THE GRAVITATING EFFECT OF GRAVITATION

BY

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ABSTRACT

The work reported in this thesis is based on the vierbein field formulation of gravitational theory, used in conjunction with the method of the compensating field. It is shown that the most general linear equations of second order for a tensor field, which are invariant under orientations of the local inertial frame and under gauge transformations of the vierbein field components are identical with Einstein's field equation written down in the weak field approximation. An attempt is made to take into account any possibly existing gravitating effect of gravitation by applying the method of the compensating field to the weak field Lagrangian, resulting in a set of nonlinear field equations. The invariance properties of the modified field equations are examined, and some special solutions are exhibited.

CONTENTS

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1. The Vierbein description of the gravitational field.

In describing the gravitational field, one usually characterizes it by a metric field, $g_{\mu\nu}$, such that the trajectories of test particles are geodesics in that field. As is well known (see, for example, Landau and Lifshitz, 1959) Einstein proposed in 1916 as field equations for the free gravitational field

$$R_{\mu\nu} = 0 \qquad (1.1)$$

This approach has the obvious disadvantage that there is no simple criterion for telling whether a given metric tensor describes a true gravitational field, a pseudo-gravitational field, or a mixture of both, where, by definition, pseudo-gravitational fields can be transformed away globally, and true gravitational fields can only be transformed away locally. A familiar example of pseudo-gravitational fields are the centrifugal and Coriolis fields encountered in coordinate systems rotating with respect to a Newtonian frame of reference. In the present work, a new approach based on an idea by Einstein (1928) will be attempted in order to avoid the above drawback and obtain, if possible, simpler field equations.

Firstly, one notes that in the theory of gravitation, one can always find at least one local inertial frame by going into free fall. This means that one can always locally transform away all but the diagonal terms of the metric tensor.

Denoting x^{μ} by coordinates of the curvilinear continuum and y^{k} by coordinates of the local inertial frame ($y^4 = it$) one can introduce now the transformation functions connecting the displacements

$$dy^{k} = h^{k}_{\alpha}(x)\, dx^{\alpha} \tag{1.2}$$

The main purpose of this first chapter is to develop a formalism enabling one to describe all the metric properties in terms of these functions h^{k}_{α}, which were given the name "vierbein" field by Einstein in his original paper on this subject.

By assuming that the determinant $|h^{k}_{\alpha}|$, does not vanish, one ensures the existence of the inverse functions $f^{\alpha}_{k}(y)$ defined by

$$dx^{\alpha} = f^{\alpha}_{k}(y)\, dy^{k} \tag{1.3}$$

which are connected, because of (1.2), with the h^{k}_{α} by

$$f^{\alpha}_{k}\, h^{k}_{\beta} = \delta^{\alpha}_{\beta} \tag{1.4}$$

Hence

$$f^{\alpha}_{k} = \frac{[h^{k}_{\alpha}]}{|h^{k}_{\alpha}|} \tag{1.5}$$

where $[h^{k}_{\alpha}]$ is the cofactor of h^{k}_{α}.

Once the vierbein field is given, one can easily determine whether it describes true gravitational fields or not, since if the h^k_α are integrable, that is if $h^k_{\alpha|\beta} = h^k_{\beta|\alpha}$ where a vertical bar denotes partial differentiation, one can find a global inertial frame $y^k = y^k(x^\alpha)$ and accordingly look upon the metric field described by the in this case as representing a pseudo-gravitational field. Any true gravitational field is then characterized by the nonintegrability of the h^k_α .

This suggests defining the functions

$$G^k_{\alpha|\beta} \equiv h^k_{\alpha|\beta} - h^k_{\beta|\alpha} \tag{1.6}$$

as the "true gravitational field strengths", and the non-integrable functions h^k_α as "gravitational potentials". This is done to keep a close analog to the well known case of electromagnetic theory, where the "field strengths"

$$F_{\alpha\beta} \equiv A_{\alpha|\beta} - A_{\beta|\alpha} \tag{1.7}$$

are derivable from the potentials A_α by differentiation.

The h^k_α are a set of sixteen independent functions consisting of four four-vectors $h^1_\alpha, h^2_\alpha, h^3_\alpha, h^4_\alpha$ whose components transform under reorientation of the local inertial frame according to

$$h^k_\alpha \rightarrow h^{k\,\prime}_\alpha = R^k_i\, h^i_\alpha \tag{1.8}$$

where R^k_i is an orthogonal matrix, that is

$$R^{i}_{k} R^{k}_{j} = \delta^{i}_{j} \tag{1.9}$$

Because of the experimentally well established isotropy of inertia (Hughes et. al. 1960, Drever 1961) one must in fact require invariance of the field equations under the transformation (1.8).

In order to eliminate pseudo-gravitational effects, one must further require invariance under the transformation

$$h^{k}_{\alpha} \longrightarrow h^{k}_{\alpha} + \Lambda^{k}{}_{|\alpha} \tag{1.10}$$

with arbitrary four-vectors Λ^{k}. This is the exact analog to the "gauge invariance" of electrodynamics that is, under transformations

$$A_{\alpha} \longrightarrow A_{\alpha} + \Lambda_{|\alpha} \tag{1.11}$$

of the potentials which leave F^{α}_{β} invariant.

Once the h^{k}_{α} are known, one can compute the metric tensor. In the following, local inertial frames will be taken to be cartesian coordinate systems, and thus one may take as metric of the local inertial frame $g_{ik} = \delta_{ik}$. The invariant line element

$$ds^{2} = \delta_{ik}\, dy^{i}\, dy^{k} \tag{1.12}$$

can then be expressed in terms of dx_α, dx_β by (1.2) as

$$ds^2 = h^k_\alpha h^k_\beta dx^\alpha dx^\beta \tag{1.13}$$

Since also

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{1.14}$$

one has, by comparison

$$g_{\alpha\beta} = h^k_\alpha h_{k\beta} \tag{1.15}$$

Similarly, one obtains

$$g^{\alpha\beta} = f^\alpha_k f^{\beta k} \tag{1.16}$$

Thus one sees that the components of $g_{\alpha\beta}$ are factorised by the functions h^k_α . Moreover, if the vierbein field is given, one can unambiguously compute the metric tensor. The converse, however, is obviously not true. The encourges one to take the view that the vierbein field is more fundamental than the metric tensor when one wishes to describe gravitational effects.

With all these tools in hand, one can now rework the theory of general relativity in terms of the vierbein field. For instance, the affinities

$$\Gamma^\tau_{\beta\alpha} = \frac{1}{2} g^{\tau\sigma} [g_{\sigma\beta|\alpha} + g_{\alpha\sigma|\beta} - g_{\alpha\beta|\sigma}] \tag{1.17}$$

read

$$\Gamma^{\tau}_{\beta\alpha} = \frac{1}{2} f^{\tau}_{k}\left[f^{\sigma}_{k} h^{i}_{\beta}\left(h^{i}_{\sigma|\alpha} - h^{i}_{\alpha|\sigma}\right) + f^{\sigma}_{k} h^{i}_{\alpha}\left(h^{i}_{\sigma|\beta} - h^{i}_{\beta|\sigma}\right) + \left(h^{k}_{\beta|\alpha} + h^{k}_{\beta|\alpha}\right)\right] \tag{1.18}$$

and hence the equation of the geodesics, for example,

$$\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}\,\dot{x}^{\beta}\,\dot{x}^{\gamma} = 0 \tag{1.19}$$

can be represented in terms of the vierbein field by (1.18) Furthermore, looking at equation (1.18) it is interesting to note that one can in fact separate out true gravitational fields from pseudo-gravitational fields in the equation of the geodesic (1.19). This discourages the view, taken by some authors, to look upon the $\Gamma^{\tau}_{\beta\alpha}$ as "field strengths" since the $\Gamma^{\tau}_{\beta\alpha}$ may actually contain both true gravitational and pseudo-gravitational fields.

As an illustration, consider the local transformations between an inertial (dx^{α}) and a rotating coordinate (dy^{k}) system given by

$$(h^{k}_{\alpha}) = \begin{pmatrix} \cos(i\omega x^4) & \sin(i\omega x^4) & 0 & i\omega[-x^1\sin(i\omega x^4) + x^2\cos(i\omega x^4)] \\ -\sin(i\omega x^4) & \cos(i\omega x^4) & 0 & -i\omega[x^1\cos(i\omega x^4) + x^2\sin(i\omega x^4)] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.20}$$

with k denoting the rows and α the columns. By direct differentiation, one finds that

$$h^{k}_{\alpha|\beta} = h^{k}_{\beta|\alpha} \tag{1.21}$$

which mean that the h^k_α are integrable and hence a global transformation between the inertial and rotating coordinate system should be possible. This is indeed the case, and the connection between the two systems is established by the well-known transformtaion equation

$$\begin{aligned} y_1 &= x^1 \cos(i\omega x^4) + x^2 \sin(i\omega x^4) \\ y_2 &= -x^1 \sin(i\omega x^4) + x^2 \cos(i\omega x^4) \\ y_3 &= x^3 \\ y_4 &= x^4 \quad . \end{aligned} \tag{1.22}$$

From equation (1.15), one can obtain the metric tensor:

$$(g_{\mu\nu}) = (h^k_\mu h_{k\nu}) = \begin{pmatrix} 1 & 0 & 0 & i\omega x^2 \\ 0 & 1 & 0 & -i\omega x^1 \\ 0 & 0 & 1 & 0 \\ i\omega x^2 & -i\omega x^1 & 0 & 1-\omega^2[(x^1)^2+(x^2)^2]^2 \end{pmatrix} \tag{1.23}$$

and notes that it does not indicate explicitly whether the field is true gravitational or pseudo-gravitational, whereas the mere fact that $h^k_{\alpha|\beta} = h^k_{\beta|\alpha}$ implies immediately that the field must be pseudo-gravitational.

For most practical applications, one is interested only in weak gravitational fields, that is, in small deviations from the Euclidean metric. This can be done by writing:

$$h^{k}_{\alpha} = \delta^{k}_{\alpha} + \frac{g}{2}\eta^{k}_{\alpha}$$
$$f^{\alpha}_{k} = \delta^{\alpha}_{k} - \frac{g}{2}\phi^{\alpha}_{k} \tag{1.24}$$

with an expansion parameter g, introducing thus new variables η^{k}_{α}.

By comparing this with

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \gamma_{\alpha\beta} \tag{1.25}$$

where $\gamma_{\alpha\beta}$ is the deviation from the pseudoeuclidean metric, one has, by (1.15), the relation

$$\gamma_{\alpha\beta} = \frac{g}{2}(\eta_{\alpha\beta} + \eta_{\beta\alpha}) + \left(\frac{g}{2}\right)^{2}\eta^{k}_{\alpha}\eta^{k}_{\beta} \tag{1.26}$$

Using (1.4), one can also express the ϕ^{α}_{k} in terms of the η^{k}_{α}.

$$\phi^{\alpha}_{k} = \eta^{\alpha}_{k} - \left(\frac{g}{2}\right)\eta^{i}_{k}\eta^{\alpha}_{i} + \left(\frac{g}{2}\right)^{2}\eta^{i}_{k}\eta^{j}_{k}\eta^{\alpha}_{j} - \cdots \tag{1.27}$$

Is is important to realise that the η^{k}_{α} do not from a symmetric coefficient scheme. However, as (1.26) shows, in linear approximation, only the symmetric part of $\eta_{\alpha\beta}$ contributes to the metric. Effects from the skew part of the vierbein field can show up only when terms quadratic in η^{k}_{α} are taken into account.

2. The method of the compensating field.

From the discussion of the preceeding chapter,it is obvious that any action principle

$$\int \mathcal{L}(h^{k}_{\alpha}, h^{k}_{\alpha|\beta})\, d^4x = \text{EXTREMUM} \tag{2.1}$$

from which the field equations follow as Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial h^{k}_{\alpha}} - \left(\frac{\partial \mathcal{L}}{\partial h^{k}_{\alpha|\mu}}\right)_{/} = 0 \tag{2.2}$$

cannot be demanded to be invariant under global Lorentz transformations of the coordinates x^{μ} , because local inertial frames in the presence of gravitation are, in general, accelerated with respect to each other. One should, however be able to insist on invariance under local Lorentz transformations characterized by six coordinate dependent parameters $\lambda^{mn}(x) = -\lambda^{nm}(x)$. This invariance requirement acts as a constraint on the possible coupling between the gravitational field and a given source field, as was first shown by Utiyama (1956).

Just as the so-called "minimal coupling" between the electromagnetic potentials A_{α} , and a source field ψ can be derived (London, 1927) by demanding that the effect of any phase transformation with coordinate dependent phase $\lambda(x)$ on the field ψ be compensated by a gauge-transformation $A_{\alpha} \to A_{\alpha} + e^{-1}\lambda(x)$ among the potentials, requiring all derivatives of ψ to occour in the conjuction $\partial_{\alpha}\psi = \psi_{|\alpha} + ieA_{\alpha}\psi$

with a coupling parameter e to be determined by experiment so the "minimal coupling" between the gravitational potentials h^k_α and a source field ψ can be derived from demanding that the relevant action principle be invariant under local Lorentz transformation, requiring all derivatives of ψ to occur in the conjunction

$$\partial_\alpha \psi = \psi_{|\alpha} - \frac{1}{2} B^{mn}_\alpha \Lambda_{mn} \psi \tag{2.3}$$

where Λ_{mn} is the appropriate operator representation of λ^{mn} (Freeman, 1967) acting on the components of ψ, and where the components of the "compensating field" $B^{mn}_\alpha(x) = -B^{nm}_\alpha(x)$ transform according to

$$B^{mn}_\alpha \rightarrow B^{mn}_\alpha + \lambda^m_k B^{kn}_\alpha + \lambda^n_k B^{mk}_\alpha + \lambda^{mn}{}_{|\alpha} \tag{2.4}$$

one can show (Kaempffer, 1965) that the derivatives (2.3) are identical with the components of the covariant derivatives of ψ in the local inertial frame, and obtain the relations

$$f^r_m h^m_{\beta|\alpha} - \Gamma^r_{\beta\alpha} = f^r_m h_{n\beta} B^{mn}_\alpha \tag{2.5}$$

connecting the fields B^{mn}_α with the quantities describing the metric field.

Using the relations (1.4), one can solve (2.5) for the fields B^{mn}_α, and upon substitution of the expression (1.18) for the affinities one obtains (Appendix A.a)

$$B^{mn}_\alpha = -\frac{1}{2}\left(f^{\beta m} f^{rn} h^k_\alpha G^k_{\beta r} + f^{\beta m} G^n_{\beta\alpha} + f^{\beta n} G^m_{\alpha\beta}\right) \tag{2.6}$$

showing nicely that only true gravitational fields (1.6) contribute to the interaction term in (2.3).

In particular, one obtains upon substitution of the tensor representation (Freeman, 1967)

$$\Lambda_{mn;\alpha\rho,}{}^{\epsilon\sigma} = \delta_\alpha{}^\epsilon(\delta_{m\beta}\delta^\sigma{}_n - \delta_{n\beta}\delta^\sigma{}_m) + (\delta_{m\alpha}\delta_n{}^\epsilon - \delta_{n\alpha}\delta_m{}^\epsilon)\delta^\sigma{}_\rho \tag{2.7}$$

into (2.3) for the case of a tensor field $T_{\alpha\beta|\tau}$ the expression (Appendix A.b)

$$\partial_\tau T_{\alpha\beta} = T_{\alpha\beta|\tau} + \tfrac{1}{2}\left[(f^{sn}f^m{}_\tau G^m{}_{\lambda s} + G^n{}_{\lambda\tau})(f^\lambda{}_\beta T_{\alpha n} + f^n{}_\alpha T_{n\beta})\right.$$

$$\left. + f^{sn}(G_{\beta,\alpha s}T_{\alpha n} + G_{\tau\alpha,\tau s}T_{n\beta})\right]. \tag{2.8}$$

By similar procedure, for the case of a vector field with the vector representation

$$\Lambda_{mn;\tau,\beta} = \delta_{m\tau}\delta_{\beta n} - \delta_{m\beta}\delta_{n\tau} \tag{2.7a}$$

one obtain the relation

$$\partial_\alpha V_\tau = V_{\tau|\alpha} + \tfrac{1}{2}(f^\sigma{}_\tau f^\rho{}_\beta h^k{}_\alpha G^k{}_{\sigma\rho} + f^\sigma{}_\tau G_{\beta,\sigma\alpha} + f^\sigma{}_\beta G_{\tau,\alpha\sigma})V_\beta \tag{2.8a}$$

The expression (2.8) should enable one to incorporate the gravitating effect of gravitation into the theory in the following manner.

Suppose one has a Lagrangian

$$L = L(\eta_{\alpha\beta|\tau};\ \eta_{\alpha\beta}) \tag{2.9}$$

yielding the correct linear field equation for weak gravitational fields. If now this field $\eta_{\alpha\beta} \equiv T_{\alpha\beta}$ is itself treated as a source of gravitation, one must apply (2.8) to $\eta_{\alpha\beta}$ and construct according to Utiyama's prescription, the Lagrangian containing the interaction term by writing

$$\mathcal{L} = h L' \tag{2.10}$$

where L' is obtained from L by replacing everywhere $\eta_{\alpha\beta|\gamma}$ by $\partial_\gamma \eta_{\alpha\beta}$. By means of the expressions (1.24) and (1.27), one should then be able, in principle, to construct the Lagrangian to any order in the coupling parameter g. To this end, one needs the expansion (Appendix A.c):

$$h = 1 + \frac{g}{2}\eta_{\sigma\sigma} + \left(\frac{g}{4}\right)^2\left[3\eta_{\sigma\sigma}\eta_{\beta\beta} - 3(\eta_{\sigma\sigma})^2 - 2\eta_{\sigma\beta}\eta_{\beta\sigma}\right] \tag{2.11}$$

and (Appendix A.d)

$$\begin{aligned}\partial_\gamma \eta_{\alpha\beta} = \eta_{\alpha\beta|\gamma} &+ \frac{g}{4}\left[\eta_{\alpha\sigma}(\eta_{\gamma,\beta\sigma} + \eta_{\beta,\gamma\sigma} + \eta_{\sigma,\beta\gamma}) + \eta_{\sigma\beta}(\eta_{\gamma,\alpha\sigma} + \eta_{\alpha,\gamma\sigma} + \eta_{\sigma,\alpha\gamma})\right.\\ &+ \frac{g^2}{8}\left[\eta_{\alpha\sigma}\eta_{\omega\beta}(\eta_{\sigma,\gamma\omega} - \eta_{\gamma,\sigma\omega}) + \eta_{\alpha\sigma}\eta_{\omega\sigma}(\eta_{\beta,\omega\gamma} + \eta_{\gamma,\omega\beta})\right.\\ &+ \eta_{\sigma\beta}\eta_{\omega\alpha}(\eta_{\sigma,\gamma\omega} + \eta_{\gamma,\sigma\omega}) + \eta_{\sigma\beta}\eta_{\omega\sigma}(\eta_{\alpha,\omega\gamma} + \eta_{\gamma,\omega\alpha})\\ &\left.+ \eta_{\omega\gamma}(\eta_{\alpha\sigma}\eta_{\omega,\beta\sigma} + \eta_{\sigma\beta}\eta_{\omega,\alpha\sigma})\right],\end{aligned} \tag{2.12}$$

where $\eta_{\alpha,\beta\gamma} \equiv \eta_{\alpha\beta|\gamma} - \eta_{\alpha\gamma|\beta}$.

which are obtained by subsititution of (1.24) and (1.27) into the expression for h and $\partial_\gamma\eta_{\alpha\beta}$, and collection of the appropriate terms.

3. Construction of field equations.

As is well-known, Maxwell's vacuum field equations are the only relativistically invariant linear equations of second order for a vector field A_α satisfying the condition of gauge invariance. Restriction to linearity and second order of the field equations implies that the Lagrangian must be bilinear in the portentials A_α and their first derivatives. Now with a vector field one can form four linearly independent invariants of this type, namely

$$I_1 = A_\alpha A_\alpha \;;\; I_2 = A_{\alpha|\beta} A_{\alpha|\beta} \;;\; I_3 = A_{\alpha|\alpha} A_{\beta|\beta} \;;\; I_4 = A_{\alpha|\beta} A_{\beta|\alpha} \tag{3.1}$$

Since I_4 differs from I_3 only by a divergence :

$$I_4 = I_3 - [A_\alpha A_{\beta|\alpha} - A_{\alpha|\alpha} A_\beta]_{|\beta} \tag{3.2}$$

its contribution to an action principle will be the same as that of I_3 , and one has thus as most general Lagrangian

$$L = c_1 I_1 + c_2 I_2 + c_3 I_3 \tag{3.3}$$

with arbitrary coefficients c_i , yielding linear field equations of second order:

$$c_1 A_\alpha - c_2 A_{\alpha|\beta\beta} - c_3 A_{\beta|\beta\alpha} = 0 \tag{3.4}$$

Imposition of invariance under gauge transformations $A_\alpha \to A_\alpha + \Lambda_{|\alpha}$

with arbitrary scalar field Λ yields the conditions $c_1 = 0$ and $c_2 = -c_3$. This reduces the field equations (3.4) to Maxwell's equations:

$$A_{\alpha|\beta\beta} - A_{\beta|\beta\alpha} \equiv F_{\alpha\beta|\beta} = 0 \tag{3.5}$$

Reduction to wave equations

$$A_{\alpha|\beta\beta} = 0 \tag{3.6}$$

in the potentials is accompanied by imposition of the tranversality condition

$$A_{\beta|\beta} = 0 \tag{3.7}$$

which eliminates the unphysical longitudinal and timelike polarization modes of the electromagnetic field in the usual fashion (Källén, 1958)

An entirely analagous treatment of a general tensor field $\eta_{\alpha\beta}$ of rank two will now be carried out, with the intention to interpret the resulting linear field equations of second order as the gravitational field equations for weak fields, which are of lowest order in the expansion parameter q introduced in chapter 1.

One can form fourteen linearly independent invariants bilinear in the fields $\eta_{\alpha\beta}$ and their first derivatives, namely:

$$\begin{aligned}
&I_1 = \eta_{\alpha\beta}\,\eta_{\alpha\beta}\ ; \quad I_2 = \eta_{\alpha\beta}\,\eta_{\beta\alpha}\ ; \quad I_3 = \eta_{\alpha\alpha}\,\eta_{\beta\beta} \\
&I_4 = \eta_{\alpha\beta|\gamma}\,\eta_{\alpha\beta|\gamma}\ ; \quad I_5 = \eta_{\alpha\beta|\gamma}\,\eta_{\beta\alpha|\gamma}\ ; \quad I_6 = \eta_{\alpha\alpha|\gamma}\,\eta_{\beta\beta|\gamma} \\
&I_7 = \eta_{\alpha\beta|\gamma}\,\eta_{\alpha\gamma|\beta}\ ; \quad I_8 = \eta_{\alpha\beta|\gamma}\,\eta_{\gamma\alpha|\beta}\ ; \quad I_9 = \eta_{\alpha\beta|\gamma}\,\eta_{\gamma\beta|\alpha} \\
&I_{10} = \eta_{\alpha\alpha|\gamma}\,\eta_{\beta\gamma|\beta}\ ; \quad I_{11} = \eta_{\alpha\beta|\beta}\,\eta_{\alpha\gamma|\gamma}\ ; \quad I_{12} = \eta_{\alpha\beta|\beta}\,\eta_{\gamma\alpha|\gamma} \\
&I_{13} = \eta_{\alpha\beta|\alpha}\,\eta_{\gamma\beta|\gamma}\ ; \quad I_{14} = \eta_{\alpha\alpha|\gamma}\,\eta_{\gamma\beta|\beta}\ .
\end{aligned} \tag{3.8}$$

Since I_{11}, I_{12}, I_{13}, I_{14}, differ from I_7, I_8, I_9, I_{10} respectively only by divergences:

$$\begin{aligned}
I_{11} &= I_7 + (\eta_{\alpha\gamma}\,\eta_{\alpha\beta|\beta} - \eta_{\alpha\beta}\,\eta_{\alpha\gamma|\beta})_{|\gamma} \\
I_{12} &= I_8 + (\eta_{\alpha\gamma}\,\eta_{\beta\alpha|\beta} - \eta_{\alpha\beta}\,\eta_{\gamma\alpha|\beta})_{|\gamma} \\
I_{13} &= I_9 + (\eta_{\gamma\beta}\,\eta_{\alpha\beta|\alpha} - \eta_{\alpha\beta}\,\eta_{\gamma\beta|\alpha})_{|\gamma} \\
I_{14} &= I_{10} + (\eta_{\alpha\alpha}\,\eta_{\gamma\beta|\beta} - \eta_{\alpha\alpha}\,\eta_{\beta\gamma|\beta})_{|\gamma}
\end{aligned} \tag{3.9}$$

they need not be taken into the action principle separately, and the most general Lagrangian leading to linear equation of second order is therefore:

$$L = \sum_{i=1}^{10} c_i\, I_i \tag{3.10}$$

with arbitrary coefficients c_i.

In order to incorporate the isotropy of inertia into the theory, one requires invariance of $h^k{}_\alpha$ under the transformation (1.8), that is :

$$h^k{}_\alpha = R^k{}_i\, h^i{}_\alpha \tag{3.11}$$

In linear approximation,

$$h^k{}_\alpha = \delta^k{}_\alpha + \frac{g}{2}\eta^k{}_\alpha \tag{3.12}$$

the coefficients of $R^k{}_i$ must, so as not to violate the linear approximation, also be expanded as

$$R^k{}_i = \delta^k{}_i + \epsilon^k{}_i(x) \tag{3.13}$$

where, as is well-known, $\epsilon^k{}_i$ is skew.
Substituting (3.12) and (3.13) into (3.11), and dropping all terms quadratic in small quantities, one obtains

$$\eta'_{k\alpha} = \eta_{k\alpha} + \epsilon_{k\alpha} \tag{3.16}$$

as the transformation law for the quantities of $\eta_{\alpha\beta}$ under reorientation of the local inertia frame. As is permissible, if one uses instead of I_{10} ,

$$I'_{10} = \frac{1}{2}(I_{10} + I_{14}) = \frac{1}{2}\eta_{\alpha\alpha|\tau}(\eta_{\beta\tau|\beta} + \eta_{\tau\beta|\beta}) \tag{3.17}$$

then the transformation (3.16) on the Lagrangian obviously leaves I_3 , I_6 , I_{10} ,invariant. The remainder gives

$$c_1 = c_2 \quad ; \quad c_4 = c_5 \quad ; \quad 2c_7 = c_8 = 2c_9 \tag{3.18}$$

These have the effect of letting $\eta_{\alpha\beta}$ appear in L only in the symmetric combination

$$\tau_{\alpha\beta} \equiv \eta_{\alpha\beta} + \eta_{\beta\alpha} \tag{3.19}$$

and thus allow one to restrict consideration to

$$L = A_1 \Upsilon_{\alpha\beta} \Upsilon_{\alpha\beta} + A_3 \Upsilon_{\alpha\alpha} \Upsilon_{\beta\beta} + A_4 \Upsilon_{\alpha\beta|\tau} \Upsilon_{\alpha\beta|\tau}$$
$$+ A_6 \Upsilon_{\alpha\alpha|\tau} \Upsilon_{\beta\beta|\tau} + A_7 \Upsilon_{\alpha\beta|\tau} \Upsilon_{\alpha\tau|\beta}$$
$$+ A_{10} \Upsilon_{\alpha\alpha|\tau} \Upsilon_{\beta\tau|\beta} \tag{3.20}$$

where A_i are arbitrary constants. This Lagrangian is identical with the one considered by Wyss (1965).
Upon variation of (3.20) with respect to $\Upsilon_{\alpha\beta}$, the field equations follow as

$$2A_1 \Upsilon_{\alpha\beta} + 2A_3 \Upsilon_{\alpha\alpha} - 2A_4 \Upsilon_{\alpha\beta|\tau\tau}$$
$$- 2A_6 \Upsilon_{\alpha\alpha|\tau\tau} - 2A_7 \Upsilon_{\alpha\tau|\beta\tau}$$
$$- A_{10} (\Upsilon_{\alpha\alpha|\tau\tau} + \Upsilon_{\alpha\beta|\alpha\tau}) = 0 \tag{3.21}$$

Imposition of invariance under gauge transformations

$$\Upsilon_{\alpha\beta} \longrightarrow \Upsilon_{\alpha\beta} + \Lambda_{\alpha|\beta} + \Lambda_{\beta|\alpha} \tag{3.22}$$

for the purpose of eliminating pseudo-gravitational fields requires that upon substitution of (3.22) into (3.21), the coefficients of $\Lambda_{\alpha|\beta}$, $\Lambda_{\alpha|\beta\tau\tau}$, $\Lambda_{\tau|\alpha\beta\tau}$ vanish separately, and this gives the conditions

$$\begin{aligned} &A_1 = A_3 = 0 \quad ; \quad 2A_4 + A_7 = 0 \\ &2A_6 + A_{10} = 0 \quad ; \quad A_7 + A_{10} = 0 \end{aligned} \tag{3.23}$$

Letting $A_4 = 1$, one has

$$A_7 = -2 \;;\quad A_6 = -1 \;;\quad A_{10} = 2 \tag{3.24}$$

Thus, the Lagrangian reduces to

$$L = \eta_{\alpha\beta|\tau}(\eta_{\alpha\beta|\tau} - 2\eta_{\alpha\tau|\beta}) - \eta_{\alpha\alpha|\tau}(\eta_{\beta\beta|\tau} - 2\eta_{\beta\tau|\beta}) \tag{3.25}$$

where $\eta_{\alpha\beta} = \eta_{\beta\alpha}$.

The corresponding reduced field equations now do not contain any arbitrary parameters and are expressible entirely in terms of the true gravitational field strengths:

$$\eta_{\alpha\beta|\tau\tau} - 2\eta_{\alpha\tau|\beta\tau} + \eta_{\alpha\beta|\alpha\tau} \equiv G_{\alpha,\beta\tau|\tau} + G_{\tau,\tau\alpha|\beta} = 0 \tag{3.26}$$

where one defines $\eta_{\alpha\beta}$ as gravitational potentials for weak fields due to the fact that

$$h_{\alpha\beta} = \delta_{\alpha\beta} + \frac{g}{2}\eta_{\alpha\beta} \tag{3.27}$$

The analogy to the corresponding Maxwell's equations (3.5) is thus brought into evidence and further justifies the usage of the terms "Gravitational potentials" for $\eta_{\alpha\beta}$ and "gravitational field strengths" for $G_{\alpha,\beta\tau}$.

One also notes that contraction of the field strengths with respect to the indices α and β immediately leads to

$$G_{\alpha,\alpha\tau|\tau} = 0 \tag{3.28}$$

4. Special solutions to the field equations.

The field equations (3.26) agree in linear approximation with those of Einstein. For example, if one takes as gravitational potentials

$$\eta_{\alpha\beta} = \begin{cases} \delta_{\alpha\beta}\phi & \alpha, \beta = 1, 2, 3 \\ \mu & \alpha = \beta = 4 \end{cases} \tag{4.1}$$

where ϕ and μ are scalar potentials, then substitution of (4.1) into the Lagrangian (3.25) and the field equations (3.26) respectively give

$$\mathcal{L} = -2\phi_{|r}\phi_{|r} - 4\mu_{|r}\mu_{|r} \tag{4.2}$$

$$\phi_{|rr} + \mu_{|rr} = 0 \tag{4.3}$$

$$\phi_{|rr} = 0 \tag{4.4}$$

Specifying to spherical symmetry, the fact that the field equations obtained by the "vierbein" field approach agree, at least in linear approximation, with Einstein's field equations enables one to apply Birkhoff's theorem, which states that all spherically symmetric gravitational fields are static (see, for example, Tolman 1958), to this case, that is

$$\phi_{|4} = 0 \tag{4.5}$$

In terms of spherical coordinates:

$$\phi_{|rr} = \nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) \tag{4.6}$$

$$\phi_{|r} = \nabla \phi = \frac{d\phi}{dr} \tag{4.7}$$

The field equations (4.3) and (4.4) reduce to a pair of non-linear ordinary differential equations that can be solved readily, giving solutions:

$$\phi = A + \frac{C}{r} \tag{4.8}$$

$$\mu = B + \frac{C}{r} \tag{4.9}$$

where A, B, C are constants.

Clearly, far away from the body producing the field,the gravitational field must be Newtonian, that is

$$\phi \rightarrow 1 \; ; \; \mu \rightarrow 1 \quad \text{as} \quad r \rightarrow \infty \tag{4.10}$$

Thus, (4.8) and (4.9) become

$$\phi = 1 + \frac{C}{r} \tag{4.11}$$

$$\mu = 1 - \frac{C}{r} \tag{4.12}$$

These solutions are clearly identical in Newtonian approximation with the solutions obtained by Schwarschild (1916) using Einstein's field equations under the same conditions.

Now, in accordance with the prescription developed in chapter 2, the gravitating effect of gravitation can be taken into account by replacing in the Lagrangian (3.25) the partial derivatives $\eta_{\alpha\beta|r}$ by $\partial_r \eta_{\alpha\beta}$, and forming

$$\mathcal{L} = h L' \tag{4.13}$$

with the result, up to order g ,

$$\mathcal{L} = (1+\tfrac{g}{2}\eta_{\sigma\sigma})L + \tfrac{g}{2}\{(\eta_{\alpha\beta|\tau} - 2\eta_{\alpha\tau|\beta})[\eta_{\alpha\sigma}(\eta_{\tau,\beta\sigma} + \eta_{\beta,\tau\sigma} + \eta_{\sigma,\beta\tau})$$
$$+ \eta_{\sigma\beta}(\eta_{\tau,\alpha\sigma} + \eta_{\alpha,\tau\sigma} + \eta_{\sigma,\alpha\tau})] + \eta_{\alpha\alpha|\tau}[\eta_{\beta\sigma}(\eta_{\beta,\tau\sigma}$$
$$+ \eta_{\tau,\beta\sigma} + \eta_{\sigma,\tau\beta}) + 2\eta_{\sigma\tau}\eta_{\beta,\beta\sigma}]\} \tag{4.14}$$

where the $\eta_{\alpha\beta|\tau}$ in L are symmetric.

The corresponding field equations are then

$$\eta_{\alpha,\beta\tau|\tau} - \eta_{\tau,\tau\alpha|\beta} = \{\delta_{\epsilon\rho}[L - \eta_{\alpha\sigma|\omega}(\eta_{\alpha,\omega\sigma} + \eta_{\omega,\alpha\sigma} + \eta_{\sigma,\omega\alpha}) - 2\eta_{\sigma\omega|\omega}\eta_{\alpha,\alpha\sigma}$$
$$- 2\eta_{\omega\sigma|\omega}\eta_{\alpha\alpha|\sigma}] + 2\eta_{\epsilon\sigma|\omega}(\eta_{\omega,\sigma\rho} + 2\eta_{\sigma,\omega\rho} - \eta_{\omega\rho|\sigma}) + 2\eta_{\sigma\rho|\omega}(\eta_{\omega,\sigma\epsilon} + \eta_{\epsilon,\sigma\omega}$$
$$+ 3\eta_{\sigma,\omega\epsilon}) - 2\eta_{\epsilon\omega|\sigma}(\eta_{\omega,\sigma\rho} + \eta_{\sigma,\omega\rho} + \eta_{\rho,\sigma\omega}) - 2\eta_{\sigma\omega|\rho}(\eta_{\sigma,\omega\epsilon} + \eta_{\omega,\sigma\epsilon}$$
$$+ \eta_{\epsilon,\sigma\omega}) + \eta_{\sigma\sigma|\rho}(2\eta_{\omega,\omega\epsilon} - \eta_{\omega\epsilon|\omega} - \eta_{\epsilon\omega|\omega}) + \eta_{\sigma\sigma|\omega}(\eta_{\epsilon,\omega\rho} + \eta_{\omega,\epsilon\rho}$$
$$+ \eta_{\rho,\omega\epsilon} - \eta_{\rho\epsilon|\omega} - \eta_{\epsilon\rho|\omega} - 2\eta_{\rho\omega|\epsilon}) - \eta_{\sigma\sigma|\epsilon}(\eta_{\rho\omega|\omega} + \eta_{\omega\rho|\omega})\}\tfrac{g}{2}$$
$$= (G_{\alpha,\beta\tau|\tau} - G_{\tau,\tau\alpha|\beta})\tfrac{2}{g} \tag{4.15}$$

The Lagrangian (4.14) is not reorientation invariant. Indeed, if one performs the transformation (3.16), then (4.14) becomes

$$\mathcal{L}(\eta_{\alpha\beta} + \epsilon_{\alpha\beta}, \eta_{\alpha\beta|\tau} + \epsilon_{\alpha\beta|\tau})$$
$$= (1+\tfrac{g}{2}\eta_{\sigma\sigma})L + \tfrac{g}{2}\{(\eta_{\alpha\beta|\tau} - 2\eta_{\alpha\tau|\beta})[\eta_{\alpha\sigma}(\eta_{\tau,\beta\sigma} + \eta_{\beta,\tau\sigma}$$
$$+ \eta_{\sigma,\beta\tau}) + \eta_{\sigma\beta}(\eta_{\tau,\alpha\sigma} + \eta_{\alpha,\tau\sigma} + \eta_{\sigma,\alpha\tau})] + \eta_{\alpha\alpha|\tau}[\eta_{\beta\sigma}(\eta_{\beta,\tau\sigma}$$
$$+ \eta_{\tau,\beta\sigma} + \eta_{\sigma,\tau\beta})] + 2\eta_{\sigma\tau}\eta_{\beta,\beta\sigma}\eta_{\alpha\alpha|\tau}\}$$
$$+ g[(\epsilon_{\alpha\beta|\tau} - 2\epsilon_{\alpha\tau|\beta})(\epsilon_{\alpha\sigma}\epsilon_{\sigma\beta|\tau} + \epsilon_{\sigma\beta}\epsilon_{\sigma\alpha|\tau})] \tag{4.16}$$

Obviously, the Lagrangian (4.14) cannot be made reorientation invariant since no arbitrary constant appear in the expression (4.16). However one can always find a gauge so as to ensure the reorientation invariance of the field equations (4.15). In other words, one has to sacrifice the gauge invariance so that reorientation invariance of the field equations (4.15) may be preserved. In this respect, one notes that Einstein's field equations are also gauge invariant only in linear approximation.

As is discussed in chapter 2, applying the idea of the compensating field to the weak field potentials $\eta_{\alpha\beta}$ should in principle enable one to construct the Lagrangian to any order of accuracy in the parameter g, thus providing a method of obtaining the field equations by successive approximations.

Consider as an example the case when $\eta_{\alpha\beta}$ is diagonal, that is

$$\eta_{\alpha\beta} = \delta_{\alpha\beta}\,\phi \tag{4.17}$$

where ϕ is a scalar potential.

Substitution of (4.16) into (4.14) and (4.15) respectively give

$$\mathcal{L} = -\phi_{|\mu}\,\phi_{|\mu}\,(1 + 2g\phi) \tag{4.18}$$

$$\phi_{|rr} + g\,\phi_{|r}\,\phi_{|r} = 0 \tag{4.19}$$

By similar procedure as that for (4.1), one obtains, as solution for the scalar potential

$$\phi = 1 - \frac{1}{g} \ln \frac{r - gA}{r} \tag{4.20}$$

where A is a constant.

Considering next the case when

$$\eta_{\alpha\beta} = \begin{cases} \delta_{\alpha\beta}\phi & \alpha, \beta = 1, 2, 3 \\ \mu & \alpha = \beta = 4 \end{cases} \tag{4.21}$$

where ϕ and μ are scalar potentials.

One finds that the field equations (4.15) become

$$4\phi_{|rr} + 4\mu_{|rr} = g\left(-3\phi_{|r}\phi_{|r} - 2\phi_{|r}\mu_{|r} - 4\mu_{|r}\mu_{|r}\right) \tag{4.22}$$

$$4\phi_{|rr} = g\left(-2\phi_{|r}\phi_{|r} - 2\phi_{|r}\mu_{|r} + \mu_{|r}\mu_{|r}\right) \tag{4.23}$$

which clearly give (4.19) when $\phi = \mu$.

Substitution of the solutions (4.11) and (4.12) into the quadratic terms of (4.22) and (4.23) then yield

$$\phi = 1 + \frac{A}{r} + \frac{g B^2}{8 r^2} \tag{4.24}$$

$$\mu = 1 - \frac{A}{r} - \frac{3 g B^2}{4 r^2} \tag{4.25}$$

where A, B, are constants.

These solutions, however, differ in order g with Schwarschild's solutions:

$$\phi_s = 1 + \frac{A}{r} + \frac{A^2}{2r} + \cdots \tag{4.26}$$

$$\mu_s = 1 - \frac{A}{r} + \frac{3A^2}{4r^2} + \cdots \tag{4.27}$$

This suggests that the Lagrangian (4.14) be further modified by repeating the method of the compensating field. The mathematical complications are, however, beyond the scope of this thesis.

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Appendix —— Derivation of some formulae in chapter 2.

a, By (2.5), one has

$$f^{\tau}_{m} h^{m}_{\beta|\alpha} - \Gamma^{\tau}_{\beta|\alpha} = f^{\tau}_{m} h_{n\beta} B^{mn}_{\alpha} \tag{A.1}$$

Since $f^{\alpha}_{k} h^{k}_{\beta} = \delta^{\alpha}_{\beta}$; $h^{i}_{\alpha} f^{\alpha}_{k} = \delta^{i}_{k}$

hence, multiplying (A.1) by h^{i}_{τ}, one obtains

$$h^{i}_{\beta|\alpha} - h^{i}_{\tau} \Gamma^{\tau}_{\beta\alpha} = h_{n\beta} B^{in}_{\alpha} \tag{A.2}$$

Multiplying (A.2) by $f^{\beta k}$, one has

$$B^{mn}_{\alpha} = f^{\beta n} h^{m}_{\beta|\alpha} - f^{\beta n} h^{m}_{\tau} \Gamma^{\tau}_{\beta\alpha} \tag{A.3}$$

Also,

$$\Gamma^{\tau}_{\beta\alpha} = \frac{1}{2} f^{\tau}_{k} \left[f^{\sigma}_{k} h^{i}_{\beta} G^{i}_{\sigma\alpha} + f^{\sigma}_{k} h^{i}_{\alpha} G^{i}_{\sigma\beta} + h^{k}_{\beta|\alpha} + h^{k}_{\alpha|\beta} \right] \tag{A.4}$$

Substituting (A.4) into (A.3)

$$B^{mn}_{\alpha} = f^{\beta n} h^{m}_{\beta|\alpha} - \frac{1}{2} f^{\beta n} h^{m}_{\tau} f^{\tau}_{k} \left[f^{\sigma}_{k} h^{i}_{\beta} G^{i}_{\sigma\alpha} + f^{\sigma}_{k} h^{i}_{\alpha} G^{i}_{\sigma\beta} + h^{k}_{\beta|\alpha} + h^{k}_{\alpha|\beta} \right]$$

After slight simplification, one obtains

$$B^{mn}_{\alpha} = -\frac{1}{2} \left[f^{\beta m} f^{\tau n} h^{k}_{\alpha} G^{k}_{\beta\tau} + f^{\beta m} G^{n}_{\beta\alpha} + f^{\beta n} G^{k}_{\alpha\beta} \right]$$

which is (2.6)

b, Letting ψ be $T_{\alpha\beta}$, one has, by (2.3)

$$\partial_{\tau} T_{\alpha\beta} = T_{\alpha\beta|\tau} - \frac{1}{2} B^{mn}_{\tau} \Lambda_{mn;\alpha\beta}{}^{\epsilon\varrho} T_{\epsilon\varrho} \tag{A.5}$$

where

$$B^{mn}_{r} = -\frac{1}{2}\left(f^{\omega m} f^{\sigma n} h^{k}_{r} G^{k}_{\omega\sigma} + f^{\omega m} G^{n}_{\omega r} + f^{\omega n} G^{m}_{r\omega}\right) \qquad (A.6)$$

and

$$\Lambda_{mn;\alpha\beta}{}^{\epsilon\varrho} = \delta^{\epsilon}_{\alpha}\left(\delta_{m\beta}\delta^{\varrho}_{n} - \delta_{n\beta}\delta^{\varrho}_{m}\right) + \delta^{\varrho}_{\beta}\left(\delta_{m\alpha}\delta^{\epsilon}_{n} - \delta_{n\alpha}\delta^{\epsilon}_{m}\right) \qquad (A.7)$$

Multiplying the first term of (A.6) by (A.7), and $T_{\epsilon\varrho}$ one has

$$f^{\omega m} f^{\sigma n} h^{k}_{r} G^{k}_{\omega\sigma} T_{\alpha\varrho} + f^{\omega\varrho} f^{\sigma}_{\beta} h^{k}_{r} G^{k}_{\sigma\omega} T_{\alpha\varrho}$$

$$= f^{\omega}_{\beta} f^{\sigma\varrho} h^{k}_{r} G^{k}_{\omega\sigma} T_{\alpha\varrho} + f^{\omega\varrho} f^{\sigma}_{\beta} h^{k}_{r} G^{k}_{\sigma\omega} T_{\alpha\varrho}$$

$$+ f^{\omega\alpha} f^{\sigma\epsilon} h^{k}_{r} G^{k}_{\omega\sigma} T_{\epsilon\beta} + f^{\omega\epsilon} f^{\sigma}_{\alpha} h^{k}_{r} G^{k}_{\sigma\omega} T_{\epsilon\beta}$$

$$= \left(f^{r}_{\beta} f^{sn} h^{m}_{r} G^{m}_{rs} T_{\alpha n} + f^{sn} f^{r}_{\beta} h^{m}_{r} G^{m}_{rs} T_{\alpha n}\right)$$

$$+ \left(f^{sn} h^{m}_{r} G^{m}_{rs} f^{r}_{\alpha} T_{n\beta} + f^{sn} h^{m}_{r} G^{m}_{rs} f^{r}_{\alpha} T_{n\beta}\right)$$

$$= 2\left(f^{sn} h^{m}_{r} G^{m}_{rs}\right)\left(f^{r}_{\beta} T_{\alpha n} + f^{r}_{\alpha} T_{n\beta}\right) \qquad (A.8)$$

From the second and third terms of (A.6), one similarly obtains

$$f^{\omega m} G^{n}_{\omega r} \Lambda_{mn;\alpha\beta}{}^{\epsilon\varrho} T_{\epsilon\varrho} = f^{\omega\epsilon} G_{\alpha,r\omega} T_{\epsilon\beta} +$$

$$+ f^{\omega}_{\beta} G^{\varrho}_{\omega r} T_{\alpha\varrho} + f^{\omega\varrho} G_{\beta,r\omega} T_{\alpha\varrho} + f^{\omega}_{\alpha} G^{\epsilon}_{\omega r} T_{\epsilon\beta} \qquad (A.9)$$

and

$$f^{\omega n} G^{m}_{r\omega} \Lambda_{mn;\alpha\beta}{}^{\epsilon\varrho} T_{\epsilon\varrho} = f^{\omega}_{\alpha} G^{\epsilon}_{\omega r} T_{\epsilon\beta} +$$

$$+ f^{\omega\varrho} G_{\beta,r\omega} T_{\alpha\varrho} + f^{\omega}_{\beta} G^{\varrho}_{\omega r} T_{\epsilon\beta} + f^{\omega\epsilon} G_{\alpha,r\omega} T_{\epsilon\beta} \qquad (A.10)$$

Since (A.9) and (A.10) are equal, thus one has, from (A.5):

$$\partial_r T_{\alpha\beta} = T_{\alpha\beta|r} + \frac{1}{2}\left[(f^{sn} h^m_r G^m_{ns})(f^n_\beta T_{\alpha n} + f^n_\alpha T_{n\beta}) \right.$$
$$\left. + f^\omega_\beta G^\rho_{\omega r} T_{\alpha\rho} + f^{\omega\rho} G_{\beta,r\omega} T_{\alpha\rho} + f^\omega_\alpha G^\epsilon_{\omega r} T_{\epsilon\beta} + f^{\omega\epsilon} G_{\alpha,r\omega} T_{\epsilon\beta}\right]$$

By renaming the running scripts and after slight rearrangements, one obtains:

$$\partial_r T_{\alpha\beta} = T_{\alpha\beta|r} + \frac{1}{2}\left[(f^{sn} h^m_r G^m_{ns} + G^n_{nr})(f^n_\beta T_{\alpha n} + f^n_\alpha T_{n\beta}) + f^{sn}(G_{\beta,rs} T_{\alpha n} + G_{\alpha,rs} T_{n\beta})\right]$$

which is (2.8)

c, Since

$$h \equiv \det |h^k_\alpha| = \begin{vmatrix} h^1_1 & h^1_2 & \cdots & h^1_n \\ \vdots & & & \vdots \\ h^n_1 & \cdots & \cdots & h^n_n \end{vmatrix}$$

Consider $n=3$, one has

$$h = h^1_1(h^2_2 h^3_3 - h^2_3 h^3_2) - h^1_2(h^2_1 h^3_3 - h^2_3 h^3_1) + h^1_3(h^2_1 h^3_2 - h^2_2 h^3_1)$$

But $h^k_\alpha = \delta^k_\alpha + \frac{g}{2}\eta^k_\alpha$,

hence expanding h up to order g^2, one obtains

$$h = (\delta^1_1 + \tfrac{g}{2}\eta_{11})\left[(\delta^2_2 + \tfrac{g}{2}\eta^2_2)(\delta^3_3 + \tfrac{g}{2}\eta^3_3) - (\delta^2_3 + \tfrac{g}{2}\eta^2_3)(\delta^3_2 + \tfrac{g}{2}\eta^3_2)\right] - \cdots$$
$$= 1 + \tfrac{g}{2}(\eta^1_1 + \eta^2_2 + \eta^3_3) + \tfrac{g^2}{2^2}(\eta^1_1\eta^2_2 + \eta^2_2\eta^3_3 + \eta^3_3\eta^1_1)$$
$$- (\tfrac{g}{2})^2(\eta^1_2\eta^2_1 + \eta^1_3\eta^3_1 + \eta^2_3\eta^3_2) - \cdots$$

One can generalise this expansion to the case for any integer n , giving

$$h = 1 + \frac{g}{2}\eta^\sigma_\sigma + (\frac{g}{2})^2(A-B)^2 \qquad \text{(A.11)}$$

where

$$q^{\sigma}_{\sigma} = \eta^{1}_{1} + \cdots + \eta^{n}_{n}$$

$$\begin{cases} A = \eta^{1}_{1}\eta^{2}_{2} + \cdots + \eta^{1}_{1}\eta^{n}_{n} + \eta^{2}_{2}\eta^{3}_{3} + \cdots + \eta^{2}_{2}\eta^{n}_{n} + \cdots \\ B = \eta^{1}_{2}\eta^{2}_{1} + \cdots + \eta^{1}_{n}\eta^{n}_{1} + \eta^{2}_{3}\eta^{3}_{2} + \cdots + \eta^{2}_{n}\eta^{n}_{2} + \cdots \end{cases}$$

Now, since

$$\eta^{\sigma}_{\sigma}\eta^{\beta}_{\beta} = 2A + (\eta^{\sigma}_{\sigma})^2 \tag{A.12a}$$

$$\eta^{\sigma}_{\beta}\eta^{\beta}_{\sigma} = A + 2B \tag{A.12b}$$

thus solving for A and B in (A.12a), (A.12b) and substituting the values for A, B into (A.11), one obtains

$$h = 1 + \frac{q}{2}\eta^{\sigma}_{\sigma} + \left(\frac{q}{4}\right)^2\left[3\eta^{\sigma}_{\sigma}\eta^{\beta}_{\beta} - 3(\eta^{\sigma}_{\sigma})^2 - 2\eta^{\sigma}_{\beta}\eta^{\beta}_{\sigma}\right]$$

One notices that the running scripts of the η's can all be made subscripts by multiplication with the appropriate δ's . Thus

$$h = 1 + \frac{q}{2}\eta_{\sigma\sigma} + \left(\frac{q}{4}\right)^2\left[3\eta_{\sigma\sigma}\eta_{\beta\beta} - 3(\eta_{\sigma\sigma})^2 - 2\eta_{\sigma\beta}\eta_{\beta\sigma}\right]$$

This completes the derivation for the expression (2.11).

d, To obtain an expression for $\partial_{\tau}\eta_{\alpha\beta}$ up to order q^2 , one first expands B^{mn}_{τ} in terms of the η's . From (1.24) and (1.27),

$$h^{k}_{\alpha} = \delta^{k}_{\alpha} + \left(\frac{q}{2}\right)\eta^{k}_{\alpha} \tag{A.13}$$

$$f^{\alpha}_{k} = \delta^{\alpha}_{k} + \left(\frac{g}{2}\right)\phi^{\alpha}_{k} = \delta^{\alpha}_{k} - \frac{g}{2}\left[\eta^{\alpha}_{k} - \left(\frac{g}{2}\right)\eta^{i}_{k}\eta^{\alpha}_{i} + \cdots\right] \qquad \text{(A.14)}$$

one notes that $G_{\alpha,\beta\tau} = \left(\frac{g}{2}\right)\eta_{\alpha,\beta\tau}$.

Since

$$B^{mn}_{\tau} = -\frac{1}{2}\left(f^{\omega m}f^{\epsilon n}h^{k}_{\tau}G^{k}_{\omega\epsilon} + f^{\omega m}G^{n}_{\omega\tau} + f^{\omega n}G^{m}_{\tau\omega}\right)$$

and

$$f^{\omega m}f^{\epsilon n}h^{k}_{\tau}G^{k}_{\omega\epsilon} = \frac{g}{2}\left(\delta_{\omega m}\delta_{\epsilon n}\delta_{k\tau}\eta_{k,\omega\epsilon}\right)$$

$$+\left(\frac{g}{2}\right)^{2}\left(-\delta_{\omega m}\delta_{k\tau}\eta_{k,\omega\epsilon}\eta_{\epsilon n} + \delta_{\omega m}\delta_{\epsilon n}\eta_{k\tau}\eta_{k,\omega\epsilon} - \eta_{\omega m}\eta_{k,\omega\epsilon}\delta_{k\tau}\delta_{\epsilon n}\right)$$

$$f^{\omega m}G^{n}_{\tau\omega} = \frac{g}{2}\left(\delta^{\omega m}\eta^{n}_{\tau\omega}\right) - \left(\frac{g}{2}\right)^{2}\left(\eta^{\omega m}\eta^{n}_{\tau\omega}\right)$$

$$f^{\omega n}G^{m}_{\tau\omega} = \frac{g}{2}\left(\delta^{\omega n}\eta^{m}_{\tau\omega}\right) - \left(\frac{g}{2}\right)^{2}\left(\eta^{\omega n}\eta^{m}_{\tau\omega}\right)$$

hence

$$B^{mn}_{\tau} = -\frac{1}{2}\left[\frac{g}{2}\left(\eta_{\tau,mn} + \eta_{m,\tau n} + \eta_{n,m\tau}\right) + \left(\frac{g}{2}\right)^{2}\left(\eta_{\tau,\epsilon m}\eta_{\epsilon n}\right.\right.$$

$$\left.\left. + \eta_{k,mn}\eta_{k\tau} + \eta_{\tau,n\omega}\eta_{\omega m} + \eta_{n,\tau\omega}\eta_{\omega m} + \eta_{m,\omega\tau}\eta_{\omega n}\right)\right] \qquad \text{(A.15)}$$

Now

$$\partial_{\tau}\eta_{\alpha\beta} = \eta_{\alpha\beta|\tau} - \frac{1}{2}B^{mn}_{\tau}\Lambda_{mn;\alpha\beta}{}^{\sigma\rho}\eta_{\sigma\rho} \qquad \text{(A.16)}$$

where

$$\Lambda_{mn;\alpha\beta}{}^{\sigma\rho} = \delta^{\sigma}_{\alpha}\left(\delta_{m\beta}\delta^{\rho}_{n} - \delta_{n\beta}\delta^{\rho}_{m}\right) + \delta^{\rho}_{\beta}\left(\delta_{m\alpha}\delta^{\sigma}_{n} + \delta_{n\alpha}\delta^{\sigma}_{m}\right) \qquad \text{(A.17)}$$

Thus

$$(\eta_{r;mn} + \eta_{n,mr} + \eta_{m,rn})\Lambda_{mn;\alpha\beta}{}^{\sigma\rho}\eta_{\sigma\rho}$$

$$= \eta_{\alpha n}\eta_{r;\beta n} + \eta_{\alpha m}\eta_{r;\beta m} + \eta_{n\beta}\eta_{r;\alpha n} + \eta_{\alpha n}\eta_{n,\beta r} + \eta_{\alpha m}\eta_{\beta,rm}$$

$$+ \eta_{m\beta}\eta_{r;\alpha m} + \eta_{n\beta}\eta_{n,\alpha r} + \eta_{m\beta}\eta_{\alpha,rm} + \eta_{\alpha n}\eta_{n,\beta r} + \eta_{\alpha n}\eta_{\beta,rn}$$

$$+ \eta_{n\beta}\eta_{n,\alpha r} + \eta_{n\beta}\eta_{\alpha,rn}$$

$$= 2\left[\eta_{\alpha n}(\eta_{r;\beta n} + \eta_{n,\beta r} + \eta_{\beta,rn}) + \eta_{n\beta}(\eta_{r;\alpha n} + \eta_{n,\alpha r} + \eta_{\alpha,rn})\right] \quad \text{(A.18)}$$

Similarly

$$(\eta_{r;\epsilon m}\eta_{\epsilon n} + \eta_{k,mn}\eta_{kr} + \eta_{r;n\omega}\eta_{\omega m} + \eta_{n,r\omega}\eta_{\omega m} + \eta_{m,\omega r}\eta_{\omega n})\Lambda_{mn;\alpha\beta}{}^{\sigma\rho}\eta_{\sigma\rho}$$

$$= 2(\eta_{\alpha n}\eta_{\omega n}\eta_{r;\omega\beta} + \eta_{\alpha n}\eta_{\omega\beta}\eta_{r;n\omega} + \eta_{n\beta}\eta_{\omega n}\eta_{r;\omega\alpha} + \eta_{n\beta}\eta_{\omega\alpha}\eta_{r;n\omega}$$

$$+ \eta_{\alpha n}\eta_{\omega r}\eta_{\omega,\beta n} + \eta_{n\beta}\eta_{\omega r}\eta_{\omega,\alpha n} + \eta_{\alpha n}\eta_{\omega\beta}\eta_{n,r\omega} + \eta_{\alpha n}\eta_{\omega n}\eta_{\beta,\omega r}$$

$$+ \eta_{n\beta}\eta_{\omega\alpha}\eta_{n,r\omega} + \eta_{n\beta}\eta_{\omega n}\eta_{\alpha,\omega r})$$

$$= 2\left[\eta_{\alpha n}\eta_{\omega n}(\eta_{r;\omega\beta} + \eta_{\beta,\omega r}) + \eta_{\alpha n}\eta_{\omega\beta}(\eta_{r;n\omega} + \eta_{n,r\omega}) + \eta_{n\beta}\eta_{\omega n}(\eta_{r;\omega\alpha} + \eta_{\alpha,\omega r})\right.$$

$$\left. + \eta_{n\beta}\eta_{\omega\alpha}(\eta_{r;n\omega} + \eta_{n,r\omega}) + \eta_{\omega r}(\eta_{\alpha n}\eta_{\omega,\beta n} + \eta_{\beta n}\eta_{\omega,\alpha n})\right] \quad \text{(A.19)}$$

Expressions (A.16), (A.18), and (A.19) immediately give the expansion (2.12).