ON THE INTERACTION OF
LOW ENERGY PIONS WITH NUCLEI

by

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ABSTRACT

The optical interaction of low energy (≤ 30 MeV) pions with nuclei is discussed. In particular, it is shown that, since the nuclear density enters the low energy pion-nucleus interaction in a very direct manner, this interaction provides a sensitive means of investigating such properties of the nuclear density as the diffuseness of the nuclear surface.

A geometric discussion of the structure of the low energy pion-nucleus interaction is given which emphasizes the analogy between adding the scattered pion waves in the nuclear medium and adding electric potentials in a classical dielectric. The parameters of the optical potential which represents the interaction are taken to be those calculated by earlier authors who have used a multiple-scattering formalism to deduce the details of the optical interaction from a microscopic point of view. The interaction is strongly momentum-dependent and the local part of the interaction is repulsive.

It is shown that in optical scattering and absorption, the resonance aspects of the problem depend mainly only upon the height of the local potential barrier (~ 15 MeV) because of the long wavelength of the pion inside the nucleus. The optical absorption of low energy pions is shown to be sensitive to the diffuseness of the nuclear surface through the strong suppression of momentum-dependent absorption near the top of the potential barrier. It is argued that low energy pions can therefore be used to resolve the conflicts which presently exist in the information available from

several experiments concerning the distribution of neutrons in the nuclear surface.

The ideas developed in discussing the optical absorption of pions are extended to the excitation by pions of rotational states in strongly deformed nuclei. It is shown that, unlike the excitation cross sections obtained with more conventional interactions (such as the Coulomb interaction), the excitation cross sections obtained from the pion-nucleus optical interaction depend strongly upon such characteristics of the nuclear density as its surface thickness.

The rotational model of strongly deformed nuclei is assumed and an analysis of the excitation cross sections is made in the Distorted Wave Born Approximation. It is shown that the sensitivity to the nuclear surface thickness in the excitation cross sections arises from the suppression of the excitation processes due to the momentum-dependent interaction near the top of the potential barrier. It is suggested that the excitation of rotational levels in strongly deformed nuclei by pions can therefore be used to carefully examine the distribution of neutrons in the surfaces of strongly deformed nuclei.

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In this thesis it is our purpose to describe the properties of the low energy ($\lesssim 30$ MeV) pion-nucleus optical interaction and to suggest and examine some of the ways in which this interaction can be used to investigate the structure of nuclei. Because a transparent connection exists between the optical interaction of a pion with a nucleus and the elementary interactions of the pion with the constituent nucleons, the properties of this optical interaction are related in a very direct way to such macroscopic nuclear properties as the nuclear density. In this thesis we examine the role of the nuclear density both in determining the optical scattering and absorption of pions in nuclei and also in determining the excitation by pions of rotational states in strongly deformed nuclei. We show that these processes depend strongly upon the details of the density of nucleons in the surface of the nucleus. We therefore suggest that pions provide an ideal probe for measuring such aspects of nuclear structure as the distribution of nucleons, in particular, neutrons in the nuclear surface.

Details of the neutron distribution in the nuclear surface are not easily available from more conventional interactions (such as the nucleon-nucleus interaction) where the connection between the microscopic and macroscopic aspects of the problem is less direct; in fact, information about the surface distribution of neutrons in nuclei currently available from several experiments is neither very well

determined nor are the results of these experiments mutually consistent. We suggest that the conflicts in this important aspect of nuclear structure can be largely resolved by means of experiments which invoke the optical properties of pions, such as those to be discussed in this thesis.

The direct manner in which the macroscopic aspects of the nucleus, such as the nuclear density, enter the pion-nucleus optical interaction arises from the transparent connection between the optical interaction and the scattering and absorption processes associated with the elementary nucleon (and two-nucleon) scatterers whose average over the scattering nucleus it purports to represent. This singular directness in the relationship of the macroscopic to the microscopic interactions of pions in nuclei makes the pion-nucleus optical interaction of interest as one of the few examples of tractable many-body scattering problems as well as an excellent tool for studying the macroscopic properties of the nucleus. The simplicity in the case of low energy pion scattering arises from the short pion-nucleon scattering lengths (~ 0.1 fm.) combined with the small mass of the pion relative to the scatterers ($\sim 1/7$). In these circumstances, the pion interacts individually with each nucleon and, because of the small energy transfers in elementary collisions, it is relatively oblivious to the dynamics which govern the motions of the scattering nucleons. This allows one to make the impulse approximation (making the scattering matrix tractable) and greatly simplifies the Green's functions which describe

the propagation of the pion in the nuclear medium; the dynamics of the problem are thus taken care of through known one-body operators and the kinematics are described by using familiar techniques for adding multiply scattered waves. This is to be contrasted with the other familiar example of a strongly interacting projectile, the nucleon, where contrary conditions hold and the connection between macroscopic and microscopic processes is clouded by the intrinsic manner in which the many-body aspects of the scatterer manifest themselves in the optical properties of the nuclear medium.

Apart from the simple connections which exist between the low energy pion-nucleus interaction and its elementary origins, the interaction itself possesses a structure and form which make it of interest in its own right. The basic structure of the interaction is readily seen by noting that, at the energies of interest, only s- and p-waves enter the elementary scattering processes. The geometric problem of adding these waves is analogous to adding the potentials arising from the charges and dipoles in a classical dielectric. The s-waves add like the potentials due to the charges in the dielectric and lead to a local contribution to the pion-nucleus optical potential. The p-waves modify the intra-nuclear momentum field of the pion in analogy to the modification of the electric field due to the dipoles in a dielectric; the result of the p-waves is therefore to contribute a momentum-dependent term to the pion-nucleus optical potential. In fact,

if we include the effects of the short-range correlations of the nucleons within the nucleus, we find that the modification of the intra-nuclear momentum field due to the p-waves is non-linear in the nuclear density, in analogy to the well-known Lorenz-Lorentz effect in dense optical media.

Since the Lorenz-Lorentz effect affects the structure of the optical interaction, it raises hopes that pion optics might provide a tool for investigating the elusive nature of the short-range correlations of the nucleons within the nucleus. In fact, these hopes are considerably minimized since the wavelength of a low energy pion is much longer than the range of the correlations so that it is an insensitive tool for measuring their structure. Thus, while the pion-nucleus interaction may be strongly affected even in its structure by the anticorrelations of nucleons at short distances (which removes the self-excitation of the scatterers), it is probably insensitive to their details. Nevertheless, careful measurements of optically scattered pions might provide information about these correlations which can presently be obtained only in very indirect ways (for instance, from the studies of nuclear matter).

While the analogy with electromagnetic theory provides the basic structure of the pion-nucleus interaction, a quantitative estimate of the macroscopic parameters from the microscopic dynamics requires a considerably more detailed calculation. A thorough study of this problem has been given by Ericson and Ericson (1966). They have used the techniques

of multiple scattering theory to calculate the macroscopic form of the interaction taking into account kinematic corrections associated with the motion of the scattering nucleons and including terms which simulate optical absorption.

The origin of the absorptive terms in the Ericsons' potential lies in real pion absorption on "elementary" two-nucleon scatterers (pions, being bosons, need not be conserved); kinematics inhibit pion absorption by a single nucleon and the small mass of the pion inhibits nuclear excitation through kinetic exchanges. The absorptive aspects of the pion-nucleus optical interaction are thus connected to elementary processes as directly as the refractive aspects; this is an enormous simplification over the absorptive aspects of the nucleon-nucleus optical interaction which arise solely from kinetic exchanges between the projectile and the scatterers (and, hence, intrinsically involve the many-body aspects of the scatterer). In addition to their contribution to absorption, the Ericsons find that these two-nucleon scatterers also make important contributions to the refractive properties of the optical interaction, particularly to the local interaction where a strong cancellation occurs in the s-wave pion-nucleon scattering lengths.

With these effects taken into account, the Ericsons find that the optical scattering of pions from nuclei can be summarized by a Schroedinger equation which has the structure predicted by the electromagnetic analogy (including the Lorenz-Lorentz effect) and whose parameters are transparently connected

to pion scattering lengths from nucleons and deuterons. In addition, their optical interaction includes optical isospin terms which account for quasi-elastic charge exchange processes and includes a hyperfine term which takes into account the spin of the target nucleus. They find that the resulting interaction is consistent with the (essentially zero energy) π -mesic x-ray data but, unfortunately, the interaction cannot be tested at higher energies until the advent of more intense pion beams (such as the Triumf Facility proposed for U.B.C.).

It is reasonable, however, to expect that the Ericsons' potential should adequately describe the pion-nucleus optical interaction up to 30 or 40 MeV where the zero-energy description of the elementary scattering processes begins to fail and where the effects of the 3-3 pion-nucleon resonance make the naive treatment of the multiple scattering problem invalid. This limit is to some extent set by the Fermi motion of the nucleons and the parameters of the potential may be somewhat energy-dependent at the higher energies. Nevertheless, one would expect that the basic structure of the interaction should be correct and in our calculations we have used the zero-energy parameters suggested by the Ericsons' analysis. This seems a reasonable procedure since, in any event, the potential has not been tested at higher energies and it would not seem worthwhile without some reasonable empirical basis to attempt to untangle the various second order effects which determine its exact details.

In the present thesis we have taken the Ericsons' form for the optical pion-nucleus interaction and we have attempted both to describe pion optics in the low energy region and to elucidate those features of nuclear structure which are susceptible to investigation with the pion probe, in particular, the density of neutrons in the nuclear surface. The basic character of pion optics resides in the mildly repulsive local potential, in the strong momentum-dependence of the interaction, and in the fact that pions are very strongly absorbed by nuclei. These characteristics lead to optical phenomena which are qualitatively quite different from those encountered with more conventional projectiles.

In particular, the resonance aspects of the problem at the energies of interest (≤ 30 MeV) are governed by the presence of a potential barrier (~ 15 MeV) which, combined with the small pion mass, yields a pion wavelength within the nucleus which is considerably greater than the nuclear dimensions; the resonance aspects of the problem are, therefore, rather insensitive to the details of the interaction and are governed mainly by the size of the nucleus and by the properties of the electrostatic barrier which surrounds it. This contrasts with the strong resonances characteristic of the attractive potentials encountered in nuclear optics. The absorption of pions also presents some unusual features: pions are absorbed not only through a local interaction (which depends primarily upon the quantity of matter present) but also through a momentum-dependent interaction (which depends sensitively

upon the distribution of this matter). Because pions are absorbed primarily in the nuclear surface, the energy dependence of pion absorption cross sections yields detailed information about the peripheral nuclear density.

The density of nucleons in the nuclear surface has long been an elusive quantity so that the relatively direct and simple technique of measurement provided by pion absorption represents one of the most useful aspects of pion optics. Of course, the issue is somewhat complicated by the fact that pion absorption takes place primarily on two-nucleon scatterers. However, provided adequate account is taken of the long-range correlations of nucleons in the nucleus (which are quite well understood), the connection between the density and the absorption rates can be obtained with considerable precision. In fact, the Ericsons have shown in their calculations that the absorption rates are simply related to the square of the density and they have found these absorption rates to be in good agreement with the absorption rates required to explain the widths of π -mesic x-rays. At any rate, optical pion measurements of the nuclear density should be a considerable improvement over measurements with nucleons (which strongly polarize the target) and measurements based on the branching ratios associated with kaonic atoms (where the analysis is confused by complicated atomic physics).

Of course, the interest in finding the nuclear surface density is to obtain separately the surface distribution of protons and neutrons. This is immediately accomplished since the charge density (or distribution of protons) is well-known

both from electron scattering experiments and, more recently, from μ -mesic x-ray analysis. The information about proton and neutron distributions then provides a new dimension in the tests of nuclear models (such as the shell model) and is also necessary information in obtaining precise rates for various processes associated with the nucleus (such as pion production rates from high energy protons).

In addition to optical pion absorption, a second technique can be used to utilize the momentum-dependence of the pion-nucleus interaction in obtaining the surface features of deformed nuclei. In the strong-coupling model, a deformed nucleus is described as the product of a rotational and intrinsic wave function; the pion field can be used to excite the rotational states of such a nucleus in the same way as both the electrostatic field (Coulomb excitation) and neutron field have been used. In fact, in the Distorted Wave Born Approximation, the pion excitation amplitudes are essentially sums of the Coulomb excitation and local excitation amplitudes along with a momentum-dependent excitation amplitude arising from the momentum-dependent interaction.

The simple DWBA integrals which relate the excitation cross sections to the interaction (and, hence, the density) then allow a direct connection between the cross sections and the properties of the intrinsic wave function of the nucleus. The Coulomb and local excitation amplitudes, being integrals over the nucleus, are only moderately sensitive to the details of the nuclear surface; the momentum-dependent excitation amplitude, however, is very sensitive to these details since,

near the top of the potential barrier, the momentum varies rapidly both with energy and across the nuclear surface. Thus the analysis of the energy dependence of the excitation by pions of rotational states in deformed nuclei yields detailed information about the surface features of the intrinsic wave function.

A more complete discussion of the topics in this Introduction is given in the remaining chapters of the thesis. A review and interpretive discussion of the construction of the pion-nucleus optical potential is given in Chapter 2. In Chapter 3 this interaction is used to analyze the elastic scattering and total absorption cross sections of pions from various spherical nuclei with particular emphasis on the role of the nuclear surface. In Chapter 4 the excitation of rotational levels in deformed nuclei by pions is discussed in DWBA as a technique for investigating the surface features of the intrinsic wave functions of deformed nuclei. Chapter 5 presents a review of the thesis and a summary of the conclusions reached in the earlier chapters.

This thesis also includes, as Appendix C, a preprint of a paper now submitted for publication of which the author of this thesis is a co-author and which represents a separate problem in which the author became interested during the period of his interest in the pion-nucleus problem presented in the body of this thesis. This problem was prepared in collaboration with Professor E. W. Vogt, the author's Research Supervisor, and with Mr. Georges Michaud, a visiting student

in astrophysics from the California Institute of Technology.

This paper investigates the optical interactions between nuclei and heavy ions and particularly examines the effects of the diffuse nuclear surface upon the scattering and absorption cross sections. These cross sections are of considerable interest in determining the rates of various processes encountered in astrophysics.

The author's interest in this problem arose from his earlier interest in alpha-decay rates and his main contributions are to the discussion of the resonance properties of real potentials in the presence of large Coulomb barriers, these being contained in Sec. 4 of the paper.

CHAPTER 2THE LOW ENERGY
PION - NUCLEUS INTERACTION

In the present chapter we discuss the construction of the low energy (≤ 30 MeV) pion-nucleus optical interaction from the elementary processes of pion-nucleon and pion-deuteron scattering. A comprehensive study of this problem and a review of the earlier work by other authors has been given in an article by M. Ericson and T. E. O. Ericson (1966) and our discussion is to a large extent a summary of their discussion and results. A more general discussion of the interaction of pions with nuclei can be found in a recent review article by Koltun (1969).

The pion-nucleus interaction is distinguished from most other complex interactions which admit an optical description --- such as the nucleon-nucleus interaction --- in that it can easily be constructed from the microscopic pion-nucleon and pion-deuteron scatterings (whose average it represents).

This simplification arises in part in that, because the pion-nucleon scatterings lengths are small (≤ 0.1 fm.) compared to the inter-nucleon separation (~ 1.4 fm.) and because the pion mass is small ($\sim 1/7$ the nucleon mass), one can assume that a pion scatters from a nucleon inside the nucleus in the same way in which it scatters from a nucleon in free space; that is, one can make the impulse approximation. In addition, the small mass of the pion allows a considerable simplification of the Green's function which describes pion propagation within the nucleus since one may then largely

neglect the effects of intermediate nuclear excitations. The problem is thus essentially reduced to the classical problem of calculating the total scattered wave from a system of massive scatterers. In fact, since at low energies scatterings are mainly s-wave scatterings (resembling electric charges) and p-wave scatterings (resembling electric dipoles), one can calculate the total scattered wave in exactly the same way in which one calculates the total electric potential in a dielectric medium. It is the purpose of this chapter to present a derivation of the low energy pion-nucleus optical potential from this point of view and to demonstrate the direct manner in which it involves the density of the scattering nucleus.

It was realized by Peaslee (1952) that the scattering of pions from complex nuclei could be at least qualitatively obtained simply by making the impulse approximation and by summing the singly scattered pion waves from the elementary scatterers. In fact, by applying his analysis to the scattering of 62 MeV pions by carbon, he inferred the 3-3 pion-nucleon resonance at 200 MeV. This led him to conclude that below this energy both s- and p-waves must be taken into account in the elementary scatterings while higher partial waves can be neglected.

His results were extended to include multiple scatterings by Kisslinger (1955) who used the multiple scattering formalism of Watson (1953) and Francis and Watson (1953) to write the pion-nucleus optical interaction in terms of a

pseudopotential. While Kisslinger neglected absorption and the effects of nucleon-nucleon correlations, his treatment showed the direct manner in which the nuclear density enters the interaction and correctly predicted the structure of the interaction showing it to consist of a local interaction --- arising from the elementary s-wave scatterings --- and a momentum dependent interaction --- arising from the elementary p-wave scatterings. The strong momentum dependence of the interaction suggested by Kisslinger was also shown by Frank et al. (1956) who evaluated both the real and absorptive terms of the potential in an infinite medium of uncorrelated nucleons.

The basic structure of the pion-nucleus interaction is elucidated by considering the analogy between adding the elementary s- and p-wave scatterings in the nuclear medium and adding electric potentials and fields in a dielectric with free charges --- the geometry of the two problems is essentially the same. Baker et al. (1958) used classical arguments essentially of this kind to present a simplified derivation of the Kisslinger potential and Kroll (1961), realizing the electromagnetic analogy, suggested that there should be a modification of the intra-nuclear pion momentum field since such an effect exists in the refractive properties of dense optical media (the Lorenz-Lorentz effect). However, the electromagnetic analogy does not seem to be written in detail in the literature and --- because of its heuristic value in revealing the structure of the pion-nucleus

interaction --- we discuss it in some detail.

A derivation of the low energy pion-nucleus interaction has recently been given by M. Ericson and T. E. O. Ericson (1966) which takes into account both absorption and nuclear correlations. They derive the multiple scattering equations for a pion in the nucleus by a method similar to Lax's treatment (1951) of multiple scattering in classical media. Their basic idea is to use a systematic expansion into higher order correlation functions between the scatterers which they have truncated at pair correlations. They find in this way that the short-range anticorrelations of nucleons lead to a modification of the intra-nuclear momentum field in analogy to the Lorenz-Lorentz effect in a dense optical medium. They make the impulse approximation taking the s- (p-) wave scattering lengths (volumes) to be those obtained from pion-nucleon scattering; they then include kinematic corrections for the binding and Fermi motion of the nucleons and also extend their calculations to take account of the effects of spin and isospin.

Their calculation is further distinguished by the fact that, in addition to accounting for the pion scattering from nucleon scatterers, they also take into account pion scattering from two-nucleon (or quasi-deuteron) scatterers. In this way they are able to describe pion optical absorption which is dominated by the absorption of a pion on two nucleons. In addition, they find that the two-nucleon scatterers make an important contribution to the local interaction where a strong

cancellation occurs in the s-wave scattering lengths. Their resulting potential --- which contains no free parameters --- is found to be in good empirical agreement with the shifts and widths of x-rays associated with π -mesic atoms.

This transparent connection which exists between the elementary scattering processes and the pion-nucleus optical interaction is to be contrasted with the nucleon-nucleus optical interaction. In the latter case, the interactions between the scattered nucleon and target nucleons within the nucleus depend strongly upon the properties of the nuclear medium, both because of the long nucleon-nucleon scattering lengths and because of the larger nucleon mass; hence, their interactions can no longer be expected to be described in such a simple way as was provided to us in the pion-nucleus case by the impulse approximation. In addition, in nucleon-nucleus scattering, we can no longer introduce the technical simplifications which were allowed us in pion-nucleus scattering by the small pion mass and, in general, both intermediate nuclear excitations and kinematic corrections for recoil of the nucleon target must be taken into account. The pion-nucleus optical interaction therefore involves the macroscopic properties of the scattering nucleus in a much more explicit manner than the nucleon-nucleus optical interaction so that pions are much better tools for investigating such properties of the nucleus as the nuclear density.

In the present chapter our purposes will be two-fold. Firstly, we will show that the structure of the pion-nucleus

interaction can be simply understood from the familiar problem of adding potentials and fields in a classical dielectric. Secondly, we will discuss the Ericsons' calculation in a manner which emphasizes the microscopic aspects of the problem. The interaction will then be investigated in later chapters with emphasis on how it can be used as a tool to investigate macroscopic properties of the nucleus, particularly the nuclear density.

2.1 Structure of the Interaction

In the present section we present a simplified classical derivation of the pion-nucleus optical interaction which emphasizes the structure of the interaction and which is based on the analogy between adding s- and p-waves in the nuclear medium and adding electric potentials and fields in a dense dielectric. The idea is that s-wave scatterings, being isotropic, look like free charges in a dielectric medium; the p-wave scatterings, being dipole in character, look like electric dipoles. The geometric problem of adding the scattered pion wave from the nuclear medium is then the same as adding the total electrostatic potential from a homogeneous and isotropic dielectric medium immersed in an electric field.

Of course, a derivation of this type possesses all the naïveté of geometrical solutions and cannot be expected to quantitatively include the dynamics or quantum aspects of the problem; in addition, the technique of solution was realized (though not specifically stated) by other authors and, hence, we present nothing really new. Nevertheless, a

solution of this type --- which specifically invokes a knowledge of elementary electromagnetic theory --- is of considerable value in understanding the structure of the interaction.

We shall first review the calculation of the electrostatic potential from a homogeneous and isotropic dielectric immersed in a constant electric field. We shall then simply identify this potential and the corresponding electric field with the pion optical wave function and the pion optical momentum field, respectively.

Let us define the macroscopic properties of the dielectric by defining it to have a free charge density, $e\rho(\underline{r})$, and a polarization $\underline{P}_E(\underline{r})$. The total potential at a point $\underline{r}$, $U(\underline{r})$, is then the sum of the potential, $U_0(\underline{r})$, associated with the external electric field plus the potential, $U_d(\underline{r})$, due to the dielectric:

$$(2-1a) \quad U(\underline{r}) = U_0(\underline{r}) + U_d(\underline{r}) .$$

The latter term is well-known from elementary electromagnetic theory:[#]

$$(2-1b) \quad U_d(\underline{r}) = \int d\underline{r}' \left[\frac{e\rho(\underline{r}')}{|\underline{r} - \underline{r}'|} + \underline{P}_E(\underline{r}') \cdot \underline{\nabla}' \frac{1}{|\underline{r} - \underline{r}'|} \right] \\ = -4\pi \int d\underline{r}' \left[e\rho(\underline{r}') g_0(\underline{r} - \underline{r}') + \underline{P}_E(\underline{r}') \cdot \underline{\nabla}' g_0(\underline{r} - \underline{r}') \right] ,$$

where we have defined the free particle Green's function,

$$(2-1c) \quad g_k(\underline{r} - \underline{r}') = -\frac{1}{4\pi} \frac{e^{ik|\underline{r} - \underline{r}'|}}{|\underline{r} - \underline{r}'|} \quad . .$$

[#]A discussion of the macroscopic properties of dielectrics can be found in most textbooks on classical electromagnetic theory (for instance, c.f. Jackson (1962)).

Now, if the dielectric were homogeneous and isotropic on the microscopic scale as well as on the macroscopic scale, we would have that the effective field at a dipole, $\underline{E}_{\text{eff}}(\underline{r})$, was simply the macroscopic electric field in the dielectric, $\underline{E}(\underline{r})$:

$$(2-2) \quad \underline{E}_{\text{eff}}(\underline{r}) = \underline{E}(\underline{r}) = -\nabla U(\underline{r}) . \quad (\text{no correlations}).$$

However, if the dipoles are anticorrelated in the dielectric --- that is, if two dipoles cannot exist at the same point --- the effective microscopic field will in general differ from $\underline{E}(\underline{r})$. This effect is essentially the well-known Lorentz-Lorentz effect (although, strictly, this term is used to refer to the resultant effect on the refractive index of the medium).

To calculate the Lorentz-Lorentz effect, let us first note that the polarization is simply related to the effective field:

$$(2-3) \quad \underline{P}(\underline{r}) = \chi \mathcal{P}_d(\underline{r}) \underline{E}_{\text{eff}}(\underline{r}) ,$$

where χ is the molecular polarizability and $\mathcal{P}_d(\underline{r})$ is the dipole density. Now the effective field is the sum of the macroscopic field plus the field arising from the polarization:

$$(2-4) \quad \underline{E}_{\text{eff}}(\underline{r}) = \underline{E}(\underline{r}) + \underline{E}_p(\underline{r}) .$$

To calculate the latter, let us choose a sphere of radius, a , surrounding the dipole so that no other dipoles are in the sphere (which we can do since the dipoles are anticorrelated) and assume that outside the sphere the medium is homogeneous

and isotropic. The electric field arising from the polarization is then easily calculated.

Spherical Cavity Surrounding
a Dipole in a Dielectric

Referring to the above sketch and noting that $-\underline{P} \cdot \underline{n}$ is the charge on the surface of the cavity ($\underline{n}$ is the outward normal), we have that

$$(2-5) \quad \underline{E}_p = \int_S \frac{(-\underline{P} \cdot \underline{n}) (-ndS)}{r^2} = \int P \cos\theta \underline{n} d\Omega = \frac{4\pi}{3} \underline{P}.$$

Thus, from Eqs. 2-3 and 2-4,

$$(2-6a) \quad \underline{E}_{\text{eff}}(\underline{r}) = \frac{1}{1 - \frac{4\pi}{3} \chi \mathcal{P}_d(\underline{r})} \underline{E}(\underline{r}); \quad (\text{zero-range anticorrelations})$$

$$(2-6b) \quad \underline{P}(\underline{r}) = \frac{\chi \mathcal{P}_d(\underline{r})}{1 - \frac{4}{3} \chi \mathcal{P}_d(\underline{r})} \underline{E}(\underline{r}).$$

Substituting these relations in Eq. 2-1a, we have an integral equation for the potential:

$$(2-7) \quad U(\underline{r}) = U_0(\underline{r}) - 4\pi \int d\underline{r}' \left[e\rho(\underline{r}') \varepsilon_0(\underline{r}-\underline{r}') + \frac{\delta \rho_d(\underline{r}')}{1 - \frac{4\pi}{3} \delta \rho_d(\underline{r}')} \nabla' U(\underline{r}') \nabla' \varepsilon_0(\underline{r}-\underline{r}') \right]$$

Eqs. 2-6 are often used in elementary optics to explain the non-linear dependence of the refractive index of an optical medium upon its density --- the classical Lorenz-Lorentz effect. This is easily seen by remembering from elementary electromagnetic theory that the molecular polarizability, χ , is defined by the equation,

$$(2-8a) \quad \underline{P} = \chi \underline{E},$$

and that the refractive index of the medium, n , is given in terms of χ by the relation,

$$(2-8b) \quad n^2 = 1 + 4\pi\chi,$$

so that (from Eq. 2-6b),

$$(2-8c) \quad n^2 = 1 + \frac{4\pi\delta\rho_d}{1 - \frac{4\pi}{3}\delta\rho_d}.$$

For our purposes, we will be interested in the geometric modification of the intra-nuclear pion momentum field suggested by Eqs. 2-6; the "refractive index" of the pion in the nucleus does not bear the same analogy to the electromagnetic case because the pion optical wave and the electric potential are not described by the same equations.

We will now show that the optically scattered pion wave from a nucleus is described by an equation like Eq. 2-7 provided

we identify the optical wave with the electrostatic potential and provided we identify the s-wave scattering with the electric charge and the p-wave scattering with the molecular polarizability (for simplicity, we will neglect the pion charge).

To see the analogy, let us first note that the scattering amplitude of a pion from a nucleon has the general form (neglecting explicit reference to the internal pion and nucleon co-ordinates and taking $\hbar = c = 1$),

$$(2-9) \quad f = b + c \underline{k}' \cdot \underline{k} \quad ,$$

where the first term is the s-wave scattering amplitude and the second term is the p-wave scattering amplitude; $\underline{k}$ is the initial pion momentum and $\underline{k}'$ the final pion momentum.

Now the wave, $d\tilde{\Phi}_{sc}(\underline{r})$, scattered from an element of nuclear density, $\rho(\underline{r}')d\underline{r}'$, at the point, $\underline{r}'$, is the product of the amplitude for finding a pion at $\underline{r}'$, described by the pion optical wave, $\tilde{\Phi}(\underline{r}')$, with the strength of the scattering, $f\rho(\underline{r}')d\underline{r}'$, and the spherically outgoing wave from the scatterer, $g_k(\underline{r}-\underline{r}')$, where g_k is the free-particle Green's function of Eq. 2-1c:

$$\begin{aligned} (2-10) \quad d\tilde{\Phi}_{sc}(\underline{r}) &= \frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \tilde{\Phi}(\underline{r}') f\rho(\underline{r}')d\underline{r}' \\ &= -4\pi g_k(\underline{r}-\underline{r}') b \tilde{\Phi}(\underline{r}') \rho(\underline{r}')d\underline{r}' \\ &\quad - 4\pi g_k(\underline{r}-\underline{r}') c \underline{k}' \cdot \underline{k} \tilde{\Phi}(\underline{r}') \rho(\underline{r}')d\underline{r}' \quad . \end{aligned}$$

Now

$$(2-11) \quad \nabla' g_k(\underline{r}-\underline{r}') = -i \underline{k}' g_k(\underline{r}-\underline{r}')$$

and we may conveniently define the effective intra-nuclear

momentum density at $\underline{r}'$

$$(2-12) \quad \underline{E}_{\text{eff}}(\underline{r}') = -i\underline{k} \underline{\Phi}(\underline{r}') .$$

$\underline{E}_{\text{eff}}(\underline{r}')$ is related to the macroscopic momentum density,

$$(2-13) \quad \underline{E}(\underline{r}) = i \nabla \underline{\Phi}(\underline{r}) ,$$

through Eqs. 2-2 and 2-6. Eq. 2-10 can then be rewritten in the form of Eq. 2-1b:

$$(2-1b') \quad \underline{\Phi}_{\text{sc}}(\underline{r}) = -4\pi \int d\underline{r}' \left[b \underline{\Phi}(\underline{r}') \underline{\mathcal{P}}(\underline{r}') \underline{g}_k(\underline{r}-\underline{r}') + \underline{P}(\underline{r}') \cdot \nabla' \underline{g}_k(\underline{r}-\underline{r}') \right]$$

with "polarization"

$$(2-3') \quad \underline{P}(\underline{r}) = -c \underline{\mathcal{P}}(\underline{r}) \underline{E}_{\text{eff}}(\underline{r}) .$$

Assuming an incident plane wave,

$$(2-14) \quad \underline{\Phi}_0(\underline{r}) = e^{i\underline{k} \cdot \underline{r}} ,$$

the total wave becomes

$$(2-15) \quad \underline{\Phi}(\underline{r}) = e^{i\underline{k} \cdot \underline{r}} - 4\pi \int d\underline{r}' \left[b \underline{\Phi}(\underline{r}') \underline{\mathcal{P}}(\underline{r}') \underline{g}_k(\underline{r}-\underline{r}') - c \underline{\mathcal{P}}(\underline{r}') \underline{E}_{\text{eff}}(\underline{r}') \nabla' \underline{g}_k(\underline{r}-\underline{r}') \right] .$$

Thus --- to the extent that we can neglect the structure and quantum aspects of the scatterer --- the pion scattering within the nuclear medium is equivalent to the classical optical scattering within a dielectric medium, provided we identify the s-wave scattering with the electric charge,

$$(2-16a) \quad e \longrightarrow b \underline{\Phi}(\underline{r}') ,$$

and provided we identify the p-wave scattering with the molecular polarizability,

$$(2-16b) \quad \chi \longrightarrow -c .$$

We can write the integral Schroedinger Equation 2-15 in conventional differential form by operating on both sides

with the operator

$$0 = \nabla^2 + k^2$$

Then

$$(2-17) \quad (\nabla^2 + k^2) \Phi(\underline{r}) = -4\pi b\rho(\underline{r})\Phi(\underline{r}) + \nabla \cdot 4\pi c\rho(\underline{r})\underline{E}_{\text{eff}}(\underline{r})$$

In an uncorrelated nuclear medium, we have from Eq. 2-2 that

$$(2-2') \quad \underline{E}_{\text{eff}}(\underline{r}) = \nabla \Phi(\underline{r}) \quad (\text{no correlations})$$

so that

$$(2-18a) \quad (\nabla^2 + k^2) \Phi(\underline{r}) = -4\pi b\rho(\underline{r})\Phi(\underline{r}) + \nabla \cdot 4\pi c\rho(\underline{r}) \nabla \Phi(\underline{r}),$$

(no correlations)

which is the form of the pion-nucleus interaction derived by Kisslinger (1955); in a nuclear medium with short-range anticorrelations we have the Lorenz-Lorentz effect,

$$(2-6') \quad \underline{E}_{\text{eff}}(\underline{r}) = \frac{1}{1 + \frac{4\pi}{3} c\rho(\underline{r})} \underline{E}(\underline{r}), \quad \begin{matrix} \text{(zero-range} \\ \text{anticorrelations)} \end{matrix}$$

so that

$$(2-18b) \quad (\nabla^2 + k^2) \Phi(\underline{r}) = -4\pi b\rho(\underline{r})\Phi(\underline{r}) + \nabla \cdot \frac{4\pi c\rho(\underline{r})}{1 + \frac{4\pi}{3} c\rho(\underline{r})} \nabla \Phi(\underline{r}),$$

(zero-range anticorrelations)

which is the modification to the Kisslinger equations suggested by Kroll (1961) and shown in detail by the Ericsons (1966).

In the present section we have demonstrated the structure of the pion-nucleus interaction by a classical and geometric technique for adding the elementary scattered waves from a (macroscopically) homogeneous and isotropic nuclear

medium.

T. E. O. Ericson (1967) has also considered the classical problem but from a point of view which emphasizes the role of the microscopic correlations of the scatterers. His technique is to expand the optical wave in terms of the higher order correlation functions between the scatterers, truncating the resulting hierarchy of equations at pair correlations. He then finds the Kisslinger equation, 2-18a, in the absence of correlations and the Lorenz-Lorentz effect, Eq. 2-18, in the case of zero-range anticorrelations. The advantage of Ericson's technique is that it generalizes in an obvious way to include the structure of the scatterer and the quantum aspects of the problem. This generalization has been made by the Ericsons (1966) and we discuss their calculation in the next section.

2.2 A Microscopic Derivation of the Interaction:

In the preceding section, we have derived the structure of the pion-nucleus interaction using arguments which involve only the geometry of adding classical waves. In the present section we will discuss a microscopic and quantum mechanical treatment of the problem which has been given by T. E. O. Ericson and (his wife) M. Ericson (1966). The important feature of their calculation is that it enables one to introduce the structure of the scatterer and the mechanism for pion absorption. We shall discuss their calculation from a point of view which emphasizes the assumptions which must be made in order to obtain the simplified potential of the

previous section and which points out the corrections by which it can be improved.

For the purposes of its interactions with nucleons, a low energy pion may be regarded as a light boson with zero spin and three charge states (π^+ , π^0 , π^-). Moreover, unlike the nucleon-nucleon interaction, the low energy pion-nucleon interaction is described by very short scattering lengths; for instance, the NN singlet scattering length is about -24 fm, and the NN triplet scattering length is about 5 fm, while the π N isospin doublet and quadruplet scattering lengths are both of the order of 0.2 fm. It is mainly this property of the pion, combined with the small pion mass ($\sim 1/7$ the nucleon mass), which allows us to simply construct the average pion-nucleus interaction from the elementary pion-nucleon scattering data; on the other hand, the corresponding problem for the nucleon optical potential is almost untractable.

Nucleons's view
of a nucleus

Pion's view
of a nucleus

(zero energy)

To see how the small pion-nucleus scattering lengths simplify the problem, it is helpful to consider the zero-energy scattering; we can then replace the actual interactions by equivalent hard sphere interactions having the radius of the scattering length (above sketch). For a nucleon projectile, the spheres are larger than the inter-nucleon separation within the nucleus (~ 1.4 fm.), while for a pion they are much smaller.

We can interpret this as meaning that a nucleon simultaneously interacts with many nucleons within a nuclear medium so that the nucleon-nucleon scattering operator is a many-body operator depending upon the positions and motions of many other nucleons. On the other hand, a low energy pion interacts with only one nucleon at a time, and it is reasonable to assume that the pion-nucleon scattering operator within a nuclear medium is the same as in free space, particularly so since the pion is light and does not feel the dynamics of the nucleus. This assumption is commonly known as the impulse approximation (and is also valid at very high energies when the particle wavelength becomes smaller than the inter-particle spacing).

The small mass of the pion also simplifies the calculation in the following two ways. Firstly, in its elementary collisions with the nucleon scatterers, the kinetic exchanges of the pion with the scatterers are small so that we can essentially neglect the effects of intermediate nuclear excitations. In fact, in this approximation we consider the

elementary collisions to be essentially elastic, which results in an enormous simplification of the Green's function describing the propagation of the pion within the nucleus. The Ericsons have estimated the corrections from intermediate excitations and find them to be negligible.

A second simplification results from considering the scatterers to be massive. This greatly simplifies the multiple scattering equations since in the scattering processes we do not have to account for the motion of the nucleons. This assumption can be improved by making the following two kinematical corrections: firstly, we must correct for the fact that the target nucleons are in motion having a kinetic energy of about 25 MeV (Fermi motion); secondly, we must correct for the recoil of the bound nucleons. The Ericsons have estimated both corrections, finding them to be of the order of 10%.

To make these ideas more explicit, we will describe the Ericsons' technique for solving the multiple scattering problem. We will not give a detailed exposition of the derivations, but will content ourselves with stating the highlights of the calculation.

We begin by noting that the pion wave, $\psi_{\underline{k}}^+$, (which, of course, is an operator in the space of the target nucleons) satisfies the equation (Goldberger & Watson (1964)),

$$(2-19a) \quad \psi_{\underline{k}}^+ = \chi_{\underline{k}} + \sum_{i=1}^A \lim_{\epsilon \rightarrow 0} \frac{1}{E_{\underline{k}} + H_N + K_i - i\epsilon} \tau_i \psi_i,$$

where the "effective wave", ψ_i , is the sum of the incident

wave, $\chi_{\underline{k}}$, and the scattered waves from all scatterers except the i^{th} :

$$(2-19b) \quad \psi_{\underline{i}} = \chi_{\underline{k}} + \sum_{j(=i)=1}^A \lim_{\eta \rightarrow 0} \frac{1}{E_{\underline{k}} + H_N + K_{\pi} - i\eta} \gamma_j \psi_j;$$

H_N is the nuclear Hamiltonian and K_{π} the pion kinetic energy; A is the atomic number of the scatterer; $\underline{k}$ is the pion wave number $E_{\underline{k}}$ the total energy of the system; γ_i is the scattering operator from the i^{th} nucleon.

We will assume the nucleus to initially be in the ground state, $|0\rangle$, and will apply the operators, Eq. 2-19, to the system in the configuration space of the pion. Eqs. 2-19 are then equivalent to the hierarchy of equations,

$$(2-20a) \quad \psi_{\underline{k}}^r(\underline{r})|0\rangle = \chi_0(\underline{r})|0\rangle + \sum_i \iint g(\underline{r}, \underline{r}') f_i(\underline{r}', \underline{r}'') \psi_i(\underline{r}'') \otimes |0\rangle d\underline{r}' d\underline{r}'';$$

$$(2-20b) \quad \psi_i(\underline{r})|0\rangle = \chi_0(\underline{r})|0\rangle +$$

$$\sum_{j \neq i} \iint g(\underline{r}, \underline{r}') f_j(\underline{r}', \underline{r}'') \psi_{j;i}(\underline{r}'')|0\rangle d\underline{r}' d\underline{r}'';$$

$$(2-20c) \quad \psi_{j;i}(\underline{r}) = \chi_0(\underline{r})|0\rangle +$$

$$\sum_{k \neq j} \iint g(\underline{r}, \underline{r}') f_k(\underline{r}', \underline{r}'') \psi_{k;i,j}(\underline{r}'')|0\rangle d\underline{r}' d\underline{r}'';$$

where we have introduced an obvious notation to keep track of the nucleon co-ordinates. Here

$$\begin{aligned} (2-21a) \quad g(\underline{r}, \underline{r}') &= \lim_{\eta \rightarrow 0} \langle \underline{r} | \frac{1}{E_{\underline{k}} + H_N + K_{\pi} - i\eta} \sum_n |n\rangle \langle n| | \underline{r}' \rangle \\ &= \sum_n \lim_{\eta \rightarrow 0} \langle \underline{r} | \frac{1}{E_{\underline{k}} + H_N + K_{\pi} - i\eta} | \underline{r}' \rangle |n\rangle \langle n| \\ &= \sum_n g_n(\underline{r}, \underline{r}') |n\rangle \langle n| \end{aligned}$$

is the total pion-nucleus Green's function, where we have defined the one-particle Green's functions,

$$\begin{aligned}
 (2-21b) \quad g_n(\underline{r}, \underline{r}') &= \lim_{\eta \rightarrow 0} \langle \underline{r} | \frac{1}{E_{\underline{k}} + E_n + K_{\pi} - i\eta} | \underline{r}' \rangle \\
 &= \frac{e^{i\mathbf{k}_n \cdot (\underline{r} - \underline{r}')}}{|\underline{r} - \underline{r}'|}
 \end{aligned}$$

and where the sum is over all nuclear states, $|n\rangle$. For simplicity, we have ignored the pion charge in the one-particle Green's function, Eq. 2-21b (and, for notational convenience, have suppressed the internal co-ordinates of the pion). In Eq. 2-20 we have also defined the matrix elements of the scattering operator,

$$(2-22) \quad f_i(\underline{r}, \underline{r}') = \langle \underline{r} | \gamma_i | \underline{r}' \rangle$$

We have mentioned in earlier paragraphs that we can simplify the Green's function, Eq. 2-21, because of the light mass of the pion and that we can simplify the scattering operator, Eq. 2-22, mainly because of the small scattering lengths which describe the pion-nucleon interaction.

Because of the small value of the pion mass, elementary kinetic exchanges are small so that the propagation of the pion is not sensitive to the intermediate states of nuclear excitation. Thus we may choose the elementary collisions to be elastic,

$$(2-23) \quad g_n(\underline{r}, \underline{r}') \sim g_0(\underline{r}, \underline{r}')$$

whence the Green's function, Eq. 2-21, simply becomes

$$(2-24) \quad g(\underline{r}, \underline{r}') = g_0(\underline{r}, \underline{r}') \sum_n |n\rangle \langle n| = g_0(\underline{r}, \underline{r}') I_N$$

where I_N is the identity operator in the space of the nucleus.

This approximation, of course, represents an enormous simplification since it largely decouples the structure of the scatterer from the optical scattering.

To see how the impulse approximation simplifies the scattering operator, $f_i(\underline{r}, \underline{r}')$, we will write its matrix elements in the configuration space of the nucleons (suppressing the intrinsic nucleon co-ordinates). From the impulse approximation we have that,

$$(2-25) \quad \langle \underline{r}; \underline{R}_1, \underline{R}_2, \dots, \underline{R}_A | T_1 | \underline{r}'; \underline{R}'_1, \underline{R}'_2, \dots, \underline{R}'_A \rangle = \\ f_i(\underline{r}, \underline{R}_1; \underline{r}', \underline{R}'_1) \delta(\underline{R}_1 - \underline{R}'_1) \delta(\underline{R}_2 - \underline{R}'_2) \dots \delta(\underline{R}_{i-1} - \underline{R}'_{i-1}) \\ \times \delta(\underline{R}_{i+1} - \underline{R}'_{i+1}) \dots \delta(\underline{R}_A - \underline{R}'_A) ,$$

where

$$f_i(\underline{r}, \underline{R}_i; \underline{r}', \underline{R}'_i) = \langle \underline{r} - \underline{R}_i | T_i | \underline{r}' - \underline{R}'_i \rangle ,$$

T_i being the free space scattering operator of the pion from the i^{th} nucleon. Hence, the many-body scattering operator is reduced to a known two-body scattering operator.

The scattering matrix is further simplified by the assumption that the nucleons are massive:

$$(2-26) \quad f_i(\underline{r}, \underline{R}_i; \underline{r}', \underline{R}'_i) = f_i(\underline{r} - \underline{R}_i; \underline{r}' - \underline{R}'_i) \delta(\underline{R}_i - \underline{R}'_i) .$$

Since in the impulse approximation the scattering operator is simply the free space scattering operator of a pion from a nucleon, it is easily evaluated because its matrix elements in momentum space are (at low energies) simply the s-wave and p-wave scattering amplitudes:

$$\begin{aligned}
 (2-27) \quad f_i(\Omega) &\equiv \langle \underline{k}' | T_i | \underline{k} \rangle \\
 &= b_0 + b_1 \underline{T} \cdot \underline{t}_i \quad (\text{s-wave}) \\
 &\quad (c_0 + c_1 \underline{T} \cdot \underline{t}_i) \underline{k} \cdot \underline{k}' + i(d_0 + d_1 \underline{T} \cdot \underline{t}_i) \underline{k}' \times \underline{k} \cdot \underline{\sigma}_i, \\
 &\quad (\text{p-wave})
 \end{aligned}$$

Here b_0 , b_1 are related to the s-wave pion-nucleon scattering lengths, and c_0 , c_1 , d_0 , d_1 are related to the p-wave scattering volumes (c.f. Ericson and Ericson (1966)); the spin-flip term has no s-wave contribution since the pion has spin zero. In Eq. 2-27, $\underline{T}$ and $\underline{t}_i$ are the isospin operators of the pion and the i^{th} nucleon, respectively. As we have often noted, it is important to retain the p-wave contributions, even at low energies, because of the lingering effects of the 3-3 resonance; the higher partial waves make negligible contributions.

Employing the assumptions in Eqs. 2-23 to 2-27, the optical pion wave,

$$(2-28) \quad \bar{\Phi}(\underline{r}) \equiv \langle 0 | \psi_{\underline{k}}^+(\underline{r}) | 0 \rangle,$$

can be evaluated in a straightforward manner from Eqs. 2-20. The Ericsons (1966) have performed the nuclear averages and find multiple scattering equations which are essentially those for multiple scattering in a classical media (Lax (1951)):

$$\begin{aligned}
 (2-29a) \quad \bar{\Phi}(\underline{r}) &= \chi_0(\underline{r}) + \int \mathcal{P}(\underline{r}_1) d\underline{r}_1 \iint g_0(\underline{r}-\underline{r}') f(\underline{r}'-\underline{r}_1; \underline{r}''-\underline{r}_1) \\
 &\quad \times \bar{\Phi}_{\underline{r}_1}(\underline{r}'') d\underline{r}' d\underline{r}'';
 \end{aligned}$$

$$\begin{aligned}
 (2-29b) \quad \bar{\Phi}_{\underline{r}_1}(\underline{r}) &= \chi_0(\underline{r}) + \int \mathcal{P}(\underline{r}_2; \underline{r}_1) d\underline{r}_2 \iint g_0(\underline{r}-\underline{r}') \\
 &\quad \times f(\underline{r}'-\underline{r}_2; \underline{r}''-\underline{r}_2) \bar{\Phi}_{\underline{r}_2; \underline{r}_1}(\underline{r}'') d\underline{r}' d\underline{r}'';
 \end{aligned}$$

$$(2-29c) \quad \bar{\Phi}_{\underline{r}_2; \underline{r}_1}(\underline{r}) = \chi_0(\underline{r}) + \int \rho(\underline{r}_3; \underline{r}_1, \underline{r}_2) d\underline{r}_3 \iint \epsilon_0(\underline{r} - \underline{r}') \\ \times f(\underline{r}' - \underline{r}_3; \underline{r}'' - \underline{r}_3) \bar{\Phi}_{\underline{r}_3; \underline{r}_1, \underline{r}_2}(\underline{r}'') d\underline{r}' d\underline{r}'' ;$$

----- ;

Here $\rho(\underline{r}_1)$ is the average density of the nucleus; $\rho(\underline{r}_1; \underline{r}_2)$ is the probability density for finding a particle at $\underline{r}_1$ when there is a particle at $\underline{r}_2$; $\bar{\Phi}_{\underline{r}_2; \underline{r}_1}(\underline{r}'')$ is the sum at $\underline{r}''$ of the incident wave and the scattered waves from all scatterers except the one at $\underline{r}_2$, subject to the knowledge that there is a scatterer at $\underline{r}_1$. The other terms have a similar meaning. These equations describe essentially the same problem that we gave a geometric solution for in Sec. 2-1 and they have been discussed in various approximations by Ericson (1967).

We shall now wish to truncate the hierarchy of Eqs. 2-29 at some point and solve the coupled equations. For instance, if we took

$$(2-30) \quad \bar{\Phi}_{\underline{r}_1}(\underline{r}) = \chi_0(\underline{r})$$

in Eq. 2-29a, it is seen that we would simply have made the first Born approximation. That this is not a good approximation can be seen from the dominance of the s- and p-waves in the pion-nucleon scattering amplitudes.

As a second attempt --- one which includes multiple scattering --- we might take

$$(2-31) \quad \bar{\Phi}_{\underline{r}_1}(\underline{r}) = \bar{\Phi}(\underline{r}) .$$

This is identical to our geometric derivation in the first section in which we neglected nuclear correlations; in fact,

this is seen explicitly to be the case since terminating the hierarchy at Eq. 2-29a is equivalent to neglecting the correlations. Of course, this approximation leads to the Kisslinger Eq. 2-18a. From our present point of view, we note that the primary handicap in this approximation is that $\bar{\Phi}_{\underline{r}_1}(\underline{r})$ differs from $\bar{\Phi}(\underline{r})$ at $\underline{r}$ by a wave which is singular at $\underline{r}$.

To avoid this self-excitation of the scatterer, we go to the next approximation and set

$$(2-32) \quad \bar{\Phi}_{\underline{r}_2; \underline{r}_1}(\underline{r}) = \bar{\Phi}_{\underline{r}_2}(\underline{r})$$

in Eq. 2-29b. Since $\underline{r}_2$ and $\underline{r}_1$ are generally far apart in the nucleus, we do not expect the wave $\bar{\Phi}_{\underline{r}_2; \underline{r}_1}(\underline{r})$ to depend very sensitively on the exact position of $\underline{r}_1$. In fact, if we write the joint probability density, $\mathcal{P}(\underline{r}_2; \underline{r}_1)$, in terms of the pair correlation function, $G(\underline{r}_1, \underline{r}_2)$,

$$(2-33) \quad \mathcal{P}(\underline{r}_2; \underline{r}_1) = \mathcal{P}(\underline{r}_2) (1 + G(\underline{r}_1, \underline{r}_2)) ,$$

we can remove the unphysical divergence at $\underline{r}_1 = \underline{r}_2$ by introducing the zero-range anticorrelation of two nucleons,

$$(2-34) \quad G(\underline{r}_1, \underline{r}_2) = -\delta_{\underline{r}_1, \underline{r}_2} .$$

In fact, it is this choice of the pair correlation function which leads to the Lorenz-Lorentz effect, as we found in our geometric derivation (Eq. 2-18b).

The Ericsons have shown that replacing the zero-range correlation function, Eq. 2-34, with a function of finite range, leads to effects of order $(\xi K)^2$ where ξ is the range of the correlation function and K is the wave number of the

pion in the nuclear medium. The structure of the correlation region is thus negligible for wavelengths appreciably larger than one fermi; on the other hand, typical wavelengths of low energy pions are several times the nuclear dimensions. Hence, we expect to learn little about the structure of the correlation function from low energy pion scattering. In fact, the Ericsons have estimated these range corrections and find them to be of the order of 10%. This is in contrast to the measurements of molecular correlations through the scattering of slow neutrons by a crystal lattice; here the neutron-nuclear scattering lengths are also much smaller than the inter-molecular spacing, but the neutron wavelength is of a comparable value (Van Hove (1954)).

We have now discussed the connections between the microscopic quantum description of the pion-nucleus interaction and the geometric derivation given in the previous section (Sec. 2-1). In addition, we have discussed the kinematic and range corrections which must be made to the pion-nucleus interaction, Eqs. 2-18. It now remains to describe the Ericsons' technique for describing optical pion absorption.

There are essentially three mechanisms by which a low energy optical pion can be absorbed: firstly, it can be inelastically scattered by the nucleus; secondly, being a boson, it can be absorbed by a single nucleon, its rest mass being transformed into the kinetic energy of the scatterer; thirdly, it can be absorbed by two nucleons. Excitation of the nucleus is suppressed due to the small mass of the pion, so that effects of the first mechanism may be neglected.

In addition, the absorption by one nucleon requires a nuclear momentum of about 500 MeV/c from kinematic considerations while the Fermi momentum of the nucleons is of the order of 250 MeV/c; hence, one-nucleon absorption is also strongly suppressed. Thus only the process of two-nucleon absorption contributes significantly to the optical absorption.

To strictly account for two-nucleon pion absorption, we should use field theoretic techniques to describe the creation and destruction of pions. In addition, we would need a detailed model of the nucleus (such as the shell model) to describe the kinematics of the absorption, which obviates the simple macroscopic description which we obtained for the refractive processes. As discussed by the Ericsons, this method of approach has not met with much success; in fact, proceeding in this manner, Spector (1964) found absorption rates nearly two orders of magnitude smaller than those observed. This then would seem sufficient justification to use a more phenomenological approach, and the Ericsons have done this by extending and reformulating the quasi-deuteron model of Brueckner et al. (1951).

In the Brueckner model, absorption is considered to take place at zero energy with the two absorbing nucleons in a relative s-state; the two-nucleon absorption amplitude is then introduced as a phenomenological quantity accounting for all the short-range absorption processes. Hence, the problem of obtaining a description which is satisfactory from a field theoretic point of view is largely transferred to a justification of these phenomenological quantities.

The Ericsons have generalized the Brueckner model (and its reformulation by Eckstein (1963)) in two ways. Firstly, they introduce an elastic scattering amplitude to describe pion scattering by a two-nucleon scatterer; following Eckstein (1963), they measure the parameters of this amplitude from considerations of pion production in NN collisions. Secondly, they extend the Brueckner model, which included only s-wave pion absorption, to include p-wave pion absorption. The result is that the two-nucleon amplitude can be written in the form,

$$(2-35) \quad f_{ij} = \int (\underline{r}_i - \underline{r}_j) \int (\underline{r} - \frac{1}{2}(\underline{r}_i + \underline{r}_j)) (B_0 + C_0 \underline{k}' \cdot \underline{k} \\ + (B_5 + C_5 \underline{k}' \cdot \underline{k}) [(\underline{T} \cdot \underline{t}_i)(\underline{T} \cdot \underline{t}_j) + (\underline{T} \cdot \underline{t}_j)(\underline{T} \cdot \underline{t}_i)]),$$

where B_m and C_m are linear combinations of the various amplitudes of the angular momentum and isospin states of the (π 2N) system. Since Eq. 2-35 is of the form of the (π N) amplitude, Eq. 2-27, we can simply account for pion optical absorption by adding the two-nucleon amplitude of Eq. 2-35 to the one-nucleon amplitude of Eq. 2-27. Of course, the two-nucleon amplitudes will also make contributions to the refractive aspects of the problem through the real parts of the parameters B_m and C_m ; these real parts are to some extent induced by the absorption, and are not very well determined. This is of particular interest to the s-wave scattering, where a strong cancellation occurs in the one-nucleon amplitude, b_0 .

Of course, for the two-nucleon processes, the two-

nucleon density replaces the one-nucleon density. To a good approximation the two-nucleon density can simply be replaced by the square of the density; however, in the case of s-waves, where the scattering lengths strongly cancel, the correction for long-range correlations should be included. The Ericsons show that the correction term is also proportional to the square of the density; for our later purposes we will only require the fact that absorptive processes are proportional to the square of the density and we will not include this correction in our subsequent discussion.

We shall now simply state the result which the Ericsons obtain for the pion-nucleus interaction. Their result is essentially that the pion wave satisfies the optical equation,

$$(2-36a) \quad (\nabla^2 + k^2) \Phi(\underline{r}) = \left[-m_1(\underline{r}) - m_2(\underline{r}) + \nabla \cdot \frac{n_1(\underline{r}) + n_2(\underline{r}) - 14\pi d_0 \sigma(\underline{r}) \underline{x}}{1 + \frac{1}{3}(n_1(\underline{r}) + n_2(\underline{r}))} \right] \nabla \Phi(\underline{r}),$$

where

$$(2-36b) \quad m_1(\underline{r}) = 4\pi(1 + \frac{m_\pi}{M_N}) (b_0 \rho(\underline{r}) + 2b_1 \frac{\underline{T} \cdot \underline{t}}{A} \rho(\underline{r})) + (\frac{m_\pi}{M_N})^2 n_1(\underline{r}) (1 + \frac{1}{3}(n_1(\underline{r}) + n_2(\underline{r})))^{-1} \langle p_N^2 \rangle;$$

$$(2-36c) \quad m_2(\underline{r}) = 4\pi(1 + \frac{m_\pi}{2M_N}) \left[B_0 \rho^2(\underline{r}) + \frac{8B_5 (\underline{T} \cdot \underline{t})(\underline{T} \cdot \underline{t}) \rho^2(\underline{r})}{A^2} \right] + (\frac{m_\pi}{2M_N})^2 n_2(\underline{r}) \left[1 + \frac{1}{3}(n_1(\underline{r}) + n_2(\underline{r})) \right]^{-1} \langle p_{2N}^2 \rangle;$$

$$(2-36d) \quad n_1(\underline{r}) = 4\pi(1 + \frac{m_\pi}{M_N}) \left[c_0 \rho(\underline{r}) + \frac{2 c_1 \underline{T} \cdot \underline{t} \rho(\underline{r})}{A} \right];$$

$$(2-36e) \quad n_2(\underline{r}) = 4\pi(1 + \frac{m_\pi}{2M_N}) \left[c_0 \rho^2(\underline{r}) + \frac{8c_5 (\underline{T} \cdot \underline{t})(\underline{T} \cdot \underline{t}) \rho^2(\underline{r})}{A^2} \right];$$

The electrostatic potential, $V_c(\underline{r})$, which we have not included in Eqs. 2-36, is to be evaluated from the nuclear charge distribution; we discuss the electrostatic potential in Sec. 3-3 of the next chapter.

Comparing Eqs. 2-36 and Eqs. 2-18, we see that the interaction retains the simple form predicted by the geometric arguments of the previous section (Sec. 2-1).

2-3 Experimental Verification

In the present section we will briefly review the experimental corroboration of the Ericsons' potential, Eq. 2-36. Virtually the only data available in the low energy region is that from the π -mesic x-ray experiments (which are at essentially zero energy). The technique is to analyze the shifts and the widths of these x-rays with the potential, Eq. 2-36, and to compare the resulting parameters with the theoretically predicted values. A general remark might be that the Ericsons (1966) and Ericson (1967) find theory to be in satisfactory agreement with experiment, the theoretical and empirical parameters generally agreeing to better than 50%. We shall only cursorily review their discussion here.

The essential ideas involved in analyzing the π -mesic x-ray data for the parameters of the pion-nucleus potential

are as follows. Since a pion is much heavier than an electron, a pion in a π -mesic atom spends much more time in the nucleus than does an electron of corresponding quantum numbers in a conventional atom. The energy levels of a π -mesic atom are thus considerably modified from those which we would expect from the purely electromagnetic forces. The effect of the real terms in the optical potential is to shift the energy levels from those values calculated from the electromagnetic interaction and the effect of the absorptive terms is to broaden the levels. Thus we can measure the real terms in the optical potential by analyzing the level shifts of the π -mesic x-rays and we can measure the imaginary terms in the optical potential by analyzing their widths.

A simple way to analyze the real level shifts and the widths, or imaginary level shifts, is to calculate the complex level shifts, ΔE_{nl} , of the level (n, l) , in first-order perturbation theory:

$$(2-37) \quad \Delta E_{nl} = \frac{\hbar^2}{2m_\pi} \int m(\underline{r}) |\phi_{nl}(\underline{r})|^2 d\underline{r} + \frac{\hbar^2}{2m_\pi} \int \frac{n(\underline{r})}{1 + \frac{n(\underline{r})}{3}} |\nabla \phi_{nl}(\underline{r})|^2 d\underline{r};$$

here $\phi_{nl}(\underline{r})$ is the wave function of the pion in the orbit with principal quantum number, n , and angular momentum, l , calculated from the electromagnetic interaction; $m(\underline{r})$ and $n(\underline{r})$ schematically represent the local and momentum-dependent potentials in Eq. 2-36 (neglecting the hyperfine term).

Now at the x-ray energies (~ 1 MeV) the intra-nuclear pion wavelength is long compared to the nuclear dimensions

and the s-wave function ($l=0$) varies slowly with radius. The complex s-wave level shifts are thus dominated by the local interaction, $m(\underline{r})$, in Eq. 2-37, since the gradient term which enters the momentum-dependent interaction is small. By analyzing the s-wave shifts we therefore obtain the local interaction, $m(\underline{r})$, and, hence, the microscopic s-wave pion-nucleon and pion-deuteron parameters, b_m and B_m , in Eq. 2-36. For higher partial waves, $\tilde{\phi}_{nl}(\underline{r})$, is a much stronger function of radius (since it looks like a spherical Bessel function of order, l) so that, from the complex level shifts of higher angular momentum states, we can obtain the momentum-dependent interaction, $n(\underline{r})$, and, hence, the microscopic p-wave pion-nucleon and pion-deuteron parameters, c_m and C_m , of Eq. 2-36.

In actual practice, the Ericsons have not used the perturbation formula, Eq. 2-37, to analyze the π -mesic x-ray data but rather have obtained analytic formulae for the level shifts (for a uniform potential) by matching wave functions calculated inside the potential with the well-known Whittaker functions (bound-state spherical Coulomb functions) which describe the pion wave outside the potential.

The validity of the Ericsons' potential is tested by comparing the pion-nucleon and pion-deuteron scattering lengths (or volumes) obtained from analyzing the π -mesic x-ray data with Eq. 2-36 with those obtained from the elementary experiments. The most important uncertainty lies in the real part of the local interaction, where a large can-

cellation (1:10) occurs in the s-wave scattering lengths. For this reason terms ordinarily of order 10% --- the Fermi motion corrections, the finite correlation length corrections, the real part of the two-nucleon scattering amplitude --- become of first order importance. Nevertheless, making these corrections, the Ericsons find remarkable agreement with the empirical values ($\sim 20\%$) as calculated from the 1s level shifts in Li and F, although this may be somewhat fortuitous. The local isotopic spin term in the local interaction is found to have a similar empirical validity. The local absorption term is measured from the 1s level widths and is found to be in agreement to about 50%. The non-local terms are measured from the 2p and 3d level shifts and widths (where momentum-dependent effects predominate) and are found in similar agreement to the local terms.

A good deal of this imprecision may lie in the rather crude data obtained from the x-ray measurements and in the only poorly determined empirical input scattering lengths rather than in the precision of the technique of calculation. At any rate, the potential would seem to be sufficiently well verified (at least at zero energy) to study its properties further. In the subsequent chapters we will investigate the properties of the optical interaction and suggest how it may be used to measure the diffuseness of the nuclear surface.

CHAPTER 3THE OPTICAL PROPERTIES OF
LOW ENERGY PIONS IN SPHERICAL NUCLEI

In the previous chapter we have shown that the structure of the pion-nucleus optical interaction is quite well understood and we have suggested that in the low energy region (≤ 30 MeV) at least the qualitative features of the interaction are known with reasonable certainty. Our purposes in the present chapter are two-fold: firstly, we wish to use this interaction to study the optical properties of low energy pions in spherical nuclei; secondly, we wish to study the effects of the diffuseness of the nuclear surface on scattering and absorption cross sections. In a later chapter we will use these results to interpret the behaviour of deformed nuclei in the presence of the optical pion field.

In this chapter our first objective is to display the various optical processes associated with the low energy pion-nucleus interaction. We shall be particularly concerned with those processes which concern the wave properties of the pion inside the nucleus, namely, resonance and absorption. Since we are at low energies, it is convenient to demonstrate these properties by making a partial wave analysis. We then follow the familiar technique of defining an interface in the region of the nuclear surface and match wave functions calculated inside the interface from the optical potential with exterior wave functions which depend only upon the electrostatic and angular momentum barriers surrounding the

nucleus (spherical Coulomb functions).

This allows us to state the partial wave phase shifts in terms of the interior logarithmic derivatives which, for the low energy pion-nucleus interaction, have a simple and instructive form: the real parts of the interior logarithmic derivatives, which describe the resonance properties of the optical interaction, are (in the low energy region) insensitive to the details of the optical potential because of the long pion wavelength inside of the nucleus; the imaginary parts of the interior logarithmic derivatives, which describe absorption, can be written in the form of a sum over local and momentum-dependent absorption so that we can explicitly state and examine the relative effects of these processes near the top of the local potential barrier where the pion momentum is small.

In Sec. 3-2 we present a qualitative discussion of the pion-nucleus interaction treating the nucleus as having a uniform density. In the low energy region (≤ 30 MeV) scattering and absorption are governed by the potential barrier (~ 15 MeV) arising from the repulsive local interaction. This barrier results in the pion wavelength in the nucleus being very long so that the resonance properties are determined mainly by the size of the scattering nucleus rather than by the details of the interaction (in contrast to nucleon-nucleus scattering where we have an attractive potential and a short nucleon wavelength inside the nucleus). In fact, the resonance properties of s-waves are quite

insensitive to the interaction while the resonance properties of higher partial waves depend mainly upon the effective mass of the pion inside the nucleus arising from the momentum-dependent interaction. This latter effect is expected since angular momentum must be conserved in crossing the nuclear interface while the mass of the pion apparently changes. Nevertheless, because of the long pion wavelength, we learn little about the structure of the nuclear interface --- for instance, the diffuseness of the nuclear surface --- from the effects of resonance upon optical scattering experiments.

Of course, we can go to higher energies where the wavelength of the pion is shorter and resonance effects are more dependent upon the details of the potential, but we then encounter two difficulties: firstly, except in very light nuclei, several partial waves enter the problem so that it is difficult to untangle these various partial waves in a convincing manner; secondly, the simple connection between microscopic processes and optical phenomena which we have discussed in Sec. 2-2 is no longer valid. However, from our discussion of the structure of the pion-nucleus interaction in Sec. 2-1, we might expect that the only effect of the latter difficulty is to make the parameters of the potential energy dependent. In fact, Kisslinger (1955) was able to show that the strong back scattering of 62 MeV pions in C^{12} could only be explained when proper account was taken of the nuclear surface. Nevertheless, as we will now show, low energy optical pion absorption cross sections should provide a much more

direct method for measuring the nuclear surface and one which can be used even in heavy nuclei.

The basic idea of using absorption cross sections to measure the nuclear diffuseness is the following. The optical absorption of pions arises both from the local interaction and the momentum-dependent interaction. The absorption due to the local interaction is sensitive only to the total amount of matter present and is rather insensitive to its distribution. On the other hand, as the pion enters the nucleus and encounters the repulsion of the local potential, the internal pion momentum becomes small so that momentum-dependent processes are both suppressed and, at the same time, very energy dependent; in fact, the pion momentum varies rapidly over the nuclear surface so that the interaction is very sensitive to the nuclear diffuseness. Since the momentum-dependent processes are comparable in magnitude to the local processes, absorption cross sections have considerable structure near the top of the potential barrier; in fact, this structure clearly mirrors the diffuseness of the nuclear surface. Of course, our arguments strictly apply only to s-waves which have no angular momentum; however, the effects of angular momentum for higher partial waves is simply to "raise" the barrier yielding a similar sort of structure only at higher energy. Since we are at low energies, only a few partial waves enter the problem and their untanglement is relatively straightforward. Optical pion absorption should therefore be a good tool for measuring the nuclear diffuseness in

a large variety of nuclei.

One of the major reasons for wishing to know the diffuseness of the nuclear surface is that it allows us to infer the neutron density in the nuclear surface. In fact, the nuclear density is simply the sum of the proton and neutron densities and, since the proton density is well-known from electron scattering and μ -mesic x-ray experiments (Devons and Duerdorth (1969), Wu (1967)), we can easily infer the latter. This elusive quantity is of considerable interest to the theory of nuclear structure (for instance, the shell-model) and results currently available from various experiments are not mutually consistent, as discussed below.

One technique for measuring the surface distribution of neutrons has been to observe the branching ratios for events involving various elementary particles. Auerbach et al. (1968) have reanalyzed the 700 MeV pion charge-exchange data of Abashian et al. (1956) and find that the rms neutron radius agrees with the rms proton radius to within a few percent (generally being smaller) unless one admits sharply peaked neutron distributions in the nuclear surface. Burhop (1967) has analyzed the decay products resulting from the capture of kaons in atoms and finds that the results can be accommodated by choosing the rms radius of the proton and neutron distributions to be the same but with the neutron surface thickness 50% greater than that of the proton distribution; however, as Bugg et al. (1969) have pointed out, these results are somewhat clouded by the complicated atomic physics which enters the problem.

Temmer (1966) suggested that the observed Coulomb displacement energies between isobaric analog states in lead isotopes was evidence for a neutron excess in heavy nuclei; Nolen et al. (1968) extended these results and found the rms neutron radius to be about a percent larger than that of the charge radius. This is in agreement with results using optical scattering of nucleons: Greenlees et al. (1966) measured the rms radius of the neutron distribution by choosing the rms radius of the mass distribution to be given by the spin-orbit term in the optical potential; Elton (1968) deduced the distributions from the shell-model potentials obtained from analysis of the single-particle and single-hole states in Pb^{208} . Both authors found that the neutron distribution is up to ten percent greater than the proton distribution. We feel that these disagreements can be largely resolved simply by measuring the optical absorption of low energy pions in heavy nuclei where the nuclear density enters the interaction in a much more direct manner.

Unfortunately, at the present time there is essentially no optical scattering data available for pions in the low energy region (< 50 MeV). We therefore thought it useful to discuss one example (the scattering of positive pions from Ca^{40}) from the point of view from which an experiment might be analyzed. We first give a numerical discussion of results for the Ericsons' potential described in Sec. 2-2 taking into account the electrostatic potential arising from the nuclear charge distribution and noting relativistic corrections which

may become important at higher energies; this discussion is contained in Sec. 3-3. While this zero-energy potential may neglect some energy dependence of the parameters of the potential at higher energies, it should provide a reasonable description of empirical processes.

In Sec. 3-4 we use this potential to construct the elastic scattering and absorption cross sections of pions from a light nucleus such as Ca^{40} . We discuss these cross sections in terms of a partial wave analysis and we use the formalism developed in Sec. 3-1 to provide an explanation of the partial wave phase shifts in terms of the internal logarithmic derivatives. We examine the optical properties of the potential for this quantitative example in terms of the qualitative features which we describe for a uniform distribution in Sec. 3-2. In addition, we suggest a procedure for obtaining the empirical parameters of the potential from the experimental cross sections. We also extend our discussion to examine the effects of the diffuseness of the nuclear surface on the optical cross sections, particularly the absorption cross sections. We then extend our discussion to the properties of pion scattering and absorption in heavy nuclei, choosing as an example the nucleus Pb^{208} .

In Sec. 3-5 we briefly mention some of the terms of order A^{-1} and higher which enter the Ericsons' potential described in Eq. 2-36. These include isospin terms, which macroscopically account for single and double charge exchange, and terms which depend on the nuclear spin and lead to

hyperfine splitting in the π -mesic x-ray spectrum. Because these terms are small and play a subordinate role in determining the optical properties of pions in nuclei, they have been omitted from the remaining discussion of the thesis.

3-1 A Formalism for Analyzing Optical Scattering and Absorption

In the present section we will describe a formalism for analyzing optical scattering and absorption cross sections by a partial wave analysis. Our technique is to first choose an interface in the region of the nuclear surface. Then, for each partial wave, we calculate the radial wave functions inside the interface from the pion-nucleus optical potential; the partial wave phase shifts are then defined by matching these interior wave functions to the incoming and outgoing spherical Coulomb functions which describe the interaction outside the interface. This procedure allows us to express the phase shifts in terms of quantities which are specific to the barrier --- the penetration factor and shift function --- and quantities which are specific to the optical potential --- the interior logarithmic derivatives. The formulae we derive are well-known for local potentials and, where necessary, we extend them to momentum-dependent potentials.

The advantage of our formulae is that they allow us direct access to the interior logarithmic derivatives. For the pion-nucleus interaction these quantities are both simple and instructive: as we show in the next section, the real parts of the logarithmic derivatives, which describe the

resonance aspects of the problem, are roughly constant because of the long pion wavelength inside the nucleus; the imaginary parts are related in a very direct way to absorption. Hence, in the discussion of the later sections, we will emphasize the role of the logarithmic derivatives in understanding the partial wave phase shifts; the connection between these concepts and the empirical data is the subject of the present section.

A spherically symmetric and momentum-dependent potential, such as that which describes the pion-nucleus optical interaction in spherical nuclei, has the general form,

$$(3-1a) \quad \mathcal{V}(r) = -\frac{\hbar^2}{2\mu} \nabla \cdot \alpha^e(r) \nabla + V_c(r) + V(r) + iW(r).$$

Here,

$$(3-1b) \quad \alpha^e(r) = \frac{\alpha(r)}{1 - \frac{\alpha(r)}{3}} = \alpha_R^e(r) + i \alpha_I^e(r)$$

is the effective strength of the momentum-dependent interaction, including the term $(1 - \frac{\alpha(r)}{3})^{-1}$ arising from the Lorentz-Lorentz effect, where $\alpha(r)$ represents terms in Eq. 2-36, $n_1(r)$ and $n_2(r)$, which are proportional to the density (or square of the density); $V_c(r)$ is the electrostatic potential, $V(r)$ is the real local potential, and $W(r)$ is the absorptive local potential. The non-relativistic Schroedinger equation which describes the pion-nucleus interaction then has the form

$$(3-2) \quad \frac{\hbar^2}{2\mu} \nabla^2 \phi(\underline{r}) + \mathcal{V}(r)\phi(\underline{r}) = E\phi(\underline{r}) ;$$

E is the relative energy (and μ the reduced mass) of the pion and the target nucleus.

Eq. 3-2 can be reduced to a set of radial equations by making the partial wave expansion,

$$(3-3) \quad \phi(\underline{r}) = \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)} i^{\ell} \frac{u_{\ell}(r)}{r} Y_{\ell}^0(\theta) ,$$

where ℓ is the orbital angular momentum. Substituting Eq. 3-3 in Eq. 3-2, we deduce the radial Schroedinger equations, for each angular momentum value, ℓ , for the radial pion waves, $u_{\ell}(r)$:

$$(3-4) \quad -\frac{\hbar^2}{2\mu} (1 + \alpha^e(r)) u_{\ell}''(r) - \frac{\hbar^2}{2\mu} \alpha_e'(r) u_{\ell}'(r) + \frac{\hbar^2}{2\mu} \frac{\ell(\ell+1)}{r^2} (1 + \alpha^e(r)) u_{\ell}(r) + \frac{\hbar^2}{2\mu} \frac{\alpha_e'(r)}{r} u_{\ell}(r)$$

$$+ [V_c(r) + V(r) + iW(r)] u_{\ell}(r) = E u_{\ell}(r) \quad (\ell = 0, 1, 2, \dots) .$$

To begin our calculation, we will first define the boundary conditions on Eq. 3-4 by assuming the usual experimental situation of a beam of particles described by an incident distorted plane wave, ϕ_c :

$$(3-5) \quad \phi_c(\underline{r}) = \frac{1}{\rho} \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)} i^{\ell} e^{i\sigma_{\ell}^c} F_{\ell}(\eta, \rho) Y_{\ell}^0(\theta) ;$$

here $F_{\ell}(\rho, \eta)$ is the regular spherical Coulomb wave function,

$$\sigma_{\ell}^c \equiv \arg \Gamma(\ell+1+i\eta) = \sigma_0^c + \sum_{s=1}^{\ell} \tan^{-1} \frac{\eta}{s}$$

is the usual Coulomb phase shift, and ρ and η are the conventional Coulomb parameters,

$$\rho = kr; \quad \eta = \frac{zZe^2}{\hbar} \sqrt{\frac{\mu}{2E}} ;$$

where k is the free space wave number,

$$k = \frac{\sqrt{2\mu E}}{\hbar}$$

Z is the charge of the target nucleus and z the charge of the pion (1, 0, or -1).

The technique is to now solve Eq. 3-4 inside some matching radius, R_0 , which is chosen sufficiently close to the nucleus so as to exclude barrier properties but sufficiently far removed so as to essentially include all aspects of the potential. Subject to Eq. 3-5, we then obtain the familiar continuity conditions at the matching radius for each partial wave and its derivative in terms of the (complex) phase shifts,

α_ℓ :

$$(3-6a) \quad I_\ell(\mathcal{P}_0) - e^{2i\alpha_\ell} O_\ell(\mathcal{P}_0) = u_\ell(\mathcal{P}_0); \quad (\mathcal{P}_0 \equiv kR_0;$$

$$(3-6b) \quad I'_\ell(\mathcal{P}_0) - e^{2i\alpha_\ell} O'_\ell(\mathcal{P}_0) = u'_\ell(\mathcal{P}_0); \quad \left(\frac{d}{d\mathcal{P}} \right)$$

where

$$(3-7a) \quad I_\ell(\mathcal{P}_0) = e^{i\sigma_\ell^c} (G_\ell(\mathcal{P}_0) - iF_\ell(\mathcal{P}_0));$$

$$(3-7b) \quad O_\ell(\mathcal{P}_0) = e^{-i\sigma_\ell^c} (G_\ell(\mathcal{P}_0) + iF_\ell(\mathcal{P}_0));$$

are the exterior spherical incoming and outgoing Coulomb waves, respectively: $G_\ell(\mathcal{P}, \eta)$ is the irregular spherical Coulomb wave function.

Of course, Eqs. 3-6 are trivially soluble for the total phase shifts, α_ℓ . The problem is to re-express the solution in a form which explicitly states the optical properties of the pion-nucleus interaction. To do this we need only note that the partial wave phase shift, α_ℓ , is the sum of two terms:

$$(3-8) \quad \alpha_\ell = \sigma_\ell^c + (\delta_\ell + i\beta_\ell),$$

where σ_ℓ^c is the Coulomb phase shift associated with the long-

range Coulomb potential and where $(\delta_\ell + i\beta_\ell)$ is the phase shift which arises from the short-range pion-nucleus potential. It is the latter phase shift which contains the optical properties of the potential and it is this phase shift which we wish to evaluate.

To perform this evaluation, it is convenient to first account for the Coulomb and angular momentum barriers which surround the target nucleus. These barriers have essentially two effects which are specific to the short-range interaction. Firstly, we must, for each partial wave, account for the quantum mechanical penetration of the pion through these barriers which we do by introducing the conventional penetration factors,

$$(3-9a) \quad P_\ell(\mathcal{P}_0) \equiv \frac{\mathcal{P}_0}{F_\ell(\mathcal{P}_0)^2 + G_\ell(\mathcal{P}_0)^2}.$$

Secondly, we must account for the phase of the waves in going through the barriers, which is taken care of by the shift functions[#],

$$(3-9b) \quad S_\ell(\mathcal{P}_0) \equiv \frac{\mathcal{P}_0 [F'_\ell(\mathcal{P}_0)F_\ell(\mathcal{P}_0) + G'_\ell(\mathcal{P}_0)G_\ell(\mathcal{P}_0)]}{F_\ell(\mathcal{P}_0)^2 + G_\ell(\mathcal{P}_0)^2}.$$

It is convenient to note that these quantities are summarized in the exterior derivative quantity,

[#]

The term "shift function" is used for a slightly different quantity in the $\mathcal{R}$ -matrix theory of nuclear reactions (Vogt (1962)). It differs from our shift function mainly by the addition of a boundary condition number.

$$(3-10) \quad L_\ell(\rho_0) \equiv \rho_0 \frac{O'_\ell(\rho_0)}{O_\ell(\rho_0)} = S_\ell(\rho_0) + iP_\ell(\rho_0) .$$

The properties of the short-range potential are entirely contained in the interior logarithmic derivatives,

$$(3-11) \quad \lambda_\ell(\rho_0) \equiv \rho_0 \frac{u'_\ell(\rho_0)}{u_\ell(\rho_0)} \equiv \sigma_\ell(\rho_0) + i\pi_\ell(\rho_0) ;$$

the real part, σ_ℓ , contains the resonance properties of the interaction and the imaginary part, π_ℓ , the absorptive properties.

The match of the interior and exterior wave functions is provided by Eqs. 3-6 which can now be conveniently rewritten in the form,

$$(3-12) \quad e^{2i\alpha_\ell} \frac{O_\ell(\rho_0)}{I_\ell(\rho_0)} = \frac{\lambda_\ell(\rho_0) - L_\ell(\rho_0)^*}{\lambda_\ell(\rho_0) - L_\ell(\rho_0)} .$$

Now, as is easily shown from Eqs. 3-7,

$$(3-13) \quad \frac{O_\ell(\rho_0)}{I_\ell(\rho_0)} = e^{-2i\sigma_\ell^c} e^{-2i\Omega_\ell} ,$$

where σ_ℓ^c is the Coulomb phase shift (Eq. 3-5 ff.) and

$$(3-14) \quad \Omega_\ell = \tan^{-1} \left(-\frac{F_\ell}{G_\ell} \right)$$

is the well-known hard-sphere phase shift which accounts for the position of the matching radius with respect to the origin. Defining the partial wave transmission functions,

$$(3-15) \quad T_\ell = 1 - e^{-4\beta_\ell} ,$$

and using Eqs. 3-10, 3-12, and 3-13, we see that

$$(3-16a) \quad \delta_\ell = \frac{1}{2} \tan^{-1} \frac{2P_\ell(\sigma_\ell - S_\ell)}{(\sigma_\ell - S_\ell)^2 + \pi_\ell^2 - P_\ell^2} + \Omega_\ell ;$$

(3-16b)

$$T_\ell = \frac{P_\ell \pi_\ell}{(\sigma_\ell - S_\ell)^2 + (P_\ell + \pi_\ell)^2}$$

The real part, δ_ℓ , of the partial wave phase shifts accounts for the reflectivity and resonance properties of the optical potential; the imaginary part, β_ℓ , or, alternatively, the transmission function, T_ℓ , accounts for absorption, stating the net flux of particles through the nuclear interface.

We will show in subsequent sections that the real part of the interior logarithmic derivative, σ_ℓ , which contains the resonance aspects of the problem, has rather simple properties: for s-waves it is nearly unity and for higher partial waves it depends sensitively only upon the effective mass of the pion inside the nucleus. The absorptive term, π_ℓ , deserves further comment.

We show in Appendix A, Sec. A-1, that π_ℓ can, for each partial wave, be written in terms of an integral over the wave functions:

$$(3-17) \quad \pi_\ell(R_0) = - \frac{2\nu R_0}{n^2} \frac{\int_0^{R_0} W(r) |u_\ell(r)|^2 dr}{|u_\ell(R_0)|^2} - \frac{\int_0^{R_0} \alpha_I^e(r) \left[\left| \frac{d}{dr} \frac{u_\ell(r)}{r} \right|^2 + \frac{\ell(\ell+1)}{r^2} \left| \frac{u_\ell(r)}{r} \right|^2 \right] r^2 dr}{|u_\ell(R_0)|^2}.$$

Clearly, the first term is the local absorption and the second term is the momentum-dependent absorption. Hence, we can easily segregate and display those absorptive effects arising from the momentum-dependent interaction.

3-2 Qualitative Features of Low Energy Pion Optics in Nuclei

In the previous section we have derived a formalism for describing the optical properties of pions in nuclei. In the present section we will use this formalism to investigate the qualitative features of the pion-nucleus interaction. These features are mainly governed by the long pion wavelength within the nucleus arising from the repulsive local potential (and the small mass of the pion) and the strong energy dependence of processes arising from the momentum-dependent interaction when we are near the top of the potential barrier.

Because of the long pion wavelength, optical resonance properties tend to depend mainly upon the size of the scatterer and the effective mass of the pion inside the nucleus rather than upon the detailed structure of the interaction in the nuclear surface. On the other hand, optical absorption exhibits considerable structure through the momentum-dependent interaction since the pion momentum in the nucleus varies rapidly near the top of the potential barrier. To make these points clearer we will in this section consider the nucleus to have a uniform density, in which case we can obtain simple analytic formulae for the scattering parameters. We will extend our results to diffuse nuclei in our numerical discussion in Sec. 3-4.

For a uniform nuclear density of radius, R_0 , the potential parameters of Eqs. 3-1 have the radial dependence,

$$(3-18a) \quad \alpha^e(r) = \begin{cases} \alpha_0^e = -\alpha_{R_0}^e - \alpha_{I_0}^e, & r < R_0; \\ 0, & r \geq R_0; \end{cases}$$

$$(3-18b) \quad V(r) = \begin{cases} V_0, & r < R_0; \\ 0, & r \geq R_0; \end{cases}$$

$$(3-18c) \quad W(r) = \begin{cases} -W_0, & r < R_0; \\ 0, & r \geq R_0. \end{cases}$$

$\alpha_{R_0}^e$, $\alpha_{I_0}^e$, V_0 , and W_0 have been chosen to be positive.

We then have from Eq. 3-4 that the wave function for the pion inside the nucleus is

$$(3-19) \quad u_\ell(r) = A_\ell r j_\ell(br), \quad r < R_0,$$

where the internal wave number, b , is given by the relation,

$$(3-20) \quad b = \frac{\sqrt{2 \frac{\mu}{1 + \alpha_0^e} (E - V_c - V_0 + iW_0)}}{\hbar}$$

(V_c being the average value of the electrostatic potential within the nucleus).

As we show in Appendix A, Sec. A-2, the logarithmic derivative at R_0 is given by the relation,

$$(3-21) \quad \lambda_\ell(R_0) = (1 + \alpha_0^e) \frac{r \frac{d}{dr} j_\ell(br)}{j_\ell(br)} \Big|_{r=R_0} + 1.$$

Now, near the top of the potential barrier, $E \sim (V_c + V_0)$, so that bR_0 is small (W_0 is small). Thus, we may make an asymptotic expansion of the spherical Bessel functions:

$$(3-22) \quad j_\ell(bR_0) \sim \frac{(bR_0)^\ell}{(2\ell+1)!!} - \frac{(bR_0)^{\ell+2}}{2(2\ell+3)!!} + \dots$$

In this case, Eq. 3-21 has the approximate form (to order $(bR_0)^2$),

$$(3-23) \quad \lambda_{\ell}(R_0) = \ell(1 + \alpha_0^e) \left(1 - \frac{(bR_0)^2}{2\ell + 3}\right) + 1,$$

or, from Eqs. 3-11, 3-18, and 3-20,

$$(3-24a) \quad \sigma_{\ell}(R_0) \doteq 1 + \ell(1 - \alpha_{R_0}^e) - \frac{2\nu(E - V_0 - V_0)R_0^2}{(2\ell + 3)\hbar^2};$$

$$(3-24b) \quad \pi_{\ell}(R_0) = \ell\alpha_{I_0}^e + \frac{2\nu W_0}{(2\ell + 3)\hbar^2}.$$

From Eq. 3-24a, it is seen that the resonance aspects of the problem, which are contained in the logarithmic derivatives, σ_{ℓ} , are insensitive to the details of the potential for s-waves and, for higher partial waves, depend only upon the real part of the effective strength of the momentum-dependent interaction, $\alpha_{R_0}^e$, inside the nucleus. We therefore do not expect to find in the case of a diffuse-edge nucleus that these quantities depend very sensitively upon the diffuseness of the nuclear surface; in fact, we will show this to be the case in Sec. 3-4.

It is also seen from Eq. 3-24b that s-wave absorption, which is described by the logarithmic derivative, π_0 , is, near the top of the potential barrier, dominated by the local interaction. This arises since, near the top of the barrier, the pion momentum is small. The momentum-dependent processes are therefore strongly energy dependent in the vicinity of the barrier and, as we will show in Sec. 3-4, the details of

this energy dependence provide a sensitive tool for measuring the diffuseness of the nuclear surface.

Since the effect of increasing the angular momentum is simply to raise the potential barrier, we expect to find that for absorption involving higher partial waves there is a similar structure to that demonstrated here for s-waves, only now at a higher energy. Unfortunately, we cannot so easily demonstrate this analytically since we can no longer use simple expansions for the spherical Bessel functions. We will therefore satisfy ourselves with demonstrating this numerically in a subsequent section, Sec. 3-4. To do this we will first require numerical values for the optical potential.

3-3 Pion-Nucleus Optical Potential

In the final sections of this chapter, we will demonstrate the optical properties of the pion-nucleus interaction in a more quantitative way. To this purpose we will require a detailed optical potential which we will take to be the Ericsons' potential of Sec. 2-2. This potential would seem from the π -mesic x-ray data to be quite good at low energies (Sec. 2-3) and at higher energies (~ 30 MeV) it should still reasonably represent the optical interaction, at least if one allows for some energy dependence in its parameters. In addition, we must include the electrostatic potential arising from the nuclear charge distribution and we also mention relativistic effects of the pion which become important at higher energies.

To consider the relativistic effects, we first note that, since the pion is spinless, its optical wave function should satisfy a Klein-Gordon equation of the form,

$$(3-25) \quad \left[\hbar^2 c^2 \nabla^2 - \mu^2 c^2 + \left[(E + \mu c^2) - V_c(r) - \mathcal{V}(r) \right]^2 \right] \phi(\underline{r}) = 0.$$

Here E is the kinetic energy of the pion, μ is its (reduced) rest mass, $V_c(r)$ is the electrostatic potential arising from the nuclear charge distribution, and $\mathcal{V}(r)$ is the momentum-dependent short-range interaction of the pion with the nucleus discussed in Chapter 2 and Eq. 3-1. Eq. 3-25 can be rewritten in the convenient form,

$$(3-26) \quad \left[\frac{\hbar^2}{2\mu} \nabla^2 + (E - V_c(r) - \mathcal{V}(r)) \left[1 + \frac{E - V_c(r) - \mathcal{V}(r)}{2\mu c^2} \right] \right] \times \phi(\underline{r}) = 0.$$

From Eq. 3-26 we see that to order,

$$\frac{E - V_c(r) - \mathcal{V}(r)}{2\mu c^2} \lesssim \frac{20 \text{ MeV}}{280 \text{ MeV}} \sim 0.1,$$

the Klein-Gordon equation reduces to the non-relativistic Schroedinger equation,

$$(3-27) \quad -\frac{\hbar^2}{2\mu} \nabla^2 \phi(\underline{r}) + (V_c(r) + \mathcal{V}(r)) \phi(\underline{r}) = E \phi(\underline{r}).$$

Hence, neglecting relativistic corrections will make parameters of the potential energy-dependent to the order of 10%. Since the parameters are probably somewhat energy dependent in any event, we have simply neglected these corrections.

The electrostatic potential, $V_c(r)$, for a uniform distribution is well-known and, as we show in Appendix B, Sec. B-2,

is given to good approximation (even for diffuse-edge nuclei) by the formula,

$$(3-28) \quad V_c(r) = \begin{cases} \frac{zZe^2}{R_c} \left(\frac{3}{2} - \frac{1}{2} \left(\frac{r}{R_c} \right)^2 \right), & r \leq R_c ; \\ \frac{zZe^2}{r}, & r \geq R_c ; \end{cases}$$

where R_c is the rms radius of the charge distribution, Z is the nuclear charge, and z is the charge of the pion.

As discussed in Chapter 2, we will take the short-range pion-nucleus optical potential, $V(r)$, to be the Ericsons' potential given in Eq. 2-36 and Eq. 3-1. We will only consider spherical nuclei having zero spin and isospin; in any event, corrections for these effects are small (of order A^{-1}) except in very light nuclei. Eq. 2-36 can then be rewritten in the form of Eq. 3-2:

$$(3-29a) \quad -\frac{\hbar^2}{2\mu} \nabla^2 \phi(\underline{r}) - \frac{\hbar^2}{2\mu} \nabla \cdot \frac{\alpha(\underline{r})}{1 - \frac{\alpha(\underline{r})}{3}} \nabla \phi(\underline{r}) \\ + [V_c(r) + V(r) + iW(r)] \phi(\underline{r}) = E\phi(\underline{r}),$$

where

$$(3-29b) \quad V(r) = V_0 \rho(r),$$

$$V_0 = -\frac{\hbar^2}{2\mu} 4\pi \left(1 + \frac{m_\pi}{M_N} \right) (b_0)_{\text{eff}} \left(\frac{m_\pi c}{\hbar} \right)^{-1};$$

$$(3-29c) \quad W(r) = -W_0 \rho^2(r),$$

$$W_0 = \frac{\hbar^2}{2\mu} 4\pi \left(1 + \frac{m_\pi}{2M_N} \right) \text{Im}(B_0) \left(\frac{m_\pi c}{\hbar} \right)^{-4};$$

$$(3-29d) \quad \alpha(r) = -\alpha_{R_0} \rho(r) - i\alpha_{I_0} \rho^2(r),$$

$$\alpha_{R_0} = 4\pi(1 + \frac{m_\pi}{M_N})(c_0)_{\text{eff}} (\frac{\hbar}{m_\pi c})^3 ;$$

$$\alpha_{I_0} = 4\pi(1 + \frac{m_\pi}{2M_N}) \text{Im}(C_0) (\frac{\hbar}{m_\pi c})^6 .$$

We have here included the conversion factors from the units $\hbar = c = 1$ used in the Ericsons' paper to c.g.s. units.

In calculations with Eq. 3-29 we have used the experimental values for the parameters given by the Ericsons (1966):

$$(3-30a) \quad (b_0)_{\text{eff}} = -0.028;$$

$$(3-30b) \quad \text{Im}(B_0) = 0.028;$$

$$(3-30c) \quad (c_0)_{\text{eff}} = 0.208;$$

$$(3-30d) \quad \text{Im}(C_0) = 0.134.$$

We have chosen the nuclear density in Eq. 3-29 to have a Saxon-Woods distribution:

$$(3-31) \quad \rho(r) = \frac{\rho_0}{1 + e^{\frac{(r-R_0)}{a}}},$$

where 'a' is the diffuseness parameter, $R_0 = 1.05 A^{1/3}$ fm., and ρ_0 is normalized to the atomic number.

With these values for the parameters, we find that typical values of the potential inside a nucleus are,

$$(3-32a) \quad V_0 \sim 15 \text{ MeV} ;$$

$$(3-32b) \quad W_0 \sim 7 \text{ MeV} ;$$

$$(3-32c) \quad \alpha_{R_0} \sim -1.5 ;$$

$$(3-32d) \quad \alpha_{I_0} \sim -0.5 .$$

Of course, the exact values depend somewhat on the particular nucleus through the choice of the density function, $\rho(r)$.

3-4 Numerical Discussion

In the previous sections we have qualitatively described the optical scattering and absorption of low energy pions in nuclei. In particular, we have noted that resonance properties tend to be rather insensitive to the details of the potential near the top of the potential barrier because of the long pion wavelength. On the other hand, we have suggested that optical absorption exhibits considerable structure near the top of the potential barrier through the momentum-dependent interaction; in fact, we have suggested that this latter effect is strongly dependent on the details of the nuclear surface and, hence, provides a sensitive tool for measuring the diffuseness of the nuclear surface.

In the present section we will make these concepts more quantitative by employing the pion-nucleus optical potential discussed in Sec. 3-3. We will first provide a rather detailed analysis of one example --- the interactions of pions with a light nucleus, such as Ca^{40} --- and from the point of view from which an experiment might be analyzed. (This seems worthwhile for descriptive purposes since there is presently essentially no experimental data in the energy region of interest ($\lesssim 30$ MeV).) We will then extend these ideas to the interactions of pions with a heavy nucleus, such as Pb^{208} .

From the experimental point of view, the raw data to be analyzed is provided by the differential elastic scattering cross sections and the total absorption cross sections. We have calculated these cross sections for the example of

positive pions on Ca^{40} employing the potential of Eqs. 3-28 and 3-29 in Eq. 3-4 to calculate the radial partial wave functions. The partial wave phase shifts, $\delta_l + i\beta_l$, are then calculated from Eqs. 3-6 (at an asymptotic matching radius) and the differential elastic scattering cross section and absorption cross section are calculated, respectively, from the formulae:

$$(3-33a) \quad \frac{d\sigma_{el}(0)}{d\Omega} = \left| -\frac{\pi}{2k\sin^2\frac{\theta}{2}} e^{-i\eta\log(\sin^2\frac{\theta}{2})} + \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i((2\sigma_l' + \delta_l) + i\beta_l)} \sin(\delta_l + i\beta_l) \times P_l(\cos\theta) \right|^2 ;$$

$$(3-33b) \quad \sigma_{abs} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) T_l .$$

Of course, in practice the experimental phase shifts are to be deduced by analyzing the empirical cross section data with Eqs. 3-33.

Since one of our aims is to display the dependence of optical cross sections on the diffuseness of the nuclear surface, we have performed our calculations for both a sharp-edge (diffuseness parameter, $a=0.1$ fm.) nucleus and for a more conventional ($a=0.5$ fm.) diffuse-edge nucleus. A plot of these two densities for a Saxon-Woods density distribution (Eq. 3-31) with the above surface parameters, and a radius appropriate to Ca^{40} ($R_0 = 1.05 A^{1/3}$ fm. = 3.59 fm.) is given in Fig. 1. We have in this figure also plotted the squares of the densities upon which the optical absorption depends.

Substituting these densities in the potential of Eq. 3-29, we have calculated the differential elastic scattering cross section and total absorption cross section for a variety of energies. The results (for the two densities of Fig. 1) are presented in Fig. 2a (differential elastic scattering cross section) and Fig. 2b (total absorption cross section).

It is seen from Fig. 2a that the differential elastic scattering cross sections possess considerable structure and are quite sensitive to the diffuseness of the nuclear surface. The strong destructive interference at higher energies between the Rutherford scattering and the scattering due to the strong part of the pion-nucleus interaction (middle and lower figures) arises from the interference between the p-wave (and higher partial wave) scattering --- which is dominated by the attractive momentum-dependent potential (c.f. Eq. 3-24) --- with the scattering due to the Coulomb potential --- which is repulsive. At lower energies the interference is constructive (top figure) since, like the Coulomb interaction, the s-wave interaction is repulsive (c.f. Eq. 3-24). These properties of the strong pion-nucleus interaction are demonstrated more explicitly in Figs. 3 where we have plotted the real partial wave phase shifts, δ_l ; the s-wave phase shifts are seen to be negative (corresponding to a repulsive interaction) and the p-wave phase shifts (and d-wave phase shifts) are seen to be positive (corresponding to an attractive interaction). It is seen that at large

FIG. 3_a

REAL PHASE SHIFTS

FIG. 3b

REAL PHASE SHIFTS

FIG. 3c REAL PHASE SHIFTS

FIG. 4

REAL
LOGARITHMIC DERIVATIVES

FIG. 5

IMAGINARY
PHASE SHIFTS. $\pi^+ + \text{Ca}^{40}$

FIG. 6

RATIOS OF
IMAGINARY
PHASE SHIFTS $\pi^+ + \text{Ca}^{40}$ $\frac{\beta_0^s}{\beta_0^d}$ $\frac{\beta_0^s}{\beta_0^d}$

angles there is also (as expected) interference between the s-wave scattering and the scatterings of the higher partial waves.

The dependence of the elastic scattering on the diffuseness of the nuclear surface in Fig. 2a does not depend on the resonance properties of the potential but rather it depends on the difference in the diffraction associated with the difference in the absorption between the diffuse-edge and sharp-edge choices for the nuclear density. This is shown in Fig. 3a, where we see that the real phase shifts associated with the two potentials are nearly the same, and in Fig. 5, where we see that the corresponding imaginary phase shifts are quite different. The strong dependence of absorption on the diffuseness parameter is also demonstrated in Fig. 2b, where we have plotted the total absorption cross sections.

We will now address ourselves to the problem of obtaining the empirical parameters of the pion-nucleus potential, Eq. 3-29, from the phase shifts which describe the elastic scattering and absorption cross sections. Let us first enumerate the essential six parameters of the potential assuming the Saxon-Woods density distribution, Eq. 3-31:

- (1) The real local interaction, V_0 ; (2) The absorptive local interaction, W_0 ; (3) The real momentum-dependent strength, α_{R0} ; (4) The absorptive momentum-dependent strength, α_{I0} ; (5) The nuclear radius parameter, R_0 ; (6) The nuclear surface parameter, a .

Before proceeding, it will be convenient to recall Eq. 3-24

in which we stated the properties of the interior logarithmic derivatives near the top of the local potential barrier for a uniform density distribution:

$$(3-24a) \quad \sigma_l(R_0) \doteq 1 + l(1 - \alpha_{R_0}^e) - \frac{2\nu(E - V_c - V_0)R_0^2}{(2l+3)\hbar^2};$$

$$(3-24b) \quad \pi_l(R_0) = l\alpha_{I_0}^e + \frac{2\nu W_0 R_0^2}{(2l+3)\hbar^2}.$$

In Eqs. 3-24, the effective strengths of the momentum-dependent interaction inside the nucleus, $\alpha_{R_0}^e$ and $\alpha_{I_0}^e$, differ from the strengths α_{R_0} and α_{I_0} (which are proportional to the density) through the Lorenz-Lorentz effect. It is the effective strengths which are important in determining the scattering, as can be seen from Eq. 3-24. Of course, the effective strengths vary in the surface through the Lorenz-Lorentz effect (Eq. 3-1b) but, to first order, the optical properties are determined by their values inside the nucleus where the density is constant (as can be seen from Eq. 3-24 and as we shall presently show). That is, the optical properties are sensitive to the total change in the effective mass of the pion in entering the nucleus which involves an integral over the nuclear surface.

For s-waves ($l=0$) it is seen from Eq. 3-24a that σ_0 is almost unity and, to first order, depends only on V_0 and R_0 (V_c is fixed by the nuclear charge distribution).[#] We

[#] This result is to be contrasted with the behaviour of logarithmic derivatives for interactions involving strongly attractive potentials (such as the nucleon-nucleus interaction) where the interior logarithmic derivatives are rapidly varying functions of the energy. In fact, it is this property of the derivatives which gives rise to the strong resonance properties characteristic of such potentials.

have plotted σ_0 for both the diffuse-edge and sharp-edge densities of Fig. 1 in Fig. 4 (upper figure). It is seen from Fig. 4 that σ_0 is rather insensitive to the diffuseness of the nuclear surface; in fact, in both cases the logarithmic derivative is seen to be in close agreement with the approximate value calculated from Eq. 3-24a (dashed).

Now the real part of the s-wave phase shift, δ_0 , depends mainly on σ_0 (and the barrier properties) since, in general, η_0 is small (c.f. Eq. 3-16a). We therefore expect that δ_0 , like σ_0 , is rather insensitive to the diffuseness of the nuclear edge; this is shown for our present example in Fig. 3a (upper figure). On the other hand we expect δ_0 to depend sensitively on V_0 (which determines the height of the potential barrier) and R_0 (which determines the penetration); this is demonstrated in Figs. 3b and 3c (upper figures), respectively. Hence, δ_0 depends sensitively only on V_0 and R_0 .

For higher partial waves, we see from Eq. 3-24a that δ_ℓ depends sensitively only $\propto \epsilon_{R_0}^e$ (through the interior logarithmic derivative, σ_ℓ) and R_0 (through barrier penetration). The insensitivity of δ_ℓ to diffuseness parameter, a , and the real potential, V_0 , is demonstrated in Figs. 3a and 3b (lower figures); the sensitivity to R_0 is demonstrated in Fig. 3c (lower figures). We have demonstrated the sensitivity of σ_1 (and, hence, δ_1) to $\epsilon_{R_0}^e$ in Fig. 4 (lower figure); it is seen that σ_1 is in close agreement with the approximate value (dashed) calculated from Eq. 3-24a for both the sharp-

edge and diffuse-edge potentials.

It is seen from the above discussion that an analysis of the real parts of the phase shifts, δ_l , yields the parameters V_0 , R_0 , and $\alpha_{R_0}^e$ of the pion nucleus potential relatively independent of the other parameters, in particular, independent of the diffuseness parameter, a .

Near the top of the potential barrier ($V_c + V_0 \sim 26$ MeV), momentum-dependent absorption processes are suppressed.

Hence, near the top of the barrier, the imaginary logarithmic derivative, π_0 and, therefore, the imaginary phase shifts,

β_0 , (Eq. 3-16), are sensitive only to the local absorption parameter, W_0 . This is demonstrated in Fig. 6 (upper figure) where we have plotted the ratios of the imaginary phase shifts appropriate to the sharp-edge and diffuse-edge densities of Fig. 1; it is seen that, near the top of the potential barrier (~ 26 MeV), $\frac{\beta_0^s}{\beta_0^d} \sim 1$. Hence, W_0 can be obtained from the absorption cross sections near the top of the barrier (which is known from analysis of the real parts of the phase shifts, δ_l).

It is therefore possible to obtain values for the parameters V_0 , R_0 , $\alpha_{R_0}^e$, and W_0 relatively independent of the nature of the nuclear surface. Now, as we shall shortly discuss in more detail and as can be seen from Fig. 5, the energy dependence of the imaginary parts of the phase shifts, β_l , is a strong function of the diffuseness parameter, a , through the momentum-dependent absorption. Hence, using the values which we have previously obtained for the other para-

meters of the interaction, we can analyze the phase shifts, β_l , for the diffuseness parameter, a , and the absorptive term in the momentum-dependent interaction, α_{l0}^e .

Of course, by following this procedure which we have outlined only very approximate values are obtained for the parameters of the potential. Of course, better values can be found by repeating the analysis with the values found from the first attempt. In any event, at least a preliminary use of these procedures would seem to be both more physically instructive and more efficient than simply attempting to simultaneously vary the several (six) parameters which enter the interaction to obtain a best fit to experimental cross sections (when such cross sections become available).

As we have noted in Figs. 5 and 6, the optical absorption described by the imaginary phase shifts, β_l , should depend strongly on the thickness of the nuclear surface. This results since, near the top of the potential barrier, the momentum-dependent absorption is strongly suppressed. Now, because a diffuse-edge nucleus contains a greater amount of matter at lower densities than a square-edge nucleus, the barrier appears lower for the former than the latter and, in fact, the structure of the imaginary phase shifts (or imaginary interior logarithmic derivatives) clearly mirrors the diffuseness of the nuclear surface. We have already seen this effect in Fig. 5 where we have plotted the imaginary phase shifts, β_l , for two possible choices for the nuclear surface thickness given in Fig. 1. We now exhibit the

S - WAVE ABSORPTION

FIG. 7a

7b

 $\pi^+ + C_a^{40}$ $a = 0.1 \text{ fm}$ $a = 0.5 \text{ fm}$ $d(\text{ABSORPTION})/dr, \text{ ARBITRARY UNITS}$ $V_c + V$

LOCAL

M.-D.

E

E
13 MeV

7c

7d

E

E
23 MeV

7e

7f

E
33 MeV

0

2

4

 $r, \text{ fm}$

0

2

4

6

 $r, \text{ fm}$

specific effect of the momentum-dependence more clearly by making use of Eq. 3-17 which allows us to segregate the effects of the local and momentum-dependent absorption:

$$(3-17) \quad \pi_q(R_0) = - \frac{2\mu R_0}{h^2} \frac{\int_0^{R_0} W(r) |u_q(r)|^2 dr}{|u_q(R_0)|^2}$$

$$= \frac{\int_0^{R_0} \propto \frac{e}{I}(r) \left[\frac{d}{dr} \left| \frac{u_\ell(r)}{r} \right|^2 + \frac{\ell(\ell+1)}{r^2} \left| \frac{u_\ell(r)}{r} \right|^2 \right] r^2 dr}{|u_\ell(R_0)|^2}$$

(Local)
(Momentum-dependent)

In Fig. 7 we have used Eq. 3-17 to plot the local (solid) and momentum-dependent (dashed) absorption for the two densities of Fig. 1 ($a = 0.5$ fm., left-hand side; $a = 0.1$ fm., right-hand side). For reference purposes, we have included the total local potential barrier, $V_c(r) + V_o(r)$, and the incident energy of the pion (marked 'E').

In Fig. 7a and 7b, we have compared s-wave absorption at an energy below the barrier for the diffuse-edge and sharp-edge nuclear densities of Fig. 1. It is seen that, for the diffuse-edge density, most pions are absorbed at nearly zero momentum so that momentum-dependent absorption is strongly suppressed (the momentum is roughly proportional to $|E - V_c(r) - V(r)|^{1/2}$); on the other hand, for the sharp-edge density, pions are being absorbed inside the barrier at higher momentum so that momentum-dependent absorption is considerably less suppressed. We see from Fig. 7c and 7d that, by similar arguments, momentum-dependent absorption is suppressed for both choices of the

density near the top of the barrier. In Fig. 7e and 7f we see that, for energies above the barrier, momentum-dependent absorption is more suppressed for the sharp-edge density (where absorption takes place inside the barrier) than for the diffuse-edge density (where, because there is more surface, absorption takes place at lower density and higher momentum).

For higher partial waves, this simple analysis becomes somewhat more complicated through the presence of angular momentum barriers. Firstly, it is seen from Eq. 3-17 that, for $l \neq 0$, there is a contribution to the absorption from the angular terms which enter the momentum-dependent interaction as well as from the radial terms; like the local absorption, these terms are insensitive to the diffuseness of the nuclear surface. In addition, the radial terms which enter the momentum-dependent interaction for the higher partial waves see a higher potential barrier through the appearance of the centrifugal potential; we therefore expect the contribution to the momentum-dependent absorption from the radial terms to exhibit a similar dependence on the potential barrier to that shown in Fig. 7 for s-waves, only now at higher energies. In fact, this is seen to be the case from Fig. 5. We will discuss these effects more explicitly in Sec. 4-3 where similar effects are found in the cross sections which describe the excitation of rotational levels in deformed nuclei.

Now, as we have previously discussed, we know the height of the potential barrier inside of the nucleus from

FIG. 8

S-WAVE IMAGINARY
PHASE SHIFTS $\pi^- + Ca^{40}$ $\pi^+ + Ca^{40}$

FIG. 9

S-WAVE IMAGINARY
PHASE SHIFTS

our analysis of the real part of the phase shifts, δ , which describe the resonance aspects of the problem. On the other hand, we see here that the imaginary part of the phase shifts, β , are determined by the behaviour of the potential barrier (and, hence, the nuclear density) in the region of the nuclear surface. It should therefore be possible from optical scattering and absorption experiments with pions to very accurately determine the diffuseness of the nuclear surface.

In the preceding discussion, we have noted that the main effect of the electromagnetic interaction upon our analysis was in adjusting the effective height of the potential barrier. In the case of Ca^{40} , the electrostatic potential has an average value inside the nucleus of about 10 MeV. We therefore would expect if we were to use negative pions rather than positive pions that the only important effect on our analysis would be that the barrier would appear 20 MeV lower. This effect is shown in Fig. 8, where we have compared the s-wave imaginary phase shifts, β_0 , for the two reactions $\pi^+ + \text{Ca}^{40}$ and $\pi^- + \text{Ca}^{40}$ employing both the sharp-edge (dashed) and diffuse-edge (solid) nuclear densities of our previous calculations (Fig. 1).

In a similar manner, we expect that if we go to heavier nuclei, the only major changes in our analysis will be due to the large electrostatic barriers (as seen from Eq. 3-29, the short-range pion-nucleus interaction is insensitive to atomic number). We have demonstrated this effect in Fig. 9

where we compare the s-wave imaginary phase shifts, β_0 , for the reactions $\pi^+ + \text{Ca}^{40}$ and $\pi^+ + \text{Pb}^{208}$. Both calculations are for Saxon-Woods densities (Eq. 3-31) with two choices for the diffuseness parameter, $a = 0.1$ fm. (dashed) and $a = 0.5$ fm. (solid). It is seen that, apart from the difference in penetration, the only effect of going to a heavier nucleus is to raise the effective repulsive barrier through the difference in the electrostatic potentials.

In fact, in heavy nuclei the electrostatic potentials are so large that, for negative pions, it is possible that the net potential might be attractive. This effect is demonstrated in Fig. 10 where we compare the reactions $\pi^+ + \text{Pb}^{208}$ and $\pi^- + \text{Pb}^{208}$. The average electrostatic potential inside the nucleus is similar to 21 Mev, whereas our choice for the short-range potential barrier is similar to 16 Mev. For the net depth of the potential suggested by these estimates (roughly, - 5MeV), it would seem that the pion mass is too light for there to be strong resonance effects at low energies. Of course, as we have discussed in Chapter 2, the parameters of the pion-nucleus potential, in particular, the height of the potential barrier, are at present not very well determined.

Another effect of the electrostatic potential in the case of negative pions is that it sufficiently reduces the angular momentum barriers so that many partial waves enter the problem. We have demonstrated this effect in Fig. 11.

for the reaction $\pi^- + \text{Pb}^{208}$ (lower figure). The untangling of the partial waves is then a formidable problem. Of course, in the case of positive pions, the converse effect applies since the effect of the electrostatic potential is to increase the penetration factors (Fig. 11, upper figure).

From the analysis presented in this chapter, we believe that we have clearly demonstrated that the nuclear surface thickness plays an important role in determining the absorption rates of pions in nuclei. In fact, we would suggest that, on the basis of these results, it is of great interest to measure low energy pion optical data, particularly pion absorption, as a means to measuring the density of nucleons in the nuclear surface. Accurate measurements of this nature should finally resolve the considerable controversy over the surface distribution of neutrons in heavy nuclei which we have discussed in the introduction to this chapter.

3-5 Charge Exchange and other Exotic Terms in the Ericsons' Potential

In the present section we will briefly mention some terms of order A^{-1} and higher which enter the Ericsons' potential, Eq. 2-36, and which we have until now omitted from our discussion. These terms consist of isospin terms, which macroscopically account for single and double charge exchange between nuclear analog states, and terms arising from the spin of the nucleus, which lead to hyperfine splitting in the π -mesic x-ray spectrum. Hence, while these terms make small contributions to the pion-nucleus optical

interaction, they lead to qualitatively different sorts of effects which are both interesting and easily measured.

The problem of charge exchange has long been of interest since the pion, unlike most other familiar elementary particles, has three charge states --- π^+ , π^0 , π^- . Hence, in addition to single charge exchange, there are the interesting processes of double charge exchange by which, for example, one can reach nuclear isobars of greater (or lesser) charge than have heretofore been available.

The first approaches to charge exchange were microscopic in nature in which, so to speak, one traces the path of the pion through the nucleus and attempts to account for the particular nucleon where charge exchange occurred. For these purposes one requires a microscopic description of the nucleus which is usually taken to be that given by the shell model. The pion-nucleon scattering operators are often taken to be the free space scattering operators (the impulse approximation) and the propagation of the pion within the nucleus is often treated by making some average of the intermediate excitation energies so that one can reduce the many-particle Green's function to a one-body Green's function for the pion (the "closure approximation"; c.f. Eq. 2-21a). Calculations of this type have been attempted by several authors; a summary of the current status of both single and double charge exchange reactions, along with a review of the earlier calculations and experiments, can be found in a recent review article by Koltun (1969)

A second manner in which one can account for charge exchange is through an optical potential, such as the Ericsons' optical potential of Eq. 2-36. Such an optical description states that charge exchange can only occur through those nucleons which contribute to the macroscopic isospin of the nucleus, that is, between isotopic analog states; charge exchange which takes place in this manner is often referred to as quasi-elastic scattering since only the charge of the nucleus is changed in the scattering process.

As an example of quasi-elastic scattering, let us consider the $A = 14$ isospin triplet, (C^{14} , N^{14*} , O^{14}). All these nuclei have closed $s_{1/2}$ orbits and $p_{3/2}$ orbits; the remaining two nucleons are in $p_{1/2}$ orbits and are coupled to spin, $J = 0$, and isospin, $T = 1$, with $M_T = -1$ for C^{14} , $M_T = 0$ for N^{14*} , and $M_T = 1$ for O^{14} . Quasi-elastic scattering can only account for changes in M_T and, hence, only involves the $p_{1/2}$ nucleons; on the other hand, we might well expect charge exchange to occur on the $p_{3/2}$ nucleons or even the $s_{1/2}$ nucleons (of course, charge exchange is more likely in open than closed shells from energetic considerations). In fact, from considerations only of quasi-elastic scattering, we would conclude that charge exchange is forbidden in nuclear states of isospin, $T = 0$, such as the ground state of C^{12} . An optical description of charge exchange, such as that given by the Ericsons, is therefore only of value in considering quasi-elastic scattering between nuclear analog states.

As can be seen from Eq. 2-36, the Ericsons' potential accounts for optical charge exchange processes arising in a variety of ways. Firstly, there are single charge exchange processes arising from both the s-wave (terms in b_1) and p-wave (terms in c_1) interactions (including a charge exchange term through the Lorenz-Lorentz effect). Double charge exchange can occur either through repeated application of these vector couplings or directly through the tensor coupling (terms in B_5 and C_5) which arise from the two-nucleon processes. Calculations with charge exchange interactions being described by an optical potential, such as the Ericsons', are found to be in moderate agreement with experiment (Koltun (1969)).

In addition to terms accounting for isospin, the Ericsons' potential also takes into account the coupling between the angular momentum of the incident pion and the nuclear spin (terms in d_0) --- the hyperfine interaction. At present pion intensities are too low to observe hyperfine splitting in the π -mesic x-ray spectra, but interesting information about these terms should be obtained once intense pion beams (such as those provided by the proposed Triumf facility) become available.

CHAPTER 4THE EXCITATION OF ROTATIONAL LEVELS
IN DEFORMED NUCLEI
BY PIONS

In the last chapter we have examined the optical properties of pions in nuclei and, in particular, we have shown that the optical absorption of pions in nuclei depends strongly upon the details of the nuclear surface. In the present chapter we will discuss a second way in which the pion optical field may be used to examine nuclear surface features, namely, by the excitation of rotational states in strongly deformed nuclei.

Conventional techniques for studying deformed nuclei --- the excitation of rotational levels by electromagnetic or nucleon fields; the electromagnetic decay of these excited levels --- essentially involve integrals over the nucleus which are sensitive only to the average nuclear deformation and which are insensitive to the diffuseness of the nuclear edge. From such experiments on deformed nuclei, we learn detailed information about the quadrupole and higher multipole deformations of the nuclear density, but we learn little about the details of its shape, such as the nuclear surface thickness. Our purpose in this chapter is to show that the excitation of rotational levels in deformed nuclei by pions --- which we will henceforward (and somewhat loosely) call "pion excitation" (in analogy with Coulomb excitation) --- does intrinsically involve the surface features of the nuclear density because of the momentum-dependent part of the pion-nucleus interaction. Our technique for demonstrating this

is to segregate the Coulomb and local excitation processes in "pion excitation" from the momentum-dependent excitation processes by employing the Distorted Wave Born Approximation.

The information obtained from pion excitation, when combined with the analysis of optical absorption experiments (as discussed in Chapter 3), should allow a comprehensive study of the nuclear surface in deformed nuclei. Since the charge distribution in deformed nuclei is becoming increasingly well-known from μ -mesic x-ray experiments, this should yield quantitative information about the surface distributions of protons and neutrons in such nuclei including, possibly, the angular dependence of these distributions. This information should be of considerable value in refining the microscopic descriptions of deformed nuclei and in comparing the extent to which these descriptions correspond to the microscopic descriptions of spherical nuclei.

Our picture of a strongly deformed nucleus is that of the strong-coupling rotational model in which the nucleus is viewed as an (almost) spheroidal body rotating in space (c.f. Sec. 4-1). The essential idea is that when the nucleus is immersed in a potential field the speed of rotation increases --- i.e. we excite a rotational degree of freedom of the nucleus; in general, as we will discuss in Sec. 4-1, we may neglect excitations of other (vibrational and intrinsic) degrees of freedom, so that we can directly connect the cross sections for exciting rotational states to such macroscopic aspects of the nucleus as the nuclear density.

For instance, the excitation of rotational levels due to the electric field between the projectile and the deformed target nucleus is the well-known phenomena of Coulomb excitation (Alder and Winther (1966)). From Coulomb excitation of rotational levels we can infer the static multipole moments of the deformed nucleus and, hence, the average deformation of the nuclear charge distribution (which we expect to be similar to the average deformation of the mass distribution); in fact, recent experiments on inelastic alpha particle scattering have allowed the evaluation of even the higher multipole moments with considerable precision (Bernstein (1969)). The excitation of rotational levels due to a local strong-interaction has also been investigated by bombarding deformed nuclei with neutrons (Chase, Wilets, and Edmonds (1958)). However, the excitations of rotational levels with either the Coulomb or neutron fields essentially involve integrals over the nucleus and, hence, are rather insensitive to the details of the nuclear surface.

In the case of the nuclear charge density this is not too disappointing since the surface details can be obtained moderately well from other sources, such as the μ -mesic x-ray experiments (Devons and Duerdath (1969); Wu (1967)). The details of the nuclear mass density, however, are not easily obtained from current experiments; in fact, as we discussed in Chapter 3, the surface distribution of neutrons is a controversial issue even in spherical nuclei. The purpose of this chapter is to demonstrate that the surface

features of the mass density (and, hence of the neutron distribution) are susceptible to investigation by pion excitation (as well as by optical absorption) because of the momentum dependent pion-nucleus interaction.

Our basic idea is the same as in our study of pion optical absorption in Chapter 3: near the top of the potential barrier arising from the local pion-nucleus interaction the momentum-dependent processes are strongly suppressed; in fact, their variation with energy in this region should strongly reflect the details of the nuclear surface because of the variation of momentum across the nuclear surface.

Of course, the "pion excitation" cross sections also include Coulomb excitation, due to the pion charge, and local excitation, due to the local pion-nucleus interaction. A graphic technique for segregating these processes from the interesting momentum-dependent processes is provided by employing the Distorted Wave Born Approximation. As we discuss in Sec. 4-1, we expect the DWBA to be reasonably good since the excitation cross sections are much less than the geometric cross section of the nucleus.

In DWBA the amplitude for excitation is proportional to integrals over the pion optical potential and the pion optical wave functions. The total amplitude is then essentially a sum of three terms arising from the three terms in the optical potential: a Coulomb excitation term, arising from the electrostatic potential; a local excitation term, arising from the local pion-nucleus interaction; a momentum-dependent term,

arising from the momentum-dependent pion-nucleus interaction. We can therefore clearly demonstrate the effects of the momentum-dependent interaction by examination of the corresponding DWBA amplitude.

In Sec. 4-1 we review the rotational model of strongly deformed nuclei and we discuss the excitation of rotational states in such nuclei by pions. In Sec. 4-2 we obtain the cross sections for these excitation processes in DWBA. In Sec. 4-3 we employ the Ericsons' potential of Chapter 2 and Sec. 3-3 to evaluate these formulae for a realistic pion-nucleus interaction and, hence, to demonstrate the strong dependence of the pion excitation cross sections on the nuclear surface parameters.

4-1 A Review of the Rotational Model and a Discussion of Pion Excitation

In the present section we first briefly review the rotational model of strongly deformed nuclei. We then discuss the excitation of rotational states in such nuclei by pions and we suggest that these processes can be well described by the techniques of DWBA.

To understand the strong-coupling or rotational model, it is worthwhile to review our basic knowledge of the interactions of a constituent nucleon in its nuclear environment. From studies of nuclear matter we know that, because of the Pauli exclusion principle, a nucleon in nuclear matter moves in the average potential of the surrounding nucleons and is rather insensitive to the details of their positions

and motions. Hence, in a finite nucleus we expect each nucleon to move essentially in a single-particle orbit. On the other hand, the average potential in which the nucleons move is determined by the nuclear density which is simply the sum over the nucleus of the nucleon densities obtained from the single-particle orbits. Hence, the problem of finding the average potential and the problem of finding the single-particle orbits are coupled. The way out of this apparent "chicken or egg" conundrum is provided by the variational principle which says that the resultant nuclear wave function must represent a state of minimum energy.

In a nucleus with many nucleons in an unfilled shell, it turns out that because of the interaction between the extra-core nucleons, which tend to polarize the nuclear core, and because the nuclear core which, like a closed shell nucleus, prefers to be spherical, the equilibrium shape of the nucleus is strongly deformed. Such a deformed nucleus exhibits collective modes of oscillation, rotations and vibrations, in addition to the single-particle excitations arising from the basic shell structure of the nucleus.

The single-particle aspects and the rotational aspects of the problem are strongly coupled since a rotation of the nucleus corresponds to a rotation of the average potential which determines the single-particle orbits. The problem is considerably simplified by noting that the nucleons orbit sufficiently quickly compared to the rotational motion that they respond adiabatically to the changing average potential.

Thus the total wave function of the nucleus may be treated adiabatically and taken to include the product of an intrinsic wave function, χ , describing the shape and structure of the nucleus, and a rotational wave function, D , describing its orientation in space. In addition to rotation, the nucleus also exhibits small collective vibrations about its equilibrium shape. Since these vibrations only slightly modify the average potential seen by the nucleons, they are only weakly couple to the single-particle and rotational aspects of the problem and we can account for them separately with, say, a vibrational wave function, ϕ_{vib} . Thus for the moment we can take the total nuclear wave function to have the schematic form,

$$(4-1) \quad \Psi = \phi_{\text{vib}} D \chi ;$$

in fact, vibrational excitations generally occur at considerably higher energies than the rotational excitations in which we shall be interested and we will simply omit them from our discussion (taking the nucleus to always be in its vibrational ground state).

The motivation of our calculation is now clear. Our purpose is to investigate the excitation of rotational states, D , through the optical pion-nucleus interaction which, from our discussion in Chapter 2, involves the nuclear mass density, $\rho(\underline{r})$, in a very direct manner. In fact, what we will show is that the excitation of these rotational states depends sensitively on the surface details of $\rho(\underline{r})$ through the momentum-dependent pion-nucleus interaction. Since $\rho(\underline{r})$

is simply the sum of the single-particle densities obtained from the intrinsic wave function, χ , and since we know the surface distribution of protons in χ from other experiments (such as the μ -mesic x-ray experiments), this allows us to state separately the surface distribution of neutrons and protons. When combined with the results of optical absorption experiments, as described in Chapter 3, this should allow a comprehensive investigation of the nuclear surface in deformed nuclei and an accompanying increase in our understanding of the microscopic aspects of the problem accounted for in the intrinsic wave function, χ .

Before proceeding, we require a more complete description of the nuclear wave function which we have schematically written in Eq. 4-1. A detailed derivation of the rotational model can be found in most elementary textbooks on nuclear physics (Preston (1962)) and we will satisfy ourselves with a brief description.

The basic postulate of the adiabatic assumption is that the total nuclear Hamiltonian, h , can be written in the form,

$$(4-2) \quad h = H_{\text{intr}}(x') + T_{\text{rot}},$$

where $H_{\text{intr}}(x')$ is, say, a shell-model Hamiltonian appropriate to a deformed average potential and depending upon the intrinsic co-ordinates, x' , and where T_{rot} is the kinetic energy of the rotational motion.

For our purposes, we will be interested in the (large) class of deformed nuclei which are axially symmetric, these exhibiting primarily spheroidal deformations. The rotation

of such a body is analogous to the classical precession of a symmetric top (Goldstein (1950)). We therefore might expect the kinetic energy operator to have the form,

$$(4-3) \quad T_{\text{rot}} = \frac{\hbar^2}{2\mathcal{J}(\beta)} \tilde{I}^2,$$

where $\mathcal{J}(\beta)$ is the "moment of inertia" of the nucleus, depending upon the average deformation, β , and where $\tilde{I}$ is the total angular momentum operator of the nucleus. The normalized eigenstates, D , of T_{rot} are then easily shown to have the form,

$$(4-4) \quad D(\alpha_k) = \left[\frac{2I+1}{8\pi^2} \right]^{\frac{1}{2}} \mathcal{D}_{MK}^I(\alpha_k),$$

where $\alpha_k = (\alpha, \beta, \gamma)$ are the Euler angles (Goldstein (1950)) which rotate the space-fixed axes, (x, y, z) , z being in the direction of the axis of quantization, into the body-fixed axes, $(1, 2, 3)$, 3 being the symmetry axis of the nucleus.

$\mathcal{D}_{MK}^I(\alpha_k)$ is the matrix element of the rotation matrix, $\mathcal{D}$, (Preston (1962)) corresponding to the eigenvalues I , M , and K of the operators $\tilde{I}$, $\tilde{I}_z$, and $\tilde{I}_3$, respectively. Clearly, these are the quantities which we expect to be constants of the motion from the analogy with the classical symmetric top.

In addition, since the nucleus is axially symmetric, we expect its spin in the 3 - direction, say, Ω , to also be a constant of the motion so that the eigenstates of the intrinsic Hamiltonian, $H_{\text{intr}}(x')$, have the form $\chi_{\alpha}^{\gamma}(x')$, where γ are the remaining quantum numbers required to specify

the intrinsic nuclear state. It can easily be shown that axial symmetry requires that $\Omega = K$. The nuclear wave function therefore has the form,

$$(4-5a) \quad \Psi_{\alpha} = \left[\frac{2I_{\alpha}+1}{8\pi^2} \right]^{\frac{1}{2}} \mathcal{D}_{MK}^{I_{\alpha}}(\alpha_k) \chi_K^r(x') \quad (K=0).$$

Actually, Eq. 4-5a correctly represents the nuclear state only if $K=0$; for higher values of K we require linear combinations of states like Eq. 4-5a in order to have states of definite parity (Preston (1962)). The result is then of the form,

$$(4-5b) \quad \begin{aligned} \Psi_{\alpha} = & \left[\frac{2I_{\alpha}+1}{16\pi^2} \right]^{\frac{1}{2}} \left[\mathcal{D}_{MK}^{I_{\alpha}}(\alpha_k) \chi_K^r(x') \right. \\ & \left. + (-)^{I_{\alpha}+2K} \mathcal{D}_{M,-K}^{I_{\alpha}}(\alpha_k) \sum_j (-)^j c_{jK} \chi_{j,-K}^r(x') \right] \\ & (K \neq 0). \end{aligned}$$

(+ or - depending on whether the state has positive or negative parity, respectively) where we have expanded χ_{-K}^r in terms of its angular momentum substates, $\chi_{j,-K}^r$.

From Eqs. 4-5 it is seen that a given intrinsic state, χ_K^r , may exist in various states of rotation as prescribed by the rotational wave functions, $\mathcal{D}_{MK}^{I_{\alpha}}$. Such a set of rotational states is called a "rotational band". It is easily shown that, if $K=0$, I_{α} increases in steps of two --- eg. $I_{\alpha} = 0^+, 2^+, 4^+, \dots$ --- while if $K \neq 0$, I_{α} increases in steps of one --- $I_{\alpha} = K, K+1, K+2, \dots$. This band structure is the signature of the energy level spectrum of a strongly deformed nucleus.

Our technique for evaluating the rotational excitation cross sections will be to use the Distorted Wave Born Approximation (c.f. Kikuchi and Kawai (1968), Messiah (1962)). In DWBA we see the nucleus in its ground state as being immersed in the elastically scattered pion field which we calculate from the pion-nucleus interaction, say that described by the Ericsons' potential discussed in Chapter 2 and Sec. 3-3. The pion field then causes the nucleus to increase its rotation, exciting it to some higher state in the rotational band. Before proceeding to evaluate the excitation cross sections in DWBA we must first ask ourselves about the validity of this approximation.

We argued in Chapter 2 that the absorptive effects due to excitation within the intrinsic wave function were small compared to real pion absorption (which is taken into account in the Ericsons' potential). In addition, we do not expect absorptive effects of vibrational states to be important since they occur at considerably higher energies than rotational states.

The question of the validity of DWBA can therefore to a large extent be rephrased by enquiring what the absorptive effects of rotational excitation are on the ground state wave function --- that is, we wish to determine the coupling of the rotational channels. If it is small, we may simply use DWBA; if it is large (as it is in the excitation of rotational states by neutrons), the distorted wave calculated from the optical pion-nucleus interaction (which does not include the effects of channel-coupling) does not accurately describe the

pion field seen by the nucleus in its ground state and we must resort to some coupled channel approximation.

The essential condition for the validity of DWBA is therefore that the excitation cross sections be much smaller than the geometric cross section $4\pi R^2$, where R is the radius of the target nucleus. As we will show in the numerical discussion (Sec. 4-2), the excitation cross sections are a few millibarns while the geometric cross section is a few barns. This condition would therefore seem to be well satisfied and DWBA should be good at least to first order. This is to be contrasted with the excitation of rotational states by neutrons where it is found that the geometric cross section is of the same order as the excitation cross sections so that one must resort to some coupled-channel approximation (Chase, Wilets, and Edmonds (1958)).

Even if it should turn out that one might later wish to take more careful account of the coupling of the rotational channels, the DWBA calculations are of considerable interest since they allow us to separately state the excitation processes due to the Coulomb, local, and momentum-dependent pion-nucleus interactions; these effects are not easily separated in, say a coupled-channel approximation.

4-2 DWBA Formulae for Pion Excitation Cross Sections

In the present section we derive in the Distorted Wave Born Approximation the formulae for the excitation of rotational states in strongly deformed nuclei by pions. A general discussion of DWBA can be found in many text books on

nuclear physics (Kikuchi and Kawai (1968), Preston (1962)) and quantum mechanics (Messiah (1962)). For our purposes, we only cursorily review the subject and from the point of view of the problem at hand. We then use the DWBA expressions to present formulae suitable for evaluating and interpreting the pion excitation cross sections. In Sec. 4-3 we will evaluate these formulae numerically for a realistic pion-nucleus potential (the Ericsons' potential of Chapter 2 and Sec. 3-3) with particular emphasis upon the role of the nuclear surface thickness.

To begin with, let us consider a pion, described by a kinetic energy operator, T_π , to be incident on a deformed target nucleus described by the Hamiltonian, h , of Eq. 4-2. We will assume that the pion interacts with the nucleus through an average potential, $\mathcal{V}(\underline{r})$, such as the momentum-dependent Ericsons' potential discussed in Chapter 2. We will assume that real pion absorption and absorptive effects due to the many-body structure of the nucleus are accounted for in the imaginary part of this potential; the only channels available to the problem are then through those collective modes of oscillation which can be excited by the average interaction, in particular, the rotational channels.

In our calculations we will assume the deformed nucleus to be axially symmetric. To evaluate the excitation cross sections it is then convenient to expand $\mathcal{V}(\underline{r})$ in the body-fixed frame in a series of spherical harmonics,

$$(4-6) \quad \mathcal{V}(\underline{r}) = \sum_{\lambda=0}^{\infty} v_{\lambda}(\underline{r}_b) Y_{\lambda}^0(\theta_b) ,$$

where the body-fixed co-ordinate system (r_b, θ_b, ϕ_b) is chosen to have its origin at the center of the nucleus and its polar axis along the nuclear axis of symmetry. We can then divide $\mathcal{V}(\underline{r})$ into the spherically symmetric potential, $\mathcal{V}_0(r)$, and the perturbing potential,

$$(4-7) \quad \mathcal{V}'(\underline{r}) = \sum_{\lambda=1}^{\infty} \mathcal{V}_{\lambda}(r_b) Y_{\lambda}^0(\theta_b),$$

which is responsible for the nuclear excitations.

The total Hamiltonian of the system can now be written in the form,

$$(4-8a) \quad H = H_1 + \mathcal{V}'(\underline{r}),$$

where the unperturbed Hamiltonian is

$$(4-8b) \quad H_1 = H_0 + \mathcal{V}_0(r),$$

and where the Hamiltonian in the absence of interaction is

$$(4-8c) \quad H_0 = T_{\pi} + h.$$

Before proceeding, it will be convenient to define the eigenstates of the Hamiltonians of Eqs. 4-8. Let us first define the eigenstates, Φ_a , of H_0 , the Hamiltonian in the absence of interaction:

$$(4-9) \quad \Phi_a = \Psi_{\alpha} e^{i \underline{k}_a \cdot \underline{r}},$$

where the nuclear states, Ψ_{α} , are given by Eq. 4-5, and $e^{i \underline{k}_a \cdot \underline{r}}$ is the pion wave function, $\underline{k}_a$ being the incident pion momentum (for notational convenience, we will treat the incident pion as a plane wave and neglect terms in $\log(2k_a r)$ which arise from the long-range effects of the pion-nucleus interaction; the generalized discussion is essentially the same).

The total state of the system is represented by the wave function, $\psi_a^{(+)}$, which is that eigenstate of the total Hamiltonian, H , of total energy, E , which asymptotically contains an incoming plane wave in the incident channel, α , with pion momentum, $\underline{k}_a$, and outgoing waves in all other channels, say γ , with corresponding pion momentum, say $\underline{k}_c$:

$$(4-10a) \quad H \psi_a^{(+)} = E \psi_a^{(+)} ;$$

$$(4-10b) \quad \psi_a^{(+)} \underset{r \rightarrow \infty}{\sim} \psi_\alpha \left[e^{i \underline{k}_a \cdot \underline{r}_\alpha} + f_{a\alpha}^{(+)}(\Omega_\alpha) \frac{e^{-i \underline{k}_a \cdot \underline{r}_\alpha}}{r_\alpha} \right] ;$$

$$(4-10c) \quad \psi_a^{(+)} \underset{r \rightarrow \infty}{\sim} \psi_\alpha f_{a\gamma}^{(+)}(\Omega_\gamma) \frac{e^{-i \underline{k}_c \cdot \underline{r}_\gamma}}{r_\gamma} \quad (\gamma \neq \alpha) .$$

Here $f_{a\beta}^{(+)}(\Omega_\beta)$ is the scattering amplitude from the incident channel, α , into the channel, β .

We can define states, $\chi_a^{(+)}$, for the unperturbed Hamiltonian, H_1 , similar to the total state, $\psi_a^{(+)}$, of Eqs. 4-10, except that $\chi_a^{(+)}$ contains only an outgoing wave in the incident channel, α , since H_1 , having spherical symmetry, contains no mechanism for exciting a rotational state. We can then write

$$(4-11a) \quad \chi_a^{(+)} = \psi_\alpha \phi_a^{(+)}(\underline{r}_\alpha) ,$$

where $\phi_a^{(+)}(\underline{r}_\alpha)$ is the optical wave function of the pion satisfying the equation,

$$(4-11b) \quad (T_\pi + V_0(r_\alpha)) \phi_a^{(+)}(\underline{r}_\alpha) = (E - \epsilon_\alpha) \phi_a^{(+)}(\underline{r}_\alpha) ,$$

and having the asymptotic form,

$$(4-11c) \quad \phi_a^{(+)}(\underline{r}_\alpha) \underset{r_\alpha \rightarrow \infty}{\sim} e^{i \underline{k}_a \cdot \underline{r}_\alpha} + f_{1a\alpha}^{(+)}(\Omega_\alpha) \frac{e^{-i \underline{k}_a \cdot \underline{r}_\alpha}}{r_\alpha} ;$$

here the pion wave number is given by the relation,

$$(4-11d) \quad k_a = \frac{2\mu(E - \epsilon_a)}{\hbar},$$

where μ is the reduced mass of the pion.

The DWBA now says the following: If the amplitudes $f_{a\gamma}^{(+)}(\Omega_\gamma)$ ($\gamma \neq \alpha$) in Eq. 4-10 b are small --- i.e., if the excitation cross sections are small --- we may omit from $\psi_a^{(+)}$ those terms which lead to asymptotically outgoing waves in an excited channel; it is then reasonable also to neglect the absorptive effects on $f_{a\alpha}^{(+)}(\Omega_\alpha)$ of the coupling between the incident channel, α , and the excited channels, γ , in $\psi_a^{(+)}$. Since the perturbation term, $V'(\underline{r})$, is small compared to $V_0(r)$, we may also neglect its effects on the elastic scattering and thus take

$$(4-12) \quad f_{a\alpha}^{(+)}(\Omega_\alpha) = f_{1a\alpha}^{(+)}(\Omega_\alpha).$$

Hence, DWBA says that, to good approximation,

$$(4-13) \quad \psi_a^{(+)} = \chi_a^{(+)}.$$

We call $\chi_a^{(+)}$ the "distorted wave".

It is to be noted that $\chi_a^{(+)}$ contains almost all the effects of the pion-nucleus interaction. We therefore expect DWBA to be a much better approximation than plane wave Born Approximation, where we take $\psi_a^{(+)} = \bar{\phi}_a$, the incident plane wave; in fact, in pion-nucleus scattering, the elastically scattered wave is strongly distorted and we do not expect the latter approximation to be good at all.

It remains now only to evaluate the excitation cross sections. From elementary quantum mechanics (Messiah (1963)),

the excitation cross section is given by

$$\begin{aligned}
 (4-14) \quad \frac{d\sigma_{a \rightarrow b}}{d\Omega_{\beta}} &= \frac{k_b}{k_a} \left| f_{a\beta}^{(+)}(\Omega_{\beta}) \right|^2 \\
 &= \frac{k_b}{k_a} \left(\frac{\mathcal{N}}{2\pi\hbar^2} \right)^2 \left| T_{a \rightarrow b} \right|^2,
 \end{aligned}$$

where the matrix elements, $T_{a \rightarrow b}$, of the scattering matrix, T , are given by the expressions,

$$\begin{aligned}
 (4-15) \quad T_{a \rightarrow b} &= \langle \Phi_b | T | \Phi_a \rangle \\
 &= \langle \Phi_b | (\mathcal{V}_0(r) + \mathcal{V}'(\underline{r})) | \psi_a^{(+)} \rangle.
 \end{aligned}$$

It is then a well-known result (Messiah (1962)) that,

$$(4-16) \quad T_{a \rightarrow b} = \langle \Phi_b | \mathcal{V}_0(r) | \chi_a^{(+)} \rangle + \langle \chi_b^{(-)} | \mathcal{V}'(\underline{r}) | \psi_a^{(+)} \rangle.$$

The distorted wave, $\chi_b^{(-)}$, in Eq. 4-16 satisfies equations analogous to Eqs. 4-11:

$$(4-17a) \quad \chi_b^{(-)} = \Psi_{\beta} \phi_b^{(-)}(\underline{r}) ;$$

$$(4-17b) \quad (T_{\beta} + \mathcal{V}_0(r_{\beta})) \phi_b^{(-)}(\underline{r}_{\beta}) = (E - \epsilon_{\beta}) \phi_b^{(-)}(\underline{r}_{\beta}) ;$$

$$(4-17d) \quad k_b = \frac{2\mathcal{N}(E - \epsilon_{\beta})}{\hbar} ;$$

except that it is asymptotic to incoming rather than outgoing scattered waves:

$$(4-17c) \quad \phi_b^{(-)}(\underline{r}_{\beta}) \underset{r_{\beta} \rightarrow \infty}{\sim} e^{i\mathbf{k}_b \cdot \underline{r}_{\beta}} + f_{b\beta}^{(-)}(\Omega_{\beta}) \frac{e^{ik_b r_{\beta}}}{r_{\beta}} ;$$

here $\mathbf{k}_b$ is the momentum in the direction of the scattered pion wave. We will defer giving explicit expressions for $\phi_a^{(+)}$ and $\phi_b^{(-)}$ until we are ready to evaluate the matrix elements of Eq. 4-16 (c.f. Eq. 4-30 ff.).

Now, if $\delta \neq \alpha$, we have from the spherical symmetry of $\psi_0(r)$ and from Eqs. 4-5 and 4-11 that in Eq. 4-16,

$$(4-18) \quad \langle \phi_c | \psi_0(r) | \chi_a^{(*)} \rangle = 0.$$

If we further make the DWBA, Eq. 4-13, we have from Eqs. 4-14 and 4-16 that

$$(4-19) \quad \frac{d\sigma_{a \rightarrow c}}{d\Omega_\gamma} = \frac{k_c}{k_a} \left(\frac{\nu}{2\pi\hbar^2} \right)^2 |\langle \chi_c^{(*)} | \psi(r) | \chi_a^{(*)} \rangle|^2$$

$$= \frac{k_c}{k_a} \left(\frac{\nu}{2\pi\hbar^2} \right)^2 |\langle \psi_c^{(*)}(r) | \psi(r) | \psi_a(r) \rangle|^2.$$

Eq. 4-19 can be evaluated in a straightforward manner and the details of the evaluation have been relegated to Appendix B of this thesis.

As is shown in Appendix B, Sec. B-1, the result for an unpolarized beam of pions (where we average over initial states and sum over final states) is that

$$(4-20) \quad \frac{d\sigma_{a \rightarrow c}}{d\Omega_\gamma} = \frac{k_c}{k_a} \left(\frac{\nu}{2\pi\hbar^2} \right)^2 \sum_{\lambda=1}^{\infty} A_{\gamma\alpha}^{\lambda} \sum_{\nu=-\lambda}^{\lambda} |e_{ac}^{\lambda\nu}|^2,$$

where $A_{\gamma\alpha}^{\lambda}$ are statistical coefficients describing the rotational band,

$$(4-21a) \quad A_{\gamma\alpha}^{\lambda} = \frac{\langle \lambda I_\alpha 0 K | I_\gamma K \rangle}{2\lambda + 1} \quad (\lambda \text{ even});$$

$$(4-21b) \quad A_{\gamma\alpha}^{\lambda} = 0 \quad (\lambda \text{ odd});$$

and where $e_{ac}^{\lambda\nu}$ are matrix elements over the pion co-ordinates,

$$(4-22) \quad e_{ac}^{\lambda\nu} = \int d\underline{r} \phi_c^{(*)}(\underline{r}) \psi_\lambda(\underline{r}) Y_\lambda^\nu(\theta, \phi) \phi_a^{(*)}(\underline{r}).$$

Before proceeding to evaluate Eq. 4-22 for the matrix

elements $\mathcal{O}_{ac}^{2\nu}$ we shall need to describe the potentials

$\mathcal{V}_0(\underline{r})$ and $\mathcal{V}'(\underline{r})$ of Eqs. 4-6 and 4-7 in more detail.

We will take the pion-nucleus interaction to be described by the Ericsons' potential of Eq. 2-36.

Now we know from electromagnetic experiments, such as Coulomb excitation, that a strongly deformed nucleus is nearly spheroidal so that we can write the nuclear density as

$$(4-23) \quad \rho(\underline{r}) = \rho \left(\frac{r}{1 + \beta Y_2^0(\theta_b)} \right),$$

where $\rho(r)$ is, say, the Saxon-Woods density of Eq. 3-31.

The average deformation parameter, β , is typically small ($\lesssim 0.3$) so that we can take

$$(4-24) \quad \rho\left(\frac{r}{1 + \beta Y_2^0(\theta_b)}\right) = \rho(r) - \beta r \rho'(r) Y_2^0(\theta_b).$$

As can be seen from Fig. 1, Sec. 3-4, $\rho(r) \sim r^2$, and we will simply make this replacement in Eq. 3-29. We can then take the pion-nucleus potential to be given by

$$(4-25) \quad \mathcal{V}(\underline{r}) = -\frac{\hbar^2}{2\nu} \nabla \cdot \frac{\alpha(\underline{r})}{1 - \frac{\alpha(\underline{r})}{3}} \nabla + V_c(\underline{r}) + V(\underline{r}) + iW(\underline{r})$$

where

$$(4-26a) \quad \alpha(\underline{r}) = \alpha_0(r) - \beta r \alpha'_0(r) Y_2^0(\theta_b);$$

$$(4-26b) \quad V(\underline{r}) = V_0(r) - \beta r V'_0(r) Y_2^0(\theta_b);$$

$$(4-26c) \quad W(\underline{r}) = W_0(r) - \beta r W'_0(r) Y_2^0(\theta_b);$$

here $\alpha_0(r)$, $V_0(r)$, and $W_0(r)$ are the potentials $\alpha(r)$, $V(r)$,

and $W(r)$ in Eqs. 3-29. The Coulomb potential, $V_c(r)$, is evaluated in Appendix B, Sec. B-2, where we find that

$$(4-27a) \quad V_c(\underline{r}) = \frac{Ze^2}{r} + \frac{3}{5} \beta \frac{Ze^2}{r^3} R^2 Y_2^0(\theta_b), \quad r \geq R;$$

$$(4-27b) \quad V_c(\underline{r}) = \frac{Ze^2}{R} \left[\frac{3}{2} - \frac{1}{2} \left(\frac{r}{R} \right)^2 \right] + \frac{3}{5} \beta \frac{Ze^2 r^2}{R^3} Y_2^0(\theta_b), \quad r \leq R;$$

(Here we have treated the charge density as uniform and of radius, R).

We therefore have that

$$(4-28) \quad \mathcal{V}_0(r) = -\frac{\hbar^2}{2\mu} \nabla \cdot \frac{\alpha_0(r)}{1 - \frac{\alpha_0(r)}{3}} \nabla + V_c(r) + V_0(r) + iW_0(r),$$

is the optical potential of Eq. 3-29 and that the perturbing potential is

$$(4-29a) \quad \mathcal{V}'(\underline{r}) = f_L(r) Y_2^0(\theta_b) + \nabla \cdot f_{LD}(r) Y_2^0(\theta_b) \nabla + f_c(r) Y_2^0(\theta_b),$$

where

$$(4-29b) \quad f_c(r) = -rV'_0(r) - irW'_0(r)$$

is the local interaction, where

$$(4-29c) \quad f_{LD}(r) = \frac{\hbar^2}{2\mu} \frac{r \alpha'_0(r)}{\left(1 - \frac{\alpha_0(r)}{3} \right)^2}$$

is the momentum-dependent strength, and where

$$(4-29d) \quad f_c(r) = -\frac{3}{5} \beta Ze^2 \frac{R^2}{r^3}, \quad r > R;$$

$$f_c(r) = -\frac{3}{5} \beta Ze^2 \frac{r^2}{R^3}, \quad r < R;$$

is the electromagnetic interaction.

Of course, we should strictly have calculated the potential harmonics, $\mathcal{V}_\lambda(r)$, from Eq. 4-6. For instance, such a calculation has been performed by Chase, Wilets, and Edmonds (1958) for the excitation of rotational states in deformed nuclei by neutrons; they find that estimates such as those made in deriving Eqs. 4-28 and 4-29 are of only moderate accuracy. Nevertheless, for our purposes we simply wish to demonstrate the effect of the nuclear surface on the excitation cross sections and for this purpose these estimates should be quite satisfactory. A more accurate calculation would, of course, take a more careful account of the potentials.

Since we are interested in pion excitation at low energies ($\lesssim 30$ MeV), only a few partial waves enter the problem (as can be seen from Figs. 3, 5, and 11 of Sec. 3-4) and it will be convenient to make a partial wave expansion of the optical waves, $\phi_a^{(+)}(\underline{r})$, and $\phi_c^{(-)}(\underline{r})$. From Eqs. 4-11 and 4-17 it can be shown (Messiah (1962)) that

$$(4-30) \quad \phi_{\underline{k}}^{(+)}(\underline{r}) = \frac{4\pi}{kr} \sum_{\ell, m} i^\ell e^{-i\beta_\ell} e^{i\delta_\ell} Y_\ell^{m*}(\hat{r}) Y_\ell^m(\hat{k}) F_\ell(k; r),$$

where $\delta_\ell + i\beta_\ell$ are the partial wave phase shifts (and $F_\ell(k; r)$ is the regular spherical Coulomb function). Hence,

$$(4-31) \quad \phi_{\underline{k}}^{(-)}(\underline{r}) = \phi_{\underline{k}}^{(+)*}(-\underline{r})$$

At the energies which we will consider (several MeV) we can neglect the effects of nuclear excitation ($\epsilon_\gamma - \epsilon_\pi \lesssim 1$ MeV) and take $k_c = k_a$. In addition, we will take $\hat{k}_a$, the beam

direction, to be along the (space-fixed) z -axis and we will henceforward suppress the indices a, c , for example, taking $\hat{k} = \hat{k}_c$ to be the direction of the scattered wave. Then,

$$(4-32a) \quad \phi_a^{(+)}(\underline{r}) \equiv \phi(\underline{r}) = \frac{\sqrt{\pi}}{k} \sum_{\ell, m} \sqrt{2\ell+1} i^{\ell+1} \frac{u_\ell(r)}{r} Y_\ell^0(\hat{r}) ;$$

$$(4-32b) \quad \phi_c^{(-)}(\underline{r}) = \phi^*(-\underline{r}) = -\frac{2\pi}{k} \sum_{\ell, m} i^{\ell+1} \frac{u_\ell^*(r)}{r} Y_\ell^m(\hat{k}) Y_\ell^{m*}(\hat{r}) ;$$

where the partial waves, $u_\ell(r)$, are solutions of the radial Schroedinger Eqs. 3-4.

Considering Eqs. 4-29 and 4-32, we can rewrite Eq. 4-20 in the convenient form,

$$(4-33a) \quad \frac{d\sigma_{\alpha \rightarrow \gamma}}{d\Omega} = \beta^2 A_{\gamma\alpha}^2 \frac{d\sigma^{\text{red}}}{d\Omega} ,$$

where

$$(4-33b) \quad \frac{d\sigma^{\text{red}}}{d\Omega} = \left(\frac{\nu}{2\pi\hbar^2} \right)^2 \sum_{\nu=-2}^2 |I^\nu|^2 ;$$

here,

$$(4-34) \quad I^\nu = \int d\underline{r} \phi(-\underline{r}) \frac{\sqrt{2(r)}}{\beta} Y_2^\nu(\theta, \phi) \phi(\underline{r}) .$$

As can be seen from Eq. 4-34, $\frac{d\sigma^{\text{red}}}{d\Omega}$ is independent of the rotational band (to the extent to which we can neglect the energies associated with nuclear excitations) and of the nuclear deformation (to the extent Eq. 4-29 correctly represents the perturbing potential) and it is $\frac{d\sigma^{\text{red}}}{d\Omega}$ which we shall consider in our calculations.

The evaluation of I^ν simply involves the substitution of the partial wave expansion, Eq. 4-32, for $\phi(\underline{r})$ and the

substitution of the perturbing potential, Eq. 4-29, in Eq. 4-34. The integration over the angular variables is then straightforward and we can conveniently state the result of the calculation by defining the following radial integrals.

Firstly, there are radial integrals

$$(4-35a) \quad F_L^{l'l} = \int dr u_{l'}(r) f_L(r) u_l(r) \quad , \#$$

arising from the local interaction; there are similar radial integrals,

$$(4-35b) \quad F_C^{l'l} = \int dr u_{l'}(r) f_C(r) u_l(r) \quad ,$$

arising from the electromagnetic interaction. There are essentially two kinds of radial integrals arising from the momentum-dependent interaction: firstly, there are radial integrals

$$(4-35c) \quad F_{MD,r}^{l'l} = \int r^2 dr \frac{d}{dr} \frac{u_{l'}(r)}{r} f_{MD}(r) \frac{d}{dr} \frac{u_l(r)}{r}$$

arising from the radial term in the momentum operator;

secondly, there are radial integrals

$$(4-35d) \quad F_{MD,\theta}^{l'l} = \int dr \frac{u_{l'}(r)}{r} f_{MD}(r) \frac{u_l(r)}{r}$$

arising from the angular terms in the momentum operator

$$(F_{MD,\phi}^{l'l} = 0).$$

With these definitions it is found that the differential cross sections, Eq. 4-33, are given in terms of I^ν by the equation

Note that, from Eq. 4-34, the integrals are over $u_{l'}(r)$, not $u_{l'}^s(r)$, as one might initially think.

$$(4-36) \quad I^\nu = (-)^{\nu+1} \frac{\pi}{k^2} \sum_{\ell', \ell} \sqrt{2\ell+1} \sqrt{2\ell'+1} (-i)^{\ell+\ell'} Y_\ell^\nu(\hat{k})$$

$$\begin{aligned} & \times \langle \ell' 00 | 20 \rangle \left[F_L^{\ell' \ell} + F_c^{\ell' \ell} - F_{MD, r}^{\ell' \ell} \right] \langle 2\ell' - \nu \nu | \ell 0 \rangle \\ & - F_{MD, \theta}^{\ell' \ell} \frac{\sqrt{\ell(\ell+1)}}{2} \left(\sqrt{\ell'(\ell'+1) - \nu(\nu-1)} \langle 2\ell' - \nu \nu - 1 | \ell - 1 \rangle \right. \\ & \left. + \sqrt{\ell'(\ell'+1) - \nu(\nu+1)} \langle 2\ell' - \nu \nu + 1 | \ell 1 \rangle \right) \end{aligned}$$

Eq. 4-33 is easily integrated to give the total excitation cross section, the result being

$$\begin{aligned} (4-37a) \quad \sigma_{red} &= \left(\frac{\nu}{2\pi h^2} \right)^2 \frac{\pi}{k^4} \sum_{\ell', \ell, \ell_1} (2\ell'+1) (-i)^{\ell} i^{\ell+\ell_1} \sqrt{2\ell+1} \\ & \times \sqrt{2\ell_1+1} \langle \ell' 00 | 20 \rangle \langle \ell' 00 | 20 \rangle \left[F_L^{\ell' \ell} + F_c^{\ell' \ell} - F_{MD, r}^{\ell' \ell} \right]^2 \delta_{\ell \ell'} \\ & - 2 \operatorname{Re} \left[(F_L^{\ell' \ell} + F_c^{\ell' \ell} - F_{MD, r}^{\ell' \ell}) F_{MD, \theta}^{\ell' \ell} \right] A^{\ell' \ell \ell_1} (1 + (-)^{\ell+\ell_1}) \\ & + F_{MD, \theta}^{\ell' \ell} F_{MD, \theta}^{\ell' \ell} B^{\ell' \ell \ell_1} (1 + (-)^{\ell+\ell_1}) \end{aligned}$$

$$(4-37b) \quad \sigma_{\alpha \rightarrow \gamma} = \beta^2 A_{\gamma \alpha}^2 \sigma_{red} ;$$

where

$$(4-38a) \quad A^{\ell' \ell \ell_1} = \frac{\sqrt{\ell_1(\ell_1+1)}}{2} \sum_{\nu=-2}^2 \sqrt{\ell'(\ell'+1) - \nu(\nu+1)} \langle 2\ell' - \nu \nu | \ell 0 \rangle$$

$$\times \langle 2\ell' - \nu \nu + 1 | \ell_1 1 \rangle ;$$

$$\begin{aligned} (4-38b) \quad B^{\ell' \ell \ell_1} &= \frac{\sqrt{\ell(\ell+1)}}{2} \frac{\sqrt{\ell_1(\ell_1+1)}}{2} \\ & \sum_{\nu=-2}^2 \sqrt{\ell'(\ell'+1) - \nu(\nu+1)} \langle 2\ell' - \nu \nu + 1 | \ell 1 \rangle \\ & \times \left[\sqrt{\ell'(\ell'+1) - \nu(\nu+1)} \langle 2\ell' - \nu \nu + 1 | \ell_1 1 \rangle \right. \\ & \left. + \sqrt{\ell'(\ell'+1) - \nu(\nu-1)} \langle 2\ell' - \nu \nu - 1 | \ell_1 - 1 \rangle \right]. \end{aligned}$$

It is seen in Eqs. 4-36 and 4-37 that we have succeeded

in separating those excitation processes which arise from the momentum-dependent interaction from those which arise from the local and electromagnetic interaction; in fact, the momentum-dependence is contained in the integrals, $F_{MD,r}^{ll}$.

In the next section we will provide numerical estimates of the integrals, Eq. 4-35, and the cross sections, Eqs. 4-33, 4-36 and 4-37, based on the Ericsons' potential of Chapter 2 and Sec. 3-3.

4-3 Numerical Discussion

In the present section we present a numerical discussion of pion excitation based on the DWBA cross section formulae and the radial integrals which we have derived in the last section. In these calculations, our main objective is to demonstrate the sensitivity of the excitation cross sections to the diffuseness of the nuclear surface. As discussed in the previous section, this sensitivity is contained in the radial integrals, $F_{MD,r}^{ll}$, which arise from the radial terms in the momentum-dependent part of the pion-nucleus interaction. To show that the experimental data can be analyzed for these quantities, we first construct the total and differential excitation cross sections and demonstrate that these cross sections are sensitive to the details of the nuclear surface. We then show that this sensitivity is contained in the radial integrals, $F_{MD,r}^{ll}$, and that it can be understood in terms of the suppression of momentum-dependent processes near the top of the potential barrier (in analogy with our discussion in Chapter 3).

For the purposes of the present discussion, we have chosen the potential to be that given by Eqs. 4-28 and 4-29 of Sec. 4-2 where we have expanded the actual potential in terms of the average nuclear deformation parameter, β , retaining only terms of order β . For the parameters of the interaction, we have used the values obtained from the Ericsons' calculation (1966) which we have discussed in Sec. 3-3 of the previous chapter. We have then introduced the diffuseness of the nuclear surface by employing a Saxon-Woods distribution (Eq. 3-31) for the nuclear density. In the Saxon-Woods density we have chosen $R_0 = 1.05 A^{1/3}$ fm. and we have performed our calculations for two choices of the surface parameter, $a = 0.5$ fm. (which in our figures we denote by solid lines) and $a = 0.1$ fm. (which in our figures we denote by dashed lines). We have performed our calculations for both a light nucleus, where we have chosen parameters appropriate to Al^{25} , and for a heavy nucleus, where we have chosen parameters appropriate to U^{238} .

In Figs. 12a and 12b we have plotted the reduced total excitation cross sections, σ_{red} , for Al^{25} and U^{238} , respectively. σ_{red} is related to the actual excitation cross sections, $\sigma_{\alpha \rightarrow \gamma}$, by Eq. 4-37b of Sec. 4-2:

$$(4-37b) \quad \sigma_{\alpha \rightarrow \gamma} = \beta^2 A_{\gamma\alpha}^2 \sigma_{red} ;$$

where β is the average nuclear deformation parameter and $A_{\gamma\alpha}^2$ is a statistical coefficient (Eq. 4-21) describing the initial and final rotational states involved in the excitation. Typically $\beta \lesssim 0.3$ and $A_{\gamma\alpha}^2$ is a number always less than unity

FIG. 12a

REDUCED TOTAL
EXCITATION CROSS SECTIONS

FIG 12b

REDUCED TOTAL
EXCITATION CROSS SECTIONS

FIG. 13a

REDUCED DIFFERENTIAL
EXCITATION CROSS SECTIONS $\pi^+ + \text{Al}^{25}$ $E = 25 \text{ MeV}$ $\frac{d\sigma_{\text{red}}}{d\Omega}, \text{ mb/sterad}$

30

20

10

0

75

50

25

0

 $a = 0.5 \text{ Fm}$ — $a = 0.1 \text{ Fm}$ - - - $E = 35 \text{ MeV}$

0°

30°

60°

90°

120°

150°

180°

 θ

FIG. 13b

REDUCED DIFFERENTIAL
EXCITATION CROSS SECTIONS

(for example, $\Lambda_{0^+ \rightarrow 2^+} = \frac{1}{\sqrt{5}}$). It is therefore seen from Figs. 12 that, typically, $\sigma_{\alpha \rightarrow \gamma} \lesssim 0.03$ barns, considerably less than the geometric cross section of a few barns. From our previous discussion, this would seem sufficient justification to employ the DWBA.

It is seen from Figs. 12 that, in both light and heavy nuclei, even the total excitation cross sections are quite sensitive to the diffuseness of the nuclear surface. In practice, the total cross sections can be measured very accurately by measuring the de-excitation γ -rays when the excited rotational state decays so that these effects can be measured with considerable precision. Although the effects of the diffuseness of the surface become greater at higher energies, the simple connection between the density and the pion-nucleus interaction may then break down, as was discussed in Chapter 2. Nevertheless, if our present understanding of the interaction is even qualitatively correct, the excitation cross sections should provide a powerful tool for investigating the details of the nuclear surface.

In Figs. 13a and 13b we have plotted the reduced differential cross sections (for the inelastically scattered pions) for Al^{25} and U^{238} corresponding to the total cross sections in Figs. 12a and 12b, respectively. Measurement of the differential cross sections involves measuring the inelastically scattered pions and is probably somewhat more difficult than measuring the total cross sections. It is seen, however, that the differential cross sections have a structure which is

strongly dependent upon the diffuseness of the nuclear surface; in Fig. 13a (upper figure) it is seen that this may be the case even when the total cross sections exhibit a much smaller sensitivity. The large back scattering arises since it is more probable for the pion to excite the nucleus in a head-on collision ($\theta = 180^\circ$) than in a grazing collision ($\theta = 0^\circ$).

Before we proceed to analyze the cross sections in more detail, it will be useful to remember from our earlier discussion (Eqs. 4-35) that pion excitation arises from essentially four separate processes: firstly, there is excitation due to the local pion-nucleus interaction, accounted for in the integrals, $F_L^{l'l}$; secondly, there is "Coulomb excitation" due to the electrostatic potential, accounted for in the integrals, $F_C^{l'l}$; thirdly, there is excitation due to the angular terms in the momentum-dependent interaction, accounted for in the integrals, $F_{MD,\theta}^{l'l}$; finally, there is excitation due to the radial terms in the momentum-dependent interaction, accounted for in the integrals, $F_{MD,r}^{l'l}$. In the subsequent discussion we will show that the first three integrals are insensitive to the details of the nuclear surface and that the sensitivity demonstrated in Figs. 12 and 13 is contained mainly in the integrals, $F_{MD,r}^{l'l}$. It will then be our objective to show that this sensitivity can easily be understood in terms of the suppression of pion momentum near the top of the potential barrier.

In Fig. 14 we have plotted the absolute values of the radial integrals for the local interaction, $|F_L^{l'l}|$, at 35 MeV

FIG. 14

RELATIVE VALUES OF
RADIAL INTEGRALS

$$|F_L^{\ell\ell}|$$

E = 35 MeV

 $\pi^+ + A^{25}$

E = 35 MeV

 $\pi^+ + U^{238}$

FIG 15_a

RADIAL INTEGRALS

FIG. 15b

RADIAL INTEGRALS

ARBITRARY

 $\pi^+ + \text{U}^{238}$ $|F_L^{II}|$ $a = 0.5 \text{ fm}$ — $a = 0.1 \text{ fm}$ - - - $|F_C^{II}|$ $|F_{MD_9\theta}^{II}|$

0 5 10 15 20 25 30 35

$E, \text{ MeV}$

FIG. 16_a

RADIAL INTEGRALS

ARBITRARY

 $\pi^+ + \text{Al}^{25}$ $|F_{MD,r}^{II}|$

$a = 0.5 \text{ fm}$ —
 $a = 0.1 \text{ fm}$ - - -

 $|F_{MD,r}^{02}|$

0 5 10 15 20 25 30 35
 $E, \text{ MeV}$

FIG. 16b

RADIAL INTEGRALS

for the allowed angular momentum transitions between the initial state partial wave of angular momentum, l , and the final state partial wave of angular momentum, l' . We have restricted our calculation to $l, l' \leq 3$ (although a more precise calculation would take into account the higher partial waves). It is seen that the main contribution to the excitation arises from the integral F_L^{11} and the secondary contribution from the integral F_L^{02} (or, equivalently, F_L^{20}). Hence, even at moderately high energies, only a few partial waves enter the problem; of course, as we go to lower energies the relative dominance of the integrals, F_L^{11} , increases. We will therefore here restrict ourselves to integrals of the form F^{11} and F^{20} ; in any event, the generalization to integrals involving partial waves of higher angular momentum will be apparent from our discussion.

In Fig. 15a and 15b we have plotted the relative magnitudes of the absolute values of the integrals, F_L^{11} , F_C^{11} , and $F_{MD,0}^{11}$, for Al^{25} and U^{238} , respectively. It is seen from Figs. 15 that these radial integrals are quite insensitive to the diffuseness parameter, a , even in the case of Al^{25} , which is mostly surface. This arises since these quantities involve integrals over the nucleus and are sensitive only to the amount of nuclear matter present rather than to its distribution. A similar insensitivity to the surface is found in general for the integrals $F_L^{l'l}$, $F_C^{l'l}$, and $F_{MD,0}^{l'l}$.

In Fig. 16a and 16b (upper figures) we have plotted

(on the same scale) the corresponding values of the integrals arising from the radial term in the momentum-dependent interaction, $F_{MD,r}^{11}$. It is seen that, unlike the other integrals, these integrals depend very strongly on the diffuseness parameter, a . In fact, as we will now show, the behaviour of these integrals can be understood in an analogous manner to the behaviour of momentum-dependent absorption discussed in Chapter 3.

In Fig. 16a we have plotted the radial integrals $|F_{MD,r}^{11}|$ (upper figure) and $|F_{MD,r}^{02}|$ (lower figure) appropriate to Al^{25} . Since the parameters of the pion-nucleus potential for Al^{25} are similar to those of Ca^{40} , which we discussed in connection with optical absorption in Sec. 3-4, we will interpret the effects found here in terms of the explanations given there.

For instance, the behaviour of $F_{MD,r}^{02}$ depends on the properties of the radial s-wave function. Now, near the top of the potential barrier, the radial momentum associated with the s-wave becomes small. For the diffuse-edge potential, most of the interaction takes place in the surface of the nucleus where the nuclear density is lower; hence, for the diffuse-edge potential, the effective barrier appears lower. Thus, we expect at lower energies to find that the value of $|F_{MD,r}^{02}|$ is less if we assume a diffuse-edge potential than if we assume a sharp-edge potential and, at higher energies, we expect to find that it is greater. This is in analogy with our discussion of momentum-dependent absorption

in Ca^{40} which we have discussed in Fig. 7. This effect is shown here in Fig. 16a (lower figure) and is to be compared with the imaginary s-wave phase shift for optical absorption in Ca^{40} (Fig. 5).

As we discussed in Sec. 3-4, the effect of increasing the angular momentum is to increase the potential barrier. We therefore expect at higher energies to see a similar behaviour for the integral, $F_{\text{MD},r}^{11}$, as was found at lower energies for the integral, $F_{\text{MD},r}^{02}$. This effect is shown in Fig. 16a (upper figure).

As we discussed for Pb^{208} in Sec. 3-4 (Fig. 8), the major effect in going to a heavy nucleus is an increase in the effective barrier due to an increase in the electrostatic potential. This effect is seen in Fig. 16b where it is shown that even at higher energies both the integrals $F_{\text{MD},r}^{11}$ and $F_{\text{MD},r}^{02}$ are more strongly suppressed for a diffuse-edge potential than for a sharp-edge potential. The attenuation at lower energies which is seen in the diagram is due, of course, to the effects of penetration.

We might now return briefly to the cross sections corresponding to these radial integrals which we have given in Figs. 12 and 13. We believe that in the discussion which we have presented here we have clearly shown that the dependence of these cross sections on the nuclear surface arises in an easily understandable way through the momentum-dependent interaction. In fact, our discussion depended only upon the qualitative features of the pion-nucleus interaction --- the

presence of a potential barrier and a strong momentum-dependent interaction. We have not investigated the structure of the cross sections in detail since, firstly, the parameters of the potential are not very well determined (and are probably somewhat energy-dependent) and, secondly, our rough treatment of the potential (Eqs. 4-28 and 4-29) is only approximately correct. Future calculations of this type will probably employ potentials obtained from scattering experiments (such as those suggested in Chapter 3) and will treat the perturbing potential in a more precise manner (such as that suggested by Chase, Wilets, and Edmonds (1958)). It is worthwhile, however, to mention one further effect which is seen in these cross sections.

It is seen in Fig. 12 that, in the case of Al^{25} , increasing the diffuseness parameter, a , decreases the total cross section while, in the case of U^{238} , the total cross section is enhanced. On the other hand, we know from Fig. 16 that the radial integrals, $F_{\text{MD},r}^{l'l}$ --- the only aspects of the problem which are sensitive to the diffuseness parameter --- are generally suppressed at low energies for a diffuse-edge potential (with the minor exception of $F_{\text{MD},r}^{02}$ in the case of Al^{25}).

The explanation of this behaviour of the cross sections is understood from the formula for the total cross section, Eq. 4-37, in which it is seen that the cross section involves interference between the amplitudes, $F_{\text{MD},r}^{l'l}$ and the other amplitudes which enter the excitation process ($F_L^{l'l}$, $F_C^{l'l}$, and

$F_{MD,0}^{l'l}$ in fact, by comparing the absolute values of these amplitudes on Figs. 15a and 15b and on Figs. 16a and 16b, respectively, (which have been drawn to the same scale) it is clear that the effects of $F_{MD,r}^{l'l}$ are seen primarily through interferences with the more dominant terms.

Hence, in Fig. 12a ($A1^{25}$) the decrease in $F_{MD,r}^{l'l}$ with increasing diffuseness parameter represents a decrease in constructive interference; in Fig. 12b (U^{238}) it represents a decrease in destructive interference and, hence, an enhancement of the cross section. It is difficult to see through the complex algebra in Eq. 4-37 and to present these arguments in a more satisfactory form. However, a more careful analysis of effects such as these will be required in any event when more satisfactory values for the pion-nucleus potential become available.

In this thesis we have attempted to delineate the important features of the low energy pion-nucleus interaction and, in particular, we have emphasized the usefulness of this interaction as a tool for investigating the structure of the nuclear surface.

In Chapter 2 we have investigated the construction of the pion-nucleus optical potential from the elementary pion-nucleon and pion-deuteron processes; we have shown that a transparent connection exists between the macroscopic and microscopic descriptions of pion interactions in nuclei and, in fact, that the potential which describes the macroscopic interaction involves the nuclear density in a very direct manner.

In Chapter 3 we have used this potential to investigate the optical properties of pions in nuclei and we have shown that the elastic scattering and absorption cross sections depend strongly upon the details of the nuclear surface through the momentum-dependent absorption. In Chapter 4 we have shown that the cross sections for the excitation of rotational states in deformed nuclei through the optical pion-nucleus interaction depends strongly on the nuclear surface through momentum-dependent excitation for analogous reasons to those which were discussed in Chapter 3 for optical absorption. We have concluded from our calculations and discussion that low energy pions provide an interesting new probe of nuclear structure; in particular, they provide a direct means of measuring the

distribution of nucleons in the nuclear surface.

In Chapter 2 we have discussed the construction of an optical potential which describes the low energy pion-nucleus optical interaction in terms of elementary pion-nucleon and pion-deuteron scattering and absorption processes. In particular, we have shown that this interaction involves the density of the nucleus in a very direct manner.

In Sec. 2-1 we have discussed a geometric technique for exhibiting the structure of the low energy pion-nucleus optical potential; we have argued that, since the elementary pion-nucleon and pion-deuteron scatterings are s-wave and p-wave scatterings, the elementary scattered waves should add to form the total scattered wave in an analogous way to that in which one adds the electric potentials of dipoles and charges in a classical dielectric. We have shown that the s-wave scatterings add like the potentials from free charges in a dielectric and lead to a local term in the pion-nucleus potential; the p-wave scatterings modify the pion momentum-field in the nucleus, in analogy to the manner in which the dipoles modify the electric field in a dielectric, and lead to a momentum-dependent term in the pion-nucleus interaction which is analogous to the Lorenz-Lorentz effect in a dense classical medium.

We have discussed the derivation of the optical potential from a microscopic and quantum mechanical point of view in Sec. 2-2 where we have reviewed an earlier discussion due to Ericson and Ericson (1966). We have shown that the many-

body aspects of the problem are reduced to tractable proportions mainly through the short pion-nucleon scattering lengths and the small mass of the pion. We have shown that these properties allow us to make the impulse approximation, so that we may use the known free space pion-nucleon scattering amplitudes, and allow us to simplify the Green's functions which describe the propagation of the pion in the nucleus, since the elementary pion collisions may be treated as being essentially elastic. From these approximations, and the assumption that the nucleon scatterers are massive, we were able to perform the nuclear averages and to obtain multiple scattering equations for the pion analogous to those describing multiple scattering in a classical medium. In fact, we were able to show that these equations could be reduced to conventional Schroedinger equations for the pion in which the optical potential involved the density of the nucleus in a very direct manner and which had the structure predicted by the electromagnetic analogy. In Sec. 2-3 we presented the rather sparse experimental verification of the parameters of this potential provided by the π -mesic x-ray experiments.

In Chapter 3 we used this potential to examine the optical properties of pions in nuclei and to investigate the role of the nuclear surface in determining elastic scattering and absorption cross sections. In Sec. 3-1 we have made a partial wave analysis of the cross sections and we have reviewed the derivation of well-known formulae which express the

partial wave phase shifts in terms of the interior logarithmic derivatives; where necessary, we have extended these formulae (which are conventionally written for local interactions) to momentum-dependent interactions.

In Sec. 3-2 we have used the formalism developed in Sec. 3-1 and the assumption of a uniform nuclear density to obtain analytic formulae which describe the optical properties of the pion-nucleus interaction. We have shown that the resonance aspects of the problem, which are contained in the real parts of the interior logarithmic derivatives, are insensitive to the details of the interaction because of the long pion wavelength near the top of the potential barrier. In fact, we have shown that the resonance aspects of the problem depend only upon the height of the potential barrier and, for higher partial waves, the real term in the momentum-dependent interaction. We have also shown that, near the top of the potential barrier, the momentum-dependent absorption processes, which are described by the imaginary parts of the logarithmic derivatives, are strongly suppressed due to the small momentum of the pion.

In Sec. 3-3 we have stated numerical values for the parameters of the pion-nucleus optical potential which we have taken to be the zero-energy values given by Ericson and Ericson (1966). In Sec. 3-4 we have used this potential assuming a diffuse-edge nuclear density to give a more quantitative discussion of the ideas developed in Sec. 3-2. We have shown that the real parts of the partial wave phase shifts

depend sensitively only upon the height of the potential barrier (through the resonance aspects of the problem) and upon the radius chosen for the nucleus (through the penetration aspects of the problem). We have used these results to suggest a procedure for analyzing empirical data.

We have made a thorough investigation of the absorptive aspects of the problem and we have shown that the imaginary part of the partial wave phase shifts, which determines absorption and diffraction, depends very sensitively upon the diffuseness of the nuclear surface. We have provided an explanation of this effect by segregating the local absorption from the momentum-dependent absorption in the imaginary part of the s-wave interior logarithmic derivative. We have shown, by comparing the absorption found with a sharp-edge nuclear density and a more conventional diffuse-edge nuclear density, that the effect of the diffuse-edge is to lower the effective potential barrier (which is proportional to the nuclear density) seen by the absorbed pions. The strong suppression of momentum-dependent absorption therefore occurs at a lower energy in a diffuse-edge nucleus than a sharp-edge nucleus. On the other hand, we know the height of the potential barrier (and, hence, the nuclear density) inside the nucleus from the real part of the partial wave phase shifts which contain the resonance aspects of the problem. We therefore have concluded that the thickness of the nuclear surface can be accurately determined from measurements of pion-nucleus elastic scattering and absorption cross sections. We have

shown that the main effect on the analysis of pion-nucleus cross sections of using positive or negative pions, or of using different nuclei, is simply to adjust the effective height of the potential barrier seen by the pion through the electrostatic interaction.

In Sec. 3-5 we have briefly discussed the isospin and hyperfine terms which enter the pion-nucleus interaction.

In Chapter 4 we have examined the excitation of rotational levels in deformed nuclei by pions and we have shown that, unlike excitation processes arising from more conventional interactions, the excitation process found here depends sensitively upon the structure of the nuclear surface. In Sec. 4-1 we have briefly reviewed the description of strongly deformed nuclei provided by the rotational model and we have discussed the validity of the Distorted Wave Born Approximation description of pion excitation processes. In Sec. 4-2 we have employed the DWBA to write the pion excitation amplitudes in a form which separately segregates the effects of Coulomb excitation, local excitation, and momentum-dependent excitation.

In Sec. 4-3 we have used the Ericsons' potential of Sec. 3-3 to calculate the excitation cross sections for a realistic diffuse-edge potential and for a sharp-edge potential. We have shown that the structure of the cross sections is strongly affected by the thickness of the nuclear surface and that this structure is mainly contained in the radial integrals which describe the momentum-dependent excitation.

We have shown that the effect of the diffuse nuclear surface is to lower the effective potential barrier so that the strong suppression of momentum-dependent excitation occurs at a lower energy for a diffuse-edge nucleus than for a sharp-edge nucleus. In fact, this is in analogy to our discussion of momentum-dependent absorption in Sec. 3-4 and we have found here, as we found there, that the main effect of considering higher partial waves is to raise the effective barrier seen by the pion through the centrifugal potential and that the main effect of considering different nuclei or pions of different charges is to change the effective barrier through the electrostatic potential. We have concluded from these results that pion excitation cross sections provide a sensitive technique for investigating the structure of the nuclear surface in deformed nuclei.

The general conclusions of this thesis are that low energy pions have optical properties in nuclei which are easily understood and which are qualitatively quite different from those encountered with more conventional interactions because of the appearance of a short-range potential barrier and a strongly momentum-dependent contribution to the interaction. These properties of the interaction, and the fact that the nuclear density enters the pion-nucleus optical potential in a very direct manner, allow one to investigate aspects of nuclear structure which have previously not been easily accessible. In particular, an analysis of pion-nucleus elastic scattering and absorption cross sections and of pion

excitation cross sections in deformed nuclei should reveal detailed information about the distribution of nucleons in the nuclear surface, information crucial to a refinement of the detailed models of nuclear structure.

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APPENDIX AEQUATIONS FOR THE
INTERIOR LOGARITHMIC DERIVATIVES

In the present appendix, we will be concerned with evaluating the interior logarithmic derivatives, λ_ℓ , given in Eq. 3-13. Firstly, we will find an integral solution for their imaginary parts, π_ℓ , in terms of the partial wave functions, $u_\ell(r)$. Secondly, we will evaluate λ_ℓ for a uniform nuclear density in terms of the interior wave functions.

A-1 Integral Equation for Absorption

In Eq. 3-4 we have written radial Schroedinger equations for the partial waves, $u_\ell(r)$:

$$\begin{aligned}
 (3-4) \quad & -\frac{\hbar^2}{2\nu} (1 + \alpha^e(r)) u_\ell''(r) - \frac{\hbar^2}{2\nu} \alpha^{e'}(r) u_\ell'(r) \\
 & + \frac{\hbar^2}{2\nu} \frac{\ell(\ell+1)}{r^2} (1 + \alpha^e(r)) u_\ell(r) \\
 & + \frac{\hbar^2}{2\nu} \frac{\alpha^{e'}(r)}{r} u_\ell(r) + [V_c(r) + V(r) + iW(r)] u_\ell(r) \\
 & = E u_\ell(r) \quad (\ell = 0, 1, 2, \dots)
 \end{aligned}$$

We will now use this equation to obtain an integral equation for the absorptive term, π_ℓ , of the logarithmic derivative, λ_ℓ .

Let us first define the quantity,

$$(A-1) \quad f_\ell(r) \equiv (1 + \alpha^e(r)) \frac{u_\ell'(r)}{u_\ell(r)}.$$

It is then easily shown from Eq. 3-4 that

$$(A-2) \quad f'_\ell(r) = (1 + \alpha^e(r)) \frac{\ell(\ell+1)}{r^2}$$

$$\frac{\alpha^{e'}(r)}{r} + \frac{2\nu}{\hbar^2} [V_c(r) + V(r) + iW(r)] - k^2$$

$$- (1 + \alpha^e(r)) \left(\frac{u'_\ell(r)}{u_\ell(r)} \right)^2$$

Let us divide $f_\ell(r)$ into its real and imaginary parts, $f_\ell^R(r)$, and $f_\ell^I(r)$:

$$(A-3) \quad f_\ell(r) = f_\ell^R(r) + i f_\ell^I(r)$$

Then, from Eq. A-2 and remembering Eq. 3-1b,

(A-4)

$$\begin{aligned} \frac{df_\ell^I(r)}{dr} &= \alpha_I^e(r) \frac{\ell(\ell+1)}{r^2} + \frac{\alpha_I^{e'}(r)}{r} + \frac{2\nu}{\hbar^2} W(r) \\ &- \frac{1}{2i} \left[(1 + \alpha^e(r)) \left(\frac{u'_\ell(r)}{u_\ell(r)} \right)^2 - (1 + \alpha^{e*}(r)) \left(\frac{u'_\ell(r)}{u_\ell(r)} \right)^* \right] \end{aligned}$$

Now, from Eq. A-1 (and suppressing r),

$$\begin{aligned} (A-5) \quad f_\ell^I \frac{d|u_\ell|^2}{dr} &= \frac{1}{2i} \left[(1 + \alpha^e) \frac{u'_\ell}{u_\ell} - (1 + \alpha^{e*}) \left(\frac{u'_\ell}{u_\ell} \right)^* \right] \left[\frac{u'_\ell}{u_\ell} + \left(\frac{u'_\ell}{u_\ell} \right)^* \right] \\ &= \frac{1}{2i} \left[(1 + \alpha^e) \left(\frac{u'_\ell}{u_\ell} \right)^2 - (1 + \alpha^{e*}) \left(\frac{u'_\ell}{u_\ell} \right)^* \right] + \alpha_I^e \left| \frac{u'_\ell}{u_\ell} \right|^2 \end{aligned}$$

Substituting Eq. A-5 in Eq. A-4, we have that (suppressing r),

$$(A-6) \quad \frac{df_\ell^I}{dr} = \alpha_I^e \frac{\ell(\ell+1)}{r^2} + \left[\frac{\alpha_I^e}{r} + \frac{2\nu}{\hbar} W \right] + \alpha_I^e \left| \frac{u'_\ell}{u_\ell} \right|^2 - \frac{d|u_\ell|^2}{dr} \frac{1}{|u_\ell|^2}$$

As is easily checked, a solution to Eq. A-6 is given by the integral equation

(A-7)

$$\begin{aligned}
f_{\ell}^I(r) = & \left[\int_0^r \frac{2\nu}{\hbar^2} W(r') |u_{\ell}(r')|^2 dr' \right. \\
& + \int_0^r \alpha_I^e(r') \left[\left| u_{\ell}(r') - \frac{u_{\ell}(r')}{r'} \right|^2 + \frac{\ell(\ell+1)}{r'^2} |u_{\ell}(r')|^2 \right] dr' \\
& \left. + \frac{\alpha_I^e(r) |u_{\ell}(r)|^2}{r} \right] \left[|u_{\ell}(r)|^2 \right]^{-1}
\end{aligned}$$

Now outside the interaction region, say at the matching radius, R_0 , $\alpha_I^e(r)$ vanishes. In addition, we have from Eq. A-1 that $f^I(r)$ is then equal to $\frac{\pi_{\ell}}{R_0}$ (Eq. 3-11). Thus π_{ℓ} is given by the relation,

$$\begin{aligned}
(A-8) \quad \pi_{\ell}(R_0) = & - \frac{2\nu R_0}{\hbar^2} \frac{\int_0^{R_0} W(r) |u_{\ell}(r)|^2 dr}{|u_{\ell}(R_0)|^2} \\
& + \frac{\int_0^{R_0} \alpha_I^e(r) \left[\left| \frac{d}{dr} \frac{u_{\ell}(r)}{r} \right|^2 + \frac{\ell(\ell+1)}{r^2} \left| \frac{u_{\ell}(r)}{r} \right|^2 \right] r^2 dr}{|u_{\ell}(R_0)|^2}
\end{aligned}$$

A-2 Uniform Distribution

We will here derive the logarithmic derivative associated with Eq. 3-4 for a uniform distribution (c.f. Eq. 3-18). Let us first define the quantity

$$(A-9) \quad g_{\ell}(r) = (1 + \alpha^e(r)) \frac{\frac{d}{dr} \frac{u_{\ell}(r)}{r}}{u_{\ell}(r)/r}$$

It is then easily shown that Eq. 3-4 can be rewritten in terms of the Ricatti equation,

$$(A-10) \quad g'_\ell(r) = (1 + \alpha^e(r)) \frac{\ell(\ell+1)}{r^2} + \frac{2\mathcal{N}}{\hbar^2} [V_c(r) + V(r) + iW(r)] \\ - k^2 - \frac{2 g_\ell(r)}{r} - \frac{g_\ell(r)^2}{1 + \alpha^e(r)}.$$

Now, from Eq. 3-4, it is seen that inside R_0 ,

$$(A-11) \quad u_\ell(r) = A_\ell r j_\ell(br),$$

where A_ℓ is a normalization constant and where

$$(A-12) \quad b = \sqrt{\frac{2}{\hbar^2} \frac{\mathcal{N}}{1 + \alpha_0^e} (E - V_c - V_0 + iW_0)}$$

From Eq. A-10 it is seen that the coefficient functions of the differential equation contain no singularities so that

$$\lim_{r \rightarrow R_0} g_\ell(r) = g_\ell(R_0).$$

Thus,

$$(A-13) \quad \frac{\frac{d}{dr} \frac{u_\ell(r)}{r}}{u_\ell(r)/r} \bigg|_{r=R_0} = \frac{1}{R_0} \lambda_\ell(R_0) - \frac{1}{R_0} \\ = (1 + \alpha_0^e) \frac{r \frac{d}{dr} j_\ell(br)}{j_\ell(br)} \bigg|_{r=R_0},$$

or, rewriting Eq. A-13 in terms of

$$(A-14) \quad \lambda_\ell(R_0) = (1 + \alpha_0^e) \frac{r \frac{d}{dr} j_\ell(br)}{j_\ell(br)} \bigg|_{r=R_0} + 1.$$

APPENDIX BNOTES ON EVALUATION OF
PION EXCITATION CROSS SECTIONS

In the present Appendix we will evaluate some quantities which are required in the evaluation of the pion excitation cross sections given in Sec. 4-2. In Sec. B-1, we provide details of the evaluation of the matrix elements which enter the pion excitation cross section formulae. In Sec. B-2, we evaluate the electrostatic potential for a uniform deformed charge distribution.

B-1 Evaluation of DWBA Matrix Elements

In the text, Sec. 4-2, we have shown that in DWBA the cross sections for the excitation of a rotational state in a strongly deformed nucleus are given by the expression,

$$(4-19) \quad \frac{d\sigma_{\alpha \rightarrow \gamma}}{d\Omega_{\gamma}} = \frac{k_c}{k_a} \left(\frac{\nu}{2\pi\hbar^2} \right)^2 \left| \langle \Psi_c^{(\gamma)}(\underline{r}) | \mathcal{V}'(\underline{r}) | \Psi_a^{(\alpha)}(\underline{r}) \rangle \right|^2,$$

where, taking the example of a $K = 0$ band (c.f. Eq. 4-5),

$$(4-5a) \quad \Psi_{\alpha} = \left[\frac{2I_{\alpha} + 1}{8\pi^2} \right]^{\frac{1}{2}} \mathcal{D}_{MO}^{I_{\alpha}}(\alpha_k) \chi_0^{\tau}(\underline{x}') \quad (K = 0);$$

here $\alpha_k = (\alpha, \beta, \gamma)$ are the Euler angles (in the conventions of Goldstein (1950)) which rotate the space-fixed axes (x, y, z) into the body-fixed axes $(1, 2, 3)$ (see text); $\phi_a^{(\alpha)}(\underline{r})$ and $\phi_c^{(\gamma)}(\underline{r})$ are solutions of Eqs. 4-11b and 4-17b, respectively; and $\mathcal{V}'(\underline{r})$ is the perturbing potential:

$$(4-7) \quad \mathcal{V}'(\underline{r}) = \sum_{\lambda=1}^{\infty} \mathcal{V}_{\lambda}(\underline{r}_b) \mathcal{Y}_{\lambda}^0(\theta_b).$$

We begin our evaluation by rewriting Eq. 4-7 in the

space-fixed co-ordinate system. To do this let us first note that

$$(B-1) \quad Y_{\lambda}^0(\theta_b, \phi_b) = \sum_{\nu=-\lambda}^{\lambda} \mathcal{Y}_{\nu 0}^{\lambda*}(\alpha_k) Y_{\lambda}^{\nu}(\theta, \phi)$$

where (r, θ, ϕ) are the spherical co-ordinates in the space-fixed system and (r_b, θ_b, ϕ_b) are the spherical co-ordinates in the body-fixed system. Both systems are chosen to have their origins at the center of the nucleus so that $r = r_b$.

Using Eq. B-1, we can write Eq. 4-7 in the form,

$$(B-2) \quad \psi'(\underline{r}) = \sum_{\lambda=1}^{\infty} \psi_{\lambda}(r) \sum_{\nu=-\lambda}^{\lambda} \mathcal{Y}_{\nu 0}^{\lambda*}(\alpha_k) Y_{\lambda}^{\nu}(\theta, \phi)$$

We shall now assume that we have an unpolarized beam in which case we average over initial states and sum over final states. To do this we see from Eq. 4-21 that we first require matrix elements of the form (taking $K = 0$):

$$(B-3a) \quad \begin{aligned} i_{M'M}^{\alpha} &= \int d\underline{r} d\alpha_k d\underline{x}' \left(\phi_c^{(\dagger)*}(\underline{r}) \left[\frac{2I_{\alpha}+1}{8\pi^2} \right]^{\frac{1}{2}} \mathcal{Y}_{M'0}^{\alpha*}(\alpha_k) \chi_0^{\tau}(\underline{x}') \right) \\ &\times \left(\sum_{\lambda=1}^{\infty} \psi_{\lambda}(r) \sum_{\nu=-\lambda}^{\lambda} \mathcal{Y}_{\nu 0}^{\lambda*}(\alpha_k) Y_{\lambda}^{\nu}(\theta, \phi) \right) \\ &\times \left(\phi_a^{(\dagger)}(\underline{r}) \left[\frac{2I_{\alpha}+1}{8\pi^2} \right]^{\frac{1}{2}} \mathcal{Y}_{M0}^{\alpha}(\alpha_k) \chi_0^{\tau}(\underline{x}') \right) \\ &= \sum_{\lambda=1}^{\infty} \sum_{\nu=-\lambda}^{\lambda} c_{\alpha; M'M}^{\lambda\nu} \mathcal{A}_{ac}^{\lambda\nu} \end{aligned}$$

where

$$(B-3b) \quad c_{\alpha; M'M}^{\lambda\nu} = \left[\frac{2I_{\alpha}+1}{8\pi^2} \right]^{\frac{1}{2}} \left[\frac{2I_{\alpha}+1}{8\pi^2} \right]^{\frac{1}{2}}$$

$$\times \int d\alpha_k \mathcal{Y}_{M'0}^{\alpha*}(\alpha_k) \mathcal{Y}_{\nu 0}^{\lambda*}(\alpha_k) \mathcal{Y}_{M0}^{\alpha}(\alpha_k) ;$$

$$(B-3c) \quad \mathcal{D}_{ac}^{\lambda\nu} = \int d\underline{r} \phi_c^{(\nu)*}(\underline{r}) \mathcal{V}_\lambda(\underline{r}) Y_\lambda^\nu(\theta, \phi) \phi_a^{(\mu)}(\underline{r})$$

Remembering the relation (c.f. Preston (1962)),

$$\int \mathcal{D}_{M_1 K_1}^{J_1} \mathcal{D}_{M_2 K_2}^{J_2} \mathcal{D}_{M_3 K_3}^{J_3} d\alpha_K = \frac{8\pi^2}{2J_3 + 1} \langle J_1 J_2 M_1 M_2 | J_3 M_3 \rangle$$

$$\times \langle J_1 J_2 K_1 K_2 | J_3 K_3 \rangle$$

we find that (after some elementary manipulations),

$$(B-4) \quad c_{\sigma\alpha}^{\lambda\nu}; M'M = (-)^{I+M'} \sqrt{\frac{2I_\alpha + 1}{2\lambda + 1}} \langle I' I - M' M | \lambda \nu \rangle$$

$$\times \langle \lambda I 0 0 | I' 0 \rangle$$

Now, averaging over initial states and summing over final states, we have that

$$(B-5) \quad \frac{d\sigma_{\alpha \rightarrow \sigma}}{d\Omega_\sigma} = \frac{k_c}{k_a} \left(\frac{\nu}{2\pi\hbar^2} \right)^2 \frac{1}{2I_\alpha + 1} \sum_{M, M'} |i_{M'M}^{\sigma\alpha}|^2$$

$$= \frac{k_c}{k_a} \left(\frac{\nu}{2\pi\hbar^2} \right)^2 \frac{1}{2I_\alpha + 1} \sum_{\substack{\lambda, \nu \\ \lambda', \nu'}} \mathcal{C}_{ac}^{\lambda'\nu'} \mathcal{C}_{ac}^{\lambda\nu} \sum_{M, M'} c_{\sigma\alpha}^{\lambda'\nu'}; M'M c_{\sigma\alpha}^{\lambda\nu}; M'M$$

$$= \frac{k_c}{k_a} \left(\frac{\nu}{2\pi\hbar^2} \right)^2 \sum_{\lambda=1}^{\infty} A_{\sigma\alpha}^\lambda \sum_{\nu=-\lambda}^{\lambda} |\mathcal{C}_{ac}^{\lambda\nu}|^2$$

where

$$(B-6) \quad A_{\sigma\alpha}^\lambda = \frac{\langle \lambda I 0 0 | I' 0 \rangle}{2\lambda + 1} \quad (K = 0)$$

For bands where $K \neq 0$ we need only replace Eq. 4-5a with Eq. 4-5b of the text in the above derivation. It is then easily shown that the generalization of Eq. B-6 is

given by the relation,

$$(B-7a) \quad A_{\lambda\alpha}^{\lambda} = \frac{\langle \lambda I O K | I^{\lambda} K \rangle}{2\lambda + 1}, \quad (\lambda \text{ even});$$

$$(B-7b) \quad A_{\lambda\alpha}^{\lambda} = 0, \quad (\lambda \text{ odd}).$$

B-2 Evaluation of Electromagnetic Potential for a Deformed Charge Distribution

In the present section, we will evaluate the electromagnetic potential, $V_c(\underline{r})$, for a deformed charge distribution of uniform charge density, $\rho(\underline{r})$. As is known from elementary electromagnetic theory,

$$(B-8) \quad V_c(\underline{r}) = \int \frac{\rho(\underline{r}') d\underline{r}'}{|\underline{r}' - \underline{r}|},$$

so that, because of the integral, we make negligible errors by neglecting the diffuseness of the nuclear edge.

We take the charge distribution to have a spheroidal deformation, as in the text (Eq. 4-23):

$$(B-9a) \quad \rho(\underline{r}) = \rho \left(\frac{r}{1 + \beta Y_2^0(\theta_b)} \right) \\ = \rho(r) - \rho r \rho'(r) \sum_{\lambda=-2}^2 \mathcal{D}_{\lambda 0}^{2*}(\alpha_k) Y_2^{\lambda}(\theta, \phi);$$

for a uniform distribution,

$$(B-9b) \quad \rho(r) = \rho_0 H(R-r),$$

where

$$(B-9c) \quad \rho_0 = Ze^2 \left(\frac{4\pi}{3} R^3 \right)^{-1},$$

and where $H(R-r)$ is the well-known Heaviside function. From

Eqs. B-9a and B-9b,

(B-10)

$$\mathcal{P}(\underline{r}) = \mathcal{P}_0 H(R-r) + \beta r \mathcal{P}_0 \mathcal{J}(R-r) \\ \times \sum_{\nu=-2}^2 \mathcal{D}_{\nu 0}^{2*}(\alpha_k) Y_2^\nu(\theta, \phi),$$

whence,

(B-11)

$$V_c(\underline{r}) = \mathcal{P}_0 \int_0^R r'^2 dr' \int \frac{d\Omega'}{|\underline{r} - \underline{r}'|} \\ + \beta \mathcal{P}_0 R^3 \sum_{\nu=-2}^2 \mathcal{D}_{\nu 0}^{2*}(\alpha_k) \int \frac{d\Omega' Y_2^\nu(\theta', \phi')}{|\underline{r} - \underline{r}'|}.$$

To evaluate the angular integrals in Eq. B-11, it is convenient to note the identity,

(B-12)

$$\frac{1}{|\underline{r} - \underline{R}'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r_{<}^{\ell}}{r_{>}^{\ell+1}} Y_{\ell}^m(\theta, \phi) Y_{\ell}^{m*}(\theta', \phi'),$$

where $r_{<}$ ($r_{>}$) is the smaller (larger) of r and R .

Using this identity, we find that

(B-13a)

$$\int \frac{d\Omega'}{|\underline{r} - \underline{r}'|} = \begin{cases} \frac{4\pi}{r'} & , \quad r < r' ; \\ \frac{4\pi}{r} & , \quad r > r' ; \end{cases}$$

and that

(B-13b)

$$\int \frac{d\Omega' Y_2^\nu(\theta', \phi')}{|\underline{r} - \underline{r}'|} = \begin{cases} \frac{4\pi}{5} \frac{r^2}{r'^2} Y_2^\nu(\theta, \phi), & r < r' ; \\ \frac{4\pi}{5} \frac{r'^2}{r^2} Y_2^\nu(\theta, \phi), & r > r' . \end{cases}$$

Substituting Eqs. B-13 in Eq. B-11, and noting Eq. B-9, we find that

(B-14a)

$$V_c(\underline{r}) = \frac{Ze^2}{R} \left[\frac{3}{2} - \frac{1}{2} \left(\frac{r}{R} \right)^2 \right]$$

$$+ \beta \frac{3}{5} Ze^2 \frac{r^2}{R^3} \sum_{\nu=-2}^2 \mathcal{D}_{\nu 0}^{2*}(\alpha_k) Y_2^\nu(\theta, \phi), \quad r < R;$$

(B-14b)

$$V_c(\underline{r}) = \frac{Ze^2}{r}$$

$$+ \beta \frac{3}{5} Ze^2 \frac{R^2}{r^3} \sum_{\nu=-2}^2 \mathcal{D}_{\nu 0}^{2*}(\alpha_k) Y_2^\nu(\theta, \phi), \quad r > R.$$

APPENDIX C

**The Nuclear Optical Model and Wave Properties:
Barrier Penetration, Reflection, Absorption and Resonance**

by

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ABSTRACT

A detailed study of the wave properties of the nuclear optical model is presented to elucidate the problem of barrier penetration by charged particles and to remove some of the mystique of optical model calculations. The wave properties and the concomitant penetration are most straightforward for square wells for which the resonance, reflection and penetration are easily ascribed to separate factors. We show that the wave properties of more general diffuse-edge optical potentials achieve a similar simplicity by the construction of an equivalent square well (ESW) which has the same resonance, penetration and absorption factors as the optical potential but which differs in its reflection factor. A general construction of the ESW is given and we apply it to the following problems: (1) the very narrow single-particle resonances of real optical potentials which occur at energies far below the Coulomb barrier; (2) the nuclear absorption cross sections in the presence of barriers; (3) the calculation of absorption cross sections at astrophysical energies (extreme barrier penetration) employing optical models fitted to data at higher energies; (4) the value of the nuclear radius and sum rule limits appropriate to the analysis of nuclear reactions. In some cases of extreme barrier penetration the equivalent square well fails to yield all the properties. For example, cases are described where the bulk of the absorption may attain in the distant "tail" of the imaginary term in the optical potential: the corresponding reaction rates can yield information about the behaviour of the nucleus at distances much beyond the normal nuclear radius.

1. Introduction

The behaviour of most nuclear reactions at low energy - particularly those of interest for astrophysical systems - is dominated by Coulomb and angular momentum barriers. Early treatments¹⁻³ of such reactions employed a simple picture, the "black nucleus" or "black box" picture. In this picture a bombarding particle was viewed as passing through known Coulomb and angular momentum barriers up to the nuclear radius. At the nuclear radius it was absorbed by the "black box". This early picture was based on the short range and great strength of the nuclear forces.

During the last two decades a microscopic model of nuclear structure has emerged which has changed our view of nuclear reactions and of barrier penetration. In spite of the short range and great strength of nuclear forces it has been found that in zero-order nuclei may be regarded as composed of neutrons and protons moving in orbits or shells and interacting with moderate potentials of finite range. The single-particle orbits or shells are those of an appropriate average potential well. The vestiges of such a single-particle picture remain in nuclear reactions. The neutrons and protons at low energy exhibit "giant" resonances^{4,5} at the position of the single-particle levels of the average potential well. The resonances are broadened by nucleon-nucleon interaction and, at high energy, the observed resonances resemble⁶ those of the Ramsauer-Townsend effect in the electron bombardment of atoms.

The optical model of nuclear reactions accounts for the observed single-particle effects.⁷ In the optical model, the interaction of a bombarding particle and a target nucleus is depicted in terms of a complex potential well. The real part of the potential refracts the incoming waves and the imaginary part absorbs them. The magnitude of cross sections is determined by the magnitude and shape of the optical potential. When Coulomb and angular momentum barriers are present the whole real potential of the system is the sum of the real part of the optical potential with the Coulomb potential and centripetal potential. Our aim in the present paper is to understand the formation of the compound nucleus in the presence of barriers and optical potentials: the absorption of waves by the imaginary part of the optical potential yields the compound nucleus formation cross section; the barriers dominate the behaviour of the wave amplitudes in the absorbing region.

The formation of the compound nucleus has the following aspects which are treated in subsequent sections of this paper.

1) On the one hand the optical model is a wave model and deals with the simple properties of wave penetration, refraction, reflection and absorption. On the other hand, the optical potentials in common usage⁷ have many phenomenological parameters. In order to understand how variations in the parameters affect cross sections it is useful to describe the relationship between the parameters and the basic wave properties.

2) Many of the reactions of interest in astrophysics occur at such low energy that they are rarely measured in the laboratory. In such cases it is tempting to use an optical model which fits the more abundant data at higher energies to calculate the desired rates. Thus the optical model serves as an extrapolation formula. Our analysis of the wave properties for the nuclear optical model allows us to understand the factors affecting the extrapolation.

3) The early black nucleus models are still frequently employed, particularly in astrophysical calculations where a very large number of reaction rates are required in a single calculation. The optical model is then often too cumbersome. It is therefore useful to know the relation between the cross section of the black nucleus model and the corresponding cross sections of the optical model. We develop such relationships (sec. 2 and sec. 3) as part of a general connection between diffuse edge potentials and square wells.

4) The theory of resonance reactions^{8,9} is a very general and powerful framework for describing nuclear reactions: it has been successfully applied to analyzing compound nucleus resonances and to the cross sections of the statistical theory of nuclear reactions¹⁰ in which averages are made over the compound nucleus resonances. It provides a justification¹¹ for the description of the average cross sections by an optical model potential. Nonetheless, the resonance theory has some of the undesirable features of the early black box picture: it relies explicitly on the use of a definite nuclear radius. Many of the common results of the resonance

theory imply a square edge to the nuclear surface. In an earlier article¹² one of us tried to show how the diffuse edge of the optical model could be accommodated in the general resonance theory for neutron reactions. That accommodation is extended in the present article (sec. 4) to cover reactions involving Coulomb and angular momentum barriers.

5) In seeking to describe all nuclear reactions with the physical ideas of the optical potential we need to re-examine the value of the nuclear radius. In the several decades during which the simple black nucleus picture was employed to describe nuclear reactions the usual choice of radius³ was $R = 1.4 (A_1^{1/3} + A_2^{1/3})$ fermis. Here A_1 is the atomic weight of the target nucleus and A_2 that of the bombarding particle. The nuclear radius of the optical model is determined quite sensitively by the fits to the diffraction patterns of elastic scattering data and it turns out to be much smaller, e.g. for nucleons its value is approximately $R = 1.25 A_1^{1/3}$ fermis. (In turn, the nuclear charge radius as measured in elastic scattering is smaller still; the charge radius is $R = 1.09 A_1^{1/3}$ fermis. The small difference of this radius from the normal radius of the nucleon-nucleus interaction lies in a number of effects; such as core polarization, which are beyond the scope of the present work). There is an overwhelming amount of recent evidence suggesting that the smaller optical model radius is the right one. Why was the earlier model wrong for several decades, particularly for reaction rates involving barriers where the cross section depends quite strongly on the choice of nuclear

radius? Our elucidation of the optical potential suggests a fairly universal explanation. The early black-nucleus model implied a square nuclear edge and therefore had an unrealistic amount of wave reflection. It compensated for this attenuation of absorption by an appropriate increase in the nuclear radius.

Some preliminary results of our present investigation were described briefly in an earlier paper.¹³ In our present paper we try to give a complete account of all aspects of barrier penetration and the other wave properties within the scope of the optical model.

We begin our analysis (Sec. 2) with a comparison of the wave properties for diffuse-edge optical potentials and for similar square wells. The motive for bringing square wells into the analysis is two-fold: first of all, the various cross sections for a square well can be written in the simple, familiar forms which make possible easy evaluation of the wave properties; secondly, for a square well the basic wave properties are easily separated - barrier penetration occurs only beyond the square well radius, reflection occurs at the square well radius and absorption and resonance within the square well radius. The vehicle for the comparison (Sec. 2) is the absorption cross section. The comparison yields the dominant result of our work: it is found that for each diffuse-edge optical potential an equivalent square well can be defined uniquely. The cross sections of the diffuse-edge potential have the same simple form as those of its equivalent square well with all the basic wave properties clearly separated. The only quantitative difference between the wave properties or cross sections of the two wells is shown to reside in the penetration and shift functions. The conventional square

well penetration and shift functions apply to the diffuse-edge well when they are multiplied by a reflection factor which depends on the surface thickness and reduced mass but not on the charge, energy or angular momentum of the bombarding particle. Thus the breakdown of the optical model into basic wave properties is achieved by means of equivalent square wells.

In Sec. 3 we carry out, for diffuse-edge optical potentials, the explicit construction of equivalent square wells and determine the corresponding reflection factors. In Sec. 4 we show that our analysis of wave properties based on the behaviour of absorption cross sections (Sec. 2) also applies to purely real wells: the sharp resonances of a diffuse-edge real potential are given approximately in terms of the sharp resonances of the equivalent square well when the penetration factor and shift function of the latter are multiplied by a known reflection factor. In Sec. 5 we describe the behaviour of general optical model absorption cross sections and the circumstances under which there are some departures of the wave properties from those which may be dealt with in terms of equivalent square wells. In Sec. 6 we use our analysis of the basic wave properties of the optical potential to discuss the uncertainties in the conventional approach of astrophysics to the problem of extreme barrier penetration. In Sec. 7 we discuss a number of conclusions resulting from our work including the place of the nuclear radius in the analysis of nuclear reaction data.

The optical model has a firm foundation only for nucleon reactions but it has also been found to be a very useful tool for describing heavy ion reactions. It is more ambiguous for heavy ions - the resonance effects which are the strongest signature of the optical

model are largely missing in this case - but it can accommodate the strong absorption and the diffuse nuclear edge which are of importance for heavy ions. Therefore the optical model is perhaps the best simple model for dealing with heavy ions. Because of the mass and charge of the heavy ions many of the basic wave properties are more complex than for nucleons. Such reactions serve as a useful measure of the success of our approach. Although we shall seek to be general most of our results will be illustrated with a particular case: the interaction of alpha particles with ^{32}S . This example offers extreme barrier penetration and is typical of the astrophysical reaction rates to which our results might be applied.

Recently a modified version of our methods was applied to an extreme case of barrier penetration - that of alpha decay in heavy nuclei.¹⁴ Other applications are suggested in the following sections.

2. Comparison of the Absorption Cross Sections of Diffuse-Edge Optical Potentials with those of Square Wells

In order to display the basic wave properties of diffuse-edge optical potentials we shall compare them with square wells for which the wave properties are much more perspicuous. We begin by choosing and parameterizing those terms of the modern nuclear optical potential which are of interest to us.

For most scattering and absorption cross sections the principal terms of the optical potential may be written

$$V(r) = - V_0 \left(1 + e^{(r-R_0)/a} \right)^{-1} + i W(r) \quad (1)$$

Here V_0 is the "depth" of the real part of the potential and has a value in the neighbourhood of 50 MeV for nucleons and considerably more for heavy ions; R_0 is the nuclear radius whose value is about $1.25 A^{1/3}$ fm for nucleons (where A is the atomic weight of the target nucleus) and a slightly larger value for heavy ions; a is the "surface thickness" which has a value of about 0.5 fm for nucleons and heavy ions; $i W(r)$ is the imaginary term of the optical potential which leads to absorption. The imaginary term is usually chosen to have a shape either like that of the real term,

$$W(r) = - W_0 (1 + e^{(r-R_0)/a})^{-1} \quad (2)$$

which is called "volume absorption" or, alternatively, $W(r)$ is chosen to be larger in the region of the nuclear surface (this choice is called "surface absorption"). In the main part of our discussion we will choose volume absorption in order to be specific: in Sec. 5 we will discuss in some detail the effect of the shape of $W(r)$ on the wave properties. The depth parameter, W_0 , typically has a value of 2-5 MeV for nucleons and somewhat larger values for heavy ions.

We have chosen to concentrate on the principal terms, (1), of the optical potential because the other terms, which we neglect, do not modify our main results about wave properties. For example, the spin-orbit coupling term which is proportional to $\underline{l} \cdot \underline{s}$ (where $\underline{l}$ is the orbital angular momentum and $\underline{s}$ the spin) makes the optical model phase shifts depend on both l and j ($j = l + s$). Our treatment below must then also be carried out separately for each value of l and j . Similarly the isotopic spin term when important makes it necessary to treat neutrons and protons

on a common footing and leads to unusual cross section contributions such as the "quasielastic" (p, n) reactions. We could also accommodate such terms in our analysis. However to clarify the wave properties we will ignore such refinements and deal only with the dominant terms of (1).

The optical potential of (1) (or Saxon-Woods potential which the particular well shape of (1) is frequently called) will be shown to exhibit all of the basic wave properties: barrier penetration, resonance, reflection and absorption. The most straightforward manifestation of these properties occurs in the absorption cross section and therefore we choose to begin our analysis with it.

For any reaction channel α (α labels the pair of reaction products and their state of excitation) described by an optical potential with phase shifts $\delta_{\alpha\lambda}$ the absorption cross section is:

$$\sigma_{\alpha}(\text{abs}) = (\pi/k_{\alpha}^2) \sum_{\lambda} (2\lambda + 1) T_{\lambda}(\alpha) \quad (3)$$

where

$$T_{\lambda}(\alpha) \equiv 1 - \left| e^{2i\delta_{\alpha\lambda}} \right|^2 \quad (4)$$

The $T_{\lambda}(\alpha)$ are called nuclear transmission functions. All of the properties of the absorption cross section are those of the transmission functions or, equivalently, those of the phase shifts. Therefore the numerical calculation of the absorption cross section is straightforward - one needs merely to integrate the Schroedinger equation containing the optical potential to a large distance where the decomposition of the wave function into incoming and outgoing waves yields the phase shift. But the numerical solution does not

display the wave properties. To see these we first examine the transmission functions of a complex square well potential.

The transmission function of a complex square well of depth $-(V_0 + i W_0)$, and radius R_α , may be written*

* For a derivation of this result and a detailed discussion of nuclear transmission functions see pages 281-295 of reference 10

$$T_\ell(\alpha) = \frac{4 P_\ell f_{\alpha\ell}^{\text{Im}}}{(1 - S_\ell f_{\alpha\ell}^{\text{Re}} + P_\ell f_{\alpha\ell}^{\text{Im}})^2 + (P_\ell f_{\alpha\ell}^{\text{Re}} + S_\ell f_{\alpha\ell}^{\text{Im}})^2} \quad (5)$$

Here P_ℓ is the usual penetration factor used in nuclear reaction studies,

$$P_\ell \equiv k_\alpha R_\alpha / \left[F_\ell^2(k_\alpha R_\alpha) + G_\ell^2(k_\alpha R_\alpha) \right] \quad (6)$$

where F_ℓ and G_ℓ are, respectively, the regular and irregular Coulomb wave functions of the channel α . The S_ℓ are the usual shift functions of nuclear reaction theory

$$S_\ell \equiv -b_{\alpha\ell} + \left\{ \frac{r F_\ell dF_\ell/dr + r G_\ell dG_\ell/dr}{F_\ell^2 + G_\ell^2} \right\} \bigg|_{r=R_\alpha} \quad (7)$$

in which the $b_{\alpha\ell}$ are boundary condition numbers determined by the resonance properties of the well (see below). Finally, we have

$f_{\alpha\ell}$ ($\equiv f_{\alpha\ell}^{\text{Re}} + i f_{\alpha\ell}^{\text{Im}}$) related to the logarithmic derivative of the wave function inside the square well, $j_\ell(K_\alpha r)$, in the following way

$$f_{\alpha\ell} \equiv \left[\frac{dj_\ell/dr}{j_\ell} \right]_{r=R_\alpha} - b_{\alpha\ell} \quad (8)$$

The $f_{\alpha\lambda}$ are complex numbers because the K_{α} are complex:

$$K_{\alpha} \equiv \left[\frac{2m_{\alpha}}{\hbar^2} (E + V_0 + i W_0) \right]^{\frac{1}{2}} \quad (9)$$

The quantities $f_{\alpha\lambda}$ refer to the properties of the square well while the quantities P_{λ} and S_{λ} refer to the external properties. The transmission functions (5) clearly exhibit absorption and barrier penetration in separate factors. The manifestation of resonance and reflection requires further analysis.

To exhibit the "resonances" of the transmission functions of the square well we take the real part of the potential and find the normal modes or resonances. The normal modes are solutions of the Schroedinger equation, which vanish at the origin and satisfy a suitable boundary condition at the square well radius. If we write the solutions as $A_{p\lambda} j_{\lambda}(K_{p\lambda} r)$ where $A_{p\lambda}$ is a normalizing constant and $K_{p\lambda} (\equiv \left[\frac{2m_{\alpha}}{\hbar^2} (E_{p\lambda} + V_0) \right]^{\frac{1}{2}})$ is the discrete value of the wave number for which the following boundary condition is satisfied:

$$r \frac{dj_{\lambda}}{dr} = b_{\alpha\lambda} j_{\lambda} \quad \Big|_{r=R_{\alpha}} \quad (10)$$

The correct value of the boundary condition number $b_{\alpha\lambda}$ is that which makes the shift function (7) vanish at the energy of interest.^{1,2} The normal modes have single particle energies

$$E_{p\lambda} = \frac{\hbar^2}{2m_{\alpha}} K_{p\lambda}^2$$

The $f_{\alpha\lambda}$ were given quite generally in terms of spherical Bessel functions of complex arguments: alternatively we can give them in terms of the normal modes or resonances of the real part of the potential. By using the completeness of the resonances we find:

$$f_{\alpha\ell}^{\text{Im}} \equiv \pi s_{\ell} = \sum_p \frac{\gamma_{\ell p}^2 W_0}{(E_{\ell p} - E_0)^2 + W_0^2} \quad (11)$$

$$f_{\alpha\ell}^{\text{Re}} = \sum_p \frac{(E_{\ell p} - E) \gamma_{\ell p}^2}{(E_{\ell p} - E)^2 + W_0^2} = P \int_{-\infty}^{+\infty} \frac{s_{\ell}(E') dE'}{E' - E} \quad (12)$$

where P stands for the principal value of the improper integral and where $\gamma_{\ell p}^2$ is the single particle reduced width. If we choose the boundary condition number to be $b_{\alpha\ell} = -\ell$ - which would be correct for neutral particles in the low energy limit - then $\gamma_{\ell p}^2 = \hbar^2/m_{\alpha} R_{\alpha}^2$. The quantity s_{ℓ} defined by (11) is usually referred to as the strength function.

The nuclear strength function of (11) clearly exhibits a resonance structure. Using the corresponding value of $f_{\alpha\ell}^{\text{Im}}$ in the transmission function (5) - or in the absorption cross section (3) - shows that these resonances also are expected to appear in the data. There is much evidence for them in low energy nucleon absorption. For heavy ions the values of W_0 are larger and the values of E_p closer together so that the data does not usually exhibit resonances.

The reflectivity of the square well can be traced to the penetration factor. For $\ell = 0$ absorption of neutral particles we have no Coulomb or angular momentum barriers and therefore the penetration factor, (6), becomes $P_0 = k_{\alpha} R_{\alpha}$. Near threshold the strength function remains finite but P_0 vanishes as $E^{\frac{1}{2}}$. This vanishing of P_0 can be traced to total reflection of the wave approaching the square well radius from either side.

Having displayed absorption, barrier penetration, resonance and reflection for square wells we now seek to compare the corresponding quantities for diffuse-edge wells with the results above. It might appear reasonable to begin with a decomposition of the transmission function as in (5).

There is no feature of the decomposition of (5) which does not apply to an arbitrary well shape. We can always choose a matching radius, evaluate P_λ and S_λ for that radius (taking account of the modification of P_λ and S_λ by the tail of the complex potential), and evaluate the logarithmic derivative, $f_{\alpha\lambda}$, for the complex potential at that matching radius. Indeed, we can even decompose the logarithmic derivative of any well to exhibit the resonances as above. But, how do we choose the matching radius? How do the penetration factors, strength functions, resonance energies, etc. depend on well properties? In our answers to these questions will reside our analysis of wave properties.

Before adapting our exposition to diffuse-edge potentials we illustrate the decomposition of the transmission function as in

(5) by the "black nucleus" model - which is not a potential model at all. In this case it is assumed that there are only incoming waves for $r \leq R_\alpha$. Replacing j_λ by e^{-iKr} in (8) and choosing $b_{\alpha\lambda} = 0$ we then have

$$f_{\alpha\lambda} = \frac{i}{KR_\alpha} \quad (13)$$

Using this value of $f_{\alpha\lambda}$ in (5) we obtain the black nucleus transmission functions

$$T_{\lambda}(\alpha) = \frac{4 P_{\lambda} / KR_{\alpha}}{(1 + P_{\lambda} / KR_{\alpha})^2 + (S_{\lambda} / KR_{\alpha})^2} \quad (14)$$

$$= \frac{4k/K}{(1 + k/K)^2} \quad \text{for s-wave neutrons.}$$

At low energies ($k \ll K$) all the transmission functions of the black nucleus model (even those for s-wave neutrons) are very small because of wave reflection or wave penetration. At higher energies ($P \rightarrow kR_{\alpha}$, $k \rightarrow K$) the black nucleus transmission functions approach their maximum value of unity.

Like the square well model, the black nucleus has more reflection than a real nucleus. To understand the wave analysis of the black nucleus we can compare it to a wave guide problem¹⁵ in classical electromagnetic theory. The bombarding wave approaches the nucleus which is treated like a resonating cavity. The cavity is tuned so that waves propagate inward at the cavity entrance, R_{α} , with the given wave propagation number K . The tuning to accomplish this is not easily duplicated by a potential well model. Just as it is difficult to build resonating cavities which are tuned in a certain way for all wavelengths, it is impossible to make optical-model potentials for which the black nucleus conditions apply at all energies. The black nucleus transmission functions completely lack the resonances of the square-well: the square well transmission functions oscillate about the black nucleus transmission function

if we choose K to have the same value for both. The two transmission functions have the same mean value. We show below that the transmission functions of a realistic nuclear optical potential have a larger mean value than both of the above models because the diffuse nuclear edge gives less reflection than a square edge. The black nucleus suffers the same unrealistic reflection because of the sudden change in the wave number at R_λ . Thus the "black nucleus" model fails to a considerable degree in its main objective of optimizing nuclear absorption.

Much of the above wave analysis can be adapted at once to a realistic nuclear optical potential. For example, the decomposition of the transmission function as in (5) still applies except that P_λ , S_λ and $f_{\alpha\lambda}$ are modified, and the choice of a matching radius at which these three quantities are to be evaluated is no longer obvious. The decomposition is valid for an arbitrary choice of matching radius but for our purposes a well-defined choice will turn out to be useful. For an arbitrary radius we can find P_λ and S_λ from the logarithmic derivative of an outgoing wave 0_λ ,

$$\begin{aligned} \frac{rd0_\lambda/dr}{0_\lambda} &= \frac{rdF_\lambda/dr + i r dG_\lambda/dr}{F_\lambda + i G_\lambda} \quad (\text{for large } r) \quad (15) \\ &= S_\lambda + b_{\alpha\lambda} + i P_\lambda \quad (\text{at the matching radius}) \end{aligned}$$

simply by making the analytic continuation of the logarithmic derivative to the matching radius as indicated. Similarly, $f_{\alpha\lambda}$ is still given by (8) if, in (8), we replace j_λ by the

appropriate regular wave function of the optical potential.

There are two important factors which narrow down the choice of matching radius in such a wave analysis. First of all, the nuclear surface of realistic potentials is still quite sharp: typically a/R_0 in (1) has a value of 0.1. Secondly, the resonance conditions themselves do not allow much freedom of choice. Considering the nucleus as a wave guide we think of setting up normal modes or resonances. If we include in our definition of the wave guide part of the transmission channels which lead into it then the resulting large - artificially large - dimension of the guide give it unusual properties. Even when our wavelength is close to that of one of the artificial normal modes the one-mode approximation is poor. In the nucleus the one-level Breit-Wigner formula breaks down. We must choose the matching radius to lie reasonably close to R_0 .

In an earlier work¹² one of us showed that for s-wave neutrons there existed a very simple correspondence between the optical potential (1) and a square well of very nearly the same radius. Fig. 1 shows the important features of the correspondence. With the radii the same, the depth of the square well is adjusted so that the resonances near zero energy coincide for the two wells. With this choice it turns out that all of the resonance energies and all of the reduced widths of the two wells are the same. The two wells have exactly the same resonance properties. But the penetration functions and shift functions of the two wells are different. For the square well the bracketed term of (7) is equal to ℓ at low energy and therefore it is natural¹² to choose $b_{\alpha\ell} = \ell$ so that $S_{\ell} = 0$. For the diffuse-edge well, one obtains

$b_{\alpha\lambda}$ by calculating the analytic continuation of the bracketed term and by choosing $b_{\alpha\lambda}$ so that S_{λ} vanishes at low energy. The correspondence of Fig. 1 is obtained with this choice. The penetration factors of the two wells differ by a constant factor, f , which is nearly energy independent. As shown first by Peaslee¹⁶, f can be well approximated by

$$f = \pi K a \coth (\pi K a) \quad (16)$$

which has a value of about 2.5 for the optical potentials in common use for nucleons. The factor f accounts for the difference in reflection of the diffuse-edge and square-edge.

For s-wave neutrons the correspondence between the optical potential and its equivalent square well extends to all cases, from real wells with narrow resonances to complex potentials with broad peaks in the transmission functions. In all cases the square well results apply to diffuse wells also if we merely multiply the square well penetration function and shift function by the reflection factor f . Thus the whole wave analysis is established for the diffuse-edge wells. The correspondence of Fig. 1 was established for purely real wells. It extends to complex potentials too because we can include an imaginary term of square shape in the real logarithmic derivative simply by replacing E by $E - i W_0$. Thus (11) and (12) apply to the diffuse-edge well also. The shape of the imaginary term is of little consequence because there is little reflection from the imaginary term (see, however, Sec. 5 below). When the transmission functions are much smaller than unity they can be well approximated by

$$T \approx 4\pi P_{\lambda} s_{\lambda} f \quad (17)$$

Here they clearly show the common resonance and penetration factors of diffuse-edge and square wells and the reflection factor which distinguishes them.

The earlier analysis applied to s-wave neutrons and it is not at all clear that in the presence of Coulomb barriers it is possible to decompose the transmission functions into wave properties in quite the same simple way. It turns out to be so for protons as we show on Fig. 2. Here transmission functions are shown well below the Coulomb barrier for d-wave protons. The radii of all the wells are the same but the well depths are adjusted* to make resonance energies coincide roughly. Although

* The adjustment procedure is the following: If $V(r)$ is the real part of the optical potential the $\int_0^{\infty} V(r) r dr$ is taken to be constant for all potentials.

there is strong barrier penetration here the relation between the various transmission functions is still that of (17).

It is not difficult to understand why the same prescription works for protons. Fig. 3 shows the sum of the nuclear and Coulomb potential for the optical model and its equivalent square well. The point is that the Coulomb potential varies little in the surface region (encircled). Within the surface region we might approximate the Coulomb potential as being constant. Then our reflection problem is exactly like that for s-wave neutrons of negative energy $E - B$ where B is the barrier height. The earlier analysis would apply to this case.

It was surprising to us that the simple relationship between diffuse-edge wells and square wells appeared to break down completely for heavy ions. Fig. 4 shows a similar ratio of transmission functions for $^{32}\text{S} + ^4\text{He}$. The expected reflection factor in this case is $f = 4.6$ according to (17). But the ratio of transmission functions is much larger. Moreover it varies strongly with the energy and depends strongly on the orbital angular momentum.

There is a simple way in which the wave analysis for heavy ions is restored to the simplicity of that for nucleons. We explained that the matching radius for the optical potential had to be chosen to be reasonably close to R_0 , the midpoint of the surface. But it does not need to be exactly R_0 . In the next section we show that if we choose $R_\alpha = R_0 + \Delta R$ with ΔR clearly defined and evaluated - we again get results as simple as those of (17). The role of ΔR is important for heavy ions and not for nucleons because the more massive heavy ions have wave functions which oscillate more rapidly - the "wave guide" commences earlier.

3. The Construction of Equivalent Square Wells. (ESW)

For arbitrary optical potentials of the kind (1) we wish to construct equivalent square wells (ESW) in order that we may display the basic wave properties of the potential. According to the approximate view of nuclear reactions depicted on Fig. 3, Coulomb barriers have no great effect on the reflection of a diffuse-edge optical potential. Neither do angular momentum

barriers as we shall show below. The problem of displaying the wave properties is then reduced to the case where we have s-wave neutral particles. According to earlier work¹² the ESW for the case of s-wave neutral particles is independent of the energy of the particles or of the absorptive (imaginary) term in the optical potential. We can therefore construct the ESW using the scattering of zero-energy s-wave neutral particles in only the real part of the optical potential. The depth, radius and reflection factor of the ESW constructed in this way are expected to apply to all the other, more complicated cases as well. In later sections we show to what extent the ESW is universal and to what extent it enables us to display the basic wave properties for all cases.

Having reduced the construction of the ESW to the case of zero-energy s-wave neutral particles, of mass m , in the real part of the optical potential we proceed, for all masses, in the way that was shown for neutrons on Fig. 1. Since the ESW is purely real it has only two parameters: its depth V_1 and its radius R_1 . On the other hand the real part of the optical potential (1) has three parameters: V_0 , R_0 and a . We need two conditions to fix V_1 and R_1 .

Both wells can have resonances at zero-energy: for example, if we keep the depth of either well fixed and vary the radius (as we would do, in proceeding through the periodic table) resonances occur at zero-energy whenever the radial wave function has zero derivative beyond the well. Since the resonances greatly affect the scattering cross section we must choose the position of the resonances in the two wells to be the same if we want the wells to have equivalent properties. Making the positions of the resonances

coincide fixes one of the two parameters of the ESW.

A second condition is obtained from the widths of the scattering resonances. A conventional diffuse-edge potential has much larger widths than a square well, a fact which we associate with reflection. For each well we can write the width Γ_{nl} , with $\lambda = 0$ in our case, where n is the number of nodes of the s -wave resonant wave function) as a product of a penetration factor and a reduced width

$$\Gamma_{no} = P_o \gamma_{no}^2 \quad (18)$$

where the two factors are each evaluated at a suitable matching radius, R . For the square well it is natural and necessary to choose $R = R_1$ and for this choice we have $P_o = kR_1$ and $\gamma_{no}^2 = \hbar^2/mR_1^2$. For the diffuse-edge potential we can easily exhibit a resonant wave function - as on Fig. 5 - by varying R_o until the wave function has zero derivative far beyond the well, but there is arbitrariness in the choice of the matching radius R . The width itself is independent of the choice of R but the penetration factor and reduced width are not. At this stage we make a choice*. We wish to associate

* The choice is not entirely an arbitrary one - no more than the choice of the matching radius in the R -matrix theory, for which the radius must be that of the actual nucleus if the resonance formulae are to have reasonable convergence properties.

the difference in widths - the reflection - with the difference in penetration of the two wells and therefore we choose a matching radius, R , such that the optical potential and its ESW have the same reduced widths, at the common matching radius. Fixing the position and reduced

width of the ESW determines V_1 and R_1 .

The application of the above two conditions to the construction of the ESW is illustrated on Fig. 5. For a given optical potential we fix V_0 and a and then vary R_0 to find all of the resonances. We then have a discrete set of values of R_0 and the corresponding resonant wave functions as shown on the figure. Next, we begin with a square well with an initial choice of R_1 ($\approx R_0$), and fix V_1 so that we also get a zero-energy resonance in the square well having the same number of nodes as the diffuse edge resonance. For this initial choice we compare reduced widths of both wells at the same matching radius, $R(= R_1)$. If $\psi_{no}(r)$ is the radial wave function of the resonance the reduced width is given by^{1,2}

$$\gamma_{no}^2 = (\hbar^2/2mR) [\psi_{no}(R)]^2 \left[\int_0^R \psi_{no}^2(r) dr \right]^{-1} \quad (19)$$

It is the square of the amplitude of the normalized wave function at the matching radius. If the two wells do not have the same reduced width for the initial choice of R_1 , we vary R_1 (always adjusting V_1 so that the resonance positions of the two wells coincide exactly) until they do. In this way we achieve a unique value of the depth and radius of the optical potential. This construction holds only for those values of R_0 which correspond to resonance, but we can interpolate the construction in between resonances.

The reflection factor is a direct result of the above construction of the ESW. It can be defined by

$$\Gamma_{no} \text{ (diffuse well)} = f \Gamma_{no} \text{ (square well)} \quad (20)$$

Since the reduced widths of the two wells are the same this yields

$$P_0 \text{ (diffuse well)} = f P_0 \text{ (square well)} = f K R_1 \quad (21)$$

The comparison of (20) and (21) employed 'natural' widths - not the observed widths of narrow resonances. We give a complete discussion in Sec. 4 of the relative values of observed and natural widths for various wells.

∴ We can evaluate the diffuse-well penetration factor directly from (6) if we replace F_λ and G_λ by $\bar{F}_\lambda$ and $\bar{G}_\lambda$ where the latter are the radial wave functions of the optical potential which behave at large distance like usual radial wave functions - F_λ and G_λ , respectively - in the absence of a potential. In our case $F_0 = \sin k r$ and $G_0 = \cos k r$ so that at zero energy $F_0 = 0$ and $G_0 = 1$. Therefore we find

$$\begin{aligned} f &= \left[\bar{G}_0(R) / \bar{G}_0(R) \right]^2 = (\bar{G}_0(\infty) / \bar{G}_0(R))^2 \\ &= (\Psi_{\text{diff}}(\infty) / \Psi_{\text{diff}}(R))^2 = (\Psi_{\text{diff}}(\infty) / \Psi_{\text{square}}(\infty))^2 \end{aligned} \quad (22)$$

since $\bar{G}_0$ is proportional to the diffuse-well resonant wave function of Fig. 5 and the resonant wave functions of the two wells are normalized to the same amplitude at the matching radius on Fig. 5. Then f is just the square of the amplitude ratio of the resonant wave functions at large distance, as indicated on the figure.

If we remember the definition (6) of the penetration factor and employ the conventional definition of an incoming wave, I_λ (see (41) below), we can give a similar physical meaning to

the reflection factor which is valid at finite energy. The conventional penetration factor may be written

$$P_{\ell} = k R (F_{\ell}^2 + G_{\ell}^2)^{-1} = k R (I_{\ell} I_{\ell}^*)^{-1}$$

For a diffuse-edge well we replace I_{ℓ} by its analytical continuation $\bar{I}_{\ell}$, yielding

$$\bar{P}_{\ell} = k R (\bar{I}_{\ell} \bar{I}_{\ell}^*)^{-1} = f k R (I_{\ell} I_{\ell}^*)^{-1}$$

where
$$f \equiv I_{\ell} I_{\ell}^* / \bar{I}_{\ell} \bar{I}_{\ell}^*$$

The intensity of the incoming wave at large distance needed to yield a given intensity at the matching radius is f times bigger for a square well than for a diffuse well. The reflection factor that we determine is related to wave reflection.

In the above construction of an ESW for a particle of mass m and an arbitrary optical potential only two of the four parameters (V_0, R_0, a, m) are independent. This can be seen from the radial equation with the optical potential V , of (1).

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dr^2} - \left[\frac{V_0}{1 + e^{(r-R_0)/a}} \right] \psi = E \quad (23)$$

At zero energy the right-hand side vanishes and we can write (23) in terms of two dimensionless parameters

$$\frac{d^2 \psi}{dx^2} = \frac{-(K_0 R_0)^2}{1 + e^{(x-1)(R_0/a)}} \psi \quad (24)$$

where $x = r/R_0$ (25)

$$K_0^2 = 2m V_0 / \hbar^2 \quad (26)$$

Thus R_0 determines only the scale of the abscissa on Fig. 5.

The two important parameters are $K_0 R_0$ and R_0/a . If we find f and the ESW for all choices of these two parameters then we have covered all cases.

We carried out the construction of the ESW and the associated reflection factor for a large range of values of $K_0 R_0$ and a/R_0 . The depth of the ESW is not important in many applications - it must always be adjusted to produce the resonant positions at the right energies (see Sec. 4) and is always found to be given very nearly by $V_1 R_1^2 \approx V_0 R_0^2$. The ESW radius is usually important and differs appreciably from R_0 . We present the results in terms of the radius difference $\Delta R/R_0$

$$\frac{\Delta R}{R_0} = \frac{R_1}{R_0} - 1 \quad (27)$$

which is also dimensionless. Fig. 6 shows the values of f and $\Delta R/R_0$ which we obtained from the zero energy resonances and interpolations between resonances. Fig. 6 is an enlargement of a small region of a corresponding map given by us in a preliminary report¹³: the enlargement given here contains the region of the map used in most applications.

Although our wave analysis of potential wells has taken the reflection factor to be energy independent, the analysis of wave properties of potentials in terms of the equivalent square well can be generalized and improved by allowing f to be energy dependent.

In Fig. (7) we see the extent to which the reflection factor is independent of energy. The reflection factor was calculated by supposing an incoming wave at infinity, and calculating the ratio

$$f = \left(\frac{I_{DW}}{I_{ESW}} \right)^2 ,$$

where I_{DW} and I_{ESW} are evaluated at $R = 0.1 R_{ESW}$.

To eliminate any possible effect of absorption in the barrier, we made our calculations by supposing that, for both the diffuse and square wells, absorption occurred only at $R = 0$ and that the incoming wave was completely absorbed there. The reflection factor is then seen to be moderately dependent of energy. The energy dependence is also a function of the reduced mass; it is larger for the heavier particles. We have also checked that the energy dependence of the reflection factor was not a function of charge. It is the same within 10% or so, whatever the Coulomb barrier, for a given reduced mass.

Before applying the ESW in some detail to obtain the basic wave properties of various problems we give a simple example and discuss some special cases. For our standard example of $^{32}\text{S} + \alpha$ we can resolve the difficulties of Fig. 4. For a known optical potential with a diffuse-edge we now construct the ESW as above, with ΔR no longer zero but fixed by Fig. 6. The results are shown

on Fig. 8. Now the ratio of transmission functions, at low energy, is roughly equal to the reflection factor as expected from (17). There are corrections however, the most important of which is absorption in the barrier. It largely cancels the energy variation of the reflection factor so that the notion of a constant reflection factor becomes a good approximation. The corrections to the reflection factor will be further discussed in Sec. 5. The prescription for the ESW is obviously quite successful in this case.

For protons, alpha particles and ^{12}C nuclei we can apply Fig. 6 to conventional optical potentials. This is done on Fig. 9 which shows the value of f and ΔR for these particles as a function of the bombarding nucleus. The results for f are also compared on the figure to the approximate reflection factor, (16), derived by Peaslee for s-wave neutrons.

4. Single Particle Resonances For Real Potentials

The most straightforward case in which the ESW constructed in Sec. 3 can be applied to display the wave properties of optical potentials is for the elastic scattering resonances of real potential wells. The ESW was constructed from a matching of the resonances of neutral particles at zero energy. We now go to finite energy and also add Coulomb barriers, much as one might do in analyzing the scattering of alpha particles from light nuclei. In later sections we go further still and add absorption to the picture (Sec. 5). The interest here is to show how well a zero-energy neutral-particle prescription applies to a realistic potential with barriers. The results will be useful in generalizing the formal many-channel theory of nuclear reactions (Sec. 7) to take into account the finite surface

thickness of nuclei.

We begin with a real nuclear potential which might be suitable for $^{32}\text{S} + \alpha$. Choosing the Saxon-Woods form (1) we expect V_0 to be about 75 MeV, R_0 to be* $1.25 A^{1/3} + 1.6$ and a

* This value of the radius is commonly used for optical model analyses of alpha-particle reactions. Its difference from $1.25 (A_1^{1/3} + A_2^{1/3})$ or from the more fundamental radius of $1.09 (A_1^{1/3} + A_2^{1/3})$ - which is suggested by electron scattering data - is neither justified by the data nor important for our discussion.

to be 0.5 fm. We fix a at this value and R_0 at 5.5685 fm. The Coulomb barrier has a height of 8.3 MeV at R_0 . Therefore any alpha-particle resonances at a few MeV energy will be very narrow. For purposes of illustration we chose a fixed resonance energy, $E_{\text{res}} = 3.0$ MeV, and varied V_0 in the vicinity of 75 MeV to obtain a single-particle, s-wave resonance. Fig. 10a shows such a resonant wave function obtained for $V_0 = 64.9$ MeV. At backward angles where Coulomb scattering is minimum the scattering cross section is that of the Breit-Wigner formula

$$\frac{d\sigma}{d\Omega} = \frac{1}{4k^2} \frac{\Gamma_{\text{no}}^2}{(E_{\text{no}} - E - \Delta_{\text{no}})^2 + \frac{1}{4} \Gamma_{\text{no}}^2} \quad (28)$$

where Δ_{no} is the level shift. Fig. 10b shows the scattering cross section near the energy of the resonant wave function of Fig. 10a. The single-particle resonance of the optical model well is 1.5 keV wide for this case.

In finding the single-particle resonance of an optical model potential we have completely specified the parameters of the potential. Therefore we can at once use Fig. 6 to construct the ESW. In the present case the ESW has a radius of 6.25 fm and a depth $V_1 = 56.3$ MeV in order that it yield an alpha-particle resonance with five nodes at 3.0 MeV. The ESW and its resonant wave function are shown on Fig. 10a and the corresponding scattering cross section on Fig. 10b.

Clearly, both the optical potential and its ESW exhibit resonances. The resonance of the ESW is considerably narrower than that of the optical model, a fact which might be ascribed to reflection. We wish to enquire in some detail how the optical model resonance properties can be described in terms of the ESW resonance properties after taking account of reflection. In order to do so we need to factor the resonance widths into penetration factors and reduced widths and we need to analyze the behaviour of the level shift. In such a factorization a part of the optical potential - the "tail" beyond the matching radius - modifies the shift and penetration factors from their conventional forms, (7) and (6) respectively. A re-derivation of the Breit-Wigner formula (see ref. 12, for example) shows that P_λ and S_λ still are given by (6) and (7) if we replace F_λ and G_λ by their analytic continuations, $\bar{F}_\lambda$ and $\bar{G}_\lambda$, to the matching radius. Denoting these by $\bar{P}_\lambda$ and $\bar{S}_\lambda$ we have

$$\Gamma_{no} = 2 \bar{P}_0 \gamma_{no}^2 \quad (29)$$

$$\Delta_{no} = - \bar{S}_0 \gamma_{no}^2 \quad (30)$$

Just as in Sec. 3, the matching radius of both wells for the factorization is the ESW radius $R_1 = 6.25$ fm. The reduced widths are obtained from the resonant wave functions by (19).

In turn, the resonant wave function is found to be that wave function which behaves like the irregular solution G_λ far beyond the potential. The boundary condition number, b_λ , of the level shift function (7) is still to be chosen to make the level shift vanish at $E = E_{\text{res}}$. However both $\bar{P}_\lambda$ and $\bar{S}_\lambda$ are functions of the energy and therefore we need to retain both Γ_{no} and Δ_{no} in the Breit-Wigner formula.

There is no doubt about the validity of the one-level approximation in our case. The $^{32}\text{S} + \alpha$ resonances are about 10 MeV apart compared to the width of 1.5 keV.

The energy dependence of the level shift can be taken into account approximately by using a Taylor series for the shift about the energy and retaining only the lowest non-zero term

$$\Delta_{\text{no}} = \left. \frac{d\Delta_{\text{no}}}{dE} \right|_{E = E_{\text{no}}} \times (E - E_0) \quad (31)$$

Combining (31) with (28) we can write the single-level resonance formula in the following way

$$\frac{d\sigma}{d\Omega} = \frac{1}{4k^2} \frac{(\Gamma_{\text{no}})^2}{\left(1 - \frac{d\Delta_{\text{no}}}{dE}\right) (E - E_{\text{no}})^2 + \frac{1}{4} \Gamma_{\text{no}}^2} \quad (32)$$

The effect of the energy dependence of the level shift on cross sections was first noted by Thomas¹⁷ and is described in detail

by Breit¹⁸ who has also computed those cases where approximation (31) fails. Thomas¹⁷ has suggested that if (31) were a good approximation (as it should be for the extremely narrow resonances within a Coulomb barrier) that the energy dependence of the level shifts be associated with the reduced widths

$$(\gamma'_{no})^2 \equiv \gamma_{no}^2 / \left[1 - \frac{d\Delta_{no}}{dE} \right]_{E = E_{no}} \quad (33)$$

In fact, he then found that the reduced widths of his calculation for a square well with Coulomb barrier were in close agreement with those appropriate to a square well without Coulomb barrier (i.e. $\hbar^2/mR^2$). With this association we may naturally define the observed widths of the calculation

$$\Gamma'_{no} \equiv 2 P_o (\gamma'_{no})^2 = \Gamma_{no} / \left[1 - \frac{d\Delta_{no}}{dE} \right]$$

so that the single-level formula attains the simple form appropriate in the absence of level shifts

$$\frac{d\sigma}{d\Omega} = \frac{1}{4k^2} \frac{(\Gamma'_{no})^2}{(E - E_{no})^2 + \frac{1}{4} (\Gamma'_{no})^2} \quad (34)$$

In comparing the resonance properties of the real optical potential and its ESW there are a number of ratios which might be related to the reflection factor of 5.2 (or more precisely 4.4 at 3 MeV from Fig. (7)) found for our problem from Fig. 6. The ratios are:

- i) $f' \equiv \Gamma'_{diff} / \Gamma'_{ESW} = f_P f_{\gamma^2} f_{\Delta} = 3.22$
- ii) $f_{\gamma^2} \equiv (\gamma_{diff})^2 / (\gamma_{ESW})^2 = 1.91$

$$\text{iii)} \quad f_p \equiv P_{\text{diff}} / P_{\text{ESW}} = 2.34$$

$$\text{iv)} \quad f_{\Delta} \equiv \frac{1 - (d\Delta_{\text{no}}^{\text{ESW}}/dE) \Big|_{E=E_{\text{no}}}}{1 - (d\Delta_{\text{no}}^{\text{diff}}/dE) \Big|_{E=E_{\text{no}}}} = 0.727 \quad (35)$$

$$\text{v)} \quad f_{(\gamma)^2} \equiv (\gamma_{\text{diff}}')^2 / (\gamma_{\text{ESW}}')^2 = 1.38$$

$$\text{vi)} \quad f \equiv \Gamma_{\text{diff}} / \Gamma_{\text{ESW}} = 4.46$$

where the numbers refer to our special case of $^{32}\text{S} + ^4\text{He}$ at 3 MeV.

The first ratio, f' , of the observed widths differs by an appreciable amount from the zero-energy, zero-barrier result of 5.2 while the ratio, f , of the one-body widths Γ_{no} is in moderately good agreement with these results and in excellent agreement with the one obtained at $E = 3.0$ MeV (from Fig. (7)). It must be remembered, however, that introduction of the surface effects into the channel widths will, in general, involve an adjustment for the energy dependence of the level shift ($f' = f_{\Delta} f$).

It is interesting to note that the factor f' enters in a simple way into the level shift*

$$d \Delta_{no}^{diff} / dE \Big|_{E = E_{no}} = f' d \Delta_{no}^{ESW} / dE \Big|_{E = E_{no}} \quad (36)$$

* To show this let

$$a_{n\ell}(R) \equiv \left\{ \frac{d}{dE} \left[R u_{n\ell}^{-1}(R) \frac{d u_{n\ell}}{dr} \right] \right\} \Big|_{E=E_{n\ell}}, \quad a_{n\ell}^{\circ}(R) \equiv \left\{ \frac{d}{dE} (R \bar{G}_{\ell}^{-1}(R) \frac{d \bar{G}_{\ell}}{dr}) \right\} \Big|_{E=E_{n\ell}}$$

where R is an arbitrary matching radius, $u_{n\ell}(r)$ is the normalized wave function appropriate to the n th ℓ -wave resonance at energy, $E_{n\ell}$, of the one-body potential, and $\bar{G}_{\ell}$ is the analytic continuation of the irregular ℓ -wave Coulomb function. Since the one-level approximation applies we have, near resonance, that $a_{n\ell}(R) = \gamma_{n\ell}^{-2}(R)$. Moreover, (suppressing the indices (n, ℓ)) it is easily shown that

$(a(R)/a(R_0)) = a^{\circ}(R)/a^{\circ}(R_0)$ where R_0 is the radius of the calculation, since $a_{(diff)}^{\circ}(R_{\infty}) = a_{(ESW)}^{\circ}(R_{\infty})$ we have (36),

$$f' = \frac{\Gamma_{diff}'}{\Gamma_{ESW}'} = \frac{\gamma_{diff}^2(R_{\infty})}{\gamma_{ESW}^2(R_{\infty})} = \frac{\gamma_{diff}^2(R_0)}{\gamma_{ESW}^2(R_0)} = \frac{a_{diff}^{\circ}(R_0)}{a_{ESW}^{\circ}(R_0)}$$

In fact in the example of $^{32}\text{S} + \alpha$ the ratio of (36) turned out to be 3.26 while f' has the value 3.22 of (35). The observed widths of the real diffuse-edge potential can then again be stated in terms of the parameters of its equivalent square well

$$\Gamma_{diff}' = \frac{f \Gamma_{ESW}}{1 - f' \frac{d \Delta_{no}^{ESW}}{dE} \Big|_{E = E_{no}}} \quad (37)$$

by the positive solution of the simple quadratic equation

$$c f^2 - f + (f - c f) = 0$$

where

$$c \equiv \left. \frac{d\Delta_{no}^{ESW}}{dE} \right|_{E = E_{no}}$$

(38)

Hence we may state the widths (37) of the diffuse-well entirely in terms of the parameters of the square interaction and of the standard reflection factor, f , obtained from the zero-energy zero-barrier case.*

* In an earlier calculation on alpha-decay rates in heavy nuclei¹⁴ it was of heuristic value to define an ESW in a slightly different manner. To have chosen the ESW to have the parameters suggested in Sec. 3 would have yielded a reflection factor consistent with that found for neutral particles at zero energy but (as we have shown in the present analysis) this reflection would have been divided between the interior aspects (reduced widths) and exterior aspects (penetrabilities) of the alpha-decay problem. Our point was that previous authors had treated the penetrabilities correctly but had not accounted for the enhancement of the wave function in the nuclear surface effected by the diffuse nuclear edge; in fact, this enhancement explained the anomalous radii which had plagued earlier alpha-decay rate calculations (which had generally taken the nuclear interior to be square). This was most easily displayed by constructing an "ESW" from first principles, that is, in the manner of Sec. 3: the radius and depth of the ESW were chosen so

that the resonant wave functions of the diffuse and square well had the same number of nodes and the same amplitude at the ESW radius. The radius of this ESW was found to be considerably larger than that of the diffuse-well, explaining much of the anomaly in the previous calculations.

5. Complex Optical Potentials

We wish to extend our treatment of barrier penetration to the case of complex optical potentials. As in the case of barriers and real optical potentials (Sec. 4) the combination of barriers with complex potentials will be shown to exhibit many of the simple wave properties present where there are no barriers at all. Again there are some straightforward modifications of the simple wave properties brought about by the introduction of barriers. As in our earlier discussion, we shall use the example $^{32}\text{S} + \alpha$ to illustrate the main points.

In the absence of barriers the addition of an imaginary term to a real optical model potential introduces absorption, it broadens the wave resonances, but has an almost negligible effect on wave reflection¹². The imaginary term for nucleons and even for heavy ions is normally chosen to have a small magnitude compared to the real term. Therefore the reflection from the imaginary term is small - indeed, the absorption and scattering cross section of a complex potential well are relatively insensitive to the shape of the imaginary term. For the same reasons we expect the same kind of changes when, in the presence of barriers, we add an imaginary term to the optical potential: absorption enters the picture, the resonances broaden but the wave reflection should remain relatively

unchanged.

At low energy, the wave reflection properties manifest themselves most directly in nuclear transmission functions. On Fig. 8 we showed a straightforward application of the zero-energy, zero-barrier results to the transmission functions for $^{32}\text{S} + \alpha$. The success of the model of Sec. 3 appears to be greater for the transmission functions than for the widths discussed in Sec. 4.

In order to elucidate to what extent the ratio of the transmission functions should be related to the reflection factor, we have calculated that ratio over a wide range of energies and for reactions involving widely different Coulomb barriers. To analyze those results, we will examine the effect of introducing a square and then a diffuse absorbing part to the potential. We shall find that the reflection properties are negligibly affected by the presence of an imaginary potential but that, if the absorbing potential has a diffuse edge, an important fraction of the absorption may take place much beyond the normal nuclear radius.

The nuclear transmission functions are calculated exactly by means of (4) from the complex phase shifts at complex potentials. In turn, the phase shifts are calculated directly from numerical integration of the wave equation to a point well beyond the nuclear radius where the numerical solution is matched on to standard Coulomb wave functions. The exact transmission functions found in this way can have their wave properties exhibited as in Sec. 2. If we adopt the form (5) for the transmission function - with the appropriate modifications made for diffuse-edge potentials,

as discussed in Sec. 2 - then a simple choice of the boundary condition numbers, b_λ , is one which makes the shift function, S_λ , vanish. With this choice we have

$$T_\lambda = \gamma_\lambda / (1 + \gamma_\lambda / 4)^2 \quad (39)$$

where γ_λ is defined by

$$\gamma_\lambda = 4 \pi f P_\lambda s_\lambda \quad (40)$$

Here P_λ is the penetration function, s_λ the strength function, and f the reflection factor discussed in Sec. 2.

The result (39) for the form of the transmission function explains the results of Fig. 8, both the simple ratio of transmission functions at low energy and also the corrections pertaining at higher energy. At low energy, far below the barrier, $\gamma_\lambda / 4$ is much smaller than unity so that $T_\lambda \approx \gamma_\lambda$. Then the ratio of the diffuse-edge transmission function to that of the ESW should be equal to the reflection factor. This is found to be so in Fig. 8. At energies approaching the top of the Coulomb barrier it is not the transmission functions themselves but rather the ratio of γ_λ which should equal f . Again this is found to be verified by Fig. 8. These results and many other similar results which we obtained but cannot show here justify our claim that the ESW should be independent of both the Coulomb and the angular momentum barriers. A comparison of Fig. 5 and Fig. 8 shows the importance of ΔR in the choice of the ESW for heavy ion reactions.

Although the dominant wave properties of complex wells are given in the discussion above, there are some minor and some

major corrections in certain cases. Two minor corrections (each less than 10% in our standard $^{32}\text{S} + \alpha$ reaction) arise from resonance effects and from reflection by the imaginary part of the potential. The former of these manifests itself in a slight energy dependence of the strength function when (39) is fitted to calculated transmission functions. Although minor for most heavy ion reactions the resonance corrections can become very large if the value of W_0 , the depth of the imaginary term in the potential, becomes smaller than the single-particle level-spacing in the real part of the well. The reflection induced by W_0 is small in all cases of practical interest. It can be shown that a rough estimate of this effect is to replace V_0 by $(V_0^2 + W_0^2)^{\frac{1}{2}}$ in Peaslee's formula, (16), for the reflection factor.

A major correction to optical model analyses concerns absorption deep within the barrier which occurs at energies very far below the Coulomb barrier. It follows from an application of Green's theorem to the wave equation¹⁹ that the transmission function, T_λ , is exactly proportional to the integral of the imaginary part of the potential

$$T_\lambda = c \int_0^\infty W(r) \Psi_\lambda \Psi_\lambda^* dr \quad (41)$$

where c is a constant and Ψ_λ is the radial wave function, obtained by numerical integration of the wave equation with the optical potential. In certain cases important contributions to the integral of (41) occur well beyond the normal nuclear radius. The point is

straightforward. Within the barrier the wave function increases exponentially:

$$\psi_l \propto \exp(Kr) \quad (42)$$

$$K \equiv \left[\frac{2m}{\hbar^2} (B - E) \right]^{\frac{1}{2}} \quad (43)$$

where B is the height of the barrier and E is the energy. The absorbing potential (2), for $r - R_0 \gg a$, falls off as $e^{-r/a}$. Therefore the contribution to the absorption from the region within the barrier becomes important at energies such that $2K$ approaches the value of a^{-1} . If we take a standard value of 0.5 fm for a we find $2Ka = 0.23\sqrt{m(B-E)}$ where m is the reduced mass (in units of the proton mass) and (B-E) is in MeV.

The absorption within the barrier is of great importance for many heavy ion reactions. Its magnitude can be estimated by means of (41). In our standard case of $^{32}\text{S} + \alpha$ at 3 MeV 16% of the absorption occurs beyond the ESW radius. This absorption doesn't obey the normal rules: it is not subject to a correction due to reflection and it tends to increase linearly with the value of W_0 . It is important to remember that the effect on the transmission functions of absorption in the barrier is opposite to that of the energy dependence of the reflection factor. As we go down in energy from the top of the barrier the absorption in the barrier tends to increase T_ℓ while the energy dependence of f decreases T_ℓ . For many commonly used optical potentials the two effects nearly cancel. More importantly, it suggests that some astrophysical

reaction rates will be dominated by the tail of the imaginary part of the optical potential. In such cases the optical model loses its utility in extrapolating from measurements at high energies to astrophysical rates far below the barrier. Conversely it raises the prospect that absorption cross section measurements at low energies can provide important information about nuclear densities far beyond the nuclear radius.

An extreme example of barrier absorption is shown on Fig. 11 for the $^{208}\text{Pb} + \alpha$ reaction at an energy of 8 MeV appropriate to the inverse of the alpha decay of ^{212}Po . Here the absorption is important at twice the nuclear radius, at values of r comparable to $R_0 + 1.5a$. Many optical model programs in current use would throw away the barrier absorption because they match the numerically integrated wave functions to Coulomb functions at too small a radius.

6. The Use of the Optical Model in Astrophysics

Current efforts by astrophysicists to determine the late stages of evolution of the stars and to determine the early history of the universe according to the "Big Bang" theory require the knowledge of a large number of charged particle reaction rates.

According to the "Big Bang" theory²⁰ matter would have emerged from thermodynamic equilibrium at very high temperatures ($T \gtrsim 10^{10}$ °K) and would have cooled down as it expanded. The amount of nuclear transmutations occurring as the matter expanded, the energy release during that time, and the temperature at which the nuclear reaction rates become too slow to allow any more nuclear reactions to occur all depend sensitively on charged particle reaction rates. On the other hand, once the stars, in their deep interiors, have burned all their hydrogen and all their helium, ^{12}C and ^{16}O are expected to burn until, after their exhaustion, ^{28}Si decomposes itself into alpha particles which, adding themselves to ^{28}Si seed nuclei, form the iron peak²¹.

The $^{12}\text{C} + ^{12}\text{C}$ and the $^{16}\text{O} + ^{16}\text{O}$ reactions have been studied in the laboratory^{22,23}, but not at quite as low an energy as they are expected to occur at in the stars. One needs an understanding of the physics of the reactions to extrapolate downwards with accuracy, or at least, to evaluate the uncertainties of the extrapolation. Many other needed charged particle reactions have not been measured, especially (α, p) , (α, n) , (α, γ) and (p, γ) reactions for target nuclei with masses around and above ^{28}Si . Some of those reactions probably will never be measured since they involve unstable target nuclei. Theoretical determinations of those rates, based on the optical model have been used by astrophysicists.

In principle they could have solved the Schrödinger equation of the Woods-Saxon optical model for each channel and for each nucleus. Due to the large number of rates needed, they have relied upon square well, black nucleus calculations. The accuracy with which the equivalent square well replaces a diffuse well vindicates their efforts, but also shows the relationship between the radius obtained by fitting scattering data to an optical model and the radius one should use in absorption cross sections. By fitting experimental reaction cross sections for (n, p) reactions (for $37 \leq A_{\text{target}} \leq 60$) and for (p, α) , (p, γ) and (α, γ) reactions (for $A \lesssim 35$) to black nuclei cross sections, Truran et al.²⁴ determined a radius $R = 1.2 (A_0^{1/3} + A_1^{1/3})$ fm where A_0 is the mass number of the projectile and A_1 that of the target. From electron and elastic scattering data, one rather expects $R_0 = 1.25 A^{1/3}$ fermis for protons and neutrons and $R_0 = 1.09 A^{1/3} + 1.6$ fm for alpha particles. From Fig. 9, one should add to obtain the ESW radius $\Delta R = 0.1$ fm for protons and neutrons and $\Delta R = 0.7$ fm for alpha particles. The radius obtained by Truran et al. is seen to be larger than the scattering one for nucleon channels and very closely the same for alpha channels. Part of the discrepancy probably comes from the artificial reflection occurring in the black nucleus. If the reflection factor is used to multiply the penetrability, the radius needed to fit the (n, p) reaction measurements is smaller. The alpha channel radius is also reduced however. We have made calculations,

using diffuse well transmission functions, of the reactions $^{27}\text{Al} (p, \alpha) ^{24}\text{Mg}$ and $^{31}\text{P} (p, \alpha) ^{28}\text{Si}$. Comparing with the experimental²⁵ cross sections points to an alpha channel radius $R_0 = 1.09 A^{1/3} + 1.6 \pm .4$ f in good agreement with the radius for particle scattering.

It is then possible to relate the parameters to be used in reaction cross section calculations to those obtained from electron, meson, p, n and ^4He scattering by nuclei. Formulas like C-60 of Fowler and Hoyle²⁶ may be used if one replaces β by $f\beta$, that is, if one multiplies the penetrability, or equivalently the strength function, by the reflection factor. The radius to be used is then the one obtained from scattering experiments plus the ΔR , from Fig. 6 or Fig. 9. One can use Eq. C-60 of Ref. 26 to estimate the uncertainty in the reaction rates caused by the uncertainty in the parameters of the optical model.

Changing the depth, V_0 , of the potential has only a second order effect: the first order effects in Eq. (C-60) are due to reflection and they are cancelled by the dependence of f on V_0 . The effect of the surface thickness, a , is more profound. Increasing it by 25% will generally increase the f factor by 25% but, more important, will sometimes, as for $^{32}\text{S} + ^4\text{He}$, increase ΔR by a factor of 2 (see Fig. 6). All important uncertainties can then be related to the radius, R_0 , of the diffuse well, and to the ΔR needed to obtain the radius of the ESW. For $^{32}\text{S} + ^4\text{He}$, at $T = 3.0 \times 10^9$ °K, it can easily be calculated, using eq. C-60,

that changing the radius from $R = 5.6$ fm to $R = 6.6$ fm increases the reaction rate by a factor of 5.0. Uncertainties in the radius and the surface thickness then seem to introduce uncertainties of a factor of 5.0 in the reaction rates involving ${}^4\text{He}$ channels.

In the preceding discussion, it was assumed that the optical model is a proper representation of the ${}^4\text{He}$ reaction channels. We can also study how sensitive the value of $\langle \sigma \rangle$ is on the underlying physics. To this end, we have used as our optical potential,

$$V(r) = -V_0 (1 + e^{(r-R_0)/a_v})^{-1} - iW_0 (1 + e^{(r-R_0)/a_w})^{-1} \quad (44)$$

and we have varied a_w while keeping a_v constant. This represents one of the ways in which the commonly used optical model can be modified to represent different hypotheses as to matter distribution and the location of the compound nucleus formation.

Numerical calculations have been carried out for ${}^{32}\text{S} + \alpha$, ${}^{44}\text{Ti} + \alpha$ and ${}^{12}\text{C} + {}^{12}\text{C}$ with $a_v \neq a_w$. The ratios $T_\lambda(a_w \neq a_v)/T_\lambda(\text{ESW})$ are presented on Fig. 12. Whereas the equivalent square well is a good approximation to the diffuse well for $a_w = a_v$ it breaks down completely if $a_w > a_v$. The cross section is then dominated by absorption in the barrier as can be seen for ${}^{12}\text{C} + {}^{12}\text{C}$ on Fig. 13. The low energy cross section obtained with the black nucleus could then easily be an underestimate by another factor of 5. This is expected to be important specially for alpha particles incident on nuclei with large Z ($Z \geq 20$). However, the data available for alpha particle channels in reaction cross sections at low energy is for

^{31}P (p, α) ^{28}Si , ^{27}Al (p, α) ^{24}Mg and ^{35}Cl (p) $^{32}\text{S}^*$. Fitting these

* Since the Q value of those reactions is positive, the transmission function for protons is approximately equal, at a given energy in the compound nucleus, to that for alpha particles. The alpha particle transmission function is reasonably large and is not expected to show absorption in the barrier,

with black nuclei cross sections and then extrapolating to higher mass numbers completely neglects the possibility of absorption in the barrier. More experimental data is needed to permit reliable estimates of alpha particle channels at $A \gtrsim 40$.

The $^{12}\text{C} + ^{12}\text{C}$ results indicate how sensitive the cross section of the $^{12}\text{C} + ^{12}\text{C}$ reaction is on the detailed shape of the optical model chosen. A 20% increase in the diffuseness parameter of the imaginary potential increases the cross section at $E \simeq 3$ MeV by a factor of 5. Only a model that would represent closely the physics of the $^{12}\text{C} + ^{12}\text{C}$ system could be hoped to permit any extrapolations. The recent results of Ref. 22 may indicate that the optical model is too crude a tool.

7. General Conclusions Concerning Nuclear Reactions

Our analysis of the role of the optical potential in barrier penetration suggests a number of general conclusions about nuclear reactions. They concern the role of the radius in resonance reactions, the value of the nuclear radius in reactions and the use of absorption cross sections far below the barrier to probe the

tail of the nuclear density distribution.

The many-channel theory of resonance reactions appears to have a much more complicated geometry than the one-channel potential-scattering problem whose wave properties we described in Sec. 4. We shall show that it is reasonable to decompose the scattering matrix in a manner such that each reaction channel possesses the one-channel potential-scattering properties. Then we can use our earlier results for the one-channel case to remove many of the artificial "square-well" aspects of the resonance theory results. A similar treatment for nuclear reactions without barriers was given in an earlier paper by Vogt.¹²

The general theory of nuclear reactions¹⁰ provides a framework in which all cross sections can be described in terms of level parameters. The relation between cross sections and level parameters is made in two steps. First the cross section, $\sigma_{cc'}$, for an initial channel c and a final channel c' , is written in terms of statistical spin factors and collision matrix components. Next the collision matrix components, $U_{cc'}$, are written in terms of level parameters

$$U_{cc'} = e^{i(\Omega_c + \Omega_{c'})} (\delta_{cc'} + i \sum_{\lambda\lambda'} \Gamma_{\lambda c}^{-\frac{1}{2}} \Gamma_{\lambda c'}^{\frac{1}{2}} A_{\lambda\lambda'}) \quad (45)$$

where the Ω_c are phase shifts and the $A_{\lambda\lambda'}$ the components of a matrix whose inverse is

$$(A^{-1})_{\lambda\lambda'} = (E_\lambda - E) \delta_{\lambda\lambda'} + \Delta_{\lambda\lambda'} - \left(\frac{i}{2}\right) \Gamma_{\lambda\lambda'} \quad (46)$$

This form of the framework is completely general: it applies to all approximate forms from the Breit-Wigner Formula to the Hauser-Feshbach theory. The question is, how do the level parameters change when a reaction channel is assumed to include an average Saxon-Woods potential?

The square well interaction is easily adapted to the ordinary reaction theories.

For a square well, both the partial widths and the level shift can be factored in a manner which separates out the many-body features of the problem and which clearly displays the wave properties associated with the average interaction:

$$\Gamma_{\lambda c} = S_{\lambda c}^p \quad \Gamma^{\ell p} = S_{\lambda c}^p \quad 2P_{\ell}^2 (\gamma^{\ell p})^2 \quad (47)$$

$$\Delta_{\lambda} = - \sum_c S_{\lambda c}^p \quad S_{\ell} (\gamma^{\ell p})^2 \quad (48)$$

The spectroscopic factors, $S_{\lambda c}^p$, are essentially statistical coefficients which measure the probability for finding the compound nucleus in the particular mode specified by the channel number, c . Since they depend on averages over the nuclear volume they are insensitive to the details of the nuclear surface. Thus the effects of the surface influence only the one-body aspects of the problem, the single particle widths $\Gamma^{\ell p}$ or the corresponding single-particle reduced widths $(\gamma^{\ell p})^2$. Therefore the transition from a square average interaction to a Saxon-Woods interaction affects only the single-particle width. The single-particle width is affected by the diffuse-edge in the manner discussed in Sec. 2 and Sec. 3. Thus the insensitivity of the spectroscopic factors to the precise value of the matching radius makes our wave analysis

apply to each reaction channel separately.

The conventional treatment of nuclear reactions by the black-box or resonance theories is essentially a square-well treatment. Because of this fact the nuclear radius required in this treatment to fit observed reaction rates was artificially large. For several decades³ the usual value of the nuclear radius for nuclear reactions was

$$R = 1.4 (A_1^{1/3} + A_2^{1/3}) \text{ fm} \quad (49)$$

where A_1 is atomic weight of the target nucleus and A_2 that of the bombarding particle. On the other hand the usual optical model radius for nucleons is

$$R = 1.25 A_1^{1/3} \text{ fm} \quad (50)$$

and for heavy ions

$$R = 1.25 (A_1^{1/3} + A_2^{1/3}) \text{ fm} \quad (51)$$

or perhaps

$$R = 1.09 (A_1^{1/3} + A_2^{1/3}) \text{ fm} \quad (52)$$

For both nucleons and heavy ions the difference in the radii is usually between one and two fermis. It is our conclusion that the larger radii were a result of the square well treatment.

Before embarking on an explanation of the differences between old and new radii we make some remarks on the current fashions. The charge radii of nuclei, as measured by electron scattering and mesic atoms are $1.09 \times A^{1/3}$ fermis. If this is taken to reflect the nuclear density as well as the charge then a nucleon should feel the

same radius - the same value should apply to the optical potential. Similarly $1.09 (A_1^{1/3} + A_2^{1/3})$ should then be the radius for heavy ion reactions. Now there are some additional effects which tend to increase the optical model radius slightly and to make it slightly dependent on the shell structure. The first is core polarization - an incoming nucleon pulls the target nucleons toward it. The second is the neutron excess in the surface of a nucleus - an excess which leads to the isotopic spin term in the optical potential. Such a term means that the density extends beyond the charge. Both of these effects can be estimated only roughly and depend on the particular nucleus involved. Their magnitude roughly justifies the small amount by which current radii exceed the charge radii. There is some experimental uncertainty in the radii as well. Although the wave resonance effects determine the product $V_o R_o^2$ very accurately - perhaps to 1% - we know of no experiment which unambiguously determines the radius of the real part of the optical potential to anything like this accuracy. Therefore the choice for nucleons of $1.25 A^{1/3}$ is not only a reasonable choice but also has about a 10% uncertainty. By the same token, an alpha particle radius of 1.6 is also reasonable.

From our analysis of wave properties we see that there are several ways in which the change from an optical potential to a square well modifies the radius. First of all, there is the difference in reflection between the wells which can be compensated for by a difference in radii; secondly, there is the wave oscillation in the tail of the real part of the optical potential which leads (Fig. 5) to a difference in radius between the optical potential and

its equivalent square well; thirdly, there is absorption in the tail of the imaginary part of the potential.

A reassessment of the old analyses needs to examine only the heavy ion absorption cross sections for which the dependence on the nuclear radius is unambiguous. For nucleons it has turned out that there are size-resonance effects which vary through the periodic table (see Fig. 2). For alpha particles and other heavy ions all the size-resonance effects are washed out - particles reaching the nucleus are absorbed. Early evidence²⁷ for the large radius came from charged particle reactions²⁷ as well as neutron reactions.

In the black-nucleus model for alpha particle absorption the strength function was shown (in Sec. 2) to be that of a square well. It has no giant resonance structure but neither does the diffuse-edge optical model potential suitable for alpha particle reactions. We might therefore be tempted to say that the black-nucleus radius should be chosen to be that of the equivalent square well of the appropriate optical potential. In such a choice the ΔR of Fig. 9 has a value of about 0.5 fermis. But such a choice ignores wave reflection. From Fig. 9 the reflection factor for alpha particles has a value between 3 and 5. The transmission functions far below the barrier are directly proportional to the reflection factor. To enhance the black-nucleus transmission functions by a factor of 3 to 5 we can increase the nuclear radius, thus increasing the penetration factor. The required increase in

the radius is about 0.5 fermis*.

* The penetration factor at an energy, E, far below the Coulomb barrier, B, depends on the nuclear radius roughly as

$$P \approx k R G^{-2} (k R) \approx k R^{-2kR}$$

where

$$k = \left[\frac{2m}{\hbar^2} (B-E) \right]^{\frac{1}{2}}$$

For alpha particles k typically has values between 1.0 and 2.0.

Thus the wave oscillation ($\Delta R \simeq 0.5$) and the wave reflection together account for the 1.0 fm difference between the black-nucleus radius and that of modern optical potentials. The early analysis of alpha decay rates in heavy nuclei required similar anomalous large radii which have been brought into agreement with modern values by our wave analysis.¹⁴ In our view there is no evidence at all that any nuclear reaction rates require anomalously large radii or that they throw into question the individual particle picture of nuclear structure.

The sum-rule limits associated with reduced widths are square well values which need to be modified to take into account the reflection factor of a diffuse-edge well. According to (20) above, for any reaction channel we can write a single particle width as

$$\Gamma^{\text{sp}} = 2P_k (\gamma_k^{\text{sp}})^2 = 2P_k^{\text{F}} (\gamma_{\text{SW}}^{\text{sp}})^2 \quad (53)$$

Where P_λ is the conventional penetration factor, (6), and $(Y_\lambda^{lp})^2$ the single-particle reduced width of the well. Equally, we can write

$$(Y_\lambda^{lp})^2 = f(Y_{SW}^{lp})^2 = f\hbar^2/mR_o^2 \quad (54)$$

where f is the reflection factor discussed above and $(Y_{SW}^{lp})^2$ is the single-particle reduced width of the equivalent square well (ESW). It is $(Y_{SW}^{lp})^2$ which has the conventional sum-rule value of $\hbar^2/mR_o^2$ where R_o is the radius of the ESW. Therefore a true single particle level should have a reduced width exceeding the conventional sum-rule limit by the reflection factor f which typically has a value between two and five. Our work shows that this factor applies to all charged particle reactions as well as to neutrons and gives quantitative estimates of f .

Far below the barrier the absorption in the tail of the optical potential can dominate the whole absorption process. We have warned of the dangers of using conventional optical models to calculate reaction rates which are dominated by their tails. We can, however, turn the argument around and suggest that the measurement of absorption cross sections far below the barrier be used to determine nuclear properties at large radii. This is a problem of considerable importance in modern nuclear physics because of fission isomerism²⁸, mesonic atoms²⁹ and the general question of nuclear clusters in regions of low density. From our study it is clear that any nuclear density (whether nucleons or

other clusters) extending to large radii can dominate the absorption process. It would be very interesting and valuable to perform nuclear reactions with intense beams at low energies to examine, say, the probability of alpha particle absorption in the bombardment of many light nuclei. Similarly, experiments with proton or heavy ion beams might also yield anomalously large cross sections. Is ^{16}O much more tightly bound than ^{20}Ne or ^{24}Mg ? Does the distant tail of the nuclear density in the latter present nucleons or alpha particle clusters? It is likely that the kind of surprises found in the $^{12}\text{C} + ^{12}\text{C}$ reactions ^{22}Ne will abound in many other cases. The results may also change many of our notions concerning stellar reaction rates.

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Figure Captions

Figure 1. Comparison of the low lying resonant states of the square well and of the corresponding diffuse edge well for s-wave neutrons. Once the depth and radius of the square well are adjusted so that both wells have a resonance with the same reduced width at $E = -0.43$ MeV, all the other low lying resonances in the two wells occur at very closely the same energy and with very nearly the same reduced width (except for the reduced width of the resonance at $E = -53$ MeV). It is then possible for a square well to reproduce the interior properties of resonance of a diffuse well. The difference between the two wells is then contained only in the penetrability and shift functions.

Figure 2. Comparison of γ_2 of protons for different values of the surface thickness a . Because the transmission functions are close to 1., it is necessary to compare γ_1 to obtain the reflection factors. The γ_2 are obtained from the transmission functions by using $T_1 = \gamma_1 / (1 + \gamma_1/4)^2$. It is seen that by multiplying γ_2 ($a = 0$) by a factor $f(a)$, independent of energy, one gets an excellent approximation to γ_2 ($a \neq 0$). The interior of the two wells must then have very similar resonance properties. Only the penetrabilities

vary. It then seems possible, as it was for neutrons, to replace a diffuse well by a square well.

Figure 3. The shape of the real potential for s-wave protons ($^{32}\text{S} + p$). The Coulomb potential (V_c) does add a radius dependent term. However its radius dependence is much weaker than the radius dependence of the nuclear potential (V_{DW}). In first approximation the Coulomb field adds only a constant term which does not change the reflection properties of the two wells. Protons should behave like negative energy neutrons. As is the case for neutrons, it should then be possible to find an equivalent square well (V_{ESW}) for charged particles. The difference between the two wells should again reside only in the reflection factor. This is vindicated by Fig. 2. In this example, the nuclear potential is chosen to have a Saxon-Woods shape with depth 50 MeV, radius 3.96fm, and surface thickness 0.5fm.

Figure 4. Ratio of transmission functions of an optical potential suitable for $^4\text{He} + ^{32}\text{S}$ for two values of the surface thickness. ($T_{\lambda=0}(a = .5\text{fm})/T_{\lambda=0}(a = 0.0)$). The two wells have the same V_0 and the same R_0 . It seems either that the interior of the two wells show different

resonance behaviour or that different phenomena occur under the Coulomb barrier. It will be seen in the next sections that most of the energy variations of the ratio comes from choosing two wells which don't have the same interior properties. Absorption in the barrier will be seen to become dominant only at very low energy ($E \lesssim 1$ MeV in this case).

Figure 5. A diffuse (SOLID) and a square (DASHED) potential well and the corresponding s-wave zero-energy wave functions. The depth and radius of the square well were chosen so that it had a resonance at $E = 0$, and that the reduced width of that resonance be the same as that of the diffuse well at the square well radius. The interior properties of the two wells will then be the same as can be seen from Fig. 1. The wave function of the diffuse well continues to rise outside of the square well radius. As discussed in the text this can be related to the penetrability of the diffuse well and the differences between the square and diffuse potential can all be related to the ratio $\left[\psi'_{DW}(\infty) / \psi'_{DW}(R_{ESW}) \right]^2$ whether we be studying narrow resonances far below the Coulomb barrier (Sec. 4) or the transmission functions of complex potentials (Sec. 5). The diffuse potential has been chosen to have a Saxon-Woods shape with interior wave number K , radius R_0 , and surface thickness a .

Figure 6. The value of $\Delta R/R_0$ and of the reflection factor f , as a function of a/R_0 and $K R_0/\pi$ ($= 2m V_0 R_0^2 / \hbar^2 \pi^2$)^{1/2}. This is the most frequently used section of Fig. 1 of Ref. 13. For a wider range of values, the reader is referred to Ref. 13. ΔR and f allow one to determine the cross sections and the resonance width of diffuse wells, once those of square wells are known.

Figure 7. The energy variation of the reflection factor was calculated from the ratio of the transmitted waves for a given incoming wave. We have made other calculations varying the Coulomb potential, but keeping the reduced mass and the nuclear potential the same. The reflection factor was then found to depend on (B-E) only, where B is the height of the barrier in the diffuse well. The reflection factor takes its maximum value at the energy $E = B_{MAX}$ where B_{MAX} is the maximum value of the diffuse barrier. Letting the reflection factor vary with energy will, in some cases, improve the accuracy of the equivalent square well model (not for the calculation of transmission functions however). The energy variation of the reflection factor depends mainly on the reduced mass and the figure may be used to estimate it, by interpolation, for cases not covered by our calculations. Note that for $^{12}\text{C} + ^{32}\text{S}$ the energy scale is given by $E_{^{12}\text{C} + ^{32}\text{S}} = E + 10.0 \text{ MeV}$.

Figure 8. The ratio of diffuse-edge optical model transmission functions to equivalent square-well transmission functions as a function of the energy for $^{32}\text{S} + \alpha$. The diffuse potential is that of Fig. 4 and the equivalent square well has been calculated from Fig. 6 ($\Delta R = 0.58$ fm). The reflection factor ($f = 4.62$) is the ratio obtained at zero energy in the absence of Coulomb and angular momentum barriers. It is seen that if one chooses a correct value of ΔR for the equivalent square well the observed reflection is insensitive to charge and centripetal barriers over a wide range of energies and is in close agreement with the reflection factor; in contrast, it was seen in Fig. 4 for a different square well ($\Delta R = 0$) that the observed reflection depended strongly on all these factors. (The dotted lines give the corresponding ratios for $\gamma_i = 2\pi \frac{\langle r_i \rangle}{0}$ connected to the transmission functions by the approximate relation $T_i \approx \gamma_i / (1 + \frac{1}{4} \gamma_i)^2$).

Figure 9. ΔR and f for p, n, ^4He and ^{12}C incident on target nuclei of different masses. The diffuse potentials used had $R_0 = 1.25 A_T^{1/3}$, $1.6 + 1.25 A_T^{1/3}$, and $1.25 (12^{1/3} + A_T^{1/3})$ and $V_0 = 50$ MeV, 75 MeV and 100 MeV, for nucleons, alpha particles and ^{12}C respectively and $a = 0.5\text{fm}$ in all cases. ΔR and f were obtained

from Fig. 6. We have also plotted with a dashed line (- - -) the approximation to f obtained by Peaslee for s-wave neutrons (16). The agreement is seen to be quite close for neutrons and protons.

Figure 10. The resonant s-wave functions (upper part of Fig. 10a) and the elastic scattering cross sections (Fig. 10b) for a diffuse potential (SOLID) appropriate to the reaction $^{32}\text{S} + \alpha$ and its equivalent square well (DASHED) (lower Fig. 10a). The ratio (3.24) of the cross sections is simply related to the wave reflection factor (5.2) calculated in Sec. 3 through the energy dependence of the level shift. It is seen from the resonant wave functions that the reflection of the diffuse well is now distributed between the interior (resonant) and exterior (penetration) properties of the well; nevertheless, the sum of the effects is what we would expect from the zero-energy barrier calculations. The diffuse well has a Saxon-Woods shape with depth - 64.9 MeV, radius 5.5685 fm., and surface thickness 0.5 fm.; the equivalent square well is found to have depth - 56.3 MeV and radius 6.25 fm.

Figure 11. Fraction of the absorption as a function of radius for diffuse potentials with parameters appropriate to the reactions $^{208}\text{Pb} + \alpha$ (SOLID) and $^{32}\text{S} + \alpha$ (DASHED). The radii used for the Saxon-Woods wells

are indicated by vertical lines. The relative amount of absorption in the barrier is seen to be greatly increased by the extreme barriers associated with heavy nuclei. In fact, at low energies ($\lesssim 7$ MeV) all absorption occurs in the barrier if we use a conventional Saxon-Woods shape for the imaginary potential (with a radius, R_0 , of 5.568 fm for $^{32}\text{S} + \alpha$ and of 8.8 fm for $^{208}\text{Pb} + \alpha$). Hence, the nature of the imaginary potential in the barrier region is critical in describing absorption cross sections. If the absorption described by the figure is, in fact, physical, the equivalent square well model would fail badly since most particles would be absorbed before reaching the nuclear surface.

Figure 12. Ratios of transmission functions for a variety of reactions. Even in the presence of moderate barriers (Fig. a) the barrier absorption is seen to depend strongly on the surface thickness of the imaginary potential. In fact, even a slight increase in the barrier (Fig. b) is seen to considerably enhance the barrier absorption; on the other hand, it is rather insensitive to the reduced mass (Fig. c). In all these cases the equivalent square well model is seen to have considerable validity provided we choose a conventional surface thickness for the imaginary

potential ($a_W = a_V = 0.5$ fm). In the presence of the extreme barriers associated with heavy ion reactions and alpha particle scattering from heavy nuclei (Fig. d) it is seen that the equivalent square well model fails badly even for conventional surface thicknesses. Hence, it is of crucial interest to understand the details of the imaginary potential in the barrier if one wishes to extrapolate high energy data to the energies of interest to astrophysics.

Figure 13. Fraction of the absorption as a function of the radius for a diffuse potential whose real and imaginary parts have Saxon-Woods shapes with parameters appropriate to the reaction $^{12}\text{C} + ^{12}\text{C}$. The comparison of the absorption for s-waves (SOLID) and g-waves (DASHED) shows that the relative amount of absorption which occurs in the barrier is increased when the barrier is heightened. It is also seen that increasing the surface thickness a_W , of the imaginary potential (DOTTED) considerably enhances the relative amount of barrier absorption. Hence, low energy absorption cross sections may yield detailed information about the shape of the absorptive potential in the nuclear surface. (The energy is 4 MeV.)

NORMALIZED RESONANCE STATES

ENERGY	LEVELS (MeV)	REDUCED WIDTH RATIO $\gamma^2/(\hbar^2/mR^2)$
261.90	259.73	0.989

191.58	190.63	0.989
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130.49	130.00	0.987
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78.12	77.92	0.987
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34.48	34.42	0.989
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-0.43	-0.43	1.000
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-26.61	-26.53	1.030
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-44.06	-43.69	1.043
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-52.79	-53.41	4.260
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SQUARE WELL	SAXON-WOODS
$V = -53.88 \text{ MeV}$	$V_0 = -51$
$R = 6.845 \text{ f}$	$R = 6.72 \text{ f}$
	$a = 0.5 \text{ f}$

$$\alpha/R_0 = 0.1$$

$$KR_0/\pi = 4.5$$

$$f = 4.9$$