\varkappa^2/4)\, h_{\mu\nu}(x)\, h_{\alpha\beta}(y)\, N\left[T_{\mu\nu}(x)\, T_{\alpha\beta}(y)\right] \\
&= -(\varkappa^2/2)\, h_{\mu\nu}(x)\, h_{\alpha\beta}(y) \\
&\quad N\Big\{\overset{1}{\Psi}{}^{\dagger -}(x)\, \sigma_\nu\, \overset{1}{\Psi}{}^{+}_{|\mu}(x)\, \overset{2}{\Psi}{}^{\dagger -}(y)\, \sigma_\beta\, \overset{2}{\Psi}{}^{+}_{|\alpha}(y) \\
&\quad -\overset{1}{\Psi}{}^{\dagger -}(x)\, \sigma_\nu\, \overset{1}{\Psi}{}^{+}_{|\mu}(x)\, \overset{2}{\Psi}{}^{\dagger -}_{|\alpha}(y)\, \sigma_\beta\, \overset{2}{\Psi}{}^{+}(y) \\
&\quad -\overset{1}{\Psi}{}^{\dagger -}_{|\mu}(x)\, \sigma_\nu\, \overset{1}{\Psi}{}^{+}(x)\, \overset{2}{\Psi}{}^{\dagger -}(y)\, \sigma_\beta\, \overset{2}{\Psi}{}^{+}_{|\alpha}(y) \\
&\quad +\overset{1}{\Psi}{}^{\dagger -}_{|\mu}(x)\, \sigma_\nu\, \overset{1}{\Psi}{}^{+}(x)\, \overset{2}{\Psi}{}^{\dagger -}_{|\alpha}(y)\, \sigma_\beta\, \overset{2}{\Psi}{}^{+}(y) \\
&\quad +\overset{2}{\Psi}{}^{\dagger -}(x)\, \sigma_\nu\, \overset{1}{\Psi}{}^{+}_{|\mu}(x)\, \overset{1}{\Psi}{}^{\dagger -}(y)\, \sigma_\beta\, \overset{2}{\Psi}{}^{+}_{|\alpha}(y) \\
&\quad -\overset{2}{\Psi}{}^{\dagger -}(x)\, \sigma_\nu\, \overset{1}{\Psi}{}^{+}_{|\mu}(x)\, \overset{1}{\Psi}{}^{\dagger -}_{|\alpha}(y)\, \sigma_\beta\, \overset{2}{\Psi}{}^{+}(y) \\
&\quad -\overset{2}{\Psi}{}^{\dagger -}_{|\mu}(x)\, \sigma_\nu\, \overset{1}{\Psi}{}^{+}(x)\, \overset{1}{\Psi}{}^{\dagger -}(y)\, \sigma_\beta\, \overset{2}{\Psi}{}^{+}_{|\alpha}(y) \\
&\quad +\overset{2}{\Psi}{}^{\dagger -}_{|\mu}(x)\, \sigma_\nu\, \overset{1}{\Psi}{}^{+}(x)\, \overset{1}{\Psi}{}^{\dagger -}_{|\alpha}(y)\, \sigma_\beta\, \overset{2}{\Psi}{}^{+}(y)\Big\}
\end{aligned}$$

Upon inspection of (6.37) one sees that the first four terms contribute to a process whose graph is



(1)

and that the last four terms contribute to a process whose graph is



(2)

The corresponding matrix elements are, in the center-of-momentum frame,

$$(6.38a)\quad s_1 = -i(\varkappa^2/V^2)(2\pi)^4\ \delta(p+q-p'-q')(1/8\vec{k}^2)\Big\{(\vec{p}\cdot\vec{p}' + 3p_o^{\,2})u^\dagger(p)\ \sigma_\mu\ u(p')\ u^\dagger(q)\ \sigma_\mu\ u(q') - (8p_o^{\,2})u^\dagger(p)\ u(p')\ u^\dagger(q)\ u(q')\Big\}$$

$$(6.38b)\quad s_2 = i(\varkappa^2/V^2)(2\pi)^4\ \delta(p+q-p'-q')(1/8\vec{s}^2)\Big\{(-\vec{p}\cdot\vec{p}' + 3p_o^{\,2})u^\dagger(q)\ \sigma_\mu\ u(p')\ u^\dagger(p)\ \sigma_\mu u(q') - (8p_o^{\,2})u^\dagger(q)\ u(p')\ u^\dagger(p)\ u(q')\Big\}$$

where p and p' denote the initial and final momenta of one neutrino and q and q' denote the corresponding quantities for the other neutrino.

7. The Cross-sections for the Collisions

For each matrix element given in Chapter 6 a quantity M is defined by

$$(7.1) \quad s = i(2\pi)^4\ \delta(p+q-p'-q')\ M\ .$$

The transition probability per unit of space-time is then

$$(7.2) \quad \omega = (2\pi)^4\ \delta(p+q-p'-q')\ |M|^2$$

and the cross-section is

$$(7.3) \quad d\sigma = (\omega/J)\ dN$$

where J is the incoming flux density and dN is the number of final states(Muirhead 1965).

Following Muirhead(1965) one obtains for the cross-section for unpolarized particles in the center-of-momentum frame

$$(7.4) \quad d\sigma/d\Omega = (V^4/(2\pi)^2)(p_o^{\ 2}/4)\ \overline{\sum_i}\ \sum_f |M|^2$$

where $\overline{\sum_i}$ indicates an average over initial spin states and $\sum_f$ indicates a sum over final spin states.

With the help of the results obtained in the previous Chapters, the cross-sections for the collisions listed in the Introduction can now be worked out.

a) The Cross-section for Scalar-Scalar Scattering

The matrix element is the sum of (6.10a), (6.10b) and (6.10c). Therefore, from (7.1),

$$(7.5)\quad M = (\varkappa^2/2V^2)\Big[\cos^2(\theta/2)/\sin^2(\theta/2) + \sin^2(\theta/2)/\cos^2(\theta/2) + (\sin^2\theta)/4\Big].$$

Since scalar particles have spin 0, there are no polarization sums. Hence, from (7.4),

$$(7.6)\quad d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)(p_o{}^2/16)\Big[\cos^2(\theta/2)/\sin^2(\theta/2) + \sin^2(\theta/2)/\cos^2(\theta/2) + (\sin^2\theta)/4\Big]^2$$

b) The Cross-section for Scalar-Photon Scattering

From (6.23) and (7.1) one obtains

$$(7.7)\quad M = (\varkappa^2/2V^2)\ (a_+ a_+'^{*} + a_- a_-'^{*})\ \cos^2(\theta/2)/\sin^2(\theta/2)\ .$$

For a photon there are two possible initial states corresponding to

$$(7.8)\quad |a_+|^2 = 1 \text{ and } |a_-|^2 = 1$$

and two final states corresponding to

$$(7.9)\quad |a_+'|^2 = 1 \text{ and } |a_-'|^2 = 1\ .$$

Therefore, from (7.4),

$$(7.10)\quad d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)\ (p_0^{\,2}/16)\ \cos^4(\theta/2)/\sin^4(\theta/2)\ .$$

c) The Cross-section for Scalar-Four Component Neutrino Scattering

From (6.23) and (7.1) one obtains

$$(7.11)\quad M = (\varkappa^2/V^2)(1/2\vec{k}^2)(\vec{p}\cdot\vec{p}\,' + 3p_o^{\,2})\ \bar{u}_{r'}(p')\ \gamma_4\ u_r(p)\ .$$

From (7.4) one has, therefore,

$$(7.12)\quad d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)(p_o^{\,2}/16\vec{k}^2)(\vec{p}\cdot\vec{p}\,' + 3p_o^{\,2})^2 \times \overline{\sum_r}\ \sum_{r'} \left|\bar{u}_{r'}(p')\ \gamma_4\ u_r(p)\right|^2 .$$

For the four component neutrino there are two initial and two final states. Hence,

$$(7.13)\quad \overline{\sum_r}\ \sum_{r'} \left|\bar{u}_{r'}(p')\ \gamma_4\ u_r(p)\right|^2$$
$$= (1/2)\sum_r \sum_{r'}\ \bar{u}_{r'}(p')\ \gamma_4\ u_r(p)\ \bar{u}_r(p)\ \gamma_4\ u_{r'}(p')$$
$$= (1/2)\sum_{r'}\ \bar{u}_{r'}(p')\ \gamma_4\ \left[-i(\gamma p)/2p_o\right]\ \gamma_4\ u_{r'}(p')$$
$$= (-1/8p_o^{\,2})\ \mathrm{Trace}\left[(\gamma p')\ \gamma_4\ (\gamma p)\ \gamma_4\right]$$
$$= \cos^2(\theta/2)$$

where the Hermitian property of γ_4 in the representation (4.4a), the property (4.17c) of the solutions $u_r(p)$, and the trace theorems for the γ matrices(Muirhead 1965) have been used. The substitution of (7.13) back into (7.12) and the use of (6.7b) yields

$$(7.14)\quad d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)(p_o^{\,2}/64)\left[\cos^4(\theta/2)/\sin^4(\theta/2)\right]\times (1 + \cos^2(\theta/2))^2$$

d) The Cross-section for Scalar-Two Component Neutrino Scattering

From (6.26) and (7.1) one obtains

$$(7.15) \quad M = -(\varkappa^2/V^2)(1/2\vec{k}^2)(\vec{p}\cdot\vec{p}\,' + 3p_o^{\,2})\; u^\dagger(p)\; u(p')\; .$$

There is no polarization sum for a two component neutrino since there is only one initial and one final state. Therefore, from (7.4),

$$(7.16) \quad d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)(p_o^{\,2}/16\vec{k}^2)(\vec{p}\cdot\vec{p}\,' + 3p_o^{\,2})\left|u^\dagger(p)u(p')\right|^2$$

Now,

$$\begin{aligned}(7.17) \quad \left|u^\dagger(p)\; u(p')\right|^2 &= u^\dagger(p)\; u(p')\; u^\dagger(p')\; u(p) \\ &= -\; u^\dagger(p)\left[(p'\bar{\sigma}\;)/2p_o\right]\; u(p) \\ &= (1/4p_o^{\,2})\mathrm{Trace}\left[(p\bar{\sigma}^T)\;(p'\bar{\sigma}^T)\right] \\ &= -(1/4p_o^{\,2})\mathrm{Trace}\left[(p\bar{\sigma}^T)\;\sigma_4\;(p'\bar{\sigma}^T)\;\sigma_4\right] \\ &= \cos^2(\theta/2)\end{aligned}$$

where the properties of σ_4 in the representation (5.2), the trace theorem (5.5b) and the property (5.12a) of the solution u(p) have been used. The substitution of (7.17) into (7.16) and the use of (6.7b) yield

$$(7.18) \quad d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)(p_o^{\,2}/64)\left[\cos^4(\theta/2)/\sin^4(\theta/2)\right]\times \\ (1 + \cos^2(\theta/2))^2\; .$$

e) The Cross-section for Photon-Four Component Neutrino Scattering

With the help of

$$(7.19)\quad \bar{u}_{r'}(p')\left[(\gamma q') + (\gamma q)\right] u_r(p) = i4\gamma_4\, p_o\, \bar{u}_{r'}(p')\, u_r(p)$$

one obtains, from (6.29) and (7.1),

$$(7.20)\quad M = i(\varkappa^2/V^2)(1/8p_o\vec{k}^2)\, e_i'^{*}\, e_j\, \bar{u}_{r'}(p')\, X_{ij}\, u_r(p)$$

where

$$(7.21)\quad X_{ij} = (4i\gamma_4 p_o)\left[\delta_{ij}(-\vec{p}\cdot\vec{p}' - 3p_o^2) + q_i q_j'\right] + (\gamma_j\, q_i + \gamma_i\, q_j')(4p_o^2)\ .$$

Therefore, from (7.4) one has

$$(7.22)\quad d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)(p_o^2/4)(1/64p_o^2\vec{k}^2)\ \overline{\sum_i}\sum_f \left|e_i'^{*} e_j\, \bar{u}_{r'}(p')\, X_{ij}\, u_r(p)\right|^2$$

Now,

$$(7.23)\quad \overline{\sum_i}\sum_f \left|\quad\right|^2 = (-1/8p_o^2)\ \overline{\sum_e}\sum_{e'} (e_i'^{*} e_j\, e_k'\, e_l^{*}) \times \mathrm{Trace}\left[(\gamma p')\, X_{ij}\, (\gamma p)\, \tilde{X}_{kl}\right]$$

where (4.17c) has been used and where

$$(7.24)\quad \tilde{X}_{ij} = \gamma_4\, X_{ij}^{\dagger}\, \gamma_4\ .$$

The trace is evaluated in Appendix B and the photon polarization sum is worked out in Appendix C. One obtains

$$(7.25)\quad \overline{\sum_i}\sum_f \left|\quad\right|^2 = 64p_o^6\left[\cos^2(\theta/2) - \cos^4(\theta/2) + 4\cos^6(\theta/2)\right].$$

The substitution of (7.25) into (7.22) and the use of (6.7b) yield

$$(7.26)\quad d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)(p_o{}^2/64)(1/\sin^4(\theta/2)) \times \left[\cos^2(\theta/2) - \cos^4(\theta/2) + 4\cos^6(\theta/2)\right]$$

f) The Cross-section for Photon-Two Component Neutrino Scattering

With the help of

(7.27) $u^\dagger(p)\left[(\sigma q') + (\sigma q)\right]u(p') = -4p_o u^\dagger(p)u(p')$

one obtains, from (6.32) and (7.1),

(7.28) $M = -(\varkappa^2/V^2)(1/8p_o\vec{k}^2)e_i'^{*}e_j\ u^\dagger(p)\ X_{ij}\ u(p')$

where

(7.29) $$X_{ij} = -4p_o\left[\delta_{ij}(-\vec{p}\cdot\vec{p}' - 3p_o^{\ 2}) + q_i q_j'\right] + (\sigma_j q_i + \sigma_i q_j')(4p_o^{\ 2})$$

Therefore, from (7.4) one has

(7.30) $$d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)(p_o^{\ 2}/4)(1/64p_o^{\ 2}\vec{k}^2)\times \overline{\sum_i}\sum_f\left|e_i'^{*}e_j\ u^\dagger(p)\ X_{ij}\ u(p')\right|^2$$

Now,

(7.31) $$\overline{\sum_i}\sum_f\left|\quad\right|^2 = (1/4p_o^{\ 2})\overline{\sum_e}\sum_{e'}(e_i'^{*}e_j\ e_k'\ e_l^{*})\times \mathrm{Trace}\left[(p\bar{\sigma}^T)\ X_{ij}\ (p'\bar{\sigma}^T)\ X_{kl}\right]$$

where the property (5.12a) of the solution u(p) and the Hermitian property of X_{ij} have been used. The trace can be evaluated with the help of the trace theorems (5.5a,b). One obtains for (7.31) the result given in Appendix B for the photon-four component neutrino case. Thus, the photon polarization sum is the same as that given in Appendix C for the photon-four component neutrino case. Hence, one obtains

$$(7.32)\quad \overline{\sum_i}\sum_f \left| \quad \right|^2 = 64p_o{}^2(\cos^2(\theta/2) - \cos^4(\theta/2) + 4\cos^6(\theta/2)).$$

The substitution of (7.32) into (7.30) and the use of (6.7b) yield

$$(7.33)\quad d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)(p_o{}^2/64)(1/\sin^4(\theta/2)) \times \left[\cos^2(\theta/2) - \cos^4(\theta/2) + 4\cos^6(\theta/2)\right].$$

g) The Cross-section for Four Component Neutrino-Four Component Neutrino Scattering

The matrix element is the sum of (6.35a) and (6.35b). From (7.1) one then obtains

$$(7.34)\quad M = (\varkappa^2/8V^2)\Big[A\bar{u}_{r'}(p')\,\gamma_\mu\, u_r(p)\,\bar{u}_{s'}(q')\,\gamma_\mu\, u_s(q)$$
$$+ B\bar{u}_{r'}(p')\,\gamma_4\, u_r(p)\,\bar{u}_{s'}(q')\,\gamma_4\, u_s(q)$$
$$- C\bar{u}_{r'}(p')\,\gamma_\mu\, u_s(q)\,\bar{u}_{s'}(q')\,\gamma_\mu\, u_r(p)$$
$$- D\bar{u}_{r'}(p')\,\gamma_4\, u_s(q)\,\bar{u}_{s'}(q')\,\gamma_4\, u_r(p)\Big]$$

where

$$(7.35)\quad A = (\vec{p}\cdot\vec{p}\,' + 3p_o{}^2)/\,\vec{k}^2$$
$$B = 8p_o{}^2/\,\vec{k}^2$$
$$C = (-\vec{p}\cdot\vec{p}\,' + 3p_o{}^2)/\,\vec{s}^2$$
$$D = 8p_o{}^2/\,\vec{s}^2$$

It is necessary to compute $\overline{\sum_i}\sum_f |M|^2$. This is done in Appendix D. The substitution of the result obtained there into (7.4) yields

$$(7.36)\quad d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)(p_o{}^2/512)\times$$
$$\Big\{\big[1+6\cos^2(\theta/2)+18\cos^4(\theta/2)+6\cos^6(\theta/2)+\cos^8(\theta/2)\big]/\sin^4(\theta/2)$$
$$+\big[1+6\sin^2(\theta/2)+18\sin^4(\theta/2)+6\sin^6(\theta/2)+\sin^8(\theta/2)\big]/\cos^4(\theta/2)$$
$$+2\big[4+9\sin^2(\theta/2)\cos^2(\theta/2)\big]/\big[\sin^2(\theta/2)\cos^2(\theta/2)\big]\Big\}$$

h) The Cross-section for Two Component Neutrino-Two Component Neutrino Scattering

The matrix element is the sum of (6.38a) and (6.38b). From (7.1) one then obtains

$$(7.37)\quad M = -(\varkappa^2/8V^2)\Big[A u^\dagger(p)\,\sigma_\mu\, u(p')\, u^\dagger(q)\,\sigma_\mu\, u(q') - B u^\dagger(p)\, u(p')\, u^\dagger(q)\, u(q') - C u^\dagger(q)\,\sigma_\mu\, u(p')\, u^\dagger(p)\,\sigma_\mu\, u(q') + D u^\dagger(q)\, u(p')\, u^\dagger(p)\, u(q')\Big]$$

where A,B,C, and D are the same as in (7.35).

It is necessary to compute $|M|^2$. This is done in Appendix E. The substitution of the result obtained there into (7.4) yields

$$(7.38)\quad d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)(p_o{}^2/512)\times \Big\{\big[1+6\cos^2(\theta/2)+18\cos^4(\theta/2)+6\cos^6(\theta/2)+\cos^8(\theta/2)\big]/\sin^4(\theta/2) + \big[1+6\sin^2(\theta/2)+18\sin^4(\theta/2)+6\sin^6(\theta/2)+\sin^8(\theta/2)\big]/\cos^4(\theta/2) + 4\big[4+9\sin^2(\theta/2)\cos^2(\theta/2)\big]/\big[\sin^2(\theta/2)\cos^2(\theta/2)\big]\Big\}$$

8. Conclusions and Discussion

In this Chapter the results of the previous Chapters are reviewed, summarized, and compared, where possible, with previously published results; a comparison with electrodynamics is made; a simple self-energy calculation is given; and finally, a recent(1970) approach, involving gravitation, to the self-energy problem in electrodynamics is mentioned.

In Chapter 1, Part a, the gravitational field variables are defined. The Lagrangian for a weak gravitational field is written down and two solutions(for a static source and for a free field) of the field equations are mentioned. In Part b of Chapter 1, a quantum theory of the weak gravitational field is developed. Ten types of gravitons arise in contrast to the Gupta formalism in which there are eleven.

In Chapter 2 the interaction Lagrangian for a massless and Hermitian scalar field is extracted from the Lagrangian which is postulated to describe the scalar field in the presence of gravitation.

In Chapter 3 the interaction Lagrangian for the photon field is extracted from the Lagrangian which is postulated to describe the photon field in the presence of gravitation.

In Chapter 4 the required properties of the four component neutrino are given. To pass to the realm of general relativity one imposes the requirement that the Lagrangian be invariant under a coordinate dependent reorientation of vierbein frames. Utiyama's prescription is used to obtain a Lagrangian which satisfies the requirement. The interaction Lagrangian is then extracted in the weak field approximation.

In Chapter 5 the required properties of the two component neutrino are given. The method of Fock is used to extend the flat space-time Lagrangian to a form which satisfies the above requirement. The interaction Lagrangian is then extracted in the weak field approximation.

In Chapter 6 the matrix elements for the collisions are given. Some of these can be compared with previously published results. Barker et al(1966) worked out the matrix elements for the gravitational interaction between photons, scalar particles with non-zero rest mass, and spin 1/2 particles with non-zero rest mass. By setting the masses to zero in their results, one may obtain some of the matrix elements given here. If one sets the masses to zero in the scalar-photon, scalar-spin 1/2, and photon-spin 1/2 matrix elements obtained by the above authors one obtains the matrix elements given here for scalar-photon, scalar-four component neutrino, and photon-four component neutrino scattering. When considering particles of the same spin, the above authors assume that the masses are different. Therefore, if one sets the masses to zero in their results for particles of the same spin, only part of the matrix element for the scattering of identical massless particles is obtained. Additional terms arise which are due to the identity of the particles. For scalar-scalar scattering, the zero mass value of their matrix element is the matrix element (6.10a) given here and corresponds to graph (1) on page 47. The other matrix elements given here, (6.10b) and (6.10c), corresponding to graphs (2) and (3) on page 47, are due to the identity of the particles. The zero mass value of the matrix element for the scattering of two massive spin 1/2 particles, given by the above authors, is just the matrix element (6.35a) given here and corresponds to graph (1) on page 58. The other matrix element given on page 58, (6.35b), is due to the identity of the neutrinos.

In Chapter 7 the cross-sections are worked out for the various collisions. These are now discussed in the order given there.

The cross-section (7.6) given here for the scattering of two massless scalar particles does not agree with the extreme relativistic limit of the cross-section for the scattering of two massive scalar particles quoted by DeWitt(1967). The reason is that the term

$$(3 - v^2)(1 + v^2) \tag{8.1}$$

in his equation (3.10) should read

$$(3 + v^2)(1 - v^2) \; . \tag{8.2}$$

The proof of this statement is given in Appendix F. As mentioned on page 44, the calculation in Appendix F can serve as the example for the calculations of the matrix elements given in Chapter 6.

The cross-section (7.10) given here for scalar-photon scattering does not agree with the extreme relativistic limit of the result given by Boccaletti et al(1969a) for the scattering of a photon by a massive scalar particle. The reason is that the factor, (8.3) $W^4/(W-K)^2$

in their equation (12), should read[1]

(8.4) W^2 .

The extreme relativistic limit of their result is then just (7.10) given here.

Upon comparison of (7.14) and (7.18) one sees that the cross-sections for scalar-four component neutrino scattering and scalar-two component neutrino scattering are identical(to this order of perturbation theory).

Upon comparison of (7.26) and (7.33) one sees that the cross-sections for photon-four component neutrino scattering and photon-two component neutrino scattering are identical(to this order of perturbation theory).

By comparing (7.36) and (7.38) one sees that the cross-section for four component neutrino-four component neutrino scattering is different from the two component neutrino-two component neutrino cross-section. The quantum mechanical exchange term(the term which comprises the last line of (7.36)) for the four component case is smaller, by a factor of one-half, than the quantum mechanical exchange term(the term which comprises the last line of (7.38)) for the two component neutrino case. The reason for this is the following.

The quantum mechanical exchange term for the four component case contains terms like

1. In the meantime an Erratum has appeared(Boccaletti et al 1969b) thus making it unnecessary to write down the proof here.

(8.5) $X = (\eta/4)\mathrm{Trace}\left[(\gamma p')\ \gamma_\mu\ (\gamma p)\ \tilde{\gamma}_\nu\ (\gamma q')\ \gamma_\mu\ (\gamma q)\ \tilde{\gamma}_\nu\right]$

where

(8.6) $\tilde{\gamma}_\nu = \gamma_4\ \gamma_\nu^\dagger\ \gamma_4$

(this follows from the expression for $\overline{\sum_i} \sum_f |M|^2$ given on page 96 and, for example, the expression for $\sum_i \sum_f Z_3$ given on page 95). The quantum mechanical exchange term for the two component case contains terms like

(8.7) $X' = \eta\mathrm{Trace}\left[(p\bar{\sigma}^\tau)\ \sigma_\mu\ (p'\bar{\sigma}^\tau)\ \sigma_\nu\ (q\bar{\sigma}^\tau)\ \sigma_\mu\ (q'\bar{\sigma}^\tau)\ \sigma_\nu\right]$

(this follows from the expression for $|M|^2$ given on page 99 and, for example, the expression for Z_3 given on page 98). η is a number which is the same in both cases and the 1/4 in (8.5) arises from the averaging over initial states. One can show that the trace in (8.5) is twice the trace in (8.7). Then, because of the 1/4 in (8.5), one has

(8.8) $X = X'/2$

which shows that the quantum mechanical exchange term for the four component neutrino case is one-half the quantum mechanical exchange term for the two component case.

In the course of performing the original calculations for the results given here, the photon-photon scattering cross-section was calculated. The result of Barker et al(1967) and Boccaletti et al(1969a) was confirmed. It is

(8.9) $$d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)(p_o{}^2/2)\ \left[1 + \cos^{16}(\theta/2) + \sin^{16}(\theta/2)\right]/\sin^4\theta\ .$$

This cross-section can dominate the electrodynamic cross-section for photon-photon scattering. The electrodynamic cross-section is

$$(8.10) \quad d\sigma/d\Omega \simeq 10^{-155}\ \omega^6$$

for very low frequencies ($\hbar\omega \ll m_e c^2$), and

$$(8.11) \quad d\sigma/d\Omega \simeq 10^{12}/\omega^2$$

for very high frequencies ($\hbar\omega \gg m_e c^2$) (Akhiezer and Berestetskii 1965). The gravitational cross-section for photon-photon scattering is, from (8.9),

$$(8.12) \quad d\sigma/d\Omega \simeq 10^{-151}\ \omega^2$$

Thus for very low frequencies ($\omega < 10$ sec^{-1}) and for very high frequencies ($\omega > 10^{41}$ sec^{-1}) the gravitational interaction dominates in the cross-section for photon-photon scattering.

For very small scattering angles all the cross-sections have the form

$$(8.13) \quad d\sigma/d\Omega = (\varkappa^4/(2\pi)^2)(p_o{}^2/\theta^4) \ .$$

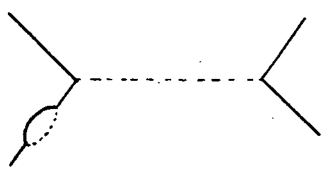
In c.g.s. units the relationship is

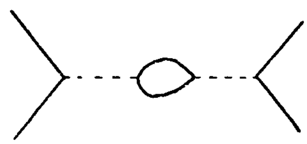
$$(8.14) \quad d\sigma/d\Omega = (8G\hbar/c^4)^2\ \omega^2/\theta^4 = 4.4 \times 10^{-151}\ \omega^2/\theta^4\ \text{cm}^2 \ .$$

The graphs for the various cross-sections are drawn on pages 82 to 84.

In quantum electrodynamics there are processes contained in S_2(the second order term in the S-matrix expansion) which lead to divergent integrals. The graphs for these processes are

where a solid line represents an electron or positron and a dashed line represents a photon. The graph on the left is called the self-energy graph of the electron and the one on the right is called the self-energy graph of the photon. The same divergent integrals which appear in the matrix elements for these simple processes also appear in higher order processes such as





Similarly, in the decomposition of S_2 for the gravitational interaction, self-energy processes appear. The question arises as to whether divergent integrals appear. For simplicity the interaction between the gravitational field and the scalar field described in Chapter 2 is considered. The graph for the process under consideration is

where a solid line represents a scalar particle and a dashed line represents a graviton.

The energy-momentum tensor is, from (2.4),

$$(8.15) \quad T_{\mu\nu} = -\phi_{|\mu}\,\phi_{|\nu} + \delta_{\mu\nu}\,\phi_{|\tau}\,\phi_{|\tau}/2 \,.$$

One then obtains for I, the integrand of the scattering matrix, defined by (6.6),

$$(8.16)\quad I = 2\kappa^2\, \underbrace{h_{\mu\nu}(x)\; h_{\alpha\beta}}(y)\times$$
$$\Big\{ \underbrace{\phi_{|\nu}(x)\;\phi}_{|\alpha}(y)\;\phi^{+}_{|\mu}(x)\;\phi^{-}_{|\beta}(y)$$
$$-(\delta_{\alpha\beta}/2)\underbrace{\phi_{|\nu}(x)\;\phi}_{|\eta}(y)\;\phi^{+}_{|\mu}(x)\;\phi^{-}_{|\eta}(y)$$
$$-(\delta_{\alpha\beta}/2)\underbrace{\phi_{|\nu}(x)\;\phi}_{|\eta}(y)\;\phi^{+}_{|\eta}(y)\;\phi^{-}_{|\mu}(x)$$
$$+(\delta_{\alpha\beta}\;\delta_{\mu\nu}/4)\underbrace{\phi_{|\tau}(x)\;\phi}_{|\eta}(y)\;\phi^{+}_{|\tau}(x)\;\phi^{-}_{|\eta}(y)\Big\}$$

where

$$(8.17)\quad \underbrace{\phi_{|\mu}(x)\;\phi}_{|\nu}(y) = \lim_{\varepsilon\to 0}(-i/(2\pi)^4)\int q_\mu q_\nu\, \frac{e^{iq(x-y)}}{q^2-i\varepsilon}\, d^4q$$

If one denotes the initial momentum by p and the final momentum by p', one obtains for the matrix element between the initial and final free particle states

$$(8.18)\quad s = \langle p'|(-1/2)\iint I dx\, dy\, |p\rangle$$
$$= (\kappa^2/2Vp_0)\;\delta(p-p')(pp')\int(1/k^2)\; d^4k$$
$$= 0$$

since $p^2=0$. The self-energy matrix element for a scalar particle is zero, in contradistinction to the matrix elements for the photon and electron self-energy graphs in electrodynamics.

In order to consider higher order processes one cannot just proceed as in electrodynamics and consider higher order terms in the S-matrix expansion. It is first necessary to carry the expansion of $g^{\mu\nu}$ and $\sqrt{g}$ to higher orders in κ. The complexity of the problem thus increases rapidly. Salam and Strathdee(1970) have stated that one should treat problems in gravitation non-perturbatively. They have given an example to indicate how gravitation might be treated non-perturbatively to suppress divergent integrals in quantum electrodynamics.

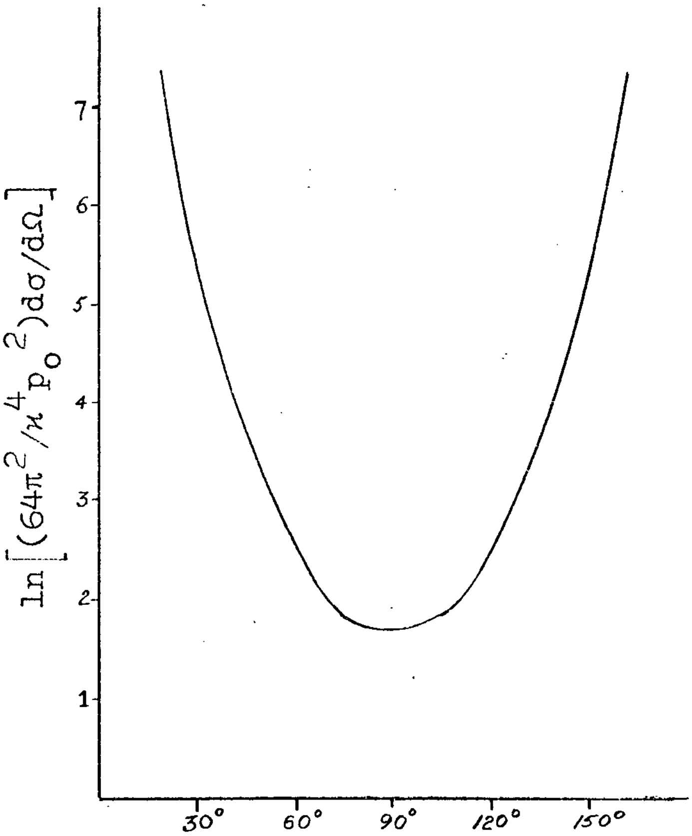


Fig. 1. Cross-section for scalar-scalar scattering

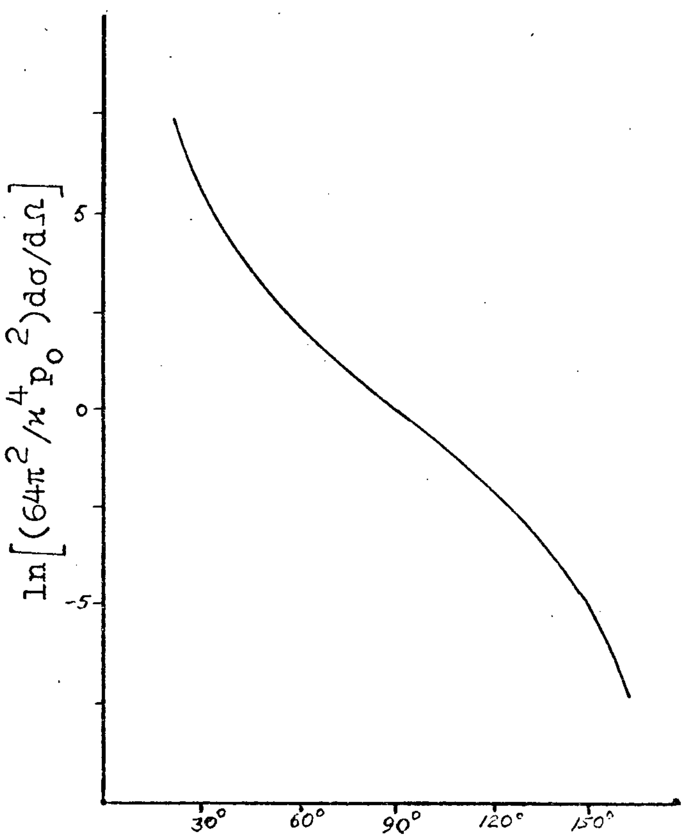


Fig. 2. Cross-section for scalar-photon scattering

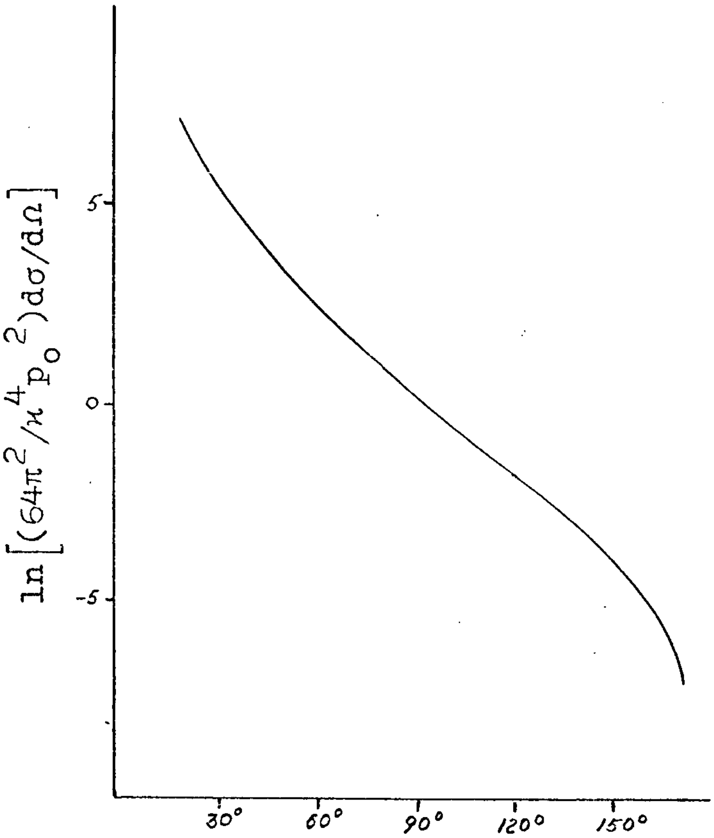


Fig. 3. Cross-section for scalar-neutrino scattering

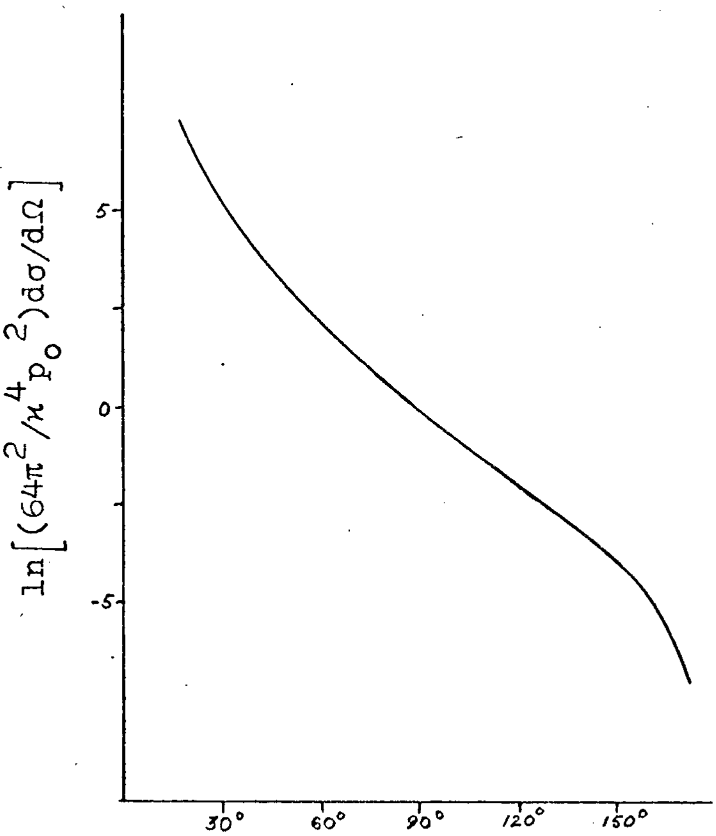


Fig. 4. Cross-section for photon-neutrino scattering

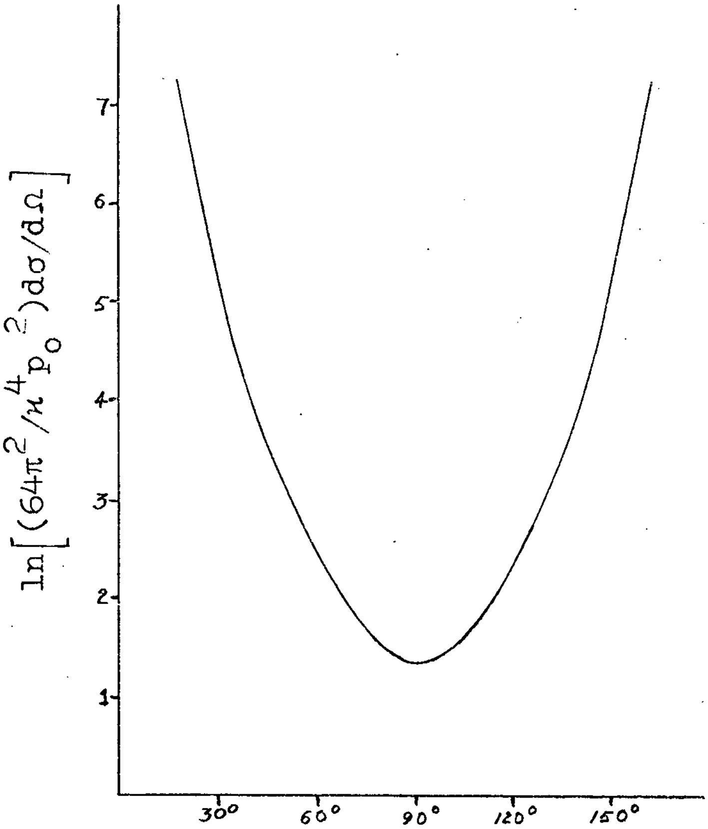


Fig. 5. Cross-section for four component neutrino-four component neutrino scattering

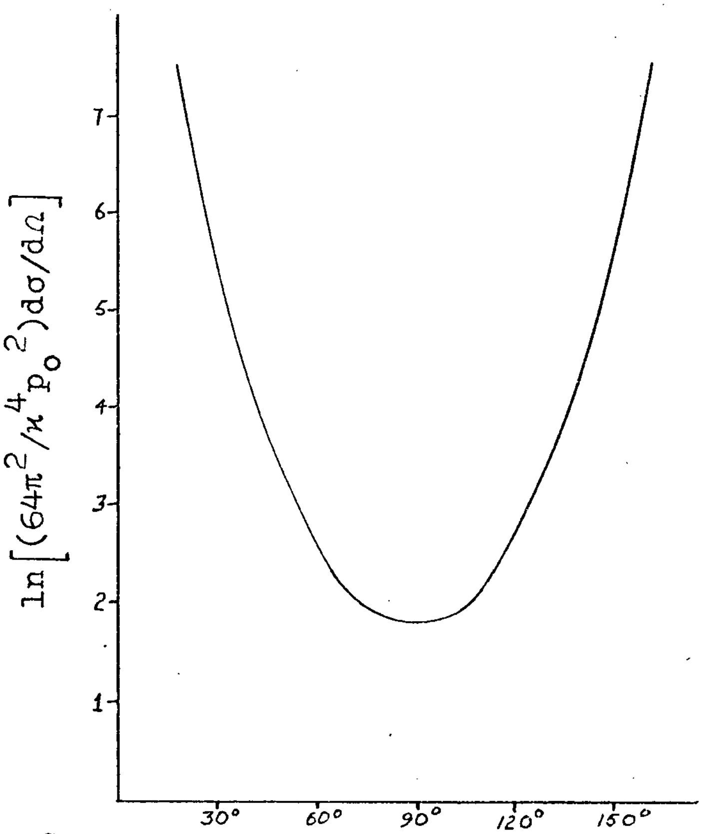


Fig. 6. Cross-section for two component neutrino-two component neutrino scattering

BIBLIOGRAPHY

Aharoni,J. 1965. The Special Theory of Relativity. (Oxford University Press, London).

Akhiezer,A.I. and Berestetskii,V.B. 1965. Quantum Electrodynamics. (Interscience Publishers, New York).

Barker,B.M., Gupta,S.N., and Haracz,R.D. 1966. Phys. Rev., 149, 1027.

Barker,B.M., Bhatia,M.S., and Gupta,S.N. 1967. Phys. Rev., 162, 1750.

Bludman,S.A. 1963. Nuovo Cimento, 27, 751.

Boccaletti,D., de Sabbata,V., Gualdi,C., and Fortini,P. 1969a. Nuovo Cimento, 60B, 320.

------1969b. Nuovo Cimento, 64B, 419.

Bogoliubov,N.N. and Shirkov,D.V. 1959. Introduction to the Theory of Quantized Fields. (Interscience Publishers, New York).

Corinaldesi,E. 1956. Proc. Phys. Soc.(London), A69, 189.

Corson,E. 1953. Intro. to Tensors, Spinors and Relativistic Wave Equations. (Blackie and Son, London).

DeWitt,B.S. 1967. Phys. Rev., 162, 1239.

Einstein,A. 1918. Sitzungsberichte der Koniglich Preussischen Akademie der Wissenschaften, 8, 153.

Einstein,A. 1956. The Meaning of Relativity. (Princeton University Press, Princeton).

Fock,V. 1929. Zeits. f. Phys., 57, 261.

Gupta,S.N. 1952. Proc. Phys. Soc.(London), A65, 161, 608.

Kerimov,B.K. and Romanov,Yu.I. 1965. Soviet Physics JETP, 20, 752.

Kobzarev,I.Yu. and Okun,L.B. 1963. Soviet Physics JETP, 16, 1343.

Kuchowicz,B. 1969. Fortschritte der Physik, 17, 517.

Lederman,L.M. 1967. Neutrino Physics, in High Energy Physics (Editor-E. Burhop), Vol. 2. (Academic Press, New York).

Lee,T.D. and Wu,C.S. 1965. Weak Interactions, in Annual Review of Nuclear Science, 15, 381.

Lurié,D. 1968. Particles and Fields. (Interscience Publishers, New York).

Marshak,R.E., Riazuddin, Ryan,C.P. 1969. Theory of Weak Interactions. (Wiley-Interscience, New York).

Muirhead,H. 1965. The Physics of Elementary Particles. (Pergamon Press, London).

Roman,P. 1960. The Theory of Elementary Particles. (North-Holland, Amsterdam).

Salam,A. and Strathdee,J. 1970. Quantum Gravity and Infinities in Quantum Electrodynamics.(International Centre for Theoretical Physics, Trieste). (preprint).

Schweber,S. 1961. An Introduction to Relativistic Quantum Field Theory. (Harper and Row, New York).

Utiyama,R. 1956. Phys. Rev., 101, 1597.

Vladimirov,Yu.S. 1964. Soviet Physics JETP, 18, 176.

Weyl,H. 1929. Proc. National Acad. of Science, 15, 323.

------ 1931. The Theory of Groups and Quantum Mechanics. (Dover Publications, Inc., New York).

Appendix A. The Reduction of the Four Component Neutrino Lagrangian

In this Appendix the term $f^{\alpha}_{\ p} B_{\alpha}^{\ rs} (\bar{M}_{rs} \gamma^p - \gamma^p M_{rs})$ which appears in equation (4.35) is shown to vanish.

$$f^{\alpha}_{\ p} B_{\alpha}^{\ rs}(\bar{M}_{rs}\gamma^p - \gamma^p M_{rs}) = \frac{\kappa}{2}\delta^{\alpha}_{\ p}(\gamma_s\gamma_r\gamma^p - \gamma^p\gamma_r\gamma_s)\delta^{r\beta}\delta^{s\sigma}(h_{\alpha\sigma|\beta} - h_{\alpha\beta|\sigma})$$

$$= \frac{\kappa}{2}(\gamma^s\gamma^r\gamma^p - \gamma^r\gamma^s\gamma^p - \gamma^p\gamma^r\gamma^s + \gamma^p\gamma^s\gamma^r)h_{ps|r}$$

$$= \frac{\kappa}{2}(\gamma^s\gamma^r\gamma^p - \gamma^r\gamma^s\gamma^p - \gamma^s\gamma^r\gamma^p + \gamma^p\gamma^s\gamma^r)h_{ps|r}$$

$$= \frac{\kappa}{2}(-\gamma^r\gamma^s\gamma^p + \gamma^p\gamma^s\gamma^r)h_{ps|r}$$

$$= \frac{\kappa}{2}(-2\delta^{rs}\gamma^p + \gamma^s\gamma^r\gamma^p + \gamma^p\gamma^s\gamma^r)h_{ps|r}$$

$$= \frac{\kappa}{2}(-2\delta^{rs}\gamma^p + 2\delta^{rp}\gamma^s - \gamma^s\gamma^p\gamma^r + \gamma^p\gamma^s\gamma^r)h_{ps|r}$$

$$= \frac{\kappa}{2}(-2\delta^{rs}\gamma^p + 2\delta^{rp}\gamma^s)h_{ps|r}$$

$$= 0$$

Appendix B. The Evaluation of a Trace for Photon-Four Component Neutrino Scattering

In this Appendix the trace in equation (7.23) is evaluated. One has, from (7.23),

$$\overline{\sum_i} \sum_f | e_i'^* e_j \bar{u}_{n'}(p') X_{ij} u_n(p) |^2 =$$

$$(-1/8p_0^2) \overline{\sum_e} \sum_{e'} (e_i'^* e_j e_k' e_\ell^*) \, Trace[(\gamma p') X_{ij} (\gamma p) \tilde{X}_{k\ell}]$$

$$e_i'^* e_j e_k' e_\ell^* \, Trace[(\gamma p') X_{ij} (\gamma p) \tilde{X}_{k\ell}] = A + B + C + D$$

where

$$A = e_i'^* e_j e_k' e_\ell^* [\delta_{ij}(-\vec{p}\cdot\vec{p}' - 3p_0^2) + q_i q_j'][\delta_{k\ell}(-\vec{p}\cdot\vec{p}' - 3p_0^2) + q_k q_\ell'](16p_0^2) \times$$

$$Trace[(\gamma p') \gamma_4 (\gamma p) \gamma_4 \gamma_4 \gamma_4]$$

$$B = e_i'^* e_j e_k' e_\ell^* [\delta_{ij}(-\vec{p}\cdot\vec{p}' - 3p_0^2) + q_i q_j'](16 i p_0^3) \times$$

$$Trace[(\gamma p') \gamma_4 (\gamma p) \gamma_4 (\gamma_\ell q_k + \gamma_k q_\ell') \gamma_4]$$

$$C = e_i'^* e_j e_k' e_\ell^* [\delta_{k\ell}(-\vec{p}\cdot\vec{p}' - 3p_0^2) + q_k q_\ell'](-16 i p_0^3) \times$$

$$Trace[(\gamma p')(\gamma_j q_i + \gamma_i q_j')(\gamma p) \gamma_4 \gamma_4 \gamma_4]$$

$$D = e_i'^* e_j e_k' e_\ell^* (16 p_0^4) \times$$

$$Trace[(\gamma p')(\gamma_j q_i + \gamma_i q_j')(\gamma p) \gamma_4 (\gamma_\ell q_k + \gamma_k q_\ell') \gamma_4]$$

The trace in (7.23) has now been broken down into four traces which can be evaluated directly.

$$Tr\,[(\gamma p')\gamma_4(\gamma p)\gamma_4] = -8p_0^2\cos^2\theta/2$$

$$\therefore\ A = [(\vec{e}'^*\cdot\vec{e})(-2p_0^2\cos^2\theta/2 - 2p_0^2) + (\vec{q}\cdot\vec{e}'^*)(\vec{q}'\cdot\vec{e})]\times$$

$$[(\vec{e}'\cdot\vec{e}^*)(-2p_0^2\cos^2\theta/2 - 2p_0^2) + (\vec{q}\cdot e')(q'\cdot e^*)](-128p_0^4\cos^2\theta/2)$$

$$e'_k e^*_\ell\ Tr[(\gamma p')\gamma_4(\gamma p)\gamma_4(\gamma_\ell q_k + \gamma_k q'_\ell)\gamma_4]$$

$$= -4(\vec{q}\cdot e')(\vec{p}'\cdot\vec{e}^*)p_4 - 4(\vec{q}'\cdot e^*)p_4(\vec{p}\cdot\vec{e}')$$

$$\therefore\ B = (64p_0^4)[(\vec{e}'^*\cdot e)(-2p_0^2\cos^2\theta/2 - 2p_0^2) + (\vec{q}\cdot\vec{e}'^*)(\vec{q}'\cdot e)]\times$$

$$[(\vec{q}\cdot\vec{e}')(\vec{p}'\cdot\vec{e}^*) + (\vec{q}'\cdot e^*)(\vec{p}\cdot\vec{e}')]$$

$$e'^*_i e_j\ Tr[(\gamma p')(\gamma_j q_i + \gamma_i q'_j)(\gamma p)\gamma_4] =$$

$$4(\vec{q}\cdot\vec{e}'^*)(\vec{p}'\cdot e)p_4 + 4(\vec{q}'\cdot\vec{e})(\vec{p}\cdot\vec{e}'^*)p_4$$

$$\therefore\ C = (64p_0^4)[(\vec{e}'\cdot e^*)(-2p_0^2\cos^2\theta/2 - 2p_0^2) + (\vec{q}\cdot\vec{e}')(\vec{q}'\cdot e^*)]\times$$

$$[(\vec{q}\cdot\vec{e}'^*)(\vec{p}'\cdot e) + (\vec{q}'\cdot\vec{e})(\vec{p}\cdot\vec{e}'^*)]$$

$$e'^*_i e_j e'_k e^*_\ell\ Tr[(\gamma p')(\gamma_j q_i + \gamma_i q'_j)(\gamma p)\gamma_4(\gamma_\ell q_k + \gamma_k q'_\ell)\gamma_4]$$

$$= -(\vec{q}\cdot\vec{e}'^*)(\vec{q}\cdot\vec{e}')\,Tr[(\gamma p')(\gamma e)(\gamma p)(\gamma e^*)]$$

$$-(\vec{q}\cdot e'^*)(\vec{q}'\cdot e^*)\,Tr[(\gamma p')(\gamma e)(\gamma p)(\gamma e')]$$

$$- (\vec{q}'\cdot\vec{e})(\vec{q}\cdot\vec{e}')\,\mathrm{Tr}[(\gamma p')(\gamma e'^{*})(\gamma p)(\gamma e^{*})]$$

$$- (\vec{q}'\cdot\vec{e})(\vec{q}'\cdot\vec{e}^{*})\,\mathrm{Tr}[(\gamma p')(\gamma e'^{*})(\gamma p)(\gamma e')]$$

$$= 4(\vec{q}\cdot\vec{e}'^{*})(\vec{q}\cdot\vec{e}')(p'p)(\vec{e}\cdot\vec{e}^{*})$$

$$-4(\vec{q}\cdot\vec{e}'^{*})(\vec{q}'\cdot\vec{e}^{*})[(\vec{p}'\cdot\vec{e})(\vec{p}\cdot\vec{e}') - (p'p)(\vec{e}\cdot\vec{e}')]$$

$$-4(\vec{q}'\cdot\vec{e})(\vec{q}\cdot\vec{e}')[-(pp')(\vec{e}'^{*}\cdot e^{*}) + (\vec{p}'\cdot e^{*})(\vec{p}\cdot\vec{e}'^{*})]$$

$$+4(\vec{q}'\cdot\vec{e})(\vec{q}'\cdot e^{*})[(pp')(\vec{e}'\cdot e'^{*})]$$

$$= D/16p_0^{4}$$

The trace in (7.23) has now been evaluated. In conclusion, therefore,

$$\overline{\sum_i}\sum_f |e_i'^{*} e_j \bar{u}_{n'}(p')X_{ij}u_n(p)|^2 = -(1/8p_0^{2})\sum_{e'}\overline{\sum_e}(A+B+C+D)$$

where $\overline{\sum_e}$ denotes an average over the initial photon states and $\sum_{e'}$ denotes a sum over the final photon states.

Appendix C. The Photon Polarization Sums for Photon-Four Component Neutrino Scattering

In this Appendix the average over initial polarization states and the sum over final polarization states of the photon is performed for photon-four component neutrino scattering. It is convenient to transform to the basis (6.16) which represents states of circular polarization. One then has,

$$(\vec{e}'^{*}\cdot\vec{e}) = (a_+ a_+'^{*})\cos^2\theta/2 + (a_+ a_-'^{*})\sin^2\theta/2 + (a_- a_+'^{*})\sin^2\theta/2 + (a_- a_-'^{*})\cos^2\theta/2$$

$$(\vec{e}'\cdot\vec{e}^{*}) = (a_+^{*} a_+')\cos^2\theta/2 + (a_+^{*} a_-')\sin^2\theta/2 + (a_-^{*} a_+')\sin^2\theta/2 + (a_-^{*} a_-'^{*})\cos^2\theta/2$$

$$(\vec{e}'^{*}\cdot\vec{q}) = \frac{i p_0}{\sqrt{2}}\sin\theta\,(a_+'^{*} - a_-'^{*})$$

$$(\vec{e}\cdot\vec{q}') = \frac{i p_0}{\sqrt{2}}\sin\theta\,(a_+ - a_-)$$

$$(\vec{q}\cdot\vec{e}') = \frac{-i p_0}{\sqrt{2}}\sin\theta\,(a_+' - a_-')$$

$$(\vec{p}'\cdot\vec{e}^{*}) = \frac{i p_0}{\sqrt{2}}\sin\theta\,(a_+^{*} - a_-^{*})$$

$$(\vec{q}'\cdot\vec{e}^{*}) = \frac{-i p_0}{\sqrt{2}}\sin\theta\,(a_+^{*} - a_-^{*})$$

$$(\vec{p}\cdot\vec{e}\,') = \frac{i p_0}{\sqrt{2}} \sin\theta\,(a'_+ - a'_-)$$

$$(\vec{p}\,'\cdot\vec{e}) = -\frac{i p_0}{\sqrt{2}} \sin\theta\,(a_+ - a_-)$$

$$(\vec{p}\cdot\vec{e}\,'^{*}) = -\frac{i p_0}{\sqrt{2}} \sin\theta\,(a'^{*}_+ - a'^{*}_-)$$

The problem is to compute

$$(-1/8p_0^2)\overline{\sum_e}\sum_{e'}(A+B+C+D)$$

where A,B,C and D are given in Appendix B.

Substitution of the results given immediately above into A,B,C and D as defined in Appendix B yields

$$\overline{\sum_e}\sum_{e'} A = \Big\{ 4p_0^4(1+\cos^2\theta/2)^2[\cos^4\theta/2 + \sin^4\theta/2]$$
$$+ 2p_0^4\sin^2\theta(1+\cos^2\theta/2)\cos\theta$$
$$+ (p_0^4/2)\sin^4\theta\Big\}(-128p_0^4\cos^2\theta/2)$$

$$\overline{\sum_e}\sum_{e'} B = \overline{\sum_e}\sum_{e'} C = \Big\{-2p_0^4\sin^2\theta[2-5\sin^2\theta/2 + 2\sin^4\theta/2]$$
$$- p_0^4\sin^4\theta\Big\}(64p_0^4)$$

$$\overline{\sum_e}\sum_{e'} D = (-2p_0^4\sin^4\theta)(64p_0^4)\,.$$

From these results one can show that

$$\left(-1/8p_0^2\right) \overline{\sum_{e}} \sum_{e'} (A+B+C+D) = 64 p_0^6 \left(\cos^2\theta/2 - \cos^4\theta/2 + 4\cos^6\theta/2\right)$$

$$\left(= \overline{\sum_{i}} \sum_{f} \left| e'^{*}_{i}\, e_j\, e'_k\, e^{*}_{\ell}\, \bar{u}_{n'}(p') X_{ij}\, u_n \right|^2 \right).$$

Appendix D. Trace Calculations for Four Component Neutrino-Four Component Neutrino Scattering

In this Appendix $\overline{\sum_i}\sum_f |M|^2$ is evaluated for four component neutrino-four component neutrino scattering. M is given by (7.34).

It is convenient to set

$$W_1 = \bar{u}_{r'}(p')\gamma_\mu u_r(p)\,\bar{u}_{s'}(q')\gamma_\mu u_s(q)$$

$$W_2 = \bar{u}_{r'}(p')\gamma_4 u_r(p)\,\bar{u}_{s'}(q')\gamma_4 u_s(q)$$

$$W_3 = \bar{u}_{r'}(p')\gamma_\mu u_s(q)\,\bar{u}_{s'}(q')\gamma_\mu u_r(p)$$

$$W_4 = \bar{u}_{r'}(p')\gamma_4 u_s(q)\,\bar{u}_{s'}(q')\gamma_4 u_r(p) \quad .$$

Then, from (7.34), one has

$$\overline{\sum_i}\sum_f |M|^2 = (\chi^4/256V^4)\sum_i\sum_f\left(\sum_{i=1}^{16} Z_i\right)$$

where

$Z_1 = A^2 W_1 W_1^\dagger$ $\quad Z_2 = AB\,W_1 W_2^\dagger$ $\quad Z_3 = -AC\,W_1 W_3^\dagger$

$Z_4 = -AD\,W_1 W_4^\dagger$ $\quad Z_5 = BA\,W_2 W_1^\dagger$ $\quad Z_6 = B^2 W_2 W_2^\dagger$

$Z_7 = -BC\,W_2 W_3^\dagger$ $\quad Z_8 = -BD\,W_2 W_4^\dagger$ $\quad Z_9 = -CA\,W_3 W_1^\dagger$

$Z_{10} = -CB\,W_3 W_2^\dagger$ $\quad Z_{11} = C^2 W_3 W_3^\dagger$ $\quad Z_{12} = CD\,W_3 W_4^\dagger$

$Z_{13} = -DA\,W_4 W_1^\dagger$ $\quad Z_{14} = -DB\,W_4 W_2^\dagger$ $\quad Z_{15} = DC\,W_4 W_3^\dagger$

$Z_{16} = D^2 W_4 W_4^\dagger$

One can show that (where $\tilde{\gamma}_\nu = \gamma_4 \gamma_\nu^+ \gamma_4$)

$$\sum_i \sum_f Z_1 = (A^2/16p_0^4)\, Tr[(\gamma p')\gamma_\mu(\gamma p)\tilde{\gamma}_\nu]\, Tr[(\gamma q')\gamma_\mu(\gamma q)\tilde{\gamma}_\nu] = 8A^2(1+\cos^4\theta/2)$$

$$\sum_i \sum_f Z_2 = (AB/16p_0^4)\, Tr[(\gamma p')\gamma_\mu(\gamma p)\gamma_4]\, Tr[(\gamma q')\gamma_\mu(\gamma q)\gamma_4] = 4AB(\cos^2\theta/2+\cos^4\theta/2)$$

$$\sum_i \sum_f Z_3 = (-AC/16p_0^4)\, Tr[(\gamma p')\gamma_\mu(\gamma p)\tilde{\gamma}_\nu(\gamma q')\gamma_\mu(\gamma q)\tilde{\gamma}_\nu] = 8AC$$

$$\sum_i \sum_f Z_4 = (-AD/16p_0^4)\, Tr[(\gamma p')\gamma_\mu(\gamma p)\gamma_4(\gamma q')\gamma_\mu(\gamma q)\gamma_4] = 4AD\sin^2\theta/2$$

$$\sum_i \sum_f Z_5 = (AB/16p_0^4)\, Tr[(\gamma p')\gamma_4(\gamma p)\tilde{\gamma}_\nu]\, Tr[(\gamma q')\gamma_4(\gamma q)\tilde{\gamma}_\nu] = 4AB(\cos^2\theta/2+\cos^4\theta/2)$$

$$\sum_i \sum_f Z_6 = (B^2/16p_0^4)\, Tr[(\gamma p')\gamma_4(\gamma p)\gamma_4]\, Tr[(\gamma q')\gamma_4(\gamma q)\gamma_4] = 4B^2\cos^4\theta/2$$

$$\sum_i \sum_f Z_7 = (-BC/16p_0^4)\, Tr[(\gamma p')\gamma_4(\gamma p)\tilde{\gamma}_\nu(\gamma q')\gamma_4(\gamma q)\tilde{\gamma}_\nu] = 4BC\cos^2\theta/2$$

$$\sum_i \sum_f Z_8 = (-BD/16p_0^4)\, Tr[(\gamma p')\gamma_4(\gamma p)\gamma_4(\gamma q')\gamma_4(\gamma q)\gamma_4] = 2BD\sin^2\theta/2\cos^2\theta/2$$

$$\sum_i \sum_f Z_9 = (-AC/16p_0^4)\, Tr[(\gamma p')\gamma_\mu(\gamma q)\tilde{\gamma}_\nu(\gamma q')\gamma_\mu(\gamma p)\tilde{\gamma}_\nu] = 8AC$$

$$\sum_i \sum_f Z_{10} = (-BC/16p_0^4)\, Tr[(\gamma p')\gamma_\mu(\gamma q)\gamma_4(\gamma q')\gamma_\mu(\gamma p)\gamma_4] = 4BC\cos^2\theta/2$$

$$\sum_i \sum_f Z_{11} = (C^2/16p_0^4)\, Tr[(\gamma p')\gamma_\mu(\gamma q)\tilde{\gamma}_\nu]\, Tr[(\gamma q')\gamma_\mu(\gamma p)\tilde{\gamma}_\nu] = 8C^2(1+\sin^4\theta/2)$$

$$\sum_i \sum_f Z_{12} = (CD/16p_0^4)\, Tr[(\gamma p')\gamma_\mu(\gamma q)\gamma_4]\, Tr[(\gamma q')\gamma_\mu(\gamma p)\gamma_4] = 4CD(\sin^2\theta/2+\sin^4\theta/2)$$

$$\sum_i \sum_f Z_{13} = (-AD/16p_0^4)\, Tr[(\gamma p')\gamma_4(\gamma q)\tilde{\gamma}_\nu(\gamma q')\gamma_4(\gamma p)\gamma_4] = 4AD\sin^2\theta/2$$

$$\sum_i \sum_f Z_{14} = (-DB/16p_0^4)\, Tr[(\gamma p')\gamma_4 (\gamma q)\gamma_4 (\gamma q')\gamma_4 (\gamma p)\gamma_4] = 2BD(\sin^2\theta/2 - \sin^4\theta/2)$$

$$\sum_i \sum_f Z_{15} = (CD/16p_0^4)\, Tr[(\gamma p')\gamma_4 (\gamma q)\tilde{\gamma}_\nu]\, Tr[(\gamma q')\gamma_4 (\gamma p)\tilde{\gamma}_\nu] = 4CD(\sin^2\theta/2 + \sin^4\theta/2)$$

$$\sum_i \sum_f Z_{16} = (D^2/16p_0^4)\, Tr[(\gamma p')\gamma_4 (\gamma q)\gamma_4]\, Tr[(\gamma q')\gamma_4 (\gamma p)\gamma_4] = 4D^2 \sin^4\theta/2 \; .$$

One also has

$$A^2 = \frac{1}{4\sin^4\theta/2}(1+\cos^2\theta/2)^2 \;, \quad AB = \frac{1}{\sin^4\theta/2}(1+\cos^2\theta/2) \;, \quad B^2 = \frac{4}{\sin^4\theta/2}$$

$$C^2 = \frac{1}{4\cos^4\theta/2}(1+\sin^2\theta/2)^2 \;, \quad CD = \frac{1}{\cos^4\theta/2}(1+\sin^2\theta/2) \;, \quad D^2 = \frac{4}{\cos^4\theta/2}$$

$$AC = \frac{1}{4\sin^2\theta/2\,\cos^2\theta/2}(1+\cos^2\theta/2)(1+\sin^2\theta/2) \;, \quad AD = \frac{1}{\sin^2\theta/2\,\cos^2\theta/2}(1+\cos^2\theta/2)$$

$$BC = \frac{1}{\sin^2\theta/2\,\cos^2\theta/2}(1+\sin^2\theta/2) \;, \quad BD = \frac{4}{\sin^2\theta/2\,\cos^2\theta/2} \; .$$

$$\therefore \overline{\sum_i} \sum_f |M|^2 = (\chi^4/256V^4) \sum_i \sum_f \{(Z_1+Z_2+Z_5+Z_6) + (Z_{11}+Z_{12}+Z_{15}+Z_{16}) + (Z_3+Z_4 + Z_9 + Z_7 + Z_8 + Z_{10} + Z_{13} + Z_{14})\}$$

$$= (\chi^4/128V^4)\{(1+6\cos^2\theta/2 + 18\cos^4\theta/2 + 6\cos^6\theta/2 + \cos^8\theta/2)/\sin^4\theta/2$$

$$+ (1+6\sin^2\theta/2 + 18\sin^4\theta/2 + 6\sin^6\theta/2 + \sin^8\theta/2)/\cos^4\theta/2$$

$$+ 2(4+9\sin^2\theta/2\,\cos^2\theta/2)/\sin^2\theta/2\,\cos^2\theta/2\}$$

Appendix E. Trace Calculations for Two Component Neutrino-Two Component Neutrino Scattering

In this Appendix $|M|^2$ is evaluated for two component neutrino-two component neutrino scattering. M is given by (7.37).

It is convenient to set

$$W_1 = u^\dagger(p)\,\sigma_\alpha\, u(p')\, u^\dagger(q)\,\sigma_\alpha\, u(q')$$

$$W_2 = u^\dagger(p)\, u(p')\, u^\dagger(q)\, u(q')$$

$$W_3 = u^\dagger(q)\,\sigma_\alpha\, u(p')\, u^\dagger(p)\,\sigma_\alpha\, u(q')$$

$$W_4 = u^\dagger(q)\, u(p')\, u^\dagger(p)\, u(q')$$

Then, from (7.37), one has

$$|M|^2 = (K^4/64V^4)\sum_{i=1}^{16} Z_i$$

where

$$Z_1 = A^2 W_1 W_1^\dagger \qquad Z_2 = -ABW_1W_2^\dagger \qquad Z_3 = -ACW_1W_3^\dagger$$

$$Z_4 = ADW_1W_4^\dagger \qquad Z_5 = -BAW_2W_1^\dagger \qquad Z_6 = B^2W_2W_2^\dagger$$

$$Z_7 = BCW_2W_3^\dagger \qquad Z_8 = -BDW_2W_4^\dagger \qquad Z_9 = -CAW_3W_1^\dagger$$

$$Z_{10} = CBW_3W_2^\dagger \qquad Z_{11} = C^2W_3W_3^\dagger \qquad Z_{12} = -CDW_3W_4^\dagger$$

$$Z_{13} = DAW_4W_1^\dagger \qquad Z_{14} = -DBW_4W_2^\dagger \qquad Z_{15} = -DCW_4W_3^\dagger$$

$$Z_{16} = D^2W_4W_4^\dagger$$

One can show that

$$Z_1 = (A^2/16p_0^4)\,Tr\,[(p\bar\sigma^T)\,\sigma_\alpha\,(p'\bar\sigma^T)\sigma_\gamma]\,Tr\,[(q\bar\sigma^T)\sigma_\alpha\,(q'\bar\sigma^T)\sigma_\gamma] = 2A^2(1+\cos^4\theta/2)$$

$$Z_2 = (+AB/16p_0^4)\,Tr\,[(p\bar\sigma^T)\,\sigma_\alpha\,(p'\bar\sigma^T)\sigma_4]\,Tr\,[(q\bar\sigma^T)\,\sigma_\alpha\,(q'\bar\sigma^T)\sigma_4] = +AB\,(\cos^2\theta/2 + \cos^4\theta$$

$$Z_3 = (-AC/16p_0^4)\,Tr\,[(p\bar\sigma^T)\sigma_\alpha\,(p'\bar\sigma^T)\sigma_\gamma\,(q\bar\sigma^T)\,\sigma_\alpha\,(q'\bar\sigma^T)\sigma_\gamma] = 4AC$$

$$Z_4 = (-AD/16p_0^4)\,Tr\,[(p\bar\sigma^T)\sigma_\alpha\,(p'\bar\sigma^T)\sigma_4\,(q\bar\sigma^T)\,\sigma_\alpha\,(q'\bar\sigma^T)\sigma_4] = 2AD\sin^2\theta/2$$

$$Z_5 = (AB/16p_0^4)\,Tr\,[(p\bar\sigma^T)\sigma_4\,(p'\bar\sigma^T)\sigma_\gamma]\,Tr\,[(q\bar\sigma^T)\,\sigma_4\,(q'\bar\sigma^T)\sigma_\gamma] = AB(\cos^2\theta/2 + \cos^4\theta/2)$$

$$Z_6 = (B^2/16p_0^4)\,Tr\,[(p\bar\sigma^T)(p'\bar\sigma^T)]\,Tr\,[(q\bar\sigma^T)(q'\bar\sigma^T)] = B^2\cos^4\theta/2$$

$$Z_7 = (-BC/16p_0^4)\,Tr\,[(p\bar\sigma^T)\sigma_4\,(p'\bar\sigma^T)\sigma_\gamma\,(q\bar\sigma^T)\,\sigma_4\,(q'\bar\sigma^T)\sigma_\gamma] = 2BC\cos^2\theta/2$$

$$Z_8 = (-BD/16p_0^4)\,Tr\,[(p\bar\sigma^T)\sigma_4\,(p'\bar\sigma^T)\,\sigma_4\,(q\bar\sigma^T)\,\sigma_4\,(q'\bar\sigma^T)\sigma_4] = BD\sin^2\theta/2\cos^2\theta/2$$

$$Z_9 = (-AC/16p_0^4)\,Tr\,[(q\bar\sigma^T)\,\sigma_\alpha\,(p'\bar\sigma^T)\sigma_\gamma\,(p\bar\sigma^T)\sigma_\alpha\,(q'\bar\sigma^T)\sigma_\gamma] = 4AC$$

$$Z_{10} = (-BC/16p_0^4)\,Tr\,[(q\bar\sigma^T)\sigma_\alpha\,(p'\bar\sigma^T)\sigma_4\,(p\bar\sigma^T)\sigma_\alpha\,(q'\bar\sigma^T)\sigma_4] = 2BC\cos^2\theta/2$$

$$Z_{11} = (C^2/16p_0^4)\,Tr\,[(q\bar\sigma^T)\sigma_\alpha\,(p'\bar\sigma^T)\sigma_\gamma]\,Tr\,[(p\bar\sigma^T)\sigma_\alpha\,(q'\bar\sigma^T)\sigma_\gamma] = 2C^2(1+\sin^4\theta/2)$$

$$Z_{12} = (CD/16p_0^4)\,Tr\,[(q\bar\sigma^T)\sigma_\alpha\,(p'\bar\sigma^T)\sigma_4]\,Tr\,[(p\bar\sigma^T)\sigma_\alpha\,(q'\bar\sigma^T)\sigma_4] = CD\,(\sin^2\theta/2 + \sin^4\theta/2)$$

$$Z_{13} = (-AD/16p_0^4)\,Tr\,[(q\bar\sigma^T)\sigma_4\,(p'\bar\sigma^T)\sigma_\alpha\,(p\bar\sigma^T)\sigma_4\,(q'\bar\sigma^T)\,\sigma_\alpha] = 2AD\sin^2\theta/2$$

$$Z_{14} = (-BD/16p_0^4)\,Tr[(q\bar{\sigma}^T)\sigma_4(p'\bar{\sigma}^T)\sigma_4(p\bar{\sigma}^T)\sigma_4(q'\bar{\sigma}^T)\sigma_4] = BD(\sin^2\theta/2\cos^2\theta/2)$$

$$Z_{15} = (CD/16p_0^4)\,Tr[(q\bar{\sigma}^T)\sigma_4(p'\bar{\sigma}^T)\sigma_\alpha]\,Tr[(p\bar{\sigma}^T)\sigma_4(q'\bar{\sigma}^T)\sigma_\alpha] = CD(\sin^2\theta/2+\sin^4\theta/2)$$

$$Z_{16} = (D^2/16p_0^4)\,Tr[(q\bar{\sigma}^T)\sigma_4(p'\bar{\sigma}^T)\sigma_4]\,Tr[(p\bar{\sigma}^T)\sigma_4(q'\bar{\sigma}^T)\sigma_4] = D^2\sin^4\theta/2$$

One also has

$$A^2 = \frac{1}{4\sin^4\theta/2}(1+\cos^2\theta/2)^2 \qquad AB = \frac{1}{\sin^4\theta/2}(1+\cos^2\theta/2) \qquad B^2 = \frac{4}{\sin^4\theta/2}$$

$$C^2 = \frac{1}{4\cos^4\theta/2}(1+\sin^2\theta/2)^2 \qquad CD = \frac{1}{\cos^4\theta/2}(1+\sin^2\theta/2) \qquad D^2 = \frac{4}{\cos^4\theta/2}$$

$$AC = \frac{1}{4\sin^2\theta/2\cos^2\theta/2}(1+\sin^2\theta/2)(1+\cos^2\theta/2) \qquad AD = \frac{1}{\sin^2\frac{\theta}{2}\cos^2\theta/2}(1+\cos^2\theta/2)$$

$$BC = \frac{1}{\sin^2\theta/2\cos^2\theta/2}(1+\sin^2\theta/2) \qquad BD = \frac{4}{\sin^2\theta/2\cos^2\theta/2}$$

$$\therefore\ |M|^2 = (K^4/64V^4)\{(Z_1+Z_2+Z_5+Z_6)+(Z_{11}+Z_{12}+Z_{15}+Z_{16})+(Z_3+Z_4$$
$$+Z_7+Z_8+Z_9+Z_{10}+Z_{13}+Z_{14})\}$$

$$=(K^4/128V^4)\{(1+6\cos^2\theta/2+18\cos^4\theta/2+6\cos^6\theta/2+\cos^8\theta/2)/\sin^4\theta/2$$

$$+(1+6\sin^2\theta/2+18\sin^4\theta/2+6\sin^6\theta/2+\sin^8\theta/2)/\cos^4\theta/2$$

$$+4(4+9\sin^2\theta/2\cos^2\theta/2)/\sin^2\theta/2\cos^2\theta/2\}$$

Appendix F. The Cross-section for the Scattering of Two Identical Massive Scalar Particles.

In this Appendix the cross-section for the scattering of two identical, massive, neutral scalar particles is worked out in detail.

The initial momenta of the particles are denoted by p and q and the final momenta by p' and q'. One must compute

$$(F1) \qquad s = \langle p'q' | S_2 | pq \rangle$$

where

$$(F2) \quad S_2 = (-\chi^2/8)\iint dx\,dy \langle 0|P(h_{\mu\nu}(x)h_{\alpha\beta}(y))|0\rangle N[T_{\mu\nu}(x)T_{\alpha\beta}(y)]$$

Here,

$$(F3) \quad T_{\mu\nu}(x) = -\sum_{i=1}^{2}\sum_{j=1}^{2}\left\{ \overset{i}{\phi}_{,\mu}\overset{j}{\phi}_{,\nu} - \frac{\delta_{\mu\nu}}{2}\overset{i}{\phi}_{,\alpha}\overset{j}{\phi}_{,\alpha} - \frac{\delta_{\mu\nu}}{2}\overset{i}{\phi}\overset{j}{\phi}m^2 \right\}$$

One decomposes $\overset{i}{\phi}$ into positive and negative frequency parts

$$(F4) \quad \overset{i}{\phi} = \overset{i}{\phi}{}^{+} + \overset{i}{\phi}{}^{-}$$

Then

$$\begin{aligned}(F5) \quad T_{\mu\nu}(x) = -\sum_{i}\sum_{j}\Big\{ & \overset{i}{\phi}{}^{+}_{,\mu}\overset{j}{\phi}{}^{+}_{,\nu} + 2\overset{i}{\phi}{}^{+}_{,\mu}\overset{j}{\phi}{}^{-}_{,\nu} + \overset{i}{\phi}{}^{-}_{,\mu}\overset{j}{\phi}{}^{-}_{,\nu} \\ & -(\delta_{\mu\nu}/2)(\overset{i}{\phi}{}^{+}_{,\alpha}\overset{j}{\phi}{}^{+}_{,\alpha} + 2\overset{i}{\phi}{}^{+}_{,\alpha}\overset{j}{\phi}{}^{-}_{,\alpha} + \overset{i}{\phi}{}^{-}_{,\alpha}\overset{j}{\phi}{}^{-}_{,\alpha}) \\ & -(\delta_{\mu\nu}/2)m^2(\overset{i}{\phi}{}^{+}\overset{j}{\phi}{}^{+} + 2\overset{i}{\phi}{}^{+}\overset{j}{\phi}{}^{-} + \overset{i}{\phi}{}^{-}\overset{j}{\phi}{}^{-})\Big\}\end{aligned}$$

where use has been made of the facts that eventually the normal ordering will be taken and that $T_{\mu\nu}$ is multiplied by $\langle 0|P(h_{\mu\nu}(x)\,h_{\alpha\beta}(y))|0\rangle$ which is symmetric in μ and ν. A similar expression holds for $T_{\alpha\beta}(y)$:

$$(F6)\quad T_{\alpha\beta}(y) = -\sum_{k=1}^{2}\sum_{l=1}^{2}\Big\{ \overset{k}{\phi}{}^{+}_{1\alpha}\overset{l}{\phi}{}^{+}_{1\beta} + 2\overset{k}{\phi}{}^{+}_{1\alpha}\overset{l}{\phi}{}^{-}_{1\beta} + \overset{k}{\phi}{}^{-}_{1\alpha}\overset{l}{\phi}{}^{-}_{1\beta}$$

$$-(\delta_{\alpha\beta}/2)\big(\overset{k}{\phi}{}^{+}_{1\eta}\overset{l}{\phi}{}^{+}_{1\eta} + 2\overset{k}{\phi}{}^{+}_{1\eta}\overset{l}{\phi}{}^{-}_{1\eta} + \overset{k}{\phi}{}^{-}_{1\eta}\overset{l}{\phi}{}^{-}_{1\eta}\big)$$

$$-(\delta_{\alpha\beta}/2)m^2\big(\overset{k}{\phi}{}^{+}\overset{l}{\phi}{}^{+} + 2\overset{k}{\phi}{}^{+}\overset{l}{\phi}{}^{-} + \overset{k}{\phi}{}^{-}\overset{l}{\phi}{}^{-}\big)\Big\}$$

It is seen that in the expressions for $T_{\mu\nu}$ and $T_{\alpha\beta}$ the terms are of the general form, neglecting differences in subscripts,

$$(F7)\quad \overset{a}{\phi}{}^{+}\overset{b}{\phi}{}^{+}\ ,\quad \overset{a}{\phi}{}^{+}\overset{b}{\phi}{}^{-}\ ,\quad \overset{a}{\phi}{}^{-}\overset{b}{\phi}{}^{-}$$

In the normal product, $N[T_{\mu\nu}(x)\,T_{\alpha\beta}(y)]$, terms of the general form

$$(F8)\quad \overset{a}{\phi}{}^{+}\overset{b}{\phi}{}^{+}\overset{c}{\phi}{}^{+}\overset{d}{\phi}{}^{+}\ ,\quad \overset{a}{\phi}{}^{+}\overset{b}{\phi}{}^{+}\overset{c}{\phi}{}^{+}\overset{d}{\phi}{}^{-}\ ,\quad \overset{a}{\phi}{}^{+}\overset{b}{\phi}{}^{-}\overset{c}{\phi}{}^{-}\overset{d}{\phi}{}^{-}\ ,\quad \overset{a}{\phi}{}^{-}\overset{b}{\phi}{}^{-}\overset{c}{\phi}{}^{-}\overset{d}{\phi}{}^{-}$$

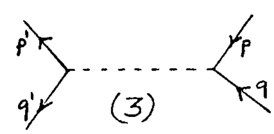
do not contribute to the process under consideration. The remaining terms in the normal product are of the following forms or can be put in one of the following forms (neglecting subscripts)

$$(F9)\quad \overset{i}{\phi}{}^{+}(x)\overset{j}{\phi}{}^{+}(x)\overset{k}{\phi}{}^{-}(y)\overset{l}{\phi}{}^{-}(y)\ ,\quad \overset{i}{\phi}{}^{+}(x)\overset{j}{\phi}{}^{-}(x)\overset{k}{\phi}{}^{+}(y)\overset{l}{\phi}{}^{-}(y).$$

The terms in the normal product of the form

$$(F10)\quad \overset{i}{\phi}{}^{+}(x)\overset{j}{\phi}{}^{+}(x)\overset{k}{\phi}{}^{-}(y)\overset{l}{\phi}{}^{-}(y),$$

given in (F9), contribute to a process whose graph is



(3)

That part of the integrand of the scattering matrix which contributes to this process (that is, that part of the integrand containing terms of the form (F10)) is

$$(F11)\quad (\chi^2/2)\langle 0|P(h_{\mu\nu}(x)h_{\alpha\beta}(y))|0\rangle N\Big\{\sum_{ijkl}\Big[{}^{i}\!\phi^{+}_{1\mu}\,{}^{j}\!\phi^{+}_{1\nu}\,{}^{k}\!\phi^{-}_{1\alpha}\,{}^{l}\!\phi^{-}_{1\beta}$$

$$-\frac{\delta_{\alpha\beta}}{2}\,{}^{i}\!\phi^{+}_{1\mu}\,{}^{j}\!\phi^{+}_{1\nu}\,{}^{k}\!\phi^{-}_{1\eta}\,{}^{l}\!\phi^{-}_{1\eta}\qquad-\frac{\delta_{\mu\nu}}{2}\,{}^{i}\!\phi^{+}_{1\tau}\,{}^{j}\!\phi^{+}_{1\tau}\,{}^{k}\!\phi^{-}_{1\alpha}\,{}^{l}\!\phi^{-}_{1\beta}$$

$$+\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{4}\,{}^{i}\!\phi^{+}_{1\tau}\,{}^{j}\!\phi^{+}_{1\tau}\,{}^{k}\!\phi^{-}_{1\eta}\,{}^{l}\!\phi^{-}_{1\eta}\qquad-\frac{\delta_{\alpha\beta}}{2}m^2\,{}^{i}\!\phi^{+}_{1\mu}\,{}^{j}\!\phi^{+}_{1\nu}\,{}^{k}\!\phi^{-}\,{}^{l}\!\phi^{-}$$

$$+\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{4}m^2\,{}^{i}\!\phi^{+}_{1\tau}\,{}^{j}\!\phi^{+}_{1\tau}\,{}^{k}\!\phi^{-}\,{}^{l}\!\phi^{-}\qquad-\frac{\delta_{\mu\nu}}{2}m^2\,{}^{i}\!\phi^{+}\,{}^{j}\!\phi^{+}\,{}^{k}\!\phi^{-}_{1\alpha}\,{}^{l}\!\phi^{-}_{1\beta}$$

$$+\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{4}m^2\,{}^{i}\!\phi^{+}\,{}^{j}\!\phi^{+}\,{}^{k}\!\phi^{-}_{1\eta}\,{}^{l}\!\phi^{-}_{1\eta}\qquad+\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{4}m^4\,{}^{i}\!\phi^{+}\,{}^{j}\!\phi^{+}\,{}^{k}\!\phi^{-}\,{}^{l}\!\phi^{-}\Big]\Big\}$$

$$=(2\chi^2)\langle 0|P(h_{\mu\nu}(x)h_{\alpha\beta}(y))|0\rangle\sum_{i=1}^{9}Z_i$$

where

$$(F12)\quad Z_1={}^{1}\!\phi^{+}_{1\mu}\,{}^{2}\!\phi^{+}_{1\nu}\,{}^{1}\!\phi^{-}_{1\alpha}\,{}^{2}\!\phi^{-}_{1\beta}\qquad Z_2=-\frac{\delta_{\alpha\beta}}{2}\,{}^{1}\!\phi^{+}_{1\mu}\,{}^{2}\!\phi^{+}_{1\nu}\,{}^{1}\!\phi^{-}_{1\eta}\,{}^{2}\!\phi^{-}_{1\eta}$$

$$Z_3=-\frac{\delta_{\mu\nu}}{2}\,{}^{1}\!\phi^{+}_{1\tau}\,{}^{2}\!\phi^{+}_{1\tau}\,{}^{1}\!\phi^{-}_{1\alpha}\,{}^{2}\!\phi^{-}_{1\beta}\qquad Z_4=\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{4}\,{}^{1}\!\phi^{+}_{1\tau}\,{}^{2}\!\phi^{+}_{1\tau}\,{}^{1}\!\phi^{-}_{1\eta}\,{}^{2}\!\phi^{-}_{1\eta}$$

$$Z_5=-\frac{\delta_{\alpha\beta}}{2}m^2\,{}^{1}\!\phi^{+}_{1\mu}\,{}^{2}\!\phi^{+}_{1\nu}\,{}^{1}\!\phi^{-}\,{}^{2}\!\phi^{-}\qquad Z_6=\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{4}m^2\,{}^{1}\!\phi^{+}_{1\tau}\,{}^{2}\!\phi^{+}_{1\tau}\,{}^{1}\!\phi^{-}\,{}^{2}\!\phi^{-}$$

$$Z_7=-\frac{\delta_{\mu\nu}}{2}m^2\,{}^{1}\!\phi^{+}\,{}^{2}\!\phi^{+}\,{}^{1}\!\phi^{-}_{1\alpha}\,{}^{2}\!\phi^{-}_{1\beta}\qquad Z_8=\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{4}m^2\,{}^{1}\!\phi^{+}\,{}^{2}\!\phi^{+}\,{}^{1}\!\phi^{-}_{1\eta}\,{}^{2}\!\phi^{-}_{1\eta}$$

$$Z_9=\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{4}m^4\,{}^{1}\!\phi^{+}\,{}^{2}\!\phi^{+}\,{}^{1}\!\phi^{-}\,{}^{2}\!\phi^{-}$$

In (F11) and (F12) the positive frequency parts are functions of x and the negative frequency parts are functions of y.

One lets

$$(F13)\quad \overset{1}{\phi}{}^{+}(x) = \frac{1}{\sqrt{V}} \sum_{\bar{p}} \frac{1}{\sqrt{2\omega_{\bar{p}}}}\, a^{+}(\bar{p}) e^{-i\bar{p}x} \qquad \overset{2}{\phi}{}^{+}(x) = \frac{1}{\sqrt{V}} \sum_{\bar{q}} \frac{1}{\sqrt{2\omega_{\bar{q}}}}\, a^{+}(\bar{q}) e^{-i\bar{q}x}$$

$$\overset{1}{\phi}{}^{-}(y) = \frac{1}{\sqrt{V}} \sum_{\bar{p}} \frac{1}{\sqrt{2\omega_{\bar{p}}}}\, a(\bar{p}) e^{i\bar{p}y} \qquad \overset{2}{\phi}{}^{-}(y) = \frac{1}{\sqrt{V}} \sum_{\bar{q}} \frac{1}{\sqrt{2\omega_{\bar{q}}}}\, a(\bar{q}) e^{i\bar{q}y} .$$

Since eventually one forms the scalar product $\langle p'q' | S_2 | pq \rangle$ one can use for purposes of computation

$$(F14)\quad \overset{1}{\phi}{}^{+} = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_{p'}}}\, e^{-ip'x} \qquad \overset{2}{\phi}{}^{+} = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_{q'}}}\, e^{-iq'x}$$

$$\overset{1}{\phi}{}^{-} = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_{p}}}\, e^{ipy} \qquad \overset{2}{\phi}{}^{-} = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_{q}}}\, e^{iqy}$$

Each derivative in $Z_1 - Z_9$ introduces the corresponding component of p, q, p' or q'. These components can be summed with

$$(F15)\quad A = (\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta})$$

which appears in $\langle 0 | P(h_{\mu\nu}(x), h_{\alpha\beta}(y)) | 0 \rangle$. One has

$$(F16)\quad Z_1 A = (p'_\mu q'_\nu p_\alpha q_\beta)(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta})\, \overset{1}{\phi}{}^{+} \overset{2}{\phi}{}^{+} \overset{1}{\phi}{}^{-} \overset{2}{\phi}{}^{-}$$

$$= [(p'p)(q'q) + (p'q)(pq') - (p'q')(pq)]\, \overset{1}{\phi}{}^{+} \overset{2}{\phi}{}^{+} \overset{1}{\phi}{}^{-} \overset{2}{\phi}{}^{-}$$

$$Z_2 A = (pq)(p'q')\, \overset{1}{\phi}{}^{+} \overset{2}{\phi}{}^{+} \overset{1}{\phi}{}^{-} \overset{2}{\phi}{}^{-}$$

$$Z_3 A = (pq)(p'q')\, \overset{1}{\phi}{}^{+} \overset{2}{\phi}{}^{+} \overset{1}{\phi}{}^{-} \overset{2}{\phi}{}^{-}$$

$$Z_4 A = -2(p'q')(pq)\, \overset{1}{\phi}{}^{+} \overset{2}{\phi}{}^{+} \overset{1}{\phi}{}^{-} \overset{2}{\phi}{}^{-}$$

$$Z_5 A = -m^2 (p'q')\, \overset{1}{\phi}{}^{+} \overset{2}{\phi}{}^{+} \overset{1}{\phi}{}^{-} \overset{2}{\phi}{}^{-}$$

$$Z_6 A = 2m^2 (p'q') \overset{1}{\phi}{}^{+} \overset{2}{\phi}{}^{+} \overset{1}{\phi}{}^{-} \overset{2}{\phi}{}^{-}$$

$$Z_7 A = -m^2 (pq) \overset{1}{\phi}{}^{+} \overset{2}{\phi}{}^{+} \overset{1}{\phi}{}^{-} \overset{2}{\phi}{}^{-}$$

$$Z_8 A = 2m^2 (pq) \overset{1}{\phi}{}^{+} \overset{2}{\phi}{}^{+} \overset{1}{\phi}{}^{-} \overset{2}{\phi}{}^{-}$$

$$Z_9 A = -2m^4 \overset{1}{\phi}{}^{+} \overset{2}{\phi}{}^{+} \overset{1}{\phi}{}^{-} \overset{2}{\phi}{}^{-}$$

Then

$$\text{(F17)} \quad \sum_{i=1}^{9} Z_i A = B \overset{1}{\phi}{}^{+} \overset{2}{\phi}{}^{+} \overset{1}{\phi}{}^{-} \overset{2}{\phi}{}^{-}$$

where

$$\text{(F18)} \quad B = \left[(p'p)(q'q) + (p'q)(pq') - (p'q')(pq) + m^2(p'q') + m^2(pq) - 2m^4\right]$$

In the center-of-momentum frame

$$\text{(F19)} \quad (pq) = (-\vec{p}^{\,2} - p_0^2) = -p_0^2(1+v^2)$$

$$(p'q') = -\vec{p}^{\,2} - p_0^2 = -p_0^2(1+v^2)$$

$$(pq') = -\vec{p}^{\,2}\cos\theta - p_0^2 = -p_0^2(1+v^2\cos\theta)$$

$$(p'q) = -\vec{p}^{\,2}\cos\theta - p_0^2 = -p_0^2(1+v^2\cos\theta)$$

$$(pp') = \vec{p}^{\,2}\cos\theta - p_0^2 = -p_0^2(1-v^2\cos\theta)$$

$$(qq') = \vec{p}^{\,2}\cos\theta - p_0^2 = -p_0^2(1-v^2\cos\theta)$$

$$(m^2) = p_0^2(1-v^2)$$

where

$$(F20)\quad v = |\vec{p}|/p_0 .$$

(F18) becomes, with the help of (F19),

$$(F21)\quad B = \{p_0^4(1-v^2\cos\theta)^2 + p_0^4(1+v^2\cos\theta)^2 - p_0^4(1+v^2)^2$$

$$- p_0^4(1-v^2)(1+v^2) - p_0^4(1-v^2)(1+v^2) - 2p_0^4(1-v^2)^2\}$$

$$= p_0^4\{1 - 2v^2\cos\theta + v^4\cos^2\theta + 1 + 2v^2\cos\theta + v^4\cos^2\theta$$

$$-1 - 2v^2 - v^4 - 2 + 2v^4 - 2 + 4v^2 - 2v^4\}$$

$$= p_0^4\{-2v^4(1-\cos^2\theta) - 3 + 2v^2 + v^4\}$$

$$= p_0^4\{-2v^4\sin^2\theta - (3 - 2v^2 - v^4)\}$$

$$= p_0^4\{-2v^4\sin^2\theta - (3+v^2)(1-v^2)\}$$

The matrix element is then

$$(F22)\quad S_3 = (-1/2)\iint(2\chi^2)\, D_F(x-y)\, B\, \overset{1}{\phi}{}^{+}\overset{2}{\phi}{}^{+}\overset{1}{\phi}{}^{-}\overset{2}{\phi}{}^{-}\, dx\, dy$$

$$= (-1/2)(2\chi^2)\frac{(-i)}{(2\pi)^4}\frac{B}{V^2}\iiint \frac{e^{ik(x-y)}}{k^2} e^{-ip'x} e^{-iq'x} e^{ipy} e^{iqy}\, dx\, dy\, dk$$

$$= -i(\chi^2/V^2)(2\pi)^4\,\delta(p+q-p'-q')\frac{B}{16p_0^4} .$$

The terms in the normal product of the form

$$(F23)\quad \overset{i}{\phi}{}^{+}(x)\overset{j}{\phi}{}^{-}(x)\overset{k}{\phi}{}^{+}(y)\overset{l}{\phi}{}^{-}(y),$$

given in (F9), contribute to processes whose graphs are

The matrix element, s_1, for process (1) can be obtained in the same manner as s_3. Or, one can obtain s_1 from the matrix element given by Corinaldesi(1956) or Barker et al(1966) for the interaction of two scalar particles with different masses. One obtains

$$(F24)\quad s_1 = i(K^2/V^2)(2\pi)^4\,\delta(p+q-p'-q')\left(1/16v^2\sin^2\tfrac{\theta}{2}\right)\Big\{(1+3v^2)(1-v^2) + 4v^2(1+v^2)\cos^2\theta/2\Big\}$$

The matrix element, s_2, for process (2) can be obtained from that for process (1) by interchanging p and q. One obtains

$$(F25)\quad s_2 = i(K^2/V^2)(2\pi)^4\,\delta(p+q-p'-q')\left(1/16v^2\cos^2\theta/2\right)\Big\{(1+3v^2)(1-v^2) + 4v^2(1+v^2)\sin^2\theta/2\Big\}$$

The complete matrix element is

$$(F26)\quad s = s_1 + s_2 + s_3$$

and M, defined by (7.1), is

$$(F27)\quad M = (K^2/V^2)\Big\{[(1+3v^2)(1-v^2) + 4v^2(1+v^2)\cos^2\theta/2]/16v^2\sin^2\theta/2$$
$$+ [(1+3v^2)(1-v^2) + 4v^2(1+v^2)\sin^2\theta/2]/16v^2\cos^2\theta/2$$
$$+ [(3+v^2)(1-v^2) + 2v^4\sin^2\theta]/16\Big\}$$

The cross-section is

(F28) $(d\sigma/d\Omega) = (\chi^4/(2\pi)^2)(P_0^2/4)\,|M|^2$

Thus it is seen that the term

(F29) $(3-v^2)(1+v^2)$

given by DeWitt(1967) in his equation (3.10) should read

(F30) $(3+v^2)(1-v^2)$.

Appendix G. Notes on Neutrinos

This Appendix complements Chapters 4 and 5 in which four and two component neutrinos are described. In particular, all the dynamical variables and constants of the motion due to internal symmetries are worked out; the reduction of the four component neutrino equation to two, two component equations is given explicitly; and, the latter two equations are obtained from first principles.

The Lagrangian which describes the four component neutrino is by definition the electron-positron Lagrangian with the mass set to zero. Thus,

$$\text{(G1)} \quad \mathcal{L} = -\bar{\psi}\gamma^{\rho}\psi_{|\rho}$$

where

$$\text{(G2)} \quad \bar{\psi} = \psi^{\dagger}\gamma^{4}$$

$$\text{(G3)} \quad \gamma^{\rho} = \begin{pmatrix} 0 & -i\sigma^{\rho} \\ i\sigma^{\rho} & 0 \end{pmatrix} \qquad \gamma^{4} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\text{(G4)} \quad \sigma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This Lagrangian differs from (4.1) in that it is not real. This does not make any difference to the following remarks.

The field equation is

$$\text{(G5)} \quad \gamma^{\rho}\psi_{|\rho} = 0$$

There are two solutions corresponding to $E = \pm|p|$.

They are

$$(G6) \quad \psi_+ = u_n(\vec{p})e^{i(\vec{p}\cdot\vec{x} - |E|t)} \qquad \psi_- = v_n(\vec{p})e^{i(\vec{p}\cdot\vec{x} + |E|t)}$$

where

$$(G7) \quad u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}\eta_1\\ \eta_1\end{pmatrix} \qquad u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}\eta_2\\ -\eta_2\end{pmatrix}$$

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}\eta_1\\ -\eta_1\end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}\eta_2\\ \eta_2\end{pmatrix}$$

with η_1 and η_2 satisfying

$$(G8) \quad \sigma\cdot\vec{n}\,\eta_1 = \eta_1 \,, \qquad \sigma\cdot\vec{n}\,\eta_2 = -\eta_2$$

$\vec{n}$ is given by (4.11). It is

$$(G9) \quad \vec{n} = \vec{P}/|E|$$

Explicit forms of η_1 and η_2 are given by (4.12). They have the normalization and orthogonality properties

$$(G10) \quad \eta_n^\dagger \eta_s = \delta_{ns}$$

The solutions u_n and v_n satisfy

$$(G11) \quad u_n^\dagger u_s = \delta_{ns} \,, \quad v_n^\dagger v_s = \delta_{ns} \,, \quad u_n v_s = 0$$

$$(G12) \quad v_1(-\vec{p}) = u_2(\vec{p}) \,, \quad v_2(-\vec{p}) = u_1(\vec{p})$$

$$(G13) \quad hu_1 = u_1 \qquad hu_2 = -u_2 \qquad hv_1 = v_1 \qquad hv_2 = -v_2$$

$$(G14) \quad \gamma^5 u_1 = -u_1 \qquad \gamma^5 u_2 = u_2 \qquad \gamma^5 v_1 = v_1 \qquad \gamma^5 v_2 = -v_2$$

where h, the helicity operator, and γ^5 , the chirality operator, are

$$(G15)\quad h = \begin{pmatrix} \sigma\cdot\vec{n} & 0 \\ 0 & \sigma\cdot\vec{n} \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$$

An arbitrary solution of the field equation can be expanded as

$$(G16)\quad \psi = \frac{1}{\sqrt{V}} \sum_{p} \sum_{n=1}^{2} a_n(\vec{p}) u_n(\vec{p}) e^{i(\vec{p}\cdot\vec{x} - |E|t)} + \tilde{b}_n^{\dagger}(\vec{p}) v_n(\vec{p}) e^{i(\vec{p}\cdot\vec{x} + |E|t)}$$

$$= \frac{1}{\sqrt{V}} \sum_{p} \sum_{n=1}^{2} a_n(\vec{p}) u_n(\vec{p}) e^{ipx} + \tilde{b}_n^{\dagger}(-\vec{p}) v_n(-\vec{p}) e^{-ipx}$$

where

$$(G17)\quad p = (\vec{p}, i|E|) .$$

If one now defines $b_n(p)$ by

$$(G18)\quad b_1(\vec{p}) = \tilde{b}_2(-\vec{p}) \quad , \quad b_2(\vec{p}) = \tilde{b}_1(-\vec{p})$$

and uses (G12), one obtains for the expansion (G16)

$$(G19)\quad \psi = \frac{1}{\sqrt{V}} \sum_{p} \sum_{n} a_n(\vec{p}) u_n(\vec{p}) e^{ipx} + b_n^{\dagger}(\vec{p}) u_n(\vec{p}) e^{-ipx}$$

One can write

$$(G20)\quad \psi = \psi_R + \psi_L$$

where

$$(G21)\quad \psi_R = \frac{1}{2}(1-\gamma^5)\psi \quad , \quad \psi_L = \frac{1}{2}(1+\gamma^5)\psi$$

so that

$$(G22)\quad \gamma^5 \psi_L = \psi_L \quad , \quad \gamma^5 \psi_R = -\psi_R$$

The letters R and L stand for right and left. This notation is used because

$$(G23) \quad h\psi_R = \psi_R \quad , \quad h\psi_L = -\psi_L \ .$$

One then has

$$(G24) \quad \psi_R = \frac{1}{\sqrt{V}} \sum_p a_1(\vec{p}) u_1(\vec{p}) e^{ipx} + b_1^{\dagger}(\vec{p}) u_1(\vec{p}) e^{-ipx}$$

$$\psi_L = \frac{1}{\sqrt{V}} \sum_p a_2(\vec{p}) u_2(\vec{p}) e^{ipx} + b_2^{\dagger}(\vec{p}) u_2(\vec{p}) e^{-ipx}$$

These forms are useful when the two component theory is considered.

If one now makes the transition to quantum field theory the a_n, b_n, $a_n^{\dagger}$, and $b_n^{\dagger}$ are interpreted as operators such that the a_n destroy neutrinos, the b_n destroy antineutrinos, the $a_n^{\dagger}$ create neutrinos, and the $b_n^{\dagger}$ create antineutrinos. The anticommutation relations are

$$(G25) \quad \{a_n, a_s^{\dagger}\} = \{b_n, b_s^{\dagger}\} = \delta_{ns} \qquad \text{all others are zero.}$$

The dynamical variables and constants of the motion due to internal symmetries are now worked out in order to characterize the various neutrinos.

The energy density is

$$(G26) \quad \mathcal{H} = :\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \dot{\psi}: = -:\psi^{\dagger} \psi_{,4}:$$

and the energy is

$$(G27) \quad H = \int \mathcal{H} d^3x = \sum_p \sum_n \omega_{\vec{p}} [\, a_n^{\dagger}(\vec{p}) a_n(\vec{p}) + b_n^{\dagger}(\vec{p}) b_n(\vec{p})]$$

The kth component of the momentum density is

$$(G28) \quad \mathcal{P}_k = -i :\bar{\psi} \gamma^4 \psi_{,k}:$$

and the kth component of momentum is

$$(G29)\quad P_k = \int \mathcal{P}_k d^3x = \sum_p \sum_\lambda p_k [a_\lambda^\dagger(\vec{p})\, a_\lambda(\vec{p}) + b_\lambda^\dagger(\vec{p})\, b_\lambda(\vec{p})]$$

The component of the spin density in the direction $\vec{n}$ is

$$(G30)\quad \mathcal{S}_{\vec{n}} = \frac{1}{2} : \psi^\dagger \hat{\sigma}\cdot\vec{n}\, \psi :$$

where

$$(G31)\quad \hat{\sigma}^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

and the component of the spin in the direction $\vec{n}$ is

$$(G32)\quad S_{\vec{n}} = \int \mathcal{S}_{\vec{n}}\, d^3x = \frac{1}{2} \sum_p [a_1^\dagger a_1 - a_2^\dagger a_2 - b_1^\dagger b_1 + b_2^\dagger b_2]$$

The Lagrangian (G1) is invariant under the transformation

$$(G33)\quad \psi \to \psi' = e^{i\alpha\gamma^5}\psi$$

The invariance leads to a current

$$(G34)\quad j^\rho = -:\frac{\partial\mathcal{L}}{\partial\psi_{,\rho}} (i\ \gamma^5)\psi :$$

The quantity which is conserved in time is

$$(G35)\quad Q = -\int j^4 d^3x = -:\int \bar{\psi}\gamma^4 (i\ \gamma^5)\psi\, d^3x :$$

$$= i \sum_p [a_1^\dagger a_1 - a_2^\dagger a_2 - b_1^\dagger b_1 + b_2^\dagger b_2]$$

which is the same as (G32).

The Lagrangian (G1) is also invariant under the transformation

$$(G36)\quad \psi \to \psi' = e^{i\alpha}\psi$$

The invariance leads to a current

$$(G37)\quad j^{\rho} = -:\frac{\partial \mathcal{L}}{\partial \psi_{,\rho}}(i)\psi: = :\bar{\psi}\gamma^{\rho}(i)\psi:$$

The conserved quantity is

$$(G38)\quad L = -i\ :\int \bar{\psi}\gamma^{4}\psi\, d^{3}\vec{x}: = (-i)\sum_{p}[a_1^{+}a_1 + a_2^{+}a_2 - b_1^{+}b_1 - b_2^{+}b_2]$$

This states that the total number of neutrinos minus the total number of antineutrinos is a constant. This is just conservation of lepton number. A suitable assignment of lepton numbers is L=+1 for neutrinos and L=-1 for antineutrinos.

This completes the specification of the various neutrinos. The properties are summarized in the following table.

Particle	ν_1	ν_2	$\bar{\nu}_1$	$\bar{\nu}_2$
Creation Operator	a_1^{+}	a_2^{+}	b_1^{+}	b_2^{+}
Destruction Operator	a_1	a_2	b_1	b_2
Energy	$\|E\|$	$\|E\|$	$\|E\|$	$\|E\|$
Momentum	$\vec{p}$	$\vec{p}$	$\vec{p}$	$\vec{p}$
Spin Projection	1/2	-1/2	-1/2	1/2
Helicity	1	-1	-1	1
Lepton No.	1	1	-1	-1
Antiparticle	$\bar{\nu}_1$	$\bar{\nu}_2$	ν_1	ν_2

The neutrinos ν_e, $\bar{\nu}_e$, ν_μ, and $\bar{\nu}_\mu$ which appear in the reactions

$$n \rightarrow p + e^{-} + \bar{\nu}_e \qquad\qquad \pi^{+} \rightarrow \mu^{+} + \nu_\mu$$

$$\pi^{-} \rightarrow \mu^{-} + \bar{\nu}_\mu \qquad\qquad \mu^{+} \rightarrow e^{+} + \nu_e + \bar{\nu}_\mu$$

may be identified with ν_1, ν_2, $\bar{\nu}_1$, and $\bar{\nu}_2$ as follows:

$$\nu_e = \nu_2 \ , \ \bar{\nu}_e = \bar{\nu}_2 \ , \ \nu_\mu = \bar{\nu}_1 \ , \ \bar{\nu}_\mu = \nu_1$$

Upon taking linear combinations of (G35) and (G38), one finds that

$$(G39a) \quad L_1 = \sum_p (a_1^\dagger a_1 - b_1^\dagger b_1)$$

$$(G39b) \quad L_2 = \sum_p (a_2^\dagger a_2 - b_2^\dagger b_2)$$

are conserved.

In conjunction with the above assignment of lepton number, the charged leptons e^- and μ^+ have lepton number +1, and the charged leptons e^+ and μ^- have lepton number -1. The formalism described here for the ν_e and ν_μ neutrinos is discussed by Bludman(1963). Kerimov and Romanov(1965) give other references. In this formalism there are two gauge transformations, (G33) and (G36), of a single four component field, which lead to two independent conservation laws (G35) and (G38).

An alternative way of describing the ν_e and ν_μ neutrinos is to define two independent four component fields ψ_1 and ψ_2. The field, ψ_{ν_e}, which describes the ν_e neutrino, and the field, ψ_{ν_μ}, which describes the ν_μ neutrino, are

$$(G40) \quad \psi_{\nu_e} = \frac{1}{2}(1+\gamma_5)\psi_1$$

$$\psi_{\nu_\mu} = \frac{1}{2}(1+\gamma_5)\psi_2$$

The equations for the two types of neutrinos are then identical. There is one conservation law for each field ψ_{ν_e} and ψ_{ν_μ}. This allows one to define an electron number

L_e which is conserved in all reactions, and a muon number L_μ which is also conserved in all reactions. This is the formalism used by Lee and Wu(1965) and by Lederman(1967).

A discussion of the two formalisms is given by Marshak, Riazuddin, and Ryan(1968). In this connection, the μ^+ in their Table 3.6 should be a μ^-.

It is now shown that ψ_R and ψ_L defined by (G21) can be reduced to a form in which they have only two non-zero components.

If one sets

$$(G41) \quad \gamma'^{\rho} = S\gamma^{\rho}S^{-1}, \quad \psi' = S\psi, \quad u_r' = Su_r, \quad v_r' = Sv_r$$

where

$$(G42) \quad S = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}$$

one obtains

$$(G43) \quad \gamma'^{k} = \begin{pmatrix} 0 & i\sigma^k \\ -i\sigma^k & 0 \end{pmatrix} \qquad \gamma'^{4} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$h' = \begin{pmatrix} \sigma\cdot\vec{n} & 0 \\ 0 & \sigma\cdot\vec{n} \end{pmatrix} \qquad \gamma'^{5} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

$$u_1' = \begin{pmatrix} \eta_1 \\ 0 \end{pmatrix} \quad u_2' = \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix} \quad v_1' = \begin{pmatrix} 0 \\ \eta_1 \end{pmatrix} \quad v_2' = \begin{pmatrix} \eta_2 \\ 0 \end{pmatrix}$$

Then,

$$(G44) \quad \psi_R' = S\psi_R = \frac{1}{\sqrt{V}}\sum_p a_1(\vec{p})u_1'(\vec{p})e^{ipx} + b_1^{\dagger}(\vec{p})u_1'(\vec{p})e^{-ipx} \equiv \begin{pmatrix} \chi_R \\ 0 \end{pmatrix}$$

$$\psi_L' = S\psi_L = \frac{1}{\sqrt{V}} \sum_p a_2(\vec{p})u_2'(\vec{p})e^{ipx} + b_2^\dagger(\vec{p})u_2'(\vec{p})e^{-ipx} \equiv \begin{pmatrix} 0 \\ \chi_L \end{pmatrix}$$

and

$$(G45) \quad \psi' = S\psi = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = S(\psi_R + \psi_L)$$

This shows that ψ_R' and ψ_L' have only two non-zero components.

The γ' representation used above is called the Kramers representation of the Dirac matrices.

The Lagrangian (G1) in the Kramers representation is

$$(G46) \quad \mathcal{L} = -\bar{\psi}'\gamma'^p \psi'_{|p} = -(\chi_L^*, \chi_R^*)\begin{pmatrix} 0 & i\sigma^k\frac{\partial}{\partial x^k} + I\frac{\partial}{\partial x^4} \\ -i\sigma^k\frac{\partial}{\partial x^k} + I\frac{\partial}{\partial x^4} & 0 \end{pmatrix}\begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix}$$

$$= -\chi_L^*\left(i\sigma^k\frac{\partial}{\partial x^k} + I\frac{\partial}{\partial x^4}\right)\chi_L - \chi_R^*\left(-i\sigma^k\frac{\partial}{\partial x^k} + I\frac{\partial}{\partial x^4}\right)\chi_R$$

This Lagrangian is invariant under the transformation

$$(G47) \quad \chi_L \to \chi_L' = \chi_L \quad , \quad \chi_R \to \chi_R' = e^{i\alpha}\chi_R$$

This yields a current

$$(G48) \quad j^p = -\frac{\partial \mathcal{L}}{\partial \chi_{R|p}}(i)\chi_R = i(i)\chi_R^*\sigma^p\chi_R$$

where

$$(G49) \quad \sigma^4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

The quantity conserved in time is

$$(G50) \quad L_1' = \int \chi_R^*\chi_R \, d^3x = \sum_p (a_1^\dagger a_1 - b_1^\dagger b_1), (= L_1)$$

which is just (G39a). The Lagrangian (G46) is also invariant under the transformation

$$(G51)\quad \psi_R \to \psi_R' = \psi_R \quad , \quad \psi_L \to \psi_L' = e^{i\alpha}\psi_L$$

which yields the conserved quantity

$$(G52)\quad L_2' = \sum_p (a_2^\dagger a_2 - b_2^\dagger b_2)$$

which is just (G39b). The sum of L_1' and L_2' then yields (G38), and the difference of L_1' and L_2' yields (G35).

The field equations for ψ_R and ψ_L follow either from the Lagrangian (G46) or from the application of the transformation (G42) to the field equations (G5). They are

$$(G53)\quad \begin{pmatrix} 0 & i\sigma^k\frac{\partial}{\partial x^k} + I\frac{\partial}{\partial x^4} \\ -i\sigma^k\frac{\partial}{\partial x^k} + I\frac{\partial}{\partial x^4} & 0 \end{pmatrix}\begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = 0$$

or

$$(G54)\quad \left(-i\sigma^k\frac{\partial}{\partial x^k} + I\frac{\partial}{\partial x^4}\right)\psi_R = 0 \quad \Leftrightarrow \quad \left(\sigma^k\frac{\partial}{\partial x^k} + iI\frac{\partial}{\partial x^4}\right)\psi_R = 0$$

$$(G55)\quad \left(i\sigma^k\frac{\partial}{\partial x^k} + I\frac{\partial}{\partial x^4}\right)\psi_L = 0 \quad \Leftrightarrow \quad \left(\sigma^k\frac{\partial}{\partial x^k} - iI\frac{\partial}{\partial x^4}\right)\psi_L = 0 \quad .$$

There are two solutions for each equation. They are

$$(G56)\quad \psi_{R_+} = u(\vec{p})\,e^{i(\vec{p}\cdot\vec{x} - |E|t)} \qquad \psi_{R_-} = v(\vec{p})\,e^{i(\vec{p}\cdot\vec{x} + |E|t)}$$

$$\psi_{L_+} = v(\vec{p})\,e^{i(\vec{p}\cdot\vec{x} - |E|t)} \qquad \psi_{L_-} = u(\vec{p})\,e^{i(\vec{p}\cdot\vec{x} + |E|t)}$$

where

$$(G57)\quad u = \eta_1 \quad , \quad v = \eta_2 \quad .$$

Arbitrary solutions of (G54) and (G55) can be expanded as

$$(G58)\quad \psi_R = \frac{1}{\sqrt{V}} \sum_p a_1(\vec{p}) u(\vec{p}) e^{ipx} + \tilde{b}_1^{\dagger}(\vec{p}) v(\vec{p}) e^{i(\vec{p}\cdot\vec{x} + |E|t)}$$

$$\psi_L = \frac{1}{\sqrt{V}} \sum_p a_2(\vec{p}) v(\vec{p}) e^{ipx} + \tilde{b}_2(\vec{p}) u(\vec{p}) e^{i(\vec{p}\cdot\vec{x} + |E|t)}$$

If one sets

$$(G59)\quad b_1(\vec{p}) = \tilde{b}_1(-\vec{p}) \quad , \quad b_2(\vec{p}) = \tilde{b}_2(-\vec{p})$$

and uses the fact that

$$(G60)\quad u(\vec{p}) = v(-\vec{p})$$

one obtains

$$(G61)\quad \psi_R = \frac{1}{\sqrt{V}} \sum_p a_1(\vec{p}) u(\vec{p}) e^{ipx} + b_1^{\dagger}(\vec{p}) u(\vec{p}) e^{-ipx}$$

$$\psi_L = \frac{1}{\sqrt{V}} \sum_p a_2(\vec{p}) v(\vec{p}) e^{ipx} + b_2^{\dagger}(\vec{p}) v(\vec{p}) e^{-ipx}$$

which agree with (G44), as they must.

The equations for ψ_R and ψ_L can be obtained from first principles. There are two, two dimensional representations of the Lorentz group which yield equations for massless particles of spin 1/2(Schweber 1961). The two field equations for the fields ξ and ζ which transform according to these two representations are

$$(G62)\quad \partial^{a\dot{b}} \xi_a = 0 \quad , \quad \partial_{\dot{a}b} \zeta^{\dot{a}} = 0 \qquad (a, b = 1, 2)$$

where

$$(G63)\quad \partial^{1\dot{1}} = -\partial/\partial x^3 - i\,\partial/\partial x^4 \qquad \partial_{\dot{1}1} = \partial/\partial x^3 - i\,\partial/\partial x^4$$

$$\partial^{1\dot{2}} = -\partial/\partial x^1 - i\,\partial/\partial x^2 \qquad \partial_{\dot{1}2} = \partial/\partial x^1 + i\,\partial/\partial x^2$$

$$\partial^{2\dot{1}} = -\partial/\partial x^1 + i\,\partial/\partial x^2 \qquad \partial_{\dot{2}1} = \partial/\partial x^1 - i\,\partial/\partial x^2$$

$$\partial^{2\dot{2}} = \partial/\partial x^3 - i\,\partial/\partial x^4 \qquad \partial_{\dot{2}2} = -\partial/\partial x^3 - i\,\partial/\partial x^4$$

The field equations become, upon substitution of (G63) into (G62),

$$\text{(G64)} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x^1} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{\partial}{\partial x^2} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial x^3} + i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial}{\partial x^4}\right]\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$$

and

$$\text{(G65)} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x^1} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{\partial}{\partial x^2} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial x^3} - i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial}{\partial x^4}\right]\begin{pmatrix} \zeta^{\dot{1}} \\ \zeta^{\dot{2}} \end{pmatrix} = 0$$

which are, respectively, the same as the equations (G54) and (G55) for χ_R and χ_L. Thus,

$$\text{(G66)} \quad \chi_R = \xi \;, \quad \chi_L = \zeta$$

The invariants are

$$\text{(G67)} \quad \mathcal{L}_R = \xi_{\dot{b}}\, \partial^{a\dot{b}}\, \xi_a = \xi^\dagger \sigma^p \xi_{|p} = \chi_R^\dagger \sigma^p \chi_{R|p}$$

where

$$\text{(G68)} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma^4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

and

$$\text{(G69)} \quad \mathcal{L}_L = \zeta^b\, \partial_{\dot{a}b}\, \zeta^{\dot{a}} = \zeta^\dagger \sigma'^p \zeta_{|p} = \chi_L^\dagger \sigma'^p \chi_{L|p}$$

where

$$(G70) \quad \sigma^{i\prime} = \sigma^{i} \;, \quad \sigma^{4\prime} = -\sigma^{4}$$

$\mathcal{L}_R$ and $\mathcal{L}_L$ are just the Lagrangian densities which appear in (G46).

One final word is necessary. In Chapter 5 the two component neutrino field Ψ is χ_R. Ψ is expanded as

$$(G71) \quad \Psi = \chi_R = \frac{1}{\sqrt{V}} \sum_{p} a(\vec{p}) u(\vec{p}) e^{ipx} + b^{\dagger}(\vec{p}) u(\vec{p}) e^{-ipx}$$

Since neutrinos, i.e. left-handed particles, are under consideration, b and $b^{\dagger}$ are used as annihilation and creation operators. These annihilate and create left-handed particles. This is the approach followed by Roman(1960). An alternative procedure is to choose Ψ as χ_L. Then Ψ is expanded as

$$(G72) \quad \Psi = \chi_L = \frac{1}{\sqrt{V}} \sum_{p} a(\vec{p}) v(\vec{p}) e^{ipx} + b^{\dagger}(\vec{p}) v(\vec{p}) e^{-ipx}$$

In this case, a and $a^{\dagger}$ are annihilation and creation operators for neutrinos, i.e. left-handed particles, for they annihilate and create left-handed particles.

The matrix elements for the processes involving two component neutrinos are independent of the formalism used. This can be seen as follows. If one takes $\Psi = \chi_L$, the terms in the matrix elements which contain the Ψ field are of the form

$$(G73) \quad \langle p' | \chi_L^{\dagger +} \sigma^{\mu\prime} \chi_L^{-} | p \rangle$$

where the $\sigma^{\mu\prime}$ are given by (G70). The matrix elements involving two component neutrinos which appear in the previous Chapters contain terms of the form

$$(G74)\quad \langle p'|N\{\chi_R^{\dagger -}\ \sigma^\mu\ \chi_R^{+}\}|p\rangle \qquad (\Psi = \chi_R)$$

where the σ^μ are given by (G68). It is necessary to show that (G73) and (G74) are equal.

One can show that $\sigma^2\chi_R^*$ satisfies the same equation that χ_L does so that one may take

$$(G75)\quad \chi_L = \sigma^2\chi_R^*$$

which implies that

$$(G76)\quad \chi_R^* = \sigma^2\chi_L\ ,\ \chi_R = \sigma^{2*}\chi_L^*\ ,\ \chi_R^\dagger = \chi_L^T\sigma^2$$

Upon substitution of (G76) into (G74), one obtains

$$(G77)\quad \langle p'|N\{\chi_R^{\dagger -}\ \sigma^\mu\ \chi_R^{+}\}|p\rangle = \langle p'|N\{\chi_L^{T-}\ \sigma^{2*}\sigma^\mu\ \sigma^{2*}\ \chi_L^{*+}\}|p\rangle$$

But,

$$(G78)\quad [\chi_L^{T-}\ \sigma^{2*}\sigma^\mu\sigma^{2*}\chi_L^{*+}] = [\chi_L^{T-}\ \sigma^{2*}\sigma^\mu\ \sigma^{2*}\chi_L^{*+}]^T$$
$$= [\chi_L^{\dagger +}\ \sigma^2\ \sigma^{\mu T}\sigma^2\ \chi_L^-]$$

Further,

$$(G79)\quad \sigma^2\sigma^{1T}\sigma^2 = -\sigma^1,\ \sigma^2\sigma^{2T}\sigma^2 = -\sigma^2,\ \sigma^2\sigma^{3T}\sigma^2 = -\sigma^3,\ \sigma^2\sigma^{4T}\sigma^2 = \sigma^4$$

Then (G77) becomes

$$(G80)\quad \langle p'|N\{\chi_R^{\dagger -}\ \sigma^\mu\ \chi_R^{+}\}|p\rangle = -\langle p'|\chi_L^{\dagger +}\ \sigma^{\mu\prime}\ \chi_L^-|p\rangle$$

Therefore, up to an unimportant sign, the matrix elements (G73) and (G74) are equal.