A QUANTUM MECHANICAL TREATMENT OF THE RELATIVISTIC SCATTERING OF LIGHT BY THE SUN

by

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B.Sc., University of Manitoba, 1967

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE
in the department
OF

PHYSICS

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

SEPTEMBER, 1969

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ABSTRACT

This thesis concerns itself with the applicability of quantum field methods, in the fixed field approximation, to problems involving a weak gravitational field. After introducing general scattering relations, various classic problems are reviewed to illustrate various approaches to solving scattering problems. Newtonian and quantum mechanical field methods are illustrated using Coulomb scattering. Classical relativity is used to solve the bending of light rays by the sun. Finally, quantum field methods are used to solve the scattering of polarized photons by the sun. The additional problems of scattering of light by a mass distribution and by a rotating mass are calculable using this method.

CONTENTS

LIST OF FIGURES

ACKNOWLEDGEMENTS

I am indebted to Dr. F. A. Kaempffer for suggesting the topic of this thesis and for his generous guidance.

I am grateful to the National Research Council for the freedom their scholarship has given me.

INTRODUCTION

In all observations of the deflection of light by the gravitational field of the sun one measures the angle of deflection suffered by the light ray emitted from a distant star as it passes near the sun's surface. Accordingly, the classical theory of this effect is based on Fermat's Principle which prescribes the geodesic as the actual path taken by light between source and observer.

In most laboratory scattering experiments, on the other hand, one measures an intensity as a function of scattering angle. Accordingly, both the classical and the quantum mechanical theory of scattering in particle physics are aimed at yielding the differential scattering cross-section, i.e. the ratio between the scattered intensity into a given solid angle and the incident intensity per unit area.

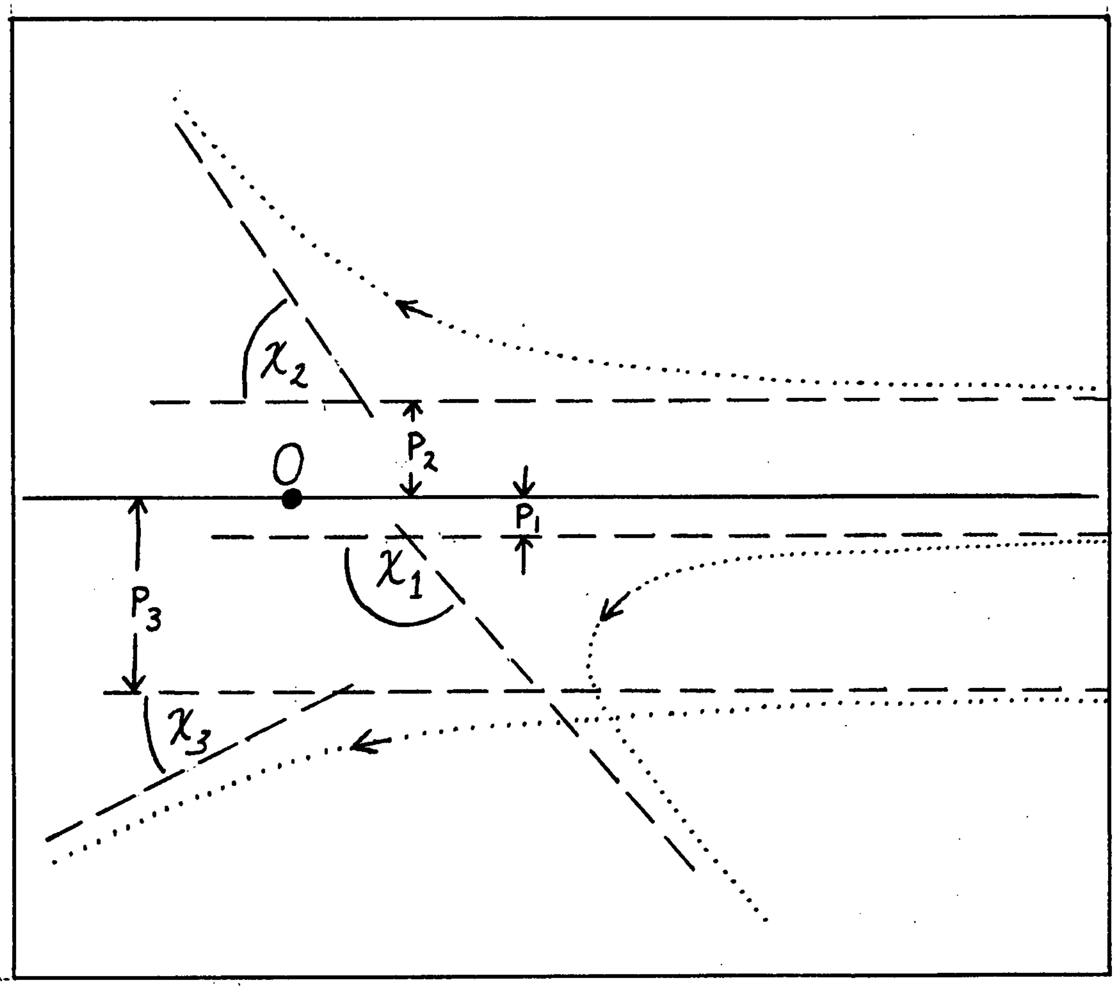
One can, of course, describe particle scattering also in terms of the scattering angle in the path of an incident particle. Even though it has no immediate experimental relevance, because it would require, for example, the experimentor to fire an α-particle with a given impact parameter past a nucleus; this alternative treatment of particle scattering is well-known and it is often used as a basis for the derivation of scattering cross-sections.

Alternatively, the gravitational deflection of light can be treated as a quantum mechanical scattering problem, in which the sun, for example, is treated as the scattering center which presents a scattering cross-section to an incident beam of photons. This alternative does not seem to be well-known, and it is the purpose of this thesis to remedy that deficiency in the literature on this subject.

To bring out the equivalence of the two alternative treatments of scattering in general, the general relations existing between the concept of the scattering angle and the concept of the scattering cross-section are exhibited in Section I. The classical example of a particle scattering is the Rutherford scattering experiment, whose theory is summarized in Section II. The classical theory is not suitable when the scattered particle possesses intrinsic attributes, such as spin, and a full quantum mechanical treatment of the scattered particle is advisable even though the scattering center may be treated in the so-called fixed field approximation, as is explained in Section III, with the scattering of a Dirac electron in a given Coulomb field serving as example. The classical Einstein theory of gravitational light scattering is reviewed in Section IV, with emphasis on the analogy to classical Rutherford scattering. Finally, the quantum mechanical treatment of gravitational photon scattering in the fixed field approximation is developed in Section V, and as in the case of the electron the intrinsic polarization property of the photon can be handled most appropriately in this way.

The present work should pave the way for the theoretical treatment of more elaborate problems, such as the effect of the solar mass distribution and of the solar rotation on the extrinsic and intrinsic attributes of incident photons, and the significance of such future work is briefly discussed in Section VI.

FIGURE 1.1



IMPACT PARAMETERS AND SCATTERING ANGLES

§I. BASIC RELATIONS BETWEEN SCATTERING CROSS-SECTIONS AND DEFLECTION ANGLES

Consider first the case in which the concept of a 'path' of a particle can be defined, and suppose one has a fixed scattering center (taken to be the origin of the coordinate system). If one is interested in the scattering of a uniform, initially parallel beam of identical, noninteracting particles incident upon the scattering region with initial velocity $\underline{v}_\infty$, then different particles in the beam will have different impact parameters p (see Figure 1.1) and hence will be scattered a different amount χ from their initial direction. The impact parameter is defined to be the perpendicular distance from the origin to the asymptotic initial path of the particle. If dN particles scatter per unit time through an angle between χ and $\chi+d\chi$, let

$$(1.1) \quad d\sigma \equiv \frac{dN}{n} ,$$

where n is the number of particles passing in unit time through a unit area of the beam cross section (assumed uniform). '$d\sigma$' is called the effective scattering cross-section. Assume that particles with impact parameters between $p_i(\chi)$ and $p_i(\chi)+dp_i(\chi)$ scatter through an angle between χ and $\chi+d\chi$. The subscript on p allows for the possibility of the relationship between p and χ not being one to one. Then

$$\text{(1.2)}\qquad dN = \sum_i 2\pi\, p_i \, dp_i \, n; \qquad d\sigma = \sum_i 2\pi\, p_i \, dp_i \, .$$

If, as usual, p and χ are one to one (which is true for the usual case of χ a monotonic, decreasing function of p),

$$\text{(1.3)}\qquad d\sigma = -2\pi p(\chi) \frac{\partial p}{\partial \chi} d\chi,$$

where the negative sign results from the decreasing nature of $\chi(p)$. Or, since the solid angle $d\Omega$ between χ and $\chi+d\chi$ is $2\pi \sin\chi \, d\chi$,

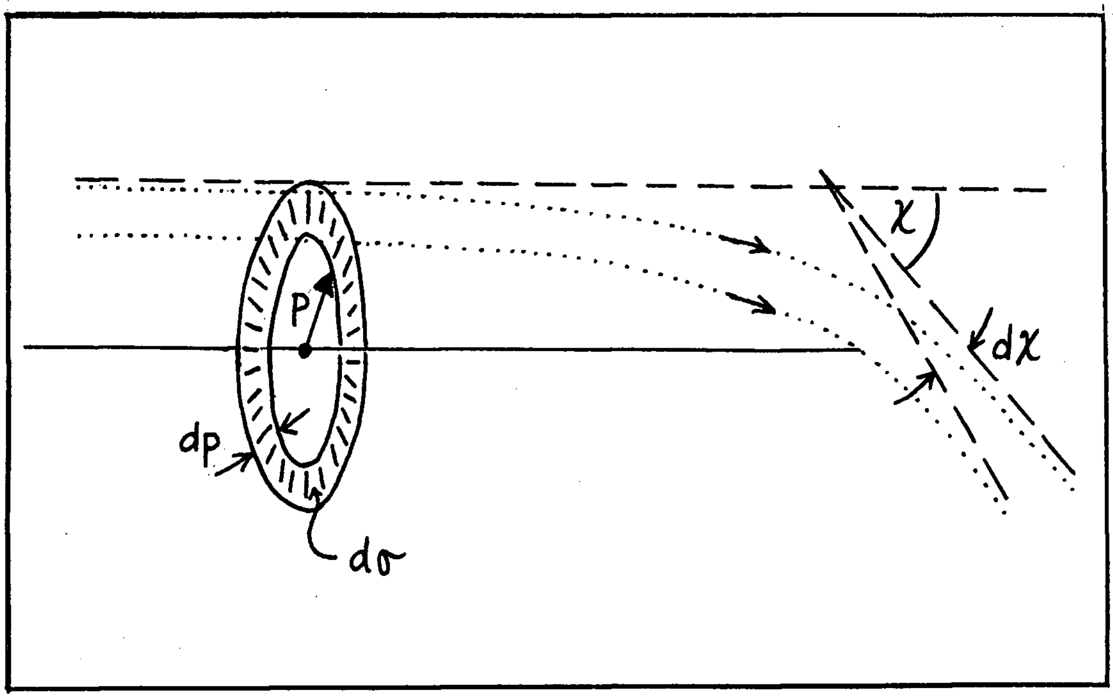
$$\text{(1.4)}\qquad d\sigma = \frac{-p(\chi)}{\sin\chi} \frac{\partial p(\chi)}{\partial \chi} d\Omega.$$

The case in which the angle of scattering is small is of interest. Suppose that the deflection angle is related to the impact parameter by

$$\text{(1.5)}\qquad \chi = \frac{A}{p},$$

where A is some constant. (This is a valid relation for large p in the case of Rutherford scattering as well as for Einstein scattering.) The particles deflected between χ and $\chi+d\chi$ are the ones that have impact parameters bet-

FIGURE 1.2



THE DIFFERENTIAL SCATTERING

CROSS - SECTION

ween p and p+dp (see Figure 1.2), where

$$(1.6) \quad p = \frac{A}{\chi}\,; \qquad |dp| = \frac{A}{\chi^2}\, d\chi\,.$$

This area, the ring between p and p+dp, is then, by definition, the differential cross-section for scattering into the angle between χ and $\chi+d\chi$; that is, into the solid angle $d\Omega = 2\pi \sin\chi \; d\chi \approx 2\pi\, \chi \; d\chi$. Thus

$$(1.7) \quad d\sigma = \left| 2\pi\, p\, dp \right| = \frac{2\pi A^2}{\chi^3}\,,$$

and

$$(1.8) \quad \frac{d\sigma}{d\Omega} = \frac{\left|2\pi p\; dp\right|}{2\pi \chi\, d\chi} = \frac{A^2}{\chi^4}\,.$$

Next consider the case of a scattered particle described quantum mechanically as an incident plane wave,

$$u(k_i, s_i)\; e^{-ik_i x}\,,$$

$$(1.9) \quad k_i x \equiv \omega_i x_0 - \underline{k}_i \cdot \underline{x}\,,$$

$$x \equiv (x_0 = t,\; \underline{x}),$$

representing a state $|k_i, s_i\rangle$, where k_i is the four-momentum ($\hbar = c = 1$) and s_i includes other quantum numbers such as spin. This wave, when entering the scattering region has a probability of being scattered into another (final) state

$\langle k_f, s_f|$ which again can be written as a plane wave. Under these conditions the amplitude for scattering from $|k_i, s_i\rangle$ to a different state $\langle k_f, s_f|$, in the first Born approximation has the form

$$(1.10)\quad \langle k_f, s_f|\, i \int \mathcal{L}_{int} d^4x \,|k_i, s_i\rangle \equiv \langle \underline{k}_f, s_f| M |\underline{k}_i, s_i\rangle\ \delta(\omega_f - \omega_i)$$

in a fixed, time-independent external field, where $\mathcal{L}_{int}$ is the Lagrangian density governing the interaction, and ω_i, ω_f are the energies of the scattered particle in the initial and final states (see Appendix A). The delta-function results from the integration over the time coordinate of the scalar product of the initial and final plane waves and ensures conservation of energy. However, if one neglects the recoil of the source, one does not have conservation of momentum. Using the relation, valid for any delta-function (see Appendix B),

$$(1.11)\quad \left[\delta(\omega_f - \omega_i)\right]^2 = \frac{\delta(\omega_f - \omega_i)}{2\pi} \int dt,$$

one has the transition probability per unit time for fixed initial and final states,

$$(1.12)\quad w_{fi} = \frac{\left|\langle k_f, s_f| \int \mathcal{L}_{int} d^4x \,|k_i, s_i\rangle\right|^2}{\int dt}$$
$$= \frac{\left|\langle \underline{k}_f, s_f| M |\underline{k}_i, s_i\rangle\right|^2}{2\pi}\ \delta(\omega_f - \omega_i)\ .$$

The incident intensity I is defined as the number of incident particles per unit time per unit area of cross section of the beam. Thus, I equals the particle density times the incident speed. If one normalizes the incident plane wave to yield one particle per volume V,

$$(1.13)\quad I = \frac{v}{V} ,$$

where v is the incident speed. The cross-section for scattering into a definite final state is

$$(1.14)\quad \frac{w_{fi}}{I} = w_{fi} \ \frac{V}{v} .$$

The differential cross-section for scattering into the set of final states between $\underline{k}_f$ and $\underline{k}_f+d\underline{k}_f$ numbering

$$(1.15)\quad dN_{\underline{k}_f} = \frac{V}{(2\pi)^3} \ d\underline{k}_f$$

is

$$(1.16)\quad d\sigma = w_{fi} \ \frac{V}{v} \ dN ,$$

$$(1.17)\quad d\sigma = \frac{\delta(\omega_f - \omega_i)}{(2\pi)^4} \ \frac{V^2}{v} \left| \langle k_f, s_f | M | k_i, s_i \rangle \right|^2 \ d\underline{k}_f .$$

If one can assume that the interaction is spherically symmetric then the cross-section for scattering into the solid angle $d\Omega$ is given by

$$(1.18)\quad \frac{d\sigma}{d\Omega} = \frac{\delta(\omega_f - \omega_i)\, V^2}{(2\pi)^4\, v} \left| \langle \underline{k}_f, s_f | M | \underline{k}_i, s_i \rangle \right|^2 \underline{k}_f^{\,2}\, d|\underline{k}_f| \ ,$$

where $d\underline{k}_f$ has been replaced by the expression integrated over $d\Omega$, i.e., $\underline{k}_f^{\,2}\, d|\underline{k}_f|\, d\Omega$ and the rest of equation (1.17) has been left unchanged due to the spherical symmetry. Since

$$(1.19)\quad \omega_f^{\,2} = m^2 + \underline{k}_f^{\,2}$$

and

$$(1.20)\quad \omega_f\, d\omega_f = |\underline{k}_f|\, d|\underline{k}_f| \ ,$$

equation (1.18) becomes,

$$(1.21)\quad \frac{d\sigma}{d\Omega} = \frac{\delta(\omega_f - \omega_i)\, V^2}{(2\pi)^4\, v} \left| \langle \underline{k}_f, s_f | M | \underline{k}_i, s_i \rangle \right|^2 |\underline{k}_f|\ \omega_f\, d\omega_f \ .$$

If the detector responds only to particles scattered into the solid angle $d\Omega(\theta,\phi)$, one must integrate over all $d\omega_f$ keeping θ, ϕ constant. Since the incident velocity may be expressed relativistically as

$$(1.22)\quad v = \frac{k}{\omega} \ ,$$

the integration over $d\omega_f$ can be carried out at once with the result

$$(1.23)\quad \frac{d\sigma}{d\Omega} = \frac{V^2\omega_i}{(2\pi)^4}\left|\langle \underline{k}_f, s_f|M|\underline{k}_i, s_i\rangle\right|^2_{\omega_f=\omega_i} \quad .$$

The above quantum mechanical analysis has been based on the stated assumption that the plane waves are normalized to yield one particle per volume V. This normalization is not Lorentz invariant since the volume V is contracted along the direction of motion relative to a frame of reference. If one wishes to have an invariant probability, it is convenient to normalize the wave functions to one electron per invariant volume Vm/ω. With this normalization the following crucial equations must be rewritten as indicated.

$$(1.24)\ \text{was}\ (1.13)\quad I = \frac{v\,\omega}{m\,V} \ .$$

$$(1.25)\ \text{was}\ (1.15)\quad dN_{\underline{k}_f} = \frac{Vm}{\omega}\frac{d\underline{k}_f}{(2\pi)^3} \ .$$

Hence, equation (1.23) becomes

$$(1.26)\quad \frac{d\sigma}{d\Omega} = \frac{V^2m^2}{(2\pi)^4}\left|\langle \underline{k}_f, s_f|M|\underline{k}_i, s_i\rangle\right|^2_{\omega_f=\omega_i} \quad .$$

The initial and final states may contain information about the polarization properties of the scattered particle. If they are not observed, the expression (1.23) has got to be subjected to appropriate sums and averages over the respective polarization states.

§II. CLASSICAL RUTHERFORD SCATTERING

The laws of motion for a mechanical system can be derived by application of Hamilton's principle (or the principle of least action). This approach necessitates that every mechanical system be characterized by a function, called the Lagrangian, which, for a system with s degrees of freedom, one writes $L(q_1, q_2, \ldots; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s; t)$, or equivalently $L(q;\dot{q};t)$ for arbitrary degrees of freedom, where the q's are the coordinates, the $\dot{q}$'s are the conjugate velocities, and t is the parameter representing time. The condition that the action,

$$(2.1) \qquad \mathscr{S} \equiv \int_{t_1}^{t_2} L(q;\dot{q};t)dt,$$

is an extremum for the actual motion, q(t), of the system in the time interval (t_1, t_2) is applied to yield the required equations of motion. One wants $\delta\mathscr{S} = 0$ for the actual path when q is varied by a small amount δq (and correspondingly $\dot{q}$ by $\delta\dot{q}$) while keeping $\delta q(t_1) = \delta q(t_2) = 0$, i.e. fixing the endpoints of the trajectory. For one degree of freedom one gets,

$$\delta\mathscr{S} = \delta\int_{t_1}^{t_2} L(q;\dot{q};t)\ dt = 0\ ,$$

$$(2.2) \qquad \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right)\ dt = 0,$$

$$\left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\ \delta q\ dt = 0,$$

where the first term is zero because the endpoints of the trajectory are fixed. Since the result is true for arbitrary δq,

$$(2.3) \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

For a system with s degrees of freedom, the s different functions $q_i(t)$ must be varied independently to yield

$$(2.4) \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \ i = 1,2,\ldots,s.$$

These are the Euler, or Lagrange's equations of motion for the system.

Some important 'integrals of motion' can be derived from the homogeneity and isotropy of space and time. The energy of the system, E,

$$(2.5) \quad E \equiv \sum_i q_i \frac{\partial L}{\partial \dot{q}_i} - L,$$

the momentum of the system, $\underline{P}$,

$$(2.6) \quad \underline{P} \equiv \sum_a \frac{\partial L}{\partial \underline{r}_a},$$

(where $\underline{r}_a$ is the position of particle 'a'), and the angular momentum, $\underline{M}$,

$$(2.7) \quad \underline{M} \equiv \sum_a \underline{r}_a \times \underline{p}_a,$$

are all conserved (see Appendix C).

For a system of two particles interacting through a central potential $U(|\underline{r}_1-\underline{r}_2|)$,

$$(2.8)\quad L = \tfrac{1}{2}m_1\dot{\underline{r}}_1{}^2 + \tfrac{1}{2}m_2\dot{\underline{r}}_2{}^2 - U(|\underline{r}_1-\underline{r}_2|).$$

Let $\underline{r} \equiv \underline{r}_1-\underline{r}_2$ and define the origin of the coordinate system such that $m_1\underline{r}_1 + m_2\underline{r}_2 = 0$. Then

$$(2.9)\quad \underline{r}_1 = \frac{m_2\underline{r}}{m_1+m_2} \quad ; \quad \underline{r}_2 = \frac{-m_1\underline{r}}{m_1+m_2} ,$$

and

$$(2.10)\quad L = \tfrac{1}{2}m\dot{\underline{r}}^2 - U(r) , \quad m \equiv \frac{m_1m_2}{m_1-m_2} .$$

Thus the problem of the motion of two particles interacting through a central force is mathematically equivalent (by transforming to center of mass coordinates) to the motion of one particle in a given central field $U(r)$.

In cylindrical coordinates r, ϕ, z, equation (2.10) becomes

$$(2.11)\quad L = \tfrac{1}{2}m\,(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - U(r).$$

Since $\underline{M}$ ($\equiv \underline{r} \times \underline{p}$) is conserved and perpendicular to $\underline{r}$, the motion is planar. Choose the z-axis such that $\dot{z} = 0$. Thus,

$$(2.12)\quad L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r).$$

Since M is the momentum conjugate to ϕ, see equation (2.6),

$$(2.13) \quad M \equiv \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}.$$

Since

$$(2.14) \quad \frac{d}{dt} \; \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} = 0,$$

$$(2.15) \quad M = mr^2\dot{\phi}$$

is constant. From the definition of E, see equation (2.5),

$$E = \dot{r}\,\frac{\partial L}{\partial \dot{r}} + \dot{\phi}\,\frac{\partial L}{\partial \dot{\phi}} - \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r) \;,$$

$$(2.16) \quad E = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}\,\frac{M^2}{mr^2} + U(r) \;,$$

$$\frac{dr}{dt} = \left(\frac{2}{m}\Big[E-U(r)\Big] \; \frac{M^2}{m^2r^2} \right)^{\frac{1}{2}} .$$

Also, from equation (2.13),

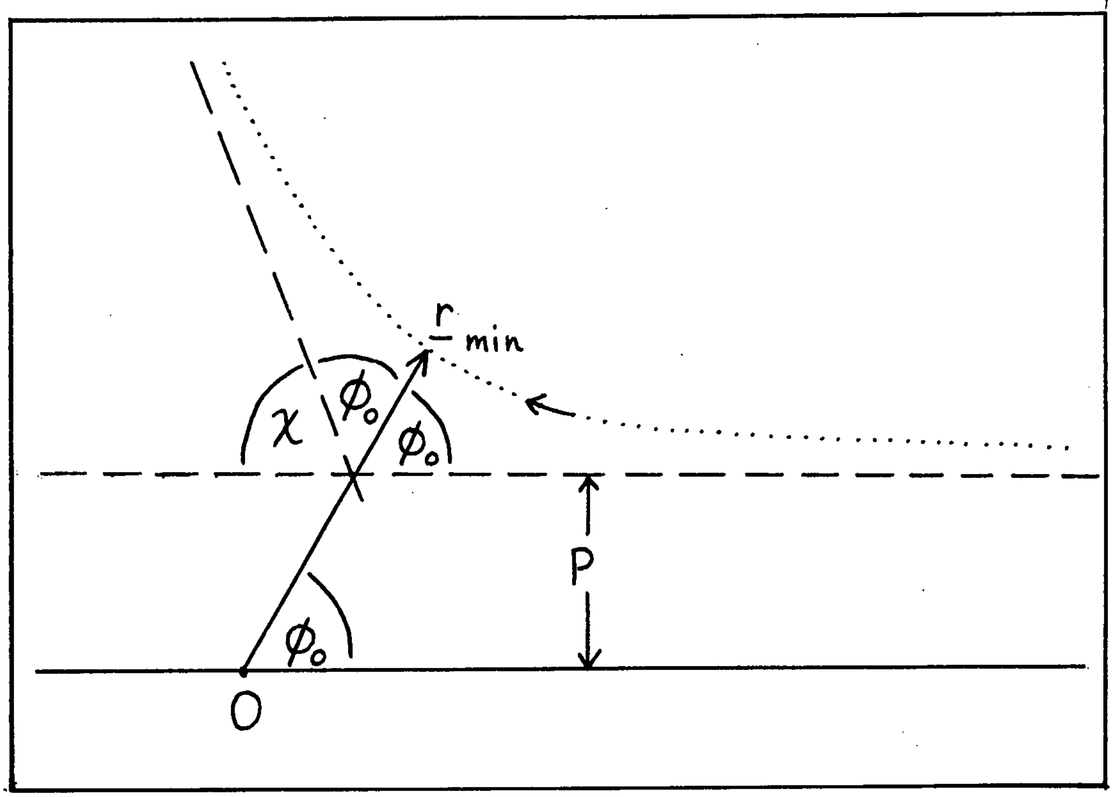
$$(2.17) \quad d\phi = \frac{Mdt}{mr^2} \;;$$

and using the third of equations (2.16), one gets

$$(2.18) \quad d\phi = \frac{M}{mr^2} \left(\frac{2}{m}\Big[E-U(r)\Big] - \frac{M^2}{m^2r^2} \right)^{-\frac{1}{2}} dr \;.$$

Thus,

FIGURE 2.1



COULOMB SCATTERING

$$(2.19)\quad \phi = \int \frac{M\,dr/r^2}{(2m\left[E-U(r)\right]-M^2/r^2)^{\frac{1}{2}}} + \text{constant.}$$

In a central field the path of a particle is symmetric about r_{min}. For an inverse square law force field the path is a conic section with the center of force as a focus, in which case the symmetry is clear. From Figure 2.1,

$$(2.20)\quad \chi = \left|\pi - 2\phi_o\right| .$$

From equation (2.19),

$$(2.21)\quad \phi_o = \int_{r_{min}}^{\infty} \frac{M\,dr/r^2}{(2m\left[E-U(r)\right]-M^2/r^2)^{\frac{1}{2}}} .$$

Since E and M remain constant, one uses their initial values,

$$(2.22)\quad \begin{aligned} E &= \tfrac{1}{2}mv_{\infty}^2 , \\ M &= mpv_{\infty} , \end{aligned}$$

where p is the impact parameter.

For Rutherford scattering the force field is Coulombic,

$$(2.23)\quad U(r) = \alpha/r , \quad \alpha = Ze^2.$$

Combining equations (2.21) to (2.23) one gets,

$$(2.24) \quad \phi_o = \int_{r_{min}}^{\infty} \frac{(p/r^2)\, dr}{\left[1-(p^2/r^2)-(2\alpha/mrv_\infty^2)\right]^{\frac{1}{2}}} ,$$

$$(2.25) \quad \phi_o = \cos^{-1} \frac{(\alpha/mv_\infty p)}{\left[1+(\alpha/mv_\infty^2 p)\right]^{\frac{1}{2}}} .$$

Or, solving equation (2.25) for p,

$$(2.26) \quad p^2 = \frac{\alpha^2}{m^2 v_\infty^4} \tan^2 \phi_o ,$$

which becomes, using equation (2.20),

$$(2.27) \quad p = \frac{\alpha}{mv_\infty^2} \cot \left(\tfrac{1}{2}|\chi|\right) .$$

Hence, applying the result of equation (1.4),

$$(2.28) \quad d\sigma = \left(\frac{\alpha}{2mv_\infty^2}\right)^2 \frac{d\Omega}{\sin^4(\frac{1}{2}|\chi|)} .$$

Equation (2.28) gives the cross-section in the frame of reference in which the center of mass is at rest. The transformation to the laboratory frame is accomplished by the formula,

$$(2.29) \quad \tan \theta_1 = \frac{m_2 \sin \chi}{m_1+m_2\cos \chi} , \quad \theta_2 = \tfrac{1}{2}(\pi - \chi),$$

where θ_1 and θ_2 are the angles between the directions of motion after the collision and the direction of impact. The subscript '2' denotes the particle which was originally at rest in the laboratory. Using the fixed field approximation, i.e., $m_1 \ll m_2$, then $m \approx m_1$, and $\chi \approx \theta_1$. Thus equation (2.28) becomes

$$(2.30) \quad \frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{4E_1}\right)^2 \frac{1}{\sin^4(\frac{1}{2}\theta_1)} ,$$

where E_1 is the incident kinetic energy. For small angle scattering one can replace sin(X) by X and hence the differential scattering cross-section in the fixed field approximation, in the limit of small angle scattering is

$$(2.31) \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} \approx \left(\frac{4Ze^2}{E_1}\right) \frac{1}{\theta_1{}^4} .$$

That is, the factor A of equation (1.8) is

$$(2.32) \quad A = \frac{4Ze^2}{E_1} .$$

For Einstein scattering (see Section IV),

$$(2.33) \quad A = 4GM ,$$

where M is the solar mass. The following 'rule of thumb' is apparent. To go from electrodynamics to gravidynamics one must replace e^2 by $GM\omega$ (in units such that $\hbar = c = 1$), or in the cgs-system of units $e^2/\hbar c$ by $GM\omega/c^3$, where ω is some characteristic frequency of the system.

Finally, it should be noted that the scattering of particles with intrinsic properties such as spin is not handled by the methods of this section. The next section illustrates a full quantum mechanical treatment of a scattering problem involving spin quantities.

§III. QUANTUM MECHANICAL RUTHERFORD SCATTERING

The classical theory of scattering, as developed in Section II, is not capable of handling the scattering of particles with intrinsic properties such as spin. In this section, the scattering of a Dirac electron by a given Coulomb field is treated, using the fixed field (or external field) approximation and the general scattering relations developed in Section I, to illustrate how these intrinsic attributes of particles are handled.

When treating a scattering problem, one generally is concerned with the interaction of two fields. The coupling of the systems is achieved by postulating the existance of a coupling term in the field equations which depends on the field variables of both fields. In some problems it is adequate to represent one field by a quantized field, involving creation and destruction (absorption) operators, while the other field is treated as a given classical function of the space-time coordinates. For the classically given field we need no equation of motion, since it is assumed to be the given space-time function; this is a great simplification.

Under certain circumstances the scattering of an electron by the field of a nucleus can be treated as the motion of a Dirac electron in a given, fixed Coulomb field. The conditions necessary to allow the concept of a fixed field to be valid can most easily be seen when one recalls the results of classical two-body elastic scattering. As in

Section II, (see equations (2.9), (2.29), and the paragraph following (2.29)), the initially stationary mass m_2 remains essentially stationary (i.e., there is a negligible momentum transfer to m_2) if $m_2 \gg m_1$. Equally, the condition that the reduced mass, $m \equiv m_1 m_2/(m_1 - m_2)$, (center of mass fixed) is approximately m_1 is that $m_2 \gg m_1$. Hence, the condition for the validity of the fixed field approximation is $m_2 \gg m_1$, when the fixed center of mass is nearly coincident with the position of m_2. In the relativistic case the condition of a relatively large target mass can often not be met at the same time as the relativistic limit condition $k^2 \gg m^2$ for the incident particle. In that case one must apply the condition of negligible momentum transfer to the target. This condition is met if the angle of scattering is small; and hence a small change of momentum occurs. In all the cases that shall be discussed, this limitation will not be serious since in all cases the scattering is very strong in the forward direction , that is, $d\sigma/d\Omega \approx A/\theta^4$.

The approximation in which the field of the nucleus is treated as a classically given Coulomb field function rather than a quantized one (involving creation and destruction operators) can be obtained from a completely quantized theory by identifying the expectation values of the field and current operators with the classical quantities (see Jauch and Rohrlich, chapter 14, 1955). The approximation is accurate when the fluctuations about the expectation values are small compared to the expectation value itself. The term 'Coulomb scattering' generally refers to the scattering of an electron by a given, fixed Coulomb field, to all orders of this field (in this Section only the first order is calculated). The effect of the presence of photons (real or virtual) is generally referred to as 'radiative correction' (corresponding to the effect of the so-called radiation field in classical electrodynamics) and is here completely neglected.

The electron is described by a plane wave solution to the Dirac equation (3.1) for a free electron.

$$(3.1) \quad (\gamma^\mu \, \partial/\partial x^\mu - m)\, \psi(x) = 0,$$

where $\sum_{\mu=0}^{3}$ is implied. The wave function $\psi(x)$ is a 4-component Lorentz spinor

$$(3.2) \quad \psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix},$$

the γ^μ, ($\mu = 0,1,2,3$) are 4x4 matrices, operating on the spinor ψ, obeying

$$(3.3) \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \quad , \quad g^{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

the rest mass of the particle is denoted by m, and the spin is represented by the operators (see A.S. Davidov, 1966)

(3.4) $\sigma_z = -i\, \gamma_x \gamma_y$ (cyclic).

Describing the electron as a quantized field, one can write $\psi(x)$ and $\psi^{\dagger}(x)$ as the Fourier series (see F. Mandl, 1959),

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_{\underline{k},r} \left(u(\underline{k},r)a(\underline{k},r)e^{+ikx} + u^{*}(\underline{k},r)b^{\dagger}(\underline{k},r)e^{-ikx}\right),$$

(3.5)

$$\psi^{\dagger}(x) = \frac{1}{\sqrt{V}} \sum_{\underline{k},r} \left(u^{*}(\underline{k},r)a^{\dagger}(\underline{k},r)e^{-ikx} + u(\underline{k},r)b(\underline{k},r)e^{+ikx}\right),$$

with $\psi(x)$ normalized to one particle per volume V. The operators $a(\underline{k},r)$ and $b(\underline{k},r)$, associated with the positive frequency part, allow interpretation as absorption (destruction) operators, while the operators $a^{\dagger}(\underline{k},r)$ and $b^{\dagger}(\underline{k},r)$, associated with the negative frequency part, become creation operators.

The following lemmas concerning the algebra of the γ-matrices are useful.

Lemma 1: Trace $\gamma^{\alpha}\gamma^{\beta}\gamma^{\delta} = 0$.

Proof: $\mathrm{Tr}\, \gamma^{\alpha}\gamma^{\beta}\gamma^{\delta} = \mathrm{Tr}\, \gamma^{\alpha}\gamma^{\beta}\gamma^{\delta}\gamma^{5}\gamma^{5}$, $\gamma^{5} \equiv i\,\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}$;

$= \mathrm{Tr}\, \gamma^{5}\gamma^{\alpha}\gamma^{\beta}\gamma^{\delta}\gamma^{5}$,

$= \mathrm{Tr}\, (-1)^{3}\, \gamma^{\alpha}\gamma^{\beta}\gamma^{\delta}\gamma^{5}\gamma^{5}$,

using the property $\gamma^{\mu}\gamma^{5} + \gamma^{5}\gamma^{\mu} = 0$.

Lemma 2: Trace $\gamma^\mu \gamma^\nu = 4\, g^{\mu\nu}$.

Proof: $\mathrm{Tr}\ \gamma^\mu \gamma^\nu = \mathrm{Tr}\ \gamma^\nu \gamma^\mu$,

$= \frac{1}{2}\ \mathrm{Tr}\ (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$,

$= 4\ g^{\mu\nu}$.

Lemma 3: Trace $\gamma^\mu \gamma^\sigma \gamma^\nu \gamma^\rho = 4(g^{\mu\sigma} g^{\nu\rho} - g^{\mu\nu} g^{\sigma\rho} + g^{\mu\rho} g^{\nu\sigma})$.

Proof: Using $\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu + 2\ g^{\mu\nu}$,

Trace $\gamma^\mu \gamma^\sigma \gamma^\nu \gamma^\rho = 2g^{\mu\sigma}\mathrm{Tr}\,\gamma^\nu\gamma^\rho - \mathrm{Tr}\ \gamma^\sigma \gamma^\mu \gamma^\nu \gamma^\rho$,

$= 8\ g^{\mu\sigma} g^{\nu\rho} - 8\ g^{\mu\nu} g^{\sigma\rho} + \mathrm{Tr}\ \gamma^\sigma \gamma^\nu \gamma^\mu \gamma^\rho$,

$= 8\ g^{\mu\sigma} g^{\nu\rho} - 8\ g^{\mu\nu} g^{\sigma\rho} + 8\ g^{\mu\rho} g^{\sigma\nu} - \mathrm{Tr}\, \gamma^\sigma \gamma^\nu \gamma^\rho \gamma^\mu$.

For purposes of this problem, this completes the quantum mechanical description of the free electron.

The field of the nucleus is described as a classical field, $A_\mu(x)$. Use is made of the Fourier series

$$(3.6)\quad A_\mu(\underline{k}_i - \underline{k}_f) = \int d^3\underline{x}\ A_\mu(x)\ e^{i(\underline{k}_i - \underline{k}_f)\cdot \underline{x}} \quad .$$

In a full quantum mechanical treatment, this plane wave decomposition (or its inverse) would be written as the sum of positive and negative frequency parts (in close analogue to equations (3.5)) and the coeffecients $A_\mu(\underline{k})$ and $A^*_\mu(\underline{k})$ would become creation and destruction operators. In the external field approximation, however, the $A_\mu(\underline{k})$'s are so-called c-numbers, not operators.

The Dirac equation for a free particle (3.1) and its adjoint equation can be derived from the Lagrangian density

$$(3.7)\quad \mathcal{L}_{\text{Dirac}} = -\psi^\dagger(x) \left[\gamma^0\, (\gamma^\mu\ \partial/\partial x^\mu\ - m)\right] \psi(x),$$

by application of the Euler (Lagrange) equations

(3.8)
$$\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi_{\alpha}(x)}{\partial x^{\nu}}\right)}\right) - \frac{\partial \mathcal{L}}{\partial \psi_{\alpha}(x)} = 0 ,$$

$$\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi^{\dagger}_{\alpha}(x)}{\partial x^{\nu}}\right)}\right) - \frac{\partial \mathcal{L}}{\partial \psi^{\dagger}_{\alpha}(x)} = 0 ,$$

which are derived in analogue to equations (2.4), treating $\psi_{\alpha}(x)$ and $\psi^{\dagger}_{\alpha}(x)$, ($\alpha$=0,1,2,3) as independent variables.

In classical electrodynamics one can derive the Lagrangian for a point charge interacting with a fixed electromagnetic field from the Lagrangian of the free point charge by substituting $p_{\mu}-eA_{\mu}$ for p_{μ} in the free Lagrangian. The Lagrangian describing the Dirac electron in a fixed electromagnetic field A_{μ} is similarily obtained by substituting $i\,\partial/\partial x^{\mu} - eA_{\mu}$ for $i\,\partial/\partial x^{\mu}$ in equation (3.7). The electromagnetic interaction term in the Lagrangian density is then

(3.9) $$\mathcal{L}_{int} = ie\, \psi^{\dagger}(x) \left[\gamma^{o}\gamma^{\mu} A_{\mu}\right] \psi(x) .$$

If one expands the Lagrangian density (3.9) according to equations (3.5) one gets two types of terms. Type one will contain one each of a creation and an absorption operator, these terms shall temporarily be retained since they describe events in which there is no change of particle number; type two will contain two operators of the same kind (i.e., both creation or both destruction). Terms of type two describe processes in which pair creation or pair annihilation take place; since these are processes not being considered in the present elastic electron scattering problem, such terms shall be neglected. Of the two terms of type one, only the one term involving electron creation and electron absorption ($a^\dagger$ and a) is retained; the term involving positron operators is of no interest. Thus the interaction Lagrangian density for elastic electron scattering can be written

$$(3.10)\quad \mathcal{L}_{int} = ie\,\frac{1}{V}\sum_{\substack{\underline{k}',r'\\ \underline{k}'',r''}} (u'^{*} a'^{\dagger}\, e^{-ik'x}\,\gamma^{o}\gamma^{\mu} A_{\mu}(x) u'' a''\, e^{+ik''x}).$$

Thus, using result (1.10), (see also Appendix A), the amplitude A_{fi} for scattering from $|k_i,s_i\rangle \equiv |\Phi_i\rangle$ to $\langle k_f,s_f| \equiv \langle\Phi_f|$ is given by

$$(3.11)\quad A_{fi} = \langle \Phi_f \,|\, i\int \mathcal{L}_{int}\, d^4x \,|\, \Phi_i\rangle$$

in the first Born approximation.

$$(3.12)\quad A_{fi} = \frac{-ie}{V}\int dx \sum_{\substack{\underline{k}',r'\\ \underline{k}'',r''}} u^{*}(\underline{k}',r')\, e^{-ik'x}\,\gamma^{o}\gamma^{\mu} A_{\mu}(x) u(\underline{k}'',r'')$$

$$e^{+ik''x}\langle\Phi_o, a^{\dagger}(\underline{k}_f,r_f)\, a^{\dagger}(\underline{k}',r')\, a(\underline{k}'',r'')\, a(\underline{k}_i,r_i)\, \Phi_o\rangle,$$

$$(3.13)\quad A_{fi} = \frac{-ie}{V} u^*(\underline{k}_i, r_i)\gamma^0\gamma^\mu A_\mu(\underline{k}_i - \underline{k}_f) u(\underline{k}_f, r_f) 2\pi\,\delta(\omega_i - \omega_f)\ ,$$

$$A_\mu(\underline{k}_i - \underline{k}_f) = \int d^3x\ A_\mu(x)\ e^{i(\underline{k}_i - \underline{k}_f)\cdot\underline{x}}\ .$$

Using relations (1.11) and (1.12),

$$(3.14)\quad w_{fi} = \frac{2\pi\ e^2}{V^2}\ (\bar{u}_f \gamma^\mu u_i \bar{u}_i \gamma^\nu u_f)\ A_\mu(\underline{k}_i - \underline{k}_f)$$
$$A_\nu(\underline{k}_i - \underline{k}_f)\ \ \delta(\omega_i - \omega_f),$$

using the fact that the A_μ are real.

Using equation (1.23), the differential scattering cross-section for electrons scattered elastically by a fixed electromagnetic field is

$$(3.15)\quad \frac{d\sigma}{d\Omega} = \frac{e^2\,\omega_i^{\,2}}{(2\pi)^2}\left[B\ A_\mu(\underline{k}_i - \underline{k}_f)\ A_\nu(\underline{k}_i - \underline{k}_f)\right]_{\omega_i = \omega_f}\ ,$$

where $B \equiv (\bar{u}_f \gamma^\mu u_i\ \bar{u}_i\ \gamma^\nu u_f)$.

If the incident beam of electrons is unpolarized, one must average over the two possible spin states s_i=1,2. If the detector does not distinquish between spin states, one must as well sum over the final spin states s_f=1,2. Thus, B of equation (3.15) becomes $\overline{B}$, B averaged and summed,

$$(3.16)\quad \overline{B} = \frac{1}{2} \sum_{s_i, s_f = 1}^{2} \bar{u}(\underline{k}_f, s_f)\ \gamma^\mu u(\underline{k}_i, s_i)\bar{u}(\underline{k}_i, s_i)\ \gamma^\nu\ u(\underline{k}_f, s_f).$$

But using

$$(3.17) \quad \sum_{r=1} \overline{u}_{\alpha}(\underline{k},r)\; u_{\beta}(\underline{k},r) = \frac{1}{2\omega}\,(\gamma^{\sigma} k_{\sigma} + m)_{\beta\alpha} \;,$$

where $\frac{1}{2\omega}(\gamma^{\sigma} k_{\sigma} + m) = \Lambda$ is the so-called energy projection operator,

$$(3.18) \quad \overline{B} = \frac{1}{8\omega^2} \sum_{\alpha,\beta} \gamma^{\mu} \left[\gamma^{\sigma} k_{i\sigma} + m\right]_{\alpha\beta} \gamma^{\nu} \left[\gamma^{\rho} k_{f\rho} + m\right]_{\beta\alpha} \;,$$

$$(3.19) \quad \overline{B} = \frac{1}{8\omega^2}\, \mathrm{Tr}\left[\gamma^{\mu}(\gamma^{\sigma} k_{i\sigma} + m)\; \gamma^{\nu}(\gamma^{\rho} k_{f\rho} + m)\right] .$$

Specializing to the case of a fixed, static, Coulomb field given by

$$(3.20) \quad A_o(x) = \frac{Ze}{|\underline{x}|} \;, \quad A_k = 0 \;, \quad (k = 1,2,3),$$

in the laboratory frame, or

$$(3.21) \quad A_o(\underline{k}_i - \underline{k}_f) = \frac{4\pi Ze}{|\underline{k}_i - \underline{k}_f|} \quad \frac{2\pi Ze}{k^2(1-\cos\theta)} \;,$$

(using $\underline{k}\cdot\underline{k} = k^2 \cos\theta$, $\omega^2 = k^2 + m^2$), equation (3.15) becomes,

$$(3.22) \quad \frac{d\overline{\sigma}}{d\Omega} = \frac{e^2 \omega_i^{\,2}}{(2\pi)^2}\, \overline{B}\, \frac{(2\pi)^2 Z^2 e^2}{k^2(1-\cos\theta)^2} \Bigg|_{\substack{\mu = \nu = 0 \\ \omega_i = \omega_f}} .$$

Using lemmas 1-3 above, equation (3.19) becomes

$$(3.23) \quad \overline{B} = \frac{1}{2\omega_i^{\,2}}\,(2\omega_i^{\,2} - k_i k_f + m^2) \;,$$

$$(3.24)\quad \overline{B} = \frac{1}{\omega_i^2}\left(m^2 + \frac{k^2}{2}(1+\cos\theta)\right),$$

$$(3.25)\quad \overline{B} = \frac{m^2}{\omega_i^2}\left(1 + \frac{k^2}{m^2}\cos^2\frac{\theta}{2}\right),$$

using $k_i k_f = \omega^2 - k^2 \cos\theta$. Thus,

$$(3.26)\quad \frac{d\overline{\sigma}}{d\Omega} = \left(\frac{Ze^2 m}{2k^2 \sin^2\frac{\theta}{2}}\right)^2 \left(1 + \frac{k^2}{m^2}\cos^2\frac{\theta}{2}\right).$$

If equations (3.5) had been normalized to ω/m particles per volume V, i.e., multiply (3.5) by an additional factor $\sqrt{\omega/m}$, one would have had to use equation (1.26) instead of (1.23), to yield in place of (3.15) the following:

$$(3.27)\quad \frac{d\sigma}{d\Omega} = \frac{e^2 m^2}{(2\pi)^2}\left[B\ A_\mu\ A_\nu\right]_{\omega_i = \omega_f}.$$

Instead of equation (3.17) one would have

$$(3.28)\quad \sum_{r=1}^{2} \overline{u}_\alpha(\underline{k},r)\ u_\beta(\underline{k},r) = \frac{1}{2m}(\gamma^\sigma k_\sigma + m).$$

Equations (3.18), (3.19), and (3.23) would contain the factor $1/2m^2$ rather than the factor $1/2\omega^2$. Beyond equation (3.25) the two normalizations lead to identical expressions.

For the non-relativistic limit, $k^2 \ll m^2$, one can write m for ω and mv for k and hence, using $E_k \equiv k^2/2m$,

$$(3.29)\quad \frac{d\overline{\sigma}}{d\Omega} = \left[\frac{Ze^2}{4E_k \sin^2\frac{\theta}{2}}\right]^2,$$

the usual classical scattering cross-section in the fixed field approximation; see also equation (2.28). For small scattering angle, the result in the non-relativistic limit is identical to the result (2.31).

The relativistic limit may also be of interest since in the case of photons being scattered this is the only case. If $k^2 \gg m^2$, ω^2 is nearly equal to k^2 and

$$(3.30) \quad \frac{d\bar{\sigma}}{d\Omega} = \left(\frac{Ze^2}{2\omega}\right)^2 \frac{\cos^2\frac{\theta}{2}}{\sin^4\frac{\theta}{2}} \, .$$

Due to the fixed field approximation one must limit the applicability of (3.30). Clearly, the condition that the mass (not rest-mass) of the incident particle be much smaller than the rest-mass of the target nucleus is not likely to be compatible with the relativistic limit condition, $k^2 \gg m^2$, for the incident particle. The condition that negligible momentum be transferred to the target nucleus must then be applied, to yield the result $\theta \ll 1$, i.e., small angle scattering implies small change in momentum. Equation (3.30) becomes in this approximation

$$(3.31) \quad \frac{d\bar{\sigma}}{d\Omega} = \left(\frac{2Ze^2}{\omega}\right)^2 \frac{1}{\theta^4} \, , \qquad \theta \ll 1.$$

§IV. THE CLASSICAL EINSTEIN EFFECT - THE GRAVITATIONAL DEFLECTION OF LIGHT

According to the general theory of relativity, the trajectory of a particle is governed by the equations for a geodesic (see Appendix C),

$$(4.1)\quad \frac{d^2x^\alpha}{ds^2} + \left\{\mu\nu,\alpha\right\} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0.$$

The metric is determined by the field equations

$$(4.2)\quad R_{\mu\nu} = -\kappa T_{\mu\nu} - \tfrac{1}{2}\kappa T\, g_{\mu\nu},$$

where the cosmological constant has been taken to be zero. A solution of equation (4.2) for the metric (or equivalently, the interval) in empty space (i.e., $T_{\mu\nu} = 0$) surrounding a gravitating point particle, the Schwarzschild metric, shall be used to describe the gravitational effect of the sun. This interval is static, spherically symmetric in space, and invariant under time reversal, and in spherical polar coordinates and time, it is written

$$(4.3)\quad ds^2 = (1-2m/r)dt^2 - \frac{dr^2}{(1-2m/r)} - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2.$$

To solve equations (4.1) with the condition (4.3), one needs the values of the 3-index symbols. All are zero except the following:

$$\{11,1\} = \frac{m/r^2}{1-2m/r}, \qquad \{12,2\} = 1/r,$$

$$\{13,3\} = 1/r, \qquad \{14,4\} = \frac{-m/r^2}{1-2m/r},$$

$$\{22,1\} = -r(1-2m/r), \qquad \{23,3\} = \cot\theta, \tag{4.4}$$
$$\{44,1\} = -(1-2m/r)(m/r^2), \{33,1\} = -r \sin^2\theta\ (1-2m/r),$$
$$\{33,2\} = -\sin\theta \cos\theta, \qquad \{\mu\nu,\alpha\} = \{\nu\mu,\alpha\}.$$

It is convenient to introduce new variables,

$$R \equiv r/2m, \text{ and } T \equiv t/2m, \tag{4.5}$$

where

$$r_s \equiv 2m \tag{4.6}$$

is the Schwarzschild radius. For the case of the sun r_s (sun) $\approx 3 \times 10^3$ meters (equivalent to about 4×10^{30} kgms.). The surface of the sun is approximately at a distance $p = 2.3 \times 10^5$ units from the origin. The characteristic time (the time taken by light to go r_s meters) is approximately 10^{-5} seconds. The time taken by a photon to travel the distance equal to the diameter of the sun is several seconds (cf. $T \approx 10^{-5}$ seconds). These two characteristic parameters determine the magnitude of the scattering effect of a photon passing the sun near the surface of the sun.

With the new variables (4.5), equation (4.3) becomes,

$$d\tau^2 = (1-1/R)dT^2 - \frac{dR^2}{1-1/R} - R^2\ (d\theta^2 + \sin^2\theta\ d\phi^2). \tag{4.7}$$

The trajectory of a photon in any plane through the origin which, because of spherical symmetry may be taken initially at $\theta = \pi/2$, is the null geodesic. Equation (4.7) gives

$$(4.8) \quad (1-1/R)\, dT^2 - \frac{dR^2}{1-1/R} - R^2\, d\phi^2 = 0,$$

where $d\theta \equiv 0$ due to the θ-component of equations (4.1), i.e.,

$$(4.9) \quad \frac{d^2\theta}{ds^2} + \frac{2}{r}\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds}\right) - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0,$$

giving $d\theta/ds \equiv 0$ if $\theta = \pi/2$ and $d\theta/ds = 0$ initially.

The angular momentum integral results from the ϕ-component of equations (4.1), yielding (see Appendix D, equation (D.10)),

$$(4.10) \quad R^2\, \dot{\phi} = J,$$

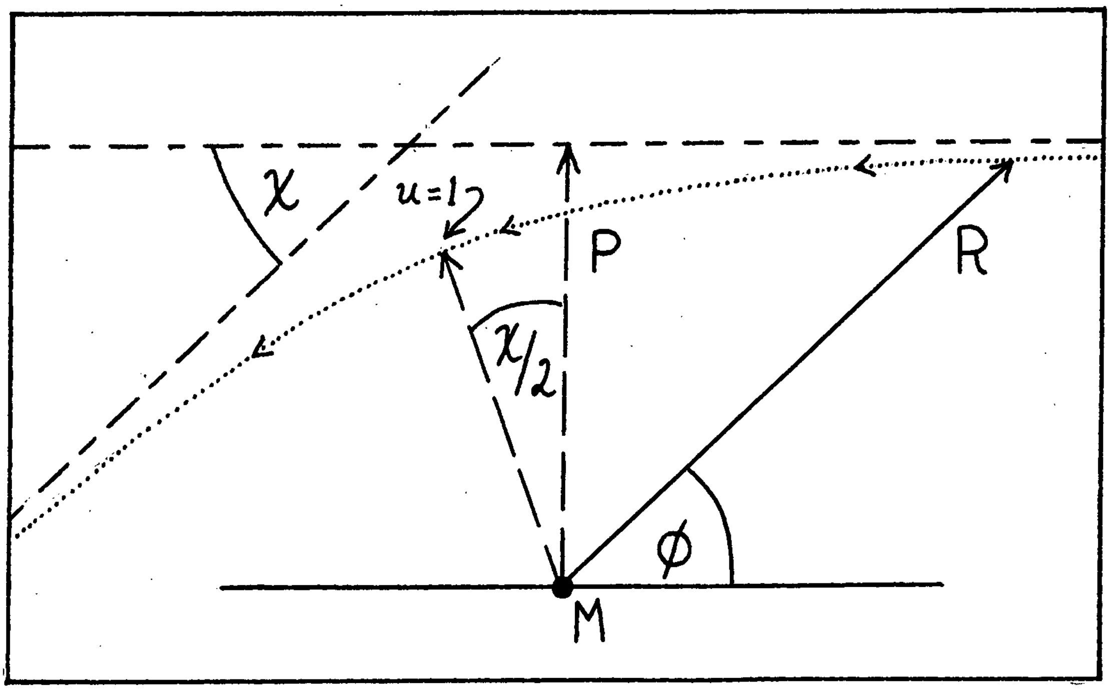
where the "dot" represents derivation with respect to τ.

The energy integral is a result of the t-component of equations (4.1), yielding (see Appendix D, equation (D.15)),

$$(4.11) \quad \dot{T}\,(1-1/R) = K.$$

It should be noted that for a photon J and K are infinite, however, their ratio is finite and that is all that is required in this problem.

FIGURE 4.1



ORBIT OF A PHOTON IN A PLANE THROUGH THE ORIGIN OF THE SCHWARZSCHILD OBJECT

Substitution of equations (4.10) and (4.11) into equation (4.8) gives,

$$(4.12)\quad \dot{R}^2 = K^2 - J^2 \frac{(R-1)}{R^3} .$$

Using a new variable, $\rho \equiv 1/R$, and dividing by $J^2 = (\dot{\phi}/\rho^2)^2$, equation (4.12) becomes

$$(4.13)\quad \left(\frac{d\rho}{d\phi}\right)^2 = \frac{1}{p^2} - \rho^2 + \rho^3 ,$$

where $p = J/K$. The cubic term is characteristic of Einstein's theory as compared to the Newtonian theory. If this term is entirely neglected, one obtains for the trajectory which begins for $\rho = 0$ (i.e., $R = \infty$) at $\phi = 0$,

$$(4.14)\quad \phi = \int_0^{\rho} (1-p^2\rho^2)^{-\frac{1}{2}}\, p\, d\rho = \arcsin\,(p\rho),$$

or

$$(4.15)\quad R \sin \phi = p.$$

This is the equation of a straight line with impact parameter p (see Figure 4.1). The exact solution of equation (4.13) is

$$(4.16)\quad \phi = \int_0^{\rho} (1-p^2\rho^2\,(1-\rho))^{-\frac{1}{2}}\, p\, d\rho .$$

Letting

(4.17) $u \equiv p\rho(1-\rho)^{\frac{1}{2}}$,

and since $p \gg 1$ for the surface of the sun, expanding in powers of p^{-1} gives

(4.18) $\rho p = u\ (1 + u/2p + \ldots)$,

(4.19) $p\ d\rho = (1 + u/p + \ldots)\ du$,

and hence equation (4.16) becomes

(4.20) $\phi = \int_0^u (1 + u/p + \ldots)\ (1 - u^2)^{-\frac{1}{2}}\ du$,

(4.21) $\phi = \arcsin u - (1/p)\ ((1-u^2)^{\frac{1}{2}} - 1) + \ldots$.

For $u = 1$, one has $du/d\phi = 0$ and thus also $d\rho/d\phi = 0$, and this corresponds to the point of closest approach (see Figure 4.1).

(4.22) $\phi(u=1) = \pi/2 + 1/p$,

corresponding to a deflection of the trajectory in the amount $1/p$ up to this point. By symmetry, the same amount of deflection will be engendered along the second half of the trajectory as it proceeds from $\phi\ (u=1)$ to $\phi = \pi + 2/p$. Hence the total deflection is

(4.23) $\chi = 2/p$.

In more usual units

(4.24) $\chi = 4MG/p$,

where M is the mass of the sun, G is the gravitational constant, and p is the impact parameter.

Numerically, for a photon travelling just past the surface of the sun $\chi \simeq 1.74$ seconds of an arc. The attempts at experimental verification of the Einstein effect and the difficulties that the detection poses are reviewed by A. A. Mikhailov, 1959.

Comparison of equation (4.24) with equations (1.5) and (1.8) yields

$$(4.25) \quad \frac{d\sigma}{d\Omega} = \frac{(4\ GM)^2}{\chi^4} \ .$$

See also equations (2.31), (2.32), and (2.33) for a comparison of the small angle scattering in the cases of Coulombic and gravitational scatterers.

The geometric method employing the notion of geodesic does not consider in its formulation the concept of polarization of photons. The prediction made is thus independent of any polarization of the photons. The quantum field method used in the next section again arrives at the result (4.25) for the unpolarized case, however, the polarization effects are calcuable as well.

§V. THE QUANTUM MECHANICAL EINSTEIN EFFECT

The classical Einstein effect, as discussed in Section IV, does not concern itself with the quantum mechanical notion of the spin of a photon. Polarization effects are thus not easily handled by this method. In analogue to Section III, this section concerns itself with the scattering of a photon by a given, weak gravitational field. A solution is obtained using the fixed field approximation, the bounds on the validity of which are as specified in Section III.

The photon is described by a plane wave solution to Maxwell's vacuum field equations, which are assumed to hold in every inertial frame,

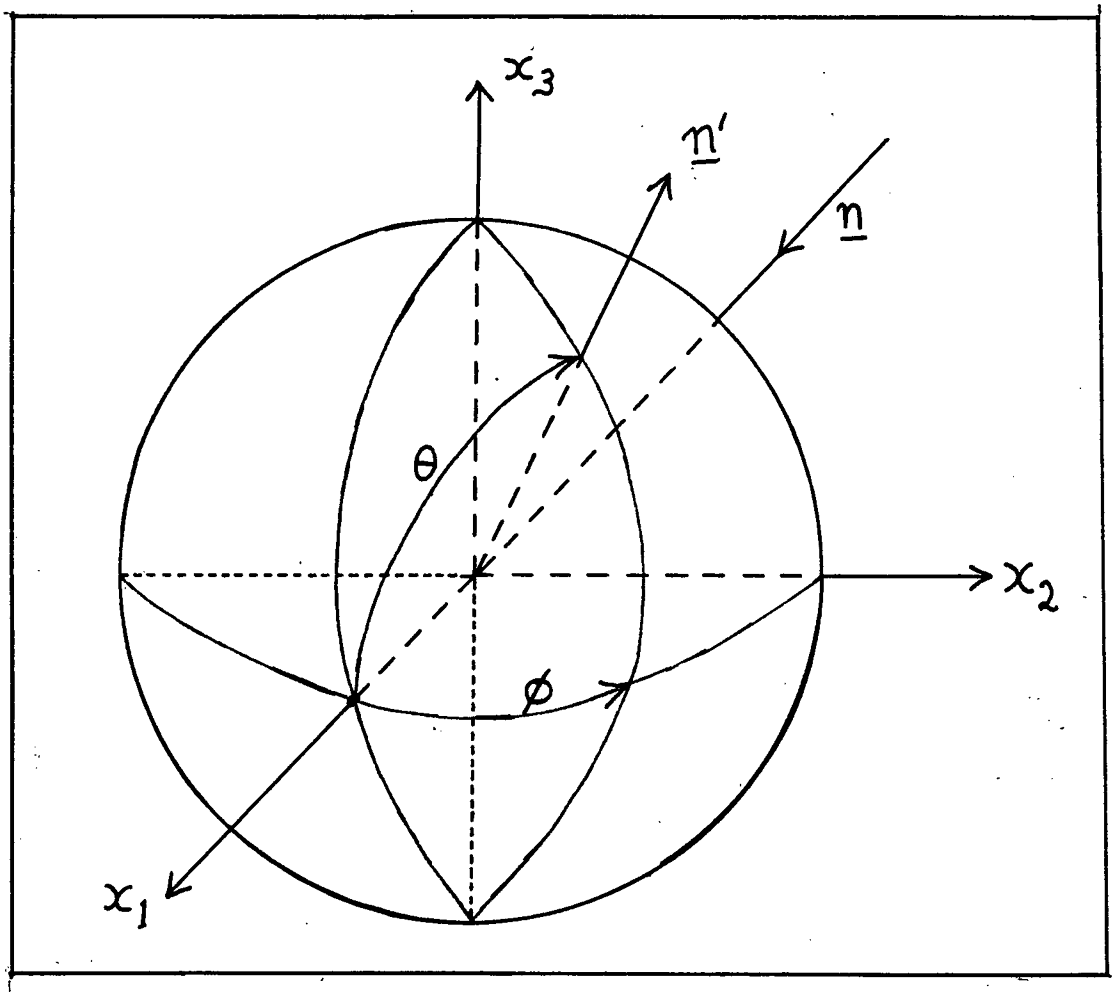
$$(5.1)\quad \frac{\partial F^{ik}}{\partial y_k} = 0,$$

where

$$(5.2)\quad F^{ik} = \frac{\partial A^i}{\partial y_k} - \frac{\partial A^k}{\partial y_i} \quad ,$$

and where $A_i(x)$, $(i = 0,1,2,3)$ is the vector potential. The plane wave solutions $A_i(x)$ can be Fourier analyzed (see Mandl, F., 1959, Chapter 9) in the usual way to describe the quantized electromagnetic field in terms of photons.

FIGURE 5.1



GEOMETRICAL IDENTIFICATION OF
INCIDENT AND SCATTERED PHOTON

$$(5.3)\quad A_i(x) = \frac{1}{\sqrt{V}} \sum_{\underline{k}} \sum_{s} \frac{1}{\sqrt{2\omega}} \Big[e_i(\underline{k},s)\, b(\underline{k},s)\, e^{-ikx} + e_i^*(\underline{k},s)\, b^\dagger(\underline{k},s)\, e^{+ikx} \Big] ,$$

with the normalization being one particle per volume V. The operators $b(\underline{k},s)$ and $b^\dagger(\underline{k},s)$ are interpreted as absorption and creation operators of photons of momentum $\underline{k}$ and in spin state s; and the polarization vectors for the photons are denoted by $e_i(\underline{k},s)$, or more simply by e_i. By introducing the unit propogation vectors n_i,

$$(5.4)\quad n_i \equiv k_i/\omega ,$$

and by combining equations (5.2) and (5.3),

$$(5.5)\quad F_{ij}(x) = \frac{-i}{\sqrt{V}} \sum_{\underline{k}} \sum_{s} \sqrt{\frac{\omega}{2}} \Big[(e_i n_j - e_j n_i)\, b(\underline{k},s)\, e^{-ikx} - (e_i^* n_j - e_j^* n_i)\, b^\dagger(\underline{k},s)\, e^{+ikx} \Big] .$$

The incident and scattered photons shall be identified as follows (see Figure 5.1):

(5.6)
- $\underline{n}$ is the direction of the incident photon,
- $\underline{n}'$ is the direction of the scattered photon,
- θ is the scattering angle, $(\underline{n}\cdot\underline{n}') = \cos\theta$,
- ϕ is the solar longitude.

Then, with the convention of Figure 5.1,

$$(5.7) \quad \underline{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; \quad \underline{n}' = \begin{pmatrix} \cos\theta \\ \sin\theta \cos\phi \\ \sin\theta \sin\phi \end{pmatrix} .$$

Since the photons are transverse one can choose (see Jauch, J.M., and Rohrlich, F., 1955)

$$(5.8) \quad e_o \equiv 0.$$

Denote by

$$(5.9) \quad \underline{e}(\underline{n},s) = \begin{pmatrix} e_1(\underline{n},s) \\ e_2(\underline{n},s) \\ e_3(\underline{n},s) \end{pmatrix}$$

the polarization vector of a photon of propogation direction $\underline{n}$ and polarization state s. If $s=1$ means 'right hand, circular polarized' and $s=2$ means 'left hand, circular polarized', then one has the standard result (see, for example, Freeman, M.J., 1967)

$$(5.10) \quad \underline{e}(\underline{n},1) = \underline{e}^*(\underline{n},2) \quad \frac{1}{\sqrt{2(1-n_3{}^2)}} \begin{pmatrix} -n_1n_3-in_2 \\ -n_2n_3-in_1 \\ 1-n_3{}^2 \end{pmatrix} .$$

Alternatively, if $s=1,2$ means the following linear polarizations, the notation will be

$$(5.11) \quad \boldsymbol{\epsilon}(\underline{n},1) \equiv \sqrt{\tfrac{1}{2}} \left[\underline{e}(1) + \underline{e}(2) \right] = \frac{1}{\sqrt{1-n_3{}^2}} \begin{pmatrix} -n_1n_3 \\ -n_2n_3 \\ 1-n_3{}^2 \end{pmatrix} ,$$

$$(5.12) \quad \boldsymbol{\epsilon}(\underline{n},2) \equiv -\sqrt{\tfrac{1}{2}}\, i \left[\underline{e}(1) - \underline{e}(2) \right] = \frac{1}{\sqrt{1-n_3{}^2}} \begin{pmatrix} -n_2 \\ n_1 \\ 0 \end{pmatrix} .$$

According to equations (5.7),

$$(5.13)\qquad \underline{\epsilon}(\underline{n},1) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \underline{\epsilon}(\underline{n},2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

that is, with $\underline{n}$ parallel to the x_1-axis, $\underline{\epsilon}(1)$ means polarization parallel to the x_3-axis and $\underline{\epsilon}(2)$ means polarization parallel to the x_2-axis. This completes the description of the photon in so far as this problem is concerned.

The gravitational field is described classically by means of the metric tensor $g^{\alpha\beta}$. In the weak field approximation, which will be used throughout,

$$(5.14)\qquad g^{\alpha\beta} = \delta^{\alpha\beta} - \gamma^{\alpha\beta} \quad , \quad g_{\alpha\beta} = \delta_{\alpha\beta} + \gamma_{\alpha\beta} \, ,$$

with

$$(5.15)\qquad \gamma^{\alpha\beta} = \gamma^{\beta\alpha} \quad , \quad |\gamma^{\alpha\beta}| << 1 \, .$$

If one specializes to the case of a static gravitational field and performs a Fourier decompostion, one gets

$$(5.16)\qquad \gamma^{ij}(x) = \gamma^{ij}(\underline{x}) = \frac{1}{\sqrt{V}} \sum_{\underline{q}} f(\underline{q})\, e^{-i\underline{q}\cdot\underline{x}}\, \epsilon^{ij} \quad ,$$

where ϵ^{ij} is the polarization tensor and $f(\underline{q})$ is a function (i.e., a c-number) not an operator, since the external field approximation is being used.

In order to be able to use the formalism developed in Section I, an interaction Lagrangian for the coupling of the photon field to the gravitational field is needed. A possible derivation of the total Lagrangian results from the so-called compensating field method (see Kaempffer, F.A., 1965).

Maxwell's vacuum field equations (5.1) can be derived from the action principle,

$$\delta \int L \, d^4y = 0 \,, \tag{5.17}$$

with

$$L = -\tfrac{1}{4} F^{ik} F_{ik} \,. \tag{5.18}$$

Using the method of the compensating field, the Lagrangian density in general space-time coordinates (not necessarily Minkowski) can be written

$$\mathcal{L}_{source} = -\tfrac{1}{4} h \, g^{\alpha\beta} g^{\rho\sigma} F_{\alpha\rho} F_{\beta\sigma} \,, \tag{5.19}$$

$$\mathcal{L}_{s} = -\tfrac{1}{4} h \, F^{\alpha\beta} F_{\alpha\beta} \,, \tag{5.20}$$

where

$$h \equiv \det \left(h^{k}{}_{\alpha}\right), \tag{5.21}$$

where

$$(5.22) \quad dy^k = h^k{}_\alpha \, dx^\alpha ,$$

where the y^k are the local inertial coordinates and the x^α are the coordinates in the underlying continuum. (For a more complete description of the 'vierbein' formalism of general relativity see Kaempffer, F.A., 1968.)

In the weak field approximation, see equations (5.14) and (5.15),

$$(5.23) \quad h = 1 - \tfrac{1}{2}\gamma^\sigma{}_\sigma .$$

Thus equation (5.19) becomes

$$(5.24) \quad \mathcal{L}_s = -\tfrac{1}{4} F^{ik}F_{ik} + \tfrac{1}{2}\gamma^{ik} (\delta^{rs}F_{ir}F_{ks} - \tfrac{1}{4}\delta_{ik}F^{rs}F_{rs}).$$

Subtracting the term (equation (5.18)) leading to the vacuum field equations in a Minkowski space, the term due to the gravitating source is

$$(5.25) \quad \mathcal{L}_{int} = \tfrac{1}{2}\gamma^{ik} (\delta^{rs} F_{ir} F_{ks} - \tfrac{1}{4} \delta_{ik} F^{rs} F_{rs}) ,$$

$$(5.26) \quad \mathcal{L}_{int} = \tfrac{1}{2}\delta^{rs} (\gamma^{ij} - \tfrac{1}{4}\gamma^{mn}\delta_{mn}\delta^{ij}) F^{ir} F_{js} .$$

For ease of notation, k shall represent both variables, the momentum $\underline{k}$ and the polarization state s. If one is interested in the scattering of a photon from an initial state $|k\rangle$ into a different (final) state $\langle k'|$ through the effect of γ^{ij}, then in the interaction Lagrangian density only terms which contain one each of creation and destruction operators need be retained; the terms dealing with the creation and destruction of photon pairs can be ignored. As in equation (1.10), the amplitude for this scattering is given by

$$(5.27)\quad \left\langle k' \left| i \int \mathcal{L}_{int}\, d^4x \right| k \right\rangle ,$$

and hence, using the normalization of one particle per volume V, equation (1.23) applies and

$$(5.28)\quad \frac{d\sigma}{d\Omega} = \frac{V^2\omega^2}{(2\pi)^4} \left| \left\langle k' \right| M \left| k \right\rangle \right|^2_{\omega=\omega'} .$$

'M', as in equation (1.10), is defined by

$$(5.29)\quad \left\langle k' \left| i \int \mathcal{L}_{int}\, d^4x \right| k \right\rangle \equiv \left\langle k' \right| M \left| k \right\rangle \delta(\omega' - \omega)$$

for a fixed, time-independent field.

Using equation (5.26) with the Fourier decompositions (5.16) and (5.5), equation (5.27) becomes,

$$(5.30)\quad \left\langle k' \left| i \int \mathcal{L}_{int} d^4x \right| k \right\rangle = \frac{Ai\sqrt{\omega\omega'}}{V(2\pi)^3} \iint f(\underline{q})\, e^{i(\underline{k}'-\underline{q}-\underline{k})\cdot\underline{x}}\, e^{i(\omega'-\omega)t}\, d\underline{q}\, d^4x ,$$

where the factor A contains all the relevant polarization tensors,

$$(5.31)\quad A = \tfrac{1}{2}\delta^{rs}\,(\epsilon^{ij}-\tfrac{1}{4}\epsilon^{mn}\delta_{mn}\delta^{ij})(e_i n_r - e_r n_i)(e^{*\prime}_j n'_s - e^{*\prime}_s n'_j).$$

Integrating equation (5.30) over d^4x yields the delta-functions, $(2\pi)^4\,\delta(\underline{k}'-\underline{q}-\underline{k})\,\delta(\omega'-\omega)$, and allows the immediate integration over $d\underline{q}$, so that,

$$(5.32)\quad \langle k'|M|k\rangle = \frac{Ai\pi}{V}\sqrt{\omega\omega'}\;\; f(\underline{k}'-\underline{k})\,.$$

Thus, equation (5.28) becomes

$$(5.33)\quad \frac{d\sigma}{d\Omega} = \left(\frac{\omega^2}{4\pi}\right)^2 \Big| A \quad f(\underline{k}'-\underline{k})\Big|^2_{\omega=\omega'}\quad .$$

The Schwarzschild metric, which (as in the last section) shall be used to represent the effect of the sun, (see equation (4.3)), can be written in isotropic coordinates,

$$(5.34)\quad \begin{aligned} r &= (1+m/2\bar{r})^2\,\bar{r} = (x^2+y^2+z^2)^{\frac{1}{2}},\\ x &= \bar{r}\sin\theta\cos\phi,\\ y &= \bar{r}\sin\theta\sin\phi,\\ z &= \bar{r}\cos\theta, \end{aligned}$$

in the form

$$(5.35)\quad ds^2 = \frac{(1-m/2r)^2}{(1+m/2r)^2}\,dt^2 - (1+m/2r)^4\,(dx^2+dy^2+dz^2),$$

or, in the weak field approximation, as

$$(5.36)\quad ds^2 = (1-2m/r)\,dt^2 - (1+2m/r)\,(dx^2+dy^2+dz^2),$$

with $m \equiv M\,G$.

In the weak field approximation

$$(5.37)\quad g^{ij} = \begin{pmatrix} +1-2m/r & 0 & 0 & 0 \\ 0 & -1-2m/r & 0 & 0 \\ 0 & 0 & -1-2m/r & 0 \\ 0 & 0 & 0 & -1-2m/r \end{pmatrix}$$

$$= \delta^{ij} - \gamma^{ij} .$$

Hence, writing $\gamma^{ij}(x)$ as $f(r)\epsilon^{ij}$ for a static, central, weak, Schwarzschild field

$$(5.38)\quad \begin{aligned} \epsilon^{ij} &= \delta^{io}\,\delta^{jo} , \\ f(r) &= 2GM/r. \end{aligned}$$

Or,

$$(5.39)\quad f(\underline{q}) \quad \frac{8\pi GM}{|\underline{q}|^2} .$$

At resonance,

$$(5.40)\quad f(\underline{k}'-\underline{k})\Big|_{\omega=\omega'} = \frac{4\pi GM}{\omega^2(1-\underline{n}\cdot\underline{n}')} .$$

Combining equations (5.4), (5.8), and (5.13)

$$(5.41)\quad |A|^2 = \left|(\underline{e}\cdot\underline{e}'^*)(1+\underline{n}\cdot\underline{n}') - (\underline{e}\cdot\underline{n}')(\underline{e}'^*\cdot\underline{n})\right|^2 .$$

Using equation (5.6),

$$(5.42) \quad |A|^2 = \left|(\underline{e}\cdot\underline{e}'^*)(1+\cos\theta) - (\underline{e}\cdot\underline{n}')(\underline{e}'^*\cdot\underline{n})\right|^2 .$$

Thus, equation (5.33) becomes

$$(5.43) \quad \frac{d\sigma}{d\Omega} = \frac{G^2M^2}{(1-\cos\theta)^2}\left|(\underline{e}\cdot\underline{e}'^*)^2(1+\cos\theta)-(\underline{e}\cdot\underline{n}')(\underline{e}'^*\cdot\underline{n})\right|^2.$$

This is the differential cross-section for scattering of photons into the solid angle $d\Omega$ at the scattering angle θ with fixed initial and final polarizations $\underline{e}$ and $\underline{e}'$.

If the incident beam of photons is unpolarized, one must average over the two possible polarization states $s = 1,2$. If the detector does not distinquish between polarization states, one must as well sum over the final spin states $s' = 1,2$. Thus, $|A|^2$ of equation (5.42) becomes $\overline{|A|^2}$, $|A|^2$ averaged and summed,

$$(5.44) \quad \overline{|A|^2} = \frac{1}{2}\sum_{s=1}^{2}\sum_{s'=1}^{2} |A|^2,$$

$$(5.45) \quad \overline{|A|^2} = \frac{1}{2}\sum_{s=1}^{2}\sum_{s'=1}^{2} (\underline{e}\cdot\underline{e}'^*)(\underline{e}^*\cdot\underline{e}')(1+\cos\theta)^2 - 2(\underline{e}\cdot\underline{e}'^*)$$

$$(\underline{e}\cdot\underline{n}')(\underline{e}'^*\cdot\underline{n})(1+\cos\theta)+(\underline{e}\cdot\underline{n}')(\underline{e}'^*\cdot\underline{n})(\underline{e}^*\cdot\underline{n}')(\underline{e}'\cdot\underline{n}).$$

Because of the spherical symmetry of this interaction one can take $\phi \equiv 0$ for this calculation and one easily obtains, using equations (5.7) and (5.10),

$$(5.46) \quad \sum_{s=1}^{2}\sum_{s'=1}^{2}(\underline{e}\cdot\underline{e}'^*)(\underline{e}^*\cdot\underline{e}') = 1 + \cos^2\theta,$$

$$(5.47)\quad \sum_{s=1}^{2}\sum_{s'=1}^{2}(\underline{e}\cdot\underline{e}'^{*})(\underline{e}\cdot\underline{n}')(\underline{e}'^{*}\cdot\underline{n}) = \cos\theta\,(\cos^2\theta-1),$$

$$(5.48)\quad \sum_{s=1}^{2}\sum_{s'=1}^{2}(\underline{e}\cdot\underline{n}')(\underline{e}'^{*}\cdot\underline{n})(\underline{e}^{*}\cdot\underline{n}')(\underline{e}'\cdot\underline{n}) = (1-\cos^2\theta)^2.$$

Finally,

$$(5.49)\quad \overline{|A|^2} = (1+\cos\theta)^2,$$

independent of ϕ, as must be because of spherical symmetry. Thus, equation (5.43) becomes

$$(5.50)\quad \frac{d\bar{\sigma}}{d\Omega} = G^2M^2\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^2 .$$

$$(5.51)\quad \frac{d\bar{\sigma}}{d\Omega} = G^2M^2\cot^4(\theta/2).$$

Thus, for small scattering angles $\theta \ll 1$,

$$(5.52)\quad \frac{d\bar{\sigma}}{d\Omega} \approx \frac{(4GM)^2}{\theta^4} ,$$

which is the same as equation (4.25) calculated by the classical method in Section IV.

Rather than considering an unpolarized beam and a detector which is unable to distinquish between the polarization states, suppose the initial beam of photons is linearly polarized in the equatorial plane of the sun. Suppose also that only scattered photons polarized in the same direction are detected. Then

$$(5.53)\quad \underline{e} = \boldsymbol{\epsilon}(\underline{n},2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

and

$$(5.54)\quad \underline{e}' = \boldsymbol{\epsilon}(\underline{n}',2) = \frac{1}{\sqrt{1-n_3'^2}} \begin{pmatrix} -n_2' \\ n_1' \\ 0 \end{pmatrix}.$$

It follows that

$$(5.55)\quad \underline{e}\cdot\underline{e}' = \frac{n_1'}{\sqrt{1-n_3'^2}},$$

$$(5.56)\quad \underline{e}\cdot\underline{n}' = n_2',$$

$$(5.57)\quad \underline{e}'\cdot\underline{n} = -\frac{n_2'}{\sqrt{1-n_3'^2}}.$$

Equation (5.42) then becomes

$$(5.58)\quad A = \frac{1}{\sqrt{1-n_3'^2}} \left[n_1'(1+\cos\theta) + n_2'^2 \right],$$

$$(5.59)\quad A = \frac{1}{(1-\sin^2\theta\sin^2\phi)^{\frac{1}{2}}} \left(\cos\theta(1+\cos\theta)+\sin^2\theta\,\cos^2\phi\right),$$

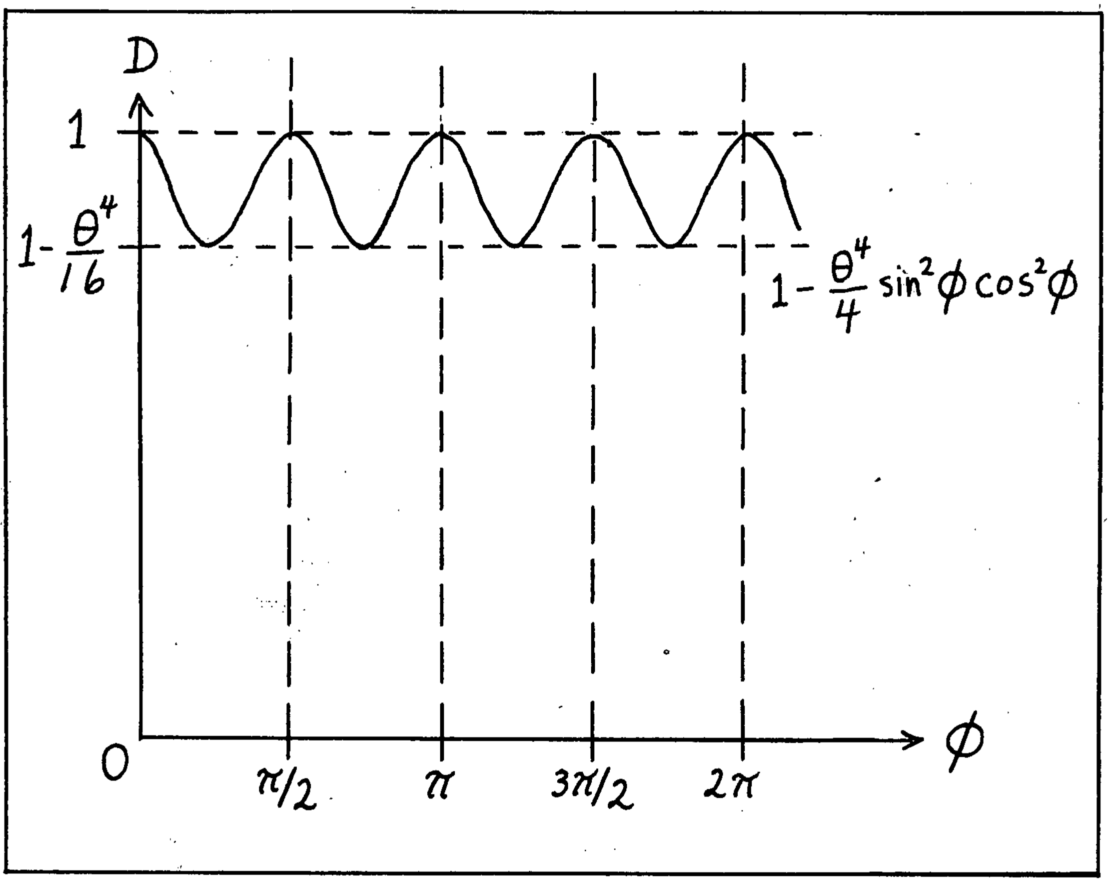
$$(5.60)\quad A = \frac{1}{(1-\sin^2\theta\sin^2\phi)^{\frac{1}{2}}} \left(\cos\theta + \cos^2\theta + \cos^2\theta\,\sin^2\phi\right),$$

using simple trigonometric identities, and hence

$$(5.61)\quad |A|^2 = \frac{(1+\cos\theta)^2}{(1-\sin^2\theta\sin^2\phi)}(1-\sin^2\phi + \cos\theta\,\sin^2\phi)^2 .$$

This result differs from the usual Einstein result by the

FIGURE 5.2



DEVIATION FROM EINSTEIN'S RESULT -

INCIDENT AND DETECTED POLARIZATIONS PARALLEL factor

(5.62) $$D = \frac{\left[1-\sin^2\phi\ (1-\cos\theta)\right]^2}{(1-\sin^2\theta\ \sin^2\phi)}$$

which, for small scattering angle, θ, becomes approximately

(5.63) $$D \simeq \frac{1-(1-\frac{\theta^2}{2}+\frac{\theta^4}{24})\sin^2\phi\ ^2}{1-(\theta-\frac{\theta^3}{6})^2\ \sin^2\phi}$$

Thus,

(5.64) $$D \simeq 1 - \frac{\theta^4}{4}\sin^2\phi\ \cos^2\phi,$$

where all terms of order θ^4 are retained. This deviation is plotted as Figure 5.2.

If the initial beam is polarized (linearly parallel to the polar axis of the sun, and the detector is effective only for photons polarized in the equatorial plane of the sun, then

(5.65) $$\underline{e} = \underline{\epsilon}(\underline{n},1) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

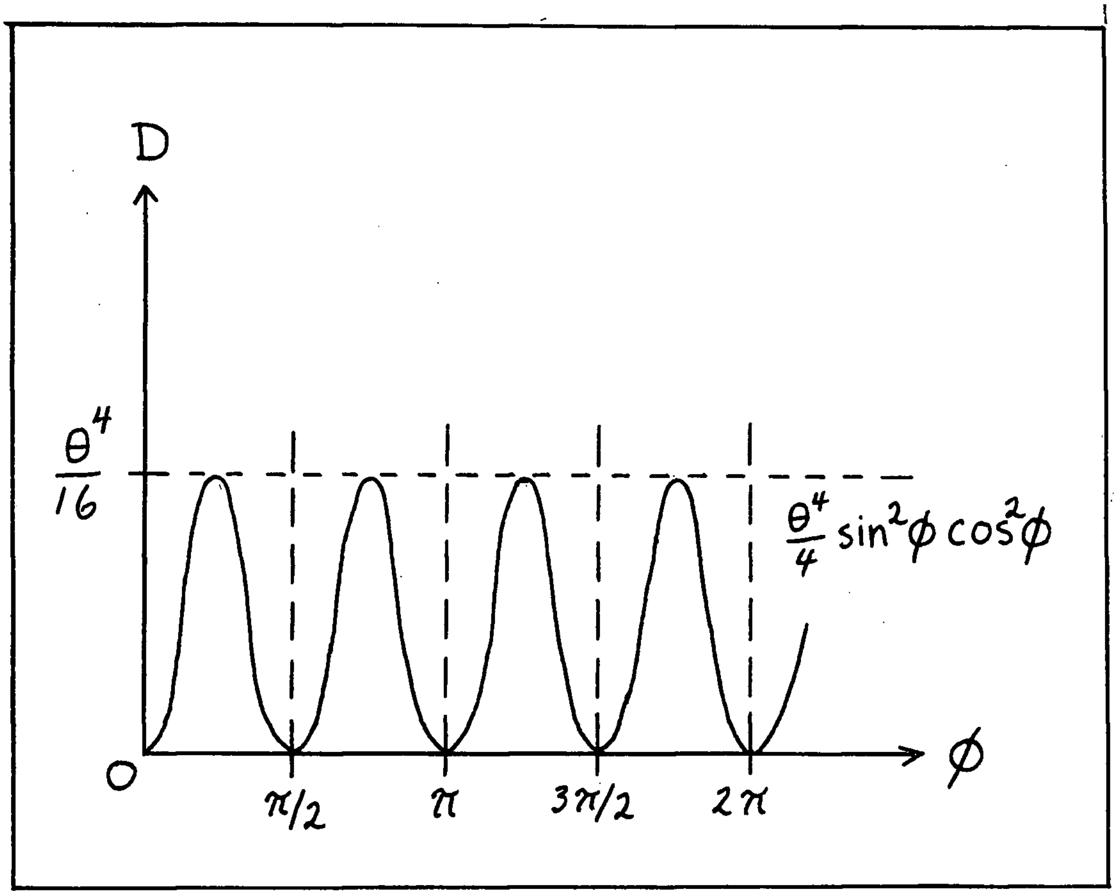
and

(5.66) $$\underline{e}' = \underline{\epsilon}(\underline{n}',2) = \frac{1}{\sqrt{1-n_3'^2}} \begin{pmatrix} -n_2' \\ n_1' \\ 0 \end{pmatrix}.$$

It follows that

(5.67) $$(\underline{e}\cdot\underline{e}') = 0,$$

FIGURE 5.3



DEVIATION FROM EINSTEIN'S RESULT -

INCIDENT AND DETECTED POLARIZATIONS PERPENDICULAR

(5.68) $(\underline{e}\cdot\underline{n}') = n_3'$,

(5.69) $(\underline{e}'\cdot\underline{n}) = -\dfrac{n_2'}{\sqrt{1-n_3'^2}}$.

Equation (5.42) thus becomes

(5.70) $A = \dfrac{n_2' n_3'}{\sqrt{1-n_3'^2}}$

and

(5.71) $|A|^2 = (1+\cos\theta)^2 \; \dfrac{(1-\cos\theta)^2 \sin^2\phi \cos^2\phi}{1-\sin^2\theta \sin^2\phi}$

which, for small scattering angle, θ, becomes approximately

(5.72) $|A|^2 = (1+\cos\theta)^2 \left[\dfrac{\theta^4}{4} \sin^2\phi \cos^2\phi\right]$.

The deviation from the Einstein result is plotted as Figure 5.3.

If one takes

(5.73) $\underline{e} = \underline{\epsilon}(\underline{n},1) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

and

(5.74) $\underline{e}' = \underline{\epsilon}(\underline{n}',1) = \dfrac{1}{\sqrt{1-n_3'^2}} \begin{pmatrix} -n_1' n_3' \\ -n_2' n_3' \\ 1-n_3'^2 \end{pmatrix}$,

then for small angle scattering

$$(5.75) \quad |A|^2 \simeq (1+\cos\theta)^2 \; (1- \frac{\theta^4}{4} \sin^2\phi \; \cos^2\phi).$$

The deviation from Einstein's result has the same graphic behavior as in Figure 5.2. That is, if the initial beam of photons is linearly polarized and only photons which are linearly polarized in the same direction are detected, then the cross-section is independent of this direction.

If one has

$$(5.76) \quad \underline{e} = \underline{\epsilon}(\underline{n},2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$(5.77) \quad \underline{e}' = \underline{\epsilon}(\underline{n}',1) = \frac{1}{\sqrt{1-n_3'^2}} \begin{pmatrix} -n_1' \; n_3' \\ -n_2' \; n_3' \\ 1-n_3'^2 \end{pmatrix} ,$$

then for small angle scattering

$$(5.78) \quad |A|^2 \simeq (1+\cos\theta)^2 \; (\frac{\theta^4}{4} \sin^2\phi \; \cos^2\phi).$$

This deviation from Einstein's result is as plotted in Figure 5.3. That is, if the initial polarization and the detected polarization are perpendicular, then the result is independent of the initial direction of polarization.

The polarization effects calculated above, simply express the geometric fact that as the orbit of the polarized photon is bent through the influence of the sun, the direction of polarization (in general) changes, since the photon must at all times remain transverse.

The maximum deviation from the clasic Einstein effect occurs for $\phi = \pi/4$, in which case, $D \simeq 1-\theta^4/16$. For $\theta \approx 1.74$ seconds of arc (as in the case of a photon travelling just past the surface of the sun), $\theta^4/16 \approx 2 \times 10^{-18}$. It is clear that these polarization effects are very small when compared to the scattering of unpolarized photons, which is itself presently just within the range of detection.

VI. CONCLUSION

That the polarization effect calculated in Section V is not in observable range is not to say that it is useful to invoke the method illustrated only to obtain results of ultra-fine detail of the already small relativistic effects. Without leaving the problem of photons scattered by the sun, it is possible to suggest several other pertinent problems.

The effect of the non-spherical nature of the sun should be calculable using the outlined method. In the weak field approximation (see Eddington, A.S., 1924), neglecting all cross-coupling terms the metric due to a mass distribution follows directly from the usual Newtonian potential of the mass distribution. In the light of Dicke's concern with the solar oblateness (see Dicke, R.H., 1967), this calculation should be carried out with the very small polarization effect again being calculable. The deviation from Einstein's result will be small since the suggested oblateness, $\Delta r/r$, is of the order 10^{-5}.

The polarization effects and the effect of the non-spherical distribution are both small refinements of the classical result. It is possible that the effect of the rotation of the sun on the photons scattering shall be of an order such that it will be more nearly measurable. Solutions of Einstein's field equations for a rotating mass distribution have been found (see Thirring, V.H., 1918; Kerr, P., 1963; Brill, D.R., and Cohen, J.M., 1966; Cohen, J.M., 1967). Using this solution for the metric, the interaction Lagrangian can be obtained and a solution for the scattering of polarized photons can be obtained following the methods of Section V.

It is suggested that by increasing the accuracy of the deflection measurement one might be able to detect an asymmetry in the scattering of photons due to the presence of a preferred direction caused by the rotation axis of the sun.

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APPENDIX A: THE SCATTERING MATRIX (THE S-MATRIX)

The amplitude for scattering from a state $|i\rangle$ to a state $\langle f|$, as used in equation (1.10), can most easily be derived as follows. In the interaction picture, one can write (see, for example, Mandl, F., 1959)

$$(A.1) \quad i \frac{\partial \Phi}{\partial t} = H_I(t)\, \Phi(t),$$

where

$$(A.2) \quad H_I(t) \equiv \int \mathcal{H}_I(t,\underline{x})\, d\underline{x}\ .$$

$\mathcal{H}_I(x)$ is the interaction Hamiltonian density. If one examines the case in which the interaction is adiabatically switched off in the remote past and in the remote future, one can write the final state $\langle f| \equiv \Phi(t \rightarrow \infty)$ in terms of the initial state $|i\rangle \equiv \Phi(t \rightarrow -\infty)$ as follows:

$$(A.3) \quad \Phi(\infty) = S\, \Phi(-\infty),$$

in which the operator S is the so-called scattering matrix. In order to solve for S and arrive at a form suitable for calculation purposes it is usually necessary to assume that the interaction is small, allowing one to use (non-rigorously) a power series in the interaction. Successive approximations to the solution of (A.1) yields upon making the identification (A.3),

$$\text{(A.4)} \quad S = 1 - i\int_{-\infty}^{+\infty} \mathcal{H}_I(x)\, d^4x + (-i)^2 \int_{-\infty}^{+\infty} \mathcal{H}_I(x) d^4x \quad H_I(t')dt' + \ldots$$

For the case in which the Lagrangian density, $\mathcal{L}(x)$, is not a function of the derivatives of the field variables, one has

$$\text{(A.5)} \quad \mathcal{H}(x) = -\mathcal{L}(x) \ .$$

To first order of approximation (Born approximation) equation (A.4) then becomes

$$\text{(A.6)} \quad S = 1 + i \int_{-\infty}^{+\infty} \mathcal{L}(x)\, d^4x \ .$$

Hence, for scattering from a state $|i\rangle$ to a different state $\langle f|$, the amplitude is

$$\text{(A.7)} \quad A_{fi} = \langle f|\, i\int \mathcal{L}(x)\, d^4x\, |i\rangle \ .$$

The probability for scattering is, in this approximation,

$$\text{(A.8)} \quad |A_{fi}|^2 \quad \left|\langle f\, |\int \mathcal{L}(x)\, d^4x\, | i\rangle\right|^2 \ .$$

The form of the S-matrix involving the Lagrangian density rather than the Hamiltonian density holds even when the restriction that $\mathcal{L}(x)$ not be a function of the derivatives of the field is removed (see Bogoliubov, N.N., and Shirkov, D.V.,1959, Section 18). The form (A.6) can be arrived at by using only the conditions of covariance, unitarity, and causality of S, together with the correspondence principle. It follows that the interaction Lagrangian must be local, Hermitean, and covariant. The usual scalar combination of field variables automatically ensures covariance and the conditions of the Hermiticity and of the local nature of $\mathcal{L}$ represent subsidiary conditions limiting the choice of a scalar $\mathcal{L}$.

An alternate approach (see Bjorken, J.D., and Drell, S.D., 1964) for the case of relativistic quantum electrodynamics, using a Green's function or Feynman propogator approach leads to similar results.

APPENDIX B. THE SQUARE OF A DELTA-FUNCTION

The delta-function has the well-known integral representation

$$\text{(B.1)}\quad \delta(\omega_f - \omega_i) = \frac{1}{2\pi} \int \exp\{i(\omega_f - \omega_i)t\}\, dt.$$

Thus, $\left[\delta(\omega_f - \omega_i)\right]\left[\delta(\omega_f - \omega_i)\right]$ can be evaluated by setting $\omega_f = \omega_i$ in one of the factors, written in the integral representation (B.1), to obtain

$$\text{(B.2)}\quad \left[\delta(\omega_f - \omega_i)\right]^2 = \frac{1}{2\pi}\delta(\omega_f - \omega_i) \int dt.$$

With the identification

$$\text{(B.3)}\quad \int dt \equiv T,$$

the total time, one has

$$\text{(B.4)}\quad \left[\delta(\omega_f - \omega_i)\right]^2 = \frac{T}{2\pi}\delta(\omega_f - \omega_i).$$

Alternately, in any physically realizable situation the limits on t are never $-\infty$ to $+\infty$. More realistically, assume the transition takes place in the time interval $(-T/2, +T/2)$. Rather than a delta-function, one then gets

$$\text{(B.5)}\quad \int_{-T/2}^{+T/2} dt(\exp i(\omega_f - \omega_i)t) = \frac{2}{2\pi}\, \frac{\sin\left[(T/2)(\omega_f - \omega_i)\right]}{(\omega_f - \omega_i)}.$$

Thus, the delta-function squared becomes

$$\text{(B.6)} \qquad \frac{4 \sin^2 \left[(T/2)(\omega_f - \omega_i)\right]}{(2\pi)^2 \, (\omega_f - \omega_i)^2} \, .$$

The area under this curve is $T/2\pi$. Thus for large but finite T, one gets

$$\text{(B.7)} \qquad \left[\delta(\omega_f - \omega_i)\right]^2 \simeq \frac{T}{2\pi} \, \delta(\omega_f - \omega_i) \; .$$

APPENDIX C. CONSERVED QUANTITIES

The Lagrangian of a closed (isolated) system, because of the assumed homogeneity of time, cannot depend explicitly on time. Thus,

$$\text{(C.1)} \quad \frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) ,$$

from which, using Euler's equations (2.4),

$$\text{(C.2)} \quad \begin{aligned} \frac{dL}{dt} &= \sum_i \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right), \\ \frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) &= 0. \end{aligned}$$

Therefore, the energy of a closed system,

$$\text{(C.3)} \quad E \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L,$$

remains constant during the motion.

The Lagrangian of a closed system, because of the homogeneity of space, must be invariant under arbitrary parallel displacement of the entire system in space. Letting $\underline{r}_a$ be the position of particle 'a', translate the system by an arbitrary, infinitesimal amount $\delta \underline{r}$ to get

$$\text{(C.4)} \quad \delta L = \sum_a \frac{\partial L}{\partial \underline{r}_a} \cdot \delta \underline{r} = 0.$$

Since $\delta \underline{r}$ is arbitrary,

$$(C.5)\qquad \sum_a \frac{\partial L}{\partial \underline{r}_a} = 0,$$

from which, using Lagrange's equations (2.4),

$$(C.6)\qquad \frac{d}{dt} \sum_a \frac{\partial L}{\partial \dot{\underline{r}}_a} = 0.$$

Therefore, the momentum of the system,

$$(C.7)\qquad \underline{P} \equiv \sum_a \frac{\partial L}{\partial \underline{r}_a} \quad ,$$

is a constant of the motion.

The Lagrangian of a closed system, because of spacial isotropy, must be invariant under arbitrary rotations of the whole system in space. Consider an arbitrary, infinitesimal rotation $\underline{\delta \phi}$ of magnitude $\delta \phi$ about an axis indicated by the direction of $\underline{\delta \phi}$, and note

$$\underline{\delta r}_a = \underline{\delta \phi} \times \underline{r}_a,$$

$$(C.8)\qquad \underline{\delta \dot{r}}_a = \underline{\delta \phi} \times \dot{\underline{r}}_a,$$

$$\delta L = \sum_a \left(\frac{\partial L}{\partial \underline{r}_a} \cdot \underline{\delta r}_a + \frac{\partial L}{\partial \dot{\underline{r}}_a} \cdot \underline{\delta \dot{r}}_a \right) = 0,$$

from which, using Lagrange's equations (2.4) and the first two of equations (C.8), (C.9) $$\underline{\delta \phi} \cdot \frac{d}{dt} \sum_a \underline{r}_a \times \underline{p}_a = 0.$$

Since $\underline{\delta \phi}$ is arbitrary, the angular momentum,

(C.10) $$\underline{M} \equiv \sum_a \underline{r}_a \times \underline{p}_a,$$

is conserved.

APPENDIX D. THE GEODESIC EQUATIONS AND INTEGRALS OF MOTION

The equations of a geodesic are determined by the condition, ds is stationary. Fixing the endpoints of the trajectory in general space-time, the path can be deformed by an infinitesimal amount dx^{σ}. Applying the stationary condition,

$$\text{(D.1)} \quad \int \delta(ds) = 0,$$

with

$$\text{(D.2)} \quad ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} ,$$

($g_{\mu\nu}$ is not the special relativistic metric of §III), results in

$$\text{(D.3)} \quad \tfrac{1}{2} \int \left\{ \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\, \delta x^{\sigma} + \left(g_{\mu\sigma}\frac{dx^{\mu}}{ds}\; g_{\sigma\nu}\frac{dx^{\nu}}{ds}\right)\frac{d}{dt}(\delta x^{\sigma}) \right\} ds = 0.$$

Integrating by parts, and setting the integrated part equal to zero (since the endpoints are fixed) gives

$$\text{(D.4)} \quad \tfrac{1}{2} \int \left\{ \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\, \delta x^{\sigma} - \frac{d}{ds}\left(g_{\mu\sigma}\frac{dx^{\mu}}{ds} + g_{\sigma\nu}\frac{dx^{\nu}}{ds}\right) \delta x^{\sigma} \right\} ds = 0.$$

Since equation (D.4) must be true for arbitrary δx^{σ}, the coefficients must be identically zero. That is,

$$\text{(D.5)} \quad \tfrac{1}{2} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} \left(\frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} \right) - g_{\epsilon\sigma} \frac{d^2 x^{\epsilon}}{d s^2} = 0.$$

Or, multiplying by $g^{\sigma\alpha}$ to get rid of $g_{\epsilon\sigma}$,

$$\text{(D.6)} \quad \tfrac{1}{2}\,\frac{dx^\mu}{ds}\,\frac{dx^\nu}{ds}\,g^{\sigma\alpha}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu}+\frac{\partial g_{\nu\sigma}}{\partial x^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right) + \frac{d^2x^\alpha}{ds^2} = 0,$$

that is,

$$\text{(D.7)} \quad \frac{d^2x^\alpha}{ds^2} + \left\{\mu\,\nu, \alpha\right\}\,\frac{dx^\mu}{ds}\,\frac{dx^\nu}{ds} = 0,$$

which are the equations determining a geodesic.

The angular momentum integral results from the ϕ-component of equations (4.1) as follows:

$$\text{(D.8)} \quad \frac{d^2\phi}{ds^2} + \frac{2}{r}\,\frac{dr}{ds}\,\frac{d\phi}{ds} = 0,$$

which has the immediate solution,

$$\text{(D.9)} \quad r^2\,\frac{d\phi}{ds} = h,$$

where h is a constant. Using the coordinates of equation (4.5) one gets,

$$\text{(D.10)} \quad R^2\,\frac{d\phi}{ds} = \frac{h}{(2m)^2} \equiv J.$$

The energy integral results from the t-component of equations (4.1) as follows: