CALCULATED CROSS-SECTIONS OF PION PRODUCTION
BY 450-MEV PROTONS ON VARIOUS NUCLEI

by

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We construct a model to explain the production of pions in the bombardment by protons of various nuclei and we use the model to calculate relative cross-sections for the process. The model assumes that the incident proton interacts with the target nucleons individually and that the proton-nucleon cross-section can be used as a free parameter. The model accounts for many important nuclear effects, some for the first time in explaining the A-dependence of the pion yield. The effects included are those due to proton and pion absorption, to the background nuclear potentials and to the struck-nucleon momentum and density distributions. We compute differential cross-sections in several special cases and compare them with experimental data at 450 MeV. Agreement is only moderate, but it is as good as any previously obtained and, unlike the earlier results, it does not depend on the assumption of an absorbing neutron blanket. Our agreement depends instead on the use of a modern nuclear radius and a reasonable treatment of pion absorption. In this respect our results confirm what earlier workers had assumed, that absorption is the dominant factor controlling the proton-nucleus production of pions. Also important is the proton-nucleon production rate, a reasonable value of which we assume. Potential effects are important because the basic production rate and pion absorption are both very energy dependent. The effects of struck-nucleon momentum and density distribution, as we calculate them, are small at the energy considered.
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CHAPTER I - INTRODUCTION

1.1 Pion production and the TRIUMF project

It is now possible to produce pions in large quantity. The most important production process brings a beam of protons from an accelerator into collision with other nucleons which may be free (a hydrogen target) or bound together as a nucleus (in a carbon target for example). The beam energy at which pion production can begin depends strongly on the kind of target being used, but the energies at which pion production is at all efficient are generally in excess of 350 MeV. The recent feasibility of high beam currents at these energies of efficient production has led to the proposed construction of meson "factories" able to turn out pions and their decay products at an unprecedented rate. It was one of these proposals, namely the TRIUMF proposal\(^1\), which stimulated the present investigation into the pion production process.

The heart of the TRIUMF project is a sector-focussed cyclotron designed to accelerate \(H^-\) ions rather than protons. A continuous proton beam is extracted by the simple and highly efficient process of mechanically stripping the two electrons from each \(H^-\) ion as it gains a predetermined orbit. In addition to providing an intense beam of relatively high energy protons the design features permit easy energy variability. This variability (though of prime interest in proton experiments) along with the freedom of choice of a pion-producing target
provides variety in the nature of the pion yield. A way to maximize the pion yield, important to many experiments, is also provided. Just how this is possible we soon make clear.

To select an upper limit to put on the proton beam energy the designers of TRIUMF balanced the two important factors of machine cost, which favours low energy, and the pion production rate, which favours high energy. In a carbon target, for example, the production rate at small angles (where the yield is always greatest) increases sharply with proton energy from 350 to 450 MeV, while beyond 450 MeV it is relatively constant. Therefore, while the mean pion energy may become ever greater with increasing proton energy, there is not a very large gain in the number of pions produced at proton energies above 450 MeV. This of course assumes that the beam current is constant with changing proton energy. Seeking a maximum pion yield rather than a high mean energy and at the same time allowing a margin for comfort the TRIUMF designers chose the value of 500 MeV for the planned maximum proton beam energy.

The choice of an operating beam energy and a pion-producing target is made complicated by several additional factors. Because the electric dissociation of an H$^-$ ion is less probable at lower energy the accelerator can tolerate larger beam currents at lower energy. Hence the pion yield, proportional to both the beam current and the production rate, can be greater at 450 MeV than at the maximum beam energy of 500 MeV. The yield may become still greater if the proton energy
is further reduced. Moreover, the pion yield may be enhanced by the use of a heavier target material since a heavier nucleus is richer in nucleons having high internal momentum which can raise the amount of energy available to the production reaction. This enhancement is especially noticed at proton energies near the pion production threshold. A competing process however is the one in which either the incident proton or the emerging pion is absorbed or scattered by the nucleus hosting the production event and in this respect a heavier host nucleus is clearly not favoured. Supporting this trend we note that a heavy nucleus also provides a large Coulomb potential which serves to lower the incident proton energy. Determining the balance of these "nuclear effects" and its dependence on both nuclear mass and beam energy is the concern of this thesis, but the reason for wanting to be able to do so is at once evident: it will permit the efficient adjustment of machine parameters to maximize the TRIUMF pion yield.

Because of the nature of the TRIUMF project we are interested in the production of charged pions at small angles in the bombardment of various nuclei by protons in the energy range around 450 MeV. In the next chapter we introduce a simple model for the production process, one which keeps separate its more important physical aspects, and in the chapter after that we describe calculations made on the basis of this model. We compute relative production cross-sections which agree with available experimental cross-sections to such an extent as to afford a fair degree of confidence in the assumptions we make in
constructing our model. We are also able to discover which of the physical aspects of the process dominate at the energies considered. We hope that subsequently refined versions of our model can then be used to predict cross-sections where none are yet available experimentally and thereby become an aid to more efficient planning in the TRIUMF or similar projects.

1.2 The basic interaction

The production of pions by proton-nucleus collisions and by proton-nucleon collisions are related processes. At energies above the pion production threshold both the incident proton wavelength and the proton-nucleon cross-section are small enough to permit the assumption that the proton interacts with but a single target nucleon to produce a pion and not with the target nucleus as a whole. In this approximation the proton-nucleus cross-section is proportional to a corresponding proton-nucleon cross-section and to the number of those nucleons in the target nucleus, although the nature of the proportionality becomes complex as one admits the operation of nuclear effects such as those mentioned in the previous section. The two-nucleon collision then is at the root of the pion production process.

The two-nucleon reactions that begin with at least one proton and produce a positive pion are as follows:

\[
\begin{align*}
  p + p & \rightarrow \pi^+ + p + n \\
  p + p & \rightarrow \pi^+ + d \\
  n + p & \rightarrow \pi^+ + n + n
\end{align*}
\]
There is only one such reaction producing a negative pion, viz:

\[ n + p \rightarrow n^- + p + p \]

The threshold energy in the laboratory system is 294 MeV for the reaction yielding deuterium and 288 MeV for the others. The differential cross-sections for the reactions are poorly known experimentally (experiments \(2-11\) are summarized in Appendix A) and the existing theories \(12-16\) which might be used to compute cross-sections are either largely phenomenological or incomplete. Consequently we are able to calculate only relative proton-nucleus cross-sections, using the proton-nucleon cross-sections as free parameters to adjust our results. It should be noted however that the combined cross-section for the first two reactions listed above is approximately ten times the cross-section for the third. This is the correct result \(17\) for pion production proceeding through an intermediate 3-3 resonance state. Hence we assume that for the production of positively-charged pions it is the protons of the target nucleus which are the most important.

The nuclear effects which influence the nature of the proportionality between the proton-nucleus cross-section and the number of target nucleons are:

1) the momentum effect wherein the momentum of a struck nucleon within its nucleus alters the amount of energy available to the production reaction and both the energy and exit angle of the pion produced. Just how the pion yield is affected
depends on the nature of the variation of the proton-nucleon cross-section with proton and pion energy and pion angle in the regions of interest. Internal momentum is especially important near the pion production threshold which, because of internal momentum, in a nucleus may be as low as 170 MeV;

2) the absorption-scattering effect wherein the non-participating nucleons of the struck nucleus, mere spectators in the production event, provide a background optical potential which alters the pion yield by allowing for absorption within the nucleus of either the incoming proton or the outgoing pion or, in the case of differential cross-sections, for scattering of pions away from the energy and angle of interest. Experimental results on the proton-nucleus production of pions as a function of increasing nuclear mass (the experiments\textsuperscript{18-30} are summarized in Appendix B) show a decrease in production efficiency, i.e., in the number of pions produced per target nucleon, consistent with the operation of an absorption effect. The nuclear cross-sections are found to increase more and more slowly with $A$ until often there is no increase at all;

3) potential effects wherein the amount of energy available for pion production and the energy of the pion as it is seen outside the nucleus are altered by the presence of background Coulomb and nuclear potentials, i.e., by the real part of the optical potential. These potentials also affect the process of absorption, which is usually energy dependent, and change the paths taken by incoming and outgoing particles (refraction);
4) density effects wherein the momentum, absorption-scattering and potential effects, as well as the simple probability of finding a nucleon to strike, each become functions of position within the target nucleus; and finally

5) miscellaneous effects which as far as our work is concerned are relatively unimportant and will not be discussed further. Included in these are the effects of correlation among the target nucleons (important in very light nuclei) and the Pauli inhibition of reactions which would leave a nucleon in an already occupied state (important at low energies and large scattering angles).

1.3 Existing models

Various models taking nuclear effects into account have been used to explain the observed A-dependence of the proton-nucleus production of pions. To compute relative pion yields they all assume corresponding proton-nucleon cross-sections. The earliest work of interest to us is that of Gasiorowicz\(^{31}\) who explains the production of low-energy pions by 240- and 340-MeV protons. He considers, in an approximate fashion, the absorption effect and assumes, furthermore, that there exists at the nuclear surface a shell or blanket of the excess neutrons which, since the majority of positive pions are produced in proton-proton collisions, encloses the effective production volume and reduces the pion yield obtained from a heavy nucleus relative to that obtained from a lighter one. His
calculated cross-sections agree well with the then-available experimental ones, but because of uncertainty in his chosen parameters he can draw no conclusion about the existence or the non-existence of a neutron blanket. Gasiorowicz discusses the effect of the Coulomb potential, but only on the relative shapes of positive and negative pion energy spectra.

Merritt and Hamlin\textsuperscript{(32)} formulate an A-dependent attenuation factor using the nuclear radius and energy-dependent proton and pion mean free paths as parameters which they adjust to obtain approximate agreement with their experimental cross-sections for positive pion production by 335-MeV protons. As for Gasiorowicz this agreement is inconclusive because the values of the parameters used are uncertain and the data points are few. They consider only the absorption effect.

Henley\textsuperscript{(33)} discusses and then neglects the effect of pion absorption on pion yield. His main concern is using the effect of struck-nucleon momentum to explain experimental pion energy spectra from carbon. Similar use of the momentum effect has been made by others, e.g. Rosenfeld\textsuperscript{(4)}.\textsuperscript{4}

Imhof et al.\textsuperscript{(34)} use the model of Gasiorowicz to explain their experimental results at 340 MeV and consider the momentum effect to explain the pion yield increasing from lithium to carbon. All calculations are crude.

Ansel'm and Shekhter\textsuperscript{(35)} derive an explicit expression for an A-dependent attenuation factor assuming that proton and
pion absorption is energy independent. They are able to fit curves to most of the then-available experimental data, but they do not consider what values for the absorption parameters might be realistic. They consider no effect other than absorption.

The most extensive and successful of the previous calculations is that of Lillethun\(^{(36)}\) who combines an improved energy-dependent treatment of the absorption effect with the concept of a neutron blanket as proposed by Gasiorowicz. Pion absorption coefficients are calculated for various nuclei from optical potentials given by Frank et al.\(^{(37)}\). Lillethun's parameters are in general quite reasonable and his calculated pion yields agree well with a large amount of experimental data at 450 MeV and this agreement, as Lillethun shows, depends very strongly on the neutron blanket assumption. Lillethun also treats the Coulomb and nuclear potentials, ignoring refraction.

In summary then the absorption effect has been most often used to obtain agreement between theory and experiment, but this agreement is never conclusive without the assumption concerning the existence of an absorbing neutron blanket. The effect of internal nucleon momentum has not been discussed in connection with the general problem of \(A\)-dependence; neither has the effect of using a nucleon density distribution that does not have square edges. Potential effects have been included in only one model, that of Lillethun. We must conclude that the roles played by the important physical factors controlling the pion production process are not well understood.
In passing we note that models not of the type outlined above have been used by Serber(38), Metropolis et al.\(^\text{(39)}\), and Margolis\(^\text{(40)}\) and that these have met with only limited amounts of success. Therefore they are of little interest to the present investigation.

1.4 Present program

It is the objective of the present investigation to calculate some relative differential cross-sections for charged pion production in the proton-nucleus reaction at energies around 450 MeV. We must assume that the free two-nucleon cross-section though not well known can be used as a parameter, with which we may normalize our results to experimental values.

The model we shall use is of the general kind we have been describing, the kind having the most success so far. It makes the basic assumption of individual proton-nucleon interaction and, unlike those described, it accounts for all the nuclear effects which might be important at the energy of interest (the effects listed as 1)-4) in section 1.2). The calculation we make attempts to sort out those effects which actually are important.

It seems particularly important to avoid the neutron-blanket assumption for two reasons. First, fair agreement with experimental results has already been obtained by Lillethun who used this assumption and, second, we feel that while certain
evidence (the isotopic spin term of the nuclear optical model potential) may indicate an excess of neutrons at the nuclear surface the actual distribution is nearer to the one with no excess than to the opposite extreme with complete proton and neutron separation. We therefore regard the hypothesis of Gasiorowicz and Lillethun as artificial.

In treating proton and pion absorption it is most convenient to use the same absorption coefficients as Lillethun, but important corrections must be made. We describe these corrections and apply them in our calculation. The absorption coefficients were computed for protons from experimental proton-proton and proton-neutron cross-sections (about the same at our energy) and for pions from mean free paths for either absorption or inelastic scattering. By using these coefficients we therefore ignore the effects of refraction (elastic scattering) and the possibility that pions may be scattered from other energies and angles into the region of interest. Multiple scattering effects are similarly ignored. The ignored interactions have small cross-sections however. The pion mean free paths were in turn computed from optical model potentials obtained from two-body scattering phenomena assuming that two-body forces are not appreciably modified within the nucleus. Hence there is some consistency in method. Pion absorption gets special attention in our calculation since it is both greater and more energy dependent than proton absorption and since, as it turns out, the effect of absorption itself is greater than the other effects.
In dealing with the momentum effect we make the assumption that the momentum distribution has a negligible effect on both the energy of the pions produced and on the angle of their emission. The justification for this seemingly bold assumption is discussed in detail at the time we make it. It is a necessary one to make and it leads us to suspect that the A-dependence of the pion yield is not as sensitive to the type of momentum distribution as it is to the mean struck-nucleon kinetic energy, at our value of proton energy anyway. In our work we therefore consider only one distribution, that of a Fermi gas.

The density effects which we consider are those of using a Saxon-Woods shape on the density distribution, i.e., of giving the nucleus a diffuse edge, and that of reducing the basic nuclear radius from the larger values used by earlier workers to the value now generally accepted. The latter consideration turns out to be an especially important one: it liberates, as we show, the model from the need of a neutron blanket. This liberation, together with the related confirmation that absorption is the dominant nuclear effect in proton-nucleus pion production at 450 MeV, constitutes the main result of the investigation to which we now turn.
CHAPTER II - THE MODEL

2.1 Theory

The theory and assumptions underlying our model are stated most directly by a mathematical expression for the proton-nucleus cross-section for pion production. This cross-section is a differential cross-section which in the laboratory system depends on the energy $E_p$ of the incident proton, on the angle $\Psi$ of pion emission relative to the incident proton beam, and on the energy $E_\pi$ of the pion emitted. The expression for it is:

\[ \mathcal{O}(E_p, \Psi, E_\pi) = \int \int \mathcal{O}_f(T_p, \Psi, \pi, k) \cdot \exp \left( -\int_{s_p} n_p ds - \int_{s_\pi} n_\pi ds \right) \cdot \rho(r) \cdot f(k, r) \cdot d^3r \cdot d^3k, \]

where $\mathcal{O}_f$ is the pion production cross-section, in the laboratory system, for protons on "free" nucleons, i.e., it is what we have been calling the corresponding proton-nucleon cross-section. It is assumed for positive pions to be the proton-proton cross-section and for negative pions the proton-neutron cross-section. If we ignore refraction effects $\mathcal{O}_f$ depends on the same angle $\Psi$ of pion emission as $\mathcal{O}$ depends on. $\mathcal{O}_f$ also depends on proton kinetic energy $T_p$ and pion kinetic energy $T_\pi$ inside the target nucleus, which differ from their energies outside by potential terms depending on position $r$ in the nucleus and the mass number $A$ of the nucleus. Finally, $\mathcal{O}_f$ depends on the momentum vector $k$ of the particular nucleon struck since we are in the laboratory frame. We have more to say about this later. The other terms
in the integrand of [1] account for the various nuclear effects mentioned in the Introduction.

The exponential term in the integrand of [1] accounts for the absorption and scattering effects. \( n_p \) and \( n_\pi \) are respectively proton and pion absorption coefficients. They depend on the corresponding kinetic energy, \( T_p \) or \( T_\pi \), and on the local nucleon density \( \rho \) at points \( s \) along \( s_p \) and \( s_\pi \) which denote in that order the paths followed by the incoming proton and the outgoing pion. Since we have ignored refraction these paths are the straight lines illustrated in Figure 1. If we adopt a spherically polar coordinate reference (Figure 1) and label the point of proton-nucleon collision by \( \mathbf{r} = (r, \theta, \phi) \) we can derive (Appendix C) explicit expressions for the distances travelled by the proton and the pion inside the target nucleus. Where \( R_{\text{max}} \) is the radius beyond which we assume negligible absorption these expressions are respectively:

\[
[2] \quad s_p = r \cdot \cos \theta + (R_{\text{max}}^2 - r^2 \sin^2 \theta)^{\frac{1}{2}}
\]

and, where \( X = \cos \theta \cdot \cos \psi + \sin \theta \cdot \sin \psi \cdot \cos \phi \),

\[
[3] \quad s_\pi = -r \cdot X + (R_{\text{max}}^2 - r^2(1 - X^2))^{\frac{1}{2}}.
\]

In our model absorption (\( n_p \) and \( n_\pi \) also account for certain kinds of scattering) is averaged over these path lengths.

The nucleon density distribution \( \rho(\mathbf{r}) \) is normalized to the total number of possible pion producers in the target nucleus, which we assume are protons for positive pion production and neutrons for negative pion production. Thus \( \int \rho(\mathbf{r}) \cdot d^3 \mathbf{r} \) is
FIGURE 1

Model of nucleus showing path of incident proton and emerging pion. Proton enters nucleus at A, travels distance $s_p$, and strikes a nucleon at $r$ where it creates a pion. The pion then travels distance $s_{\pi}$ and leaves nucleus at angle $\Psi$. Paths remain in plane parallel to that of paper. Coordinate system is indicated.
put equal to either \( Z \) or \( N = A - Z \) accordingly. The nucleon momentum distribution \( f(k,r) \) is assumed to depend on position \( r \) through the local nucleon density \( \rho(r) \). \( f(k,r) \) is always normalized to unity, i.e., \( \int f(k,r) \cdot d^3k = 1 \).

Before describing a simplification we make of \([1]\) we review the approximations which distinguish our model from a proper theory and comment on the extent to which our model is an improvement over its earlier versions. The basic approximations inherent to \([1]\) are those due to:

1) the assumption that pions are produced in single proton-nucleon collisions (the high energy of the proton and the small cross-sections for proton-nucleon interaction at this energy justify this approximation);

2) the assumption that only the protons of a target nucleus can contribute to the production of positive pions (this we assume because the neutron contribution is known to be down by one order of magnitude);

3) the neglect of nuclear detail which we assume when we use target-nucleon momentum and density distributions (many similar calculations, e.g. on electron scattering, meet with a remarkable amount of success using this treatment);

4) the neglect of incoming and outgoing particle refraction which we assume when we take \( s_p \) and \( s_\pi \) to be straight lines and when we ignore elastic scattering in computing \( n_p \) and \( n_\pi \) (this approximation is partially justified by the high energy of both the proton and the pion and by the small cross-section
for elastic scattering);  

5) the assumption that \( \sigma_f \) is a known function of \( T_p, \Psi, T_\pi, \) and \( k \) (we do not in fact know this function, but in the next section we describe how, by making several approximations not listed here, we can move \( \sigma_f \) outside the integral sign in [1] and treat the calculation phenomenologically); and  

6) the neglect of such things as multiple scattering and those effects mentioned as miscellaneous in section 1.2 (we simply consider these unimportant to our calculation).

Our results will indicate that a more accurate treatment may not be warranted. The pion production process, as our results confirm, is dominated by pion absorption and the basic production rate, both of which are accounted for by our model.

The earlier models (described in section 1.3) make essentially the same approximations as we list above. The most developed of the earlier models is that of Lillethun who ignored the momentum of the struck nucleons (we can do this by putting \( f(k) = \delta^3(0) \) into [1]). He limited their density distribution to having a square shape (he put \( \rho(\vec{r}) \) equal to a constant up to a nuclear radius and to zero beyond) and chose to use a square-shaped neutron blanket which is not a part of our model. In a number of other cases his choice of parameters differed quite significantly from ours, which we discuss in a later section.
2.2 An essential simplification

Our model as expressed by [1] cannot be used to calculate a cross-section until the dependence of $\sigma_f$ on $T_p$, $\Psi$, $T^\pi$, and $k$ is known. It is because this dependence is not known that we must make the following simplification.

Consider the momentum integral of [1], viz:

$$\sigma_k(r) = \int \sigma_f(T_p, \Psi, T^\pi, k) \cdot f(k) \cdot d^3k.$$  

In addition to angle the free cross-section, $\sigma_f$, depends on the kinetic energies $T_p$ and $T^\pi$ and on the momentum $k$, all of which we assumed were measured relative to a laboratory frame. Alternatively, the free cross-section depends on an angle $\Psi'$ and on kinetic energies $T'_p$ and $T'^\pi$, all measured in the rest frame of the struck nucleon. We can write

$$\sigma_f(T_p, \Psi, T^\pi, k) = \sigma_f(T'_p, \Psi', T'^\pi, 0),$$

showing that the free cross-section does not, in the moving frame, depend explicitly on momentum since by definition $k' = 0$. This does not help us integrate however since there is an implicit $k$-dependence in the kinetic energies $T'_p$ and $T'^\pi$, i.e., in the transformation back to the laboratory frame. The transformation is given by the following relativistic relations:

$$T'_p = \frac{T_p T - k_p \cdot k}{m} + T + T_p$$

and

$$T'^\pi = \frac{T^\pi T - k^\pi \cdot k}{m} + \frac{\mu}{m} T + T^\pi,$$
which are derived in Appendix D. $T$ is the struck-nucleon kinetic energy corresponding to $k$ and $k_i$ is the momentum which corresponds to $T_i$, $i = p$ or $\pi$. All unprimed quantities are in the laboratory system; $m$ is the nucleon rest mass (938 MeV) and $\mu$ is the pion rest mass (140 MeV).

Let us introduce two more approximations:

1) the neglect of the effect of the struck-nucleon momentum, $k$, on the angle of pion emission, which lets us write $\psi' = \psi$ (this approximation is at least partially justified by the high incident proton energy, which keeps small the angular spread caused by changing $k$, and by the free cross-section, which varies only moderately with production angle); and

2) the neglect of the effect of the struck-nucleon momentum, $k$, on the energy of the pion emitted, which lets us put $T'_\pi = T_\pi$ (this approximation has a justification similar to that of the first). The second approximation should be discussed.

The spread in pion kinetic energy caused by changing $k$ is not too large if compared to the corresponding spread caused in effective proton energy. It can be seen (by using numbers) that for the high proton energy being considered and for a reasonably high pion energy (80 MeV say) $T'_\pi$ is always better approximated by $T_\pi$ (equation [7]) than $T'_p$ is by $T_p$ (equation [6]) even though in the extreme cases of head-on and tail-on collision at high $T$ (where neither approximation is very good) the difference is not always great. The difference becomes more important however when we note that the free cross-section for
pion production is, but not without exception, more sensitive to a small change in $T_p$ than it is to a similar one in $T_{\pi}$. Thus our model will, by using the above approximations, account for the momentum effect only as it alters the amount of energy available to the production reaction. The two approximations listed in this section do not affect the model in any other way.

Putting $\Psi' = \Psi$, $T'^{\pi} = T_{\pi}$, and expanding [5] by a Taylor series in $T_p$ about the point $T_p$ gives us

\[ \sigma_f(T_p, \Psi, T_{\pi}, k) = \sigma_f(T_p, \Psi, T_{\pi}, 0) \]
\[ = \sigma_f(T_p, \Psi, T_{\pi}, 0) + \frac{d\sigma_f(T_p, \Psi, T_{\pi}, 0)}{dT_p}(T_p' - T_p) + \cdots \]

Alternatively, we can expand [5] in $T_p'$, $\Psi'$, and $T_{\pi}'$ and, in the approximation described above, drop terms containing $d\sigma_f/d\Psi$, $(\Psi' - \Psi)$, $d\sigma_f/dT_{\pi}$, or $(T_{\pi}' - T_{\pi})$. The result is the same. If we drop high-order terms and put [8] into [4], using [6] to note

\[ \int (T_p' - T_p) \cdot f(k) \cdot d^3k = \gamma \overline{T}, \]

where $\gamma$ is the factor $(T_p + m)/m$ and $\overline{T}$ is the local mean value of the struck-nucleon kinetic energy, we get the result that

\[ \sigma_k(r) = \sigma_f(T_p, \Psi, T_{\pi}) + \frac{d\sigma_f(T_p, \Psi, T_{\pi})}{dT_p} \cdot \gamma \overline{T}. \]

We have stopped indicating explicitly the fact that $k' = 0$. As our last approximation we remove the $r$-dependence from all terms.
in [10] except $\bar{T}$ by assuming that, for the purpose of computing $T_p$ and $T_\pi$ from given values of $E_p$ and $E_\pi$ only, the nuclear potentials depend just on mass number $A$. We finally find that [11] can be written in the following manner:

$$\mathcal{O}(E_p, V, E_\pi) = \mathcal{O}_f(T_p, V, T_\pi) \cdot I_1 + \frac{d\mathcal{O}_f(T_p, V, T_\pi)}{dT_p} \cdot I_2,$$

where

$$I_1 = \int \exp(-\int_{s_p}^{n_p} ds - \int_{s_\pi}^{n_\pi} ds) \cdot \phi(r) \cdot d^3r,$$

and

$$I_2 = \int \bar{T}(r) \cdot \exp(-\int_{s_p}^{n_p} ds - \int_{s_\pi}^{n_\pi} ds) \cdot \phi(r) \cdot d^3r.$$

This is the expression on which our calculations are based.

The proton-nucleus cross-section for pion production is now expressed in terms of mean struck-nucleon kinetic energy rather than in terms of a particular momentum distribution. This simplified version of our model must be used until such time as we have more information about the basic proton-nucleon production process. In [11] the proton-nucleon cross-section for pion production and its first derivative with respect to proton energy appear as coefficients to integrals which can be easily evaluated. In our calculation we evaluate both $I_1$ and $I_2$ and use the coefficients to adjust results to experimental data on the proton-nucleus production of pions. The $A$-dependence of our calculated cross-sections enters only through $I_1$ and $I_2$: we cannot take into account the $A$-dependence entering through the coefficients appearing with them. We cannot take into account any dependence on incident proton energy since it enters only through the coefficients. Our lack of knowledge concerning the
free production process also keeps us from knowing just when the assumptions we make in going from [1] to [11] might be invalid.

If we wish to ignore the momentum effect, as well as the approximations made in this section, we need only set \( \bar{T} = 0 \). The result will be the same model we would get by putting \( f(k) = \delta^3(0) \) into [1]. With the appropriate choice of \( \rho(r) \) we would have the model of Lillethun, but without the neutron blanket.

2.3 The parameters chosen

To calculate a proton-nucleus cross-section for pion production using [11] we must specify absorption coefficients, a density distribution, a momentum distribution (a mean kinetic energy at least), and a free proton-nucleon cross-section. The free cross-section we cannot specify exactly hence we calculate only relative proton-nucleus cross-sections using the free cross-section as a normalizing parameter. When discussing our results we comment on how the value we must assume for the free cross-section compares with the poorly known experimental one.

The density distribution that we choose has the familiar Saxon-Woods form, viz:

\[
\rho(r) = \rho_o \cdot (1 + \exp((r - R)/a))^{-1},
\]

in which the nuclear radius \( R \) (for nucleon density and the optical potentials of protons and pions) is the usual optical model radius \( R = r_o \cdot A^{1/3} \) with \( r_o = 1.25 \) fermis. This is the
value used in our work. Lillethun and earlier workers used a larger value \( r_0 = 1.35 \text{ fm} \) and because the pion production rate in a heavy nucleus is largely controlled by pion absorption the difference is significant. Our results show this. \( R \) is the radius where the density is 50% of its central value and is not to be confused with \( R_{\text{max}} \), the radius beyond which absorption is assumed negligible. The two coincide only in a square-edge nucleus. For our diffuse-edge nucleus we arbitrarily set \( R_{\text{max}} \) to be the radius where the density has fallen to 10% of its central value so that \( R_{\text{max}} \) and \( R \) together define the value \( S = 2 \cdot (R_{\text{max}} - R) \) which in turn defines the surface thickness constant \( a = S/(4 \cdot \ln 3) \). For \( a \) we have chosen a standard value of 0.55 fm. Our results are not very sensitive to \( a \). The normalization constant \( o_0 \) is varied according to the context of its use. For use in the integrands of [11] \( o_0 \) takes the form \( o_0 = o_0^! \cdot Z \) or \( o_0 = o_0^! \cdot N \) depending on whether positive or negative pion production is being considered, where \( o_0^! \) has the standard value

\[ [13] \quad o_0^! = (3/(4\pi R^3)) \cdot (1 + \pi^2 a^2/R^2). \]

When used alone \( o_0^! \) normalizes the density distribution to unity. For use with optical potentials and absorption coefficients, in equations like \( V(r) = o(r) \cdot V_o \) and \( n(r) = o(r) \cdot n_o \), \( o \) takes the form \( o = o_0^! \cdot \text{Vol} \), where \( \text{Vol} \) is the nuclear volume \((4/3)\pi R^3\).

For the local mean kinetic energy of the target nucleons we choose a value corresponding to the momentum distribution of a Fermi gas appropriate to the density \( o(r) \), viz:
The proton absorption is relatively weak at the energies of interest to us: we have chosen $n_p$ to be 0.182 fm$^{-1}$ in all our calculations. This is also the value used by Lillethun. It was computed by him from total proton-proton and proton-neutron cross-sections. Pion absorption on the other hand is strong and highly energy dependent.

Pion absorption coefficients depend on pion kinetic energy inside the target nucleus. At any point $r$ inside the nucleus the pion kinetic energy, $T_\pi$, is related to its total energy, i.e., its kinetic energy outside the nucleus, $E_\pi$, by

$$E_\pi = T_\pi + V_c(r) + V_r(r, T_\pi),$$

where $V_c(r)$ is the Coulomb potential and $V_r(r, T_\pi)$ is the real part of the pion optical potential, depending on pion kinetic energy. In our calculation $V_c(r)$ is computed under the assumption that $Z$ protons are distributed uniformly inside the charge radius $R_c = 1.07A^{1/3}$ fm. We therefore take

$$V_c(r) = \frac{Ze^2}{4\pi\varepsilon_0} \cdot \frac{1}{2R_c} \cdot (3 - r^2/R_c^2) \quad \text{for} \quad r < R_c$$

and

$$V_c(r) = \frac{Ze^2}{4\pi\varepsilon_0} \cdot \frac{1}{r} \quad \text{for} \quad r > R_c,$$

where $e^2/(4\pi\varepsilon_0)$ has the numerical value of 1.44.
Values of $V_\pi(r, T_\pi)$ vs. $T_\pi$ have been computed by Frank et al. We use their values corrected by a factor of $(r'_0/r_0)^3$, where $r'_0$ is the basic nuclear radius assumed by Frank et al. and $r_0$ is our value of 1.25 fm. Frank et al. implicitly assume a value of 1.41 fm. for $r'_0$ when they set $\Lambda = 1$ (cf. the caption to their Table II). $\Lambda$ is defined by the relation (their equation (3)) $R = \Lambda' A^{1/3}$, where $\Lambda'$ is the Compton wavelength for pions (1.41 fm.). The correction which we make to $V_\pi$ is appropriate since $\Lambda$ enters into the expression giving $V_\pi$ as an inverse cube (cf. Frank et al., their equation (13)). We also apply a correction for local nucleon density, $\rho(r)$, normalized to nuclear volume (cf. their equation (1)) and by manipulating [15] compute the pion kinetic energy, $T_\pi$, at $r$ from the given pion total energy, $E_\pi$.

Having thus counted $T_\pi$ at a point $r$ we can interpolate for $n_\pi$ using the values of $n_\pi$ vs. $T_\pi$ also given by Frank et al. (they give values of pion mean free path vs. $T_\pi$, but these are just reciprocals of the absorption coefficients). Three corrections must be applied to the interpolated value: 1) a correction for units, inverse Compton wavelengths to inverse fermis (cf. Frank et al., their Table I); 2) a correction for local nucleon density, $\rho(r)$, normalized to volume; and 3) the correction mentioned in connection with $V_\pi$ concerning the basic nuclear radius, $r_0$. The last correction, applied to $n_\pi$, is again one of $(r'_0/r_0)^3$ obtained from an inspection of Frank et al., equations (6), (7) and (9). Their equation (9), giving $n_\pi$, is
is derived using a model of Brueckner et al.\(^{(41)}\) which assumes that pion absorption by nuclear matter is proportional to the pion capture cross-section of a deuteron. We re-derive this equation in Appendix E to show its inverse cube dependence on \(\lambda\), not shown explicitly by Frank et al.

Alternatively, we have taken for our pion absorption coefficients those values listed by Lillethun (in his Table V), who also uses the data of Frank et al., but without applying the corrections described above. Lillethun uses an interpolation technique which seems to be different from ours since, if for no other reason, his computed values of \(n_\pi\) are not at all smooth functions of \(A\) and we think they should be.

Nuclear size is important to the absorption effect in two ways: 1) the distance a particle travels inside a nucleus depends directly on the nuclear size; and 2) the absorption coefficient depends on local nucleon density, which in turn depends on the nuclear size. The first dependence is roughly linear in the nuclear radius and the second is one involving the inverse cube of the nuclear radius. Hence, since total absorption depends on the product of distance travelled and absorption coefficient, we have the seemingly curious result that the smaller of two equally massive nuclei is the stronger absorber. It must not be forgotten however that the smaller nucleus is also the smaller target, i.e., that the integrations of \([11]\) are over a smaller volume. The net balance of these and other effects on pion yield we calculate in a special case.
CHAPTER III - THE CALCULATION

3.1 The integration

Our calculations required the numerical evaluation of the integrals \( I_1 \) and \( I_2 \) of equation [11]. This we did using the University of British Columbia's IBM 7044 computer and a FORTRAN IV program called PIPROD, the logic of which is outlined in Appendix F. In this program the integrands of \( I_1 \) and \( I_2 \) are each evaluated throughout the nucleus over a network of points, none of which is separated from its nearest neighbour by more than a fraction of a fermi \((10^{-13} \text{ cm.})\). The integration is completed by an appropriate number of Simpson's rule approximations. The error introduced by each approximation is estimated by re-calculating the particular integral over a coarser network of points obtained by doubling the separation distance. In our calculation the coarse and fine integrals differed by an amount that was usually less than two percent and very seldom more than five percent of the first, i.e., the fine integral. The true error, i.e., the difference between the fine and an infinitely fine integral, at each stage should always be much smaller than the one estimated in the above manner.

At each point of the network the paths \( s_p \) and \( s_\pi \) have to be determined. At each of several points (separated by less than fermi) along \( s_\pi \) the pion optical potential, \( V_\pi \), and the pion absorption coefficient, \( n_\pi \), have to be determined. This we did by fitting a third order polynomial to the known data of
Frank et al. (different at each point since we include a density correction) and interpolating at the pion kinetic energy of interest (which also varies from point to point) The averaging of \( n_\pi \) along \( s_\pi \), like the integration of \( I_1 \) and \( I_2 \), was done in the Simpson's rule approximation with an error estimation.

Variations run on the PIPROD program are not discussed in this thesis. Variations were run however and their outcomes constitute the subject matter of section 3.3.

3.2 Results compared to experiment

A typical result of our main calculation is illustrated in Figure 2, for the case in which we have incident protons of 450 MeV producing pions of 83 MeV at an angle of 21.5\(^\circ\) in the laboratory system. The experimental cross-sections are those of Lillethun. The calculated curves have been normalized to the experimental value at \(^{27}\text{Al}\). The curve marked \( L \) is Lillethun's published result, calculated from his model with an absorbing neutron blanket. We were able to duplicate this result in our work by using Lillethun's neutron blanket and parameters, these including his calculated absorption coefficients.

The curve marked \( I_1 \) in Figure 2 is our result, calculated using our model as expressed by [11] and the parameters and corrections described in section 2.3. We have plotted the integral \( I_1 \) only, thereby ignoring the effect of a momentum distribution. The integral \( I_2 \), which allows us to take a
FIGURE 2: Cross-sections for the production of 83-MeV positive pions at 21.5° by 450-MeV protons on various nuclei.
momentum effect into account, coincides almost exactly with the plotted curve, I₁, when normalized at $^{27}$Al. The ratio $I_2/I_1$ for the momentum distribution of a Fermi gas is shown as an inset to Figure 2. The indication is that the effect of a momentum distribution on the relative pion yield is small, although the effect may be important when choosing normalizing factors $\sigma_f$ and $d\sigma_f/dT_p$ to compare with experimental data on the proton-nucleon production of pions. Our model is not yet able to treat pion production at proton energies near the production threshold where the momentum effect is certain to be more important, but at the energy of 450 MeV it suggests that we may neglect this effect.

We have thus ignored the effect of a momentum distribution in plotting our results. There are two other A-dependent effects which we have ignored, one in deriving [11] and one in plotting our curves. They are: 1) the effect of target-nucleus neutrons on positive pion production (we could estimate this effect by including in our result the factor $(10\cdot Z + N)/11\cdot A\cdot Z$ which removes the normalization imposed on $\sigma(r)$ and weights the proton-proton and proton-neutron free cross-sections in accordance with pion production proceeding through an intermediate 3-3 resonance state: it would raise all pion yields, but would especially favour high values of A, by 5% in $^{208}$Pb as shown in Figure 2 for example); and there is 2) the effect due to the coefficients $\sigma_f$, $d\sigma_f/dT_p$ and $\gamma$, which have an A-dependence that enters through the potential terms used to relate kinetic energy in the nucleus with total energy outside (this effect is hard to
even estimate, but it probably lowers slightly the relative yield from a heavy nucleus at our energies).

The value for $\sigma_f$ which we implicitly assume in normalizing our result of Figure 2 to the experimental cross-section of $^{27}$Al is 31.0 microbarns/MeV/steradian. Putting the integral $I_2$ into our result we can lower this value to nearly 25 $\mu$b/MeV/st. by taking for $d\sigma_f/dT_p/\sigma_f$ the value 0.015, which is the value of the logarithmic derivative at 450 MeV taken from data on the total cross-sections $\sigma(pp\to\pi^+d)$ and $\sigma(pp\to\pi^+pn)$ vs. proton energy as summarized by McIlwain et al.\(^{(42)}\). Pondrom\(^{(5)}\) uses 450-MeV protons to obtain differential cross-sections at 20°27' for the reaction $pp\to\pi^+pn$. His value is almost 8 $\mu$b/MeV/st. at the pion energy of 122 MeV, this being the kinetic energy needed by a positive pion inside an $^{27}$Al nucleus if it is to be seen outside with an energy of 83 MeV. Gell-Mann and Watson\(^{(14)}\) indicate that 70% of the positive pions coming from proton-nucleon reactions at 300-400 MeV are due to the reaction $pp\to\pi^+d$. If we assume that this is also true at 450 MeV and that ten times the number that come from $pn\to\pi^+nn$ come from the two $pp$ reactions, then we can estimate a possible experimental value for $\sigma_f$ of just over 31 $\mu$b/MeV/st. This last value is very uncertain, but it is nonetheless close to our assumed value and for that reason it is encouraging.

Illustrated in Figure 3 are some results on positive pion production at 450 MeV, but at other pion energies and pion scattering angles as indicated. The experimental points and the
FIGURE 3 More cross-sections for positive pion production.
curves marked L are once again those of Lillethun. What we have said about the result in Figure 2 can also in general be said about the results in Figure 3.

Some results on negative pion production at 450 MeV are given in Figure 4. Pion energies and angles are indicated and again the experimental points and the curves marked L are due to Lillethun. In the case of negative pions the remark we made in connection with the result in Figure 2 concerning a correction factor for the target-nucleus neutron contribution to the pion yield no longer applies since the neutrons of the target, and only the neutrons, are considered. There is no proton contribution whatever. The remark made about the A-dependence of the normalizing parameters still applies.

Agreement between our calculated curves and the data of Lillethun is seldom excellent and it is never really poor. It is often as good as any previously obtained. We show in the next section how the absorption effect, especially pion absorption, dominates the proton-nucleus production process. It therefore seems likely that the use of different, more modern data on pion absorption might lead to different and perhaps better agreement than we obtain. Hence agreement between our calculated results and those of experiment should not be viewed as being critical to our model. The importance of our results is of another kind: good agreement previously depended on the neutron blanket assumption and now it does not. We show why in the next section.
FIGURE 4 Some cross-sections for negative pion production.
3.3 Results with parameters varied

In order to determine the manner in which the various physical factors controlling the proton-nucleus pion production process affect the A-dependence of the pion yield, we calculated cross-sections using parameters other than those described in section 2.3. The results of these calculations are illustrated in Figure 5. As for Figure 2 we have considered 450-MeV protons producing 83-MeV positive pions at $21.5^\circ$ and again curve L is Lillethun's calculated result, which we were able to duplicate using a neutron blanket and his parameters, these including his calculated absorption coefficients.

Curve I is the result of using a neutron blanket and Lillethun's parameters, but using our interpolation technique to calculate pion absorption coefficients. Curve II shows the effect of removing the neutron blanket, i.e., the effect of considering the protons and neutrons uniformly distributed within the nuclear radius of Lillethun, using the same parameters and interpolation technique as we used for curve I. Curve III, on the other hand, was obtained by using a value of 1.25 fm. for $r_0$, a Saxon-Woods density distribution, and our interpolation technique to compute pion absorption. The corrections described in section 2.3 were all applied. Hence curve III is our result in Figure 2, the momentum effect still being ignored. Curve IV is obtained when, still ignoring the momentum effect, we remove the diffuse edge from the model for curve III by setting $a = 0$. 
FIGURE 5 Cross-sections obtained by varying parameters.
Otherwise the models used for curves III and IV are identical. The suggestion here is that, like the effect of a momentum distribution, the effect of a density distribution is secondary when compared to that of absorption. The importance of the absorption effect is indicated by the fact that without absorption the pion yield becomes directly proportional to Z (curve Z in Figure 5). The importance of nuclear size to the absorption effect is indicated by the fact that the reduction of the basic nuclear radius, \( r_0 \), from 1.35 fm. (curve II) to 1.25 fm. (curve IV) lowers the relative pion yield in a heavy nucleus by almost exactly the same amount as does the introduction of an absorbing neutron blanket (curve I). That our agreement with experiment is not as good as Lillethun's therefore seems to be due only to the fact that the method we use to interpolate for pion absorption coefficients is different than the one used by him. Were we to use Lillethun's method in calculating our curve III we would obtain his result. And without a neutron blanket.

3.4 Conclusion

We have constructed a model to explain the production of pions in the bombardment by protons of various nuclei and we have used the model to calculate relative cross-sections for the process. The model assumes, among other things, that the incident proton interacts with but a single target nucleon to produce a pion and not with the target nucleus as a whole. It assumes that the proton-nucleus cross-section for pion production
can be calculated knowing the basic proton-nucleon cross-section
for pion production and the number of target nucleons, provided
certain important nuclear effects are taken into account. The
nuclear effects accounted for by our model are those due to
proton and pion absorption, to nuclear potentials, and to the
struck-nucleon density and momentum distributions.

Because our knowledge of the basic proton-nucleon
production rate is limited, we can only calculate relative cross-
sections and use the basic rate as a free parameter. We do this
in several special cases and compare our results with data from
experiment at 450 MeV. Agreement is only moderate, but it is as
good as any previously obtained and, unlike the earlier results,
it does not depend on the rather artificial assumption of an
absorbing neutron blanket. Our agreement depends instead on the
use of a modern nuclear radius and on a reasonable treatment of
pion absorption. In this respect our results confirm what the
earlier workers had assumed, that absorption, especially pion
absorption, is the dominant physical factor controlling the
proton-nucleus production of pions. At high proton energies we
find that the effects of a struck-nucleon momentum distribution
are all but negligible in determining the A-dependence of the
pion yield, although they may become more important at lower
energies near the pion production threshold where we hope to
extend the present investigation. Density distribution effects
we find are likewise almost negligible at high energies. These
are our conclusions.
REFERENCES

Reference (1) describes the TRIUMF (Tri-University Meson Facility) proposal, which we mention in Chapter I.


References (2-11) describe experiments on the proton-nucleon production of pions, which we summarize in Appendix A.

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References (12-16) describe attempts at explaining
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References (18-30) describe experiments on the proton-
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(20) R. Sagane and W. F. Dudziak, Phys. Rev. 92, 212 (1953)
(23) M. M. Block et al., see reference (2).
(24) A. H. Rosenfeld, see reference (4).
(26) E. Heer et al., see reference (7).
(27) H. Helfer et al., see reference (8).
A. G. Meshkovskii, I. S. Pligin, I. I. Shalamov, and V. A.
References (31-36) describe attempts at explaining the proton-nucleus production of pions using models, which we describe in section 1.3, similar to ours. References (38-40) attempt the same explanation using other models.

(32) J. Merritt and D. A. Hamlin, see reference (22).
(33) E. M. Henley, Phys. Rev. 85, 204 (1952).
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The calculations described in the text of this thesis are discussed in two other works, which we have not mentioned. They are the following:

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APPENDIX A

EXPERIMENTS ON PROTON-NUCLEON PION PRODUCTION

We list below the values of $E_p$, $\psi$, and $E_\pi$ (as defined in the text) at which a differential cross-section for a reaction like $(p + \text{nucleon} \rightarrow \pi^\pm + \text{nucleons})$ has been measured. All values are in the laboratory frame unless otherwise indicated.

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Year</th>
<th>Ref.</th>
<th>Reaction</th>
<th>$E_p$ MeV</th>
<th>$\psi$ deg.</th>
<th>$E_\pi$ MeV</th>
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<tr>
<td>Block et al.</td>
<td>1952</td>
<td>(2)</td>
<td>pp $\rightarrow$ $\pi^+$</td>
<td>381</td>
<td>90</td>
<td>0-95</td>
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<tr>
<td>March</td>
<td>1960</td>
<td>(3)</td>
<td>pp $\rightarrow$ $\pi^+$@</td>
<td>420</td>
<td>65</td>
<td>23-76</td>
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<td>1954</td>
<td>(4)</td>
<td>pp $\rightarrow$ $\pi^-$</td>
<td>440</td>
<td>55</td>
<td>50-75$^c$</td>
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<td></td>
<td></td>
<td></td>
<td>&quot;</td>
<td></td>
<td>90</td>
<td>15-58</td>
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<td>124</td>
<td>12-23</td>
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<tr>
<td>Pondrom</td>
<td>1959</td>
<td>(5)</td>
<td>pp $\rightarrow$ $\pi^+\text{pn}$</td>
<td>450</td>
<td>$30^\circ14'$</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>&quot;</td>
<td></td>
<td>$20^\circ27'$</td>
<td>40-147</td>
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<td>$13^\circ14'$</td>
<td>45-152</td>
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<tr>
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<td>1967</td>
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<td>Simonov</td>
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<td>Heer et al.</td>
<td>1966</td>
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<td>all</td>
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<td>Year</td>
<td>Ref.</td>
<td>Reaction</td>
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</tr>
<tr>
<td>Vovchenko</td>
<td>1966</td>
<td>(9)</td>
<td>$pp \rightarrow \pi^+$</td>
<td>655</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yale report</td>
<td>1964</td>
<td>(10)</td>
<td>$pp \rightarrow \pi^+$</td>
<td>660</td>
<td>19.5</td>
<td></td>
</tr>
<tr>
<td>Haddock et al.</td>
<td>1964</td>
<td>(11)</td>
<td>$pp \rightarrow \pi^+$</td>
<td>725</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

@ 62% polarized protons

‡ centre-of-mass system

$ from a CH$_2$-C subtraction

& from a CD$_2$-CH$_2$ subtraction

# in H$_2$ and D$_2$ both

% in D$_2$

Some early references are listed by Gell-Mann and Watson(14) and by Mandelstam(15).
APPENDIX B

EXPERIMENTS ON PROTON-NUCLEUS PION PRODUCTION

We list below the values of $E_p$, $\psi$, and $E_\pi$ (as defined in the text) at which a differential cross-section for a reaction like $(p + \text{nucleus} \rightarrow \pi^\pm + \text{nucleus'})$ has been measured. All values are in the laboratory frame unless otherwise indicated.

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Year</th>
<th>Ref.</th>
<th>Target(s)</th>
<th>$E_p$</th>
<th>$\psi$</th>
<th>$\pi$</th>
<th>$E_\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clark</td>
<td>1952</td>
<td>(18)</td>
<td>Be to Pb</td>
<td>240</td>
<td>135°</td>
<td>± 40</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>35°</td>
<td>± 40</td>
<td></td>
</tr>
<tr>
<td>Imhof et al.</td>
<td>1957</td>
<td>(19)</td>
<td>Li to Pb</td>
<td>340</td>
<td>135</td>
<td>± 36</td>
<td></td>
</tr>
<tr>
<td>Sagane and</td>
<td>1953</td>
<td>(20)</td>
<td>Be to Pb</td>
<td>340</td>
<td>90</td>
<td>± 12.5-33</td>
<td></td>
</tr>
<tr>
<td>Dudziak</td>
<td></td>
<td></td>
<td>Li to C</td>
<td></td>
<td>135</td>
<td>± 63</td>
<td></td>
</tr>
<tr>
<td>Hamlin et al.</td>
<td>1951</td>
<td>(21)</td>
<td>Be to Pb</td>
<td>340</td>
<td>0</td>
<td>± 53</td>
<td></td>
</tr>
<tr>
<td>Merritt et al.</td>
<td>1955</td>
<td>(22)</td>
<td>Be to Pb</td>
<td>335</td>
<td>0</td>
<td>± 34-129</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>± 52-147</td>
<td></td>
</tr>
<tr>
<td>Block et al.</td>
<td>1952</td>
<td>(23)</td>
<td>D to Pb</td>
<td>381</td>
<td>90</td>
<td>± 20-120</td>
<td></td>
</tr>
<tr>
<td>Rosenfeld</td>
<td>1954</td>
<td>(24)</td>
<td>C only</td>
<td>440</td>
<td>90</td>
<td>± 30-125</td>
<td></td>
</tr>
<tr>
<td>Lillethun</td>
<td>1961</td>
<td>(25)</td>
<td>Be to U</td>
<td>450</td>
<td>21.5</td>
<td>± 83-236</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>C only</td>
<td></td>
<td>21.5</td>
<td>± 132-200</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Be to U</td>
<td></td>
<td></td>
<td></td>
<td>21.5</td>
<td>± 144-max</td>
<td></td>
</tr>
<tr>
<td></td>
<td>C to U</td>
<td></td>
<td></td>
<td></td>
<td>60</td>
<td>± 99</td>
<td></td>
</tr>
<tr>
<td></td>
<td>C only</td>
<td></td>
<td></td>
<td></td>
<td>60</td>
<td>± 44-max</td>
<td></td>
</tr>
<tr>
<td>Author(s)</td>
<td>Year</td>
<td>Ref.</td>
<td>Reaction</td>
<td>$E_p$ MeV</td>
<td>$\psi$ deg.</td>
<td>$\sigma$ MeV</td>
<td></td>
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<tr>
<td>-----------------</td>
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<td>--------------</td>
<td>----------</td>
<td>-------------</td>
<td>--------------</td>
<td></td>
</tr>
<tr>
<td>Heer et al.</td>
<td>1966</td>
<td>(26)</td>
<td>Be to Pb</td>
<td>600</td>
<td>0</td>
<td>± 100-max</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>&quot;</td>
<td>&quot;</td>
<td>21.5</td>
<td>± 100-max</td>
<td></td>
</tr>
<tr>
<td>Helfer et al.</td>
<td>1961</td>
<td>(27)</td>
<td>C only</td>
<td>654</td>
<td>56</td>
<td>± all</td>
<td></td>
</tr>
<tr>
<td>Meshkovskii et al.</td>
<td>1959, 1958, 1957</td>
<td>(28)</td>
<td>C only Li to Cu</td>
<td>660</td>
<td>19.5</td>
<td>± 100-max</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>&quot;</td>
<td>45</td>
<td>± 70-max</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>&quot;</td>
<td>45</td>
<td>- 90-max</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Meshcheriakov et al.</td>
<td>1957, 1956</td>
<td>(29)</td>
<td>C Ag, Pb</td>
<td>660</td>
<td>24</td>
<td>± 50-max</td>
<td></td>
</tr>
<tr>
<td>Haddock et al.</td>
<td>1964</td>
<td>(30)</td>
<td>C</td>
<td>725</td>
<td>0</td>
<td>± 50-max</td>
<td></td>
</tr>
</tbody>
</table>

@ everything between 130 and 150
@ everything between 30 and 50

Summaries are also given by Lillethun(25) and by Heer et al.(7). A rather detailed reference list appears in the Yale report(10).
APPENDIX C

DERIVATION OF PATH LENGTHS IN A NUCLEUS

In Figure 6 A is the point of proton entry and B = r is the point of pion production. The origin O is at the nuclear center. Hence OA = R_{max}, OB = r, and AB = s_p and we have

\[ s_p = r \cos \theta + R_{\text{max}} \cos \alpha, \]

where \( \alpha \) is the angle AOD. From the triangle AOD, since AD = \( r \cdot \sin \theta \), we get

\[ R_{\text{max}}^2 = (r \cdot \sin \theta)^2 + (R_{\text{max}} \cdot \cos \alpha)^2. \]

Solving \[ C.2 \] for \( R_{\text{max}} \cdot \cos \alpha \) (positive root) and putting the result into \[ C.1 \] then gives us

\[ s_p = r \cdot \cos \theta + (R_{\text{max}}^2 - r^2 \sin^2 \theta)^{\frac{1}{2}}. \]

If we denote the angle COE by \( \beta \), then CE is given by

\[ R_{\text{max}} \cdot \sin \beta = r \cdot \sin \theta \cdot \cos \phi + s_{\pi} \cdot \sin \psi. \]

Since OF = r \cdot \cos \theta + s_{\pi} \cdot \cos \psi \ triangle FOE establishes

\[ (R_{\text{max}} \cdot \cos \beta)^2 = (r \cdot \sin \theta \cdot \sin \phi)^2 + (r \cdot \cos \theta + s_{\pi} \cdot \cos \psi)^2. \]

Squaring \[ C.4 \] and adding the result to \[ C.5 \] gives us, since \( \sin^2 \beta + \cos^2 \beta = 1 \), a quadratic equation in \( s_{\pi} \) having the roots

\[ s_{\pi} = -r \cdot X \pm (R_{\text{max}}^2 - r^2 (1 - X^2))^{\frac{1}{2}}, \]

where \( X = \cos \theta \cdot \cos \psi + \sin \theta \cdot \sin \psi \cdot \cos \phi. \)
Figure 6

Geometry of path lengths in a nucleus.
It is easily shown that only the root in \([C,6]\) using the + sign always gives \(s_{\pi} \geq 0\). With this in mind we have the desired results ([2] and [3] of the text) in \([C.3]\) and \([C.6]\).
APPENDIX D
TRANSFORMATION OF ENERGY FROM LABORATORY FRAME
TO REST FRAME OF A STRUCK NUCLEON

If we let \( P = (p, iE_p) \) and \( K = (k, iE_k) \) represent the four-momenta of particles having rest masses of \( m_p \) and \( m_k \) respectively, then we can write

\[
[D.1] \quad (P + K)^2 = (p + k)^2 - (E_p + E_k)^2
\]

\[
= (p + k)^2 - (T_p + T_k + (m_p + m_k))^2,
\]

where \( T_p \) and \( T_k \) are kinetic energies. Assume that the quantities on the right-hand side of [D.1] are measured in the laboratory frame. Then if primed quantities are measured in the rest frame of the particle described by \( K \) we can also write

\[
[D.2] \quad (P + K)^2 = (p' + k')^2 - (E'_p + E'_k)^2
\]

\[
= p'^2 - (T'_p + (m_p + m_k))^2,
\]

since by definition \( k' = T_k = 0 \).

If we equate [D.1] and [D.2] and expand terms using standard relations like \( p^2 = T_p^2 + 2m_p T_p \) we get directly that

\[
[D.3] \quad T'_p = T_p + (T_p T_k - p \cdot k)/m_k + (m_p/m_k)T_p,
\]

which has [6] and [7] of the text as special cases.
APPENDIX E
DERIVATION OF AN EXPRESSION USED TO COMPUTE
PION ABSORPTION COEFFICIENTS

To explain the absorption of pions in nuclear matter Brueckner et al.\(^{(41)}\) assume that the absorption per nucleon in any nucleus is proportional to the known capture cross-section of pions by deuterons. Letting \(\Gamma\) be the proportionality constant we can then write the nuclear cross-section for absorption as

\[
\sigma = n \cdot \Gamma \cdot \sigma_d,
\]

where \(n\) is the number of absorbing nucleons in the nucleus and \(\sigma_d\) is the deuteron pion-capture cross-section. If we assume (with Brueckner et al.) that the basic pion absorption reactions are just the inverses of the basic pion production reactions (some of which we list in the text, section 1.2) then for positive pions we have \(\sigma_d \geq \sigma(\pi^+d \to pp)\) and \(n \geq N\), i.e., we have absorption by neutrons mostly. For negative pions we have \(\sigma_d \geq \sigma(\pi^-d \to nn)\) and \(n \geq Z\), absorption by protons mostly.

Using the model described above we can write the pion absorption coefficient, defined as absorption per unit volume, as

\[
\eta_a = \frac{n \cdot \Gamma \cdot \sigma_d}{(4/3)\pi R^3}.
\]

To use \([E.2]\) in computing values of \(\eta_a\), Frank et al.\(^{(37)}\) take for \(\sigma_d\) (cf. their footnote (13)) the values
where \( q \) is the maximum center-of-mass momentum available to the pion, in units of \( \mu \cdot c \). This they obtain from the semi-empirical formula of Gell-Mann and Watson\(^{(14)}\). It can also be obtained from Rosenfeld's\(^{(43)}\) expression for \( \sigma_{10} = \sigma(pp \rightarrow \pi^+d) \) and the principle of detailed balance in a low energy approximation.

Frank et al. assume, in accordance with Brueckner et al., that \( \Gamma = 4 \) and they have used (their equation (13)) for the nuclear radius \( R = (\hbar/(\mu \cdot c)) \cdot A^{1/3} \). Putting the above expressions for \( \sigma_d \) and \( R \) into (E.2) gives us an expression for the pion absorption coefficient, viz.

\[
\text{[E.4]} \quad n_a = \frac{1}{\lambda_a} = \frac{\mu \cdot c}{h} \cdot 0.107 \cdot (0.14 + q^2),
\]

which is the Frank et al. expression (their equation (9)) with the \( 1/\lambda^3 \) factor made explicit. Frank et al. implicitly set \( \lambda = 1 \) when they compute values of \( n_\pi = n_a + n_s \), where \( n_s \) is the coefficient for inelastic pion scattering (also depending on a factor of \( 1/\lambda^3 \)).

In deriving [E.4] we considered only positive pion absorption. Brueckner et al. quote references which explain why we may expect the absorption of negative pions to be similar.
APPENDIX F
OUTLINE OF COMPUTER PROGRAM PIPROD

The following is a sketch of a FORTRAN IV program written to numerically integrate equation [11] of the text. The symbols used below are defined in the text.

PIPROD
Read constants $E_f$, $\Psi$, $n_p$, $r_o$, $a$, etc.
Read Frank data on mfp vs. $T_f$ and $V_r$ vs. $T_f$ for a range of $T_f$.
Correct Frank data for units (mfp) and for $(r'_o/r_o)^3$ (mfp, $V_r$).
Calculate $n_f = 1/mfp$ vs. $T_f$ for range of $T_f$.
DO for each $A$ available:
  Read $A$ and $Z$.
  Calculate $R$, $R_{\text{max}}$, $R_c$, and $\rho'_o$.
  Calculate number of $r$-steps and exact step $\Delta r$.
  Set $r = 0$.
  DO for each $r$ up to $R_{\text{max}}$:
    Calculate number of $\theta$-steps and exact $\Delta \theta$.
    Set $\theta = 0$.
    DO for each $\theta$ up to $\pi$:
      Calculate number of $\phi$-steps and exact $\Delta \phi$.
      Set $\phi = 0$.
      DO for each $\phi$ up to $\pi$:
        Calculate $s'_\pi$.
        Calculate number of $s$-steps and exact $\Delta s$.
        Set $s = 0$.
DO for each s up to $s_\pi$:

- Calculate $r$, $\phi$, and $V_c$.
- Calculate $E_\pi = T_\pi + V_r(T_\pi) \cdot \phi + V_c$ vs. $T_\pi$ for range of $T_\pi$.
- Interpolate for $T_\pi$ at $E_\pi$ of interest.
- Interpolate for $n_\pi$ at this $T_\pi$.
- Correct this $n_\pi$ by $\phi$.
- Store corrected $n_\pi$ as, say, $A$ vs. $s$.
- Change $s$ by $\Delta s$.

Integrate $\int A(s) ds$.

Check error and write if critical.

Calculate $\exp(\int A(s) ds)$ and store as, say, $B$ vs. $\phi$.

- Change $\phi$ by $\Delta \phi$.

Integrate $\int B(\phi) d\phi$.

Check error and write if critical.

Calculate $s_\varphi$.

Calculate $2\sin \theta \cdot \exp(-s_\varphi \cdot n_\pi) \cdot \int B(\phi) d\phi$, and store as, say, $C$ vs. $\theta$.

- Change $\theta$ by $\Delta \theta$.

Integrate $\int C(\theta) d\theta$.

Check error and write if critical.

Calculate $\alpha$, $\overline{T}$.

Calculate $r^2 \cdot \int C(\theta) d\theta$ and $r^2 \cdot \int C(\theta) d\theta \cdot \overline{T}$ and store as, say, $D$ and $E$ vs. $r$.

- Change $r$ by $\Delta r$.

Integrate $I_1 = \int D(r) dr$ and $I_2 = \int E(r) dr$.

Calculate $I_2/I_1$.

Store each of $I_1$, $I_2$, and $I_2/I_1$ vs. $A$. 

1
Calculate normalization constants if A appropriate.
Change A.

Normalize I₁, I₂.
Write I₁, I₂, and normalized I₁, I₂ vs. A.
Write Eᵦ, U, normalization constants, and other data of interest.

END

Subroutines:
SIMPLE - Simpson's rule integration.
TINT - third order polynomial interpolation.

Comments:
1) For the first interpolation (for Tπ in the s-loop) check that Tπ vs. Eπ is single-valued. It is for the Frank et al. data.
2) The φ-integration is only between 0 and π because of symmetry in φ (hence the factor 2 in C(θ)).
3) Error in integrals is estimated by re-integration using a doubled step-length.