ON FINITE AMPLITUDE PLANETARY WAVES

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department

of

Physics

Institute of Oceanography

We accept this thesis as conforming to the

required standard

THE UNIVERSITY OF BRITISH COLUMBIA

August, 1970

Department of Physics

The University of British Columbia
Vancouver 8, Canada

Date August 6, 1970

Abstract

Finite amplitude planetary waves are studied on a homogeneous fluid on both the rotating sphere and on a mid-latitude β-plane. The integrated equations of motion are rederived both on the rotating sphere, in a spherical polar co-ordinate system whose axis is tilted relative to the rotation axis, and on a mid-latitude β-plane. The linear solutions are re-examined and the errors associated with the non-divergent and the β-plane approximations are each shown to be about 10 to 15% for waves of a few thousand kilometers wavelength.

Using the integrated equations of motion both on the sphere and on the β-plane, the linear non-divergent Rossby wave solutions are shown to be exact finite amplitude solutions. An exact topographic wave solution is also given for the case of an exponential depth profile. Such behaviour is not found for the divergent waves. Using a Stokes-type expansion in terms of an amplitude parameter, the second order solution for divergent Rossby waves is obtained, and it is found that, as in surface gravity wave theory, the first order correction to the phase velocity is zero.

It is also shown that the linear non-divergent Rossby wave solution on a uniformly sheared zonal current is not a finite amplitude solution, and the second order solution is then calculated. Once again, the phase speed is correct to the first order.

A class of long waves of permanent form analogous to the solitary and cnoidal waves of surface wave theory is obtained for a β-plane channel of either constant or exponentially varying depth. Such waves are found to exist in the divergent case in the absence of any zonal current; however, if the divergence is weak, or if the non-divergent approximation is made , then it is found, as it was by Larsen (1965), that these waves will exist only in the presence of a weakly sheared zonal current. On the exponential depth profile, such waves exist in the absence of a sheared zonal current, even if the non-divergent approximation is made. It is suggested that such waves may also exist trapped along long ocean ridges or scarps.

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Acknowledgements

I gratefully acknowledge the advice and suggestions of Dr. P.H. LeBlond, under whose supervision this work was done. I also thank the director, Dr. G.L. Pickard, the faculty, and fellow graduate students of the Institute of Oceanography for many amiable discussions and much cooperation. I must also acknowledge with gratitude the financial assistance of the Institute of Oceanography, the University of British Columbia, and the National Research Council of Canada. I am especially grateful to Sandra Clarke for typing this thesis.

I. Introduction

1.1 Aims of this study

In recent years, a body of literature (see §1.3) has grown up concerning linear planetary waves both in the oceans and the atmosphere and their importance in an understanding of the dynamics of both. The purpose of this present investigation is to study the properties of finite amplitude planetary waves, particularly with reference to oceanic scales. To introduce these investigations, the physical mechanisms governing planetary waves will be discussed briefly in §1.2, the existing literature outlined in §1.3, and the reported observations of planetary waves in the oceans listed in §1.4.

The finite amplitude effects are investigated by first rederiving in Chapter II the integrated equations of motion for a homogeneous, inviscid fluid on the rotating sphere. These equations are further simplified by assuming that the pressure field is hydrostatic and that the flow is barotropic, that is that the horizontal velocity components are independent of depth.

The effects of the traditional non-divergent and β-plane approximations are then determined from the existing linear theory in Chapter III in order to determine their importance in the evaluation of the finite amplitude solutions in Chapter IV and the long wave (solitary and cnoidal wave) solutions in Chapter V. Some remarks concerning the importance of these solutions are made in the concluding chapter.

The terminology used to describe planetary waves is far from standard. In these investigations a planetary wave is any wave motion in a rotating fluid which, if the rotation is allowed to go to zero, reduces to a steady current. A Rossby wave is a planetary wave on a fluid of constant depth with non-uniform rotation; a topographic wave is a planetary wave on a fluid of variable depth and either uniform or non-uniform rotation.

1.2 Physical mechanisms

A planetary wave is, therefore, defined by the physical mechanisms which drive it. In a shallow, rotating, inviscid, homogeneous fluid, the equations of motion can be integrated over the depth of the fluid, then cross-differentiated to obtain the result that the potential vorticity of a fluid column, $\left(\dfrac{\tilde{\xi} + 2\tilde{\Omega}}{\eta + H}\right)\cdot\hat{k}$, is conserved where $\tilde{\xi}$ is the relative vorticity, $2\tilde{\Omega}$ twice the rotation vector, H the depth of the fluid, η the surface elevation, and $\hat{k}$ a unit vector along the local vertical [Greenspan, (1968), p.236].

In the case of non-uniform rotation and constant depth, a water column moving with some steady velocity along a line of constant rotation may be displaced initially by an external force into a region of higher rotation. In order to conserve potential vorticity, either the water column's relative vorticity must decrease or the depth of the water must increase through a rise in surface elevation. It is found that the first effect predominates for short wavelength waves, the second for long waves. Both effects cause the path line to turn back towards regions of lower rotation and, hence, a restoring force is provided and a wave motion is set up about the undisturbed position of the path line. These waves are the Rossby waves.

If, on the other hand, the rotation is uniform, but the depth is variable, a water column moving steadily along an isobath may be displaced into a region of decreased depth. This can be seen to have exactly the same effect as had moving the column into a region of increased rotation. The fluid responds by decreasing its relative vorticity or increasing its surface elevation so as to keep its potential vorticity constant. These effects again accelerate the fluid column towards its undisturbed isobath and so provide a restoring force for the topographic wave.

In both these cases, the planetary waves can exist only in a rotating fluid. In the absence of rotation, the water columns have no potential vorticity in a uniform undisturbed flow; hence, if they are deflected, the vorticity remains zero and conservation of potential vorticity provides no restoring force to return them to their original positions. In the limit of small rotation, therefore, planetary waves reduce to steady currents.

Veronis (1967a,b) has discussed the analogous behaviour of slow steady flows in rotating and stratified fluids. This analogy is extended in Appendix II, to show that planetary and internal waves exhibit analogous behaviour. This analogy will be particularly useful in predicting the behaviour of topographic waves on bathymetries similar to density profiles for which internal wave solutions have already been found.

1.3 Historical background

The study of planetary waves in geophysical fluids was initiated by C.G. Rossby (1939) in his study of time-dependent motions in the atmosphere. Using linearized equations of motion on the β-plane, he was able to show that a homogeneous fluid could support long barotropic waves whose wavelengths and phase speeds were of the same magnitude as disturbances observed in upper atmosphere meteorological charts.

This theory was given a firm mathematical basis by Haurwitz (1940a,b), who solved the linearized equations both on the sphere and on the β-plane and showed that the various approximations introduced by Rossby had only small effects on the magnitude of the resulting solutions. In these studies, Haurwitz also pointed out that his solution on the sphere had, in fact, been obtained previously by Margules (1892).

Planetary wave theory was applied to oceanic problems by Arons and Stommel (1956) in an investigation of the free periods of meridional and zonal oceans on the β-plane. For Rossby waves they showed that although the phase velocity is always to the west, the group velocity may be in any direction, and therefore, stationary wave solutions may be constructed between meridional boundaries. The amplitudes of these solutions, however, increase without limit northward and southward.

Veronis and Stommel (1956) investigated the response of an unbounded two-layer β-plane ocean to moving wind systems. They found solutions for both barotropic and baroclinic internal free Rossby waves and showed that the frequency of baroclinic Rossby waves went through a minimum value for wavelengths of the order of several hundred kilometers. This investigation suggested that for mid-latitudes most of the energy from fluctuating winds of periods of one to seven weeks enters the ocean in the form of barotropic Rossby waves. For longer periods, increasing energy appears in baroclinic motions until for very long periods (at least 100 years), the response is purely baroclinic. Lighthill (1969), investigating the response of the Indian Ocean to the onset of the monsoon, found that close to the equator the baroclinic response was much quicker (of the order of one month).

Other studies of time-dependent motion in a two-layered, mid-latitude, β-plane ocean were reported by Fofonoff (1962) and Rattray (1964). Their studies clearly show that the frequencies for the internal modes are very much less than those of the barotropic modes. Fofonoff (1962, p.387) finds that for a difference in density between the layers of $2 \times 10^{-3} g/cm^3$, the minimum periods for internal and barotropic waves are about 7 months and 3.6 days respectively. The periods of the baroclinic modes are so long that it seems likely that frictional effects must be important.

Longuet-Higgins in a series of papers (1964a,b; 1965a; 1966) has extensively treated the linear problem of barotropic Rossby waves in a homogeneous fluid both on the surface of the sphere and on the β-plane. In these papers he obtains solutions for both non-divergent and divergent free waves in an unbounded ocean, and for their reflection along solid boundaries; using the reflection properties, he found it possible to sum linear solutions to find the eigensolutions for variously shaped ocean basins.

The effect of bathymetry on planetary wave solutions was investigated by Veronis (1966). He showed that over most of the ocean, the topographic effects were more important than the β-effect, and he also linked the theory of topographic waves to that of Rossby waves.

Topographic wave solutions on different bathymetries appear in the literature under several different names. Reid (1956) found edge wave solutions, which he called edge waves of the second class; these are topographic waves on a sloping shelf. His investigations were continued by Robinson (1964), Hamon (1966), and Mysak (1967) under the name continental shelf waves.

Topographic waves along discontinuities in depth have been called double Kelvin waves or sea-scarp waves and have been investigated by Longuet-Higgins (1968a,b), Rhines (1969a), and Mysak (1969). Rhines (1969a) also studied the reflection of Rossby waves by submarine ridges and found by calculation that the Mid-Atlantic Ridge is sufficiently broad to reflect all but the lowest mode Rossby wave in the North Atlantic.

In addition to Veronis and Stommel (1956), other investigators have studied the response of the ocean to fluctuating or moving pressure or wind systems, notably Longuet-Higgins (1965b), Pedlosky (1967), and Lighthill (1969). Their studies, all for constant depth oceans, confirm the important role Rossby waves must play in the time-dependent response of the ocean. Hamon (1966) and Mysak (1969) have discussed the generation of continental shelf waves and double Kelvin waves respectively by moving or time-dependent weather systems.

Planetary lee waves, generated by steady eastward flowing currents passing over bottom topography have also been investigated. Warren (1963) demonstrated the role topography plays in the generation of the

Gulf Stream meanders. By integrating the vorticity equation numerically over a bottom topography similar to that north of Cape Hatteras, he obtained for a variety of initial flows, a variety of meander patterns with similar shapes, amplitudes and wavelengths to those actually observed. These investigations have been continued by Niiler and Robinson (1967), and Robinson and Niiler (1967).

Porter and Rattray (1964) obtained solutions for finite amplitude Rossby lee wave patterns on steady uniform eastward flows passing over bottom discontinuities aligned north to south. A generalization of their model by Clarke and Fofonoff (1969) allowed the consideration of bottom topography aligned in any direction. This model gave a finite-amplitude lee Rossby wave solution which increased in amplitude downstream if an eastward flow crossed a southeast to northwest step. Such growth of amplitude is a consequence of the unboundedness of the model β-plane ocean.

McIntyre (1968), using a Laplace transform technique, investigated the linear problem of either eastward or westward uniform flow over a single small step. For an unbounded ocean he showed that the assumption of no upstream influence was correct for eastward flows but incorrect for flows to the west. If the ocean is bounded, as for example, the case of a zonal channel, McIntyre shows that the assumption of no upstream influence can never be made. This result is analagous to that obtained by Benjamin (1970) in his investigation of upstream influence for a body moving along the rotation axis of a fluid contained in a tube. In this study, Benjamin also found that upstream influences were always present, although this cannot be predicted on the basis of energy consideration alone.

1.4 Oceanic observations of planetary waves

The planetary waves of §1.2 take the form of disturbances (in time or space) of current speed or direction. In the upper troposphere, such waves are easily observed as wave-like disturbances on charts of isobaric surfaces. In fact, it was to explain these features that Rossby (1939) first studied these waves that bear his name.

In the ocean, planetary waves should appear as periodic fluctuations in long time series measurements of velocity at single points or as long wavelength meanders of well-defined currents if observations are completed in a time much shorter than the periods of these waves. Few long time series records of velocity are available and typical techniques in synoptic oceanographic sampling of large areas obscure the nature of the phenomena; hence, oceanic observations which may be interpreted as planetary waves are rare.

Longuet-Higgins (1965a,p.62) suggests that certain deep velocity measurements north of Bermuda by Swallow (1961) could be evidence of the presence of internal Rossby waves. He further argues from the magnitude of the velocities observed (38 cm/s) that if these were Rossby waves, their amplitudes would be such that the waves would be subject to considerable non-linearities.

Wunsch (1967) found some evidence of Rossby waves in his analysis of tidal records at island stations and suggested that these were generated by the fornightly and monthly tidal potentials. Hamon (1966) observed topographic waves in the form of continental shelf waves in his analysis of tidal records. Finally Thompson (1969) has found evidence of topographic Rossby waves from the long term current records taken at Woods Hole Oceanographic Institution's Site D.

In charts of transport streamlines for regions such as the Antarctic Ocean [Sverdrup et al., (1963), p.606] and the western boundary regions [Warren, (1963)] wave-like patterns appear which may be planetary lee wave patterns. The amplitude of the excursions of the streamlines in these stationary waves appears to be of sufficient magnitude to expect non-linear effects to be important.

Even though Rossby waves have not been unequivocally observed in the deep ocean, theoretical evidence mentioned in the previous section suggests that they should be generated in the oceans by moving or fluctuating atmospheric systems. The lack of definite observations of oceanic planetary waves may be ascribed to the great effort and expense required to make the necessary measurements, rather than to the fact the planetary waves do not exist in oceans. An account of some of these observational difficulties is given by Thompson (1969).

1.5 Non-linear effects

It appears from a few of these observations that the magnitudes of the planetary waves in the ocean may be such that the linearized theory may not be applicable and that non-linearity must be considered. Finite amplitude solutions already exist in the form of the lee wave solutions of Porter and Rattray (1964), and Clarke and Fofonoff (1969). These solutions have been shown to give reasonable agreement with meander patterns observed in the Antarctic Circumpolar Current. This present investigation will look at finite amplitude free waves of the same form.

It has also been recognized for some time that the interaction of planetary waves with ocean currents is important. Keller and Veronis (1969) investigated the effect of random currents on planetary waves; however, their study includes only the advection of the wave by the currents. In this investigation, the interaction of planetary waves with sheared zonal currents will be studied.

Finally, studies of solitary and cnoidal waves by Lax (1968) have shown that any solution of the time-dependent Korteweg-deVries equation,

$$u_t + u\,u_x + u_{xxx} = 0 \tag{1.1}$$

tends asymptotically to a sum of solitary waves; hence, solitary waves, where they exist, are an important limiting case to finite amplitude wave motions. In the final chapter, solitary and cnoidal planetary wave solutions will be described. In analogy to the surface gravity waves, solitary planetary waves, if they exist, could be an important wave form in the ocean.

II. The Equations of Motion

In the following investigations, wave solutions are sought for a homogeneous, inviscid fluid on the surface of a rotating sphere and on a β-plane. Furthermore, the waves will be long with respect to the depth of the fluid; therefore the motion will be considered to be two-dimensional (independent of z). These waves are the planetary waves and are of two classes; the first, the Rossby waves, and the second, the topographic waves.

In this chapter the equations of motion will be developed in a general form both on the sphere and on the β-plane and these will form a basis for the investigations to follow.

On the sphere, the equations are derived in a co-ordinate system (see Figure 1) which rotates about the axis of rotation of the sphere with an angular velocity, α, relative to the surface of the sphere. For a wave of permanent form and phase speed α, this frame of reference is one in which the motion is steady.

The use of spherical co-ordinates presents some difficulties because special assumptions not required by the physics of the flow, must be made at the poles of the co-ordinates in order that the mathematical solutions remain well-behaved. It is then difficult in the final solutions to separate the singularities near the poles that are due to the mathematics from those due to the physics. This problem is discussed in greater detail in Appendix I.

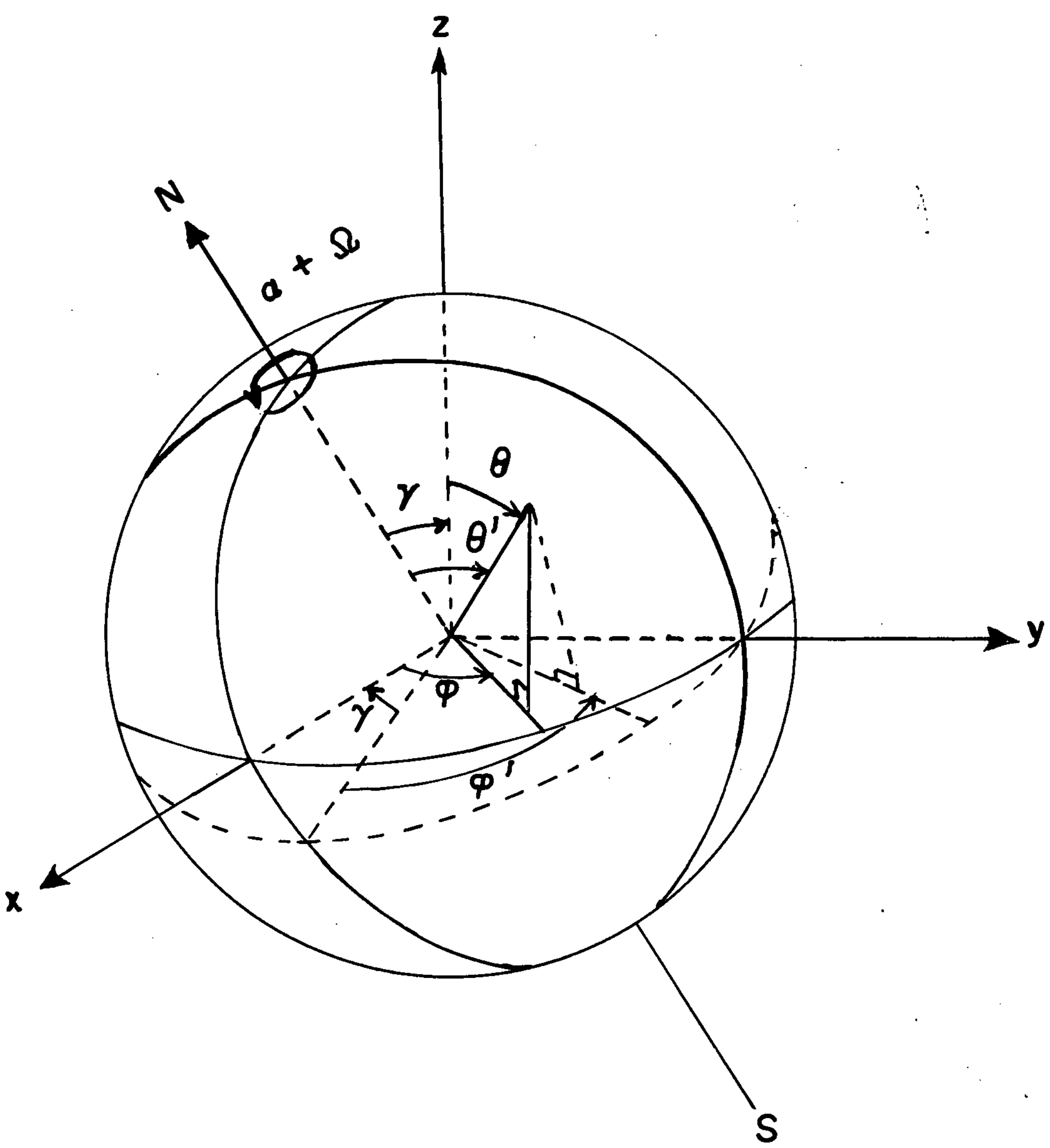

Figure 1. Co-ordinate system on the sphere tilted relative to the rotation axis.

In the development of these equations, the axis of the co-ordinate system is tilted relative to the axis of rotation by an angle γ in order that any unique behavior at the poles of rotation may be separated from the behavior at the axis of the co-ordinates. This angle γ may take any value.

Assuming that v_θ, and v_ϕ are not functions of r, the equations of motion are

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \tag{2.1}$$

$$+ 2(\alpha + \Omega)\left\{\sin\gamma \sin\phi\, v_\theta + (\sin\gamma \cos\phi \cos\theta - \sin\theta \cos\gamma)v_\phi\right\}$$

$$+ g = -\frac{1}{\rho}\frac{\partial p}{\partial r} ,$$

$$\frac{\partial v_\theta}{\partial t} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r} \tag{2.2}$$

$$- 2(\alpha + \Omega)\left\{\sin\gamma \sin\phi\, v_r + (\sin\gamma \sin\theta \cos\phi + \cos\gamma \cos\theta)v_\phi\right\}$$

$$= -\frac{1}{r\rho}\frac{\partial p}{\partial \theta} ,$$

$$\frac{\partial v_\phi}{\partial t} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi \cot\theta}{r} \tag{2.3}$$

$$+ 2(\alpha + \Omega)\left\{(\sin\gamma \sin\theta \cos\phi + \cos\gamma \cos\theta)v_\theta\right.$$

$$\left. - (\sin\gamma \cos\theta \cos\phi - \sin\theta \cos\gamma)v_r\right\}$$

$$= -\frac{1}{r\rho\sin\theta}\frac{\partial p}{\partial \phi} .$$

The equation of continuity is

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\ v_\theta) + \frac{\partial v_\phi}{\partial\phi}\right] = 0 \tag{2.4}$$

and the boundary conditions at the lower and upper boundaries respectively, are

$$v_r = \frac{\partial r_2}{\partial t} + \frac{v_\theta}{r_2}\frac{\partial r_2}{\partial\theta} + \frac{v_\phi}{r_2\sin\theta}\frac{\partial r_2}{\partial\phi} \quad \text{at } r = r_2 = R - H(\theta,\phi,t) \tag{2.5}$$

$$\left.\begin{aligned} v_r &= \frac{\partial r_1}{\partial t} + \frac{v_\theta}{r_1}\frac{\partial r_1}{\partial\theta} + \frac{v_\phi}{r_1\sin\theta}\frac{\partial r_1}{\partial\phi} \\ p &= p_0 = \text{constant} \end{aligned}\right\} \text{at } r = r_1 = R + \eta(\theta,\phi,t)\ , \tag{2.6}$$

where the various symbols are defined in the Glossary of Symbols contained in Appendix III.

In equation (2.5) the fluid depth H is written as a function of time since the frame of reference rotates relative to the sphere and, therefore, any depth variation along the direction of rotation appears in this frame as a time-dependent depth.

These equations must be further simplified before they are in a form in which they may be solved. If $\Omega \simeq \alpha \simeq 10^{-4}s^{-1}$, $v_\theta \simeq v_\phi \simeq$ 1 m/s, $R \simeq 10^6$m, and $r_1 - r_2 \simeq 10^3$m, then in (2.1) the acceleration terms are about $10^{-9}m/s^2$, the centrifugal terms $10^{-6}m/s^2$, and the Coriolis terms $10^{-4}m/s^2$ compared to $g = 9.8\ m/s^2$. Therefore to a high degree of approximation the pressure field is hydrostatic and equation (2.1) may be integrated over r from the free surface r_1

downwards to give at r

$$p(r,\theta,\phi,t) \quad = \quad p_0 \quad + \quad g\rho(R + \eta - r) \ . \tag{2.7}$$

If all the terms of the continuity equation (2.4) are to be of the same magnitude, then $v_r/v_\theta \simeq H/R \simeq 10^{-3}$. Using this value of v_r, it is seen that in (2.2) and (2.3) the centrifugal and the Coriolis terms in which v_r appears may be neglected relative to the other centrifugal and Coriolis terms.

The continuity equation (2.4) may be integrated over the depth of the fluid and the boundary conditions (2.5) and (2.6) applied. Making the approximation that $r_1 \simeq r_2 \simeq R >> H >> \eta$, the integrated continuity equation is

$$R \sin\theta \frac{\partial}{\partial t}(\eta + H) \quad + \quad \frac{\partial}{\partial\theta}[(\eta + H)\sin\theta \ v_\theta] \quad + \quad \frac{\partial}{\partial\phi}[(\eta + H) \ v_\phi] \quad = \quad 0 \ . \tag{2.8}$$

A fuller description of these approximations is given by Phillips (1966). Substituting for the pressure from (2.7) and making the approximation that $r \simeq R$, equations (2.2) and (2.3) may be rewritten as

$$\frac{\partial v_\theta}{\partial t} + \frac{v_\theta}{R}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\phi}{R\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{v_\phi^2 \cot\theta}{R} \tag{2.9}$$

$$- \ 2(\alpha + \Omega)(\sin\gamma \ \sin\theta \ \cos\phi + \cos\gamma \ \cos\theta \ )v_\theta \quad = \quad - \frac{g}{R}\frac{\partial\eta}{\partial\theta}$$

$$\frac{\partial v_\phi}{\partial t} + \frac{v_\theta}{R}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{R\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_\theta \cot\theta}{R} \tag{2.10}$$

$$+ \; 2(\alpha + \Omega)(\sin\gamma \,\sin\theta\, \cos\phi \; + \; \cos\gamma\, \cos\theta) v_\phi \; = \; - \frac{g}{R\sin\theta}\frac{\partial \eta}{\partial \phi} \; .$$

Equations (2.8) to (2.10), known as the integrated equations of motion, form the basis for the following investigations of planetary wave motions on a rotating sphere.

Planetary waves of importance in theoretical studies of the generation of time-dependent motions in the oceans have wavelengths considerably shorter than the width of the ocean basins. Since most ocean basins have dimensions less than the earth's radius, such waves have wavelengths considerably shorter than the earth's radius. For such waves, it was shown by Rossby (1939) that the surface of the sphere could be mapped onto a tangent plane, the effect of rotation being retained in a Coriolis parameter linear in y, the north-south co-ordinate. Such a transformation allows the use of Cartesian co-ordinates and, therefore, greatly simplifies the analysis. The effects of making the β-plane transformation have been examined in some detail by Veronis (1963).

Equations (2.8), (2.9) and (2.10) may be transformed to their corresponding β-plane equations by first setting α and γ to zero, then allowing $R \to \infty$ in such a way that $\frac{1}{R}\frac{\partial}{\partial\theta} \to -\frac{\partial}{\partial y}$, $\frac{1}{R\sin\theta}\frac{\partial}{\partial\phi} \to \frac{\partial}{\partial x}$,

$v_\theta \to -v$, $v_\phi \to u$, and $2\Omega\cos\theta \to f$. The equations on the β-plane are then,

$$\frac{\partial}{\partial t}(\eta + H) + \frac{\partial}{\partial x}[u(\eta + H)] + \frac{\partial}{\partial y}[v(\eta + H)] = 0 \tag{2.11}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x} \tag{2.12}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y} \tag{2.13}$$

where

$$\begin{aligned} f &= 2\Omega[\sin(y_0/R) + (y/R)\cos(y_0/R)] \\ &= f_0 + \beta y \end{aligned} \tag{2.14}$$

and y_0/R is the latitude at which the β-plane is tangent to the sphere.

Equations (2.8) to (2.10) and (2.11) to (2.13) describe the depth averaged flow of a shallow, inviscid and homogeneous fluid over a rough bottom both on the rotating sphere and on the β-plane respectively. Using these as a basis, in the following thesis, finite amplitude planetary waves will be investigated in a variety of cases.

III. Results from Linear Theory

3.1 Rossby waves on the sphere

3.1.1 Introduction

The linear theory of planetary waves has been well developed by Haurwitz (1940), Longuet-Higgins (1964b, 1965a, 1966, 1968a,b), Veronis (1966), and Rhines (1969a,b), as well as others. In this chapter the results of all these authors are summarized and the effects of the various approximations commonly used is discussed. In particular, the linear theory will show the magnitude and importance of the errors introduced by the β-plane and non-divergent approximations. A knowledge of the effects of such approximations is necessary if the non-linear solutions to be obtained later are to be interpreted.

In the following section the solutions for Rossby waves in an ocean of constant depth completely covering the surface of a rotating sphere will be given, following Longuet-Higgins (1964b, 1965a). The solutions are obtained first making the non-divergent approximation, then dropping this approximation for the divergent case.

The basic equations of motion are given by (2.8), (2.9), and (2.10). For constant depth, α and γ zero, and the velocities and surface elevations small, these equations may be linearized to give

$$\frac{\partial \eta}{\partial t} + \frac{H}{R \sin\theta}\left[\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{\partial v_\phi}{\partial \phi} \right] = 0 \qquad (3.1)$$

$$\frac{\partial v_\theta}{\partial t} - 2\Omega \cos\theta \, v_\phi = -\frac{g}{R}\frac{\partial \eta}{\partial \theta} \tag{3.2}$$

$$\frac{\partial v_\phi}{\partial t} + 2\Omega \cos\theta \, v_\theta = -\frac{g}{R \sin\theta}\frac{\partial \eta}{\partial \phi} \tag{3.3}$$

For wave solutions rotating about the axis of rotation of the sphere the dependent variables may have their ϕ and t dependence expressed by exp $i(s\phi - \sigma t)$. Substituting this into equations (3.1) to (3.3), the partial differential equations are reduced to a set of ordinary differential equations. By defining the following variables

$$D = (1 - \mu^2)\frac{\partial}{\partial \mu}, \quad \mu = \cos\theta, \quad \lambda = \sigma/2\Omega, \quad \bar{\delta} = \frac{4\Omega^2 R^2}{gh} \tag{3.4}$$

Longuet-Higgins (1965a) reduced these ordinary differential equations to the single equation

$$\left[\nabla^2 - s' - \frac{2\bar{\delta}\mu(D - s')}{s'^2 - \bar{\delta}(1 - \mu^2)} + \bar{\delta}(\lambda^2 - \mu^2)\right](v_\theta \sin\theta) = 0 \tag{3.5}$$

where

$$s' = s/\lambda, \quad \nabla^2 = \frac{\partial}{\partial \mu}\left\{(1 - \mu^2)\frac{\partial}{\partial \mu}\right\} - \frac{s^2}{1 - \mu^2}. \tag{3.6}$$

3.1.2 The non-divergent approximation

Equation (3.5) still includes in it the effects of divergence, and, therefore, it may be simplified by making the non-divergent approximation. This approximation assumes that the first term of (3.1) is much smaller than the other two and, hence, may be neglected.

In terms of the non-dimensional parameters defined by (3.4), this assumption implies that $\bar{\delta} << s'$. If this is true, then (3.5) reduces to

$$[\nabla^2 - s'](v_\theta \sin\theta) = 0 , \tag{3.7}$$

which for

$$s' = 2\Omega s/\sigma = - n(n + 1) \tag{3.8}$$

has as solutions the sperical harmonics of degree n and order s, where $s \leq n$.

These non-divergent solutions were first obtained by Haurwitz (1940). Longuet-Higgins (1964b) generalized these results by showing that the axis of the spherical harmonics could be rotated through an arbitrary angle away from the axis of rotation of the sphere. Providing these spherical harmonics rotated about the axis of rotation with an angular velocity of $- 2\Omega/n(n+1)$, the non-divergent linear equations are still satisfied. That is to say, in the co-ordinate system described by Figure 1 (p. 12), the linear non-divergent solution consists of spherical harmonics of degree n and order s where $\alpha = - 2\Omega/n(n+1)$, and γ is arbitrary. Therefore, while the angular phase speed of these waves must be about the axis of rotation of the sphere, the tilting of the co-ordinate axis shows that the poles of the co-ordinates (at which the fluid velocity due to the waves is zero) do not necessarily coincide with the poles of rotation.

For values of H, R, and Ω corresponding to those of real oceans on the earth, Longuet-Higgins (1965a) gives values of $\bar{\delta}$ ranging from 15 to 150. For an ocean of 4 km depth, $\bar{\delta} = 22$; therefore, it appears that, in order that the non-divergent approximation be valid on the sphere,

s' must be very large. From (3.8) $s' = O(n^2)$; therefore, the non-divergent approximation is valid only for large n.

Ocean basins have horizontal dimensions much less than the earth's circumference. In many physical oceanic problems such as air-sea energy exchanges with atmospheric disturbances, the wavelengths of interest must be much smaller than the width of the ocean; hence, s, the number of wavelengths around the equator, must be large. Since $n \geq s$, then n is indeed large; therefore, the non-divergent Rossby wave solutions may be useful in the examination of oceanic phenomena.

On the other hand, for the investigations of stationary Rossby waves in large enclosed basins such as the Pacific Ocean, waves whose wavelengths are the same magnitude as the width of the basin will be important. In this case the non-divergent approximation is not likely to be applicable, and a more reasonable assumption would be that $\bar{\delta}/s' \leq O(1)$.

3.1.3 The divergent solution

If $\bar{\delta} = O(s')$ and all terms of O(1) or less are neglected, then equation (3.5) is approximated by the spheroidal wave equation

$$\left[\frac{d}{d\mu}\left((1-\mu^2)\frac{d}{d\mu}\right) - \left(\frac{s^2}{1-\mu} + \bar{\delta}\mu^2 + s'\right)\right](v_\theta \sin\theta) = 0 . \quad (3.9)$$

This equation was first obtained for Rossby waves by Longuet-Higgins (1965); its solutions are given by the spheroidal wave functions $S_n^s(\sqrt{\bar{\delta}},\mu)$ where

$$s' = -A_{sn}(\sqrt{\bar{\delta}}). \quad (3.10)$$

The function A_{sn} is given by the solution of a transcendental equation involving continued fractions. Values of $A_{sn}(c)$ are tabulated in Stratton, Morse, Chu, Little and Corbato (1956) for values of s, n, and c, all ranging form 0 to 8. These ranges cover most of the expected variation of $\bar{\delta}$; however, the tables do not extend to large enough values of n and s.

In the non-divergent limit as $\bar{\delta} \to 0$, $A_{sn}(\sqrt{\bar{\delta}})$ may be expressed in terms of a power series in $\bar{\delta}$ [Stratton et al., (1956)] given by

$$A_{sn}(\sqrt{\bar{\delta}}) = n(n + 1) + \frac{\bar{\delta}}{2}\left[1 - \frac{(2s - 1)(2s + 1)}{(2n - 1)(2n + 3)}\right] + 0(\bar{\delta}^2) \quad (3.11)$$

Therefore, in the limit of small $\bar{\delta}$, (3.10) reduces to (3.8) given by the non-divergent analysis.

The shape of the waves will be changed from that given by the non-divergent solutions if the divergence terms are included; however, since these waves are unlikely to be observed in detail, such differences are not of much interest. Of more interest are the differences in the dispersion relations between the two cases.

From the definition of s' given by (3.4) it is seen that the angular phase speed of the wave about the axis of rotation is given by $2\Omega/s'$. Hence, for the non-divergent case, the phase speed is independent of s, the longitudinal wave number, while for the divergent case the phase speed is a function of both n and s. This difference in dispersion relations has an important effect on the combination of the wave solutions. In the non-divergent case, waves of the same degree n but different orders s may be summed to form new linear wave solutions. Since the phase speeds are all the same, these solutions will not disperse as the wave travels around the globe. Of course, non-linear interactions between the solutions can be expected to disperse the wave eventually.

On the other hand, for the divergent case, the phase speed is different for each different value of n or s; hence, no such superposition of solutions is possible. Any two solutions of the same degree but different order will slowly disperse as the wave moves around the sphere, independently of non-linear effects.

The magnitude of this dispersion can be estimated using the tabulated values of A_{sn} [Stratton et al., (1956)]. Using these tables, Table I was drawn up to give the difference between the non-divergent and divergent phase speed as a percentage of the divergent phase speed for $\bar{\delta} = 64$. This value of $\bar{\delta}$ is the highest for which $A_{sn}(\sqrt{\bar{\delta}})$ is tabulated and represents a value that is larger than those calculated for most of the world's oceans. Hence, the differences shown in the table are larger than what might be expected for an ocean of average depth 4 km.

In Table I it is seen that the percentage differences in phase speed between the two cases decrease with increasing s, and after an initial increase also decrease with increasing n, except for s = 0, which shows no initial increase. The minimum percentage differences for each n occur along the diagonal given by n = s, and these minimum values also decrease with increasing n. The maximum value of s for which a tabulated value of A_{sn} was given is s = 8. This represents a wave

TABLE I

THE PERCENTAGE DIFFERENCE BETWEEN DIVERGENT AND NON-DIVERGENT PHASE SPEEDS OF ROSSBY WAVES ON THE SPHERE

n(n+1)	$100[A_{sn} - n(n+1)]/[n(n+1)]$								
	s = 0	1	2	3	4	5	6	7	8
2	870.	275.							
6	495.	290.	91.7						
12	298.	212.	127.	40.8					
20	190.	152.	111.	67.0	22.0				
30	125.	109.	89.0	65.3	40.0	13.0			
42	85.2	80.0	69.8	56.7	41.9	25.7	8.3		
56	61.8	59.8	54.6	47.3	38.6	28.6	17.5	5.7	
72	47.2	44.4	43.5	39.2	33.8	27.4	20.3	12.5	4.2

whose wavelength at the equator is approximately 5×10^3 km. or about a third the width of the Pacific Ocean. Even for this large value of $\bar{\delta}$, the error in phase speed caused by the non-divergent approximation is only of the order of 20% if $(n - s) \leq 2$, $n = 8$.

From Table I an estimate can also be made of the magnitude of the difference in phase speed between two divergent Rossby waves of the same degree n but different orders. The percentage difference in angular phase speed, $(\bar{\alpha}_{sn} - \bar{\alpha}_{\bar{s}n})\, 100/\bar{\alpha}_{\bar{s}n}$, is approximately equal to $(A_{\bar{s}n} - A_{sn})\, 100/[n(n + 1)]$, the difference between any two columns of Table I. For n = 8, and for a difference in s of 1, this percentage difference ranges from 3% to 8% as s increases from 0 to 8. This percentage difference also appears to decrease as n increases, and so for the range of large n and s, which is of the most interest, the effect is expected to be negligibly small. However, the fact that such a difference in dispersive behaviour does exist between divergent and non-divergent solutions indicates that their non-linear behaviour, which will be studied in later chapters, may also be different.

3.1.4 Properties of the solutions

Following Longuet-Higgins (1965a), the spheroidal wave equation (3.9) can be put into the standard Liouville form

$$\frac{d^2V}{d\theta^2} - [(s^2 - \frac{1}{4})\csc^2\theta + \bar{\delta}\cos^2\theta + (s' - \frac{1}{4})]\,V = 0 \quad (3.12)$$

through the transformation

$$v_\theta = (\sin\theta)^{-\frac{1}{2}}\,V(\theta). \quad (3.13)$$

Setting

$$(s^2 - \frac{1}{4})\csc^2\theta + \bar{\delta}\cos^2\theta + (s' - \frac{1}{4}) = -\nu^2 \quad (3.14)$$

in equation (3.12), it can be seen that the character of the solution of (3.12) changes from sinusoidal to exponential as ν^2 goes from positive to negative values. For large s, the first two terms of the left-hand side of (3.14) are both positive and monotonically increasing as $\left|\theta - \frac{\pi}{2}\right|$ increases. Therefore, ν^2 is positive only if s' is both large and negative, and then only if θ lies between $\pi - \theta_0$ and θ_0 where

$$(s^2 - \frac{1}{4}) \csc^2\theta_o + \bar{\delta} \cos^2\theta_\delta + (s' - \frac{1}{4}) = 0 \quad (3.15)$$

The effect of the non-divergent approximation is to change the range of θ over which the solution is sinusoidal as well as to change the shape of the solution. It was also shown by Longuet-Higgins (1964b) that for the non-divergent case, the poles of the spherical harmonics which make up the solution do not have to coincide with the sphere's poles of rotation. Hence, the equatorial belt in which the waves are sinusoidal is a belt surrounding the equator of a co-ordinate system, whose axis, as in Chapter II, may be tilted at an arbitrary angle γ from the axis of rotation providing it rotates about that axis with angular velocity $-2\Omega/n(n+1)$.

Since the equations are linear, the sum of solutions is also a solution. Therefore, it is possible to sum many solutions of the same n but different s and different orientations to give a resultant solution that is periodic in θ everywhere. This sum is not possible in the divergent case as the waves of different orders each move with a different phase speed.

In conclusion the linear solutions show that on the sphere, the errors introduced by the non-divergent approximation decrease with increasing wave number. For wave numbers around 8, the error introduced in the phase speed is about 10 to 20%. The non-divergent approximation eliminates the variation of phase speed with the longitudinal wave number found for the divergent solutions; however, this dispersion is found to be small for n = 8 and appears also to decrease with both increasing n and decreasing $\bar{\delta}$. For the wavelengths of interest in the world oceans s and n are both greater than 8 and the error introduced by the non-divergent approximation in the dispersion relation is, therefore, less than 20%. There is some indication that the non-divergent approximation may have a large effect when it comes time to investigate the non-linearities of the solutions in later chapters.

3.2 The β-plane solutions

It has already been stated that the solutions of most interest in the study of oceanic problems are those which have wavelengths smaller than the dimensions of the world oceans. In these cases it has been shown that the errors introduced by the non-divergent approximation are not serious. For the same range of wavelengths, that is, those smaller than the earth's radius, it seems likely that the β-plane approximation may also be used to simplify the solutions still further.

The β-plane equations are obtained in Chapter II by mapping a restricted area on the surface of the sphere onto a tangent plane, and are given by (2.11), (2.12), and (2.13). If these equations are linearized and the depth held constant, they reduce to

$$\frac{\partial \eta}{\partial t} + H\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = 0 \tag{3.16}$$

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial \eta}{\partial x} = 0 \tag{3.17}$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial \eta}{\partial y} = 0 . \tag{3.18}$$

Following Longuet-Higgins (1965a) these equations may be further reduced to the single equation

$$\left[gH\left(\frac{\partial}{\partial t}\nabla^2 + \beta\frac{\partial}{\partial x}\right) - \left(\frac{\partial^3}{\partial t^3} + f^2\frac{\partial}{\partial t}\right)\right]\frac{\partial v}{\partial t} = 0 \ . \tag{3.19}$$

If it is assumed, as it was on the the sphere, that $\sigma << 2\Omega$, where σ is the radian frequencey, and also that f^2 in (3.19) may be treated as a constant, then (3.19) has a simple sinusoidal solution given by

$$v = v_\circ \exp i\,(kx + \ell y - \sigma t) \tag{3.20}$$

where the dispersion relation is

$$\frac{\sigma}{k} = -\frac{\beta}{k^2 + \ell^2 + f_\circ^2/gH} \ . \tag{3.21}$$

If the non-divergent approximation is made by neglecting the first term of (3.16), then (3.16) to (3.18) reduce to

$$\left[\frac{\partial}{\partial t}\nabla^2 + \beta\frac{\partial}{\partial x}\right] v = 0 \tag{3.22}$$

whose solutions is also (3.20). However, for the non-divergent case, the dispersion relation is given by

$$\frac{\sigma}{k} = -\frac{\beta}{k^2 + \ell^2} \ . \tag{3.23}$$

Since the β-plane approximation is valid only over distances which are short relative to the earth's raduis, the β-plane solutions should show reasonable agreement with the solutions on the sphere only for the short wavelength cases. On the sphere it was for these short wavelength cases that the non-divergent approximation was valid. Comparing (3.21) to (3.23), it is seen that this is also the case on the β-plane. For large k and ℓ (short wavelength), the per cent error in the zonal phase speed, σ/k, introduced by the non-divergent approximation is approximately $100f^2/gH(k^2 + \ell^2)$. For a wavelength of about 1000 km, this error is about 10% and will decrease with decreasing wavelength.

Near the equator, the phase speed of the Rossby wave solution on the sphere is given by $-2\Omega R/A_{sn}$ and the longitudinal wavelength by $2\pi R/s$. If $n = s$, the wave crests are aligned along the meridians of longitude and the corresponding wave on the β-plane is given by (3.20) where $\ell = 0$ and $k = s/R$. Since the assumption that f^2 is constant is not valid near the equator, and since for large n and s and for $n = s$, the non-divergent approximation is valid, the non-divergent solutions on the sphere and on the β-plane are compared. Comparing their phase speeds, it is found that, for $n = s$, the percentage difference is approximately $100/s$. For $s \geq 10$, that is, for wavelengths less than 1000 km, the error in phase speed introduced by the β-plane approximation is about 10%, and this is the same order as the errors introduced by the non-divergent approximations. For longer wavelengths, Longuet-Higgins (1966) shows very good agreement between β-plane solutions and spherical solutions for a hemispherical ocean basin centered around the equator.

In discussing the form of the solutions on the sphere it was pointed out that these solutions are sinusoidal in θ only for a range of co-latitudes on each side of the equator of the co-ordinate system. At first sight this behaviour does not seem to be reproduced by the β-plane solutions, which seem to remain periodic in both x and y regardless of the latitude. This is not entirely true for the divergent solutions, since the assumption that f^2 may be treated as a constant is valid over different ranges of y for different latitudes.

For a β-plane taken around the equator, $f = \beta y$, and a solution of (3.19) for $\sigma \ll f$, is given in terms of Parabolic Cylinder functions by

$$v = v_o \exp i(kx - \sigma t) \left[A\, U(\lambda, \sqrt{\frac{2\beta}{gH}}\, y) + B\, V(\lambda, \sqrt{\frac{2\beta}{gH}}\, y)\right] \tag{3.24}$$

where

$$\lambda = \frac{\sqrt{gH}}{2} (k^2 + \frac{k\beta}{\sigma}) \quad . \tag{3.25}$$

Since the parameter λ is related to the wavelength of the wave in the north-south direction, (3.25) is the dispersion relation for the wave. Even though in (3.24) the solution varies in a non-sinusoidal fashion with latitude, in contrast to the solutions on the sphere, it still remains periodic in y over any range of y. This difference in behaviour is due to the fact that the β-plane, while restricted in the range over which it is valid, is actually treated mathematically as being unbounded.

In discussing the non-divergent solutions on the sphere, it was noted that the belt of co-latitudes for which the solutions were periodic in two dimensions could be tilted at any angle to the axis of rotation of the sphere. Therefore, anywhere on the surface of the sphere, it is possible for non-divergent waves that are doubly-periodic to exist. What these solutions do require is that these doubly-periodic waves are of finite lateral extent. It is this finite lateral extent that is missing from the β-plane solutions. However, if the width of the equatorial belt on the sphere is large, then it may exceed the range over which the β-plane approximations is valid; hence, within their range of applicability the β-plane solutions are a good approximation to the solutions on the sphere.

Summarizing briefly, the linear solutions indicate that for short wavelengths (less than 1000 km.) both the β-plane and non-divergent approximations may be made, the errors from each not exceeding 10%.

3.3 Topographic waves

In each of the preceding sections the fluid depth has been held constant and the resulting solutions have been referred to as Rossby waves. In §1.2, it was shown that variations in depth will support a class of planetary waves known as topographic waves in the same way a non-uniform rotation field supports Rossby waves. Veronis (1966) showed that for typical oceanic values, this topographic effect is much more important than the β-effect.

The basic equations are obtained by linearizing (2.11), (2.12), and (2.13) to give

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uH) + \frac{\partial}{\partial y}(vH) = 0 \tag{3.26}$$

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial \eta}{\partial x} = 0 \tag{3.27}$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial \eta}{\partial x} = 0\ . \tag{3.28}$$

As with Rossby waves, the non-divergent approximation is made by neglecting the first term in (3.26). Following Veronis (1966), the non-divergent approximation is made, $H = h(y)$, and $v \propto \exp i(kx-\sigma t)$; then under these conditions, equations (3.26), (3.27), and (3.28) may be reduced to

$$\left[\frac{1}{h}(hv)_y\right]_y - \left[\left(\frac{f}{h}\right)_y \frac{kh}{\sigma} + k^2\right] v = 0 \ . \tag{3.29}$$

In equation (3.29) the depth h plays a dual role. In the second term, the variation of h plays the same role for topographic waves as does the variation of f for β-plane Rossby waves. However, unlike f in the Rossby wave case, h also appears in the first term. This occurs because, independent of the vorticity effects, the velocity must increase or decrease with increasing or decreasing depth in order that mass conservation be satisfied.

For Rossby waves on a mid-latitude β-plane, f is always a linear function of latitude; however, for topographic waves a whole range of different depth profiles may be chosen, all of which model actual oceanic bathymetries. A simple profile, studied by Veronis (1966), is the exponential profile, $h = h_\circ \exp(-\Lambda y)$. For this profile Veronis gives as a solution to (3.29)

$$v = v_\circ \exp(\tfrac{1}{2}\Lambda y) \exp i(kx + \ell y - \sigma t) \tag{3.30}$$

where

$$c = \frac{\sigma}{k} = -\frac{f\Lambda}{k^2 + \ell^2 + \frac{1}{4}\Lambda^2} \tag{3.31}$$

and where f has been held constant. The first factor in this solution for v is a growth factor required by the presence of h in the first term of (3.29)

It was pointed out by Rhines (1969a) that for real oceanic slopes, $\Lambda \ll k$, and, hence, the variation of h may be neglected in the first term of (3.29). Such an approximation has its analogue in the theory of internal gravity waves. There it is traditional to make the Boussinesq approximation in which one neglects the variation of ρ where it appears as an inertial mass but retains its variation where it appears multiplied by g and, hence, as part of the body forces on the fluid.

Veronis (1966) also treats the case, in which both f and h are allowed to vary, and outlines the difficulties which one may encounter, if the assumption that f may be treated as a constant except under differentiation is made without due care. He shows that if any terms are neglected, care must be taken to neglect all other terms of the same magnitude lest terms are retained that may indicate that the solutions is growing in time.

Considering the case of $h = h_o \exp(-\Lambda x)$, equations (3.26) to (3.28) may be solved by making the non-divergent approximation, then defining a transport stream function, ψ, by

$$uh = -\frac{\partial\psi}{\partial y}, \qquad vh = \frac{\partial\psi}{\partial x}. \tag{3.32}$$

In terms of this stream function Veronis (1966) obtained as a solution to these equations

$$\psi = \psi_o \exp(-\frac{\Lambda x}{2}) \exp i[(\ell y + \int^y \frac{f\Lambda}{2\sigma}\, dy) + (k - \frac{\beta}{2\sigma})x - \sigma t] \tag{3.33}$$

where

$$\sigma^2 = \frac{f_o^2\Lambda^2 + \beta^2}{4(k^2 + \ell^2 + \Lambda^2/4)} \quad . \tag{3.34}$$

Locally equation (3.33) may be approximated by

$$\psi = \psi_o \exp(-\frac{\Lambda}{2}x) \quad \exp i(\bar{\ell}y + \bar{k}x - \sigma t) \tag{3.35}$$

where

$$\bar{\ell} = \ell + f_o\Lambda/2\sigma \ , \qquad \bar{k} = k - \beta/2\sigma \tag{3.36}$$

and therefore the dispersion relation may be written

$$\sigma = \frac{f_o\Lambda\bar{\ell} - \beta\bar{k}}{\bar{k}^2 + \bar{\ell}^2 + \Lambda^2/4} \tag{3.37}$$

which is similar in form to equation (3.31).

The form of equation (3.37) allows the effects of bottom topography to be compared with those of non-uniform rotation since both appear. The most level areas of the ocean floor, an abyssal plain, have slopes of about 10^{-4}; for continental slopes, rises and shelves and for the mid-ocean ridges the slopes are typically an order or two greater in magnitude. Even for a slope of 10^{-4} and $h \simeq 10^3$ m, $\Lambda \simeq 10^{-7}$ m^{-1} and, hence, for mid-latitudes $\Lambda f_o \simeq 10^{-11}$ m^{-1} s^{-1}, the same magnitude as $\beta \simeq 10^{-11}$ m^{-1} s^{-1}. Therefore, it is seen that over much of the ocean basins, the topographic effect will be equal to or dominate over the β-effect. Furthermore, (3.37) shows that for given σ and if $\Lambda f_o > \beta$, the wavelength of the topographic waves is shorter than that of the Rossby waves, and also that this wavelength decreases with increasing bottom slope. Therefore, it appears that for the same range of frequencies the non-divergent and β-plane approximations may be made with greater confidence for the topographic waves than for the Rossby waves.

If equation (3.29) is written in terms of the transport $V = vh$, it takes the form

$$h\left[\frac{1}{h} V_y\right]_y - \left[\frac{kh}{\sigma}\left(\frac{f}{h}\right)_y + k^2\right] V = 0 \ . \qquad (3.38)$$

In this equation an analogy, discussed in greater detail in Appendix II, may be clearly seen between the behaviour of planetary waves and internal gravity waves. For internal gravity waves on a density distribution that varies only with depth, the vertical velocity is governed by the equation [Krauss, (1965)].

$$\rho\left[\frac{W_z}{\rho}\right]_z - k^2\left[\frac{g\rho}{\sigma^2}\left(\frac{1}{\rho}\right)_z + 1\right] W = 0 \qquad (3.39)$$

where

$$w(x,y,z) = W(z) \exp i(kx - \sigma t) \ . \qquad (3.40)$$

If, in the internal wave case, the depth is constant and also if the upper surface is assumed rigid then the boundary condition requires that w be zero at both boundaries. In a fluid of infinite depth this condition is replaced by the condition that the vertical velocities tend to zero as $z \to \pm\infty$.

It is easily seen that if f is held constant in (3.38), then the two equations are identical in form. Hence, all the solutions for internal waves on various density profiles will have analogous solutions for topographic waves on depth profiles of the same form.

In particular, Rhines (1969a) and Longuet-Higgins (1968a,b) have found planetary wave solutions which consist of waves trapped along depth profiles such as ocean ridges and sea scarps. These solutions are analogous to the internal wave solutions given by Groen (1948) and Krauss (1965) for waves trapped on a pycnocline in a fluid of infinite depth.

Solutions to (3.38) have been given for a variety of profiles by various authors. Rhines (1969b) has given solutions for waves trapped around islands and sea mounts; and Mysak (1967) has obtained shelf wave solutions, planetary waves trapped on a sloping continental shelf. For other bathymetries, still more types of topographic waves can undoubtedly be found.

IV. Finite amplitude planetary waves

4.1 Introduction

Finite amplitude effects for planetary waves, as for any other wave governed by non-linear equations of motion, can be investigated by two fundamentally different techniques. In the first, the entire non-linear set of equations is manipulated, making only those approximations necessary to find an "exact" solution. Having obtained such solutions, it is then possible, a posteriori, to determine whether such solutions have any physical significance. Such a method has been used with great success by Yih (1960) in his studies of stratified flows over obstacles.

With the second technique, the investigator must begin with a physical concept of the phenomena of interest so that the terms of the non-linear equations may be properly scaled and solved using a perturbation expansion in some small parameter. Both of these techniques will be used in this chapter to determine the finite amplitude effects on the linear solutions outlined in Chapter III.

The first technique is used in sections 4.2 to 4.4 and exact non-divergent solutions are obtained for a constant depth ocean on the rotating sphere and for a β-plane channel both for uniform depth and for an exponential depth profile. These Rossby wave solutions on the sphere and the β-plane are shown to be identical to the linear non-divergent Rossby wave solutions obtained by Longuet-Higgins (1964b, 1965a).

If the non-divergent approximation is not made, the equations cannot be reduced to a form that can be solved exactly. In §4.5, the finite amplitude divergent Rossby wave solutions on the β-plane are found to the second order in wave amplitude using a perturbation expansion.

In §4.6, the important problem of the interaction of Rossby waves with shear currents is investigated. In particular, it is shown that the linear non-divergent solution is no longer a solution to the non-linear equation of motion in the presence of a weakly sheared zonal current.

4.2 Rossby waves on the sphere

4.2.1 The equations

The relevant equations of motion for inviscid flow on the sphere have been developed in Chapter II and are given by (2.8) to (2.10). For free waves of permanent form rotating about the axis of rotation of the sphere with angular phase speed α, the motion is steady in the frame of reference described in Chapter II, that is, a frame rotating with angular velocity, $(\alpha + \Omega)$, around the rotation axis of the sphere. In order that the motion be steady it is also necessary that the depth be a function of θ'only, where θ' is the co-latitude relative to the axis of rotation.

Under these conditions (2.8) is written

$$\frac{\partial}{\partial\theta}\left[(\eta + H)\sin\theta\, v_\theta\right] + \frac{\partial}{\partial\phi}\left[(\eta + H)\, v_\phi\right] = 0 . \tag{4.1}$$

Equation (4.1) allows the definition of a stream function, $\psi(\theta,\phi)$, such that

$$v_\theta = \frac{1}{(\eta + H)\sin\theta}\frac{\partial\psi}{\partial\phi}, \qquad v_\phi = -\frac{1}{\eta + H}\frac{\partial\psi}{\partial\theta}. \tag{4.2}$$

In terms of this stream function (2.9) and (2.10) are given by

$$\frac{1}{(\eta + H)\sin\theta}\frac{\partial\psi}{\partial\phi}\frac{\partial}{\partial\theta}\left[\frac{1}{(\eta + H)\sin\theta}\frac{\partial\psi}{\partial\phi}\right] - \frac{1}{(\eta + H)\sin^2\theta}\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial\phi}\left[\frac{1}{\eta + H}\frac{\partial\psi}{\partial\phi}\right]$$

$$- \frac{\cot\theta}{(\eta + H)^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2 + \frac{2(\Omega + \alpha)R}{\eta + H}(\sin\gamma\,\sin\theta\,\cos\phi + \cos\gamma\,\cos\theta)\frac{\partial\psi}{\partial\theta}$$

$$= -g\frac{\partial\eta}{\partial\theta}, \tag{4.3}$$

and

$$\frac{1}{(\eta + H)\sin\theta}\frac{\partial\psi}{\partial\phi}\frac{\partial}{\partial\theta}\left[\frac{1}{\eta + H}\frac{\partial\psi}{\partial\theta}\right] - \frac{1}{(\eta + H)\sin\theta}\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial\phi}\left[\frac{1}{\eta + H}\frac{\partial\psi}{\partial\theta}\right]$$

$$+ \frac{\cot\theta}{(\eta + H)^2\sin\theta}\frac{\partial\psi}{\partial\phi}\frac{\partial\psi}{\partial\theta} - \frac{2(\Omega + \alpha)R}{(\eta + H)\sin\theta}(\sin\gamma\,\sin\theta\,\cos\phi + \cos\gamma\,\cos\theta)\frac{\partial\psi}{\partial\phi}$$

$$= \frac{g}{\sin\theta}\frac{\partial\eta}{\partial\phi}. \tag{4.4}$$

4.2.2 Non-divergent solutions

Since it seems impossible to manipulate (4.3) and (4.4) in order to get a single equation in either ψ or η, we shall make the non-divergent approximation. This approximation, discussed previously in Chapter III, is made here by neglecting η except where it is multiplied by g. Then η may be eliminated by cross-differentiation between (4.3) and (4.4) to give

$$J\left[\frac{1}{H\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{H\sin\theta}\frac{\partial\psi}{\partial\phi}\right) + \frac{1}{H\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{H}\frac{\partial\psi}{\partial\theta}\right)\right.$$

$$\left. - \frac{2(\Omega + \alpha)R}{H}(\sin\gamma\sin\theta\cos\phi + \cos\gamma\cos\theta), \psi\right] = 0 \tag{4.5}$$

where J(a,b) is the Jacobian $\frac{\partial(a,b)}{\partial(\theta,\phi)}$. This can be integrated once to give

$$\frac{\partial}{\partial\phi}\left(\frac{1}{H}\frac{\partial\psi}{\partial\phi}\right) + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{H}\frac{\partial\psi}{\partial\theta}\right)$$

$$- 2(\Omega + \alpha)R\sin^2\theta(\sin\gamma\sin\theta\cos\phi + \cos\gamma\cos\theta) = F(\psi)H\sin^2\theta \tag{4.6}$$

where $F(\psi)$ appears as an arbitrary integration function.

Physically, equation (4.6) is an expression of the conservation of potential vorticity of a fluid column. Since, in steady flow, the streamlines coincide with pathlines, the potential vorticity field is a function of the stream function only. The integration function $F(\psi)$ is, therefore, the distribution of potential vorticity.

In order to solve (4.6), the function $F(\psi)$ must first be determined. Since $F(\psi)$ is both the vorticity distribution due to the wave plus that due to a basic flow, its form for any particular case is not immediately apparent. For want of any information of the shape of $F(\psi)$, it may be assumed that it is at most a linear function of ψ and all possible solutions resulting from such an assumption, determined. From these solutions the basic flows for which this linear function is the vorticity distribution can then be found.

Since one can always add an arbitrary constant to a stream function there are only two possible cases for which $F(\psi)$ is linear in ψ, these being

$$\text{Case I} \qquad F(\psi) = d_o \tag{4.7}$$

$$\text{Case II} \qquad F(\psi) = -d_1\psi \ . \tag{4.8}$$

The solutions of interest are free waves in an ocean of constant depth which completely covers the surface of the sphere. For H constant and $F(\psi)$ given by (4.7), equation (4.6) then becomes

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{4.9}$$

$$= d_o H^2 + 2(\Omega + \alpha)RH(\sin\gamma\,\sin\theta\,\cos\phi + \cos\gamma\,\cos\theta) \ .$$

This equation has no solutions periodic in ϕ which are also finite over the entire sphere; therefore, Case I gives no wave solutions. An equation of the same form as (4.9) but with $\alpha = 0$ may have some importance in the study of steady flows in channels on the sphere.

Turning to Case II, for H constant and $F(\psi)$ given by (4.8), equation (4.6) becomes

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + d_1 H^2\psi \tag{4.10}$$

$$= 2(\Omega + \alpha)RH\,(\sin\gamma\,\sin\theta\,\cos\phi + \cos\gamma\,\cos\theta) \ .$$

The solution of (4.10) , finite over the entire sphere, is given by

$$\psi = \sum_{m=0}^{n} A_n^m \cos m\phi \; P_n^m(\cos\theta) \tag{4.11}$$

$$+ \frac{2(\Omega + \alpha)RH}{n(n+1) - 2} (\sin\gamma \sin\theta \cos\phi + \cos\gamma \cos\theta)$$

where

$$d_1 H^2 = n(n+1) . \tag{4.12}$$

4.2.3 Properties of the solutions

Recalling from Figure 1 (p.12) that

$$\cos\theta' = \cos\gamma \cos\theta + \sin\gamma \sin\theta \cos\phi , \tag{4.13}$$

where θ' is the co-latitude relative to the rotation axis, it can be seen that the stream function as given by (4.11) consists of the sum of surface harmonics of degree n (and of any orientation) plus a steady flow which is zonal relative to the rotation axis. If all the wave amplitudes A_n^m are set to zero, equation (4.11) is reduced to the stream function for the undisturbed flow. This steady zonal flow is given in the rotating frame by

$$v'_{\phi'} = \frac{2(\Omega + \alpha)R}{n(n+1) - 2} \sin\theta' . \tag{4.14}$$

Relative to the surface of the solid sphere, rather than to the rotating frame, the zonal velocity is

$$\bar{v}_{\phi'} = \frac{2\Omega + n(n+1)\alpha}{n(n+1) - 2} R \sin\theta' . \tag{4.15}$$

If this basic zonal current relative to the sphere is set to zero, then from (4.15)

$$\alpha = -\frac{2\Omega}{n(n+1)} . \tag{4.16}$$

The phase speed of a linear non-divergent wave on the sphere is given by equation (3.8) of the previous chapter as

$$\frac{\sigma}{s} = -\frac{2\Omega}{n(n+1)} ; \tag{4.17}$$

therefore, for the case of zero basic flow, the linear and non-linear dispersion relations are identical. Furthermore, the form of the non-linear solution, being the sum of surface harmonics, is identical to that of the linear solution. Hence the linear non-divergent on the sphere is, in fact, an exact solution. This result was previously obtained by Neamtan (1946) and by Barrett (1958); however, it does not appear to be well-known in the literature of oceanic planetary waves. For this reason the analysis has been repeated here and later for Rossby waves on the β-plane. An extension of Neamtan's analysis in §4.4 allows the examination of non-linear topographic wave solutions.

Haurwitz (1940b) showed that a zonal wind of the form $V_\circ R \sin\theta'$ could be added to the linear equations without changing the form of the solutions. If the zonal current given by (4.15) is set equal to $V_\circ R \sin\theta'$, then

$$V_\circ\left[1 - \frac{2}{n(n+1)}\right] - \alpha = \frac{2\Omega}{n(n+1)} . \tag{4.18}$$

Since n >> 1, then (4.18) may be approximated by

$$V_\circ - \alpha = \frac{2\Omega}{n(n+1)} . \tag{4.19}$$

For the case of an undisturbed zonal current of the form $V_\circ R \sin\theta'$, the linear solution is then the full solution. In order to get any higher order interactions it will be necessary to change the undisturbed vorticity field by the addition of a sheared basic zonal current. This would be equivalent to changing the form of $F(\psi)$ in equation (4.6).

On the other hand, if the non-divergent approximation is not made it appears impossible to reduce (4.3) and (4.4) to a simple equation in only one of ψ and η. Hence, it is very unlikely that the linear divergent solutions would be exact solutions of the non-linear equation. This can be tested by direct substitution, but for ease of calculation this test will be made only on the β-plane.

4.3 Rossby waves in a β-plane channel, I. Exact solutions

4.3.1 The non-divergent solution

As discussed earlier, the β-plane approximation involves mapping the surface of the sphere onto a tangent plane, and therefore, is a valid approximation only for horizontal scales much less than the radius of the earth. In order to establish a horizontal scale, the problem is treated in a zonal channel of width L.

If there exist planetary waves of permanent form which have a phase velocity, c, in the x direction, then in a frame moving at this phase velocity relative to the earth, the motion is steady. For such waves to exist, $H = H(y)$ only, since if the depth varies with x, time-dependent terms must enter into the equations as the frame moves from one depth to another. Transforming to such a frame through the transformation

$$s = x - ct \tag{4.20}$$

equations (2.11) to (2.13) become

$$\frac{\partial}{\partial s}\Big[(u - c)(\eta + H)\Big] + \frac{\partial}{\partial y}\Big[v(\eta + H)\Big] = 0 \tag{4.21}$$

$$(u - c)\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial \eta}{\partial s} = 0 \tag{4.22}$$

$$(u - c)\frac{\partial v}{\partial s} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial \eta}{\partial y} = 0 \tag{4.23}$$

with boundary conditions given by

$$v = 0 \qquad \text{at } y = 0, L . \tag{4.24}$$

A transport stream function satisfying (4.12) may be defined by

$$(u - c) = \frac{1}{\eta + H}\frac{\partial \psi}{\partial y} , \qquad v = -\frac{1}{\eta + H}\frac{\partial \psi}{\partial s} . \tag{4.25}$$

Substituting (4.25) into (4.22) and (4.23) and cross-differentiating between the resulting two equations gives

$$J\left(\frac{1}{\eta + H}\frac{\partial}{\partial s}\left[\frac{1}{\eta + H}\frac{\partial \psi}{\partial s}\right] + \frac{1}{\eta + H}\frac{\partial}{\partial y}\left[\frac{1}{\eta + H}\frac{\partial \psi}{\partial y}\right] - \frac{f}{\eta + H} , \psi\right) = 0 \tag{4.26}$$

which can be immediately integrated once to give

$$\frac{1}{\eta + H}\frac{\partial}{\partial s}\left[\frac{1}{\eta + H}\frac{\partial \psi}{\partial s}\right] + \frac{1}{\eta + H}\frac{\partial}{\partial y}\left[\frac{1}{\eta + H}\frac{\partial \psi}{\partial y}\right] - \frac{f}{\eta + H} = F(\psi) \tag{4.27}$$

where again $F(\psi)$ is an arbitrary integration function specifying the distribution of potential vorticity in the fluid. In the same way as on the sphere, $F(\psi)$ is chosen to be a linear function of ψ. Since ψ is defined only up to an arbitrary constant, there are only two cases for which $F(\psi)$ is linear, these corresponding to (4.7) and (4.8) given previously for the solutions on the sphere.

If the non-divergent approximation is made by neglecting η relative to H, and if H is held constant, then equation (4.27) is

for Case I

$$\frac{\partial^2\psi}{\partial s^2} + \frac{\partial^2\psi}{\partial y^2} - fH = d_o H^2 \quad , \tag{4.28}$$

and for Case II

$$\frac{\partial^2\psi}{\partial s^2} + \frac{\partial^2\psi}{\partial y^2} + d_1 H^2\psi - fH = 0 \quad , \tag{4.29}$$

with the boundary condition that

$$\frac{\partial\psi}{\partial s} = 0 \quad \text{at} \quad y = 0, L \; . \tag{4.30}$$

As on the sphere, there are no solutions for case I satisfying the boundary conditions which are also periodic in s. On the other hand, case II has a wave solution given by

$$\psi = \sum_{m=1}^{M} A_n \sin\frac{m\pi y}{L} \cos k_m s + \frac{f}{d_1 H} + B\cos(\sqrt{d_1}Hy + b_1) \quad , \tag{4.31}$$

where

$$k_m^2 + \frac{m^2\pi^2}{L^2} = d_1 H^2 = \kappa^2 \tag{4.32}$$

and

$$M^2 \leq \frac{d_1 H^2 L^2}{\pi^2} \leq (M+1)^2 \quad . \tag{4.33}$$

4.3.2 Properties of the solution

As on the sphere, this solution of the non-linear equations is identical to to linear solution as given in chapter III. These results were previously obtained by Neamtan (1946) and applied to atmospheric processes. The solutions are here rederived in order that they may be compared to divergent Rossby wave solutions and topographic wave solutions which shall be obtained in later sections.

Replacing d_1H^2 by κ^2, the total wave number, and allowing the wave amplitudes to go to zero, the zonal velocity is given by

$$u - c = \frac{\beta}{\kappa^2} - \frac{\beta\kappa}{H} \sin (\kappa y + b_1) \quad . \qquad (4.34)$$

From equation (4.31) it appears that a non-divergent Rossby wave can exist in the presence of a sheared basic zonal current providing that current is of the form (4.34). However, in order that the wave be periodic in s, $\kappa^2 > \frac{m^2\pi^2}{L^2}$ for some $m \leq M$; therefore, this basic zonal current, if it is to be sheared, must have at least as many zeros across the channel as does the wave solution itself. In the real ocean or atmosphere such a complex basic zonal flow is unlikely to exist; hence, here B will be set to zero.

If B were non-zero, the total wave number, κ, in this solution would be determined by the wave number of the basic flow; however, for a uniform basic flow, (B = 0), κ is unspecified and waves of any total wave number may exist. Equation (4.31) shows that waves of the same κ, though of different m and k, can be summed together with no interactions; however, non-linear interactions may occur between two waves of different κ.

A case of interest in Chapter V is the case of a weakly sheared zonal basic current. This may be modelled here by looking at the solution for small κ. For $\kappa L \ll 1$, (4.34) gives a weakly linearly sheared basic current,

$$u - c \approx \frac{\beta}{\kappa^2} - \frac{B\kappa}{H} [\kappa y \cos b_1 + \sin b_1] \quad ; \qquad (4.35)$$

however, (4.32) shows that for $m \geq 1$,

$$k_m^2 = \kappa^2 - \frac{m^2\pi^2}{L^2} \simeq - \frac{m^2\pi^2}{L^2} < 0 . \tag{4.36}$$

Therefore, it appears that a weakly sheared current will not support a wave of the form, $\sin\left(\frac{m\pi}{L}y\right) \cos k_m s$, as a wave of permanent form. The question of whether any finite amplitude wave of permanent form can exist in this case will be discussed in §4.6 and Chapter V.

Such simple solutions of the non-linear equations of motion are possible only in the case of non-divergent motions. The presence of divergence brings into play a whole new set of non-linear interaction terms, and it is no longer possible to find simple solutions to the full equations of motion.

While it is possible to write an equation such as (4.27) which formally appears to be linear in ψ, it is not possible to separate out η without introducing new non-linearities into the equations. In the linear solutions, it was shown that for suitably short wavelengths there were negligible differences between the divergent and non-divergent solutions. However, in considering the full equations, we see that their non-linear behaviour is much different. While the non-divergent linear solutions were shown to be exact solutions, no such behaviour is indicated for the divergent solutions.

This difference in behaviour was also suggested by the linear solutions on the sphere. There, all non-divergent solutions of the same degree moved with the same phase speed, suggesting that superposition of solutions to form a wave of permanent form was possible. On the other hand, the phase speed of the divergent solutions varied with both degree and order; therefore, any superposition of solutions would disperse in time unless non-linear interactions worked to exactly cancel this dispersion. If such a wave of permanent form does exist, it will be a solitary or a cnoidal wave; such waves will be investigated in Chapter V.

4.4 Finite amplitude topographic waves

4.4.1 The exponential profile

In Chapter III, on the linear planetary waves, it was shown that gradients of depth may act in the same way as gradients of f to support planetary wave motions. In the non-linear case, equation (4.27), for $H = H(y)$, will also give wave solutions, even in the case of uniform f.

As in the theory of internal waves where one finds that the mean density profile determines many of the properties of the wave solutions, in the study of topographic waves the choice of depth profile has similar consequences. Many different depth profiles may be chosen, but here the problem will be solved only for the exponential profile,

$$H = H_\circ \exp(-\Lambda y) \quad . \tag{4.37}$$

Once again, making the non-divergent approximation and setting $F(\psi) = -\frac{\kappa^2}{H_\circ^2}\psi$, equation (4.27) becomes

$$\psi_{ss} + e^{-\Lambda y}[e^{\Lambda y}\psi_y]_y + \kappa^2 e^{-2\Lambda y}\psi = fH_\circ e^{-\Lambda y} \tag{4.38}$$

which has as a solution

$$\psi = \zeta^{\frac{1}{2}} \left\{ B_1 \, J_\nu(\lambda\zeta) + B_2 \, Y_\nu(\lambda\zeta) \right\} \sin ks \tag{4.39}$$

$$+ \frac{f_o H_o}{\Lambda\kappa} \int_1^\zeta \frac{\sin\lambda(\zeta - t)}{t} \, dt + D_1 \sin \lambda\zeta + D_2 \cos \lambda\zeta$$

$$- \frac{\beta H_o}{2\Lambda^3} \int_1^\zeta \ln^2 t \; \cos\lambda(t - \zeta) \; dt$$

where

$$\zeta = \exp(-\Lambda y) \, , \tag{4.40}$$

$$\lambda = \frac{\kappa}{\Lambda} \, , \tag{4.41}$$

$$\nu^2 = \frac{k^2}{\Lambda^2} + \frac{1}{4} \, , \tag{4.42}$$

and D_1 and D_2 are arbitrary constants.

4.4.2 Properties of the solution

The integrals which make up the solution (4.39) cannot be evaluated analytically; however, since for all finite y, $\zeta > 0$, then the integrands are finite, and therefore, the integrals themselves are finite.

Once again, it is necessary to apply some bounds to the ocean within which the β-plane approximation remains valid. For a zonal channel, equation (4.39) must satisfy the boundary conditions given by (4.30). These are satisfied if

$$J_\nu(\lambda) \, Y_\nu(\lambda e^{-\Lambda L}) - Y_\nu(\lambda) \, J_\nu(\lambda e^{-\Lambda L}) = 0 \, . \tag{4.43}$$

If ν is real, and a and b are positive, Gray and Matthews

(1922, p.82) show that $J_\nu(ax)\,Y_\nu(bx) - J_\nu(bx)\,Y_\nu(ax)$ is a single-valued, even function of x whose zeros are all real and simple. In (4.42), ν may be chosen to be the positive root, and since κ is an arbitrary constant, it may be chosen of the same sign as Λ so that λ, as given by (4.41), is always positive. Hence, (4.43) has a sequence of real roots, $\{\lambda_\nu^r\}$.

Abramowitz and Stegun (1965, p.374) give an asymptotic formula for determining the r^{th} zero of the cross-products if r is large. In terms of the variables used here, this is

$$e^{-\Lambda L}\lambda_\nu^r \sim a_1 + \frac{p}{a_1} + \frac{q - p^2}{a_1^3} + \frac{d - 4pq + 2p^3}{a_1^5} \tag{4.44}$$

where

$$a_1 = \frac{r\pi}{e^{\Lambda L} - 1}, \qquad p = \frac{4\nu_r^2 - 1}{8e^{\Lambda L}},$$

$$q = \frac{(4\nu_r^2 - 1)(4\nu_r^2 - 25)(e^{3\Lambda L} - 1)}{6(4e^{\Lambda L})^3(e^{\Lambda L} - 1)}, \tag{4.45}$$

$$d = \frac{(4\nu_r^2 - 1)(16\nu_r^4 - 456\nu_r^2 + 1073)(e^{5\Lambda L} - 1)}{5(4e^{\Lambda L})^5(e^{\Lambda L} - 1)}.$$

Since (4.40) gives a relation between ν and k, therefore from (4.44) a dispersion relation giving k in terms of κ, Λ, L, and r may be obtained. However, because (4.44) is a transcendental relation valid only for large r, the actual dispersion relation cannot be obtained analytically.

The roots represented by (4.44) have been shown by Kline (1948) to reduce in the limit as $e^{-\Lambda L} \to 0$ to the r^{th} zero of $J_\nu(x)$. However, in this limit the non-divergent approximation, which requires that

$\eta << H$, would not be valid. In any practical problem the first few roots of (4.43) would have to be calculated numerically for the actual values of κ, Λ, L.

The case of uniform rotation may be investigated by setting $\beta = 0$ in (4.39). This eliminates the last integral, and therefore, simplifies the solution somewhat. Also if f is constant, the solution will hold for a channel of any orientation.

The solution given by (4.39) expressed in terms of the zonal velocity is

$$\begin{aligned} u - c = & -\frac{\Lambda}{H_\circ}\left[\left(\nu + \frac{1}{2}\right)\zeta^{-\frac{1}{2}}\{B_1 J_\nu(\lambda\zeta) + B_2 Y_\nu(\lambda\zeta)\}\right. \\ & \left. - \lambda\zeta^{\frac{1}{2}}\{B_1 J_{\nu+1}(\lambda\zeta) + B_2 Y_{\nu+1}(\lambda\zeta)\}\right] \sin ks \\ & - \frac{f_\circ}{\Lambda}\int_1^\zeta \frac{\cos\lambda(\zeta - t)}{t}\,dt - D_1\frac{\kappa}{H_\circ}\cos\lambda\zeta + D_2\frac{\kappa}{H_\circ}\sin\lambda\zeta \\ & + \frac{\beta}{2\Lambda^2}\ln^2\zeta + \frac{\beta\lambda}{2\Lambda^2}\int_1^\zeta \ln^2 t\,\sin\lambda(t - \zeta)\,dt \quad . \end{aligned} \tag{4.46}$$

Equation (4.46) may be averaged over a wavelength in s; however the resulting zonal flow still remains a very complicated function of y, much more complicated, in fact, than one would expect to exist as a real ocean flow. While (4.39) is an exact solution to the non-divergent equations, it is too complicated to interpret or be useful as an approximation to real oceanic flow.

In summary, it has been found that a wave of permanent form will exist as an exact solution of the non-divergent equations of motion for the case of a channel with bottom profile $H = H_\circ e^{-\Lambda y}$ on the β-plane, and furthermore, that such a solution will exist even if the rotation is uniform. However, the basic zonal flow required in order that this wave exist is so complicated that it is unlikely that the solution represents a wave likely to be observed in either the ocean or the atmosphere.

Since, in topographic waves, the wave properties depend to a large extent on the properties of the topography, it is possible that for a different topography, a simple wave of permanent form may exist without requiring such a complex basic zonal current; however, such an inverse problem would be very difficult to solve.

The solution described above was obtained by requiring $F(\psi)$ in equation (4.27) to be a linear function of ψ. Since $F(\psi)$ is the distribution of potential vorticity, and since for the exponential profile the potential vorticity due to the rotation of the fluid, f, is distributed exponentially with y, in order that the basic zonal flow be simple (that is, at most, linear in y) it would seem likely that $F(\psi)$ should be some exponential function of ψ. In this case though, (4.27) is a non-linear equation and direct solution would be very difficult, particularly since, for non-linear $F(\psi)$, (4.27) is no longer separable.

4.5 Rossby waves in a β-plane channel, II. Perturbation expansions

4.5.1 The perturbation equations

In this section, the finite amplitude effects on divergent Rossby waves in a β-plane channel will be investigated using a Stokes-type perturbation expansion. The basic equations governing waves of permanent form in a β-plane channel are given by (4.21), (4.22), (4.23), and the boundary conditions by (4.24) For divergent Rossby waves in a channel of constant depth, the variables will be non-dimensionalized through the transformation,

$$(u,\ v,\ c) \quad = \quad \beta L^2(u',\ v',\ c')\ , \tag{4.47}$$

$$(s,\ y) \quad = \quad L(s',\ y')\ , \quad f \quad = \quad \beta L f' \quad = \quad \beta L(f_o' \ + \ y')\ ,$$

$$(\eta,\ H) \quad = \quad H(\delta\eta',\ 1)$$

where $\delta = \beta^2 L^2/gH$, a non-dimensional divergence parameter.

Substituting these non-dimensional variables into (4.21) to (4.24), the non-dimensional equations of motion become (on dropping the primes)

$$(u - c)\ u_s \ + \ v\ u_y \ - \ fv \ + \ \eta_s \quad = \quad 0 \tag{4.48}$$

$$(u - c)\ v_s \ + \ v\ v_y \ + \ fu \ + \ \eta_y \quad = \quad 0 \tag{4.49}$$

$$[(u - c)(1 + \delta\eta)]_s \ + \ [v(1 + \delta\eta)]_y \quad = \quad 0 \tag{4.50}$$

$$v \ = \ 0 \quad \text{at} \quad y \ = \ 0,\ 1\ . \tag{4.51}$$

The various variables may be expanded in powers of ε, an amplitude parameter, as follows

$$\begin{aligned} u &= u_o(y) + \varepsilon u_1(s,y) + \varepsilon^2 u_2(s,y) + \ldots \\ v &= \qquad\qquad \varepsilon v_1(s,y) + \varepsilon^2 v_2(s,y) + \ldots \\ \eta &= \eta_o(y) + \varepsilon \eta_1(s,y) + \varepsilon^2 \eta_2(s,y) + \ldots \\ c &= c_o \quad + \varepsilon c_1 \quad + \varepsilon^2 c_2 \quad + \ldots\ . \end{aligned} \tag{4.52}$$

On substitution of these expansions into the non-dimensional equations and on separation of terms in powers of ε, the equations of motion are to zeroth order,

$$fu_\circ + \eta_{\circ y} = 0 , \tag{4.53}$$

to first order

$$(u_\circ - c_\circ)u_{1s} + v_1 u_{\circ y} - fv_1 + \eta_{1s} = 0 \tag{4.54}$$

$$(u_\circ - c_\circ)v_{1s} + fu_1 + \eta_{1y} = 0 \tag{4.55}$$

$$(1 + \delta\eta_\circ)u_{1s} + [(1 + \delta\eta_\circ)v_1]_y + \delta(u_\circ - c_\circ)\eta_{1s} = 0 \tag{4.56}$$

$$v_1 = 0 \quad \text{at} \quad y = 0, 1 , \tag{4.57}$$

to second order

$$(u_\circ - c_\circ)u_{2s} + v_2 u_{\circ y} - fv_2 + \eta_{2s} = -(u_1 - c_1)u_{1s} - v_1 u_{1y} \tag{4.58}$$

$$(u_\circ - c_\circ)v_{2s} + fu_2 + \eta_{2y} = -(u_1 - c_1)v_{1s} - v_1 v_{1y} \tag{4.59}$$

$$(1 + \delta\eta_\circ)u_{2s} + [(1 + \delta\eta_\circ)v_2]_y + \delta(u_\circ - c_\circ)\eta_{2s} \tag{4.60}$$

$$= -\delta[(u_1 - c_1)\eta_1]_s - \delta[v_1\eta_1]_y$$

$$v_2 = 0 \quad \text{at} \quad y = 0, 1 . \tag{4.61}$$

4.5.2 The first order solutions

In this case of $\delta = 0(1)$, the equations will be simplified by setting the basic current, $u_\circ$, to zero. Under these circumstances, the first order equations can be reduced to a single equation in v_1 by first eliminating η_1 between (4.54) and (4.55), and between (4.54) and (4.56) to give

$$fu_{1s} + c_o u_{1sy} - c_o v_{1ss} + (fv_1)_y = 0 \tag{4.62}$$

and

$$(1 - \delta c_o^2)u_{1s} + v_{1y} - \delta c_o f v_1 = 0 \tag{4.63}$$

respectively, then eliminating u_1 between these two equations to leave

$$(1 - \delta c_o^2)v_{1ss} + v_{1yy} - \frac{1}{c_o}(1 + \delta c_o f^2)\, v_1 = 0 \ . \tag{4.64}$$

Equation (4.64) is a non-dimensional form of the linear equation for divergent Rossby waves (3.19) as obtained by Longuet-Higgins (1965a). A solution to (3.19), given by (3.20) and (3.21), is obtained by making an approximation equivalent in (4.64) to neglecting δc_o^2 with respect to 1, and by treating f^2 as a constant. Making these approximations the solution to (4.64) is given by

$$v_1 = \sin m\pi y \ \cos ks \ , \tag{4.65}$$

and

$$c_o = -\,[m^2\pi^2 + k^2 + \delta f_o^2]^{-1} \ . \tag{4.66}$$

For a mid-latitude channel such that $L \simeq 10^6$m, $H \simeq 10^3$m, $\beta \simeq 10^{-11}$m^{-1}s^{-1}, and $f_o \simeq 10^{-4}$s^{-1}, then (4.47) gives $\delta \simeq 10^{-2}$, and (4.66) gives $c_o \simeq 10^{-1}$. Hence, a posteriori, it is seen that the error that these approximations introduce into (4.64) is approximately

1% for treating f^2 as constant, 10^{-2}% for neglecting $\delta c_\circ^2$ in the first term.

Equation (4.64) can be solved exactly, its solution being given by

$$v_1 = Y(y) \cos ks \tag{4.67}$$

where

$$Y_{yy} - \left[\frac{1}{c_\circ} + \delta f^2 + (1 - \delta c_\circ^2)k^2\right] Y = 0 \ . \tag{4.68}$$

The transformation

$$\zeta = \sqrt[4]{4\delta}\ f(y) \tag{4.69}$$

transforms (4.68) into

$$Y_{\zeta\zeta} - \left[\frac{1}{2c_\circ\sqrt{\delta}} + \frac{(1 - \delta c_\circ^2)k^2}{2\sqrt{\delta}} + \frac{\zeta^2}{4}\right] Y = 0 \tag{4.70}$$

which has as solutions [Abramowitz and Stegun, (1965)] the Parabolic Cylinder functions $U(\lambda,\zeta)$, $V(\lambda,\zeta)$ where

$$\lambda = \frac{1}{2c_\circ\sqrt{\delta}} + \frac{(1 - \delta c_\circ^2)k^2}{2\sqrt{\delta}} \ . \tag{4.71}$$

The boundary condition (4.57) is satisfied if

$$U(\lambda,\zeta_1)\ V(\lambda,\zeta_2) - U(\lambda,\zeta_2)\ V(\lambda,\zeta_1) = 0 \tag{4.72}$$

where

$$\zeta_1 = \sqrt[4]{4\delta}\ f_\circ \ , \text{ and } \quad \zeta_2 = \sqrt[4]{4\delta}\ (f_\circ + 1) \ , \tag{4.73}$$

in which case the solution is given by

$$Y = V(\lambda,\zeta_1)\,U(\lambda,\zeta) - U(\lambda,\zeta_1)\,V(\lambda,\zeta) \quad . \tag{4.74}$$

A search of the literature was carried out, but no tables of zeros of these cross-products nor any information on their properties were found. Tables of values of $U(\lambda,\zeta)$ and $V(\lambda,\zeta)$ are given in Abramowitz and Stegun (1965) for $-5 \leq \lambda \leq 5$, $0 \leq \zeta \leq 5$. From these tables it is seen that $V(\lambda,\zeta)$ is monotonic increasing for $\lambda > -1.5$, and $U(\lambda,\zeta)$ is monotonic decreasing for $\lambda > -0.5$; therefore, for any ζ_1, ζ_2 such that $\zeta_2 > \zeta_1$, the cross-product will be positive for $\lambda \geqslant -0.5$. Thus, a necessary condition for (4.72) to be satisfied is that $\lambda < -0.5$. If, as seems likely, $\delta c_o^2 << 1$, then the second term of (4.71) is positive and so, in order that $\lambda < -0.5$, $c_o < 0$. Therefore, in common with the approximate linear solutions for divergent Rossby waves, the phase velocity of these solutions is always toward the west.

Both $U(\lambda,\zeta)$ and $V(\lambda,\zeta)$ are oscillatory in λ if $\lambda < 0$ and $|\zeta| < 2\sqrt{|\lambda|}$; therefore, there will exist an infinite sequence, $\{\lambda_m\}$, of eigenvalues for which (4.72) is satisfied. For $|\lambda| >> \zeta^2$ Abramowitz and Stegun (1965, p.690) give the expansions,

$$U(\lambda,\zeta) \sim \frac{\Gamma\left(\frac{1}{4} - \frac{\lambda}{2}\right)\left(1 - \frac{\zeta^2}{16\lambda}\right)}{\sqrt{\pi}\; 2^{\frac{1}{2}(\lambda+\frac{1}{2})}} \cos\left[\left(\frac{\lambda}{2} + \frac{1}{4}\right)\pi + \sqrt{|\lambda|}\;\zeta - \frac{\zeta^3}{24\sqrt{|\lambda|}}\right]$$

(4.75)

$$V(\lambda,\zeta) \sim \frac{\Gamma\left(\frac{1}{4} - \frac{\lambda}{2}\right)\left(1 - \frac{\zeta^2}{16\lambda}\right)}{\Gamma\left(\frac{1}{2} - \lambda\right)\sqrt{\pi}\; 2^{\frac{1}{2}(\lambda+\frac{1}{2})}} \sin\left[\left(\frac{\lambda}{2} + \frac{1}{4}\right)\pi + \sqrt{|\lambda|}\;\zeta - \frac{\zeta^3}{24\sqrt{|\lambda|}}\right] \quad .$$

Substituting these expansions into (4.72), it is found that the eigenvalues, $\{\lambda_m\}$, are given by

$$\lambda_m = -\frac{m^2\pi^2}{2\sqrt{\delta}} - \frac{\sqrt{\delta}}{2} f_o^2 + \ldots \tag{4.76}$$

where $|\lambda_m| >> \zeta_2^2 > \zeta_1^2$ for $m >> 1$. Substituting from (4.76) for λ, equation (4.71) becomes

$$c_o = -[\, m^2\pi^2 + (1 - \delta c_o^2)k^2 + \delta f_o^2 \,]^{-1} \tag{4.77}$$

which, if $\delta c_o^2 << 1$, is the phase speed given by (4.66). Since $|\lambda_m| >> \zeta^2$ implies that m is large and, further, that $m^2\pi^2 >> \delta f_o^2$, it is seen that for large m and k (short wavelengths), (4.65) and (4.66) are good approximations to the first order solutions. For this case the non-divergent solutions may also be valid.

4.5.3 The second order solutions

In the same manner as were the first order equations, the second order equations (4.58) to (4.61) are reduced to a single equation in v_2. First η_2 is eliminated between (4.58) and (4.59), and between (4.58) and (4.60) to give

$$f u_{2_s} + c_o u_{2_{sy}} - c_o v_{2_{ss}} + (f v_2)_y \tag{4.78}$$

$$= -[(u_1 - c_1)v_{1_s} + v_1 v_{1_y}]_s + [(u_1 - c_1)u_{1_s} + v_1 u_{1_y}]_y$$

and

$$(1 - \delta c_o^2)u_{2_s} + v_{2_y} + \delta c_o f v_2 \tag{4.79}$$

$$= -\delta\{[(u_1 - c_1)\eta_1]_s + [v_1\eta_1]_y + c_o[(u_1 - c_1)u_{1_s} + v_1 u_{1_y}]\}$$

respectively, then u_2 is eliminated between these two to give

$$c_o(1 - \delta c_o^2)v_{2ss} + c_o v_{2yy} - (1 + \delta c_o f^2)v_2 \tag{4.80}$$

$$= (1 - \delta c_o^2)\{[u_1 v_{1_s} + v_1 v_{1_y}]_s - [u_1 u_{1_s} + v_1 u_{1_y}]_y\}$$

$$- f\delta\{(u_1\eta_1)_s + (v_1\eta_1)_y + c_o(u_1 u_{1s} + v_1 u_{1y})\}$$

$$- c_o\delta\{(u_1\eta_1)_{sy} + (v_1\eta_1)_{yy} + c_o(u_1 u_{1s} + v_1 u_{1y})_y\}$$

$$+ c_1\{(1 - \delta c_o^2)(u_{1sy} - v_{1ss}) + \delta f(\eta_{1s} + c_o u_{1s})$$

$$+ \delta c_o(\eta_{1sy} + c_o u_{1sy})\} \;.$$

This equation may be simplified by making the approximation that $\delta c_o^2 << 1$, and then substituting for v_1, u_1, and η_1 from (4.67), (4.63), and (4.54). After some manipulation (4.80) may be written as

$$c_o(v_{2ss} + v_{2yy}) - (1 + \delta c_o f^2)v_2 \tag{4.81}$$

$$= \delta f[(3 + c_o k^2)Y^2 - fY\, Y_y - c_o Y_y^2]\,\frac{\sin ks \quad \cos ks}{k}$$

$$- \frac{c_1}{c_o}\, Y \cos ks \quad .$$

The form of equation (4.81) suggests the solution for v_2 is

$$v_2(s,y) = Z_1(y) \cos ks + Z_2(y) \sin 2ks \tag{4.82}$$

where $Z_1(y)$ and $Z_2(y)$ satisfy the equations

$$Z_{1yy} - \left(\frac{1}{c_o} + \delta f^2 + k^2\right)Z_1 = -\frac{c_1}{c_o^2}\, Y \tag{4.83}$$

$$Z_{2yy} - \left(\frac{1}{c_o} + \delta f^2 + 4k^2\right) Z_2 = \frac{\delta f}{2kc_o}\left[(3 + c_o k^2)Y^2 \;\; fYY_y - c_o Y_y^2\right] \tag{4.84}$$

as well as the boundary conditions

$$Z_1(0) = Z_2(0) = Z_1(1) = Z_2(1) = 0 \; . \tag{4.85}$$

If (4.83) is multiplied through by Y, then integrated over y from 0 to 1, it is found that the left-hand side is identically zero and the right-hand side reduces to

$$c_1 \int_0^1 Y^2 \, dy = 0 \; . \tag{4.86}$$

The integrand of (4.86) is always positive; therefore, in order that the equation be satisfied, $c_1 \equiv 0$. There is, therefore, no first order correction to the phase speed of the waves and $Z_1(y)$ is zero.

The solution to (4.84) is formally written as

$$\begin{aligned} Z_2 = & - \sqrt{\frac{\pi}{2}}\, U(\kappa,\zeta) \int_a^{\zeta} V(\kappa,t)\; S(t)\; dt \\ & + \sqrt{\frac{\pi}{2}}\, V(\kappa,\zeta) \int_b^{\zeta} U(\kappa,t)\; S(t)\; dt \end{aligned} \tag{4.87}$$

where ζ is defined by (4.69),

$$\kappa = \frac{1 + 4c_o k^2}{2c_o\sqrt{\delta}} \tag{4.88}$$

$$S(\zeta) = \frac{\sqrt{\delta}\;\zeta}{4kc_o\sqrt[4]{4\delta}} \left[(3 + c_o k^2)Y^2 - \zeta YY_\zeta - 2c_o\sqrt{\delta}\; Y_\zeta^2\right] \tag{4.89}$$

and a and b are chosen to satisfy the boundary conditions. Since Parabolic Cylinder functions are not easy to manipulate, equation (4.87), while formally representing the second order correction to the solution, will have to be simplified in order that the solution be interpreted.

If δ is small or if m and k are large, it has been shown that (4.65) is a good approximation to the first order solution. Therefore, (4.84) may be solved by making the same approximation, that is, that δf^2 may be treated as a constant in the left-hand side of (4.84). On substituting from (4.65) for Y, and from (4.66) for c_o, (4.84) becomes

$$Z_{2yy} - (3k^2 - m^2\pi^2) Z_2 = \frac{\delta f}{4c_o k}\Big[(\delta c_o f_o^2 - 2)\cos 2m\pi y - m\pi f \sin 2m\pi y - (2m^2\pi^2 c_o + \delta c_o f_o^2 - 2)\Big] . \tag{4.90}$$

The solution to this is

$$\begin{aligned} Z_2(y) = {} & -\frac{\delta f}{4c_o k} A_1 \cos 2m\pi y + \frac{m\pi\delta}{12 c_o k(k^2+m^2\pi^2)}\Big[f^2 + A_2\Big]\sin 2m\pi y \\ & + \frac{\delta f(2m^2\pi^2 c_o + \delta c_o f_o^2 - 2)}{4c_o k(3k^2-m^2\pi^2)} \\ & - \frac{\delta f_o}{4c_o k}\left\{\frac{2m^2\pi^2 c_o + \delta c_o f_o^2 - 2}{3k^2 - m^2\pi^2} - A_1\right\}\left\{\cos \lambda_1 y + \left[\frac{1+f_o}{f_o} - \cos\lambda_1\right]\frac{\sin\lambda_1 y}{\sin\lambda_1}\right\} \end{aligned} \tag{4.91}$$

where

$$\lambda_1^2 = m^2\pi^2 - 3k^2 \neq n^2\pi^2 \tag{4.92}$$

$$A_1 = \frac{1}{3(m^2\pi^2 + k^2)}\left[\delta c_o f_o^2 - 2 - \frac{8m^2\pi^2}{3(m^2\pi^2 + k^2)}\right] \tag{4.93}$$

$$A_2 = \frac{2}{3(m^2\pi^2 + k^2)} \left[2\delta c_\circ f_\circ^2 - 5 + \frac{6k^2 - 10m^2\pi^2}{3(m^2\pi^2 + k^2)} \right] . \tag{4.94}$$

If $\lambda_1^2 < 0$, then the solution is given by

$$\begin{aligned} Z_2(y) = & - \frac{\delta f}{4c_\circ k} A_1 \cos 2m\pi y + \frac{\delta m\pi}{12c_\circ k(m^2\pi^2+k^2)} \left[f^2 + A_2 \right] \sin 2m\pi y \\ & + \frac{\delta f(2m^2\pi^2 c_\circ + \delta c_\circ f_\circ^2 - 2)}{4c_\circ k(3k^2-m^2\pi^2)} \qquad (4.95) \\ & - \frac{\delta f_\circ}{4c_\circ k} \left\{ \frac{2m^2\pi^2 c_\circ + \delta c_\circ f_\circ^2 - 2}{3k^2 - m^2\pi^2} - A_1 \right\} \left\{ \cosh \lambda_2 y \right. \\ & \left. + \left[\frac{1+f_\circ}{f_\circ} - \cosh \lambda_2 \right] \frac{\sinh \lambda_2 y}{\sinh \lambda_2} \right\} \end{aligned}$$

where

$$\lambda_2^2 = 3k^2 - m^2\pi^2 . \tag{4.96}$$

4.5.4 Properties of the solutions

The first important property of these solutions to the second order is that there is no first order correction to the phase velocity. Hence, for large m and k, the phase velocity is given by

$$c = - (m^2\pi^2 + k^2 + \delta f_\circ^2)^{-1} + O(\varepsilon^2) . \tag{4.97}$$

This result is similar to that found in the Stokes-expansion of surface gravity waves on a fluid of infinite depth [Lamb, (1945), p.417], and in the second order expansions of internal gravity waves on a linear density profile [Thorpe (1968), p.589]; in each of these cases first order correction for the phase speed is zero.

This result shows that for divergent Rossby waves, the dispersion relation obtained from the linear equations of motion is much more accurate than previously suspected, having errors of $O(\varepsilon^2)$ rather than of $O(\varepsilon)$.

Although the phase velocity is not changed by the second order solution, the wave profile is. In terms of the cross channel velocity, the wave solution, given by

$$v = \varepsilon \sin m\pi y \cos ks \tag{4.98}$$
$$+ \varepsilon^2 \left(\frac{m\pi\delta[f^2 + A_2]}{12c_o k(k^2+m^2\pi^2)} \sin 2m\pi y - \frac{\delta f A_1}{4c_o k} \cos 2m\pi y \right.$$
$$+ \frac{\delta f(2m^2\pi^2 c_o + \delta c_o f_o^2 - 2)}{4c_o k(3k^2-m^2\pi^2)}$$
$$- \frac{\delta f_o}{4c_o k} \left\{ \frac{2m^2\pi^2 c_o + \delta c_o f_o^2 - 2}{3k^2 - m^2\pi^2} - A_1 \right\} \left\{ \cos \lambda_1 y \right.$$
$$\left. \left. + \left[\frac{1+f_o}{f_o} - \cos \lambda_1 \right] \frac{\sin \lambda_1 y}{\sin \lambda_1} \right\} \right) \sin 2ks \quad ,$$

differs from the linear solution, which is $O(\varepsilon)$, with terms of $O(\varepsilon^2)$.

Any programme attempting to measure Rossby waves in the ocean would probably involve measurements of velocity at fixed points over a period of time. On such a record, a wave profile such as (4.98) would appear as

$$v(t) = \varepsilon D_1 \cos(-kc_o t) + \varepsilon^2 D_2 \sin(-2kc_o t) + O(\varepsilon^3) \tag{4.99}$$

where D_1 and D_2 at any fixed point are constants of order unity, provided $y \neq n/m$. Therefore, the current record will appear as a sinusoidal wave of angular frequency kc_o which is steepened at either the leading or trailing edge. On the nodal surface, $y = n/m$, $D_1 = 0$ and the current record appears as a sinusoidal wave of amplitude $O(\varepsilon^2)$ and angular frequency $2kc_o$.

Equations (4.91) and (4.95) are solutions of (4.90) only if $m^2\pi^2 - 3k^2 \neq n^2\pi^2$. In the special case for which $\lambda_1 = n\pi$, the question arises as to whether (4.90) will have solutions which satisfy both boundary conditions. Supposing that such solutions exist, equation (4.90) may be multiplied through by cos $n\pi y$, then integrated over y from 0 to 1, with the result that the left-hand side is identically zero. If, on the right-hand side f is held constant, then the integrated equation gives

$$\frac{\delta f_o^2 m^2}{2c_o k}\left[\frac{(-)^n - 1}{4m^2 - n^2}\right] = 0 \ . \tag{4.100}$$

If n is odd, then to this order of approximation, no second order solution can exist which satisfies both boundary conditions. If n is even, then (4.100) is satisfied and the solution to (4.90), if f is held constant, is

$$Z_2(y) = \frac{\delta m\pi(f_o^2 + A_2)}{12c_o k(k^2 + m^2\pi^2)} \sin 2m\pi y - \frac{\delta f_o}{4c_o k} A_1 \cos 2m\pi y \tag{4.101}$$

$$+ \frac{\delta f(2m^2\pi^2 c_o + \delta c_o f_o^2 - 2)}{4c_o k(3k^2 - m^2\pi^2)}$$

$$- \frac{\delta f_o}{4c_o k}\left[\frac{2m^2\pi^2 c_o + \delta c_o f_o^2 - 2}{3k^2 - m^2\pi^2} - A_1\right] \cos 2p\pi y$$

where

$$4p^2\pi^2 = m^2\pi^2 - 3k^2 \quad (p \text{ is an integer}) \ . \tag{4.102}$$

Originally, this problem was solved for a β-plane channel only in order that the width of the channel provide a horizontal scale, L, within which the β-plane approximation remains valid. A solution like

(4.98) is periodic in y, and therefore, the boundary condition at $y = 0,1$ can be reinterpreted as a periodicity condition and the solution considered to be a two dimensional wave periodic in both x and y in an unbounded ocean. Such an interpretation is valid only if $\lambda_1^2 > 0$, as the solution (4.95) for $\lambda_1^2 < 0$ is no longer periodic in y, and, in fact, increases exponentially with y outside of the dimensions of the channel.

Returning once more to (4.81), in the non-divergent limit as $\delta \to 0$, the right-hand side goes to zero; hence, there is no second order correction. This is consistent with the results of §4.3 which show that, for the constant zonal current case, the linear non-divergent Rossby wave solution is an exact solution.

The fact that there exists a second order correction to the linear divergent Rossby wave solution demonstrates that, unlike the non-divergent case, the linear divergent solutions are not exact solutions of the equations of motion. In this way the non-divergent Rossby waves are fundamentally different from the divergent solutions.

4.6 Rossby waves in a β-plane channel, III. Uniformly sheared current

4.6.1 The perturbation expansions

It has been shown previously for mid-latitude channels of width 10^3 km, and depth 1 km, that $\delta \simeq 10^{-2}$. In view of the complexity of the perturbation equations for a sheared basic current and $\delta = 0(1)$, perhaps a new expansion in which $\delta = 0(\varepsilon)$ would be appropriate.

Setting

$$\delta = \mu\varepsilon \quad , \qquad (4.103)$$

where $\mu = O(1)$, and using the expansions for u, v, η, and c in terms of ε, given by (4.52), equations (4.48) to (4.51) may be separated in powers of ε to give to the zeroth order (4.53), to the first order (4.54), (4.55), (4.57) plus

$$u_{1_s} + v_{1_y} = 0 \tag{4.104}$$

and to the second order (4.58), (4.59), (4.61), plus

$$u_{2_s} + v_{2_y} = -\mu[(u_\circ - c_\circ)\eta_{1_s} + (v_1\eta_\circ)_y] . \tag{4.105}$$

4.6.2 The first order solutions

The reduction of the first order equations to a single equation in v_1 is easily accomplished. First η_1 is eliminated between (4.54) and (4.55) to give

$$[(u_\circ - c_\circ)u_{1s}]_y - fu_{1s} + [(u_{\circ_y} - f)v_1]_y - (u_\circ - c_\circ)v_{1ss} = 0 \tag{4.106}$$

then u_1 is eliminated between this and (4.104) to leave

$$(u_\circ - c_\circ)[v_{1ss} + v_{1yy}] + [1 - u_{\circ_{yy}}]v_1 = 0 . \tag{4.107}$$

If the basic current is uniformly sheared, that is if

$$u_\circ = W_\circ + ay \tag{4.108}$$

where a and $W_\circ$ are both constants, then the solution for v_1 is

$$v_1 = \Phi(y) \sin ks \tag{4.109}$$

where

$$\Phi_{yy} + \left[\frac{1}{u_\circ - c_\circ} - k^2\right] \Phi = 0 . \tag{4.110}$$

Setting

$$\zeta = 2\left|\frac{k}{a}\right| (W_\circ - c_\circ + ay) \tag{4.111}$$

and

$$\Phi = \zeta \exp(-\zeta/2)\, \bar{\Phi}(\zeta) \tag{4.112}$$

equation (4.110) may be transformed into

$$\zeta\bar{\Phi}_{\zeta\zeta} + (2 - \zeta)\bar{\Phi}_\zeta - (1 - \frac{1}{2|ka|})\bar{\Phi} = 0 , \tag{4.113}$$

the confluent hypergeometric equation, the solutions of which, in the notation of Slater (1960), are given by the confluent hypergeometric functions, ${}_1F_1(1 - \frac{1}{2|ka|}, 2, \zeta)$ and $U(1 - \frac{1}{2|ka|}, 2, \zeta)$. The boundary condition (4.57), in terms of $\bar{\Phi}$, is given by

$$\bar{\Phi}(\zeta_\circ) = \bar{\Phi}(\zeta_1) = 0 , \tag{4.114}$$

and is satisfied if

$$ {}_1F_1(A_3,2,\zeta_\circ)\, U(A_3,2,\zeta_1) - {}_1F_1(A_3,2,\zeta_1)\, U(A_3,2,\zeta_\circ) = 0 \tag{4.115}$$

where

$$\zeta_\circ = 2\left|\frac{k}{a}\right| (W_\circ - c_\circ) , \quad \zeta_1 = 2\left|\frac{k}{a}\right| (W_\circ - c_\circ + a) , \quad A_3 = 1 - \frac{1}{2|ka|} . \tag{4.116}$$

For $-n < A_3 < -n+1$, ${}_1F_1(A_3,2,\zeta)$ and $U(A_3,2,\zeta)$ each have n positive real zeros [Slater, (1960), pp.102-106]; hence, k and $c_\circ$ may be chosen such that (4.115) is satisfied. The zeros of these functions are not tabulated, and the calculation of the actual dispersion relation is not of sufficient importance to warrent their calculation here.

If the shear is weak, that is, if $a \ll 1$, then (4.110) may be solved using perturbation expansions of Φ and $c_\circ$ in terms of powers of a. These are given by

$$\begin{aligned} \Phi &= \Phi_\circ + a\Phi_1 + a^2\Phi_2 + \ldots \\ c_\circ &= c_{\circ\circ} + ac_{\circ 1} + a^2 c_{\circ 2} + \ldots \quad . \end{aligned} \tag{4.117}$$

On substitution of these expansions and separation in powers of a, equation (4.110) is to the zeroth order in a

$$\Phi_{\circ yy} + \left[\frac{1}{W_\circ - c_{\circ\circ}} - k^2\right]\Phi = 0 \ , \tag{4.118}$$

to the first order

$$\Phi_{1yy} + \left[\frac{1}{W_\circ - c_{\circ\circ}} - k^2\right]\Phi_1 = \frac{y - c_{\circ 1}}{(W_\circ - c_{\circ\circ})^2}\Phi_\circ \ , \tag{4.119}$$

etc. These equations may be easily solved subject to the boundary conditions

$$\Phi_\circ(0) = \Phi_\circ(1) = \Phi_1(0) = \Phi_1(1) = 0 \tag{4.120}$$

to give

$$\Phi = \left[1 + \frac{ay(k^2 + \ell^2)^2}{4\ell^2}\right]\sin \ell y + \frac{a(y - y^2)(k^2 + \ell^2)^2}{4\ell}\cos \ell y + O(a^2) \ , \tag{4.121}$$

and

$$c_\circ = W_\circ - \frac{1}{k^2 + \ell^2} + \frac{a}{2} + O(a^2) \tag{4.122}$$

where

$$\ell = m\pi \ . \tag{4.123}$$

4.6.3 The second order solution

In the same way, η_2 may be eliminated from (4.58) and (4.59) to leave

$$\begin{aligned}(u_\circ - c_\circ)v_{2ss} + fu_{2s} - [(u_\circ - c_\circ)u_{2s}]_y - [(u_{\circ y} - f)v_2]_y \\ = - [u_1 v_{1s} + v_1 v_{1y}]_s + [u_1 u_{1s} + v_1 u_{1y}]_y \\ + c_1(v_{1ss} - u_{1sy}) \ .\end{aligned} \tag{4.124}$$

Eliminating u_2 from (4.124) using (4.105) and also substituting for u_1, v_1, and η_1 in terms of Φ, gives

$$\begin{aligned}(u_\circ - c_\circ)(v_{2ss} + v_{2yy}) + v_2 \\ = - \frac{c_1 \Phi}{u_\circ - c_\circ} \sin ks \\ + \mu\{[(u_\circ - c_\circ)(2a^2 - af + c_\circ) + c_\circ f(a - f)]\ \Phi \\ + [\eta_\circ(f - a) - 2a(u_\circ - c_\circ)^2 + 2fu_\circ(u_\circ - 2c_\circ)]\ \Phi_y \\ - (u_\circ - c_\circ)[\eta_\circ + (u_\circ - c_\circ)^2]\Phi_{yy}\} \sin ks \\ + \frac{1}{2k}(\Phi\Phi_{yyy} - \Phi_y\Phi_{yy})\sin 2ks \ .\end{aligned} \tag{4.125}$$

The solution for v_2 is therefore of the form

$$v_2 = Z_1(y)\sin ks + Z_2(y)\sin 2ks \tag{4.126}$$

where Z_1 and Z_2 are functions which satisfy

$$\begin{aligned}(u_\circ - c_\circ)Z_{1_{yy}} &+ [1 - k^2(u_\circ - c_\circ)]Z_1 \\ &= -\frac{c_1\Phi}{u_\circ - c_\circ} \\ &\quad + \mu\{[(u_\circ - c_\circ)(2a^2 - af + c_\circ) + c_\circ f(a - f)]\Phi \\ &\qquad + [\eta_\circ(f - a) - 2a(u_\circ - c_\circ)^2 + 2fu_\circ(u_\circ - 2c_\circ)]\Phi_y \\ &\qquad - (u_\circ - c_\circ)[\eta_\circ + (u_\circ - c_\circ)^2]\Phi_{yy}\}\end{aligned} \tag{4.127}$$

and

$$(u_\circ - c_\circ)Z_{2_{yy}} + [1 - 4k^2(u_\circ - c_\circ)]Z_2 = \frac{1}{2k}\left(\Phi\Phi_{yyy} - \Phi_y\Phi_{yy}\right) . \tag{4.128}$$

A necessary and sufficient condition that (4.127) have solutions that satisfy the boundary conditions,

$$Z_1(0) = Z_1(1) = 0, \tag{4.129}$$

is obtained by multiplying (4.127) through by $\Phi/(u_\circ - c_\circ)$, then integrating over y from 0 to 1. From the boundary conditions in Φ, the left-hand side is identically zero, and the right-hand side is zero if

$$c_1 \int_0^1 \frac{\Phi^2}{(u_\circ - c_\circ)^2} dy = \mu \int_0^1 \{- [\eta_\circ + (u_\circ - c_\circ)^2]\Phi\Phi_{yy} \tag{4.130}$$

$$+ [\eta_\circ(f - a) - 2a(u_\circ - c_\circ)^2 + 2fu_\circ(u_\circ - 2c_\circ)] \frac{\Phi\Phi_y}{u_\circ - c_\circ}$$

$$+ [(u_\circ - c_\circ)(2a^2 - af + c_\circ) + c_\circ f(a - f)] \frac{\Phi^2}{u_\circ - c_\circ} \} dy \quad .$$

The proof that this condition is a sufficient condition for which (4.127) will have solutions satisfying (4.129) is given by Courant and Hilbert (1953, p.359). Equation (4.130) may be integrated to give

$$c_1 = \frac{\mu}{(k^2 + \ell^2)^2} \left[\frac{- k^2}{(k^2 + \ell^2)^2} + f_\circ^2 + f_\circ + \frac{1}{3} - \frac{1}{2\ell^2} \right. \tag{4.131}$$

$$- W_\circ \left\{ \left(2f_\circ^2 + f_\circ + \frac{1}{3} + 8W_\circ - \frac{1}{2\ell^2}\right)\left(\frac{k^2 + \ell^2}{4}\right) \right.$$

$$\left. \left. + \frac{\ell^2}{6}(3f_\circ + 1) + \frac{7}{4} \right\} + O(a) \right] \quad .$$

Obtaining an actual solution to (4.127) by substituting for c_1 would be a tedious task, giving in return, only the term which is of the same zonal wave number as the basic wave.

On the other hand, (4.128) may be easily solved, to give

$$Z_2 = \frac{a(k^2 + \ell^2)^2}{12k\lambda_1^2} \{3(k^2 + \ell^2) + \lambda_1^2 \cos 2\ell y \tag{4.132}$$

$$- 4\ell^2[\cos \lambda_1 y + \frac{1 - \cos \lambda_1}{\sin \lambda_1} \sin \lambda_1 y]\} + O(a^2)$$

where again $\lambda_1^2 = \ell^2 - 3k^2 \neq n^2\pi^2$ [see (4.92)] .

If $\lambda_1^2 = (2p+1)^2\pi^2$, there is no solution to (4.128) which will also satisfy the boundary conditions. On the other hand, if $\lambda_1^2 = 4p^2\pi^2$, then

$$Z_2 = \frac{a(k^2+\ell^2)^2}{12k\lambda_1^2}\left[3(k^2+\ell^2) + \lambda_1^2 \cos 2\ell y - 4\ell^2 \cos 2p\pi y\right] + O(a^2) \quad . \tag{4.133}$$

4.6.4 Properties of the solutions

Equation (4.132) shows that $Z_2(y)$ is non-zero only if the basic current is uniformly sheared; this is true despite the fact that δ, the divergence parameter, is non-zero. Since $Z_2(y)$ is the coefficient of the "sin 2ks" term, only if it is non-zero will there be any deviation of the wave profile along the axis of the channel from the linear solution, at least at $O(\varepsilon^2)$.

In the previous section, a second order term of wave number 2k was obtained when there was no sheared current present; however, if $\delta = O(\varepsilon)$ in (4.91) it is seen that these terms are then $O(\varepsilon^3)$. Therefore, if $\delta = O(\varepsilon)$ and if the basic current is zero or uniform, one must look at third order terms in order to find non-linearities in the wave profiles.

For the case of $W_\circ = 0$, $a = 0$, and short wavelengths, $f_\circ^2 \gg (f_\circ, 1, \ell^{-2}, k^{-2})$; hence, the phase velocity given by (4.131) and (4.122) may be approximated by

$$c = -\frac{1}{k^2+\ell^2}\left[1 - \frac{\delta f_\circ^2}{k^2+\ell^2}\right] \quad . \tag{4.134}$$

Equation (4.134) is exactly the first two terms of the binomial expansion for (4.97) where $\delta = \varepsilon\mu << 1$. Hence, for $\delta = O(\varepsilon)$, the solutions obtained in §4.5 reduce to the solutions obtained here.

If $\mu = 0$, the solutions reduce to the non-divergent solution for a uniformly sheared zonal current. For $\mu = 0$, (4.130) gives $c_1' = 0$; hence, there is no contribution to the solutions from equation (4.127). The non-divergent solution to the second order is then given by

$$v = \left\{\left[1 + \frac{ay(k^2 + \ell^2)^2}{4\ell^2}\right]\varepsilon \sin \ell y - \frac{ay(y-1)(k^2+\ell^2)^2}{4\ell}\varepsilon \cos \ell y + O(\varepsilon^2)\right\} \sin ks + \frac{\varepsilon^2 a(k^2+\ell^2)^2}{12k\lambda_1^2}\{3(k^2+\ell^2) + \lambda_1^2 \cos 2\ell y - 4\ell^2[\cos \lambda_1 y + \frac{1-\cos\lambda_1}{\sin\lambda_1}\sin\lambda_1 y]\} \sin 2ks \quad . \tag{4.135}$$

and the phase speed is given by (4.122) to the second order in ε.

Since, for a constant zonal current, the linear or first order non-divergent solution has been already shown to be an exact solution to the non-divergent equations of motion, it is not surprising that for $a' = 0$, (4.135) reduces to the linear solution.

For non-zero a, the second order term introduces a non-linearity to the wave profile along the axis. Depending on the signs of the coefficients of the first and second order terms, this non-linearity appears as a steepening of the leading (or trailing) edge of the wave.

The fact that even a weak uniform shear should have such a marked effect on the non-divergent wave is due in part to the change that such a shear makes in the vorticity field in which the wave finds itself, and also in part to the physical distortion such a shear current causes by moving some parts of the wave relative to other parts. Thinking of a typical ocean situation with random currents and random shears, it seems likely that any observed Rossby wave field will be very much altered from that theoretically predicted in such a simple model as a channel with a uniformly sheared current. This model is valuable, however, in suggesting the importance of the interactions with currents.

The phase speed, as given by the linear theory, is correct to $O(\varepsilon^2)$ for non-divergent waves in a uniformly sheared current. Therefore, although the presence of real ocean currents will greatly distort the wave fields, the theoretical dispersion relations given by the linear theory will give accurate results. This effect has an analogue in surface gravity waves where it is found that dispersion relations for linear surface wave theory give accurate results when applied to actually observed wave fields.

4.7 Summary

It has been shown that, in the presence of a uniform or zero zonal current, the linear non-divergent Rossby wave solutions are exact solutions of the non-divergent equations of motion both on the sphere and on the β-plane. Furthermore, linear non-divergent solutions of the same total wave number, and hence, of the same phase speed, may be summed together to form new linear solutions; these new solutions are also exact solutions of the non-divergent equations of motion.

This behaviour is markedly changed in the presence of a sheared zonal current. Even in the simplest case of a weak uniform shear, the non-divergent solutions exhibit non-linearities in the wave profile at $O(\varepsilon^2)$, although the linear dispersion relation is unaffected at $O(\varepsilon)$. Keller and Veronis (1969) have previously shown that random currents may scatter Rossby waves or cause them to grow. Here, however, it is shown that the presence of current shear can cause energy of a single non-divergent Rossby wave to be fed into higher wave numbers. Since the real ocean situation consists of many currents in different directions, this interaction between Rossby waves and currents should be very important in understanding oceanic dynamics.

The linear divergent solutions are shown not to be exact solutions of the β-plane equations. In the absence of a basic zonal current, the divergent solutions exhibit non-linearities at $O(\varepsilon^2)$. Once again the linear dispersion relation is correct to $O(\varepsilon)$.

An exact solution for non-divergent topographic waves on the β-plane was found; however, this solution requires a very complex basic current pattern in order to exist. It is felt that such a complex solution is not of much applicability to real ocean situations.

V. Long Planetary Waves in a Zonal Channel

5.1 The scaled equations

A class of long non-linear waves, the solitary and cnoidal waves, has long been known and investigated for the case of surface gravity waves [Korteweg and deVries, (1895); Keulegan and Patterson, (1940); Benjamin and Lighthill, (1957)] and more recently for the case of internal gravity waves [Keulegan (1953); Benjamin (1966); Benney (1966)]. These are waves of permanent form whose wavelengths along the channel are long relative to the width of the channel. Since, as shown in Appendix II, there is a restricted analogy in the behaviour of planetary and internal waves, the question arises whether an analogous class of waves exists for planetary motions.

Using the non-divergent approximation Larsen (1965) showed that solitary and cnoidal waves could exist in a zonal channel, providing there was also present a basic zonal current with a weak uniform shear. The fact that Larsen found that non-divergent solitary and cnoidal waves could not exist if the basic current was uniform is not surprising in the light of the results obtained in the previous chapter. Since the linear solution on a uniform current is an exact solution to the non-divergent equations of motion in this case, they already form a class of solutions of permanent form.

In the previous chapter it was also shown that the non-linear behaviour of divergent waves is much more complex than that of the non-divergent waves. In particular, the linear solutions for divergent waves are not exact solutions nor were any exact solutions found. Furthermore, it was shown in Chapter IV that the non-divergent approximation was valid only for short wavelengths. For these reasons Larsen's theory will be extended and solitary and cnoidal wave solutions sought in the divergent case.

Again the fluid is assumed to be inviscid and homogeneous, the motion barotropic and hydrostatic, and the solution a wave of permanent form moving in the x-direction along the axis of the zonal channel. The wavelength of the disturbance will be assumed to be short enough that the β-plane approximation remains valid while, at the same time, being long with respect to the width, L, of the channel. The full unscaled equations for this case have been discussed, and are given in Chapter IV by (4.21) to (4.24). Non-dimensional variables are defined by

$$(s,\ y) \ = \ L\,(s',\ y')\ ,\qquad (u.\ v,\ c) \ = \ \beta L^2(u',v',c') \qquad (5.1)$$

$$f \ = \ \beta L f' = \beta L(f'_o \ + \ y'),\qquad (\eta,\ z) \ = \ H(\delta\eta',z')$$

where L, H are the width and the depth of the channel, and $\delta = \dfrac{L^4\beta^2}{gH}$, the divergence parameter. On substitution from (5.1), (4.21) to (4.24) become (on dropping the primes)

$$(u - c)u_s \ + \ vu_y \ - \ fv \ = \ -\,\eta_s \qquad (5.2)$$

$$(u - c)v_s \ + \ vv_y \ + \ fu \ = \ -\,\eta_y \qquad (5.3)$$

$$(1 + \delta\eta)(u_s + v_y) \ + \ \delta[(u - c)\eta_s + v\eta_y] \ = \ 0 \qquad (5.4)$$

$$v \ = \ 0 \ \text{ at } \ y \ = \ \pm\,1\ . \qquad (5.4)$$

Since the wave solutions are long with respect to the width of the channel, therefore, following Larsen (1965), the s co-ordinate is stretched relative to the y co-ordinate through the transformation

$$\xi \quad = \quad \varepsilon^{\frac{1}{2}} s \tag{5.6}$$

where ε is the amplitude-ordering parameter of the wave and $\varepsilon << 1$. The dependent variables and parameters are expressed in terms of the following perturbation expansions:

$$\begin{aligned} u \quad &= \quad u_o(y) \; + \; \varepsilon u_1(\xi,y) \; + \; \varepsilon^2 u_2(\xi,y) \; + \; \ldots \\ v \quad &= \quad \varepsilon^{\frac{3}{2}} v_1(\xi,y) \; + \; \varepsilon^{\frac{5}{2}} v_2(\xi,y) \; + \; \ldots \\ c \quad &= \quad c_o \; + \; \varepsilon c_1 \; + \; \varepsilon^2 c_2 \; + \; \ldots \\ \eta \quad &= \quad \eta_o(y) \; + \; \varepsilon \eta_1(\xi,y) \; + \; \varepsilon^2 \eta_2(\xi,y) \; + \; \ldots \; , \end{aligned} \tag{5.7}$$

where the form of the expansion for v is chosen in order that, if the flow is non-divergent, that is, if $\delta = 0$, the two remaining terms of the continuity equation (5.4) are of the same order of magnitude.

After substitution for s, u, v, c, and η from (5.6) and (5.7), equations (5.2) to (5.5) are ordered in powers of ε to give to zeroth order:

$$f u_o \quad = \quad - \eta_{o_y} \; , \tag{5.8}$$

to the first order:

$$(u_o - c_o) u_{1_\xi} \; + \; v_1 u_{o_y} \; - \; f v_1 \; + \; \eta_{1_\xi} \quad = \quad 0 \tag{5.9}$$

$$fu_1 + \eta_{1_y} = 0 \tag{5.10}$$

$$(1 + \delta\eta_o)u_{1_\xi} + [(1 + \delta\eta_o)v_1]_y + \delta(u_o - c_o)\eta_{1_\xi} = 0 \tag{5.11}$$

$$v_1 = 0 \quad \text{at} \quad y = \pm 1 , \tag{5.12}$$

and to second order:

$$(u_o - c_o)u_{2_\xi} + (u_{o_y} - f)v_2 + \eta_{2_\xi} = -(u_1 - c_1)u_{1_\xi} - v_1 u_{1_y} \tag{5.13}$$

$$fu_2 + \eta_{2_y} = -(u_o - c_o)v_{1_\xi} \tag{5.14}$$

$$(1 + \delta\eta_o)u_{2_\xi} + [(1 + \delta\eta_o)v_2]_y + \delta(u_o - c_o)\eta_{2_\xi} \tag{5.15}$$

$$= -\delta\{[(u_1 - c_1)\eta_1]_\xi + [v_1\eta_1]_y\}$$

$$v_2 = 0 \quad \text{at} \quad y = \pm 1 . \tag{5.16}$$

These equations are in their most general form, and their solution without further approximations would be quite complicated. Larsen (1965), setting $\delta = 0$ obtained solitary and cnoidal wave solutions for the non-divergent case in which the basic current is weakly sheared. As might be predicted from the results of Chapter IV, if the basic current is not sheared, Larsen's analysis gives the linear Rossby wave solution.

Chapter IV indicates that retaining a non-zero δ in these equations should give significantly different results from Larsen's non-divergent analysis. In particular, it should be possible to obtain solitary and cnoidal wave solutions even for the case of the zonal current zero. If $\beta L \simeq 10^{-5}\ s^{-1}$, $L \simeq 10^6$m, $gH \simeq 10^4$ m/s, then $\delta \simeq 10^{-2}$; however, its magnitude is very sensitive to variations in the magnitude of L. Here, as in the previous chapter, two cases are considered: $\delta = 0(1)$ and $\delta = 0(\varepsilon)$. In the first case, in order to make the calculations more manageable, the zonal current is set to zero. This case will show that, if the divergent terms are retained, then solitary and cnoidal waves can exist independently of a zonal sheared current. In the second case, a basic current with a uniform but weak shear will be retained. The solutions that are obtained will be compared to the non-divergent solutions of Larsen (1965). Furthermore, for the second case, the effects of bottom topography will be included in the equations.

5.2 The case $\delta \simeq 0(1)$

5.2.1 Derivation of the long wave equation

Setting u_o and η_o equal to zero, η_1 is eliminated between first (5.9) and (5.10), and then between (5.9) and (5.11) to give

$$(c_o\frac{\partial}{\partial y} + f)u_{1_\xi} + (fv_1)_y = 0 \tag{5.17}$$

$$(1 - \delta c_o^2)u_{1_\xi} + v_{1_y} - \delta c_o f v_1 = 0 \quad , \tag{5.18}$$

and then u_1 is eliminated to give

$$c_o v_{1_{yy}} - (1 + \delta c_o f^2)v_1 = 0 \; . \tag{5.19}$$

Equation (5.19) together with the boundary condition (5.12) determines the variation of v_1^1 across the channel but leaves its variation along the channel completely unspecified. Therefore, it is possible to define a function $g(\xi)$ such that

$$v_1 = g_\xi(\xi)\ \Phi(y) \tag{5.20}$$

where (5.19) and (5.12) require that

$$\Phi_{yy} - (1/c_\circ + \delta f^2)\Phi = 0 \tag{5.21}$$

$$\Phi(1) = \Phi(-1) = 0 \quad . \tag{5.22}$$

In terms of $g(\xi)$ and $\Phi(y)$, u_1 and η_1 are then

$$u_1 = \frac{\delta c_\circ f\Phi - \Phi_y}{1 - \delta c_\circ^2}\ g(\xi) \tag{5.23}$$

$$\eta_1 = \frac{f\Phi - c_\circ\Phi_y}{1 - \delta c_\circ^2}\ g(\xi) \quad . \tag{5.24}$$

In order to determine $g(\xi)$, the second order equations must be investigated. In the same manner as were the first order equations, these equations are reduced to a single equation in v_2. First η_2 is eliminated from (5.13) to (5.15) to give

$$(c_\circ\frac{\partial}{\partial y} + f)u_{2_\xi} + (fv_2)_y = -c_1 u_{1_{\xi y}} + (u_1 u_{1_\xi} + v_1 u_{1y})_y + c_\circ v_{1_{\xi\xi}} \tag{5.25}$$

and

$$(1 - \delta c_\circ^2)u_{2_\xi} + v_{2_y} - \delta c_\circ f v_2 = \delta c_1(\eta_{1_\xi} + c_\circ u_{1_\xi}) - \delta c_\circ(u_1 u_{1_\xi} + v_1 u_{1_y}) - \delta[(u_1\eta_1)_\xi + (v_1\eta_1)_y] \tag{5.26}$$

then u_2 is eliminated between these to leave

$$v_{2_{yy}} - (1/c_\circ + \delta f^2)v_2 \tag{5.27}$$

$$= -(1-\delta c_\circ^2)v_{1_{\xi\xi}} - \frac{1-\delta c_\circ^2}{c_\circ}(u_1 u_{1_\xi} + v_1 u_{1_y})_y$$

$$+ \frac{c_1}{c_\circ}[(1-\delta c_\circ^2)u_{1_{\xi y}} + (f + c_\circ\frac{\partial}{\partial y})\delta(\eta_{1_\xi} + c_\circ u_{1_\xi})]$$

$$- \frac{\delta}{c_\circ}(f + c_\circ\frac{\partial}{\partial y})[(u_1\eta_1)_\xi + (v_1\eta_1)_y + c_\circ u_1 u_{1_\xi} + c_\circ v_1 u_{1_y}]$$

which, in terms of Φ and g, may be rewritten as

$$v_{2_{yy}} - (1/c_\circ + \delta f^2)v_2 \tag{5.28}$$

$$= -(1-\delta c_\circ^2)\Phi g_{\xi\xi\xi} - \frac{c_1\Phi}{c_\circ^2}g_\xi$$

$$+ \delta\left\{f\left[3 + \delta c_\circ^2 + \frac{\delta c_\circ f^2(1+\delta c_\circ^2)}{1-\delta c_\circ^2}\right]\Phi^2 + \frac{c_\circ f(3-\delta c_\circ^2)}{1-\delta c_\circ^2}\Phi_y^2\right.$$

$$\left. - \left[c_\circ(4+\delta c_\circ f^2) + \frac{f^2(1+3\delta c_\circ^2)}{1-\delta c_\circ^2}\right]\Phi\Phi_y\right\}\frac{gg_\xi}{c_\circ(1-\delta c_\circ^2)} .$$

If equation (5.28) is multiplied through by Φ, then integrated over y from y = -1 to y = 1, its left-hand side is identically zero, and its right-hand side then gives an equation in $g(\xi)$,

$$e_1 g_{\xi\xi\xi} + e_2 c_1 g_\xi + e_3 g g_\xi = 0 , \tag{5.29}$$

where

$$e_1 = c_\circ(1-\delta c_\circ^2)\int_{-1}^{1}\Phi^2\,dy \tag{5.30}$$

$$e_2 = (1/c_\circ)\int_{-1}^{1}\Phi^2\,dy \tag{5.31}$$

$$e_3 = -\frac{1}{1-\delta c_\circ^2}\int_{-1}^{1}\delta f\left[3 + \frac{5\delta c_\circ^2}{3} - \frac{1}{2}\left(\frac{5}{3} + \delta c_\circ f^2\right)\frac{1-3\delta c_\circ^2}{1-\delta c_\circ^2}\right]\Phi^3\,dy. \tag{5.32}$$

Equation (5.29) is the Korteweg-deVries equation, well-known in the treatment of solitary and cnoidal surface wave [Benjamin and Lighthill, (1954)]. If it is possible to solve the eigenvalue equation (5.21), the coefficients may be determined from (5.30) to (5.32), and hence, solutions for (5.29) may be obtained.

5.2.2 The transverse eigenfunctions

Through the transformation,

$$\zeta = \sqrt{2}\,\sqrt[4]{\delta}\, f(y) \quad , \tag{5.33}$$

equation (5.21) becomes Weber's equation,

$$\Phi_{\zeta\zeta} - \left[\frac{1}{2c_\circ\sqrt{\delta}} + \frac{\zeta^2}{4}\right]\Phi = 0 \quad , \tag{5.34}$$

whose solutions are the Parobolic Cylinder functions $U(\kappa,\zeta)$, and $V(\kappa,\zeta)$ [Abramowitz and Stegun, (1965)] where

$$\kappa = \frac{1}{2c_\circ\sqrt{\delta}} \quad . \tag{5.35}$$

The boundary condition (5.22) is satisfied if

$$U(\kappa,\zeta_1)\,V(\kappa,\zeta_2) - V(\kappa,\zeta_1)\,U(\kappa,\zeta_2) = 0 \tag{5.36}$$

where

$$\zeta_{1,2} = \sqrt{2}\,\sqrt[4]{\delta}\,(f_\circ \pm 1) \quad . \tag{5.37}$$

This is the same condition as was found to be required in Chapter IV for the divergent plane Rossby wave solution in a β-plane channel. (See p.58.) Although the zeros of the cross-products of

Parabolic Cylinder functions are not tabulated, it was earlier shown that for any values of ζ_1 and ζ_2 (5.36) is satisfied only if $\lambda < -0.5$. In terms of $c_\circ$, this condition requires that $c_\circ < -\delta^{-\frac{1}{2}} < 0$. Therefore, the wave's phase moves in the negative x-direction. Traditionally, more manageable solutions to (5.21) are obtained by holding f^2 constant except where it is differentiated. Applying this approximation, the solution to the eigenvalue equation may be written

$$\Phi = A \sin \frac{m\pi}{2} (y + 1) \tag{5.38}$$

where

$$c_\circ = -\left(\frac{m^2\pi^2}{4} + \delta f_\circ^2\right)^{-1} . \tag{5.39}$$

Using this simple solution, the coefficients for equation (5.29) may be evaluated from the integrals given by (5.30) to (5.32). Since f^2 was held constant when the eigenvalue equation was solved, to be consistent, f^2 must be held constant during these integrations. Under these conditions the coefficients are given by

$$\begin{aligned} e_1 &= c_\circ(1 - \delta c_\circ^2)\, A^2 \\ e_2 &= A^2/c_\circ \\ e_3 &= \begin{cases} \dfrac{A^3 E}{m\pi} & m \text{ even} \\ -\dfrac{A^3 f_\circ E}{m\pi} & m \text{ odd} \end{cases} , \end{aligned} \tag{5.40}$$

where

$$E = \frac{8\delta}{3(1-\delta c_o^2)}\left[3 + \frac{5\delta c_o^2}{3} - \frac{(5 + 3\delta c_o f_o^2)(1 - 3\delta c_o^2)}{6(1-\delta c_o^2)}\right] . \tag{5.41}$$

5.2.3 Solutions to the Korteweg-deVries equation

The Korteweg-deVries equation (5.29) has been the subject of considerable recent research, notably studies by Miura et al. (1968), Miura (1968), and Lax (1968). In this study, the solutions of (5.29) will be given following the work of Keulegan and Patterson, (1940). Equation (5.29) may first be integrated twice to give

$$\frac{e_1}{2} g_\xi^2 + \frac{e_3}{6} g^3 + \frac{e_2 c_1}{2} g^2 + \bar{e}_4 g + \bar{e}_5 = 0 \tag{5.42}$$

where $\bar{e}_4$ and $\bar{e}_5$ are arbitrary integration constants.

For the non-divergent case, Larsen (1965) was able to show that $\bar{e}_4$ depended on the energy of the basic flow and the momentum flux. He obtained these results using a quasi-Lagrangian co-ordinate system, also described in Clarke and Fofonoff, (1969). This co-ordinate system uses the stream function and time as independent variables in place of the usual space co-ordinates. In the divergent case, however, the two-dimensional stream function can not be defined, and therefore, such an approach is no longer feasible. Hence, $\bar{e}_4$ and $\bar{e}_5$ will be treated here as unknown constants with the results of Larsen's non-divergent analysis being used to suggest the physical processes from which they arise.

Equation (5.42), multiplied through by $2/e_1$, becomes

$$g_\xi^2 + \frac{e_3}{3e_1} g^3 + \frac{c_1}{c_o(1-\delta c_o^2)} g^2 + e_4 g + e_5 = 0 \tag{5.43}$$

where $e_4 = 2\bar{e}_4/e_1$, $e_5 = 2\bar{e}_5/e_1$. If g_ξ is to be real when g is zero, then e_5 must be negative. The solutions of (5.43) are to be periodic in ξ; hence, g_ξ must be zero for at least two different and real values of g. When g_ξ is zero, (5.43) is a cubic in g. Recalling that $e_5 < 0$, then the three roots of this cubic fall in one of the following three cases:

(a) all positive and real roots

(b) one positive and two negative real roots

(c) one positive real and two complex conjugate roots.

Case (c) is not applicable here since for a periodic solution, g must be zero for at least two different real values of g. Both case (a) and case (b) should lead to long planetary wave solutions.

5.2.4 The solitary wave

The simplest solution to (5.43) corresponds to the solitary wave. Lax (1968) showed that, in the limit of long time, any solution of the time-dependent Korteweg-deVries equation [see p.10, (1.1)] tends asymptotically to a sum of solitary waves. In general, for non-linear equations, new solutions cannot be created by summing together other solutions; hence, this special feature of the Korteweg-deVries equation was both unexpected and surprising. The solitary wave solution arises when $e_4 = e_5 = 0$ and is given by

$$g(\xi) = \operatorname{sech}^2\sqrt{\frac{e_3}{12e_1}}\;\xi \qquad (5.44)$$

and

$$c_1 \quad = \quad - c_\circ^2(1 - \delta c_\circ^2) \frac{e_3}{3e_1} \tag{5.45}$$

where the amplitude of g is arbitrarily set to unity. In order that the solution be real, e_3/e_1 must be positive. For the same scales that were used to estimate δ, (that is, $\beta L \simeq 10^{-5}\ s^{-1}$, $f_\circ \simeq 10^{-4}\ s^{-1}$, $L \simeq 10^6$ m, $gh \simeq 10^4\ m^2/s^2$), $c_\circ \simeq -10^{-1}$ m/s; therefore, the sign of e_3/e_1 is the same as that of $A(-)^m/c_\circ$. Since $c_\circ$ is negative, the wave amplitude, A, is negative if m is even, positive if m is odd. On setting

$$\left|\frac{Am\pi}{2(1 - \delta c_\circ^2)}\right| \quad = \quad 1 \tag{5.46}$$

the wave profile in terms of the zonal velocity is

$$u \quad = \quad (-)^m\ \varepsilon[\cos \frac{m\pi}{2}(y + 1) \quad - \quad \frac{2\delta c_\circ f}{m\pi} \sin \frac{m\pi}{2}(y + 1)]\ \mathrm{sech}^2\sqrt{\frac{e_3}{12e_1}}\ \xi \tag{5.47}$$

and the phase speed is

$$c \quad = \quad - \left(\frac{m^2\pi^2}{4} \quad + \quad \delta f_\circ^2\right)^{-1} \quad - \quad \varepsilon(1 - \delta c_\circ^2) c_\circ^2 \frac{e_3}{3e_1} \quad . \tag{5.48}$$

At the southern boundary, for m even, the zonal velocity is in the opposite direction to the phase velocity; for m odd the zonal velocity is in the same direction as the phase velocity. At the northern boundary the velocity is westward in both cases. The phase velocity is increased in its westerly direction by an amount proportional to the wave amplitude.

5.2.5 The cnoidal waves

Again, following Keulegan and Patterson (1940), a more general solution to (5.43) may be obtained by solving for the roots of the cubic,

$$\frac{e_3}{3e_1} g^3 + \frac{c_1}{c_o^2(1 - \delta c_o^2)} g^2 + e_4 g + e_5 = 0 \quad . \tag{5.49}$$

If these roots are g_1, g_2, g_3, where $g_1 \geq g_2 \geq g_3$, then, since at least one of the roots must be positive, $g_1 > 0$. The remaining two roots may be either both positive or both negative. This is unlike the case of surface gravity waves; there, if the roots are all positive, then the surface elevation is always positive. However, the surface elevation is usually defined as the deviation from the average surface level; hence, such solutions are physically unrealistic. In the case of planetary waves, the roots all positive requires only that at a given latitude the zonal velocity is always in the same direction. Therefore, for planetary waves no distinction need be made between cases (a) and (b) of §5.2.3 .

A general solution to (5.43) is given by

$$g(\xi) = g_i + B_i \, \mathrm{cn}^2(P_i \xi / n) \tag{5.50}$$

where i may be any one of 1, 2, or 3, and

$$P_i^2 = e_3 B_i / (12 e_1 n) \tag{5.51}$$

$$c_1 = - c_o^2(1 - \delta c_o^2)(g_i + 4[2n - 1]P_i^2) \tag{5.52}$$

$$e_4 = \frac{e_3^3}{e_1}\{g_i^2 + \frac{B_i}{3n}[(1 - n)B_i + g_i(2n - 1)]\} \quad . \tag{5.53}$$

In order that the solution be real, P_i must be real. This will also mean that the wavelength of the disturbance,

$$\lambda \quad = \quad (2/P_i)\, K(n) \ , \tag{5.54}$$

must also be real, where the wavelength is defined by the requirement that $g(\xi + \lambda) \ = \ g(\xi)$. In §5.2.4 it was shown that the sign of e_3/e_1 is that of $(-)^m A/c_o$; hence, for P_i to be real, AB_i must be negative for m even, positive for m odd. Setting

$$\left|\frac{m\pi AB_i}{2(1 - \delta c_o^2)}\right| \quad = \quad 1 \ , \tag{5.55}$$

then the zonal velocity due to the waves is

$$u \quad = \quad (-)^{m+1}\ \varepsilon\left[\cos\frac{m\pi}{2}(y+1) - \frac{2\delta c_o f}{m\pi}\sin\frac{m\pi}{2}(y+1)\right]\left[\frac{g_i}{B_i} + cn^2(P_i\xi/n)\right] \tag{5.56}$$

and

$$P_i^2 \quad = \begin{cases} -\dfrac{E}{12c_o m^2\pi^2 n} & \text{m even} \\[2ex] -\dfrac{Ef_o}{12c_o m^2\pi^2 n} & \text{m odd} \ . \end{cases} \tag{5.57}$$

An interesting property of this solution arises from the fact that both A and B_i are unspecified parameters. Equation (5.55) specifies one of them as a function of the other, and (5.53) connects them with a third parameter e_4, also unspecified. Hence, in (5.56), B_i, where it appears, may be any non-zero number. At first sight this appears to have the effect of permitting a steady zonal current of the form $\cos\frac{m\pi}{2}(y+1) - \frac{2\delta c_o f}{m\pi}\sin\frac{m\pi}{2}(y+1)$ and of any amplitude to be added to the solution without altering the form of the wave. This is not true since the root g_i is determined in part by the constant e_4 which in turn is specified when B_i is specified.

Larsen (1965) gives a series of solutions, similar in form to (5.50), but with B_i specified as particular combinations of the roots of the cubic. Such a procedure has the formal advantage of permitting P_i, c_1, and e_4 to be determined in terms of the roots g_i and n only. However, since the roots of the cubic will, in general, be quite complicated expressions, having the solution parameterized in terms of them seems of little advantage.

In order to study these solutions in greater detail it is necessary to find a special case in which the roots of the cubic, or at least one of them, is of a simple analytical form. One such case occurs if e_5 is zero, in which case one of the roots is zero and the two remaining roots are given by the roots of a quadratic. Not only does this case give simple expressions for the solution, but it also contains the solitary wave solutions as a special case.

5.2.6 A special case: one root zero

If one of the roots of (5.49) is zero, then one solution of (5.43) is

$$g(\xi) = B_1 \, cn^2(P_1\xi/n) \tag{5.58}$$

which is simply (5.50) with $g_i = 0$. Normalizing the amplitude through the condition (5.55) where $B_i = B_1$; the zonal velocity is given by

$$u = (-)^{m+1} \varepsilon \left[\cos \frac{m\pi}{2}(y+1) - \frac{2\delta c_o f}{m\pi} \sin \frac{m\pi}{2}(y+1)\right] cn^2(P_1\xi/n) \quad (5.59)$$

where

$$P_1^2 = \begin{cases} -\dfrac{E}{12c_o m^2\pi^2 n} & m \text{ even} \\ \\ -\dfrac{Ef_o}{12c_o m^2\pi^2 n} & m \text{ odd} \end{cases}, \quad (5.60)$$

$$c_1 = -4c_o^2(1 - \delta c_o^2)(2n - 1)P_1^2 \quad , \quad (5.61)$$

and the wavelength of the disturbance is given by (5.54) in which P_i is replaced by P_1.

Tables of elliptic functions such as cn(x/n), are given only for $0 \leq n \leq 1$; therefore, for n > 1, the wave profile is given by

$$u = (-)^{m+1} \varepsilon\left[\cos \frac{m\pi}{2}(y+1) - \frac{2\delta c_o f}{m\pi} \sin \frac{m\pi}{2}(y+1)\right]\left[\frac{n-1}{n} + \frac{1}{n}\, cn^2(\sqrt{n}P_1\xi/\tfrac{1}{n})\right], \quad (5.62)$$

and the wavelength by

$$\lambda = \frac{2}{\sqrt{n}P_1} K(n^{-1}) \quad . \quad (5.63)$$

For n = 1, solution (5.58) becomes

$$g(\xi) = B_1 \, sech^2(P_1\xi) \quad (5.64)$$

which is the solitary wave solution given previously in §5.2.4 .

In Figure 2, $2\lambda^{-1}[P_1(1)]^{-1}$ and $c_1(n)/c_1(1)$ are plotted against n. The wavelength, λ, increases steadily without limit from zero as n goes from zero to one, then decreases again to an asymptotic limit as n increases beyond one. The solitary wave, at n = 1, is, therefore, the limit of long wavelengths. Since $c_1(1) < 0$, the phase speed

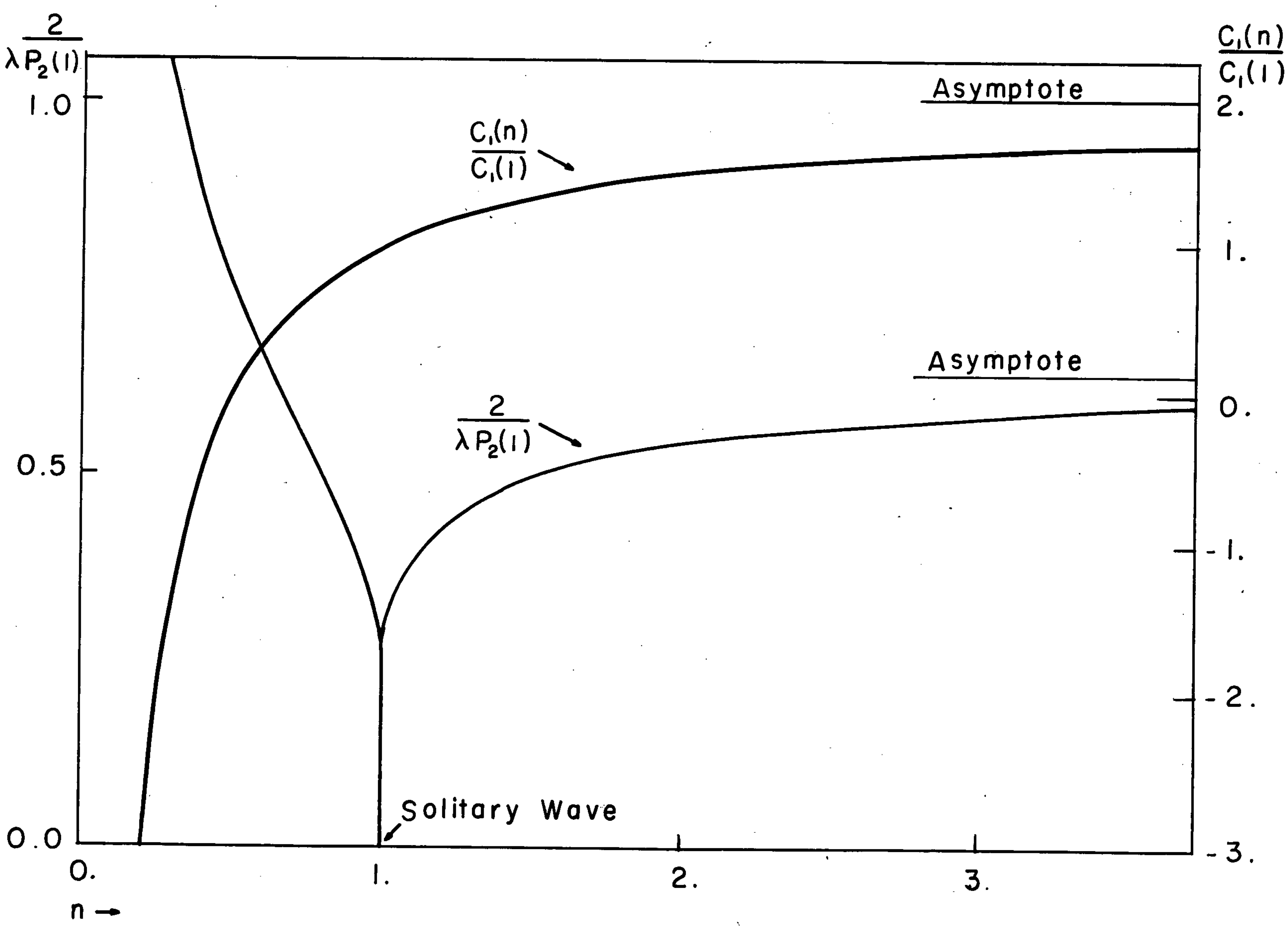

Figure 2. Wavelength and phase speed for cnoidal waves.

correction, $c_1(n)$, is negative for $n > \frac{1}{2}$, positive for $n < \frac{1}{2}$. Since the basic phase speed c_o is negative, $c_1 < 0$ represents an increase in the magnitude of the phase speed.

5.2.7 The non-divergent limit

Solutions similar to (5.59) and (5.47) were given by Larsen (1965) for the non-divergent case. These non-divergent solutions may be obtained by setting δ to zero in equations (5.59) to (5.61). As $\delta \to 0$, $E \to 0$ and hence, $e_3 \to 0$. Since P_1 must be non-zero in order that the wavelength remain finite, therefore, from (5.51), $n \to 0$. Thus, in this limit of $\delta = 0$, $n = 0$, (5.59) becomes

$$u = (-)^{m+1} \varepsilon[\cos \frac{m\pi}{2}(y+1)] \cos^2(P_1\xi) \tag{5.65}$$

or on rewriting in terms of s

$$u = (-)^{m+1} \tfrac{1}{2}\varepsilon \cos \frac{m\pi}{2}(y+1) \; [1 + \cos (2P_1\sqrt{\varepsilon}\, s)] \quad . \tag{5.65a}$$

If

$$k = 2P_1\sqrt{\varepsilon} \quad , \tag{5.66}$$

then

$$\varepsilon c_1 = c_o^2 k^2 \tag{5.67}$$

and

$$c = -\frac{4}{m^2\pi^2} + \frac{16k^2}{m^4\pi^4} \quad . \tag{5.68}$$

These results are identical to those obtained by Larsen (1965) when he allowed the basic zonal current in his non-divergent analysis to go to zero. In Chapter IV it was shown that

$$u = \cos \frac{m\pi}{2}(y+1) \cos ks \tag{5.69}$$

is an exact solution of the non-divergent equations where

$$c = -\left[\frac{m^2\pi^2}{4} + k^2\right]^{-1} . \tag{5.70}$$

If $k << \frac{m\pi}{2}$, equation (5.68) is the first two terms of the binomial expansion of (5.70). Therefore, applying the non-divergent approximation, the cnoidal wave solutions with no basic zonal current reduce to an expansion of the exact solutions given in Chapter IV.

5.3 The case of $\delta = O(\varepsilon)$

5.3.1 Introduction

The results so far have shown that if divergence is retained, cnoidal and solitary waves may exist in a long channel without the presence of a sheared basic zonal current. This is in contrast to the non-divergent results of Larsen (1965), who showed that for solitary and cnoidal waves to exist, there must also be present a zonal current with at least a weak shear.

The equations for the divergent case are too complicated to include a zonal current and still get simple enough expressions to interpret; however, recalling the scaling done previously, a reasonable simplifying assumption is that $\delta = O(\varepsilon)$. Under this assumption, the equations are sufficiently simple that the effects of a sheared basic current may be studied. Furthermore, in the previous chapters, it was shown that planetary waves can exist in a fluid of uniform rotation with bathymetry; here, the equations are generalized to include variable bottom topography.

5.3.2 The equations

The basic non-dimensionalized equations are (5.2), (5.3), and (5.5), plus a new continuity equation

$$[(u - c)(h + \delta\eta)]_s + [v(h + \delta\eta)]_y = 0 \tag{5.71}$$

where $h = h(y)$ is the non-dimensionalized depth, and H in (5.1) is redefined to be the average depth of the channel. Defining

$$\mu = \delta/\varepsilon = O(1) \tag{5.72}$$

the transformation (5.6) and the perturbation expansions (5.7) are once more applied to equations (5.2), (5.3), (5.5) and (5.71) to give to zeroth order in ε:

$$fu_o + \eta_{o_y} = 0 , \tag{5.73}$$

to the first order:

$$(u_o - c_o)u_{1_\xi} + v_1 u_{o_y} - fv_1 + \eta_{1_\xi} = 0 \tag{5.74}$$

$$fu_1 + \eta_{1y} = 0 \tag{5.75}$$

$$hu_{1_\xi} + (hv_1)_y = 0 \tag{5.76}$$

$$v_1 = 0 \text{ at } y = \pm 1 , \tag{5.77}$$

and to the second order:

$$(u_\circ - c_\circ)u_{2_\xi} + (u_{\circ_y} - f)v_2 + \eta_{2_\xi} = -(u_1 - c_1)u_{1_\xi} - v_1 u_{1y} \qquad (5.78)$$

$$fu_2 + \eta_{2y} = -(u_\circ - c_\circ)v_{1_\xi} \qquad (5.79)$$

$$hu_{2_\xi} + (hv_2)_y = -\mu[(u_\circ - c_\circ)\eta_{1_\xi} + \eta_\circ u_{1_\xi} + (\eta_\circ v_1)_y] \qquad (5.80)$$

$$v_2 = 0 \quad \text{at} \quad y = \pm 1 \ . \qquad (5.81)$$

The analysis follows that of §5.2. If the transports are defined by $V_1 = v_1 h$ and $U_1 = u_1 h$, then the first order equations are reducible to

$$(u_\circ - c_\circ)V_{1yy} - (u_\circ - c_\circ)\frac{h_y}{h} V_{1_y} + \left[(1 - u_{\circ_{yy}}) + (u_{\circ_y} - f)\frac{h_y}{h}\right]V_1 = 0 \ . \qquad (5.82)$$

If the solution is of the form

$$V_1 = g_\xi(\xi)\ \Phi(y) \ , \qquad (5.83)$$

then

$$(u_\circ - c_\circ)\left(\frac{\Phi_y}{h}\right)_y + \left(\frac{f - u_{\circ_y}}{h}\right)_y \Phi = 0 \qquad (5.84)$$

where

$$\Phi(1) = \Phi(-1) = 0 \qquad (5.85)$$

and

$$U_1 = -g(\xi)\ \Phi_y(y) \qquad (5.86)$$

$$\eta_1 = \frac{(f - u_{\circ_y})\Phi + (u_\circ - c_\circ)\Phi_y}{h}\ g(\xi) \ . \qquad (5.87)$$

Similarly, u_2 and η_2 may be eliminated from the second order equations, and u_1, v_1, and η_1 substituted for, in terms of Φ and g, to give

$$(u_\circ - c_\circ)\left[\frac{(v_2 h)_y}{h}\right]_y + \left[(1 - u_{\circ yy}) + (u_{\circ y} - f)\frac{h_y}{h}\right] v_2 \tag{5.88}$$

$$= -\frac{u_\circ - c_\circ}{h}\Phi g_{\xi\xi\xi} + c_1\left(\frac{\Phi_y}{h}\right)_y g_\xi$$

$$- \mu\left[(u_{\circ y}-f) + (u_\circ - c_\circ)\frac{\partial}{\partial y}\right]\left[\frac{(u_\circ-c_\circ)^2}{h^2}\Phi_y - \frac{\eta_\circ\frac{h_y}{h} + u_{\circ y}(u_\circ-c_\circ) + fc_\circ}{h^2}\Phi\right] g_\xi$$

$$+ \left[\left(\frac{\Phi_y}{h}\right)^2 - \frac{\Phi}{h}\left(\frac{\Phi_y}{h}\right)_y\right]_y g g_\xi \ .$$

If (5.88) is multiplied through by $\Phi/(u_\circ - c_\circ)$, then integrated over y from -1 to 1, the left-hand side is identically zero and the right-hand side gives

$$b_1 g_{\xi\xi\xi} + b_2 g_\xi + b_3 g g_\xi = 0 \tag{5.89}$$

where

$$b_1 = \int_{-1}^{1} \frac{\Phi^2}{h}\, dy \tag{5.90}$$

$$b_2 = -c_1\int_{-1}^{1} \Phi\left[\frac{1}{h}\left(\frac{\Phi}{u_\circ - c_\circ}\right)_y\right]_y dy \tag{5.91}$$

$$-\mu\int_{-1}^{1}\left(\frac{(u_\circ-c_\circ)^2}{h^2}\Phi_y^2 + \left\{\frac{u_{\circ y}-f}{u_\circ-c_\circ}\,\frac{\eta_\circ\frac{h_y}{h} + u_{\circ y}(u_\circ-c_\circ) + fc_\circ}{h^2} + \left[\frac{\eta_\circ\frac{h_y}{h} + 2u_{\circ y}(u_\circ-c_\circ) + 2fc_\circ - fu_\circ}{2h^2}\right]_y\right\}\Phi^2\right) dy$$

$$b_3 = \int_{-1}^{1} \left(\frac{\Phi}{u_\circ - c_\circ}\right)_y \left[\left(\frac{\Phi_y}{h}\right)^2 - \frac{\Phi}{h}\left(\frac{\Phi_y}{h}\right)_y\right] dy \quad . \tag{5.92}$$

Equation (5.89) is again the Korteweg-deVries equation. The major difference between these equations and the corresponding ones obtained in §5.2 is that b_2, the coefficient of g_ξ, contains, in addition to the term in c_1, a second term which arises from the divergence terms. Solutions to (5.89) are first obtained by setting $h = 1$; these are then compared with those obtained by Larsen with no divergence. The case of topographic waves will be treated in §5.4 .

5.3.3 The case of uniform shear

For a uniformly sheared zonal velocity given by

$$u_\circ = W_\circ + ay \tag{5.93}$$

where $a << 1$, equation (5.84) can be solved exactly in terms of confluent hypergeometric functions [see equation (4.110)]; however, an approximate solution gives more workable results. Using perturbation expansions for both $c_\circ$ and Φ in the terms of a, equation (5.84) gives as a solution

$$\Phi = A\left\{\sin \ell(y+1) + a\left[\frac{\ell^3}{4}(y-y^2)\cos \ell(y+1) + \frac{\ell^2}{4} y \sin \ell(y+1)\right]\right\} + O(a^2) \tag{5.94}$$

where

$$\ell = \frac{m\pi}{2} \tag{5.95}$$

and the phase speed is

$$c_\circ = W_\circ - \frac{1}{\ell^2} + \frac{a}{2} - \frac{a^2\ell^2}{16}\left(1 - \frac{3}{\ell^2}\right) + O(a^3) \quad . \tag{5.96}$$

From (5.90) to (5.92), the coefficients b_1, b_2, and b_3 are

$$b_1 = A^2\left(1 - \frac{a\ell^2}{4}\right) + O(a^2) \tag{5.97}$$

$$b_2 = A^2[\ell^4 c_1 - \frac{\mu W_\circ}{2} + \mu\ell^2(W_\circ - \ell^2)\left(\frac{1}{3} + f_\circ^2\right) + O(a)] \tag{5.98}$$

$$b_3 = \frac{2a\ell^5A^3}{3}[1 - \cos 2\ell + \frac{a\ell^2}{2}\left(6 + \cos 2\ell\right)] + O(a^2) \quad . \tag{5.99}$$

Both b_1 and b_3 and the expansions for Φ and $c_\circ$ are identical to the coefficients that were obtained by Larsen (1965) for the non-divergent equations of motion. The effect of the weak divergence is felt entirely in the b_2 coefficient which, as pointed out earlier, contains terms which do not contain c_1 as a factor. In the strongly divergent case, these terms appear in the expression for $c_\circ$ since they are of zeroth order. For $\mu = 0$, the non-divergent case is recovered.

In the case of $a = 0$, that is, for uniform zonal current, the b_3 coefficient is zero and equation (5.89) is linear with general solution

$$g(\xi) = A + B \cos P_2\xi \tag{5.100}$$

where

$$P_2^2 = \ell^4 c_1 - \mu\left[\frac{W_\circ}{2} + (1 - \ell^2 W_\circ)(\tfrac{1}{3} - f_\circ^2)\right] . \tag{5.101}$$

Transforming ξ back to s , (5.100) becomes

$$g(s) = A + B\cos\sqrt{\varepsilon}P_2 s = A + B\cos ks \tag{5.102}$$

where $k = \sqrt{\varepsilon}P_2$ and

$$c = W_\circ - \frac{1}{\ell^2} + \frac{k^2}{\ell^4} + \frac{\delta}{\ell^4}\left[\frac{W_\circ}{2} + (1 - \ell^2 W_\circ)(\tfrac{1}{3} + f_\circ^2)\right] . \tag{5.103}$$

For the non-divergent case ($\delta = 0$), this reduces to the solution obtained by Larsen, and the phase speed is the first few terms of the binomial expansion of the phase speed for the linear Rossby wave solution, where both k^2 and $\delta f_\circ^2$ are small with respect to ℓ^2 and $\delta/3$, respectively.

The coefficient of the non-linear term of (5.89) is non-zero only if a is non-zero; hence, the solitary and cnoidal wave solution can exist only if a sheared zonal current is present. This result is identical to that obtained in the non-divergent case by Larsen.

Solutions of (5.89) take the same form as in §5.2 . Again, a general form of solution may be written but this is not of much interest analytically because it involves the roots of a cubic equation as well as the introduction of two arbitrary integration constants. Therefore, the discussion will be restricted to two solutions, the solitary wave solution and the simple cnoidal wave solution for which one root of the cubic is zero.

5.3.4 The solitary wave

The solitary wave solution in terms of the zonal velocity is

$$u = W_\circ - \varepsilon[\mathrm{sgn}(a)]^m \cos\ell(y+1)\,\mathrm{sech}^2(P_2\xi) + O(a) \tag{5.104}$$

where

$$P_2^2 = \begin{cases} \dfrac{|a|\ell^4}{9} & m \text{ odd} \\ \\ \dfrac{7a^2\ell^6}{36} & m \text{ even} \end{cases} \tag{5.105}$$

and the phase speed is

$$c = W_\circ - \frac{1}{\ell^2} - \frac{4\varepsilon P_2^2}{\ell^4} + \frac{\delta}{\ell^4}\left[\frac{W_\circ}{2} + (1 - \ell^2 W_\circ)(f_\circ^2 + \tfrac{1}{3})\right] + O(a) \quad . \tag{5.106}$$

The only difference between this solitary wave solution and that for the non-divergent case is that the phase speed is increased or decreased by the term containing δ as a factor. If $W_\circ$ is set to zero and if 1/3 is neglected relative to $f_\circ^2$, then (5.106) reduces to an expansion of (5.47) for δ small, where (5.47) gives the phase speed for solitary waves in the strongly divergent case.

5.3.5 A cnoidal wave

The second solution of interest is chosen so that it contains the solitary wave as a special case. This solution is given by

$$u = W_\circ - \varepsilon[\mathrm{sgn}(a)]^m \cos \ell(y+1)\ \mathrm{cn}^2(P_2\xi/n) + O(a) \tag{5.107}$$

where

$$P_2^2 = \begin{cases} \dfrac{|a|\ell^4}{9n} & m \text{ odd} \\ \\ \dfrac{7a^2\ell^6}{36n} & m \text{ even} \end{cases} \tag{5.108}$$

and

$$c = W_\circ - \frac{1}{\ell^2} - \frac{4\varepsilon}{\ell^4}(2n-1)P_2^2 + \frac{\delta}{\ell^4}\left[\frac{W_\circ}{2} + (1 - \ell^2 W_\circ)(f_\circ^2 + \tfrac{1}{3})\right] + O(a) . \qquad (5.109)$$

Again, except for the additional terms in the expression for the phase velocity, the solution is the same as that obtained from the non-divergent equations. The variation of wavelength and phase speed with n is that given in Figure 2, except that a term independent of n must be added to each value of c_1.

Because of the close resemblance of these solutions to Larsen's non-divergent solutions, a modification of quasi-Lagrangian analysis might offer some explanation of the possible physical processes which determine these waves. Such a modification, however, lies beyond the scope of this work. In his analysis, Larsen obtains an equation of the form of (4.42) in which all the coefficients are defined in terms of the energy and momentum of the basic flow. His analysis showed that the generation of the solitary wave requires no additional energy over that of the basic flow. Waves of moduli $n > 1$ are associated with a loss of energy from the basic flow; those of moduli $n < 1$, with a gain of energy. This argument may also hold for the weakly divergent waves.

5.4 Topographic waves

As in the previous chapters, topographic waves will be investigated only for the exponential profile given by

$$h = \exp(-\Lambda y) \qquad (5.110)$$

where $\Lambda \ll 1$. If it is assumed that the topographic effect is much more important than the β-effect, f can be treated as a constant. Once again the basic current is given by (5.93), that is a steady zonal current with a weak uniform shear; and (5.84) is solved using a perturbation expansion for Φ and c_o in powers of a. The solution satisfying both boundary conditions, (5.85), is given by

$$\Phi = A\left[1 + \frac{a(\ell^2 + \Lambda^2/4)^2 y}{4\ell^2\Lambda f}\right] \exp(-\tfrac{1}{2}\Lambda y) \sin \ell(y+1) \tag{5.111}$$

$$+ \frac{aA(\ell^2 + \Lambda^2/4)^2}{4\ell\Lambda f}(1 - y^2) \exp(-\tfrac{1}{2}\Lambda y) \cos \ell(y+1) + O(a^2)$$

and

$$c_o = W_o - \frac{\Lambda f}{\ell^2 + \Lambda^2/4} + \frac{a\Lambda}{\ell^2 + \Lambda^2/4} + O(a^2) \tag{5.112}$$

where again

$$\ell = \frac{m\pi}{2} . \tag{5.113}$$

Using this solution, the coefficients of the Korteweg-deVries equation (5.89) may be calculated from the integrals given by (5.90) to (5.92). The coefficients are then given by

$$b_1 = A^2 + O(a) \tag{5.114}$$

$$b_2 = c_1 A^2 \frac{4\ell^2 + \Lambda^2}{4\Lambda f} \tag{5.115}$$

$$+ \mu A^2 \left\{\frac{\ell^2 f^2}{16(\ell^2 + \Lambda^2/4)} [27\Lambda^4 + 40\ell^2\Lambda^2 - 16\ell^4]\right.$$

$$\left. + \frac{\ell^2 f W_o (16\ell^4 - 5\Lambda^4)}{8\Lambda(\ell^2 + \Lambda^2/4)^2}\right\} \sinh \Lambda$$

$$- \mu A^2 \frac{\ell^2 f W_o (4\ell^2 - 3\Lambda^2)}{4\ell^2 + \Lambda^2} \cosh \Lambda + O(a)$$

$$b_3 = \frac{24\ell^3(4\ell^2 + \Lambda^2)A^3}{f(36\ell^2 + \Lambda^2)} \begin{cases} \sinh \Lambda & m \text{ even} \\ \\ -\cosh \Lambda & m \text{ odd} \end{cases} + O(a) \quad . \tag{5.116}$$

Having these coefficients, the solutions to (5.89) follow in the same way as they did in the previous two cases. For $a = 0$, all of the coefficients remain non-zero; hence, solitary and cnoidal wave solutions will exist even for a uniform basic zonal flow. In fact, solitary and cnoidal waves will exist for the exponential depth profile even for the non-divergent case, $\mu = 0$, and no basic flow, $W_o = a = 0$. It should again be emphasized that with topographic waves, the properties of the waves are strongly dependent on the character of the topography. Hence, any property of the waves discussed here is likely to be a property only of waves over an exponential depth profile.

For $W_o = 0$ and $a = 0$, the simple cnoidal wave solution corresponding to (5.108) is given by

$$u_1 = (-)^m \varepsilon \mathrm{cn}^2(P_2\xi/n)[\cos \ell(y+1) - \frac{\Lambda}{2\ell} \sin \ell(y+1)]\exp(\tfrac{1}{2}\Lambda y) \tag{5.117}$$

where

$$P_2^2 = \frac{2\ell^2(4\ell^2 + \Lambda^2)}{f(36\ell^2 + \Lambda^2)n} \begin{cases} \sinh \Lambda & m \text{ even} \\ \\ \cosh \Lambda & m \text{ odd} \end{cases} \tag{5.118}$$

$$c_1 = \frac{4\Lambda f P_2^2(1-2n)}{(\ell^2 + \Lambda^2/4)^2} - \mu \frac{\ell^2 \Lambda f^3 [27\Lambda^4 + 40\Lambda^2\ell^2 - 16\ell^4]}{16(\ell^2 + \Lambda^2/4)^5} \quad . \tag{5.119}$$

The solitary wave is contained as the special case of n = 1, in the above equations. Solutions of the same form in ξ have been previously discussed in §5.2.5, §5.2.6, §5.3.4, and §5.3.5 . It should be noted that the effect of the weak divergence is felt only in the phase speed, and that if μ is zero, a cnoidal wave solution still exists.

Equations (5.83) to (5.87), and (5.89) to (5.92) hold for an arbitrary depth profile. For any depth profile for which solutions of the transverse eigenfunction equation (5.84) exist, solitary and cnoidal wave solutions should also exist.

Benjamin (1967) found that internal solitary and cnoidal waves of a new form could exist on density profiles in fluids of infinite depth, provided that the density varies only in a layer whose thickness is much smaller than the depth of the fluid. Topographic waves, analogous to these solutions, include shelf waves and double Kelvin waves. A possible extension to this present study would be to investigate the existence of such solitary and cnoidal topographic waves on an isolated topographic feature in an unbounded ocean.

5.5 Summary

In summary, it has been shown that a class of long waves analogous to the solitary and cnoidal waves of surface wave theory exist in a channel on the β-plane or in a channel with cross-channel bathymetry for a uniformly rotating fluid. In the Rossby wave case, it was shown that if the non-divergent approximation is made, or that if the magnitude of the divergence terms is of the same order as that of the inertial terms, then solitary and cnoidal waves will exist only in the presence of a steady sheared current along the channel. Such a current is not necessary for cnoidal and solitary waves to exist in the divergent case or for topographic waves on an exponential profile.

In all cases, the wave profiles along the axis of the channel are given by the solutions to the Korteweg-deVries equation. In free surface flows, it is found that solitary and cnoidal waves are a preferred form of disturbance in that they show a remarkable persistence of form. Although solitary wave disturbances as discussed here have not been described in observation of either the ocean or the atmosphere, by analogy to surface waves, it is felt that these solutions may also represent a preferred form of disturbance.

VI. Concluding Remarks

In this work some of the finite amplitude behaviour of planetary waves has been explored in order to have some understanding of possible non-linear time-dependent motions in the ocean. With this in mind, the linear planetary wave solutions were closely examined both on the sphere and the β-plane in order to determine the magnitude of errors associated with the non-divergent and β-plane approximations.

For wavelengths of the order of a few thousand kilometers and less, the error in the phase speed associated with the non-divergent approximation both on the sphere and on the β-plane is about 15%, decreasing with decreasing wavelength. For the same range of wavelengths, the error associated with the β-plane is also about 10%.

The linear non-divergent solutions on the sphere exhibit the interesting property that their phase speed depends only on the degree of the spherical harmonic and is independent of the order. This means that any linear superposition of waves of the same degree will stay together because they all move with the same angular phase speed. Since each of the spherical harmonics making up this sum of solutions may have a different axis, this solution may become very complex, yet still be non-dispersive, at least to the limits of linear theory.

This property is not exhibited by the divergent waves on the sphere. Here, the phase speed depends on both the degree and order of spheroidal harmonics, and therefore, such a linear superposition of solutions will disperse in time.

On the β-plane, the non-divergent solutions are doubly-periodic sinusoidal waves whose phase speed depends only on the total wave number. In contrast to this, the divergent solutions of the β-plane, are sinusoidal in the direction of the waves but their variation in y, normal to this direction, is in the form of Parabolic Cylinder functions. For short wavelengths in y, these solutions may be approximated by a doubly-periodic sinusoidal wave, whose phase speed is a function of the total wave number only.

The errors associated with both the β-plane and non-divergent approximations are smaller for the shorter wavelength cases than for the longer. It is shown that for bottom slopes commonly found in the oceans, topographic waves will predominate over Rossby waves, and further, for the same range of frequencies the wavelengths associated with topographic waves will be much shorter than those for Rossby waves. For this reason the β-plane and non-divergent approximation may be used with greater accuracy with topographic waves than with Rossby waves. An exception to this is the continental shelf waves, where, because the depth of the fluid goes to zero, the non-divergent approximation may not be used.

The full non-divergent equations of motion on the sphere and on the β-plane give the linear non-divergent Rossby wave solutions as exact solutions. Furthermore, since these exact solutions consist of an arbitrary sum of the linear solutions of the same phase speed there is no non-linear interaction between linear non-divergent solutions of the same phase speed.

Such behaviour is not found for the divergent wave solutions nor for the linear topographic wave solutions nor for non-divergent Rossby wave solutions in the presence of a uniformly sheared current. In all of these cases, the wave profiles exhibit non-linearities to $O(\varepsilon^2)$ where ε is an amplitude parameter; however, similar to surface gravity waves, there is no first order correction to the phase speed. Such fundamental differences in non-linear behaviour of divergent and non-divergent waves was not expected. These results suggest that if one is studying non-linear effects or interactions between Rossby waves, the non-divergent approximation should be made only with a great deal of caution.

These results also suggest that in the mid-ocean, where barotropic currents are perhaps very weak, the depth nearly constant and deep, the Rossby waves once generated will interact with each other only very weakly as the motion will be nearly non-divergent. As these waves move toward the western boundary region they will experience sheared currents, bathymetry and increased divergence due to the decreasing depth. All these effects work to make the non-linear interaction terms more important. Therefore, these results suggest that the western boundary region is a region of intensified non-linear effects for Rossby waves. Such an effect at the western boundary of the Indian Ocean is also suggested by Lighthill (1969).

Finally, it is shown that a class of long divergent Rossby waves exists, analogous to the solitary and cnoidal waves of surface wave theory. Larsen's (1965) conclusion, that such waves could exist in the non-divergent case only in the presence of a sheared basic current, is confirmed and explained in light of the exact non-divergent solutions found here. Since these waves are to be long, relative to their lateral dimension, it seems reasonable to expect that the divergence terms should be retained.

It is also shown that solitary and cnoidal waves can exist on an exponential depth profile, even in the non-divergent case. A useful extension of this work would be to investigate solitary and cnoidal waves on discontinuous depth profiles in an unbounded ocean. If such waves can exist, they may be found trapped along oceanic features such as ridges and sea scarps, and play an important role in the dynamics of such regions.

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Appendix I

The problem of transformations in certain co-ordinate systems

In the mathematical solution of physical problems certain systems of co-ordinates introduce in the solution features which may be difficult to separate from the physical properties of the solution itself. In particular, special conditions are imposed on the solution at the poles or axis of spherical polar, cylindrical, or circular polar co-ordinate systems.

To illustrate this problem, consider a fluid confined between two concentric and rotating spheres. A natural choice of co-ordinates are the spherical polar co-ordinates whose axis is the axis of rotation; however, the choice of the axis of the co-ordinates is, in fact, completely arbitrary. Having chosen the axis of the co-ordinates, both the zonal, v_ϕ, and the meridional, v_θ, velocity components must be zero at the poles in order that both be single valued and continuously differentiable in a neighbourhood of the poles.

The point made here is that the velocity does not have to go to zero by any physical grounds but rather is required mathematically in order that the co-ordinate system work. Physically one can certainly allow a velocity at the pole although mathematically this could only be described if one allows v_ϕ and v_θ to be both multivalued and discontinuous at the pole.

The problems involved become very real and very clear if one tries to transform from one co-ordinate system to a second, both on the sphere. Suppose in the first system the solution is a steady zonal flow described by $v_\phi = \alpha R \sin\theta$, $v_\theta = 0$. We transform to a new co-ordinate system (θ',ϕ') whose axis lies in the $\phi = 0$ plane of the first system and is inclined at an angle γ to the original axis. (See Figure 1, p.12). In this co-ordinate system, the velocity components at the poles are given by

$$v'_{\theta'} = - \alpha R \sin\gamma \sin\phi' \tag{A.1}$$

$$v'_{\phi'} = \alpha R[\cos\gamma \sin\theta' - \sin\gamma \cos\theta' \cos\phi'] \tag{A.2}$$

Although physically the velocity field is continuous and continuously differentiable, the mathematical description of it is not. Both $v'_{\phi'}$ and $v'_{\theta'}$ take on all values between $\pm\, \alpha R \sin\gamma$ at each pole $(\theta' = 0,\pi)$; therefore, one cannot really speak of a value of either velocity component at the poles. The velocity field is known to be continuous and single-valued everywhere. Therefore, the singular behaviour at the poles must arise from the behaviour of the co-ordinate system alone. In this way the choice of the co-ordinate systems may have an effect on the solution which must not be interpreted as a physical effect.

Appendix II

Analogous behaviour of internal and planetary waves

In his treatise on non-homogeneous fluids, Yih (1965, ch.VI) gives a general discussion of the similarity between the flow of stratified fluids and fluid flow in an accelerating or rotating frame. This analogy has been discussed in greater detail with references to slow steady flows by Veronis (1967a,b). Here the analogy will be extended to wave motion and hence a parallel will be developed between internal waves and non-divergent planetary waves. This parallel has been found to be a useful tool in suggesting the existence or the form of planetary wave solutions for bathymetries of the same shape as density profiles for which internal wave solutions are known.

For an incompressible stratified fluid of constant depth, the equation governing infinitesmal amplitude internal gravity waves in two dimensions is given by Lamb (1945, p.378), by

$$\nabla^2\psi_{tt} - \Gamma\left\{\psi_{ztt} - g\psi_{xx}\right\} = 0 \tag{B.1}$$

where z is increasing upwards

$$u = \psi_z , \quad w = -\psi_x , \tag{B.2}$$

and

$$\Gamma = -\frac{1}{\rho_0}\frac{d\rho_0}{dz} . \tag{B.3}$$

For waves propagating horizontally,

$$\psi = W(z) \exp i(kx - \omega t) , \tag{B.4}$$

hence (B.1) becomes

$$W_{zz} - \Gamma W_z + k^2\left(\frac{g\Gamma}{\omega^2} - 1\right) W = 0 \tag{B.5}$$

subject to the boundary condition that

$$W = 0 \text{ at a boundary or at } z = \pm \infty. \tag{B.6}$$

Solutions are also possible for discontinuous density profiles. At a discontinuity of either ρ or Γ, both the vertical velocity and the pressure are continuous across the interface; hence,

$$\begin{aligned} W_1 &= W_2 \\ \rho_2 W_{2z} - \rho_1 W_{1z} &= \frac{gk^2}{\omega^2}(\rho_2 - \rho_1) W_1 \end{aligned} \tag{B.7}$$

where the subscripts refer to values on either side of the discontinuity.

In Chapter III, the linear equations for planetary waves are developed. In particular, equation (3.38) governs the y dependence of the zonal transport of a planetary wave propagating east-west parallel to the depth contours. This equation is given by

$$V_{yy} - \frac{h_y}{h} V_y - \left[\frac{kh}{\sigma}\left(\frac{f}{h}\right)_y + k^2\right] V = 0 \tag{B.8}$$

with the boundary conditions that

$$V = 0 \text{ at a boundary or at } y = \pm \infty. \tag{B.9}$$

At a discontinuity of h or f or their derivatives, transport and pressure must be continuous; hence,

$$V_1 = V_2$$

$$\frac{V_1}{h_1}y - \frac{V_2}{h_2}y = \frac{fk}{\omega}\frac{(h_1 - h_2)}{h_1 h_2} V_1 \qquad \text{(B.10)}$$

where again the subscripts refer to values on either side of the discontinuity.

The similarity between (B.8) and (B.5) plus their boundary conditions is at once apparent. If the Boussinesq approximation is made in each of (B.5) and (B.8) (neglect of the first derivative term) it is apparent that Γ and $-h\left(\frac{f}{h}\right)_y$ play exactly equivalent roles in internal and in planetary waves, respectively. It then follows that solutions of (B.8) will have the same y dependence, W(y), as solutions of (B.5) if $-h\left(\frac{f}{h}\right)_y$ has the same functional dependence as Γ.

The simplest case for which this analogy holds is between internal waves in a fluid with a weakly exponential density profile contained between rigid horizontal planes and Rossby waves in a β-plane channel. In both cases (the Boussinesq's approximation being made for the internal wave case) the wave form is given by $\sin \frac{n\pi}{L}z$ ($\sin \frac{n\pi}{L}y$ for Rossby waves) where

$$\frac{n^2\pi^2}{L^2} = k^2\left(\frac{g\Gamma}{\omega^2} - 1\right)$$

$$\frac{n^2\pi^2}{L^2} = k^2\left(-\frac{\beta}{\sigma k} - 1\right) \qquad \text{(B.11)}$$

respectively.

A further example of this parallel is the analogy between internal waves on the boundary between two unbounded fluids [Lamb, (1945), p.370] and the non-divergent limit of the double Kelvin wave along a depth discontinuity [Rhines, (1969a)]. In both cases the wave amplitudes decay exponentially away from the discontinuity.

The extension of this analogy to include the finite amplitude cases is far more tenuous. For internal waves, the motion itself changes the density profile while for non-divergent planetary waves the (f/h) profiles are independent of the waves. There are, however, still similarities between the non-linear equations. For example, the equation for planetary waves of permanent form (4.26) given by

$$J\left(\frac{1}{\eta + H}\frac{\partial}{\partial s}\left[\frac{1}{\eta + H}\frac{\partial\psi}{\partial s}\right] + \frac{1}{\eta + H}\frac{\partial}{\partial y}\left[\frac{1}{\eta + H}\frac{\partial\psi}{\partial y}\right] - \frac{f}{\eta + H}\,,\ \psi\right) = 0 \qquad (B.12)$$

is similar in form to an equation obtained by Magaard (1965) for progressive internal waves of permanent form

$$J(\nabla^2\Psi - gz\frac{d\rho}{d\Psi}\,,\ \Psi) = 0 \qquad (B.13)$$

where

$$w = -\frac{1}{\sqrt{\rho}}\Psi_s\,, \quad u - c = \frac{1}{\sqrt{\rho}}\Psi_z, \quad s = x - ct\,. \qquad (B.14)$$

Solutions to (B.13) have been given by Magaard (1965) and Yih (1965, ch. III) and these solutions suggested the procedure leading to the possible solutions to (B.12) which were obtained in §4.2 .

Further extensions of the analogy to include interaction of waves with currents or long wave solutions are tenuous in the extreme. The analogy serves to suggest, from research already conducted for internal waves, directions in which investigations of planetary waves might proceed.

Appendix III

Glossary of symbols

α		angular phase speed on the sphere
β		$= \frac{df}{dy}$ on the β-plane
γ		angle between the axis of rotation and the axis of the co-ordinate system on the sphere
$\bar{\delta}$	in §3.1	$= 4\Omega^2R^2/gH$, the divergence parameter
δ		$= \beta^2L^2/gH$, the divergence parameter
ε		amplitude parameter
ζ	in §4.4	$= \exp(-\Lambda y)$
	in §4.5, §5.2	$= \sqrt[4]{4\delta}\ (f_\circ + y)$
	in §4.6	$= 2\left\|\frac{k}{a}\right\| (W_\circ - c_\circ + ay)$
η		surface elevation
θ,ϕ		co-latitude and longitude in the tilted co-ordinate system
θ',ϕ'		co-latitude and longitude relative to the axis of rotation
κ	in §4.5	$= (1 + 4c_\circ k^2)/(2c_\circ\sqrt{\delta})$
	in §5.2	$= 1/(2c_\circ\sqrt{\delta})$
		total wave number on the β-plane
λ	in §3.1	$= \sigma/2\Omega$, the non-dimensionalized frequency
	in §3.2	$= \frac{gH}{2\beta}(k^2 + \frac{k\beta}{\sigma})$
	in §4.4	$= \kappa/\Lambda$
	in §4.5	$= \frac{1}{2c_\circ\sqrt{\delta}} + \frac{(1 - \delta c_\circ)k^2}{2\sqrt{\delta}}$
	in §5.2, §5.3, §5.4	wavelength

λ_1, λ_2	in §4.5, §4.6	$= \pm (m^2\pi^2 - 3k^2)$
μ	in §3.1	$= - \cos\theta$
	in §4.6, §5.3, §5.4	$= \delta/\varepsilon$
ν		$= \frac{k^2}{\Lambda^2} + \frac{1}{4}$
$\tilde{\xi}$	in §1.2	relative vorticity vector
ξ	in ch. V	$= \sqrt{\varepsilon}\, s$
ρ		density
σ		radian frequency of planetary waves
ψ		stream function
ω		radian frequency of internal waves
Γ		$= - \frac{1}{\rho_\circ}\frac{d\rho_\circ}{dz}$
Λ		$= - \frac{1}{h}\frac{dh}{dy}$
Φ	in §4.6	$v_1 = \Phi(y) \sin ks$
	in ch. V	$v_1 = \Phi(y)\, g_\xi(\xi)$
$\bar{\Phi}$	in §4.6	$\Phi = \zeta \exp(-\zeta/2)\, \bar{\Phi}(\zeta)$
$\tilde{\Omega}$		the rotation vector of the earth
a	in §4.6, §5.3	$= \frac{du_\circ}{dy}$
a_i		integration and phase constants
b_i	in §5.3	coefficients of the Korteweg-deVries equation
c		phase speed on β-plane
d_i	in §4.2, §4.3	constants
e_i	in §5.2	coefficients of the Korteweg-deVries equation
f		$= f_\circ + \beta y$, the Coriolis parameter
g		gravitational acceleration

$g(\xi)$	in ch. V	$v_1 = \Phi(y)\ g_\xi(\xi)$
h		depth of fluid
k, ℓ		wave numbers in x and y directions
m, n		integers
p		pressure
r		radius
s	in §3.1	zonal wave number
	in ch.IV, ch.V	$= x - ct$, transformed x co-ordinate
s'	in §3.1	$= 2\Omega s/\sigma$
t		time
u, v, w		velocity components on β-plane
v_θ, v_ϕ, v_r		velocity components on the sphere
v'_θ, v'_ϕ, v_r		velocity components relative to the rotation axis on the sphere
x, y, z		co-ordinates on the β-plane
A_i, B_i		amplitude constants
$F(\psi)$		distribution of potential vorticity
H		depth
L		width of the β-plane channel
R		radius of the earth
V_o, W_o		basic uniform zonal flow
$W(z)$		z-dependence of the vertical velocity of internal waves
$Y(y)$	in §4.5	first order y-dependence of north-south velocity
$Z_i(y)$	in §4.5, §4.6	second order y-dependence of north-south velocity