ON THE SYMMETRIC S- AND D-STATE COMPONENTS OF
THE TRITON WAVE FUNCTION

by

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ABSTRACT

Approximate forms for the symmetric S- and D- state components of the triton wave functions are found using the equivalent two - body approximation of Feshbach and Rubinow. Two coupled, ordinary differential equations for the components are obtained and, for comparison with previous work, are solved numerically with the Feshbach - Pease two nucleon potentials. A further approximation involving one variational parameter is shown to yield good results.

Detailed expressions for the symmetric S- and D-state contributions to the charge form factor of the triton are found and the symmetric S-state contribution is compared to the results of Schiff.

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CHAPTER 1

INTRODUCTION

The triton is the bound state of two neutrons and one proton; the experimental value of its energy is -8.492 Mev. The central theoretical problem concerning this system, of course, is to find the triton wave function and to compute its energy. We are immediately faced then with trying to solve a three - body problem. Major steps in this direction were taken by Derrick and Blatt (1958, 1960a, 1960b) who have constructed a complete set of states in terms of which the triton wave function may be expanded, and derived a set of sixteen partial differential Schrödinger equations in three variables for the expansion coefficients.

The work of Derrick and Blatt, while providing a great simplification of the nuclear problem, still clearly leaves one with an extremely difficult mathematical problem. Indeed, relatively little has been done in finding its solution. Instead the common procedure here has been to compute triton wave functions and energies using a variational parameter approach. That is, a form of the wave function containing various parameters has been assumed, and the expectation of the Hamiltonian of the system has been computed and minimized with respect to these parameters. This approach has obvious disadvantages: there is no guarantee that the assumed function approximates the actual wave function, and there is no direct correspondence between the wave function parameters and the parameters appearing in the nucleon - nucleon potential.

Some work on a non-variational approach to finding a triton wave function and energy has been done, in particular, by Feshbach and Rubinow (1955) and by McMillan (1965). Indeed, this thesis is an extension of the

of the work done by McMillan (1965) on the symmetric S-state component of the triton wave function.

The Feshbach and Rubinow (1955) approach is employed in these calculations. We note that, as has been stressed, for example, by McMillan (1965), this has the advantage over the variational parameter approach of allowing the calculation of the functional form of the approximate wave function.

If one were to continue the Feshbach - Rubinow procedure, one would end up with a set of coupled, second order, ordinary differential equations. However, the large number of components, and hence equations, makes the solution difficult, so further approximations must be made concerning the relative importance of the triton components. For example, we could omit some of the components, or modify the Feshbach - Rubinow procedure to allow the introduction of some variational parameters.

Blatt et al (1962) have shown that the symmetric S-state is the dominant component of the triton wave function followed by the D-state. Using the Derrick and Blatt (1958) kinetic energy argument, one may expect that the symmetric D-state component be the next important, followed by the mixed symmetric components. (We adhere here to the notation of Derrick (1960b) for the D-states.) As pointed out by McMillan (1966b), the Feshbach - Pease (1952) results also indicate this ordering. We shall retain here only the symmetric S- and D- state components of the triton wave function.

Throughout this thesis, the Derrick + Blatt expansion of the triton wave function, rather than the Sachs (1953) expansion, is used. The reasons

are twofold

- 1) the Sachs (1953) expansion is incomplete, as has been pointed out by McMillan (1966a); and
- 2) the Sachs basis functions are not orthonormal, whereas the Derrick - Blatt functions are.

Chapter 2 is a resume of some aspects of the Derrick and Blatt (1958) and Derrick (1960a, 1960b) classification of the components of the triton wave function. Chapter 3 gives a derivation of the coupled, ordinary differential equations which result when the Feshbach - Rubinow approximation is used. Chapter 3 also gives an ordinary differential equation which results when an additional approximation, relating the symmetric S- and D- state components of the triton wave function through a single variational parameter, is introduced. Chapter 4 contains the numerical methods of solving the above two equations and also the numerical results.

Chapter 5 gives detailed expressions for the charge form factor of the triton when the symmetric S- and D- states and the mixed S- state components of the triton wave function are retained. The symmetric S-state contribution to the form factor is evaluated numerically to provide a check on the approximate wave functions developed in chapter 3 and 4.

Chapter 6 gives the conclusions drawn from the calculations, while appendices A to D give detailed expansions of the Derrick - Blatt wave functions.

CHAPTER 2

THE DERRICK - BLATT CLASSIFICATION OF TRITON WAVE FUNCTIONS

The triton (H^3) consists of a bound state of two neutrons and one proton. Experimentally, the triton has total angular momentum $J = \frac{1}{2}$ and even parity. The third component of the isospin is $M_T = -\frac{1}{2}$ and it is assumed to have the good isospin quantum number $T = \frac{1}{2}$.

A complete expansion of the ground state wave function of H^3 has been given by Derrick and Blatt (1958) and Derrick (1960a). (We shall not give a complete account of this work here, but shall refer the reader to the original papers for all details.) Their construction proceeds in a manner analogous to the Blatt and Weisskopf (1952) construction of the 3S_1 and 3D_1 spin - angle functions which are relevant to the deuteron wave function. Since the triton has total momentum $J = \frac{1}{2}$, and since the total spin angular momentum of three nucleons may be $\frac{1}{2}$ or $\frac{3}{2}$, the expansion contains $^2S_{\frac{1}{2}}$, $^2P_{\frac{1}{2}}$, $^4P_{\frac{1}{2}}$ and $^4D_{\frac{1}{2}}$ - states.

The wave function of H^3 consists of a linear combination of functions which depend on spin, isospin, and spatial co-ordinates. These functions consist of one part, called total angular momentum - isospin functions, which depends on spin, isospin, and on spatial co-ordinates through the Euler angles which specify the orientation in space of the triangle formed by the three particles,¹ and on one part, called internal wave functions, which depends on the three interparticle distances. Further,

¹ See appendix C for the Euler angle wave functions.

the symmetric¹ group of order 3 has three irreducible representations, and thus the expansion contains functions which transform according to the symmetric, antisymmetric, or mixed representation of $S_{(3)}$ under a permutation of the three particles. These functions are given explicitly in appendices B to D and Derrick and Blatt (1958). Table 1 gives the permutation properties of these functions. These functions are labelled using the Derrick (1960a) notation. The notation of table 1 is that of Derrick and Blatt (1958) and Derrick (1960a, 1960b) with the addition that (m,k) refers to a function belonging to the $\kappa^{\underline{m}}$ row of the mixed representation, $-(m \ 1)$ means that $-y_{6,2}$, $-y_{7,2}$, and $-y_{10,2}$ are the relevant partner functions under joint spin-isospin permutations.

¹ See appendix A for the properties of the symmetric group of order 3.

TABLE 1 PERMUTATION PROPERTIES OF THE
WAVE FUNCTIONS

Spectroscopic Classification	Total Angular Momentum - Isospin Functions	Permutation Symmetry			Internal Functions
		Euler Angles	Spin- Isospin	Euler Angles Spin-Isospin	
${}^2S_{\frac{1}{2}}$	y_1	s	a	a	s
	y_2	s	s	s	a
	$y_{3,1}$	s	m,1	m,1	m,2
	$y_{3,2}$	s	m,2	m,2	m,1
${}^2P_{\frac{1}{2}}$	y_4	a	s	a	s
	y_5	a	a	s	a
	$y_{6,1}$	a	m,2	m,1	m,2
	$y_{6,2}$	a	-(m,1)	m,2	m,1
	$y_{7,1}$	a	m,2	m,1	m,2
	$y_{7,2}$	a	-(m,1)	m,2	m,1
${}^4D_{\frac{1}{2}}$	$y_{8,1}$	s	m,1	m,1	m,2
	$y_{8,2}$	s	m,2	m,2	m,1
	$y_{9,1}$	s	m,1	m,1	m,2
	$y_{9,2}$	s	m,2	m,2	m,1
	$y_{10,1}$	a	m,2	m,1	m,2
	$y_{10,2}$	a	-(m,1)	m,2	m,1

The total wave function of the triton must be antisymmetric. Thus the internal wave functions which Derrick and Blatt (1958) define must transform according to the last column of table 1 under permutations of the three particles. The total wave function (ψ) is given in Derrick and Blatt (1958) and Derrick (1960a) and, for convenience, is written here:

$$\begin{aligned}
 (2-1) \quad \psi = & f_1 y_1 + f_2 y_2 + 2^{-\frac{1}{2}} (f_{3,1} y_{3,2} - f_{3,2} y_{3,1}) + f_4 y_4 + f_5 y_5 + \\
 & + 2^{-\frac{1}{2}} (f_{6,1} y_{6,2} - f_{6,2} y_{6,1}) + 2^{-\frac{1}{2}} (f_{7,1} y_{7,2} - f_{7,2} y_{7,1}) + 2^{-\frac{1}{2}} (f_{8,1} y_{8,2} - f_{8,2} y_{8,1}) \\
 & + 2^{-\frac{1}{2}} (f_{9,1} y_{9,2} - f_{9,2} y_{9,1}) + 2^{-\frac{1}{2}} (f_{10,1} y_{10,2} - f_{10,2} y_{10,1})
 \end{aligned}$$

Derrick (1960a, 1960b) has further defined a new set of momentum-isospin wave functions for the D-states as follows:¹

$$(2-2a) \quad y_{11,1} = R^2 (3^{-\frac{1}{2}} y_{8,1} - \sin \lambda y_{9,1}) \quad , \quad y_{11,2} = R^2 (3^{-\frac{1}{2}} y_{8,2} - \sin \lambda y_{9,2})$$

$$(2-2b) \quad y_{12} = F (3^{-\frac{1}{2}} y_{8,1} - \csc \lambda y_{9,1} - \cot \lambda y_{10,1}) + G (3^{-\frac{1}{2}} y_{8,2} - \csc \lambda y_{9,2} - \cot \lambda y_{10,2})$$

$$(2-2c) \quad y_{13} = -G (3^{-\frac{1}{2}} y_{8,1} - \csc \lambda y_{9,1} - \cot \lambda y_{10,1}) + F (3^{-\frac{1}{2}} y_{8,2} - \csc \lambda y_{9,2} - \cot \lambda y_{10,2})$$

$$(2-2d) \quad y_{14,1} = F (3^{-\frac{1}{2}} y_{8,1} - \csc \lambda y_{9,1} + \cot \lambda y_{10,1}) - G (3^{-\frac{1}{2}} y_{8,2} - \csc \lambda y_{9,2} + \cot \lambda y_{10,2})$$

$$y_{14,2} = -G (3^{-\frac{1}{2}} y_{8,1} - \csc \lambda y_{9,1} + \cot \lambda y_{10,1}) - F (3^{-\frac{1}{2}} y_{8,2} - \csc \lambda y_{9,2} + \cot \lambda y_{10,2})$$

where

$$(2-3) \quad R^2 = n_{12}^2 + n_{13}^2 + n_{23}^2$$

$$F = 3^{-\frac{1}{2}} (n_{23}^2 + n_{13}^2 - 2n_{12}^2)$$

$$G = n_{23}^2 - n_{13}^2$$

$$\cos \lambda = \frac{4\sqrt{3} \Delta}{R^2}$$

$$0 \leq \lambda \leq \frac{\pi}{2}$$

$$\Delta = \frac{1}{2} | \underline{n}_{13} \times \underline{n}_{12} |$$

¹ Derrick (1960b) labels these λ_{11} to λ_{14} .

and $(y_{11,1}, y_{11,2})$, $(y_{14,1}, y_{14,2})$ are pairs of functions which transform according to the mixed representation of S_3 (see appendix A), y_{12} transforms according to the antisymmetric representation of S_3 , and y_{13} transforms according to the symmetric representation of S_3 .

It is convenient for later work to record here the following quantities defined by Derrick (1960a, 1960b):

$$(2-4) \quad \Lambda = \sqrt{R^4 - 48\Delta^2}$$

$$3(F^2 + G^2) = \Lambda^2$$

$$a = \frac{3^{\frac{1}{2}}}{\sin \lambda} \left(\frac{R^2 \cos^2 \lambda}{3 R_{12}^2} - 1 \right)$$

$$b = \frac{G \cot \lambda}{R_{12}^2}$$

$$\cos \theta_3 = \frac{R_{23}^2 + R_{13}^2 - R_{12}^2}{2 R_{23} R_{13}}$$

$$N_3 = \frac{4}{3}(G^2 - F^2), \quad N_4 = \frac{8}{3}FG, \quad N_5 = R^4 \left(\frac{2}{3} - \frac{2}{9} \sin^2 \lambda \right)$$

Using the new Euler angle-spin-isospin D-states and the definition for the total D-state wave function in Derrick (1960b) one can rewrite the internal D-state wave functions $(f_{8,1}, \dots, f_{10,2})$ in terms of the new internal D-state wave functions $(f_{11,1}, \dots, f_{14,2})$ as

$$(2-5a) \quad f_{8,1} = \left(\frac{2}{3}\right)^{\frac{1}{2}} \left[R^2 f_{11,1} + G f_{12} + F f_{13} + G f_{14,2} - F f_{14,1} \right]$$

$$(2-5b) \quad f_{8,2} = -\left(\frac{2}{3}\right)^{\frac{1}{2}} \left[-R^2 f_{11,2} + F f_{12} - G f_{13} - F f_{14,2} - G f_{14,1} \right]$$

$$(2-5c) \quad f_{9,1} = -2^{\frac{1}{2}} \left[R^2 \sin \lambda f_{11,1} + G \csc \lambda f_{12} + F \csc \lambda f_{13} + G \csc \lambda f_{14,2} - F \csc \lambda f_{14,1} \right]$$

$$(2-5d) \quad f_{9,2} = -2^{\frac{1}{2}} \left[R^2 \sin \lambda f_{11,2} - F \csc \lambda f_{12} + G \csc \lambda f_{13} + F \csc \lambda f_{14,2} + G \csc \lambda f_{14,1} \right]$$

$$(2-5e) \quad f_{10,1} = 2^{\frac{1}{2}} \left[-G \cot \lambda f_{12} - F \cot \lambda f_{13} + G \cot \lambda f_{14,2} - F \cot \lambda f_{14,1} \right]$$

$$(2-5f) \quad f_{10,2} = 2^{\frac{1}{2}} \left[-F \cot \lambda f_{12} + G \cot \lambda f_{13} - F \cot \lambda f_{14,2} - G \cot \lambda f_{14,1} \right]$$

CHAPTER 3

INTERNAL WAVE EQUATIONS¹

3.1 The variational principle

Derrick (1960a, 1960b) has given the set of coupled partial differential equations for the internal wave functions in the Derrick - Blatt (1958, 1960a, 1960b) expansion of the $J = T = \frac{1}{2}$ three nucleon wave function. In the Feshbach - Rubinow approach, however, the dynamical statement is a variational principle from which differential equations are obtained using the Euler - Lagrange equations. In the problem at hand where, as explained in the introduction, only the symmetric S- and D- state components (i.e., components 1 and 13 in the Derrick (1960a, 1960b) notation) of the triton are retained, the relevant variational principle² is

$$(3-1) \quad \delta \int d\tau L = 0$$

where

$$(3-2) \quad \int d\tau = \int_0^\infty dr_{23} \int_0^\infty dr_{13} \int_{|r_{23}-r_{13}|}^{r_{23}+r_{13}} dr_{12} r_{23} r_{13} r_{12}$$

¹ This chapter is essentially sections 2 and 3 of Best and McMillan (1966) and is based on work done by McMillan.

² That this is a suitable variational principle may be checked by noting that the corresponding Euler - Lagrange equations are the dynamical equations given by Derrick (1960a, 1960b).

and

$$(3-3) \quad L = \sum_{\text{cyclic}} \left\{ -\frac{\hbar^2}{M} \left[\left(\frac{\partial f_1}{\partial r_{12}} \right)^2 + \cos \theta_3 \frac{\partial f_1}{\partial r_{23}} \frac{\partial f_1}{\partial r_{13}} + N_5 \left\{ \left(\frac{\partial f_{13}}{\partial r_{12}} \right)^2 + \cos \theta_3 \frac{\partial f_{13}}{\partial r_{23}} \frac{\partial f_{13}}{\partial r_{13}} \right\} \right] \right. \\ \left. - \frac{1}{2} \left[V^s(r_{12}) + V^{ct}(r_{12}) \right] f_1^2 - \left[N_5 V^{ct}(r_{12}) + (F^2 + G^2) \left(2 \cdot 3^{-\frac{1}{2}} a \csc \lambda + \frac{1}{3} - \csc^2 \lambda - \cot^2 \lambda \right) V^t(r_{12}) \right] f_{13}^2 \right. \\ \left. - 2 \left[F(a \csc \lambda - 3^{-\frac{1}{2}}) + G b \cot \lambda \right] V^t(r_{12}) f_1 f_{13} \right\} + E \left[f_1^2 + N_5 f_{13}^2 \right]$$

where throughout the Derrick (1960a, 1960b) notation has been adhered to and the above functions are defined in equation (2 - 4). In addition the two-nucleon potential has been written as follows¹

$$(3-4a) \quad V_{12} = \frac{1}{4} (1 - \underline{\sigma}_1 \cdot \underline{\sigma}_2) V^s(r_{12}) + \frac{1}{4} (3 + \underline{\sigma}_1 \cdot \underline{\sigma}_2) V^{ct}(r_{12}) + S_{12} V^t(r_{12})$$

where $V^s(r_{12})$ is the singlet spin potential, $V^{ct}(r_{12})$ is the central triplet potential, and $V^t(r_{12})$ is the tensor potential with S_{12} the two particle tensor operator:

$$(3-4b) \quad S_{12} = \frac{3(\underline{\sigma}_1 \cdot \underline{r}_{12})(\underline{\sigma}_2 \cdot \underline{r}_{12})}{r_{12}^2} - \underline{\sigma}_1 \cdot \underline{\sigma}_2$$

The functions² f_1 and f_{13} are both completely symmetric functions of the three interparticle distances and are normalized according to

$$(3-5) \quad \int d\tau \left[f_1^2 + N_5 f_{13}^2 \right] = 1$$

¹ The Feshbach - Pease (1952) potentials are of this form.

² Note here that it follows from the work of McMillan (1966a) that the Derrick functions f_1 and f_{13} are proportional to the Sachs (1953) functions f_1 and f_7 respectively.

3.2 The coupled Feshbach - Rubinow equations

We now come to the central approximation: we assume that f_1 and f_{13} depend only on the single symmetric variable¹

$$(3-6) \quad r = \frac{1}{2} (r_{12} + r_{13} + r_{23})$$

i.e., we assume that

$$(3-7a) \quad f_1 = f_1(r)$$

$$(3-7b) \quad f_{13} = f_{13}(r)$$

Feshbach and Rubinow (1955) use approximation (3-7a); it may be regarded as a generalization of the trial exponential function used by Blatt and Weisskopf (1952). Approximation (3-7b) may, in the same way, be regarded as a generalization of the dominant part of the symmetric D-state component of the Feshbach - Pease (1952) trial function²

When approximation (3-7) is made, some of the integrals in (3-1) may be easily evaluated to yield

$$(3-8) \quad \delta \int_0^\infty dr \left\{ -\frac{\hbar^2}{M} \left[\left(\frac{du}{dr} \right)^2 - \frac{5u}{r} \left(\frac{du}{dr} \right) + \frac{25}{4} \frac{u^2}{r^2} + \left(\frac{dw}{dr} \right)^2 - \frac{9w}{r} \left(\frac{dw}{dr} \right) + \frac{81}{4} \frac{w^2}{r^2} \right] \right. \\ \left. - V^+(r) u^2 - [V^t(r) - \tilde{V}^t(r)] w^2 - 2 \tilde{V}^t(r) u w + \frac{14}{15} E (u^2 + w^2) \right\} = 0$$

¹ Feshbach and Rubinow (1955) and McMillan (1965) denote this variable by R . We refrain from doing so here, in order to avoid confusion with the Derrick (1960a, 1960b) R .

² One may consult McMillan (1966b) to find the Feshbach - Pease functions written in the Derrick - Blatt notation.

where we have defined u and w by¹

$$(3-9a) \quad f_1(r) = \sqrt{\frac{30}{7}} \frac{u(r)}{r^{\frac{5}{2}}}$$

$$(3-9b) \quad f_{13}(r) = \frac{9}{\sqrt{22}} \frac{w(r)}{r^{\frac{7}{2}}}$$

so that the normalization integral (3-5) becomes

$$(3-10) \quad \int_0^\infty (u^2(r) + w^2(r)) dr = 1$$

where the potential terms are

$$(3-11) \quad V^+(r) = 24 \int_0^1 dz z^2 \left(1 - z + \frac{z^2}{6}\right) \left\{ \frac{1}{2} [V^s(rz) + V^{ct}(rz)] \right\}$$

$$(3-12) \quad V^{ct}(r) = \frac{2016}{55} \int_0^1 dz z^2 \left(1 - 3z + \frac{35}{6} z^2 - \frac{20}{3} z^3 + 4z^4 - \frac{7}{6} z^5 + \frac{3}{28} z^6\right) V^{ct}(rz)$$

$$(3-13) \quad V^t(r) = \frac{672}{5} \int_0^1 dz z^4 \left(1 - 2z + \frac{59}{385} z^2 - \frac{212}{385} z^3 + \frac{9}{154} z^4\right) V^t(rz)$$

$$(3-14) \quad \tilde{V}^t(r) = \frac{336}{5} \sqrt{\frac{21}{55}} \int_0^1 dz z^4 \left(1 - z + \frac{z^6}{6}\right) V^t(rz)$$

The Euler - Lagrange equations which follow from the variational principle (3-8) are

$$(3-15a) \quad -\frac{\hbar^2}{M} \left(\frac{d^2 u}{dr^2} - \frac{15u}{4r^2} \right) + V^+(r)u + \tilde{V}^t(r)w = \frac{14}{15} E u$$

$$(3-15b) \quad -\frac{\hbar^2}{M} \left(\frac{d^2 w}{dr^2} - \frac{63w}{4r^2} \right) + [V^{ct}(r) - V^t(r)]w + \tilde{V}^t(r)u = \frac{14}{15} E w$$

¹ McMillan (1965) has shown also that the Feshbach - Rubinow approximation (3-9a) is more tenable than the Morpurgo (1952) approximation

$$f_1 = f_1 \left(\sqrt{r_{12}^2 + r_{13}^2 + r_{23}^2} \right)$$

Thus, assumption (3-7) has allowed us to reduce the problem of finding the symmetric S- and D- state components of the triton wave function to the solving of two coupled, ordinary differential equations. Equations (3-15) are called "the coupled Feshbach - Rubinow equations". It is worth stressing that the employment of variational parameters has not been resorted to, rather the triton energy appears as the eigenvalue of the coupled Feshbach - Rubinow equations.

The similarity of the coupled Feshbach - Rubinow equations to the well-known pair of equations for the deuteron 3S_1 and 3D_1 components should be noted. In particular, one sees that the triton $^2S_{\frac{1}{2}}$ and $^4D_{\frac{1}{2}}$ components are also coupled by the tensor force (as, of course, is well-known), and that the potential term in the D-state equation (3-15b) involves the difference of central-triplet and tensor potentials. On the other hand, the potential terms in the S-state equation (3-15a) involves a singlet spin potential (which, of course, does not appear in the corresponding deuteron problem), and also the "centrifugal barrier terms" involve $(\frac{3}{2} + \ell)(\frac{3}{2} + \ell + 1)$ for $\ell = 0, 2$ rather than $\ell(\ell + 1)$.

We close this section by giving expressions for the D-state probability (P_D), and for the coulomb energy ($E_{coul.}$) of He . It follows from normalization integral (3-10) that

$$(3-16) \quad P_D = \int_0^{\infty} w^2(r) dr$$

and from Derrick (1960a, 1960b) that¹

$$E_{\text{coul.}} = e^2 \int d\tau \left[\frac{f_1^2}{r_{12}} + (N_5 + N_3) \frac{f_{13}^2}{r_{12}} \right]$$

which when approximation (3-7) and definitions (3-9) are used becomes

$$(3-17) \quad E_{\text{coul.}} = \frac{25}{14} e^2 \left[\int_0^\infty \frac{u^2(r)}{r} dr + \frac{243}{275} \int_0^\infty \frac{w^2(r)}{r} dr \right]$$

¹ N_5 and N_3 are defined in equation(2-4). On comparing (3-17) and (3-5) one may think a term involving N_3 should appear in both equations. We note however, that

$$\int d\tau N_3 f_{13}^2 = 0$$

3.3 The modified Feshbach - Rubinow equation

The eigenvalue of the coupled Feshbach - Rubinow equations (3-15) will, because the equations follow from variation principle (3-8) without approximation, be better (i.e., lower) than a triton energy computed in any other way with f_1 and f_2 of the form given by equations (3-7). Nevertheless, there is some value in considering further approximations at this point. Indeed, if one were to take into account the remaining components of the triton wave function and further employ the Feshbach - Rubinow single symmetric¹ variable approximation, one would, by following the procedure of the preceeding section, arrive at a set of coupled, ordinary differential equations for the components, the number of equations depending on the number of components retained. The solution of a large number of coupled eigenvalue equations would be difficult. We now give a modification of the procedure used in the preceeding section which lends itself to a considerable simplification, and which yields good results as we show in the next section.

We assume that the functions u and w defined by equations (3-9) are related as follows

$$(3-18) \quad w(r) = \alpha u(r)$$

where α is a parameter. When approximation (3-18) is used, variational

¹ In this connection, we note that all of the Derrick - Blatt internal functions can be expressed in terms of symmetric functions, as has been pointed out by McMillan (1966a)

principle (3-8) contains a single function to be determined, and the Euler - Lagrange equation reads

$$(3-19) \quad -\frac{\hbar^2}{M} \left[(1+\alpha^2) \frac{d^2 u}{dr^2} - \frac{15+63\alpha^2}{4} \frac{u}{r^2} \right] + \left[V(r) + 2\alpha \tilde{V}(r) + \alpha^2 (V^c(r) - V(r)) \right] u = \frac{14}{15} E (1+\alpha^2) u$$

Thus, assumption (3-18) has allowed us to reduce the problem of finding the symmetric S- and D- state components to the solving of a single ordinary differential equation. Equation (3-19) is called the "modified Feshbach - Rubinow equation"¹. It will be noted that the eigenfunction u and the eigenvalue E depend upon the parameter α , and that the best triton energy provided with approximation (3-18) is the minimum E with respect to α .

The normalization integral, D-state probability, and the coulomb energy of He in this case follow immediately from equations (3-10) (3-16) and (3-17) and approximation (3-18).

One has

$$(3-20) \quad \int_0^\infty u^2(r) dr = \frac{1}{1+\alpha^2}$$

$$(3-21) \quad P_D = \frac{\alpha^2}{1+\alpha^2}$$

$$(3-22) \quad E_{coul.} = \frac{25e^2}{14} \left(1 + \frac{243}{275} \alpha^2 \right) \int_0^\infty \frac{u^2(r)}{r} dr$$

¹ Setting $\alpha=0$ in this equation yields the Feshbach - Rubinow (1955) equation (also see McMillan(1965)).

CHAPTER 4

NUMERICAL SOLUTIONS TO THE INTERNAL WAVE EQUATIONS

4.1 The modified Feshbach - Rubinow equation (3-19)

For the purpose of the numerical integration, equation (3-19) has been rewritten in the following form:

$$(4-1) \quad \frac{d^2 u}{dn^2} - \frac{c}{1+\alpha^2} \left[Y_3(n) + \alpha^2 Y_7(n) + \alpha Y_c(n) + (1+\alpha^2) \lambda \right] u = 0$$

where use has been made of equations (f - 1)

For all the numerical calculations, the potentials appearing in (3-4) are assumed to be of the Yukawa type:

$$(4-2a) \quad V^s(n) = \frac{V_{os} e^{-\mu_s n}}{\mu_s n}$$

$$(4-2b) \quad V^{ct}(n) = \frac{V_{ot} e^{-\mu_t n}}{\mu_t n}$$

$$(4-2c) \quad V^t(n) = \frac{V_{ote} e^{-\mu_{te} n}}{\mu_{te} n}$$

The parameters defined in equations (4-2) will be those used by Feshbach - Pease (1952) and are listed in table 2. The corresponding functions $Y_3(n)$, $Y_7(n)$, $Y_c(n)$ for the FP No. 1 potential are, for illustrative purposes, shown in figure 1.

TABLE 2 PARAMETERS FOR THE FESHBACH - PEASE (1952)
POTENTIALS

Potential	V_{os} (mev)	V_{ot} (mev)	V_{ote} (mev)	μ_s (F ⁻¹)	μ_t (F ⁻¹)	μ_{te} (F ⁻¹)
FP No. 1	-55.0323	-65.1839	-26.3907	.84459	.84459	.47125
FP No. 2	-55.0323	-59.3456	-34.7892	.84459	.84459	.54735
FP g= 0	-55.0323	-55.0323	-43.2731	.84459	.84459	.59067
FP No. 3	-55.0323	-48.4469	-49.5169	.84459	.84459	.65232

Equation (4-1) is an ordinary differential equation with eigenvalue λ . In order to determine this eigenvalue, one makes use of the condition that the eigenfunction $u(r)$ and $u'(r)$, its derivative with respect to r , be continuous wherever the potentials $Y(r)$ are finite. As pointed out in appendix F, the potentials defined in (f-1) are all infinite at the origin. Thus, the integration outwards from the origin must be started at a small finite value of r , which we have taken to be 0.00001F. The explicit form of the boundary conditions for small r is given by equation (f-2). Similarly, the explicit form of the boundary conditions for large r is given by equation (f-4); these were applied at $r = 30F$.

For a fixed value of the parameter α , the procedure used to find the eigenvalue λ and the eigenfunction $u(r)$ is:

1. Applying the boundary conditions (f-2) at $r = 0.00001F$, equation (4-1) is integrated outwards using the Runge - Kutta method to $r = 3F$. (the matching point) and the logarithmic derivative of $u(r)$ at the matching point is found.

2. Applying the boundary conditions (f-4) at $r = 30F$ for some arbitrary non-zero value of λ , equation (4-1) is integrated inwards using the Runge - Kutta method to $r = 3F$ and the logarithmic derivative of $u(r)$ at the matching point is found.
3. One now computes the difference in the two logarithmic derivatives calculated above; the two steps are then repeated for different values of λ until the difference in the logarithmic derivatives is arbitrarily small (we chose this to be less than 10^{-5}). The value of λ thereby obtained is the eigenvalue of (4-1) for that fixed value of α chosen.
4. The eigenfunction $u(r)$ is made continuous by multiplying all the values of $u(r)$ from $3F$ to $30F$ by the ratio of $u(3)$ calculated from the integration outwards to $u(3)$ calculated from the integration inwards. The continuity of $u(r)$ then follows from the continuity of the logarithmic derivative. Finally, one normalizes the eigenfunction $u(r)$ according to (3-20).

The above procedure is repeated for a range of values of the parameter α . One then plots the energy eigenvalue E verses the parameter α . The minimum value of E verses α is the triton binding energy. Figure 2 shows a typical graph of E verses α for the potential FP No.1. To show the dependance of the eigenfunction $u(r)$ on the parameter α , figure 3 gives a graph of $u(r)$ for $\alpha = 0$ and $u(r)$ for the value of α which gives a minimum

¹ All numerical calculations were performed on the IBM 7040 digital computer at the U.B.C. Computing Center.

value of E versus r for the FP No.1 potential. Table 3 summarizes the results of these calculations for the 4 Feshbach - Pease (1952) potentials. This table contains the minimum value of energy, the D-state probability (3-21), and the Coulomb energy of He^3 (3-22). Also the eigenvalues for the case $\alpha = 0$ are included.

4.2 The coupled Feshbach - Rubinow equations (3-15)

Again for the purposes of the numerical integration, equations (3-15) have been rewritten in the following form

$$(4-3a) \quad \frac{d^2 u}{dr^2} - c \left[(\lambda + \gamma_3(r)) u + \frac{\gamma_c(r)}{2} w \right] = 0$$

$$(4-3b) \quad \frac{d^2 w}{dr^2} - c \left[(\lambda + \gamma_7(r)) w + \gamma_c(r) u \right] = 0$$

The potentials used were again those defined in equations (4-2) with the parameters listed in table 2.

Equations (4-3) are a pair of coupled, ordinary differential equations with eigenvalue λ . To find the eigenvalue λ , one makes use of the condition that the eigenfunctions $u(r)$ and $w(r)$, and their derivatives with respect to r , $u'(r)$ and $w'(r)$, are continuous wherever the potentials $Y(r)$ are finite. As with equation (4-1), equations (4-3) have their integrations outward starting at a small finite value of r , which we have taken at $r = 0.01F$. The explicit form of the boundary conditions for small r are given by equations (f-5). Similarly, the explicit forms of the boundary conditions for large r are given by equations (f-7); these were applied at $30F$.

One sees from the boundary conditions (f-5) and (f-7) that there are two unknown parameters, $AVAL$ and β , besides the eigenvalue λ .

The procedure used to find the eigenvalue λ is as follows:

1. For fixed β and λ and applying boundary conditions (f-5) with an arbitrary value of AVAL at $r = 0.01F$, equations (4-3) are integrated outwards using the Runge - Kutta method to a matching point of $r = 3F$. and the logarithmic derivatives of $u(r)$ and $w(r)$ at the matching point are found.
2. For the same fixed values of β and λ and applying boundary conditions (f-7) at $r = 30F$, equations (4-3) are integrated inwards using the Runge - Kutta method to the matching point and the logarithmic derivatives of $u(r)$ and $w(r)$ at the matching point are found.
3. Again with the same β and λ , one now computes the difference in the logarithmic derivatives of $u(r)$ and $w(r)$ calculated above. Applying the continuity condition for $u(r)$ and $u'(r)$ at the matching point, the above two steps are repeated, with the same β and λ , for different values of AVAL until the difference in the logarithmic derivatives of $u(r)$ (UM) is arbitrarily small (we chose this to be less than 10^{-3}). One now makes $u(r)$ continuous, as in step 4 of section 4.1, by multiplying all the values of $u(r)$ from $r = 3F$. to $r = 30F$. by the ratio of $u(3)$ calculated from the integration outwards to $u(3)$ calculated from the integrations inwards. As before, the continuity of $u'(r)$ then follows. At the same time $w(r)$ and $w'(r)$ from $r = 3F$. to $r = 30F$. are also multiplied by the above ratio. In general, when this is done the difference in the logarithmic derivatives of $w(r)$ (WM) and the difference in the values of $w(r)$ from the inwards and outwards integration (WC) will not be arbitrarily small.
4. One now varies β and λ , each time repeating the above three steps, until both WM and WC are arbitrarily small. In this case the smallness

of w_m and w_{cc} depends on the accuracy of the graphs used when plotting w_m and w_{cc} verses β for fixed λ and $|u_m| < 10^{-3}$. For illustrative purposes, a typical graph of w_m and w_{cc} verses β for $|u_m| < 10^{-3}$ and fixed λ for the FP No.1 potential is given in figure 4. Two points should be noted from this graph. First, the accuracy of the eigenvalue depends on the accuracy of the graphical plots and second, there exists a unique set of parameters $AVAL$, β , and λ which makes $u(r)$, $w(r)$ and $u'(r)$, $w'(r)$ continuous. This value of λ is the eigenvalue for equations (4-3).

5. The eigenfunctions $u(r)$ and $w(r)$ are next normalized according to equation (3-10).

The matching point and step size of the numerical integration were varied by changing the matching point to $r = 2.5F$. to see the change in the eigenvalue λ and the D-state probability (3-16). No significant change in these quantities or in the form of the eigenfunctions was found. The final value of $AVAL$ changed, however, and we conclude that this reflects the invalidity of equations (f-5). We attach no importance to this fact; indeed, we feel now that detailed expansions of the functions near the origin is unnecessary and that it is sufficient to use some parameter involving the ratio of u and w . (The boundary conditions (f-7) were changed by leaving off the last terms and again no significant change was noted).

Finally, the eigenvalue E , D-state probability (3-16), and Coulomb energy of He (3-17) are tabulated in table 3.

4.3 Numerical results

From table 3, the similarity between the eigenvalue of the Feshbach - Rubnow equation and the energy calculated with the best trial

exponential function used by Blatt and Weisskopf (1952)¹, and between the eigenvalues of the coupled and modified Feshbach - Rubinow equations has encouraged one to calculate various overlap integrals. More specifically, we define

$$(4-4) \quad I_1 = \frac{4K^3}{\sqrt{7}} \int d\tau e^{-K\tau} f_1(\tau) = \sqrt{\frac{8}{15}} \int_0^\infty \tau^{\frac{5}{2}} e^{-K\tau} u_0(\tau) d\tau$$

$$(4-5) \quad I_2 = \frac{\int_0^\infty \tilde{u}(\tau) u(\tau) d\tau}{\int_0^\infty u^2(\tau) d\tau}$$

$$(4-6) \quad I_3 = \frac{\int_0^\infty \tilde{w}(\tau) w(\tau) d\tau}{\int_0^\infty w^2(\tau) d\tau}$$

The eigenfunctions $\tilde{u}(\tau)$ and $\tilde{w}(\tau)$ are the solutions to (4-1) for the best value of α , the eigenfunctions $u(\tau)$ and $w(\tau)$ are the solutions to (4-3), u_0 is the solution to (4-1) with $\alpha = 0$. The solutions to (4-1) are normalized according to (3-20) and the solutions to (4-3) are normalized according to (3-10). The exponential function is normalized according to

$$(4-7) \quad N^2 \int_0^\infty \tau^5 e^{-2K\tau} d\tau = 1$$

where² $N = \sqrt{\frac{8}{15}} K^3$. The value of K for the potentials defined by the parameters in table 2 are found from equations (2.4) and (2.8) of Blatt and Weisskopf (1952):

$$(4-8) \quad K = \sqrt{\frac{14}{15} \frac{T(S)}{b^2}}$$

¹ This similarity has also been noted by Feshbach and Rubinow (1952).

² The normalization (2.3) in Blatt and Weisskopf (1952) on page 196 seems incorrect. The right hand side should be $\frac{4}{\sqrt{7}} K^3$

TABLE 3

The first column designates the Feshbach - Pease (1952) potentials; the columns headed "FP" contain the final Feshbach - Pease (1952) results; those headed "coupled FR", the results of solving (4-2); those headed "modified FR", the results of solving (4-1), for the best α . The second column is for the best trial exponential function; the third is the results of solving (4-1) for $\alpha = 0$. The experimental results for the triton energy and the coulomb energy of He^3 are $E = -8.492$ mev and $E_{\text{coul.}} = 0.764$ mev.

Potential	E (mev)					P_D (%)			$E_{\text{coul.}}$ (mev)		
	Symmetric S-State only		S + D			FP	coupled FR	modified FR	FR	coupled FR	modified FR
	Exp. Function	FR	FP	coupled FR	modified FR						
FP No.1	-5.45	-5.89	-10.05	-8.10	-8.03	2.2	1.9	1.6	1.088	1.061	1.059
FP No.2	-3.27	-3.77	- 9.06	-6.29	-6.24	2.8	2.8	1.9	1.059	0.997	0.995
FP $g = 0$	-2.01	-2.47	-8.40	-5.37	-5.32	3.1	3.1	2.4	1.038	0.961	0.960
FP No.3	-0.2	-0.92	- 7.50	-3.26	-3.22	3.6	3.6	2.4	1.008	0.861	0.861

where b , the intrinsic range parameter defined on page 56 of Blatt and Weisskopf (1952) is equal to 2.5096 for all four potentials in table 2, and where $T(S)$ verses S can be found on page 198 of Blatt and Weisskopf (1952). The parameter S is related to the Yukawa potential depth as follows:

$$(4-9) \quad S = \frac{0.0201255}{2} [V_{0s} + V_{0t}]$$

where we have used equation (2.17) on page 56 of Blatt and Weisskopf (1952), and the fact that the singlet and central triplet potentials have the same range. Table 4 gives the overlap integrals for the four Feshbach - Pease (1952) potentials.

TABLE 4 OVERLAP INTEGRALS

Potential	I_1	I_2	I_3
FP No.1	.81	.97	.89
FP No.2	.83	.96	.91
FP $g = 0$	.85	.95	.93
FP No.3	.86	.93	.95

Table 4, along with table 3, provide a measure of the relative merits of the approximate internal wave functions. Thus, the nearness of I_1 to unity provides a measure of the merit of the best trial exponential wave function and the solution to (4-1) with $\alpha = 0$, and the nearness of I_2 and I_3 to unity provides a measure of the relative merit of the solutions of (4-1) and (4-3).¹

The eigenfunctions of (4-3) are plotted in figures 5 to 8. Figures 9 and 10 show the functions whose overlap I_1 and I_3 appear in the first line of table 4. The eigenfunctions $u(r)$ and $\tilde{u}(r)$ are identical

to two figures of accuracy and have not been graphed.

¹ No compensation has been attempted for the fact that the various functions correspond to slightly different energies.

CHAPTER 5

THE CHARGE FORM FACTOR OF H^3

5.1 Introduction

Electron scattering from H^3 yields information about the charge and magnetic moment distribution of the triton. The experimental results are presented as functions which depend on the momentum transferred from the electron to the triton and are called the charge and magnetic form factors.

Expressions for the charge form factor of the triton are developed in sections 5.2 to 5.4. These expressions contain the charge form factor of the proton and neutron and integrals which involve the internal wave functions. Hence, we have a means of testing the reasonableness of the approximate internal wave functions developed in chapters 3 and 4.

The contributions to the form factor from the antisymmetric S-state and from the P-states are completely neglected since these states occur in the ground state with negligible probabilities (see Blatt et al, 1962).

The calculation of the charge form factor is done using the Derrick and Blatt (1958) expansion of the triton wave function, whereas Schiff (1964) and Gibson and Schiff (1964) use the Sachs (1953) expansion. It is shown in appendix G that the S-state contribution to the charge form factor given by Schiff (1964) is identical to ours given in section 5.3; no comparison of D-state contributions has been made however.

Finally it is worth pointing out that the original Derrick - Blatt functions are all orthonormal while the D-state functions used by Gibson and Schiff (1964) are not orthogonal. Using the Derrick - Blatt expansion, it is relatively simple to compute the charge form factor while it appears that the computation involved using the Sachs (1953) expansion is rather complicated.

5.2 Definition of the charge form factor

If $\underline{P}_i$, $\underline{P}_i^e$ are the initial momentum of the triton and electron respectively and $\underline{P}_f$, $\underline{P}_f^e$ are the final momentum of the triton and electron respectively, then the change in wave number of the electron ($\underline{q}$) and the triton ($\underline{K}$) are

$$(5-1a) \quad \underline{q} = \frac{1}{\hbar} (\underline{P}_f^e - \underline{P}_i^e)$$

$$(5-1b) \quad \underline{K} = \frac{1}{\hbar} (\underline{P}_f - \underline{P}_i)$$

The charge form factor of the triton $\tilde{F}_c(H^3)$, as defined in Schiff (1964), is

$$(5-2) \quad \tilde{F}_c(H^3) = \int e^{i\underline{q} \cdot \underline{r}} \langle \rho_c \rangle d^3r$$

where $\langle \rho_c \rangle$ is the expectation value of the charge density operator

$$(5-3) \quad \rho_c = \frac{1}{2} \sum_{K=1}^3 \left[(1 + \tau_{Kz}) f_c^p(\underline{r} - \underline{r}_K) + (1 - \tau_{Kz}) f_c^n(\underline{r} - \underline{r}_K) \right]$$

and

$$\tau_{Kz} = 2T_{Kz}$$

(see section B-2) and operates on the K^{th} particle in isospin space,

f_c^p and f_c^n are the charge density distributions of the proton and neutron respectively. Figure 11 shows the vectors $\underline{r}, \underline{r}_1, \underline{r}_2, \underline{r}_3$.

The triton wave function ψ defined in equation (2-1) does not include the center of mass motion. However, for the calculation of the charge form factor, this center of mass motion is necessary.

We assume the center of mass wave function is given by

$$(5-4) \quad \chi = (2\pi)^{\frac{3}{2}} e^{i\underline{K}_G \cdot \underline{R}_G}$$

FIGURE 11 - THE POSITION VECTORS

where $\underline{K}_G$ is the wave number of the center of mass motion and $\underline{R}_G$ is the center of mass vector. Hence, the total triton wave function including the center of mass motion is

$$(5-5) \quad \Psi = \chi \psi$$

Rewriting (5-2) in terms of (5-5) yields

$$(5-6) \quad \tilde{F}_c(H^3) = \frac{1}{2} \sum_{k=1}^3 \iint e^{i\underline{q} \cdot \underline{r}} \Psi^* [(1+\tau_{k3})f_c^p(\underline{p}-\underline{r}_k) + (1-\tau_{k3})f_c^n(\underline{p}-\underline{r}_k)] \Psi dV d^3\underline{p}$$

where

$$(5-7) \quad \int dV = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d^3\underline{r}_1 d^3\underline{r}_2 d^3\underline{r}_3$$

The operator ρ_c does not change the total spin of the triton wave function hence the form factor can be written as¹

$$(5-8) \quad \tilde{F}_c(H^3) = \tilde{F}_c^S(H^3) + \tilde{F}_c^D(H^3)$$

where

$$(5-9a) \quad \tilde{F}_c^S(H^3) = \frac{1}{2} \sum_{k=1}^3 \iint e^{i\underline{q} \cdot \underline{r}} \Psi_S^* [(1+\tau_{k3})f_c^p(\underline{p}-\underline{r}_k) + (1-\tau_{k3})f_c^n(\underline{p}-\underline{r}_k)] \Psi_S dV d^3\underline{p}$$

$$(5-9b) \quad \tilde{F}_c^D(H^3) = \frac{1}{2} \sum_{k=1}^3 \iint e^{i\underline{q} \cdot \underline{r}} \Psi_D^* [(1+\tau_{k3})f_c^p(\underline{p}-\underline{r}_k) + (1-\tau_{k3})f_c^n(\underline{p}-\underline{r}_k)] \Psi_D dV d^3\underline{p}$$

¹ Only S- and D- states are considered here since S- and D- state components alone were calculated in chapters 3 and 4.

and we have defined¹

$$(5-10a) \quad \Psi_s = \chi \left[f_1 y_1 + 2^{-\frac{1}{2}} (f_{3,1} y_{3,2} - f_{3,2} y_{3,1}) \right] = \chi \Psi_s$$

$$(5-10b) \quad \Psi_D = \chi \left[2^{-\frac{1}{2}} (f_{8,1} y_{8,2} - f_{8,2} y_{8,1}) + 2^{-\frac{1}{2}} (f_{9,1} y_{9,2} - f_{9,2} y_{9,1}) + 2^{-\frac{1}{2}} (f_{10,1} y_{10,2} - f_{10,2} y_{10,1}) \right] \\ = \chi \Psi_D$$

5.3 The S-state contribution to the form factor

Consider $\widetilde{F}_C^S (H^3)$ and make the change of variable $\underline{u}_K = \underline{p} - \underline{r}_K$

With the aid of

$$(5-11a) \quad F_C^p = \int e^{i\underline{q} \cdot \underline{u}} f_C^p(\underline{u}) d^3 \underline{u}$$

$$(5-11b) \quad F_C^n = \int e^{i\underline{q} \cdot \underline{u}} f_C^n(\underline{u}) d^3 \underline{u}$$

which define the proton (F_C^p) and neutron (F_C^n) form factors respectively, equation (5-9a) becomes

$$(5-12) \quad \widetilde{F}_C^S(H^3) = \frac{F_C^p}{2} \sum_{K=1}^3 \int e^{i\underline{q} \cdot \underline{r}_K} \Psi_s^*(1+\tau_{K3}) \Psi_s dV + \frac{F_C^n}{2} \sum_{K=1}^3 \int e^{i\underline{q} \cdot \underline{r}_K} \Psi_s^*(1-\tau_{K3}) \Psi_s dV$$

Changing variables to $\underline{r}_K = \underline{R}_G + \underline{r}_K'$ and using

$$(5-13) \quad \delta(\underline{q} + \underline{k}) = \frac{1}{8\pi^3} \int e^{i(\underline{q} + \underline{k}) \cdot \underline{R}_G} d^3 \underline{R}_G$$

equation (5-12) becomes

$$(5-14) \quad \widetilde{F}_C^S(H^3) = \delta(\underline{q} + \underline{k}) F_C^S(H^3)$$

¹ The antisymmetric S-state has been neglected.

² We use y_8, y_9 and y_{10} instead of $y_{11}, y_{12}, y_{13}, y_{14}$ for the reason given in section 5.1.

with

$$(5-15) \quad F_c^S(H^3) = \frac{F_c}{2} \sum_{K=1}^3 \int e^{i\mathbf{q} \cdot \mathbf{R}_K'} \psi_S^*(1+\tau_{K3}) \psi_S dS + \frac{F_c}{2} \sum_{K=1}^3 \int e^{i\mathbf{q} \cdot \mathbf{R}_K'} \psi_S^*(1-\tau_{K3}) \psi_S dS$$

where

$$(5-16) \quad \int dS = \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\delta \int_0^{\pi} \sin\beta d\beta \int_0^{\infty} dR_{13} \int_0^{\infty} dR_{23} \int_{|R_{13}-R_{23}|}^{R_{13}+R_{23}} dR_{12} R_{13} R_{23} R_{12}$$

Figure 12 shows geometrically the new variable

It is interesting to note that equation (5-14) implies that the momentum transferred from the electron to H^3 is equal and opposite to the momentum transferred from the triton to the electron (i.e., $\mathbf{K} = -\mathbf{q}$).

Equation (5-15) is easily simplified with the help of the isospin matrix elements $\langle \lambda | k_j \rangle$ defined in appendix E. Performing the integration over Euler angles (5-15) becomes

$$(5-17) \quad F_c^S(H^3) = \frac{F_c}{2} \left[\sum_{K=1}^3 \int e^{i\mathbf{q} \cdot \mathbf{R}_K'} \left\{ f_1^2 + \frac{f_{3,1}^2 + f_{3,2}^2}{2} + \frac{f_1^2}{2} [\langle 1|K1 \rangle + \langle 2|K2 \rangle] \right. \right. \\ \left. \left. + \frac{f_1 f_{3,1}}{\sqrt{2}} [\langle 1|K1 \rangle - \langle 2|K2 \rangle] - \frac{f_1 f_{3,2}}{\sqrt{2}} [\langle 1|K2 \rangle + \langle 2|K1 \rangle] + \frac{f_{3,1}^2 + f_{3,2}^2}{4} [\langle 1|K1 \rangle + \langle 2|K2 \rangle] \right\} d\tau \right] \\ + \frac{F_c}{2} \left[\sum_{K=1}^3 \int e^{i\mathbf{q} \cdot \mathbf{R}_K'} \left\{ f_1^2 + \frac{f_{3,1}^2 + f_{3,2}^2}{2} - \frac{f_1^2}{2} [\langle 1|K1 \rangle + \langle 2|K2 \rangle] - \frac{f_1 f_{3,1}}{\sqrt{2}} [\langle 1|K1 \rangle - \langle 2|K2 \rangle] \right. \right. \\ \left. \left. + \frac{f_1 f_{3,2}}{\sqrt{2}} [\langle 1|K2 \rangle + \langle 2|K1 \rangle] - \frac{f_{3,1}^2 + f_{3,2}^2}{4} [\langle 1|K1 \rangle + \langle 2|K2 \rangle] \right\} d\tau \right]$$

1 The Euler angles and the Euler angle wave functions are defined in appendix C.

FIGURE 12 - THE CHANGE OF VARIABLE $L_k = R_G + L'_k$

where $\int d\tau$ is the integral in (5-16) when the integration over the angles (α, β, γ) has been done.

Performing the summation in (5-16) and using the table E-1 for the isospin matrix elements, equation (5-16) becomes

$$(5-18) \quad F_c^S(H^3) = \frac{F_c^P + 2F_c^N}{3} \int [e^{iq \cdot \underline{R}_1'} + e^{iq \cdot \underline{R}_2'} + e^{iq \cdot \underline{R}_3'}] (f_1^2 + \frac{f_{3,1}^2 + f_{3,2}^2}{2}) d\tau \\ + \frac{F_c^N - F_c^P}{3} \int [(e^{iq \cdot \underline{R}_1'} + e^{iq \cdot \underline{R}_2'}) \frac{f_1 f_{3,1}}{\sqrt{2}} + \sqrt{3} (e^{iq \cdot \underline{R}_1'} - e^{iq \cdot \underline{R}_2'}) \frac{f_1 f_{3,2}}{\sqrt{2}} - 2e^{iq \cdot \underline{R}_3'} \frac{f_1 f_{3,1}}{\sqrt{2}}] d\tau$$

The transformation properties of the functions f_1 , $f_{3,1}$ and $f_{3,2}$ may be found from appendix A. As an example of the use of these permutation properties to simplify equation (5-18), consider

$$\int e^{iq \cdot \underline{R}_2'} (f_1^2 + \frac{f_{3,1}^2 + f_{3,2}^2}{2}) d\tau$$

From figure 12, it is clear that changing $\underline{R}_2'$ to $\underline{R}_1'$ is equivalent to interchanging particles 1 and 2. However, $d\tau$ is invariant under any permutation of the three particles and the combination

$$f_1^2 + \frac{f_{3,1}^2 + f_{3,2}^2}{2}$$

transforms according to the symmetric representation of S_3 (see appendix A).

Hence the following identity holds:

$$(5-19) \quad \int e^{iq \cdot \underline{R}_1'} (f_1^2 + \frac{f_{3,1}^2 + f_{3,2}^2}{2}) d\tau = \int e^{iq \cdot \underline{R}_2'} (f_1^2 + \frac{f_{3,1}^2 + f_{3,2}^2}{2}) d\tau$$

Using arguments like this about the permutation properties of the functions f_1 , $f_{3,1}$ and $f_{3,2}$, one can rewrite equation (5-18) in the

simpler form

$$(5-20) \quad F_c^S(H^3) = (F_c^P + 2F_c^n) F_1(q^2) + \frac{2}{3} (F_c^n - F_c^P) F_2(q^2)$$

where

$$(5-21a) \quad F_1(q^2) = \int e^{i\vec{q} \cdot \vec{R}_1'} \left(f_1^2 + \frac{f_{3,1}^2 + f_{3,2}^2}{2} \right) d\tau$$

$$(5-21b) \quad F_2(q^2) = \int \left[\left(e^{i\vec{q} \cdot \vec{R}_1'} - e^{i\vec{q} \cdot \vec{R}_3'} \right) \frac{f_1 f_{3,1}}{\sqrt{2}} + \sqrt{3} e^{i\vec{q} \cdot \vec{R}_2'} \frac{f_1 f_{3,2}}{\sqrt{2}} \right] d\tau$$

Although equation (5-21b) appears to differ from that of Schiff (1964), appendix G shows that they can be brought into identical form by using the relations between the Schiff (1964) and Derrick - Blatt (1958) internal functions.

Experimentally, only the average value of $\underline{q}$ over all orientations in space is measured. If $\overline{F_1}(q^2)$ is the average value of $F_1(q^2)$ when averaged over all orientations of the vector $\underline{q}$ with respect to the vector $\underline{R}_1'$ and similarly for $F_2(q^2)$, then equations (5-21) become

$$(5-22a) \quad \overline{F_1}(q^2) = \int \frac{\sin q R_1'}{q R_1'} \left(f_1^2 + \frac{f_{3,1}^2 + f_{3,2}^2}{2} \right) d\tau$$

$$(5-22b) \quad \overline{F_2}(q^2) = \int \left[\left(\frac{\sin q R_1'}{q R_1'} - \frac{\sin q R_3'}{q R_3'} \right) \frac{f_1 f_{3,1}}{\sqrt{2}} + \sqrt{3} \frac{\sin q R_2'}{q R_2'} \frac{f_1 f_{3,2}}{\sqrt{2}} \right] d\tau$$

Since the dominant component of the triton wave function is the symmetric S-state component, it is useful to consider

$$(5-23) \quad \overline{F_1^S}(q^2) = \int \frac{\sin q R_1'}{q R_1'} f_1^2 d\tau$$

where $\overline{F}_1^S(q^2)$ is the contribution of the symmetric S-state component to the charge form factor. From figure 12, it is clear that

$$(5-24) \quad r_1' = \frac{1}{3} |\underline{r}_{12} + \underline{r}_{13}| = \frac{1}{3} \sqrt{2r_{12}^2 + 2r_{13}^2 - r_{23}^2}$$

In order to make use of the functions calculated in chapters 3 and 4, one makes the following change of variables

$$(5-25) \quad r = \frac{1}{2}(r_{12} + r_{13} + r_{23}), \quad R_2 = r_{13}, \quad R_3 = r_{12}$$

Applying this change of variables¹ to equation (5-24) yields

$$(5-26) \quad \overline{F}_1^S(q^2) = 2 \int_0^\infty dr \int_0^r dR_2 \int_{r-R_2}^R dR_3 \frac{R_2 R_3 (2r - R_2 - R_3) \sin \left[\frac{q}{3} \sqrt{R_2^2 + R_3^2 + 4r(R_2 + R_3) - 2R_2 R_3} \right]^{\frac{1}{2}}}{\frac{q}{3} (R_2^2 + R_3^2 + 4r(R_2 + R_3) - 2R_2 R_3)^{\frac{1}{2}}}$$

Applying the changes of variables

$$(5-27a) \quad R_2 = w r, \quad R_3 = t r$$

$$(5-27b) \quad x = 2^{-\frac{1}{2}}(t+w), \quad y = 2^{-\frac{1}{2}}(t-w)$$

one after the other to equation (5-26) and using the fact that the integrand is an even function of y, equation (5-26) becomes

$$(5-28) \quad \overline{F}_1^S(q^2) = 4 \left(\frac{7}{30} \right) \int_0^\infty u^2(r) dr \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}-x+\sqrt{2}} dx \int_0^{\sqrt{2}-x+\sqrt{2}} dy \frac{(x^2 - y^2) \left(1 - \frac{yx}{\sqrt{2}}\right) \sin \left[\frac{q}{3} \sqrt{2y^2 + \frac{8x}{\sqrt{2}}} - 4 \right]}{\frac{q}{3} \sqrt{2y^2 + \frac{8x}{\sqrt{2}}} - 4}$$

¹ The Jacobbian of this transformation is 2.

where

$$(5-29) \quad f_1(r) = \sqrt{\frac{30}{7}} \frac{u(r)}{r^{5/2}}$$

as in equation (3-9a).

5.4 The D-state contribution to the form factor

As in the case of the S-state contribution, equation (5-9b) can be rewritten as

$$(5-30) \quad \widetilde{F}_C^D(H^3) = \delta(\underline{q} + \underline{k}) F_C^D(H^3)$$

with

$$(5-31) \quad F_C^D(H^3) = \frac{F_C}{2} \sum_{k=1}^3 \int e^{i \underline{q} \cdot \underline{R}_k} \psi_D(1 + \tau_{k3}) \psi_D dS + \frac{F_C}{2} \sum_{k=1}^3 \int e^{i \underline{q} \cdot \underline{R}_k} \psi_D(1 - \tau_{k3}) \psi_D dS$$

Using equation (5-10b), the matrix elements in appendix E and integrating over Euler¹ angles, equation (5-31) becomes

$$(5-32) \quad F_C^D(H^3) = \frac{F_C}{2} \left[\sum_{k=1}^3 \int e^{i \underline{q} \cdot \underline{R}_k} \left\{ \frac{f_{8,1}^2 + f_{9,1}^2 + f_{10,2}^2 + f_{8,2}^2 + f_{9,2}^2 + f_{10,1}^2}{2} + \frac{1}{2} (f_{8,1}^2 + f_{9,1}^2 + f_{10,2}^2) \langle 2112 \rangle \right. \right. \\ \left. \left. + \frac{1}{2} (f_{8,2}^2 + f_{9,2}^2 + f_{10,1}^2) \langle 1111 \rangle - \frac{1}{2} (f_{8,1} f_{8,2} + f_{9,1} f_{9,2} - f_{10,1} f_{10,2}) [\langle 1112 \rangle + \langle 2111 \rangle] \right\} d\tau \right] + \\ + \frac{F_C}{2} \left[\sum_{k=1}^3 \int e^{i \underline{q} \cdot \underline{R}_k} \left\{ \frac{f_{8,1}^2 + f_{9,1}^2 + f_{10,2}^2 + f_{8,2}^2 + f_{9,2}^2 + f_{10,1}^2}{2} - \frac{1}{2} (f_{8,1}^2 + f_{9,1}^2 + f_{10,2}^2) \langle 2112 \rangle \right. \right. \\ \left. \left. - \frac{1}{2} (f_{8,2}^2 + f_{9,2}^2 + f_{10,1}^2) \langle 1111 \rangle + \frac{1}{2} (f_{8,1} f_{8,2} + f_{9,1} f_{9,2} - f_{10,1} f_{10,2}) [\langle 1112 \rangle + \langle 2111 \rangle] \right\} d\tau \right]$$

¹ See page C-5.

Performing the above summation and using table E-1 yields¹

$$\begin{aligned}
 (5-33) \quad F_c^D(H^3) = & \frac{F_c^P + F_c^N}{4} \int [e^{i\mathbf{q} \cdot \mathbf{R}_1'} + e^{i\mathbf{q} \cdot \mathbf{R}_2'}] (f_{8,1}^2 + f_{9,1}^2 + f_{10,2}^2) d\tau \\
 & + \frac{F_c^P + 5F_c^N}{12} \int [e^{i\mathbf{q} \cdot \mathbf{R}_1'} + e^{i\mathbf{q} \cdot \mathbf{R}_2'}] (f_{8,2}^2 + f_{9,2}^2 + f_{10,1}^2) d\tau \\
 & + \frac{F_c^P - F_c^N}{2\sqrt{3}} \int [e^{i\mathbf{q} \cdot \mathbf{R}_2'} - e^{i\mathbf{q} \cdot \mathbf{R}_1'}] (f_{8,1}f_{8,2} + f_{9,1}f_{9,2} - f_{10,1}f_{10,2}) d\tau \\
 & + \frac{F_c^N + 2F_c^P}{6} \int e^{i\mathbf{q} \cdot \mathbf{R}_3'} (f_{8,2}^2 + f_{9,2}^2 + f_{10,1}^2) d\tau + \frac{F_c^N}{2} \int e^{i\mathbf{q} \cdot \mathbf{R}_3'} (f_{8,1}^2 + f_{9,1}^2 + f_{10,2}^2) d\tau
 \end{aligned}$$

As for the S-state contribution in the preceding section, some of the integrals may be simplified using the permutation properties of the internal wave functions to give

$$\begin{aligned}
 (5-34) \quad F_c^D(H^3) = & \frac{F_c^P + F_c^N}{2} \int e^{i\mathbf{q} \cdot \mathbf{R}_1'} (f_{8,1}^2 + f_{9,1}^2 + f_{10,2}^2) d\tau + \frac{F_c^P + 5F_c^N}{6} \int e^{i\mathbf{q} \cdot \mathbf{R}_1'} (f_{8,2}^2 + f_{9,2}^2 + f_{10,1}^2) d\tau \\
 & + \frac{F_c^N - F_c^P}{\sqrt{3}} \int e^{i\mathbf{q} \cdot \mathbf{R}_1'} (f_{8,1}f_{8,2} + f_{9,1}f_{9,2} - f_{10,1}f_{10,2}) d\tau + \\
 & + \frac{F_c^N}{2} \int e^{i\mathbf{q} \cdot \mathbf{R}_3'} (f_{8,1}^2 + f_{9,1}^2 + f_{10,2}^2) d\tau + \frac{F_c^N + 2F_c^P}{6} \int e^{i\mathbf{q} \cdot \mathbf{R}_3'} (f_{8,2}^2 + f_{9,2}^2 + f_{10,1}^2) d\tau
 \end{aligned}$$

¹ A comparison with the work of Gibson and Schiff (1965) would require the technique used in appendix G plus the results of McMillan (1966b).

Averaging (5-34) over all orientations of $\underline{q}$, as in section (5.3) yields¹

$$\begin{aligned}
 (5-35) \quad \overline{F_c^D}(H^3) &= \frac{F_c^P + F_c^n}{2} \int \frac{\sin q_{n_1}'}{q_{n_1}'} (f_{8,1}^2 + f_{9,1}^2 + f_{10,1}^2) d\tau + \frac{F_c^P + 5F_c^n}{6} \int \frac{\sin q_{n_1}'}{q_{n_1}'} (f_{8,2}^2 + f_{9,2}^2 + f_{10,1}^2) d\tau \\
 &+ \frac{F_c^n - F_c^P}{\sqrt{3}} \int \frac{\sin q_{n_1}'}{q_{n_1}'} (f_{8,1} f_{8,2} + f_{9,1} f_{9,2} - f_{10,1} f_{10,2}) d\tau \\
 &+ \frac{F_c^n}{2} \int \frac{\sin q_{n_3}'}{q_{n_3}'} (f_{8,1}^2 + f_{9,1}^2 + f_{10,2}^2) d\tau + \frac{F_c^n + 2F_c^P}{6} \int \frac{\sin q_{n_3}'}{q_{n_3}'} (f_{8,2}^2 + f_{9,2}^2 + f_{10,1}^2) d\tau
 \end{aligned}$$

One can also write (5-35) in terms of the six Derrick (1960a, 1960b) internal functions $f_{11,1}$, $f_{11,2}$, f_{12} , f_{13} , $f_{14,1}$, $f_{14,2}$ using equations (2-5). Keeping only the symmetric D-state component, (5-35) becomes¹

$$(5-36) \quad \overline{F_c^D}(H^3) = (F_c^P + 2F_c^n) \overline{F_3^S}(q^2) + F_c^n \overline{F_4^S}(q^2)$$

where

$$(5-37a) \quad \overline{F_3^S}(q^2) = \frac{1}{3} \int \left[\left(\frac{\sin q_{n_3}'}{q_{n_3}'} + 2 \frac{\sin q_{n_1}'}{q_{n_1}'} \right) N_5 - \sqrt{3} N_4 \frac{\sin q_{n_1}'}{q_{n_1}'} + \left(\frac{\sin q_{n_3}'}{q_{n_3}'} - \frac{\sin q_{n_1}'}{q_{n_1}'} \right) N_3 \right] f_{13}^2 d\tau$$

$$(5-37b) \quad \overline{F_4^S}(q^2) = \int \left[\left(\frac{\sin q_{n_1}'}{q_{n_1}'} - \frac{\sin q_{n_3}'}{q_{n_3}'} \right) N_3 + \sqrt{3} N_4 \frac{\sin q_{n_1}'}{q_{n_1}'} \right] f_{13}^2 d\tau$$

¹ The expressions N_3 , N_4 , N_5 are defined in chapter 1.

Thus in the symmetric S- and D- state approximation for the triton charge form factor developed in chapters 3 and 4, the total charge form factor of H^3 is

$$(5-38) \quad \overline{F_C(H^3)} = (F_C^P + 2F_C^N) [\overline{F_1^S}(q^2) + \overline{F_3^S}(q^2)] + F_C^N \overline{F_4^S}(q^2)$$

As a check on the symmetric S- and D- state calculations we consider the case of zero momentum transfer from the electron to the triton (i.e., $q = 0$). It is easy to see that¹

$$(5-39a) \quad \overline{F_1^S}(0) + \overline{F_3^S}(0) = \int [f_1^2 + N_5 f_{13}^2] d\tau$$

$$(5-39b) \quad \overline{F_4^S}(0) = 0$$

where (5-39a) is just the right hand side of equation (3-5). Noting that

$$F_C^P(0) = 1, \quad F_C^N(0) = 0$$

for $q = 0$, (see Schiff (1964)), and using (3-5), equation (5-38) becomes

$$(5-39) \quad \overline{F_C(H^3)} = 1$$

as required.

5.5 Numerical results

The calculation of the contribution to the charge form factor from the S- and D- states of the triton wave function provides a step in checking the approximate internal wave functions of chapters 3 and 4.

However, only the component $\overline{F_1^S}(q^2)$ has been computed numerically up to the present time. As pointed out in section 5.3, this is the dominant

¹ We note that $\int N_4 f_{13}^2 d\tau = \int N_3 f_{13}^2 d\tau = 0$

contribution to the charge form factor since the integral in (5-23) depends on the internal S-state wave function (f_1) which is well known to be the dominant part of the triton wave function. The solutions to equation (4-1) with $\alpha = 0$ and normalized according to equation (3-20) (u_0) and the best trial exponential functions normalized according to equation (4-7) for the four Feshbach - Pease (1952) potentials were used to calculate equation (5-28).

Figures 13 to 16 are graphs of $\overline{F_1^S}(q^2)$ verses q for the best trial exponential function and the eigenfunction u_0 for all four Feshbach - Pease (1952) potentials. We see that as the triton binding energy decreases (i.e., as we go from the FP No.1 to the FP No.3 potentials), $\overline{F_1^S}(q^2)$ decreases for each fixed value of $q \neq 0$. Similarly, for approximately equal values of the triton energy but different wave functions (i.e., a comparison of the best trial exponential function and the function u_0 for any particular Feshbach - Pease (1952) potential) $\overline{F_1^S}(q^2)$ is larger for the best trial exponential function for each fixed value of $q \neq 0$.

Indeed, the observations above are not unexpected, for as one sees from (5-28) $\overline{F_1^S}(q^2)$ depends most on the wave function near its maximum as a function of r and not on the wave function tail. From figures 5 to 8, we observe that the maximum values of $u(r)$ decrease as we go from the FP No.1 to the FP No.3 potentials and from figure 9, the best trial exponential function has a higher maximum than the corresponding function u_0 for the FP No.1 potential.

For completeness, table 5 gives the values of $\overline{F_1^S}(q^2)$ verses q for the best trial exponential and the function u_0 for all four Feshbach - Pease (1952) potentials.

TABLE 5

The function $\overline{F}_1^s(q^2)$ for the four Feshbach - Pease (1952) potentials for the eigenfunction U_0 and the best trial exponential.

q	FP No.1		FP No.2		FP $g = 0$		FP No.3		Experiment	
	U_0	Exp. K=1.040	U	Exp. K=0.931	U	Exp. K=0.856	U	Exp. K=0.746	q	$\overline{F}_1^s(q^2)$
0.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000		
0.25	0.975	0.981	0.968	0.978	0.960	0.974	0.934	0.968		
0.50	0.910	0.934	0.886	0.920	0.858	0.907	0.784	0.881		
0.75	0.816	0.863	0.773	0.833	0.728	0.808	0.618	0.758		
1.00	0.710	0.774	0.651	0.730	0.595	0.691	0.473	0.622		
1.25	0.603	0.677	0.536	0.620	0.476	0.572	0.357	0.490	1.22	0.518
1.50	0.503	0.580	0.434	0.513	0.375	0.461	0.268	0.374	1.41	0.404
1.75	0.415	0.487	0.348	0.417	0.294	0.363	0.201	0.280	1.58	0.328
2.00	0.340	0.403	0.278	0.333	0.230	0.282	0.151	0.206	1.73	0.284
2.25	0.277	0.330	0.221	0.263	0.179	0.216	0.114	0.150	1.87	0.227
2.50	0.225	0.268	0.176	0.206	0.140	0.165	0.087	0.109	2.12	0.158
2.75	0.183	0.216	0.140	0.161	0.110	0.126	0.066	0.079	2.24	0.134
3.00	0.149	0.173	0.112	0.126	0.087	0.096	0.051	0.058		
3.25	0.121	0.139	0.090	0.098	0.069	0.073	0.040	0.043		
3.50	0.099	0.111	0.073	0.076	0.055	0.056	0.031	0.032		
3.75	0.080	0.088	0.058	0.059	0.044	0.043	0.025	0.024		
4.00	0.065	0.069	0.047	0.046	0.035	0.033	0.020	0.018		
4.25	0.053	0.054	0.038	0.035	0.028	0.025	0.015	0.013		
4.50	0.043	0.042	0.030	0.026	0.022	0.018	0.012	0.009		
4.75	0.034	0.033	0.023	0.020	0.018	0.013	0.008	0.006		
5.00	0.028	0.026	-	0.015	0.013	0.009	-	0.004		

CHAPTER 6

CONCLUSIONS

We have in this thesis extended the equivalent two - body method of Feshbach and Rubinow (1955) to include also one triton D-state component. The triton energy and wave function follow from the solution of eigenvalue equations, and not from the Ritz minimization procedure which is more customary in the triton problem. Indeed, the coupled Feshbach - Rubinow equations (3-15) which we have derived and solved are, as we have pointed out, very similar to the well-known pair of equations for the deuteron wave function components.

A comparison of our calculated triton binding energies and those obtained by Feshbach and Pease (1952) with the Ritz procedure, shows, however, that ours are smaller for all four Feshbach - Pease potentials. We attribute this to the fact that we included only the symmetric D-state component of the triton wave function along with the symmetric S-state component, whereas Feshbach and Pease include, as pointed out by McMillan (1966b), all the Derrick-Blatt D-state components. (We consider our assumptions (3-7) and (3-18) to be of lesser importance in this comparison since, as we have pointed out, the dominant parts of the Feshbach - Pease approximate components are of this form.) The D-state probabilities which we have calculated are major fractions of the Feshbach - Pease quantities, which further bears out our assumption that the symmetric D-state is the dominant D-state, but for accurate results it is not sufficient to include only this component. Even though the D-states occur in the triton ground state with small probabilities, their contribution to the triton binding energy is appreciable.

The general similarity between the results obtained from the coupled and modified Feshbach - Rubinow equations indicates that the approximation (3-18) is quite good. As a matter of fact, this approximation leads to far less error than the replacement of the solution of the Feshbach - Rubinow equation with the best trial exponential function. In particular, one sees that the computed functions as well as the calculated energies are in good agreement. One is thus encouraged to write the remaining D-states in terms of symmetric functions as, for example, generalizations of the Feshbach - Pease (1952) functions given by McMillan (1966b), apply approximation (3-7) to these functions, and finally apply approximations similar to (3-18) to each of these components. It is felt that in view of the calculations reported here, the results obtained with the last approximation should be quite close to those obtained from solving the coupled, ordinary differential equations which would be obtained when the Feshbach - Rubinow approximation is applied.

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APPENDIX A THE SYMMETRIC GROUP OF ORDER $3(S_3)$

A.1 Irreducible representations of S_3

The symmetric group of order 3 is the group of all permutations on three objects. This group has 3 classes of conjugate elements; hence it has 3 irreducible representations. S_3 is isomorphic to the crystallographic group D_3 , the group of all proper rotations which leaves an equilateral triangle invariant, and thus the irreducible representations can be found from D_3 . The 3 irreducible representations of S_3 are labelled 1T_1 (symmetric representation), 1T_2 (antisymmetric representation), and 2T_3 (mixed representation) where the upper left index gives the dimension of the representation.

The 6 elements of S_3 are labelled

$G_1 = (1)$, $G_2 = (132)$, $G_3 = (123)$, $G_4 = (12)$, $G_5 = (31)$, $G_6 = (23)$ where the cyclic notation of Wigner (1959) is used.

We shall also use the notation

$$P_{12} = G_4, \quad P_{13} = G_5, \quad P_{23} = G_6$$

TABLE A-1 IRREDUCIBLE MATRICES OF S_3

Group Element	1T_1	1T_2	2T_3
G_1	1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
G_2	1	1	$\frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$
G_3	1	1	$\frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$
$G_4 = P_{12}$	1	-1	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$G_5 = P_{13}$	1	-1	$\frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$
$G_6 = P_{23}$	1	-1	$\frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

A.2 Permutation properties of functions with S_3 symmetry

Let $\phi(1,2,3)$ be an arbitrary function of the three indices $(1,2,3)$.

Define P_{G_K} operating on $\phi(1,2,3)$ by

$$(a-1) \quad P_{G_K} \phi(1,2,3) = \phi\{G_K(1,2,3)\}$$

where $G_K \in S_3$ and $G_K(1,2,3)$ is one of the 6 permutations of the indices

$(1,2,3)$. Only those functions $\phi(1,2,3)$ with the following transformation properties will be considered:

(1) $\phi(1,2,3)$ transforms according to 1T_1 (see equation a-2) under a permutation of the indices $(1,2,3)$. In this case ϕ is labelled $\phi_1^S(1,2,3)$ and is called a symmetric function.

$$(a-2) \quad P_{G_K} \phi_1^S(1,2,3) = \phi_1^S(1,2,3) \quad K=1, \dots, 6$$

(2) $\phi(1,2,3)$ transforms according to 1T_2 (see equation a-3) under a permutation of the indices $(1,2,3)$. In this case ϕ is labelled $\phi_1^a(1,2,3)$ and is called an antisymmetric function.

$$(a-3) \quad P_{G_K} \phi_1^a(1,2,3) = \begin{cases} \phi_1^a(1,2,3) & K=1,2,3 \\ -\phi_1^a(1,2,3) & K=4,5,6 \end{cases}$$

(3) The pair of functions $\phi_1^m(1,2,3)$ and $\phi_2^m(1,2,3)$ transform according to 2T_3 (see equation a-3) under a permutation of the indices $(1,2,3)$. The lower index refers to the row of the representation 2T_3 and these functions are called mixed-one and mixed-two functions respectively.

$$(a-3'a) \quad P_{G_K} \phi_1^m(1,2,3) = \begin{cases} \phi_1^m(1,2,3) & K=1,4 \\ -\frac{1}{2}\phi_1^m(1,2,3) - \frac{\sqrt{3}}{2}\phi_2^m(1,2,3) & K=2,5 \\ -\frac{1}{2}\phi_1^m(1,2,3) + \frac{\sqrt{3}}{2}\phi_2^m(1,2,3) & K=3,6 \end{cases}$$

$$(a-3'b) \quad P_{G_1} \phi_2^m(1,2,3) = \phi_2^m(1,2,3)$$

$$P_{G_2} \phi_2^m(1,2,3) = \frac{\sqrt{3}}{2} \phi_1^m(1,2,3) - \frac{1}{2} \phi_2^m(1,2,3)$$

$$P_{G_3} \phi_2^m(1,2,3) = -\frac{\sqrt{3}}{2} \phi_1^m(1,2,3) - \frac{1}{2} \phi_2^m(1,2,3)$$

$$P_{G_4} \phi_2^m(1,2,3) = -\phi_2^m(1,2,3)$$

$$P_{G_5} \phi_2^m(1,2,3) = -\frac{\sqrt{3}}{2} \phi_1^m(1,2,3) + \frac{1}{2} \phi_2^m(1,2,3)$$

$$P_{G_6} \phi_2^m(1,2,3) = \frac{\sqrt{3}}{2} \phi_1^m(1,2,3) + \frac{1}{2} \phi_2^m(1,2,3)$$

A.3 Permutation addition coefficients

Suppose $\phi_{K_a}^{P_a}(1,2,3)$, with P_a one of the 3 irreducible representations of $S(3)$ and K_a the row label of the representation P_a , transforms according to the irreducible representation ${}^{n_a}T_a$ of $S(3)$ and $\phi_{K_b}^{P_b}(1',2',3')$, with P_b and K_b defined as above, transforms according to the irreducible representation ${}^{n_b}T_b$ of $S(3)$; then how do the product functions transform under joint permutations of the indices $(1,2,3)$ and $(1',2',3')$? One approach to this is to consider the direct product decomposition of two irreducible representations of $S(3)$.

$$(a-4) \quad T' = {}^{n_a}T_a \otimes {}^{n_{a'}}T_{a'} = c_1 {}^1T_1 + c_2 {}^2T_2 + c_3 {}^3T_3$$

The coefficients c_j are determined from the formula

$$(a-5) \quad c_\alpha = \frac{1}{6} \sum_{j=1}^3 h_j \chi_j(T') \chi_\alpha^{j*}({}^{n_a}T_a)$$

which can be found in Wigner (1959). h_j is the number of elements in the j^{th} conjugate class, $\chi_\alpha^j({}^{n_a}T_a)$ is the character of the j^{th} conjugate

class for the representation $n_a T_a$, and $\chi^j(T') = \chi^j(n_a T_a) \chi^j(n'_a T'_a)$ is the character of the j^{th} conjugate class for the direct product representation T' . Table A-2 summarizes the results. From this table, it is clear that one can write (see Derrick and Blatt (1958))

$$(a-6) \quad \phi_K^{R_a R'_a P} (1, 2, 3) = \sum_{K_a} \sum_{K'_a} \left(\begin{matrix} R_a R'_a P \\ K_a K'_a K \end{matrix} \right) \phi_{K_a}^{R_a} (1, 2, 3) \phi_{K'_a}^{R'_a} (1, 2, 3)$$

where the coefficients $\left(\begin{matrix} R_a R'_a P \\ K_a K'_a K \end{matrix} \right)$ are called the permutation addition

coefficients for the group S_3 . They are unique up to a phase factor which one chooses to make them real. The permutation addition coefficients form a unitary matrix, hence an orthogonal matrix since the coefficients are real, which transforms the direct product space into a direct sum space. The orthogonality condition above is written

$$(a-7) \quad \sum_{K_a} \sum_{K'_a} \left(\begin{matrix} R_a R'_a P \\ K_a K'_a K \end{matrix} \right) \left(\begin{matrix} R_a R'_a P' \\ K_a K'_a K' \end{matrix} \right) = \delta_{PP'} \delta_{KK'}$$

The coefficients may be evaluated using equation (a-7) and table A-2 and have been given by Derrick and Blatt (1958). For convenience, we have included them here in table A-3. Table A-4 lists the 16 linear combinations of the direct product space.

TABLE A-2 DIRECT PRODUCT DECOMPOSITION OF S_3

${}^1T_1 \otimes {}^1T_1 = {}^1T_1$	${}^1T_2 \otimes {}^1T_2 = {}^1T_1$
${}^1T_1 \otimes {}^1T_2 = {}^1T_2$	${}^1T_2 \otimes {}^2T_3 = {}^2T_3$
${}^1T_1 \otimes {}^2T_3 = {}^2T_3$	${}^2T_3 \otimes {}^2T_3 = {}^1T_1 \oplus {}^1T_2 \oplus {}^2T_3$

TABLE A-3

NON-ZERO PERMUTATION ADDITION COEFFICIENTS

$\begin{pmatrix} s s s \\ 1 1 1 \end{pmatrix} = \begin{pmatrix} s q a \\ 1 1 1 \end{pmatrix} = \begin{pmatrix} a s a \\ 1 1 1 \end{pmatrix} = \begin{pmatrix} a a s \\ 1 1 1 \end{pmatrix} = 1$
$\begin{pmatrix} s m m \\ 1 k \lambda \end{pmatrix} = \begin{pmatrix} m s m \\ k 1 \lambda \end{pmatrix} = \sqrt{2} \begin{pmatrix} m m s \\ k \lambda 1 \end{pmatrix} = \delta_{k\lambda}$
$\begin{pmatrix} a m m \\ 1 k \lambda \end{pmatrix} = \begin{pmatrix} m a m \\ k 1 \lambda \end{pmatrix} = -\sqrt{2} \begin{pmatrix} m m a \\ k \lambda 1 \end{pmatrix} = \begin{cases} 0 & k=\lambda \\ 1 & k=1, \lambda=2 \\ -1 & k=2, \lambda=1 \end{cases}$
$-\begin{pmatrix} m m m \\ 1 1 1 \end{pmatrix} = \begin{pmatrix} m m m \\ 1 2 2 \end{pmatrix} = \begin{pmatrix} m m m \\ 2 1 2 \end{pmatrix} = \begin{pmatrix} m m m \\ 2 2 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$

TABLE A-4

DIRECT PRODUCT FUNCTIONS

SYMMETRIC	$\phi_1^{ss's} = \phi_1^s \phi_1^{s'}$ $\phi_1^{aa's} = \phi_1^a \phi_1^{a'}$ $\phi_1^{mm's} = \frac{1}{\sqrt{2}} \{ \phi_1^m \phi_1^{m'} + \phi_2^m \phi_2^{m'} \}$
ANTISYMMETRIC	$\phi_1^{as'a} = \phi_1^a \phi_1^{s'}$ $\phi_1^{sa'a} = \phi_1^s \phi_1^{a'}$ $\phi_1^{mm'a} = \frac{1}{\sqrt{2}} \{ \phi_1^m \phi_2^{m'} - \phi_2^m \phi_1^{m'} \}$
MIXED	$\phi_1^{sm'm} = \phi_1^s \phi_1^{m'}, \quad \phi_1^{ms'm} = \phi_1^m \phi_1^{s'}$ $\phi_1^{am'm} = \phi_1^a \phi_2^{m'}, \quad \phi_1^{ma'm} = \phi_2^m \phi_1^{a'}$ $\phi_1^{mm'm} = \frac{1}{\sqrt{2}} \{ \phi_2^m \phi_2^{m'} - \phi_1^m \phi_1^{m'} \}$ $\phi_2^{sm'm} = \phi_1^s \phi_2^{m'}, \quad \phi_2^{ms'm} = \phi_2^m \phi_1^{s'}$ $\phi_2^{am'm} = -\phi_1^a \phi_1^{m'}, \quad \phi_2^{ma'm} = -\phi_1^m \phi_1^{a'}$ $\phi_2^{mm'm} = \frac{1}{\sqrt{2}} \{ \phi_1^m \phi_2^{m'} + \phi_2^m \phi_1^{m'} \}$

APPENDIX B

SPIN - ISOSPIN FUNCTIONS

B.1 Spin functions

The total spin operator for a system of three indistinguishable particles of spin $\frac{1}{2}$ is $\underline{s} = \underline{s}_1 + \underline{s}_2 + \underline{s}_3$ where $\underline{s}_i$ is the spin operator for the i^{th} particle. Label the eigenstates of $\underline{s} \cdot \underline{s} = s^2$ and $s_z = s_{1z} + s_{2z} + s_{3z}$ by $|SM_S P_S K_S\rangle$ and the eigenstates of s_i^2 and s_{iz} by $|s_i m_i\rangle_i$, with $s_i = \frac{1}{2}$, $m_i = \pm \frac{1}{2}$. For simplicity, let $|\frac{1}{2} - \frac{1}{2}\rangle_i = \beta(i)$ and $|\frac{1}{2} \frac{1}{2}\rangle_i = \alpha(i)$. There are eight eigenstates of s^2 and s_z since $s = \frac{1}{2}$ or $s = \frac{3}{2}$, and these are obtained from the double Clebsch - Gordan series.

$$(b-1) \quad |S M_S P_S K_S\rangle = \sum_{m_1} \sum_{m_2} \sum_{m_3} \langle s_1 s_2 m_1 m_2 | s' m' \rangle \langle s' s_3 m' m_3 | S M_S \rangle |s_1 m_1\rangle_1 |s_2 m_2\rangle_2 |s_3 m_3\rangle_3$$

Table B-1 gives these 8 eigenstates when the Clebsch - Gordan coefficients are evaluated using the phase convention of Condon and Shortley (1935). The eigenstates q_1, q_2 and q_3 are identical to those defined in Derrick and Blatt (1958). The spin functions are also functions of the

TABLE B-1 EIGENSTATES OF S^2 AND S_z

$S = \frac{3}{2}$	$S = \frac{1}{2}$
$q_6 = \frac{3}{2} \frac{3}{2} S\rangle = \alpha(1)\alpha(2)\alpha(3)$	$q_1 = \frac{1}{2} \frac{1}{2} m_1\rangle = 6^{-\frac{1}{2}} [\beta(1)\alpha(2)\alpha(3) + \alpha(1)\beta(2)\alpha(3) - 2\alpha(1)\alpha(2)\beta(3)]$
$q_3 = \frac{3}{2} \frac{1}{2} S\rangle = 3^{-\frac{1}{2}} [\alpha(1)\alpha(2)\beta(3) + \beta(1)\alpha(2)\alpha(3) + \alpha(1)\beta(2)\alpha(3)]$	$q_2 = \frac{1}{2} \frac{1}{2} m_2\rangle = 2^{-\frac{1}{2}} [\beta(1)\alpha(2)\alpha(3) - \alpha(1)\beta(2)\alpha(3)]$
$q_7 = \frac{3}{2} -\frac{1}{2} S\rangle = 3^{-\frac{1}{2}} [\beta(1)\beta(2)\alpha(3) + \alpha(1)\beta(2)\beta(3) + \beta(1)\alpha(2)\beta(3)]$	$q_4 = \frac{1}{2} -\frac{1}{2} m_1\rangle = 6^{-\frac{1}{2}} [\alpha(1)\beta(2)\beta(3) + \beta(1)\alpha(2)\beta(3) - 2\beta(1)\beta(2)\alpha(3)]$
$q_8 = \frac{3}{2} -\frac{3}{2} S\rangle = \beta(1)\beta(2)\beta(3)$	$q_5 = \frac{1}{2} -\frac{1}{2} m_2\rangle = 2^{-\frac{1}{2}} [\alpha(1)\beta(2)\beta(3) - \beta(1)\alpha(2)\beta(3)]$

indices (1,2,3). (i.e., $q_i = q_{\lambda}(1,2,3)$). Using the definition of P_{G_K} given in equation (a-1), the commutation relations

$$(b-2a) \quad [S^2, P_{G_K}] = 0 \quad K=1, \dots, 6$$

$$(b-2b) \quad [S_3, P_{G_K}] = 0 \quad K=1, \dots, 6$$

follow. Thus simultaneous eigenstates of P_{G_K} , S^2 , and S_3 can be constructed.

From equations (b-2a) and (b-2b), it is clear that

$$(b-3) \quad P_{G_K} q_{\lambda}(1,2,3) = \sum_{j=1}^8 D_{ji}(G_K) q_j(1,2,3)$$

where $D_{ji}(G_K)$ are coefficients to be determined. These coefficients form an 8 dimensional matrix for each $G_K \in S_3$. The matrices are easily calculated using equations (b-2) and table B-1, and the set of six matrices obtained from (b-3) form an 8 dimensional representation of S_3 which we label 8T . By inspection the decomposition of 8T into a direct sum of irreducible representations of S_3 is

$$(b-4) \quad {}^8T = 4 \, {}^1T_1 \oplus 2 \, {}^2T_3$$

The four spin functions ($S = \frac{3}{2}$) each generate the representation 1T_1 while the two pairs of spin functions ($S = \frac{1}{2}$) generate the representation 2T_3 . Hence the reason for the label $|S M_S P_S K_S\rangle$; P_S is the representation under which the functions transform, while K_S is the corresponding row number. Thus q_6, q_3, q_7, q_8 , are symmetric functions under a permutation of (1,2,3) while the pairs $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ and $\begin{pmatrix} q_4 \\ q_5 \end{pmatrix}$ are mixed functions under a permutation of (1 2 3).¹

¹ $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ and $\begin{pmatrix} q_4 \\ q_5 \end{pmatrix}$ correspond to $\begin{pmatrix} \phi_1^m \\ \phi_2^m \end{pmatrix}$ of section A.2.

B.2 Isospin functions

The total isospin operator for three particles of isospin $\frac{1}{2}$ is $\underline{T} = \underline{T}_1 + \underline{T}_2 + \underline{T}_3$ where $\underline{T}_i$ is the isospin operator of the i^{th} particle. Label the eigenstates of $\underline{T}^2$ and T_3 by $|T M_T\rangle$ and those of $\underline{T}_i^2$ and T_{i3} by $|T_i M_i\rangle$ where $|\frac{1}{2}, \frac{1}{2}\rangle_i = \pi(i)$ and $|\frac{1}{2}, -\frac{1}{2}\rangle_i = \nu(i)$.

The eight eigenstates obtained using an equivalent equation to (a-1) have the same permutation properties as the corresponding q's.

The eigenstates are labelled 1P_1 to P_8 and are obtained from table B-1 by replacing $\alpha(i)$ by $\pi(i)$, $\beta(i)$ by $\nu(i)$, and q_i by p_i .

B.3 Spin - Isospin functions

The spin - isospin functions are labelled $|TSM_T M_S P_t K_t\rangle =$

$V_{M_T M_S K_t}(P_t, T, S)$ and correspond to definite values of S , T , S_3 and T_3 .

The notation used for the V's is similar to Derrick and Blatt (1958) except here the value of M_T is included. They transform according to the representation P_t of $S(3)$ and K_t is the corresponding row number of P_t . One finds these functions by using the permutation addition coefficients of table A-3. For $T = \frac{3}{2}$, $S = \frac{3}{2}$ the spin - isospin functions are

$$(b-5) \quad V_{M_T M_S 1}(S, \frac{3}{2}, \frac{3}{2}) = q_i \cdot p_j$$

where $i, j = 3, 6, 7, 8$ depending on the values of M_T and M_S . The spin - isospin functions are listed in table B-2 for $T = \frac{1}{2}$, $S = \frac{3}{2}$ and $T = \frac{1}{2}$, and $S = \frac{1}{2}$. To find $T = \frac{3}{2}$, $S = \frac{1}{2}$, replace p by q and q by p for the case $T = \frac{1}{2}$, $S = \frac{3}{2}$. We should note that the V's are orthonormal as are the q's and the p's.

¹ In Derrick and Blatt (1958), the functions called v_1 and v_2 are here called p_4 and p_5 respectively.

TABLE B-2

SPIN - ISOSPIN FUNCTIONS

$T = \frac{1}{2} \quad S = \frac{3}{2} \quad M_T = \frac{1}{2}$	$T = \frac{1}{2} \quad S = \frac{3}{2} \quad M_T = -\frac{1}{2}$
$V_{\frac{1}{2}\frac{3}{2}1} (m, \frac{1}{2}, \frac{3}{2}) = q_6 p_1$	$V_{-\frac{1}{2}\frac{3}{2}1} (m, \frac{1}{2}, \frac{3}{2}) = q_6 p_4$
$V_{\frac{1}{2}\frac{3}{2}2} (m, \frac{1}{2}, \frac{3}{2}) = q_6 p_2$	$V_{-\frac{1}{2}\frac{3}{2}2} (m, \frac{1}{2}, \frac{3}{2}) = q_6 p_5$
$V_{\frac{1}{2}\frac{1}{2}1} (m, \frac{1}{2}, \frac{3}{2}) = q_3 p_1$	$V_{-\frac{1}{2}\frac{1}{2}1} (m, \frac{1}{2}, \frac{3}{2}) = q_3 p_4$
$V_{\frac{1}{2}\frac{1}{2}2} (m, \frac{1}{2}, \frac{3}{2}) = q_3 p_2$	$V_{-\frac{1}{2}\frac{1}{2}2} (m, \frac{1}{2}, \frac{3}{2}) = q_3 p_5$
$V_{\frac{1}{2}-\frac{1}{2}1} (m, \frac{1}{2}, \frac{3}{2}) = q_7 p_1$	$V_{-\frac{1}{2}-\frac{1}{2}1} (m, \frac{1}{2}, \frac{3}{2}) = q_7 p_4$
$V_{\frac{1}{2}-\frac{1}{2}2} (m, \frac{1}{2}, \frac{3}{2}) = q_7 p_2$	$V_{-\frac{1}{2}-\frac{1}{2}2} (m, \frac{1}{2}, \frac{3}{2}) = q_7 p_5$
$V_{\frac{1}{2}-\frac{3}{2}1} (m, \frac{1}{2}, \frac{3}{2}) = q_8 p_1$	$V_{-\frac{1}{2}-\frac{3}{2}1} (m, \frac{1}{2}, \frac{3}{2}) = q_8 p_4$
$V_{\frac{1}{2}-\frac{3}{2}2} (m, \frac{1}{2}, \frac{3}{2}) = q_8 p_2$	$V_{-\frac{1}{2}-\frac{3}{2}2} (m, \frac{1}{2}, \frac{3}{2}) = q_8 p_5$
$T = \frac{1}{2} \quad S = \frac{1}{2} \quad M_T = \frac{1}{2}$	$T = \frac{1}{2} \quad S = \frac{1}{2} \quad M_T = -\frac{1}{2}$
$V_{\frac{1}{2}\frac{1}{2}1} (s, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_1 p_1 + q_2 p_2]$	$V_{-\frac{1}{2}\frac{1}{2}1} (s, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_1 p_4 + q_2 p_5]$
$V_{\frac{1}{2}\frac{1}{2}1} (a, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_2 p_1 - q_1 p_2]$	$V_{-\frac{1}{2}\frac{1}{2}1} (a, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_2 p_4 - q_1 p_5]$
$V_{\frac{1}{2}\frac{1}{2}1} (m, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_2 p_2 - q_1 p_1]$	$V_{-\frac{1}{2}\frac{1}{2}1} (m, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_2 p_5 - q_1 p_4]$
$V_{\frac{1}{2}\frac{1}{2}2} (m, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_2 p_1 + q_1 p_2]$	$V_{-\frac{1}{2}\frac{1}{2}2} (m, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_2 p_4 + q_1 p_5]$
$V_{\frac{1}{2}-\frac{1}{2}1} (s, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_4 p_1 + q_5 p_2]$	$V_{-\frac{1}{2}-\frac{1}{2}1} (s, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_4 p_4 + q_5 p_5]$
$V_{\frac{1}{2}-\frac{1}{2}1} (a, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_5 p_1 - q_4 p_2]$	$V_{-\frac{1}{2}-\frac{1}{2}1} (a, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_5 p_4 - q_4 p_5]$
$V_{\frac{1}{2}-\frac{1}{2}1} (m, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_5 p_2 - q_4 p_1]$	$V_{-\frac{1}{2}-\frac{1}{2}1} (m, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_5 p_5 - q_4 p_4]$
$V_{\frac{1}{2}-\frac{1}{2}2} (m, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_5 p_1 + q_4 p_2]$	$V_{-\frac{1}{2}-\frac{1}{2}2} (m, \frac{1}{2}, \frac{1}{2}) = 2^{-\frac{1}{2}} [q_5 p_4 + q_4 p_5]$

APPENDIX C

EULER ANGLE WAVE FUNCTIONS

C.1 The co-ordinate system

The Euler angles (α, β, γ) ¹ defined by Derrick and Blatt (1958) depend upon the following definition of the body frame (x', y', z') . The origin of this frame is at the center of mass of the three particles. One chooses the x' - axis, up to its direction, along the principal axis of the largest moment of inertia of the triangle formed by the three particles. The z' - axis is chosen so that a rotation from particle 1 to particle 2 is a right handed rotation about the z' - axis. One chooses the y' - axis to make the body frame right-handed. Thus, once the direction of the x' - axis is chosen the body frame is fixed.

The space frame (X, Y, Z) has its origin at the center of mass of the three particles. This is a fixed, right-handed frame and the three angles which takes this space frame into the body frame are the Euler angles (α, β, γ) defined by the convention of Goldstein (1959).

It is clear from the above definition that an even permutation of the 3 particles does not affect the body frame while an odd permutation changes the direction of the z' - axis and hence the y' - axis. The x' - axis is invariant under permutations of the particles since the three particles are assumed to be of equal mass.

¹ The angles (α, β, γ) are identified with Goldstein's angles (ψ, θ, ϕ) as pointed out by Derrick (1960a).

C.2 Irreducible representations of the group of proper rotations

The angular dependence of the triton wave function (i.e., the dependence of the triton wave function on the Euler angles α, β, γ) has been given by Derrick and Blatt (1958) and are called the Euler angle wave functions. One defines the Euler angle wave functions in terms of the representation coefficients of the irreducible representations of the group of proper rotations.

$$(C-1) \quad D_{\mu M_L}^L(\alpha, \beta, \gamma) = i^{\mu - M_L} e^{i\mu\alpha} d_{\mu M_L}^L(\beta) e^{iM_L\gamma}$$

$$(C-2) \quad d_{\mu M_L}^L(\beta) = \sum_K \frac{(-1)^K f(L, \mu, M_L)}{K! (L + M_L - K)! (L - \mu - K)! (K + \mu - M_L)!} \left(\cos \frac{\beta}{2} \right)^{2L + M_L - \mu - 2K} \left(\sin \frac{\beta}{2} \right)^{2K + \mu - M_L}$$

$$(C-3) \quad f(L, \mu, M_L) = f(L, -\mu, M_L) = \sqrt{(L + \mu)! (L - \mu)! (L + M_L)! (L - M_L)!}$$

where the sum in equation (C-2) is over the zeros of the denominator. L is the orbital angular momentum, M_L is the z - component of orbital angular momentum in the space frame and μ is the z - component of angular momentum in the body frame.

C.3 Permutation properties of $D_{\mu M_L}^L(\alpha, \beta, \gamma)$

From section C.1 it is clear that only odd permutations of the three particles will affect the Euler angles and hence the representation coefficients $D_{\mu M_L}^L(\alpha, \beta, \gamma)$. An odd permutation of the three particles transforms the angles (α, β, γ) into the angles $(-\alpha, \pi + \beta, \gamma)$. Define $\bar{0}$

1

¹ It is worth recording here that McMillan (private communication) has shown that

$$D_{\mu M_L}^L(\psi, \theta, \phi) = D^{(L)}(\{\alpha, \beta, \gamma\})_{\mu M_L}$$

where the right hand side is the function given in (15.27) of Wigner (1959). The relationship between the Goldstein Euler angles (ψ, θ, ϕ) and the (α, β, γ) defined on Page 90 of Wigner is

$$\alpha = \psi + \frac{\pi}{2}$$

$$\beta = \theta$$

$$\gamma = \phi - \frac{\pi}{2}$$

operating on the representation coefficients by

$$(C-4) \quad \bar{O} D_{\mu M_L}^L(\alpha, \beta, \gamma) = D_{\mu M_L}^L(-\alpha, \pi + \beta, \gamma)$$

Hence, $\bar{O}$ is the operator acting on $D_{\mu M_L}^L(\alpha, \beta, \gamma)$ which corresponds to the operation of performing an odd permutation of the particles.

$$\text{Lemma 1:} \quad \bar{O} d_{\mu M_L}^L(\beta) = (-1)^L i^{-2\mu} d_{-\mu M_L}^L(\beta)$$

Proof: By definition (C-4), $\bar{O} d_{\mu M_L}^L(\beta) = d_{\mu M_L}^L(\pi + \beta)$ and hence using equation (C-2) we have

$$\begin{aligned} d_{\mu M_L}^L(\pi + \beta) &= \sum_K \frac{(-1)^K f(L, \mu, M_L)}{K! (L + M_L - K)! (L - \mu - K)! (K + \mu - M_L)!} \left[\cos\left(\frac{\beta + \pi}{2}\right) \right]^{2L + M_L - \mu - 2K} \left[\sin\left(\frac{\beta + \pi}{2}\right) \right]^{2K + \mu - M_L} \\ &= \sum_K \frac{(-1)^K f(L, \mu, M_L) (-1)^{2L + M_L - \mu - 2K}}{K! (L + M_L - K)! (L - \mu - K)! (K + \mu - M_L)!} \left(\sin \frac{\beta}{2} \right)^{2L + M_L - \mu - 2K} \left(\cos \frac{\beta}{2} \right)^{2K + \mu - M_L} \end{aligned}$$

Let $\lambda = L + M_L - K$. We note that λ is an integer since L is an orbital angular momentum. Thus $d_{\mu M_L}^L(\pi + \beta)$ becomes

$$\begin{aligned} d_{\mu M_L}^L(\pi + \beta) &= \sum_{\lambda} \frac{(-1)^{L + M_L - \lambda} (-1)^{M_L - \mu} f(L, -\mu, M_L)}{\lambda! (L + M_L - \lambda)! (L + \mu - \lambda)! (\lambda - \mu - M_L)!} \left(\cos \frac{\beta}{2} \right)^{2L + M_L + \mu - 2\lambda} \left(\sin \frac{\beta}{2} \right)^{2\lambda - \mu - M_L} \\ &= (-1)^L i^{-2\mu} d_{-\mu M_L}^L(\beta) \end{aligned}$$

where we have used equation (C-3).

$$\text{Theorem:} \quad \bar{O} D_{\mu M_L}^L(\alpha, \beta, \gamma) = (-1)^L D_{-\mu M_L}^L(\alpha, \beta, \gamma)$$

$$\text{Proof:} \quad \bar{O} D_{\mu M_L}^L(\alpha, \beta, \gamma) = i^{\mu - M_L} e^{-i\mu\alpha} (-1)^L d_{\mu M_L}^L(\beta) e^{iM_L\alpha} = (-1)^L D_{-\mu M_L}^L(\alpha, \beta, \gamma)$$

C.4 Euler angle wave functions

The effect of the parity operator Π acting on the representation coefficients $D_{\mu M_L}^L(\alpha, \beta, \gamma)$ is

$$(C-5) \quad \Pi D_{\mu M_L}^L(\alpha, \beta, \gamma) = (-1)^\mu D_{\mu M_L}^L(\alpha, \beta, \gamma)$$

Experimentally, the parity of the triton is even and hence only those representation coefficients with even μ are allowed for the triton wave function. Combining this with the result that the allowed values of L are 0, 1 and 2 (since the total spin is $\frac{1}{2}$ or $\frac{3}{2}$ (see appendix A) and the total angular momentum is $\frac{1}{2}$, experimentally), there are 5 Euler angle wave functions. They are given in Derrick and Blatt (1958) where we use their notation for the functions (i.e. $Y_M^L(P_e, |\mu|)$) where L is the orbital angular momentum, μ is the body z - component of orbital angular momentum, M_L is the space z - component of orbital angular momentum, and P_e is the permutation symmetry which is either symmetric or antisymmetric) We include a complete list of the Euler angle wave functions for convenience. A glance at the theorem of section C.3 shows how the permutation properties of these functions arise. Finally we note that these Euler angle wave functions are orthonormal.

S - STATE

$$Y_0^0(s, 0) = \frac{2}{4\pi}$$

P - STATE

$$Y_1^1(q, 0) = \frac{i6}{4\pi}^{\frac{1}{2}} D_{01}^1(\alpha, \beta, \gamma)$$

M=1

$$Y_1^1(q, 0) = -\frac{6}{4\pi}^{\frac{1}{2}} e^{i\gamma} \cos \frac{\beta}{2} \sin \frac{\beta}{2}$$

M=0

$$Y_0^1(q, 0) = \frac{i6}{4\pi}^{\frac{1}{2}} \left[\cos^2 \frac{\beta}{2} - \sin^2 \frac{\beta}{2} \right]$$

M=-1

$$Y_{-1}^1(q, 0) = \frac{6}{4\pi}^{\frac{1}{2}} e^{-i\gamma} \cos \frac{\beta}{2} \sin \frac{\beta}{2}$$

D - STATE

M=0, SYMMETRIC

$$Y_0^2(s, 0) = \frac{10}{4\pi}^{\frac{1}{2}} D_{00}^2(\alpha, \beta, \gamma)$$

M=2

$$Y_2^2(s, 0) = -\frac{60}{4\pi}^{\frac{1}{2}} e^{2i\gamma} \cos^2 \frac{\beta}{2} \sin^2 \frac{\beta}{2}$$

M=1

$$Y_1^2(s, 0) = \frac{i60}{4\pi}^{\frac{1}{2}} e^{i\gamma} \left[\cos^3 \frac{\beta}{2} \sin \frac{\beta}{2} - \sin^3 \frac{\beta}{2} \cos \frac{\beta}{2} \right]$$

M=0

$$Y_0^2(s, 0) = \frac{10}{4\pi}^{\frac{1}{2}} \left[\cos^4 \frac{\beta}{2} - 4 \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2} + \sin^4 \frac{\beta}{2} \right]$$

M=-1

$$Y_{-1}^2(s, 0) = \frac{i60}{4\pi}^{\frac{1}{2}} e^{-i\gamma} \left[\cos^3 \frac{\beta}{2} \sin \frac{\beta}{2} - \sin^3 \frac{\beta}{2} \cos \frac{\beta}{2} \right]$$

M=-2

$$Y_{-2}^2(s, 0) = -\frac{60}{4\pi}^{\frac{1}{2}} e^{-2i\gamma} \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2}$$

D - STATE

M=2, SYMMETRIC

$$Y_0^2(s, 2) = -\frac{5}{4\pi}^{\frac{1}{2}} \left[D_{20}^2(\alpha, \beta, \gamma) + D_{-20}^2(\alpha, \beta, \gamma) \right]$$

M=2

$$Y_2^2(s, 2) = -\frac{5}{4\pi}^{\frac{1}{2}} e^{2i\gamma} \left[e^{2i\alpha} \cos^4 \frac{\beta}{2} + e^{-2i\alpha} \sin^4 \frac{\beta}{2} \right]$$

M=1

$$Y_1^2(s, 2) = -\frac{2i5}{4\pi}^{\frac{1}{2}} e^{i\gamma} \left[e^{2i\alpha} \cos^3 \frac{\beta}{2} \sin \frac{\beta}{2} - e^{-2i\alpha} \sin^3 \frac{\beta}{2} \cos \frac{\beta}{2} \right]$$

M=0

$$Y_0^2(s, 2) = \frac{30}{4\pi}^{\frac{1}{2}} \left[e^{2i\alpha} \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2} + e^{-2i\alpha} \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2} \right]$$

M=-1

$$Y_{-1}^2(s, 2) = -\frac{2i5}{4\pi}^{\frac{1}{2}} e^{-i\gamma} \left[-e^{2i\alpha} \cos^3 \frac{\beta}{2} \sin \frac{\beta}{2} + e^{-2i\alpha} \sin^3 \frac{\beta}{2} \cos \frac{\beta}{2} \right]$$

M=-2

$$Y_{-2}^2(s, 2) = -\frac{5}{4\pi}^{\frac{1}{2}} e^{-2i\gamma} \left[e^{2i\alpha} \sin^4 \frac{\beta}{2} + e^{-2i\alpha} \cos^4 \frac{\beta}{2} \right]$$

D - STATE

$|\mu| = 2$, ANTISYMMETRIC

$$Y_{2m}^2(a, 2) = \frac{-i 5^{\frac{1}{2}}}{4\pi} \left[D_{2/m}^2(\alpha, \beta, \gamma) - D_{-2/m}^2(\alpha, \beta, \gamma) \right]$$

$M = 2$

$$Y_2^2(a, 2) = \frac{-i 5^{\frac{1}{2}}}{4\pi} e^{2i\gamma} \left[e^{2i\alpha} \cos^2 \frac{\beta}{2} - e^{-2i\alpha} \sin^2 \frac{\beta}{2} \right]$$

$M = 1$

$$Y_1^2(a, 2) = \frac{2 \cdot 5^{\frac{1}{2}}}{4\pi} e^{i\gamma} \left[e^{2i\alpha} \cos \frac{\beta}{2} \sin \frac{\beta}{2} + e^{-2i\alpha} \sin \frac{\beta}{2} \cos \frac{\beta}{2} \right]$$

$M = 0$

$$Y_0^2(a, 2) = \frac{i 30^{\frac{1}{2}}}{4\pi} \left[e^{2i\alpha} \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2} - e^{-2i\alpha} \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2} \right]$$

$M = -1$

$$Y_{-1}^2(a, 2) = \frac{-2 \cdot 5^{\frac{1}{2}}}{4\pi} e^{-i\gamma} \left[e^{2i\alpha} \cos \frac{\beta}{2} \sin \frac{\beta}{2} + e^{-2i\alpha} \cos \frac{\beta}{2} \sin \frac{\beta}{2} \right]$$

$M = -2$

$$Y_{-2}^2(a, 2) = \frac{-i 5^{\frac{1}{2}}}{4\pi} e^{-2i\gamma} \left[e^{2i\alpha} \sin^2 \frac{\beta}{2} - e^{-2i\alpha} \cos^2 \frac{\beta}{2} \right]$$

APPENDIX D

TOTAL ANGULAR MOMENTUM - ISOSPIN
WAVE FUNCTIONS

The spin - isospin wave functions $V_{M_T M_S K_t}(P_e, T, S)$ are now combined with the Euler angle wave functions $Y_M^L(P_e, |\mu|)$ to form total angular momentum - isospin wave functions labelled $Y_{M_J M_T K}(J, L, S, P, P_e, P_t, |\mu|)$. These Y 's are identical to those defined in Derrick and Blatt (1958). J is the total angular momentum of the system, M_J is the z - component of total angular momentum, L is the orbital angular momentum, S is the spin angular momentum, and μ is the body z - component of orbital angular momentum. The functions Y transform according to one of the irreducible representations of S_3 under permutations of the particles and the label P indicate this representation. As usual K is the corresponding row number of the representation P .

$$(d-1) Y_{M_J M_T K}(J, L, S, P, P_e, P_t, |\mu|) = \sum_{M_L} \sum_{M_S} \sum_{K_t} \begin{pmatrix} P_e & P_t & P \\ 1 & K_t & K \end{pmatrix} \langle L S M_L M_S | J M_J \rangle Y_{M_L}^L(P_e, |\mu|) V_{M_T M_S K_t}(P_t, T, S)$$

The coefficients $\begin{pmatrix} P_e & P_t & P \\ 1 & K_t & K \end{pmatrix}$ are the permutation addition coefficients defined in section A.3 and the coefficients $\langle L S M_L M_S | J M_J \rangle$ are Clebsch Gordan coefficients as defined by Condon and Shortley (1935).

In the case of the triton, $J = T = -T = \frac{1}{2}$. Without any loss of generality, one can assume $M_J = \frac{1}{2}$. Counting a pair of mixed states as only 1 state, there are 10 states for the triton wave function. They are given explicitly below and their permutation properties are listed in table 1. The functions $Y_1, \dots, Y_{10,2}$ given below are identical to those referred to by Derrick (1960b).

$$y_1 = \gamma_0^0(s, 0) V_{-\frac{1}{2}, \frac{1}{2}}(a, \frac{1}{2}, \frac{1}{2})$$

$$y_2 = y_0^0(s, 0) V_{-\frac{1}{2}, \frac{1}{2}, 1}(s, \frac{1}{2}, \frac{1}{2})$$

$$y_{3,1} = y_0^0(s,0) V_{-\frac{1}{2},\frac{1}{2},1}(m, \frac{1}{2}, \frac{1}{2})$$

$$y_{3,2} = y_0^0(s,0) V_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}, \frac{1}{2}}(m, \frac{1}{2}, \frac{1}{2})$$

$$y_4 = 3^{-\frac{1}{2}} \left\{ 2^{\frac{1}{2}} \gamma_1'(a, 0) V_{-\frac{1}{2}, -\frac{1}{2}}(s, \frac{1}{2}, \frac{1}{2}) - \gamma_0'(a, 0) V_{-\frac{1}{2}, \frac{1}{2}}(s, \frac{1}{2}, \frac{1}{2}) \right\}$$

$$\gamma_5 = 3^{-\frac{1}{2}} \left\{ 2^{\frac{1}{2}} \gamma_1'(q, 0) V_{-\frac{1}{2}, -\frac{1}{2}}(q, \frac{1}{2}, \frac{1}{2}) - \gamma_0'(q, 0) V_{-\frac{1}{2}, \frac{1}{2}}(q, \frac{1}{2}, \frac{1}{2}) \right\}$$

$$y_{6,1} = 3^{-\frac{1}{2}} \left\{ 2^{\frac{1}{2}} Y_1'(q, 0) V_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}}(m, \frac{1}{2}, \frac{1}{2}) - Y_0'(q, 0) V_{-\frac{1}{2}, \frac{1}{2}}^{\frac{1}{2}}(m, \frac{1}{2}, \frac{1}{2}) \right\}$$

$$y_{1,2} = -3^{-\frac{1}{2}} \{ 2^{\frac{1}{2}} Y_1'(q,0) V_{-\frac{1}{2},1}(m, \frac{1}{2}, \frac{1}{2}) - Y_0'(q,0) V_{-\frac{1}{2},1}(m, \frac{1}{2}, \frac{1}{2}) \}$$

$$y_{71} = 6^{-\frac{1}{2}} \left\{ 3^{\frac{1}{2}} y_{-1}'(q, 0) V_{-\frac{1}{2} \frac{3}{2} 2} \left(m, \frac{1}{2}, \frac{3}{2} \right) - 2^{\frac{1}{2}} y_0'(q, 0) V_{\frac{1}{2} \frac{1}{2} 2} \left(m, \frac{1}{2}, \frac{1}{2} \right) - y_1'(q, 0) V_{-\frac{1}{2} -\frac{1}{2} 2} \left(m, \frac{1}{2}, \frac{3}{2} \right) \right\}$$

$$y_{7,2} = -6^{-\frac{1}{2}} \left\{ 3^{\frac{1}{2}} \chi'_1(a,0) V_{\frac{1}{2}, \frac{3}{2}} \left(m, \frac{1}{2}, \frac{3}{2} \right) - 2^{\frac{1}{2}} \chi'_0(a,0) V_{-\frac{1}{2}, \frac{1}{2}} \left(m, \frac{1}{2}, \frac{1}{2} \right) - \chi'_1(a,0) V_{-\frac{1}{2}, -\frac{1}{2}} \left(m, \frac{1}{2}, \frac{3}{2} \right) \right\}$$

$$y_{8,1} = 10 \left\{ 2 Y_{\frac{1}{2},(s,0)}^2 V_{-\frac{1}{2},-\frac{3}{2}} \left(m, \frac{1}{2}, \frac{3}{2} \right) - 3^{\frac{1}{2}} Y_{1,(s,0)}^2 V_{-\frac{1}{2},-\frac{1}{2}} \left(m, \frac{1}{2}, \frac{3}{2} \right) + 2^{\frac{1}{2}} Y_{0,(s,0)}^2 V_{-\frac{1}{2},\frac{1}{2}} \left(m, \frac{1}{2}, \frac{3}{2} \right) - Y_{\frac{1}{2},(s,0)}^2 V_{-\frac{1}{2},\frac{3}{2}} \left(m, \frac{1}{2}, \frac{3}{2} \right) \right\}$$

$$y_{8,2} = 10^{-\frac{1}{2}} \left\{ 2 Y_{\frac{1}{2}}^2(s,0) V_{-\frac{1}{2}-\frac{3}{2}}^{\frac{1}{2}}(m, \frac{1}{2}, \frac{3}{2}) - 3^{\frac{1}{2}} Y_{1,0}(s,0) V_{-\frac{1}{2}-\frac{3}{2}}^{\frac{1}{2}}(m, \frac{1}{2}, \frac{3}{2}) + 2^{\frac{1}{2}} Y_{0,0}(s,0) V_{-\frac{1}{2}-\frac{3}{2}}^{\frac{1}{2}}(m, \frac{1}{2}, \frac{3}{2}) - Y_{1,1}(s,0) V_{-\frac{1}{2}-\frac{3}{2}}^{\frac{1}{2}}(m, \frac{1}{2}, \frac{3}{2}) \right\}$$

$$y_{q,1} = \frac{1}{10} \left\{ 2Y_2^{(s,2)} V_{-\frac{1}{2}, -\frac{3}{2}} \left(m, \frac{1}{2}, \frac{3}{2} \right) - 3Y_1^{(s,2)} V_{-\frac{1}{2}, -\frac{1}{2}} \left(m, \frac{1}{2}, \frac{3}{2} \right) + 2Y_0^{(s,2)} V_{-\frac{1}{2}, \frac{1}{2}} \left(m, \frac{1}{2}, \frac{3}{2} \right) - Y_{-1}^{(s,2)} V_{-\frac{3}{2}, \frac{3}{2}} \left(m, \frac{1}{2}, \frac{3}{2} \right) \right\}$$

$$y_{q,2} = 10^{-\frac{1}{2}} \left\{ 2 Y_2(s,2) V_{-\frac{1}{2}-\frac{3}{2}2} \left(m, \frac{1}{2}, \frac{3}{2} \right) - 3^{\frac{1}{2}} Y_1(s,2) V_{-\frac{1}{2}-\frac{1}{2}2} \left(m, \frac{1}{2}, \frac{3}{2} \right) + 2^{\frac{1}{2}} Y_0(s,2) V_{-\frac{1}{2}-\frac{1}{2}2} \left(m, \frac{1}{2}, \frac{3}{2} \right) - Y_1(s,2) V_{-\frac{1}{2}-\frac{3}{2}2} \left(m, \frac{1}{2}, \frac{3}{2} \right) \right\}$$

$$y_{10,1} = 10^{-\frac{1}{2}} \left\{ 2 Y_2^2(q, 2) V_{-\frac{1}{2}, -\frac{3}{2}}(m, \frac{1}{2}, \frac{3}{2}) - 3 Y_1^2(q, 2) V_{-\frac{1}{2}, -\frac{1}{2}}(m, \frac{1}{2}, \frac{3}{2}) + 2 Y_0^2(q, 2) V_{-\frac{1}{2}, \frac{1}{2}}(m, \frac{1}{2}, \frac{3}{2}) - Y_1^2(q, 2) V_{-\frac{3}{2}, \frac{3}{2}}(m, \frac{1}{2}, \frac{3}{2}) \right\}$$

$$y_{9,2} = -10^{-\frac{1}{2}} \left\{ 2Y_2^2(q,2) \frac{V}{-\frac{1}{2}-\frac{3}{2}}, (m, \frac{1}{2}, \frac{3}{2}) - 3^{\frac{1}{2}} Y_1^2(q,2) \frac{V}{-\frac{1}{2}-\frac{1}{2}}, (m, \frac{1}{2}, \frac{3}{2}) + 2^{\frac{1}{2}} Y_0^2(q,2) \frac{V}{-\frac{1}{2}-\frac{1}{2}}, (m, \frac{1}{2}, \frac{3}{2}) - Y_1^2(q,2) \frac{V}{-\frac{3}{2}-\frac{1}{2}}, (m, \frac{1}{2}, \frac{3}{2}) \right\}$$

$$y_{9,2} = -10^{-\frac{1}{2}} \left\{ 2Y_2^2(q,2) \frac{V}{-\frac{1}{2}-\frac{3}{2}}, (m, \frac{1}{2}, \frac{3}{2}) - 3^{\frac{1}{2}} Y_1^2(q,2) \frac{V}{-\frac{1}{2}-\frac{1}{2}}, (m, \frac{1}{2}, \frac{3}{2}) + 2^{\frac{1}{2}} Y_0^2(q,2) \frac{V}{-\frac{1}{2}-\frac{1}{2}}, (m, \frac{1}{2}, \frac{3}{2}) - Y_1^2(q,2) \frac{V}{-\frac{3}{2}-\frac{1}{2}}, (m, \frac{1}{2}, \frac{3}{2}) \right\}$$

APPENDIX E

EVALUATION OF THE ISOSPIN
MATRIX ELEMENTS

The isospin matrix elements needed for the form factor calculations are those involving $T = \frac{1}{2}$, $M_T = -\frac{1}{2}$. These are the P_4 and P_5 functions given in table B-2. To simplify writing the matrix elements of the operator τ_{kz} between these two isospin functions, the following convention is used.

$$(E-1) \quad \langle P_4 | \tau_{kz} | P_4 \rangle = \langle 1 | k | 1 \rangle, \quad \langle P_4 | \tau_{kz} | P_5 \rangle = \langle 1 | k | 2 \rangle$$

$$\langle P_5 | \tau_{kz} | P_4 \rangle = \langle 2 | k | 1 \rangle, \quad \langle P_5 | \tau_{kz} | P_5 \rangle = \langle 2 | k | 2 \rangle$$

Evaluation of these twelve matrix elements is straight forward and the results are tabulated in table E-1.

TABLE E-1

ISOSPIN MATRIX ELEMENTS

$\langle 1 1 1 \rangle = -\frac{2}{3}$	$\langle 1 2 1 \rangle = -\frac{2}{3}$	$\langle 1 3 1 \rangle = \frac{1}{3}$
$\langle 1 1 2 \rangle = 3^{-\frac{1}{2}}$	$\langle 1 2 2 \rangle = -3^{-\frac{1}{2}}$	$\langle 1 3 2 \rangle = 0$
$\langle 2 1 1 \rangle = 3^{-\frac{1}{2}}$	$\langle 2 2 1 \rangle = -3^{-\frac{1}{2}}$	$\langle 2 3 1 \rangle = 0$
$\langle 2 1 2 \rangle = 0$	$\langle 2 2 2 \rangle = 0$	$\langle 2 3 2 \rangle = -1$

APPENDIX F

INTERNAL WAVE FUNCTION BOUNDARY CONDITIONS

F.1 The triton potentials

For convenience in the numerical computations described in chapter 4, we define the following quantities:

$$(f-1a) \quad Y_3(r) = \frac{15}{4cr^2} + V^+(r)$$

$$(f-1b) \quad Y_7(r) = \frac{63}{4cr^2} + [V^{ct}(r) - V^t(r)]$$

$$(f-1c) \quad Y_c(r) = \tilde{V}^t(r)$$

$$(f-1d) \quad \lambda = -\frac{14}{15} E, \quad C = \left(\frac{\hbar^2}{M}\right)^{-1}$$

where $V^+(r)$, $V^t(r)$, $V^{ct}(r)$, $\tilde{V}^t(r)$ and E are defined in equations (3-11) to (3-15). The asymptotic forms of the potentials for small r are

$$(f-1'a) \quad Y_3(r) = \frac{15}{4cr^2} + O\left(\frac{1}{r}\right)$$

$$(f-1'b) \quad Y_7(r) = \frac{63}{4cr^2} + O\left(\frac{1}{r}\right)$$

$$(f-1'c) \quad Y_c(r) = O\left(\frac{1}{r}\right)$$

and for large r are

$$(f-2'a) \quad Y_3(r) = \frac{15}{4cr^2} + O\left(\frac{1}{r^3}\right)$$

$$(f-2'b) \quad Y_7(r) = \frac{63}{4cr^2} + O\left(\frac{1}{r^3}\right)$$

$$(f-2'c) \quad Y_c(r) = O\left(\frac{1}{r^3}\right)$$

F.2 The modified Feshbach - Rubinow equation (4-1)

The eigenfunction, $u(r)$, of equation (4-1) with fixed parameter α has, for small values of r , the following form

$$(f-2) \quad u(r) = a r^n$$

where

$$(f-3) \quad n = \frac{1}{2} + \left[\frac{1}{4} + \frac{3.75}{1+\alpha^2} (1 + 4 \cdot 2\alpha^2) \right]^{\frac{1}{2}}$$

For large values of r , $u(r)$ has the form

$$(f-4) \quad u(r) = b e^{-\sqrt{\epsilon \lambda} r}$$

That these are the asymptotic forms of $u(r)$ for large and small r is clear from looking at equations (f-1') and (f-2').

F.3 The coupled Feshbach - Rubinow equations (4-3)

For small values of r , the eigenfunctions $u(r)$ and $w(r)$ to equations (4-3) are assumed to be

$$(f-6a) \quad u(r) = r^\mu \sum_{n=0}^{\infty} (a_n + c_n \ln r) r^n$$

$$(f-6b) \quad w(r) = r^\mu \sum_{n=0}^{\infty} (b_n + d_n \ln r) r^n$$

Expanding the potentials in equations (4-3), substituting equations (f-6) into (4-3), and equating coefficients to zero yields

$$(f-5a) \quad u(r) = a_0 r^{\frac{5}{2}} [1 + A r + G r^2 + \dots]$$

$$(f-5b) \quad w(r) = a_0 B r^{\frac{5}{2}} \left[r + D r^2 \ln r + \frac{AVAL}{B} r^2 + \dots \right]$$

where

$$(f-5c) \quad A = \frac{C}{2}(V_{os} + V_{ot}) \quad , \quad B = -\frac{83.0477}{180} C V_{ote}$$

$$D = 0.7C(V_{os} + V_{ot}) + 1.127271C V_{ot} - 1.107532C V_{ote} + 1.14285\mu_{te}$$

$$G = \frac{1}{12} [5A^2 - 7B^2 - 12C(\mu_s V_{os} + \mu_t V_{ot}) - 12C\lambda]$$

Using a similar technique for large λ yields

$$(f-7a) \quad u(\lambda) = a e^{-\sqrt{C\lambda}\lambda} \left[1 + \frac{15}{8} \left(\frac{1}{\sqrt{C\lambda}\lambda} \right) + \left(A\sqrt{C\lambda} + \frac{105}{128} \right) \left(\frac{1}{\sqrt{C\lambda}\lambda} \right)^2 + \dots \right]$$

$$(f-7b) \quad w(\lambda) = a \beta e^{-\sqrt{C\lambda}\lambda} \left[1 + \frac{63}{8} \left(\frac{1}{\sqrt{C\lambda}\lambda} \right) + \left(B\sqrt{C\lambda} + \frac{55(63)}{128} \right) \left(\frac{1}{\sqrt{C\lambda}\lambda} \right)^2 + \dots \right]$$

where

$$(f-8) \quad A = 3C \left(\frac{V_{os}}{\mu_s^2} + \frac{V_{ot}}{\mu_t^2} \right) \quad , \quad B = \frac{35.6545}{4} C \left(\frac{V_{ot}}{\mu_t^2} \right)$$

APPENDIX G EQUIVALENCE BETWEEN EQUATION (5.21b)
AND THE SCHIFF S - STATE FORM FACTOR

The isospin functions (n_1, n_2) for $T = \frac{1}{2}$, $M_T = -\frac{1}{2}$ and the spin functions (χ_1, χ_2) for $S = \frac{1}{2}$, $M_S = \frac{1}{2}$ defined by Schiff (1964) differ from the functions (P_4, P_5) and (q_1, q_2) respectively, defined in appendix B, by the interchange of the indices 1 and 3. All those functions of Schiff (1964) differ from the Derrick and Blatt (1958) functions by the interchange of indices 1 and 3. Thus the functions u , v_1 , v_2 defined by Schiff (1964) correspond to

$$(g-1a) \quad u = f_1$$

$$(g-1b) \quad v_1 = - P_{13} \frac{f_{3,1}}{\sqrt{2}}$$

$$(g-1c) \quad v_2 = - P_{13} \frac{f_{3,2}}{\sqrt{2}}$$

where $f_{3,1}$, $f_{3,2}$ are defined in chapter 2, P_{13} is defined in appendix A, and the minus sign appears from the comparison of equation (2.1) to equation (6) of Schiff (1964).

Rewriting equation (5-21b) interchanging indices 1 and 3 gives

$$(g-2) \quad F_2(q^2) = \int \left[(e^{iq \cdot \underline{r}_3} - e^{iq \cdot \underline{r}_1}) f_1 \frac{P_{13} f_{3,1}}{\sqrt{2}} - \sqrt{3} e^{iq \cdot \underline{r}_2} \frac{f_1 f_{3,2}}{\sqrt{2}} \right] d\tau$$

One now applies equations (g-1) and uses the permutation properties defined in equation (3) of Schiff (1964) on the functions (v_1, v_2) to yield

$$(g-3) \quad F_2(q^2) = \int \left[(e^{iq \cdot \underline{r}_1} - e^{iq \cdot \underline{r}_2}) u v_1 + \sqrt{3} e^{iq \cdot \underline{r}_2} u v_2 \right] d\tau$$

which is identical to the expression defined in equation (10) of Schiff (1964).