# MODELING AND STABILITY ANALYSIS OF STRUCTURALLY-VARYING DYNAMIC SYSTEMS WITH APPLICATION TO MECHANICAL PROCESSING

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## Abstract

With the ever-increasing sophistication of engineering practices, more time-variant and structurally-varying dynamic systems are required to accomplish various demanding tasks. The research on various aspects of the dynamics of these complex systems is therefore becoming increasingly active. The current project, which is the modeling and stability analysis of structurally-varying dynamic system, is initiated in such a context. A structurally-varying system is defined as a dynamic system that consists of a number of structural subsystems and are interconnected together through a finite set of constraints, which are time varying. The dynamics of the overall system depends on not only the dynamics of the subsystems but also the interconnections between them. The focus of this research is the stability analysis of linear, structurally-varying dynamic systems. Since the system has a time-varying structure, the stability condition of the system is generally changing with time (or more accurately with the constraints between subsystems). For different situations, test criteria for evaluating the stability of structurally-varying systems are developed. The relationship between the system stability and time-varying constraints is investigated. The relationship of the subsystem dynamics and the overall system dynamics is also studied. The main purposes of this study can be summarized as follows: first is to evaluate the stability of the system under certain constraints, and second is to deliberately change these constraints according to a set of desired criteria (such as change the constraints of the system in order to stabilize it) to maintain the system within a desired operating region, which may serve design and development purposes.

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# Chapter 1

# Introduction

## 1.1 Literature Review

### 1.1.1 Modeling of Large-Scale Dynamic Systems

Since the early 1960s, the field of dynamic modeling and analysis of large complex structures has been a very active research area [Hurty, 1971], [Meirovitch, 1980]. A number of important techniques, such as component-mode synthesis (CMS) [Hurty, 1965], branch-mode analysis (BMA) [Gladwell,1964], and component-mode substitution [Hurty, 1971], have been developed for that purpose and gradually improved over the past several decades. These techniques, in common, make use of the information collected from the substructure (sometimes called component) analysis to study the overall structure and they are closely related to experimental modal analysis described in [de Silva, 1984]. The basic idea is to treat a complex structure as an assemblage of connected substructures, or components. Each subsystem is analyzed or treated separately to derive an appropriate dynamic model. Then the dynamic model for the overall connected structure is formulated on the basis of the individual substructure models using the constraint conditions which are derived from the connections among the substructures. A considerable amount of literature on these techniques is available, but they have almost exclusively considered systems with time-invariant constraints, due to the fact that the techniques were originally developed for the modal analysis of large systems with time-invariant structures [Gladwell,1964], [Hurty, 1965], [Smet *et al*, 1989]. Lately, Peng and de Silva extended the CMS technique to systems with time-varying constraints [Peng and de Silva,1991].

This decomposition-aggregation approach is also widely used by electrical engineers [Bondi etc., 1980], [Hsu etc., 1980], [Vidyasagar, 1981] in the study of the theory of dynamic systems. Instead of direct analysis of the whole system, analysis is carried out on an aggregate model which consists of subsystems and interconnections [Michailesco, 1980], [Lunze, 1985]. This actually brings about conceptual simplifications because the dynamic analysis of the subsystems, which usually have lower order, is often simpler. Comprehensive study of the dynamics of large scale, interconnected systems can be found in [Siljak, 1978], [Vidyasagar, 1981] and [Grujić,1987].

### 1.1.2 Stability and Stabilization of Dynamic Systems

As a very important characteristic of dynamic systems, stability has been another active research area for centuries. Early work of the stability of mechanical systems dated back to the eighteenth century when Euler considered the eigenvalue problem of column buckling. Since this pioneering groundwork, a great number of mathematicians and engineers have been working on this fascinating subject for generations. The stability theory, as Leipholz pointed out, has experienced a dramatic development toward a certain degree of perfection [Leipholz, 1987]. Numerous approaches have been developed for the determination of stability of various dynamic systems. Through the investigation of vibration of a dynamic system, in 1788 Lagrange demonstrated a theory of stability for vibratory motion of mechanical systems about an equilibrium position [Lagrange, 1788]. Energy criteria were used to determine the stability of the equilibrium positions. Routh later extended this method to the stability analysis of perturbed motions. He tried to apply the energy criteria to the investigation of the stability of states of motion. The most prominent theory of stability in the nineteenth century was developed by A.M. Liapunov in his famous doctoral dissertation published in 1892 [Liapunov, 1949]. Liapunov attempted to establish a stability theory for general motions. Among the numerous contributions, the most notable one is his second method, which is also called the *Liapunov Direct Method* [LaSalle and Lefschetz, 1961]. It provides a method to solve the stability problem of state differential equation of a system *without actually solving the equation.* Information about the stability property of the system is deduced directly from its model, although the stability definition is phrased in terms of the system motions. Inspired by Liapunov's work in the late nineteenth century, Poincaré started qualitative analysis of nonlinear differential equations as a result of his investigation of orbital stability. Many good books are available today on the classic theory of stability such as [Bellman, 1953], [LaSalle and Lefschetz, 1961], [Hahn, 1967], [Leipholz, 1987] and [Vidyasagar, 1993], which provide a thorough, comprehensive study of stability of dynamic systems.

For the analysis of the stability of dynamic systems, there are several methods which are often used. The *Liapunov function method* [LaSalle and Lefschetz, 1961] is one of the most widely used methods of stability analysis of dynamic systems. The broadness of this principle constitutes a difficult, often impossible task: finding of a Liapunov function. The method itself does not suggest a way to construct the Liapunov function. One has to analyze a variety of trial functions for the kind of dynamic systems that are under investigation in order to determine a Liapunov function, which is one of the drawbacks of the Liapunov method. In spite of the drawbacks, this method is a very popular approach for the stability analysis of linear, time-varying systems. New theories for stability analysis of time-varying systems are actively being pursued by many researchers [Ljung, 1982], [Kosut and Anderson, 1985], [Wittenmark, 1990]. The *Comparison method* [Grujić,1987] is a similar method and is basically an extension of the Liapunov function method. A comparison function has to be constructed. Although it is generally not easy to find a comparison function, there are several common ways to initiate an attempt. After a comparison function is constructed, the stability analysis is carried out based on it. From the analysis of the comparison function, a *sufficient* condition for stability of the original system can be determined. Also as a general theorem, *small gain theory* gives sufficient conditions of bounded-input-bounded-output (**BIBO**) stability of dynamic systems [Desoer and Vidyasagar, 1975]. The *Perturbation method* is another widely used method. The stability of a perturbed system is based on the stability of its unperturbed counterpart and perturbation properties.

The recent trend of stability analysis has been on finding a *qualitative* measure of a system stability property for various dynamic systems [Lunze, 1988]. Topological and functional analytical methods for the treatment of the operator equation have been used particularly in the development of stability theories, such as $L_p$-space method [Vidyasagar, 1981], in which the formation of the dynamic system description and the definition of system stability are based on the concepts of operator and mapping in the linear space theory. Very recently, a new algebraic factorization approach is innovated by Youla [Youla, 1976] and Vidyasagar [Vidyasagar, 1985]. The latest $H_\infty$ optimal control theory is also developed based on the concept of factorization [Francis, 1987]. The central idea of the so-called "factorization" approach is that of "factorizing" the transfer matrix of a system as the "ratio" of two stable rational matrices. Based on this factored system transfer matrix, a simple parameterization of all compensators that stabilize a given plant can be obtained. One could then, in principle, choose the best compensator for various applications [Youla, 1976]. The factorization approach is a very general framework, which encompasses continuous-time systems as well as discrete-time systems, lumped as well as distributed systems, one-dimensional as well as multidimensional systems. However, the factorization approach is a computation-intensive approach. For systems which have a time-varying structure, required "refactorizing" in real-time may limit its application.

Robust control is another new approach for stabilization and control of uncertain systems with unknown, sometimes time-varying uncertainty. To date, the field of robust control is still in a stage of intensive research [Lunze, 1988]. Numerous criteria have been derived to characterize the uncertainty such that the stability is guaranteed if the criteria are satisfied. A robust controller can sometimes be designed to stabilize the system for a given uncertainty bound [Jabbari, 1991]. The literature concerning this problem is quite extensive [Lunze, 1988], [Qu and Dorsey, 1991], [Olas, 1991], [Haddad etc., 1992], and [Bauer, 1992]. Jabbari [Jabbari, 1991] developed a state feedback controller based on the Lyapunov technique. The time domain framework is preserved along with the ability to readily incorporate the time-varying uncertainty. The uncertainty, which is described as a perturbation to the state space model of the system, is assumed to satisfy certain matching conditions. Robust control without the matching conditions has been studied recently by Chen [Chen, 1990], Qu and Dorsey [Qu and Dorsey, 1991]. It is shown in their work that a general control law can be designed to guarantee the stability of the uncertain system if the nominal system can be stabilized with an arbitrarily large convergence rate. The Riccati approach is another widely used method in robust control [Petersen, 1986], [Schmitendorf, 1988], in which the bound of the uncertainties does not enter explicitly into the control scheme but appears implicitly in an associated Riccati equation for solution of the feedback control gain. Instead of the matching conditions, the uncertainty functions are assumed to be linear combinations of unknown parameter variations with constant bounds and weighting matrices. In the development of a state feedback control law, prior knowledge of the structure of the uncertainties is used. The Liapunov and Riccati equation approaches have been shown to be very effective in analysis and synthesis of the systems. Much research has been done via these tools especially for robust stability and stabilization for finite dimensional time invariant systems.

For linear systems with varying structure (which may include variations of both system parameters and system order), the usual way to estimate the parameters is through some kind of estimation algorithm. Usually a certain form of the system model is assumed [Niu etc, 1982], [Guo etc., 1982]. If information is not sufficient in order to assume a reasonably good model, an artificially selected black-box parametrization for a system is sometimes used [Ljung, 1982]. In this case, the system is parameterized according to input-output properties instead of physical insight [Guo etc., 1982]. In some sense, it is a pure mathematical practice and system physics is either completely neglected, or not used. In some applications, not only the system parameters, but also the system order have to be identified simultaneously [Niu etc, 1982]. Usually a great deal of computation time is required. Recently, a different approach is adopted by Peng and de Silva in the stability analysis of systems with time-varying structure [Peng and de Silva,1993], [Peng and de Silva,1992]. Based on known subsystem models and constraints between them, the model of the overall system can be updated in real-time by a recursive algorithm. The stability of the system can then be determined. General discussion on dynamics and stability analysis of adaptive control systems can be found in [Egardt, 1979], [Anderson etc., 1986] and [Astrom and Wittenmark, 1990].

In the stability analysis of mechanical systems, Walker and Schmitendorf proposed an approach to evaluate the stability of a linear, time-invariant system without actually solving the equation of motion [Walker and Schmitendorf, 1973]. The *asymptotic stability* of a mechanical system is determined by evaluating the rank of a special evaluation matrix constructed from the parameter matrices of the system. The stability of systems with uncertain, linear and time-varying parameter perturbations was studied by Chen and Hsu [Chen, 1988]. Sufficient stability conditions for such systems are derived by using the possible *bound* of the perturbation in conjunction with the classical Liapunov approach. More recently, Lin [Lin *et al.*, 1991] studied the stability of a system subjected to parameter perturbations and model uncertainties. Asymptotic stability and bounded-input-bounded-output (**BIBO**) stability for a class of lumped-parameter systems under nonlinear time-varying perturbations are analyzed. The stability analysis is carried out based on the analysis of time domain response. The final stability criterion is stated in terms of a perturbation bound and several matrix norms.

It has to be pointed out that the multilateral meanings of the concept of stability have led to various methods for stability analysis which have been formulated separately according to different stability definitions. However, there are some common features of all stability definitions and the associated analysis methods. In general, the most crucial issue in the stability analysis of dynamic systems is to determine the characteristics used to define the stability of a system. Certain quantities, such as norms of the state vector, are sometimes emphasized, and used to characterize the system state response at any desired time. Other methods include total energy function or trajectories in phase space of the system. Although there has been a desire and effort to unify these concepts, apparently none has been satisfactory.

## 1.2 Objectives of the Proposed Research

With the increasing complexity of process control problems, more sophisticated and efficient control strategies and theories are required in order to manipulate the operation of the process effectively and economically. This research is initiated under such a situation. The main objective of the work is to develop modeling methodology and an approach for stability analysis for a class of time-varying dynamic systems that are termed structurally-varying systems, (**SVS** for short). A majority of the system analysis and control theory procedures developed to date is limited to linear and time-invariant systems or *structure-fixed* systems, which constitute only a small portion of real systems. For the analysis and control of more complicated time-varying or structure-varying systems, new approaches are needed.

In the work to be presented, the dynamic modeling and stability analysis of an **SVS** will be investigated. We assume that an **SVS** consists of a number of *subsystems*, which are connected together. By subsystems, we mean *physical entities* to be identified by a suitable partitioning method. Some knowledge of the dynamics of the subsystems, which may be linear, is assumed. It is believed that the behavior of the overall system can be predicted from the dynamics of the subsystems and the constraint conditions among the subsystems. This idea is based on the hypothesis that the dynamic characteristics of a system are solely determined by its own structure. If the subsystems are known, and also the constraints among them are known, it can be said that the overall system will be known. Hence, the dynamics of the overall system can be synthesized from that of the subsystems and the constraint conditions among the subsystems. Besides, the stability condition of an **SVS** will usually change if the structure of the **SVS** changes due to the variations of constraint condition between subsystems. The relationship between the system characteristics, particularly the time-varying model and the stability of the overall system, and its structural variations will also be studied.

The proposed research has a variety of practical applications. One of them can be in the building and deployment of a space station. In the mission of building a space station, all materials have to be moved out into space by a space shuttle. The space station may be assembled piece by piece by either astronauts or robots controlled from the space shuttle. The structure of the space station being built keeps varying, which has to be maintained stable at any instant of time. The widely used pick-and-place operations carried out by industrial robots in factories are another example of an **SVS**. A direct application of the proposed research will be in the design of a robotic fish processing workcell. During the overall working period, the architecture of the workcell may vary at different stages of operation. Hence a proper control strategy has to be developed to deal with the variations of the system structure. In our research, we will concentrate on the theoretical aspects, especially the modeling and the stability analysis of such types of systems. Some illustrative examples will also be given in the process of development of the theory in order to demonstrate the effectiveness of the theory and the procedure to apply the theory.

## 1.3 Motivations for the Proposed Research

Factory automation is widely recognized as an important goal for remaining competitive in the manufacturing sector both nationally and internationally. Robotics is one of the major research areas in manufacturing automation, which has been motivated both by economic objectives (i.e., enhances productivity, profitability, and quality) and sociological objectives (i.e., a desire to improve the quality of human life by releasing humans from repetitive, hazardous, or strenuous tasks).

The fish processing industry is an old one. The technology used in current fish processing is rather outdated. The majority of the work is done manually. With today's outdated methods of fish processing, considerable wastage is inevitable. Upgrading the fish processing technology will result in improved raw product recovery; it is estimated that recovering an additional one per cent of the raw product through improved processing would result in as much as $5-million annual savings for the Canadian fish processing industry [de Silva, 1990]. With modern robot technology, we could even go beyond that goal. There is a further promise of recovering anywhere from three-to-five per cent of the raw product, and furthermore, productivity can be increased by speeding up the whole plant process. On the other hand, fish cutting is a boring and tedious job. The sharp blade of a cutter is a potential danger to the workers, especially in long workshifts and considering the fact that the environment is very unpleasant and slippery. Developing fish processing technology relieves the humans of such hazardous work, can enhance productivity and keep fish processing economically viable.

This thesis is organized into five chapters. In Chapter two, we will propose some new definitions of stability for an **SVS**. The basic concepts associated with the definitions and terminology to be used in the later study will also be presented. Chapter three and Chapter four concentrate on the development of dynamic models and stability analysis for an **SVS**. Major results and contributions of the research will be summarized in Chapter five.

# Chapter 2

# Basic Concepts and Definitions

It has been known that although the research on stability of dynamic systems has been an active area for centuries, there is hardly a universal definition of this important concept [Leipholz, 1987]. This is due to the fact that the types of dynamic systems considered vary and also the performance requirements can be specified in many different ways. However, this fact has not prevented the theory on stability of dynamic systems from evolving. In fact it has provided a fertile subject for analytical research. The usual practice of the studies of stability has been that for the kind of dynamic systems which are of interest to us, the definition of stability is first tailored to the particular needs of the problem, and then the relevant stability theory *in that particular sense* is developed.

In this chapter, we will first discuss the dynamic system which we are interested in, specifically a structurally-varying system or **SVS**, and then we will provide an appropriate definition of stability for the **SVS**.

## 2.1 Structurally-Varying Systems

For the purposes of the present development, an **SVS** is assumed to be composed of a number of linear, deterministic and lumped-parameter subsystems. Lumped-parameter systems are those for which all energy storage or dissipation can be lumped into a finite number of discrete spatial locations. They are described by ordinary differential equations. The way these subsystems are structurally integrated is time-variant. But the dynamics of the subsystems are assumed to be time-invariant.

A simple example of an **SVS** is presented in Figure 2.1. The system consists of two subsystems. There is a *dynamic constraint* or *dynamic connection* between them which is assumed to be time-varying. $m_i, k_i$ and $c_i$ are the subsystem parameters, and $kc_i$ and $cc_i$ are parameters of the dynamic connection. In the real-time operation, the two constraints are released sequentially. From the subsystem point of view, the boundary condition is time-varying. On the other hand, the structure of the *overall* system is also time-varying due to the structural perturbation. Part (a) shows the *fully* constrained system configuration and part (b) shows the system configuration after the system is completely disintegrated into two subsystems. It is not difficult to see that the system order is a constant in this type of **SVS**. The variation of the structure will only change the parameters of the overall system.

Figure 2.2 provides an example of another type of **SVS**. Two subsystems are connected with each other through two mass nodes $m_1$ and $m_2$, which may be called a *rigid constraint* or *rigid connection.* Each of them can be considered as a combination of two smaller masses, $m_{11}$, $m_{12}$ and $m_{21}$, $m_{22}$ respectively. A rigid connection is assumed to be in one of the two states, either connected or disconnected (binary constraint model). This two-state constraint model can also be called the *static* constraint model. The meaning of the term *static* can be interpreted as the dynamics of the constraint being negligible. In this case, the connection between two subsystems is *rigid*, or in other words, each constraint has *infinite* stiffness. In this type of **SVS**, the system order will change when the constraint condition between the two subsystems changes. The system can be considered to be *growing* bigger in the sense that the order of the overall system increases when the constraints of the subsystems are being released. On the other hand, the system can be considered to be *shrinking* when a new constraint is applied to the subsystems since the order of the overall system will decrease. These two types of **SVS** will be studied in Chapter 3 and Chapter 4 respectively.

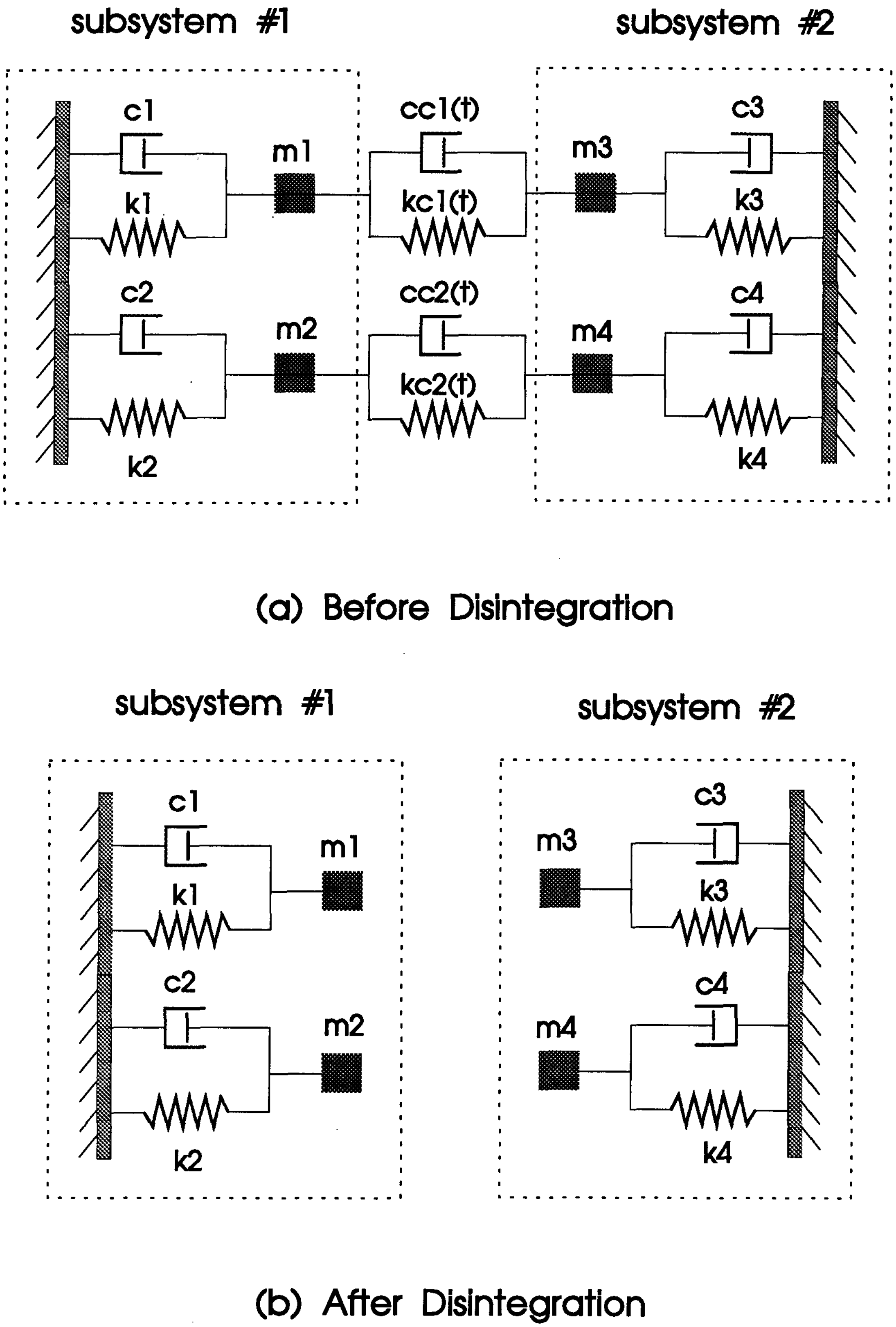


Figure 2.1: An Example of a Fixed-Order SVS

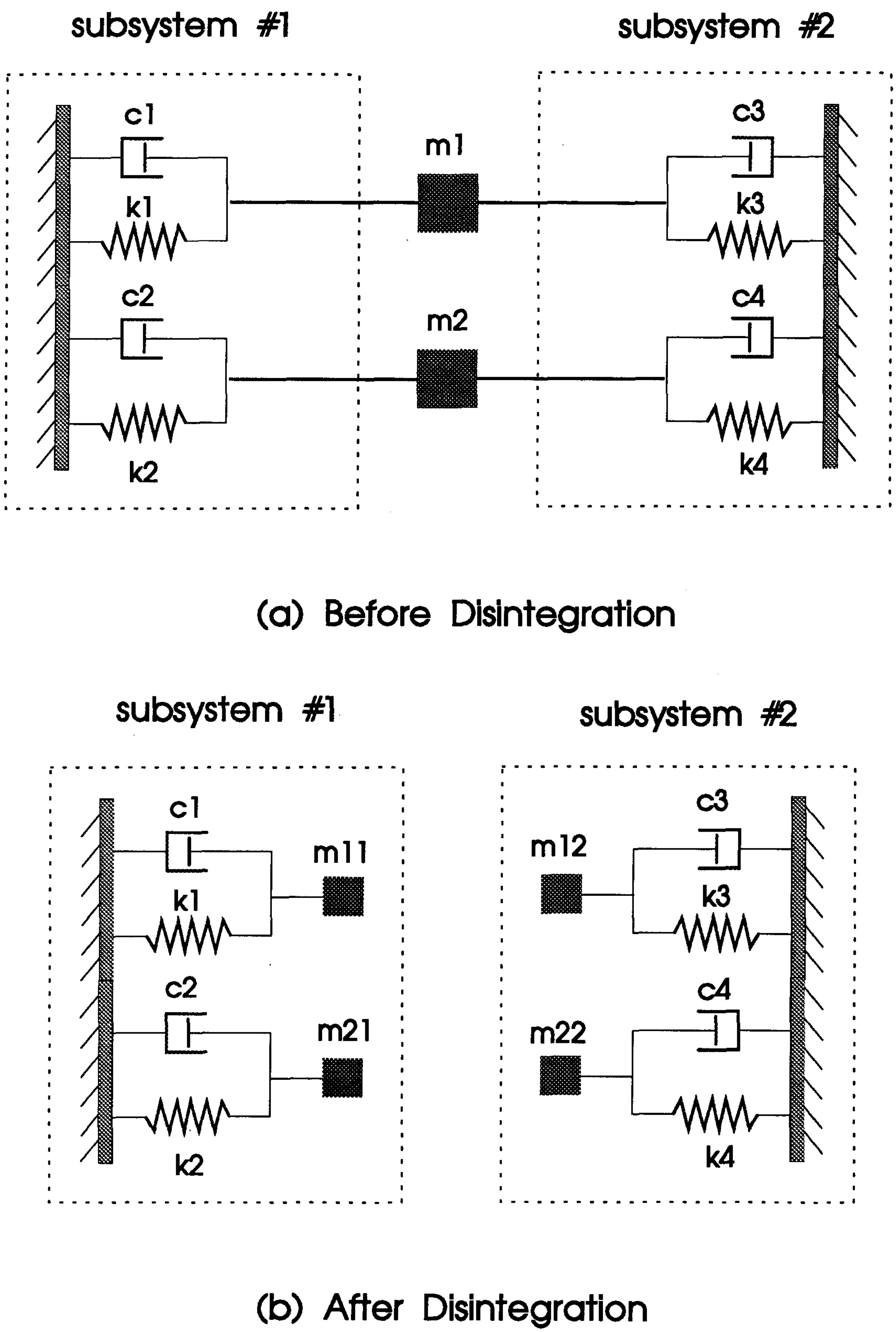


Figure 2.2: An Example of a Varying-Order SVS

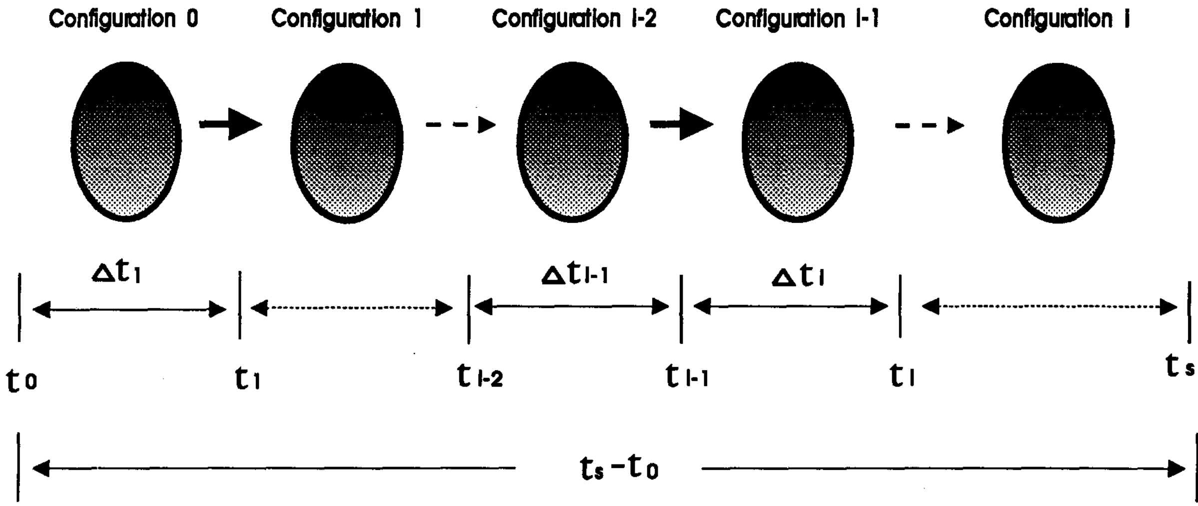


Figure 2.3: Diagram of Structure of the SVS

In general, any **SVS** can be modeled by $s$ system *configurations* and $s$ *switching instants* over a period of time, which is of interest to us. Figure 2.3 describes this model. An **SVS** has an *invariant* or *fixed* system structure between any two instants of structural variations, which are called the *structural switching instants.* The system structure takes a new configuration after a structural variation. It can be seen that the structural variation of the **SVS** is of a discrete nature. The configurations of the **SVS** are connected to each other through the structural switching instants, and conversely, the structural switching instants are related to each other through system configurations.

## 2.2 Review of Concepts of Stability of Dynamic Systems

Before we start to discuss the concepts of stability for an **SVS**, some conventional definitions of stability are reviewed. Although there are a variety of definitions of stability, they can in general be grouped into two categories, i.e., *perturbation* definition and *response* definition. A system is said to be stable if when a small disturbance is applied, the motion of the system will return to its initial equilibrium point after a period of time. If the system is not able to return to its initial equilibrium point under a small disturbance, the system is said to be unstable. This definition can be considered as a perturbation definition of stability. The stability of a system can also be defined from its response performance. If a *well-behaved* excitation produces a *desired* response over a time interval, the system can be considered stable. If a well-behaved excitation does not produce a desired response, the system is considered unstable. Here, by "well-behaved" we mean the excitation is applied *within a certain range*. The definition of the range is determined by the particular problem we are facing. In addition, by "desired" we mean that the response of the system is what we actually want and this response meets the special requirements of the particular task. These two categories can be unified if we look at them from another perspective. They all use system response over a period of time as a *measuring variable* or *evaluation function*. If the response of the system satisfies certain requirements, the system is said to be stable. Otherwise, the system is said to be unstable. Usually, the requirements include convergence rather than divergence of the response of the system over a certain period of time.

Generally, the definitions of stability of dynamic systems consist of four elements: convergence, bounds, time interval and the input. From the definitions of stability, it is usually possible to relate certain system dynamics to its stability. The stability condition of a system can then be expressed in terms of the particular dynamic characteristics of the system, for instance, the eigenvalues of the system. In order to examine the stability of dynamic systems, a measuring variable or evaluation function has to be selected which allows us to examine the dynamic characteristics of the system. This measuring variable or evaluation function carries the information of the dynamic characteristics of the system from which the stability can be determined. For instance, given a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

with $\mathbf{u}(t)$ as the input vector, $\mathbf{y}(t)$ as an output vector and $\mathbf{x}(t)$ as a state vector, we have the following definition of stability:

- A system is said to be bounded-input-bounded-output (**BIBO**) stable if for each admissible bounded input $\mathbf{u}(t)$, the output $\mathbf{y}(t)$ is also bounded.

This definition is based on the system response, which is the measuring variable. The stability of this system can also be restated in terms of system eigenvalues. If all eigenvalues of the system have negative real parts, the system is said to be stable. In the later case, however, the eigenvalues are the parameters which describe the special dynamic characteristics of the system.

## 2.3 Definition of Concepts of Stability for An SVS

One reason to discuss the concept of stability for an **SVS** is that an **SVS** has some special dynamic features, which can make most of the popular stability definitions neither applicable nor appropriate in the stability analysis of this class of systems. Most of the research on stability study of a dynamic system has focused upon the dynamic performance of the system over an *infinite-time* period and the criteria are consequently infinite-time ones. It is unrealistic to ascertain the stability condition of a system during the structure-varying period merely from an infinite-time criterion since we are interested not only in the system stability after the system operates for a long period of time, which in other words, can be mathematically interpreted as the system stability as time $\to \infty$, but also in the system stability in a relatively short period of time. Also, the time dependence of the structure of an **SVS** is of discrete nature. Each system configuration can be *considered as* a time-invariant system. Its stability is also of interest to us. Practically, it is often true that the stability during a finite period of operation would be of interest to us. Most important of all, the influence of continuous structural perturbations of an **SVS** cannot be studied by the conventional infinite-time criteria. Hence the concepts of conventional stability have to be modified in order to accommodate these special features of an **SVS**.

The stability of an **SVS** can be studied in terms of *fixed-structure* stability i.e., *static* stability or *varying-structure* stability, i.e, *dynamic* stability. Static stability describes the system stability condition at each fixed configuration, and dynamic stability describes the variation of system stability condition when the system structure changes due to perturbations.

By fixed-structure stability or static stability, we mean the stability of a particular system configuration. We look at the stability of each system configuration individually. When the time interval between two switching instants of an **SVS** is *large enough,* the conventional stability theory can be adopted to study the static stability of the **SVS** in that time interval. The static stability is appropriate and it has some physical meaning in this situation. However, if the time interval between two switching instants of an **SVS** is not large enough, the stability analysis results using the conventional stability theory would be inappropriate. Since each system configuration is of time-invariant structure, the conventional stability definitions and theories, such as **BIBO** stability, can be applied directly to its stability analysis. The analysis of static stability of an **SVS** will reveal information on the stability of individual system configurations. Basically, each system configuration is placed and analyzed on an *infinite* time scale, as the system is a time-invariant one. The time scale is stretched from a finite period of time to an infinite period of time. Analysis of the static stability is in fact no different to stability analysis of ordinary dynamic systems. The definition of static stability could be considered as one of the conventional definitions of stability.

On the other hand, the dynamic stability of an **SVS** is meant to represent the change of the stability condition from one system configuration to the next due to perturbations, which could be structural or state-variable-related. All together, they can be called *system perturbations.* The system configurations are linked to each other through the system perturbations. We mainly look at the change of the evaluation function of stability from one system configuration to the next rather than the stability condition of individual configurations. Therefore, the influence of the structural variation on the stability of an **SVS** is important. It is evident that the dynamic stability of an **SVS** investigates the system stability on a *finite* time interval only. The differentiation or the change is emphasized. At different system configurations, the dynamic stability of an **SVS** is generally different. Therefore, the dynamic stability for every configuration has to be studied in order to determine the dynamic stability of an **SVS**. This issue will be the major topic of the present research.

Based on our previous discussion, the definition of dynamic stability of an **SVS** is given here:

1. **Definition of Dynamic Stability**: The time span which we are interested in is divided into a number of equal segments. Each segment (such as the time interval $[t_i, t_{i+1}]$) corresponds to a configuration of the **SVS** and $t_i$ and $t_{i+1}$ are the time instants when the structural changes of the **SVS** occur. The **SVS** is said to be dynamically stable in the time interval $[t_i, t_{i+1}]$ if

$$D_i = \Delta x_i^{max} = x_i^{max} - x_{i-1}^{max} \leq 0$$

where $D_i$ is the change of the *maximum state response* (in the sense of a suitable norm) in two consecutive system configurations. $x_i^{max} = \mathbf{max}\{\| \mathbf{x}(t) \|; t \in [t_i, t_{i+1}]\}$, $x_{i-1}^{max} = \mathbf{max}\{\| \mathbf{x}(t) \|; t \in [t_{i-1}, t_i]\}$, and $\| \quad \|$ represents a suitable norm.

In this definition, the state response of the **SVS** is used as the evaluation function. The dynamic stability of an **SVS** can also be defined by using an energy function as the evaluation function.

2. **Definition of Dynamic Stability**: The time span which we are interested in is divided into a number of equal segments. Each segment (such as the time interval $[t_i, t_{i+1}]$) corresponds to a configuration of the **SVS** and $t_i$ and $t_{i+1}$ are the time instants when the structural changes of the **SVS** occur. The **SVS** is said to be dynamically stable in the time interval $[t_i, t_{i+1}]$ if

$$D_i = \Delta E_i^{max} = E_i^{max} - E_{i-1}^{max} \leq 0$$

where $D_i$ is the change of the maximum value of an energy function in two consecutive system configurations. $E_i^{max} = \mathbf{max}\{E(t); t \in [t_i, t_{i+1}]\}$, $E_{i-1}^{max} = \mathbf{max}\{E(t); t \in [t_{i-1}, t_i]\}$.

It is not difficult to see that the concept of dynamic instability of an **SVS** can also be defined by using either system state response or system energy function as the evaluation function.

3. **Definition of Dynamic Instability**: The time span which we are interested in is divided into a number of equal segments. Each segment (such as the time interval $[t_i, t_{i+1}]$) corresponds to a configuration of the **SVS** and $t_i$ and $t_{i+1}$ are the time instants when the structural changes of the **SVS** occur. The **SVS** is said to be dynamically unstable in the time interval $[t_i, t_{i+1}]$ if

$$D_i = \Delta x_i^{max} = x_i^{max} - x_{i-1}^{max} > 0$$

where $D_i$ is defined in **(1)**.

4. **Definition of Dynamic Instability**: The time span which we are interested in is divided into a number of equal segments. Each segment (such as the time interval $[t_i, t_{i+1}]$) corresponds to a configuration of the **SVS** and $t_i$ and $t_{i+1}$ are the time instants when the structural changes of the **SVS** occur. The **SVS** is said to be dynamically unstable in the time interval $[t_i, t_{i+1}]$ if

$$D_i = \Delta E_i^{max} = E_i^{max} - E_{i-1}^{max} > 0$$

where $D_i$ is defined in **(2)**.

In figure 2.4, two examples are provided which illustrate the concepts of dynamic stability and instability of an **SVS**.

The above concepts define the stability (and instability) of an **SVS** at a particular time interval. Since an **SVS** is classified as a time-varying system, its stability condition is generally changing with time. The system stability over a period of time $T$ can be known if the stability condition of the **SVS** over every time interval (such as $[t_i, t_{i+1}]$) is known. It can be observed that the excitation to the **SVS** is not included in the definitions of the stability. However, we assume that there exists an external force which is applied at the structural switching instant and causes the variation of the constraint between subsystems.

It should be noted that the definition of dynamic stability is designed for investigation of system dynamic performance of an **SVS**, either state response or energy value, during a finite period of time. It is different from conventional definitions of stability such as Liapunov stability or asymptotic stability which consider the dynamic response of a system in an infinite time scale.

Also, it has to be pointed out that the theories of static and dynamic stability deal with different dynamic aspects of an **SVS**. They are *independent* of each other. Static

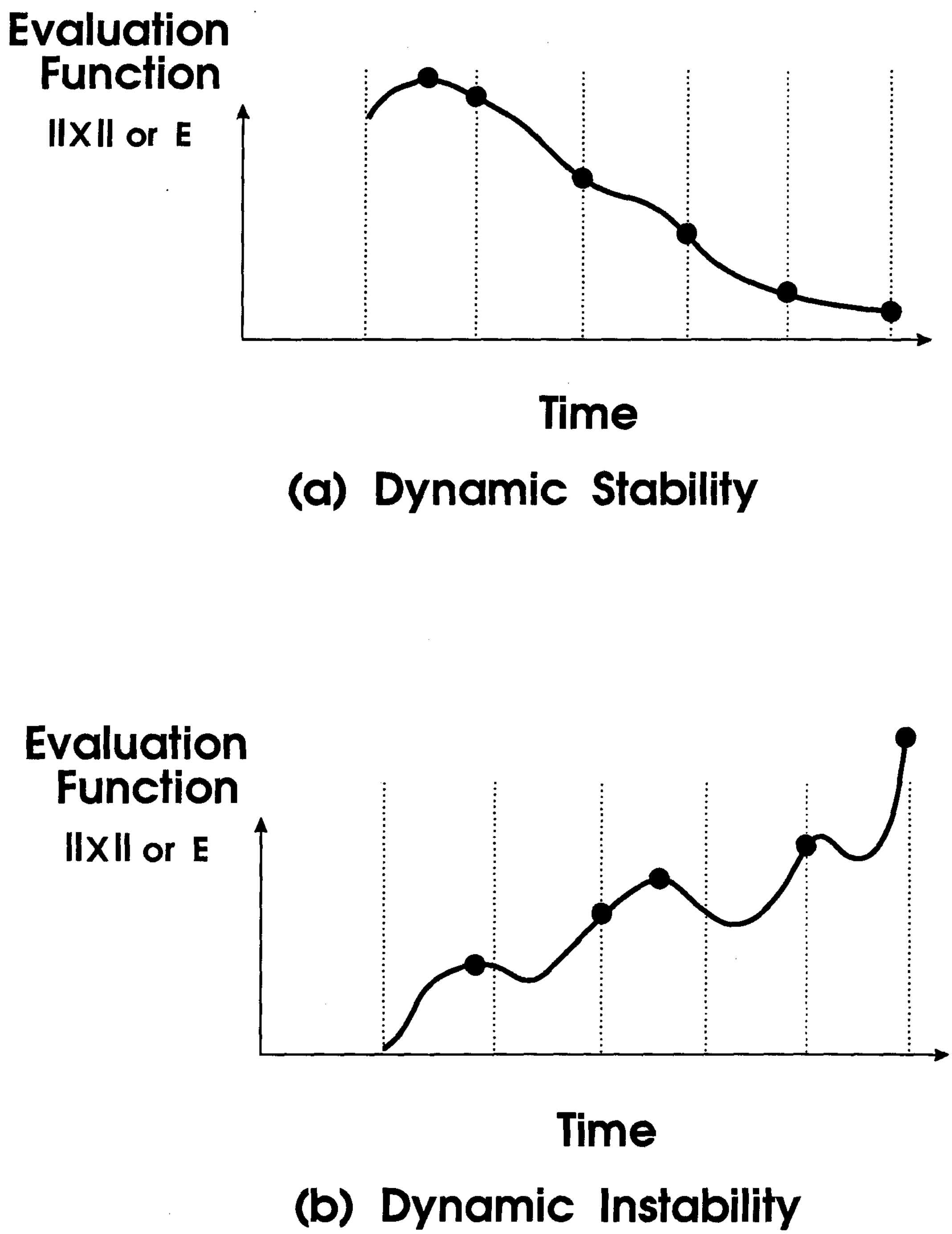


Figure 2.4: Example of Dynamic Stability and Instability of an SVS

stability of an **SVS** does not assure the dynamic stability of the **SVS** and *vice versa.* In other words, even when every configuration is statically stable, it is still possible that the **SVS** is dynamically unstable. On the other hand, a dynamically stable **SVS** may have statically unstable configurations. Therefore, in order to determine the stability of an **SVS** comprehensively, the analysis of both static and dynamic stability has to be carried out.

## 2.4 Summary

Since the dynamics of an **SVS** has some special features, the concepts of static and dynamic stability are introduced and defined in order to study the stability of the **SVS** comprehensively. They are modifications of some of the conventional stability concepts and designed particularly for the special dynamic characteristics of the **SVS**. Based on these new concepts, the stability of the **SVS** can be analyzed more thoroughly and the influence of the discrete structural variations on the stability of the **SVS** can be studied.

# Chapter 3

## Stability of Structurally-Varying Systems With Fixed Order

### 3.1 Introduction

It has been known that there are two types of **SVS**, an **SVS** with fixed order and an **SVS** with varying order. In this chapter, the stability analysis for an **SVS** with fixed order is carried out. The **SVS** to be studied is assumed to have only *dynamic connections* between subsystems, which implies that the connection between any two subsystems only consists of a spring with *finite* stiffness and a damper, as is shown in figure (2.1). There is no mass coupling between subsystems. This type of **SVS** has a constant order throughout the entire time period of operation regardless of the perturbations on the stiffness and damping matrices of the system. There is no perturbation on the mass matrix of the system. Two evaluation functions, state response function and energy function will be employed to carry out the stability analysis. A number of criteria for the evaluation of stability of an **SVS** will be derived for both static stability and dynamic stability.

### 3.2 Modeling of Switching Instants

It is known from Chapter 2 that an **SVS** can be modeled by a series of configurations and switching instants. Each configuration between two switching instants can be considered as a time-invariant system and the structural variation occurs only at the switching instant. In this chapter, we will study an **SVS** which has only flexible connections between subsystems, which means a connection is composed of either a spring or a damper. There is no mass coupling between subsystems. As a result, the total number of the mass nodes is a constant over the period of time of interest. Hence, the system order is maintained.

A schematic diagram of the switching instant is given in figure (3.1). There are two boundary mass nodes, $m_1$ and $m_2$. Dotted lines represent the connections of these two mass nodes to other parts of the systems. The connection between these two mass nodes consists of a spring and a damper. The connection between subsystems is emulated by a switch. At the time of structural change, the switch is turned off instantly so that the two mass nodes are disconnected. The connection components can be considered *removed* from the system.

Since the forces applied on the mass nodes by the spring and the damper are finite, the application or removal of them will change the system structure only and will not cause any sudden change of motion of the mass nodes, which means neither displacement nor velocity vector has a sudden change at the switching instant. If we define

$$\mathbf{d} = \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} \qquad \dot{\mathbf{d}} = \begin{bmatrix} \dot{d}_1(t) \\ \dot{d}_2(t) \end{bmatrix} \tag{3.1}$$

we can have

$$\mathbf{d}(t_i^+) = \mathbf{d}(t_i^-) \tag{3.2}$$

and

$$\dot{\mathbf{d}}(t_i^+) = \dot{\mathbf{d}}(t_i^-) \tag{3.3}$$

where $t = t_i$ is the instant of a structural variation. In other words, there is no perturbation on displacement and velocity due to the structural variation. It has been shown previously that the dimension of the displacement and velocity vectors will not change either. Therefore, we have

$$\mathbf{d}_{i+1}(t_i) = \mathbf{d}_i(t_i) \qquad \dot{\mathbf{d}}_{i+1}(t_i) = \dot{\mathbf{d}}_i(t_i) \tag{3.4}$$

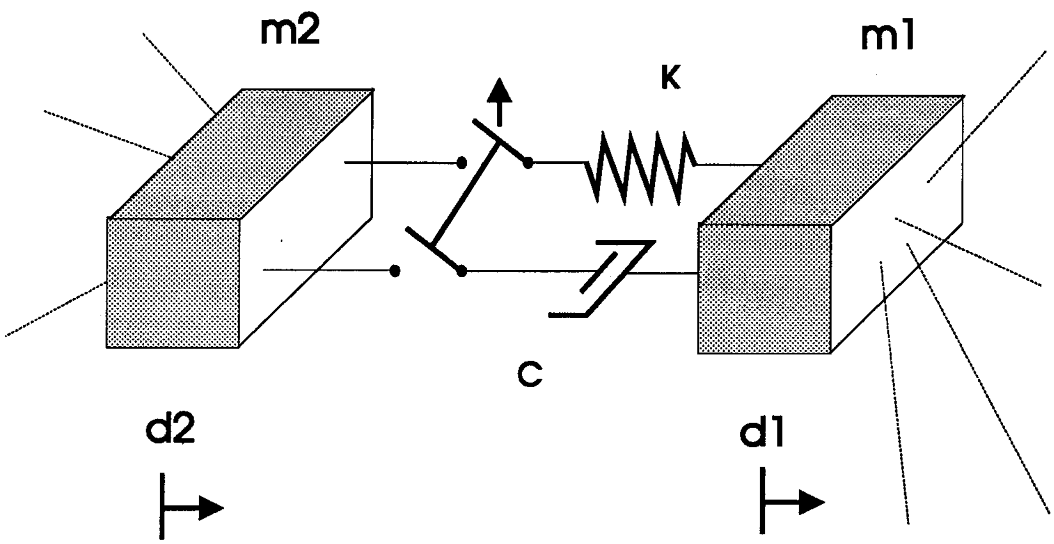


Figure 3.1: Illustration of Switching Instants

at the structural switching instants. The change of the system dynamics can then be determined based on the structural perturbation only.

However, it should be pointed out that this model of the switching instant is an ideal one. It assumes that the connection components, which could be either a spring or a damper, are massless. In reality, this assumption may not be right. There may be some mass and energy associated with the connection components. When they are disconnected from the system, the mass and energy may go away with them. The dynamic model of the switching instant may have to be modified if this factor is taken into consideration.

## 3.3 State Space Approach

We first use state response as the evaluation function. The state space model of an **SVS** will be developed and then the stability analysis will be carried out based on the state space model.

### 3.3.1 Recursive State Space Model of a Structurally-Varying System

As has been discussed in Chapter 2, the systems being studied are confined to a certain class of dynamic systems. The system is composed of a number of smaller systems, or subsystems. Each of them is modeled by

$$\mathbf{M}_j\,\ddot{\mathbf{d}}_j + \mathbf{C}_j\,\dot{\mathbf{d}}_j + \mathbf{K}_j\,\mathbf{d}_j = 0 \tag{3.5}$$

where $\mathbf{M}_j$, $\mathbf{C}_j$ and $\mathbf{K}_j$ are the mass, damping and stiffness matrices, respectively, for the $jth$ subsystem, and $\mathbf{d}_j$ is the displacement vector for the $jth$ subsystem. The overall system model can generally be assembled from the subsystem models

$$\mathbf{M}(t)\,\ddot{\mathbf{d}} + \mathbf{C}(t)\,\dot{\mathbf{d}} + \mathbf{K}(t)\,\mathbf{d} = 0 \tag{3.6}$$

where $\mathbf{M}(t)$, $\mathbf{C}(t)$ and $\mathbf{K}(t)$ are the mass matrix, damping matrix and the stiffness matrix, respectively, for the overall system. $\mathbf{d}$ is the displacement vector for the overall system. All three parameter matrices are composed of the corresponding parameter matrices of subsystems and *constraint parameter matrices*

$$\begin{aligned}\mathbf{M}(t) &= \mathbf{M}^o + \mathbf{M}^c(t)\\ \mathbf{C}(t) &= \mathbf{C}^o + \mathbf{C}^c(t)\\ \mathbf{K}(t) &= \mathbf{K}^o + \mathbf{K}^c(t)\end{aligned} \tag{3.7}$$

where $\mathbf{M}^o = diag\{\mathbf{M}_j\}$, $\mathbf{C}^o = diag\{\mathbf{C}_j\}$ and $\mathbf{K}^o = diag\{\mathbf{K}_j\}$. Superscript $o$ denotes *original* and $c$ denotes *constraint.* Subscript $j$ denotes the subsystem number. $\mathbf{M}_j$, $\mathbf{C}_j$

and $\mathbf{K}_j$ are the parameter matrices of the *jth* subsystem. $\mathbf{M}^c(t)$, $\mathbf{C}^c(t)$ and $\mathbf{K}^c(t)$ are constraint parameter matrices, which describe the dynamic connections between subsystems, and

$$\mathbf{d}(t) = \begin{bmatrix} \mathbf{d}_1(t) \\ \mathbf{d}_2(t) \\ \vdots \\ \mathbf{d}_m(t) \end{bmatrix}$$

The symmetry of parameter matrices M(t), $\mathbf{C}(t)$ and $\mathbf{K}(t)$ are ensured by *Maxwell's reciprocity theorem* if all subsystem parameter matrices and coupling constraint matrices $\mathbf{M}^c(t)$, $\mathbf{K}^c(t)$ and $\mathbf{C}^c(t)$ are symmetric [Meirovitch, 1986].

Since we assume that the subsystems are time-invariant and the constraints which connect the subsystems are also time-varying, the configurations of an **SVS** in two separate constraint conditions are generally different. For any configuration $i$, the system model can be written as

$$\mathbf{M}_i\,\ddot{\mathbf{d}} + \mathbf{C}_i\,\dot{\mathbf{d}} + \mathbf{K}_i\,\mathbf{d} = 0 \tag{3.8}$$

To derive the state space model of the **SVS**, we assume

$$\mathbf{x} = \begin{bmatrix} \mathbf{d} \\ \dot{\mathbf{d}} \end{bmatrix} \tag{3.9}$$

then,

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{d}} \\ \ddot{\mathbf{d}} \end{bmatrix} \tag{3.10}$$

Since

$$\ddot{\mathbf{d}} \;=\; -\mathbf{M}_i^{-1}\,(\mathbf{C}_i\,\dot{\mathbf{d}} + \mathbf{K}_i\,\mathbf{d})$$

$$= \begin{bmatrix} -\mathbf{M}_i^{-1}\,\mathbf{K}_i & -\mathbf{M}_i^{-1}\,\mathbf{C}_i \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \dot{\mathbf{d}} \end{bmatrix}$$

$$= \begin{bmatrix} -\mathbf{M}_i^{-1}\,\mathbf{K}_i & -\mathbf{M}_i^{-1}\,\mathbf{C}_i \end{bmatrix} \mathbf{x} \tag{3.11}$$

and

$$\dot{\mathbf{d}} = \begin{bmatrix} 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \dot{\mathbf{d}} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{I} \end{bmatrix} \mathbf{x} \tag{3.12}$$

Combining equations (3.11) and (3.12), we obtain the state space model for the configuration $i$,

$$\dot{\mathbf{x}} = \mathbf{A}_i\,\mathbf{x} \qquad \textit{and} \qquad \mathbf{x}(t_i) = \mathbf{x}_i \tag{3.13}$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{d} \\ \dot{\mathbf{d}} \end{bmatrix}$$

$$\mathbf{A}_i = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}_i^{-1}\,\mathbf{K}_i & -\mathbf{M}_i^{-1}\,\mathbf{C}_i \end{bmatrix}$$

and $\mathbf{x}_i$ is the initial condition of configuration $i$. Similarly, we can have the model for configuration $i+1$

$$\mathbf{M}_{i+1}\,\ddot{\mathbf{d}} + \mathbf{C}_{i+1}\,\dot{\mathbf{d}} + \mathbf{K}_{i+1}\,\mathbf{d} = 0 \tag{3.14}$$

Therefore, the state space model for configuration $i+1$ can be written as

$$\dot{\mathbf{x}} = \mathbf{A}_{i+1}\,\mathbf{x}, \qquad \textit{and} \qquad \mathbf{x}(t_{i+1}) = \mathbf{x}_{i+1} \tag{3.15}$$

where

$$\mathbf{A}_{i+1} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}_{i+1}^{-1}\,\mathbf{K}_{i+1} & -\mathbf{M}_{i+1}^{-1}\,\mathbf{C}_{i+1} \end{bmatrix}$$

The change of parameter matrices due to the variation of the system constraint condition at time $t = t_{i+1}$ is modeled by

$$\begin{aligned}\Delta\mathbf{M}_{i+1} &= \mathbf{M}_{i+1} - \mathbf{M}_i \\ \Delta\mathbf{C}_{i+1} &= \mathbf{C}_{i+1} - \mathbf{C}_i \\ \Delta\mathbf{K}_{i+1} &= \mathbf{K}_{i+1} - \mathbf{K}_i\end{aligned} \tag{3.16}$$

Since we assume that the system only has a spring-damper connection, we have $\mathbf{M}_i = \mathbf{M}_{i+1}$, i.e., $\Delta\mathbf{M}_i = 0$, which implies that the variations of constraints do not incur any perturbation on the mass matrix. Substituting equation (3.16) into equation (3.15), we get

$$\begin{aligned}\mathbf{A}_{i+1} &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_i^{-1}(\mathbf{K}_i + \Delta\mathbf{K}_{i+1}) & -\mathbf{M}_i^{-1}(\mathbf{C}_i + \Delta\mathbf{C}_{i+1}) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_i^{-1}\mathbf{K}_i - \mathbf{M}_i^{-1}\Delta\mathbf{K}_{i+1} & -\mathbf{M}_i^{-1}\mathbf{C}_i - \mathbf{M}_i^{-1}\Delta\mathbf{C}_{i+1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_i^{-1}\,\mathbf{K}_i & -\mathbf{M}_i^{-1}\,\mathbf{C}_i \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}_i^{-1}\,\Delta\mathbf{K}_{i+1} & -\mathbf{M}_i^{-1}\,\Delta\mathbf{C}_{i+1} \end{bmatrix}\end{aligned}$$

that is

$$\mathbf{A}_{i+1} = \mathbf{A}_i + \Delta\mathbf{A}_{i+1} \tag{3.17}$$

where

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_i^{-1}\,\mathbf{K}_i & -\mathbf{M}_i^{-1}\,\mathbf{C}_i \end{bmatrix}$$

$$\Delta\mathbf{A}_{i+1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}_i^{-1}\,\Delta\mathbf{K}_{i+1} & -\mathbf{M}_i^{-1}\,\Delta\mathbf{C}_{i+1} \end{bmatrix}$$

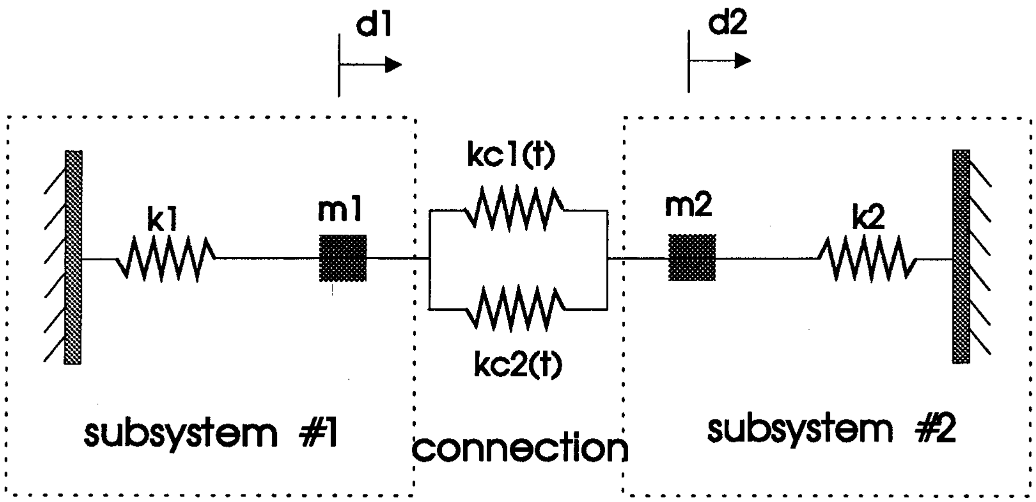


Figure 3.2: Example of a fixed order **SVS**

$\mathbf{A}_{i+1}$ is separated into two terms. The first term $\mathbf{A}_i$ is the model of the previous configuration, and the second term is the *model perturbation* due to the variation of constraints at time $t_{i+1}$. Equation (3.17) is the recursive state space model of the **SVS**.

- **Example:** Consider the system described in figure 3.2. The initial connection between two subsystems consists of a spring with stiffness $k_{c1}$. We assume that the system stiffness increases at time $t_1$ by the value $k_{c2}$. Using the precedure developed previously, we can have

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\mathbf{M}(t) = \mathbf{M}^o = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$\mathbf{K}(t) = \mathbf{K}^o + \mathbf{K}^c(t) = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} + \mathbf{K}^c(t)$$

When $t < t_1$,

$$\mathbf{K}^c(t) = \begin{bmatrix} k_{c1} & -k_{c1} \\ -k_{c1} & k_{c1} \end{bmatrix}$$

and

$$\mathbf{K}_1 = \mathbf{K}(t) = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} + \begin{bmatrix} k_{c1} & -k_{c1} \\ -k_{c1} & k_{c1} \end{bmatrix} = \begin{bmatrix} k_1 + k_{c1} & -k_{c1} \\ -k_{c1} & k_2 + k_{c1} \end{bmatrix}$$

When $t \geq t_1$,

$$\mathbf{K}^c(t) = \begin{bmatrix} k_{c1} + k_{c2} & -k_{c1} - k_{c2} \\ -k_{c1} - k_{c2} & k_{c1} + k_{c2} \end{bmatrix}$$

Hence,

$$\begin{aligned} \mathbf{K}_2 = \mathbf{K}(t) &= \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} + \begin{bmatrix} k_c & -k_c \\ -k_c & k_c \end{bmatrix} \\ &= \begin{bmatrix} k_1 + k_{c1} + k_{c2} & -k_{c1} - k_{c2} \\ -k_{c1} - k_{c2} & k_2 + k_{c1} + k_{c2} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \Delta\mathbf{K}_2 = \mathbf{K}_2 - \mathbf{K}_1 &= \begin{bmatrix} k_1 + k_{c1} + k_{c2} & -k_{c1} - k_{c2} \\ -k_{c1} - k_{c2} & k_2 + k_{c1} + k_{c2} \end{bmatrix} \\ &\quad - \begin{bmatrix} k_1 + k_{c1} & -k_{c1} \\ -k_{c1} & k_2 + k_{c1} \end{bmatrix} = \begin{bmatrix} k_{c2} & -k_{c2} \\ -k_{c2} & k_{c2} \end{bmatrix} \end{aligned}$$

The state space model of the overall system can then be derived;

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}_1\mathbf{x}, \qquad 0 \le t < t_1 \\ \dot{\mathbf{x}} &= \mathbf{A}_2\mathbf{x}, \qquad t_1 \le t < \infty\end{aligned}$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{d} \\ \dot{\mathbf{d}} \end{bmatrix}$$

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -(\mathbf{M}^o)^{-1}\,\mathbf{K}_1 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} \\ -(k_1 + k_{c1})/m_1 & k_{c1}/m_1 & \mathbf{0} \\ k_{c1}/m_2 & -(k_2 + k_{c1})/m_2 & \mathbf{0} \end{bmatrix}$$

$$\Delta\mathbf{A}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -k_{c2}/m_1 & k_{c2}/m_1 & \mathbf{0} \\ k_{c2}/m_2 & -k_{c2}/m_2 & \mathbf{0} \end{bmatrix}$$

$$\begin{aligned}\mathbf{A}_2 = \mathbf{A}_1 + \Delta\mathbf{A}_2 &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -(\mathbf{M}^o)^{-1}\,\mathbf{K}_2 & \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} \\ -(k_1 + k_{c1} + k_{c2})/m_1 & (k_{c1} + k_{c2})/m_1 & \mathbf{0} \\ (k_{c1} + k_{c2})/m_2 & -(k_2 + k_{c1} + k_{c2})/m_2 & \mathbf{0} \end{bmatrix}\end{aligned}$$

In order to investigate the relation between overall system stability and subsystem stability, we rewrite equation (3.6) as

$$\mathbf{M}^o\,\ddot{\mathbf{d}} + \mathbf{C}^o\,\dot{\mathbf{d}} + \mathbf{K}^o\,\mathbf{d} + \mathbf{M}^c(t)\,\ddot{\mathbf{d}} + \mathbf{C}^c(t)\,\dot{\mathbf{d}} + \mathbf{K}^c(t)\,\mathbf{d} = \mathbf{0} \tag{3.18}$$

where $\mathbf{M}^o$, $\mathbf{C}^o$ and $\mathbf{K}^o$ are assembled from the dynamic models of the subsystems and are time-invariant. $\mathbf{M}^c(t)$, $\mathbf{C}^c(t)$ and $\mathbf{K}^c(t)$ describe the *coupling constraints* between subsystems and are determined solely by the constraint conditions at time $t$.

The model for configuration $i$ can then be written as

$$\mathbf{M}^o\,\ddot{\mathbf{d}} + \mathbf{C}^o\,\dot{\mathbf{d}} + \mathbf{K}^o\,\mathbf{d} + \mathbf{M}_i^c(t)\,\ddot{\mathbf{d}} + \mathbf{C}_i^c(t)\,\dot{\mathbf{d}} + \mathbf{K}_i^c(t)\,\mathbf{d} = \mathbf{0}$$

The system matrix can be rewritten as

$$\mathbf{A}_i = \mathbf{A}_i^o + \mathbf{A}_i^c \tag{3.19}$$

where

$$\begin{aligned} \mathbf{A}_i^o &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -(\mathbf{M}^o)^{-1}\,\mathbf{K}^o & -(\mathbf{M}^o)^{-1}\,\mathbf{C}^o \end{bmatrix} \\ \mathbf{A}_i^c &= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}^o)^{-1}\,\mathbf{K}_i^c(t) & -(\mathbf{M}^o)^{-1}\,\mathbf{C}_i^c(t) \end{bmatrix} \end{aligned}$$

Since $\mathbf{A}_i^o$ is time-invariant, the subscript $i$ can be dropped. So

$$\mathbf{A}_i = \mathbf{A}^o + \mathbf{A}_i^c \tag{3.20}$$

Since the recursive constraint model for the **SVS** is

$$\begin{aligned} \mathbf{M}_i^c &= \mathbf{M}_{i-1}^c \\ \mathbf{C}_i^c &= \mathbf{C}_{i-1}^c + \Delta\mathbf{C}_i^c \\ \mathbf{K}_i^c &= \mathbf{K}_{i-1}^c + \Delta\mathbf{K}_i^c \end{aligned} \tag{3.21}$$

where $\Delta\mathbf{M}_i^c = 0$ is used, the constraint matrix $\mathbf{A}_i^c$ can be further expressed in a recursive form

$$\mathbf{A}_i^c = \mathbf{A}_{i-1}^c + \Delta\mathbf{A}_i^c \tag{3.22}$$

where

$$\mathbf{A}_{i-1}^c = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}^o)^{-1}\,\mathbf{K}_{i-1}^c(t) & -(\mathbf{M}^o)^{-1}\,\mathbf{C}_{i-1}^c(t) \end{bmatrix}$$

$$\Delta\mathbf{A}_i^c = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}^o)^{-1}\,\Delta\mathbf{K}_i^c & -(\mathbf{M}^o)^{-1}\,\Delta\mathbf{C}_i^c \end{bmatrix}$$

For various constraint conditions, we have different $\mathbf{K}^c(t)$ and $\mathbf{C}^c(t)$. In general, the characteristics of the $\mathbf{K}^c(t)$ and $\mathbf{C}^c(t)$ are dependent on the physical properties of the connection between subsystems. $\mathbf{A}_{i-1}^c$ is the constraint matrix for configuration $i-1$ and $\Delta\mathbf{A}_i^c$ is the constraint variation matrix at time $t = t_i$. Substituting equation (3.22) into equation (3.20) yields

$$\begin{aligned} \mathbf{A}_i &= \mathbf{A}^o + \mathbf{A}_{i-1}^c + \Delta\mathbf{A}_i^c \\ &= \mathbf{A}^o + \sum_{j=0}^{i} \mathbf{\Delta A}_j^c \end{aligned} \tag{3.23}$$

It can be seen that $\mathbf{A}^o$ is determined solely by the dynamics of the original subsystem, which is the time-invariant part of $\mathbf{A}_i$. $\mathbf{A}_i^c$, on the other hand, is determined by coupling constraints between subsystems, which is the time-varying part of $\mathbf{A}_i$. The model for configuration $i$ of the **SVS** can then be expressed as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}_i(t)\,\mathbf{x}(t) \\ &= (\mathbf{A}^o + \sum_{j=0}^{i} \mathbf{\Delta A}_j^c)\,\mathbf{x}(t) \end{aligned} \tag{3.24}$$

where each $\mathbf{\Delta A}_j^c$ describes a structural variation of the **SVS**.

### 3.3.2 Analysis of Static Stability of an SVS

From the definition given in Chapter 2, we know that the static stability can be considered as an extension of the conventional stability concepts to an **SVS**. For each of the system configurations, we can find out if the configuration is stable (in the conventional sense, such as *bounded-input-bounded-state* (**BIBS**) stable or *asymptotically stable*) when its model is known. In this section, we are not studying the static stability of each system configuration separately. Plenty of work has been done in this area before. We are investigating how the static stability of an **SVS** changes from one system configuration to the next due to the structural variation. To a degree, this problem is similar to the robustness problem of dynamic systems [Haddad etc., 1992].

It has to be pointed out that although the stability analysis is carried out based on the state space model of the **SVS** which is composed of subsystems modeled by the second-order-matrix-equation, the results derived here are not limited to the system of this category. The theory to be developed can be applied to any dynamic system as long as its state space model is available.

The solution of equation (3.15) can be written as

$$\mathbf{x}(t) = \mathbf{\Phi}_{i+1}(\Delta t)\, \mathbf{x}(t_{i+1}) \qquad t \in [t_{i+1}, t_{i+2}] \tag{3.25}$$

where $\mathbf{\Phi}_{i+1}(\Delta t)$ is the state transition matrix for configuration $i$, $i = 0, 1, 2, \cdots, m$ and $\mathbf{\Phi}_{i+1}(\Delta t) = e^{\mathbf{A}_{i+1}\,\Delta t}$, $\Delta t = t - t_{i+1}$. This group of equations determines the time history of the state response at any time instant for the **SVS**. Substituting equation (3.17) into $\mathbf{\Phi}_{i+1}(\Delta t)$ yields

$$\begin{aligned}\mathbf{\Phi}_{i+1}(\Delta t) &= e^{(\mathbf{A}_i + \Delta\mathbf{A}_{i+1})\,\Delta t} \\ &= e^{\mathbf{A}_i\,\Delta t} \cdot e^{\Delta\mathbf{A}_{i+1}\,\Delta t} \\ &= \mathbf{\Phi}_i(\Delta t) \cdot \Delta\mathbf{\Phi}_{i+1}(\Delta t)\end{aligned} \tag{3.26}$$

where

$$\mathbf{\Phi}_i(\Delta t) = e^{\mathbf{A}_i\,\Delta t}$$

$$\Delta\mathbf{\Phi}_{i+1}(\Delta t) = e^{\Delta\mathbf{A}_{i+1}\,\Delta t}$$

and it can be observed that $\mathbf{\Phi}_i(\Delta t)$ is the state transition matrix of configuration $i$, which determines the system stability at configuration $i$.

The condition of the static stability of an **SVS** can then be derived according to equation (3.26). They are presented in the following theorems.

**Theorem 3.1:** If the system configuration $i$ is statically stable in the sense of **BIBS** for the time period $[t_i, \infty)$, the system configuration $i+1$ will also be static stable in the sense of **BIBS** if $\parallel \Delta\mathbf{\Phi}_{i+1}(\Delta t) \parallel$ is bounded.

**Proof:** Suppose the configuration $i$ is stable for $[t_i, \infty)$. Since $\parallel \mathbf{\Phi}_i(\Delta t) \parallel$ is bounded,

$$\parallel \mathbf{\Phi}_i(\Delta t) \parallel < \rho_i, \qquad t \in [t_i, \infty)$$

where $\rho_i \in \mathbf{R}$. The notation $\parallel \mathbf{\Phi}_i(\Delta t) \parallel$ refers to the norm of the linear transformation $\mathbf{x} \to \mathbf{\Phi}_i(\Delta t)\mathbf{x}$, $\mathbf{x} \in \mathbf{R^n}$, which is induced by the standard Euclidean norm on $\mathbf{R^n}$. We know that

$$\mathbf{\Phi}_{i+1}(\Delta t) = \mathbf{\Phi}_i(\Delta t) \cdot \Delta\mathbf{\Phi}_{i+1}(\Delta t)$$

Therefore

$$\parallel \mathbf{\Phi}_{i+1}(\Delta t) \parallel = \parallel \mathbf{\Phi}_i(\Delta t)\, \Delta\mathbf{\Phi}_{i+1}(\Delta t) \parallel \leq \parallel \mathbf{\Phi}_i(\Delta t) \parallel \cdot \parallel \Delta\mathbf{\Phi}_{i+1}(\Delta t) \parallel$$

If $\parallel \Delta\mathbf{\Phi}_{i+1}(\Delta t) \parallel$ is bounded

$$\parallel \Delta\mathbf{\Phi}_{i+1}(\Delta t) \parallel < \rho_\Delta$$

where $\rho_\Delta \in \mathbf{R}$, then

$$\parallel \mathbf{\Phi}_{i+1}(\Delta t) \parallel < \rho_i\, \rho_\Delta = \rho_{i+1}$$

and $\rho_{i+1} \in \mathbf{R}$. Therefore, the system configuration $i+1$ is **BIBS** stable, i.e., statically stable. This concludes the proof of **Theorem 3.1**.

Now we take a further look at the condition that $\| \Delta\mathbf{\Phi}_i(\Delta t) \|$ is bounded. It is known that

$$\Delta\mathbf{\Phi}_i(\Delta t) = e^{\Delta\mathbf{A}_i \, \Delta t}$$

The main concern here is to find out what condition $\Delta\mathbf{A}_i$ has to satisfy in order to keep the system statically stable. The result of the study is summaried in the following theorem.

**Theorem 3.2:** The $\| \Delta\mathbf{\Phi}_i(\Delta t) \|$ will be bounded if and only if the eigenvalues of $\Delta\mathbf{A}_i$ have negative real parts. The proof for a similar theorem can be found in [Chen, 1988].

**Theorem 3.1** and **Theorem 3.2** present the sufficient conditions of the static stability of a configuration. The following theorem gives a condition to make a statically unstable configuration statically stable after a structural perturbation.

**Theorem 3.3:** If the system configuration $i$ is statically unstable in the sense of **BIBS** for the time period $[t_i, \infty)$, the system configuration $i+1$ will be statically stable in the sense of **BIBS** if $\| \Delta\mathbf{\Phi}_{i+1}(\Delta t) \| = 0$.

**Proof**: Suppose the configuration $i$ is unstable for $[t_i, \infty)$, which implies that $\| \mathbf{\Phi}_i(\Delta t) \| > M$ for any $M \in \mathbf{R}$. Since

$$\| \mathbf{\Phi}_{i+1}(\Delta t) \| = \| \mathbf{\Phi}_i(\Delta t)\, \Delta\mathbf{\Phi}_{i+1}(\Delta t) \| \leq \| \mathbf{\Phi}_i(\Delta t) \| \cdot \| \Delta\mathbf{\Phi}_{i+1}(\Delta t) \|$$

If $\| \Delta\mathbf{\Phi}_{i+1}(\Delta t) \| = 0$, we have

$$\| \mathbf{\Phi}_{i+1}(\Delta t) \| \leq 0 < \rho_{i+1}$$

where $\rho_{i+1} \in \mathbf{R}$. Hence, the configuration $i+1$ is **BIBS** stable, i.e., statically stable in the sense of **BIBS**.

As a theoretical result, **theorem 3.3** provides the design method to stabilize a statically unstable configuration through modifying its constraint condition. However, it has to be modified slightly in order to be applied in engineering practice because the condition $\| \Delta\mathbf{\Phi}_i(\Delta t) \| = 0$ hardly has any practical meaning. This situation results from the assumption that a statically unstable system has the transition matrix $\| \mathbf{\Phi}_i(\Delta t) \| \rightarrow \infty$, which is also of little practical significance. In reality, the system would be considered unstable if its output exceeds a certain accepted level and tends to diverge. The output will usually saturate at the physical limits of the system rather than reach infinity. The physical limit of the system is usually the maximum output the system could reach.

The previous study has focused on the relation between two consecutive configurations of the **SVS**. The following theorem provides the criterion for the evaluation of static stability of the overall system based on the stability of subsystems.

**Theorem 3.4:** If all *unconstrained* subsystems are **BIBS** stable, the constrained system will be statically stable in the sense of **BIBS** if every constraint applied in the time interval of interest is stable.

For all constraints to be stable, we mean $\| \mathbf{\Delta\Phi}_i^c(\Delta t) \| < \rho_i$, $\rho_i \in \mathbf{R}$ and for every $i$, $i = 1, 2, \cdots, m$.

**Proof**: Suppose the unconstrained subsystems are **BIBS** stable. We know from equation (3.20)

$$\mathbf{A}_i = \mathbf{A}^o + \mathbf{A}_i^c$$

Hence, the solution to equation (3.24) can be written as

$$\begin{aligned}
\mathbf{x}(t) &= exp\{(\mathbf{A}^o + \sum_{j=0}^{i} \mathbf{\Delta A}_j^c)\Delta t\}\, \mathbf{x}(t_i) \\
&= exp\{\mathbf{A}^o \Delta t\} \cdot \exp\{(\sum_{j=0}^{i} \mathbf{\Delta A}_j^c)\Delta t\}\, \mathbf{x}(t_i) \\
&= \mathbf{\Phi}_i(\Delta t)\, \mathbf{x}(t_i) \\
&= \mathbf{\Phi}^o(\Delta t)\, \mathbf{\Phi}_i^c(\Delta t)\, \mathbf{x}(t_i) \qquad t \in [t_i, t_{i+1}] \qquad (3.27)
\end{aligned}$$

where

$$\begin{aligned}\mathbf{\Phi}^o(\Delta t) &= exp\{\mathbf{A}^o\} \\ \mathbf{\Phi}^c_i(\Delta t) &= \prod_{j=0}^{i} \Delta\mathbf{\Phi}^c_j(\Delta t)\end{aligned}$$

Since all unconstrained subsystems are **BIBS** stable, we have

$$\| \mathbf{\Phi}_o(\Delta t) \| < \rho_o, \qquad t \in [t_i, \infty)$$

where $\rho_o \in \mathbf{R}$. We also know

$$\mathbf{\Phi}_i(\Delta t) = \mathbf{\Phi}^o(\Delta t) \cdot \mathbf{\Phi}^c_i(\Delta t)$$

Therefore

$$\begin{aligned}\| \mathbf{\Phi}_i(\Delta t) \| &= \| \mathbf{\Phi}^o(\Delta t)\, \Delta\mathbf{\Phi}^c_i(\Delta t) \| \\ &\leq \| \mathbf{\Phi}^o(\Delta t) \| \cdot \| \Delta\mathbf{\Phi}^c_i(\Delta t) \| \\ &\leq \| \mathbf{\Phi}^o(\Delta t) \| \cdot \prod_{j=0}^{i} \| \Delta\mathbf{\Phi}^c_j(\Delta t) \|\end{aligned}$$

If every $\| \Delta\mathbf{\Phi}^c_j(\Delta t) \|$ is bounded, i.e.,

$$\mathbf{\Phi}^c_j(\Delta t) \leq \rho_j \leq \rho_{max}$$

where $\rho_{max} = max\{\rho_j, j = 0, 1, 2, \cdots, m\}$. We have

$$\| \mathbf{\Phi}_i(\Delta t) \| \leq \rho_o\, \rho^m_{max}$$

Therefore, the overall constrained system is **BIBS** stable, i.e., statically stable in the sense of **BIBS**.

- **Example:** Assume that a system has the model

$$\dot{\mathbf{x}} = \mathbf{A}^o\mathbf{x}$$

with

$$\mathbf{A}^o = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Its eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -2$. It can be shown that it is **BIBS** stable. If at the time $t = t_1$ a coupling constraint is applied to the system, which has the model

$$\mathbf{A}^c = \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix}$$

Since

$$\| \mathbf{\Phi}^c(\Delta t) \|_2 = \| exp\{ \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} \Delta t\} \|_2 < 1$$

The constraint is stable. Hence, according to **Theorem 3.4**, we know that the overall constrained system is statically stable, which can be verified. The model of the overall constrained system is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

with

$$\mathbf{A} = \mathbf{A}^o + \mathbf{A}^c = \begin{bmatrix} 0 & 1 \\ -2 & -8 \end{bmatrix}$$

Its eigenvalues are $\lambda_1 = -7.74$, $\lambda_2 = -0.26$ and it is **BIBS** stable, i.e., statically stable in the sense of **BIBS**.

### 3.3.3 Analysis of Dynamic Stability Via State Space Model

Although an **SVS** can be considered as a special type of time-varying system, the time dependency of its system structure is not continuous in general. In order to examine the stability of this type of system, we select an evaluation function which is associated with the system configuration. When the system model changes, the value of this function will also change. In this section, the state response of an **SVS** is selected as the evaluation function first. The change of the maximum state response of consecutive system configurations is used to determine the stability of an **SVS**.

The concept of dynamic stability has been discussed in Chapter 2. It has been shown that dynamic stability studies a different aspect of stability of an **SVS** from what static stability does. It is determined by comparison of the values of evaluation functions of different configurations of the **SVS**. It is not closely related to the static stability of the **SVS**. A system configuration can be dynamically stable even when it is statically unstable.

We start the analysis from equation (3.13). The solution to equation (3.13) is

$$\begin{aligned}\mathbf{x}(t) &= exp[\mathbf{A}_i(\Delta t)]\,\mathbf{x}(t_i) && (3.28)\\ &= \mathbf{\Phi}_i(\Delta t)\,\mathbf{x}(t_i) \qquad t \in [t_i, t_{i+1}] && (3.29)\end{aligned}$$

where $\mathbf{\Phi}_i$ is the state transition matrix for each configuration, $i = 1, 2, \cdots, m$ and $\Delta t = t - t_i$. This group of equations determines the time history of the state response at any time instant for the **SVS**.

It is also known that the change of the constraint condition at the time $t = t_i$ can be modeled by a perturbation on system matrix $\mathbf{A}_{i-1}$. Therefore, a recursive state space model can be determined for the system. We can write

$$\mathbf{A}_i = \mathbf{A}_{i-1} + \Delta\mathbf{A}_i \qquad (3.30)$$

Then we can determine a recursive relation for the state transition matrix as

$$\begin{aligned}\mathbf{\Phi}_i(\Delta t) &= exp[(\mathbf{A}_{i-1} + \Delta\mathbf{A}_i)\,\Delta t] \\ &= \mathbf{\Phi}_{i-1}(\Delta t)\cdot\Delta\mathbf{\Phi}_i(\Delta t) \qquad (3.31)\end{aligned}$$

where

$$\begin{aligned}\mathbf{\Phi}_{i-1}(\Delta t) &= exp[\mathbf{A}_{i-1}\,(\Delta t)] \\ \Delta\mathbf{\Phi}_i(\Delta t) &= exp[\Delta\mathbf{A}_i\,(\Delta t)]\end{aligned}$$

The derivation of stability criteria is based on the recursive state space model. It is known [Vidyasagar, 1978] that

$$\begin{aligned}\|\,\mathbf{x}(t)\,\| &\leq exp(\int_{t_i}^{t}\mu[\mathbf{A}_i]d\tau)\,\|\,\mathbf{x}(t_i)\,\| \\ &= exp\{\mu[\mathbf{A}_i]\,\Delta t\}\,\|\,\mathbf{x}(t_i)\,\| \\ &= \gamma_i(\mathbf{A}_i,\Delta t)\,\|\,\mathbf{x}(t_i)\,\| \qquad (3.32) \\ & \quad t\in[t_i,t_{i+1}]\end{aligned}$$

where $\gamma_i(\mathbf{A}_i,\Delta t) = exp\{\mu[\mathbf{A}_i]\,\Delta t\}$, $\mu[\mathbf{A}_i]$ is the *matrix measure* of $\mathbf{A}_i$, $\Delta t = t - t_i$ and $\|\,\mathbf{x}(t)\,\|$ is a suitable norm of the state vector $\mathbf{x}(t)$. The computation of $\gamma_i(\mathbf{A}_i,\Delta t)$ consists of algebraic calculation only and is usually very simple. This feature distinguishes itself and makes this approach very suitable for real-time applications. The definitions of the mathematical concepts and their properties are given in the Appendix. The following example illustrates how to calculate the matrix measure of a matrix.

- **Example:** For a system

  $$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

  with

  $$\mathbf{x}(0) = \begin{bmatrix} 10 \\ 4 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The matrix measure of $\mathbf{A}$ is

$$\begin{aligned}\mu_\infty[\mathbf{A}] &= \mathbf{max}_i\{a_{ii} + \sum_{j\neq i} \mid a_{ij} \mid\} \\ &= 2\end{aligned}$$

Hence,

$$\gamma_i(\mathbf{A}, t) = exp[\int_0^t 2d\tau] = exp(2t)$$

So

$$\begin{aligned}\| \mathbf{x} \|_\infty &\leq \gamma_i(\mathbf{A}, t) \ \| \mathbf{x}(0) \|_\infty \\ &= exp(2t) \ \| \mathbf{x}(0) \|_\infty \\ &= 10 \ exp(2t)\end{aligned}$$

which gives the upper limit of $\| \mathbf{x} \|_\infty$ at any time $t$.

**Theorem 3.5:** An **SVS** is dynamically stable during the time interval $[t_i, t_{i+1}]$ if $\gamma_i(\mathbf{A}_i, \Delta t) \leq 1$.

**Proof:** From equation (3.32), we have

$$\frac{\| \mathbf{x}(t) \|}{\| \mathbf{x}(t_i) \|} \leq \gamma_i(\mathbf{A}_i, \Delta t) \qquad t \in [t_i, t_{i+1}]$$

If $\gamma_i(\mathbf{A}_i, \Delta t) \leq 1$, we have

$$\frac{\| \mathbf{x}(t) \|}{\| \mathbf{x}(t_i) \|} \leq 1$$

Equivalently,

$$\| \mathbf{x}(t) \| \leq \| \mathbf{x}(t_i) \|$$

On the other hand,

$$\| \mathbf{x}(t_i) \| \leq \mathbf{x}_{i-1}^{max} \qquad t \in [t_{i-1}, t_i]$$

where $\mathbf{x}_{i-1}^{max}$ is the maximum value of $\| \mathbf{x}(t) \|$ in the time interval $[t_{i-1}, t_i]$. Accordingly,

$$\mathbf{x}_i^{max} = \| \mathbf{x}(t) \| \leq \mathbf{x}_{i-1}^{max}$$

which proves that the system is dynamically stable in time interval $[t_i, t_{i+1}]$.

### 3.3.4 Recursive Algorithm for Estimation of $\gamma_i(\mathbf{A}_i, \Delta t)$

Since $\gamma_i(\mathbf{A}_i, \Delta t)$ is different for every configuration, an efficient algorithm is needed if $\gamma_i(\mathbf{A}_i, \Delta t)$ is used to evaluate the system stability. We define

$$\Delta\gamma_{i+1}(\mathbf{A}_{i+1}, \Delta t) = \gamma_{i+1}(\mathbf{A}_{i+1}, \Delta t) - \gamma_i(\mathbf{A}_i, \Delta t) \tag{3.33}$$

By substituting equation (3.30) into (3.32) we obtain

$$\begin{aligned} & \Delta\gamma_{i+1}(\mathbf{A}_{i+1}, \Delta t) \\ = \;& exp\{\mu[\mathbf{A}_{i+1}]\Delta t\} - exp\{\mu[\mathbf{A}_i]\Delta t\} \\ = \;& exp\{\mu[\mathbf{A}_i + \mathbf{\Delta A}_{i+1}]\Delta t\} - exp\{\mu[\mathbf{A}_i]\Delta t\} \\ \leq \;& (exp\{\mu[\mathbf{\Delta A}_{i+1}]\Delta t\} - 1) \cdot exp\{\mu[\mathbf{A}_i]\Delta t\} \\ = \;& \beta_{i+1}\, \gamma_i(\mathbf{A}_i, \Delta t) \end{aligned} \tag{3.34}$$

where

$$\begin{aligned} \gamma_i(\mathbf{A}_i, \Delta t) &= exp\{\mu[\mathbf{A}_i]\Delta t\} \\ \beta_{i+1} &= exp\{\mu[\mathbf{\Delta A}_{i+1}]\Delta t\} - 1 \end{aligned} \tag{3.35}$$

The recursive algorithm for estimating $\gamma_{i+1}(\mathbf{A}_{i+1}, \Delta t)$ can be obtained as

$$\begin{aligned} \gamma_{i+1}(\mathbf{A}_{i+1}, \Delta t) &= \gamma_i(\mathbf{A}_i, \Delta t) + \Delta\gamma_{i+1}(\mathbf{A}_{i+1}, \Delta t) \\ &\leq (1 + \beta_{i+1})\gamma_i(\mathbf{A}_i, \Delta t) \\ &= \alpha_{i+1}\, \gamma_i(\mathbf{A}_i, \Delta t) \end{aligned} \tag{3.36}$$

where $\alpha_{i+1} = 1 + \beta_{i+1}$ conveys the information of current structural perturbation and is determined by $\mathbf{\Delta A}_{i+1}$ only. Using this recursive algorithm, $\gamma_{i+1}(\mathbf{A}_{i+1}, \Delta t)$ can be estimated recursively. The dynamic stability of the **SVS** can then be determined by **Theorem 3.5**.

- **Example:** For a system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

with

$$\mathbf{x}(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \qquad \mathbf{A}_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

We assume that the system model varies by $\Delta\mathbf{A}_1$ at time $t = t_1$ and

$$\Delta\mathbf{A}_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

which gives

$$\mathbf{A}_1 = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

Since

$$\mu_\infty[\mathbf{A}_1] = -1$$

Hence,

$$\gamma_1(\mathbf{A}_1, \Delta t) = exp(-\Delta t) < 1$$

According to **Theorem 3.5**, we know that the system is dynamically stable in the time interval $(t_1, \infty)$.

In **Theorem 3.5**, the stability condition has been stated through $\gamma_i(\mathbf{A}_i, \Delta t)$, which is an overall system parameter and the subsystem dynamics is not reflected in it. From equation (3.20), we have

$$\begin{aligned} \mathbf{\Phi}_i(\Delta t) &= exp[(\mathbf{A}^o + \mathbf{A}_i^c)\,\Delta t] \\ &= \mathbf{\Phi}^o(\Delta t) \cdot \mathbf{\Phi}_i^c(\Delta t) \end{aligned} \tag{3.37}$$

where

$$\begin{aligned} \mathbf{\Phi}^o(\Delta t) &= exp[\mathbf{A}^o\,(\Delta t)] \\ \mathbf{\Phi}_i^c(\Delta t) &= exp[\mathbf{A}_i^c\,(\Delta t)] \end{aligned}$$

It can be seen that the state transition matrix $\mathbf{\Phi}_i$ is determined by (1) the dynamics of unconstrained subsystems which is described by time-invariant $\mathbf{\Phi}^o(\Delta t)$, and (2) constraint-dynamic characteristics which is described by $\mathbf{\Phi}_i^c(\Delta t)$.

If an *artificial system* which has the system matrix $\mathbf{A}^o$ is created, the system state transition matrix would be $\mathbf{\Phi}^o(\Delta t)$ and the stability condition of $\mathbf{\Phi}^o(\Delta t)$ does not change because $\mathbf{A}^o$ is time-invariant. In the analysis of the **SVS**, this is due to the fact that the stability condition of $\mathbf{\Phi}^o(\Delta t)$ is determined solely by *unconstrained* subsystems, which are assumed time-invariant. Their stability can be studied separately using conventional analysis tools, such as modal analysis, at the subsystem level. Based on that, the relation between the stability of the overall system and that of unconstrained subsystems can be determined.

**Theorem 3.6:** An **SVS** is dynamically stable if

$$\gamma_i^c(\mathbf{A}_i^c, \Delta t) \leq \frac{1}{\gamma^o(\mathbf{A}^o, \Delta t)}$$

where

$$\gamma^o(\mathbf{A}^o, \Delta t) = exp\{\mu[\mathbf{A}^o]\,\Delta t\}$$

$$\gamma_i^c(\mathbf{A}_i^c, \Delta t) \quad = \quad exp\{\sum_{j=0}^{i} \mu[\Delta \mathbf{A}_i^c] \, \Delta t\}$$

$\gamma^o(\mathbf{A}^o, \Delta t)$ is solely dependent on the subsystem dynamics and it is independent of the system configuration. However, $\gamma_i^c(\mathbf{A}_i^c, \Delta t)$ is determined by the constraint dynamics and it is dependent on the system configuration.

**Proof:** It is known that $\gamma_i(\mathbf{A}_i, \Delta t) = exp\{\mu[\mathbf{A}_i] \, \Delta t\}$. By substituting equation (3.23) into $\gamma_i(\mathbf{A}_i, \Delta t)$ we get

$$\begin{aligned} & \gamma_i(\mathbf{A}_i, \Delta t) \\ = \; & \gamma_i(\mathbf{A}^o, \mathbf{A}_i^c, \Delta t) \\ = \; & exp\{\mu[\mathbf{A}^o + \sum_{j=0}^{i} \mathbf{\Delta A}_j^c] \Delta t\} \\ \leq \; & exp\{\mu[\mathbf{A}^o] \Delta t\} \cdot exp\{\sum_{j=0}^{i} \mu[\mathbf{\Delta A}_j^c] \Delta t\} \end{aligned}$$

It follows that,

$$\begin{aligned} \| \mathbf{x}(t) \| \; & \leq \; exp\{\mu[\mathbf{A}^o] \, \Delta t\} \cdot exp\{\sum_{j=0}^{i} \mu[\Delta \mathbf{A}_i^c] \, \Delta t\} \cdot \| \mathbf{x}(t_i) \| \\ & = \; \gamma^o(\mathbf{A}^o, \Delta t) \cdot \gamma_i^c(\mathbf{A}_i^c, \Delta t) \cdot \| \mathbf{x}(t_i) \| \end{aligned}$$

If

$$\gamma^o(\mathbf{A}^o, \Delta t) \cdot \gamma_i^c(\mathbf{A}_i^c, \Delta t) \leq 1$$

we get

$$\mathbf{x}_i^{max} = \| \mathbf{x}(t) \| \leq \| \mathbf{x}(t_i) \| \qquad t \in [t_i, t_{i+1}]$$

Note that

$$\| \mathbf{x}(t_i) \| \leq \| \mathbf{x}(t) \| = \mathbf{x}_{i-1}^{max} \qquad t \in [t_{i-1}, t_i]$$

Hence

$$\mathbf{x}_i^{max} = \| \mathbf{x}(t_i) \| \leq \mathbf{x}_{i-1}^{max}$$

and

$$D_i \leq 0$$

Therefore, the system is dynamically stable during the time interval $[t_i, t_{i+1}]$. From **Theorem 3.6** the stability of the **SVS** can be analyzed from its subsystem stability and its constraint conditions.

## 3.4 Energy Function Approach

In this section, the dynamic stability of the **SVS** is studied by using the energy function as the evaluation function. This approach is especially appropriate when the system model is given in the form of a second-order-matrix-equation since the energy function is readily available in this case. The dynamic stability criteria are given based on the change of the energy function due to the structural perturbation.

### 3.4.1 Analysis of Dynamic Stability Via Energy Function

It is known from section 3.3.1 that the models of the configuration $i$ and $i+1$ of the **SVS** can be written respectively as

$$\mathbf{M}_i(t)\,\ddot{\mathbf{d}}_i + \mathbf{C}_i(t)\,\dot{\mathbf{d}}_i + \mathbf{K}_i(t)\,\mathbf{d}_i = 0$$

and

$$\mathbf{M}_{i+1}(t)\,\ddot{\mathbf{d}}_{i+1} + \mathbf{C}_{i+1}(t)\,\dot{\mathbf{d}}_{i+1} + \mathbf{K}_{i+1}(t)\,\mathbf{d}_{i+1} = 0$$

The corresponding energy functions are

$$E_i(t) = \frac{1}{2}\dot{\mathbf{d}}_i^T\mathbf{M}_i\dot{\mathbf{d}}_i + \frac{1}{2}\mathbf{d}_i^T\mathbf{K}_i\mathbf{d}_i$$

$$E_{i+1}(t) = \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T\mathbf{M}_{i+1}\dot{\mathbf{d}}_{i+1} + \frac{1}{2}\mathbf{d}_{i+1}^T\mathbf{K}_{i+1}\mathbf{d}_{i+1}$$

respectively. At the instant of a structural variation, we have

$$\begin{aligned}
\mathbf{M}_i \to \mathbf{M}_{i+1} &= \mathbf{M}_i \\
\mathbf{C}_i \to \mathbf{C}_{i+1} &= \mathbf{C}_i + \Delta\mathbf{C}_{i+1} \\
\mathbf{K}_i \to \mathbf{K}_{i+1} &= \mathbf{K}_i + \Delta\mathbf{K}_{i+1}
\end{aligned}$$

and

$$\mathbf{d}_i \to \mathbf{d}_{i+1}$$

Substituting these relation into $E_{i+1}(t)$ expression, we have

$$\begin{aligned}
E_{i+1}(t) &= \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T\mathbf{M}_i\dot{\mathbf{d}}_{i+1} + \frac{1}{2}\mathbf{d}_{i+1}^T(\mathbf{K}_i + \Delta\mathbf{K}_{i+1})\mathbf{d}_{i+1} \\
&= \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T\mathbf{M}_i\dot{\mathbf{d}}_{i+1} + \frac{1}{2}\mathbf{d}_{i+1}^T\mathbf{K}_i\mathbf{d}_{i+1} + \frac{1}{2}\mathbf{d}_{i+1}^T\Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1}
\end{aligned}$$

or

$$E_{i+1}(t) = E_i^\star(t) + \Delta E_i(t)$$

where

$$\begin{aligned}
E_i^\star(t) &= \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T\mathbf{M}_i\dot{\mathbf{d}}_{i+1} + \frac{1}{2}\mathbf{d}_{i+1}^T\mathbf{K}_i\mathbf{d}_{i+1} \\
\Delta E_{i+1}(t) &= \frac{1}{2}\mathbf{d}_{i+1}^T\Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1}
\end{aligned}$$

It has to be pointed that although $E_i^\star(t)$ and $E_i(t)$ have the same form, they are not equal over the time interval $[t_{i+1}, t_{i+2}]$ because in general $\mathbf{d}_{i+1}(t) \neq \mathbf{d}_i(t)$ for $t \in [t_{i+1}, t_{i+2}]$. However, at the instant of the structural variation $t = t_{i+1}$, we do have $\mathbf{d}_{i+1}(t_{i+1}) = \mathbf{d}_i(t_{i+1})$. Therefore

$$\begin{aligned}
E_{i+1}(t_{i+1}) &= E_i^\star(t_{i+1}) + \Delta E_{i+1}(t_{i+1}) \\
&= E_i(t_{i+1}) + \Delta E_{i+1}(t_{i+1})
\end{aligned}$$

It is known that when the damping matrix $\mathbf{C}_i$ is positive definite, the system will dissipate energy. The total system energy will decrease with time [Meirovitch, 1980]. Hence, $\mathbf{max}\{E_i(t)\} = E_i(t_i), t \in [t_i, t_{i+1}]$ and the final value $E_i(t_{i+1})$ will be the minimum one. If we can calculate the $\mathbf{max}\{E_i(t)\}$ from the available information, we will be able to find the difference of maximum energy of two consecutive configurations. Therefore, the stability in the sense of energy change can be determined.

It is known that the derivative of $E_i(t)$ can be computed as

$$\begin{aligned} \dot{E}_i(t) &= \frac{1}{2}[\ddot{\mathbf{d}}_i^T \mathbf{M}_i \dot{\mathbf{d}}_i + \dot{\mathbf{d}}_i^T \mathbf{M}_i \ddot{\mathbf{d}}_i] + \frac{1}{2}[\dot{\mathbf{d}}_i^T \mathbf{K}_i \mathbf{d}_i + \mathbf{d}_i^T \mathbf{K}_i \dot{\mathbf{d}}_i] \\ &= \frac{1}{2}[\ddot{\mathbf{d}}_i^T \mathbf{M}_i + \mathbf{d}_i^T \mathbf{K}_i]\dot{\mathbf{d}}_i + \frac{1}{2}\dot{\mathbf{d}}_i^T [\mathbf{M}_i \ddot{\mathbf{d}}_i + \mathbf{K}_i \mathbf{d}_i] \end{aligned} \tag{3.38}$$

Since $\mathbf{M}_i, \mathbf{K}_i, \mathbf{C}_i$ are symmetric, we can have

$$\mathbf{M}_i\ \ddot{\mathbf{d}}_i + \mathbf{K}_i\ \mathbf{d}_i = -\mathbf{C}_i\ \dot{\mathbf{d}}_i \tag{3.39}$$

and

$$(\mathbf{M}_i\ \ddot{\mathbf{d}}_i + \mathbf{K}_i\ \mathbf{d}_i)^T = \ddot{\mathbf{d}}_i^T\ \mathbf{M}_i + \mathbf{d}_i^T\ \mathbf{K}_i = -\dot{\mathbf{d}}_i^T\ \mathbf{C}_i \tag{3.40}$$

Hence

$$\begin{aligned} \dot{E} &= \frac{1}{2}[-\dot{\mathbf{d}}_i^T\ \mathbf{C}_i]\dot{\mathbf{d}}_i + \frac{1}{2}\dot{\mathbf{d}}_i^T[-\mathbf{C}_i\ \dot{\mathbf{d}}_i] \\ &= -\dot{\mathbf{d}}_i^T\ \mathbf{C}_i \dot{\mathbf{d}}_i \end{aligned} \tag{3.41}$$

It can be seen that the rate of energy decay is determined by the damping matrix. According to Liapunov's stability theorem, we know that the static stability of the configuration is assured when $\mathbf{C}_i$ is positive definite.

Considering the fact that

$$\begin{aligned} \mathbf{max}\{E_{i+1}(t)\} &= E_{i+1}(t_{i+1}) \\ &= E_i^*(t_{i+1}) + \Delta E_{i+1}(t_{i+1}) \\ &= E_i(t_{i+1}) + \Delta E_{i+1}(t_{i+1}) \qquad [t_{i+1}, t_{i+2}] \end{aligned} \tag{3.42}$$

The energy change of the system due to the structural variation is fully described by $\Delta E_{i+1}(t)$. If $\Delta E_{i+1}(t) > 0$, we have $E_{i+1}(t) > E_i(t)$. It implies that the structural variation increases the overall system energy level. Hence, the system is dynamically unstable in the time period $[t_i, t_{i+1}]$ according to the definition given in Chapter 2. On the other hand, if $\Delta E_{i+1}(t) \leq 0$, we have $E_{i+1}(t) \leq E_i(t)$. It can be shown that the structural variation causes the overall system energy level to drop, which implies that the system is dynamically stable in the time period $[t_i, t_{i+1}]$. Since

$$\begin{aligned} \mathbf{min}\{E_i(t)\} &= E_i(t_{i+1}) \\ &= \mathbf{max}\{E_i(t)\} - \int_{t_i}^{t_{i+1}} \dot{\mathbf{d}}_i^T\ \mathbf{C}_i \dot{\mathbf{d}}_i dt \end{aligned} \tag{3.43}$$

Substituting equation (3.43) into equation (3.42), we have

$$\begin{aligned} \mathbf{max}\{E_{i+1}(t)\} &= E_{i+1}(t_{i+1}) \\ &= E_i^*(t_{i+1}) + \Delta E_{i+1}(t_{i+1}) \\ &= \mathbf{max}\{E_i(t)\} - \int_{t_i}^{t_{i+1}} \dot{\mathbf{d}}_i^T\ \mathbf{C}_i \dot{\mathbf{d}}_i dt + \Delta E_{i+1}(t_{i+1}) \\ & \quad [t_{i+1}, t_{i+2}] \end{aligned} \tag{3.44}$$

Hence,

$$\begin{aligned} \Delta\mathbf{max}\{E_{i+1}\} &= \mathbf{max}\{E_{i+1}(t)\} - \mathbf{max}\{E_i(t)\} \\ &= \Delta E_{i+1}(t_{i+1}) - \int_{t_i}^{t_{i+1}} \dot{\mathbf{d}}_i^T\ \mathbf{C}_i \dot{\mathbf{d}}_i dt \\ &= \frac{1}{2}\mathbf{d}_{i+1}^T \Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1} - \int_{t_i}^{t_{i+1}} \dot{\mathbf{d}}_i^T\ \mathbf{C}_i \dot{\mathbf{d}}_i dt \end{aligned} \tag{3.45}$$

It can be seen that $\Delta\mathbf{max}\{E_{i+1}\}$ consists of two terms which have the quadratic form. The sign of the $\Delta\mathbf{max}\{E_{i+1}\}$ is determined by the positive-definiteness of the matrices $\Delta\mathbf{K}_{i+1}$ and $\mathbf{C}_i$. It is usually true that $\mathbf{C}_i$ is positive definite. It is evident that if $\Delta\mathbf{K}_{i+1}$ is negative-definite, $\Delta\mathbf{max}\{E_{i+1}\} < 0$. Therefore, the dynamic stability criterion can be stated in the following theorem.

**Theorem 3.7**: For any **SVS**, if (1) $\Delta\mathbf{K}_{i+1}$ is negative semi-definite, (2) $\mathbf{C}_i$ is positive definite, the system will be dynamically stable during the time period $[t_i, t_{i+1}]$.

The proof of the **Theorem 3.7** is straightforward based on the derivation process done before. Hence, it is not rewritten.

By **Theorem 3.7**, the dynamic stability of an **SVS** during a certain time interval can be examined using the variational parameter matrices and the damping matrix. Especially, if the system is a conservative one, the condition provided in the **Theorem 3.7** becomes a necessary one.

**Theorem 3.8**: For any conservative **SVS**, iff $\Delta\mathbf{K}_{i+1}$ is positive semi-definite, the system will be dynamically stable during the time period $[t_i, t_{i+1}]$.

**proof**: Since the system is conservative, $\mathbf{C}_i = \mathbf{0}$ for $i = 1, 2, \cdots, m$. Therefore, equation (3.45) can be rewritten as

$$\begin{aligned} \Delta\mathbf{max}\{E_{i+1}\} &= \mathbf{max}\{E_{i+1}(t)\} - \mathbf{max}\{E_i(t)\} \\ &= \Delta E_{i+1}(t_{i+1}) \\ &= \frac{1}{2}\mathbf{d}_{i+1}^T \Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1} \end{aligned} \tag{3.46}$$

If $\Delta\mathbf{K}_{i+1}$ is negative semi-definite, we will have $\mathbf{d}_{i+1}^T\Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1} \leq 0$. Therefore, $\Delta\mathbf{max}\{E_{i+1}\} \leq 0$ and the system is dynamically stable. On the other hand, if the system is dynamically stable, we must have $\Delta\mathbf{max}\{E_{i+1}\} \leq 0$, which implies that $\mathbf{d}_{i+1}^T\Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1} \leq 0$. Therefore, $\Delta\mathbf{K}_{i+1}$ has to be negative semi-definite. This proves the sufficient and necessary conditions of the dynamic stability of the **SVS**.

- **Example:** From the previous example shown in figure 3.2, we know that initially $k_c = k_{c1}$. If the system stiffness is increased by $k_{c2}$ at time $t = t_1$

$$\Delta\mathbf{K} = \begin{bmatrix} k_{c2} & -k_{c2} \\ -k_{c2} & k_{c2} \end{bmatrix}$$

Since

$$| \lambda \mathbf{I} - \Delta \mathbf{K} |= \begin{bmatrix} \lambda - k_{c2} & k_{c2} \\ k_{c2} & \lambda - k_{c2} \end{bmatrix} = \lambda^2 - 2k_{c2}\lambda = 0$$

The eigenvalues of $\Delta\mathbf{K}$ are $\lambda_1 = 0$, $\lambda_2 = 2k_{2c}$. $\Delta\mathbf{K}$ is positive semi-definite. According to **Theorem 3.8**, we know that the system is dynamically unstable after the structural variation.

On the other hand, if initially $k_c = k_{c1} + k_{c2}$, and the system stiffness is decreased by $k_{c2}$ at time $t = t_1$, we have

$$\begin{aligned} \Delta\mathbf{K} &= \begin{bmatrix} k_1 + k_{c1} & -k_{c1} \\ -k_{c1} & k_2 + k_{c1} \end{bmatrix} - \begin{bmatrix} k_1 + k_c & -k_c \\ -k_c & k_2 + k_c \end{bmatrix} \\ &= \begin{bmatrix} -k_{c2} & k_{c2} \\ k_{c2} & -k_{c2} \end{bmatrix} \end{aligned}$$

where $k_c = k_{c1} + k_{c2}$. Since

$$| \lambda \mathbf{I} - \Delta \mathbf{K} |= \begin{bmatrix} \lambda + k_{c2} & -k_{c2} \\ -k_{c2} & \lambda + k_{c2} \end{bmatrix} = \lambda^2 + 2k_{c2}\lambda = 0$$

The eigenvalues of $\Delta\mathbf{K}$ are $\lambda_1 = 0$, $\lambda_2 = -2k_{2c}$. If $k_{2c} > 0$, we have $\lambda_2 < 0$. Therefore, $\Delta\mathbf{K}$ is negative semi-definite. According to **Theorem 3.8**, we know that the system is dynamically stable after the structural variation.

## 3.5 Summary

In this chapter, both static and dynamic stability of the fixed order **SVS** has been studied. System state response and energy function have been employed respectively as the evaluation functions. The stability analysis has been carried out based on the recursive state space model developed for the fixed order **SVS**. A number of criteria for evaluating the static and dynamic stability have been derived. In particular, the relation between the subsystem stability and that of the overall system has been studied.

# Chapter 4

# Stability of Structurally-Varying Systems With Time-Varying Order

## 4.1 Introduction

In this chapter, we study the stability of an **SVS** which has a time-varying order. The order of a system of this type will change when a variation of constraints of the system occurs. The system can be considered *growing* when the order of the system increases, or *shrinking* when the order of the system decreases. A new approach has to be developed in order to accommodate the order variation of the **SVS**.

In the previous study, we have assumed that the subsystems are connected to each other through springs and dampers and there is no mass coupling between the subsystems. In that situation, the structural variations of the **SVS** can be characterized analytically by the structural perturbations or the change of the system stiffness and damping matrices alone. The system mass matrix remains virtually unchanged. Most important of all, the dimensions of parameter matrices are kept unchanged for every system configuration regardless of constraint conditions among subsystems. Therefore, comparison of the parameter matrices and the state variables, which is a crucial step in predicting the change of the system dynamics, can be made. If this assumption is dropped, i.e., the connection between subsystems is composed of not only springs and dampers but also mass elements, the order of the overall system will consequently change whenever the constraint condition among subsystems changes, as will the dimensions of the parameter matrices. Hence, simple direct comparison or algebraic operation of any of the parameter matrices becomes unfeasible due to the incompatibility of their dimensions. Therefore, the stability analysis of an order-varying **SVS** becomes more difficult than that of a fixed order **SVS**.

The approach adopted to attack this problem is to find a descriptive *scalar* variable, which should carry the stability information and be determined by structural properties of the **SVS**. This variable has to be easily computable and physically meaningful considering the fact that the algorithm for the computation of the stability condition may be used in real-time applications. From this variable, the system stability can be predicted.

## 4.2 Energy Function Approach

It has been known that in the analysis of the dynamic stability of the **SVS**, the key issue is to look at how the evaluation function would change, if it does, with the variation of the system structure. In the previous stability study of the fixed order **SVS**, the state response was used as the evaluation function. Basically, we carried out analysis on the state transition matrix. In the case of order-varying **SVS**, this approach becomes difficult to apply since the state transition matrices for different system configurations have different dimensions, which make the comparison of the system matrices of different configurations impossible in a meaningful way. Also, the dimension of the system state variable will change when an **SVS** moves from one configuration to another, i.e., the dimension of the system state variable will either *grow* or *shrink*. Therefore, the approach to use the state response as the evaluation function becomes inappropriate in the present case.

In such a context, we start by considering using the energy function as the evaluation function in the stability analysis of an order-varying **SVS**. It is known (from the definitions given in Chapter 2) that if the energy in a system *grows* over a significant time interval, the system may be considered unstable. On the other hand, if the energy remains unchanged or is even *diminishing* in a system, the system may be considered stable. In the stability analysis of the order-varying **SVS**, we will be focusing on the change of the energy function due to the structural perturbations, i.e., dynamic stability rather than static stability since the static stability is assured by the positive-definiteness of the damping matrix **C**. In other words, we will study how the energy function varies when the system takes a new configuration instead of looking at the changing rate of the energy function within a fixed configuration.

The approach we adopted is to calculate and compare the *maximum value* of the energy functions $\mathbf{max}\{E_i(t)\}$ and $\mathbf{max}\{E_{i+1}(t)\}$ of two consecutive configurations of the **SVS**. Then, we use the difference $\Delta\{E_{max}\} = \mathbf{max}\{E_{i+1}(t)\} - \mathbf{max}\{E_i(t)\}$ as the measuring variable to determine the dynamic stability of the **SVS** over this time interval. For the kind of systems we are studying, it is known that every system configuration is statically stable, i.e., for any system configuration $i$, we have $\dot{E}_i(t) < 0$ over the time period $[t_i, t_{i+1}]$, which has been proven in the previous chapter. In other words, the system is dissipating energy during each time interval in which the system structure is fixed. Therefore, $\mathbf{max}\{E_i(t)\} = E_i(t_i)$, $[t_i, t_{i+1}]$, which implies that at the initial instant of each system configuration, the energy function $E_i(t)$ assumes its highest value over the period. Then the energy keeps dissipating, as shown in Figure 4.1.

The criteria of the dynamic stability can then be developed by comparing the initial values of energy function of consecutive system configurations. In general, we have

$$\mathbf{max}\{E_i(t)\} = \mathbf{max}\{E_i(t_m) : \dot{E}_i(t_m) = 0;\ E_i(t_i);\ E_i(t_{i+1});\ t \in [t_i, t_{i+1}]\}$$

depending on the characteristics of the energy function.

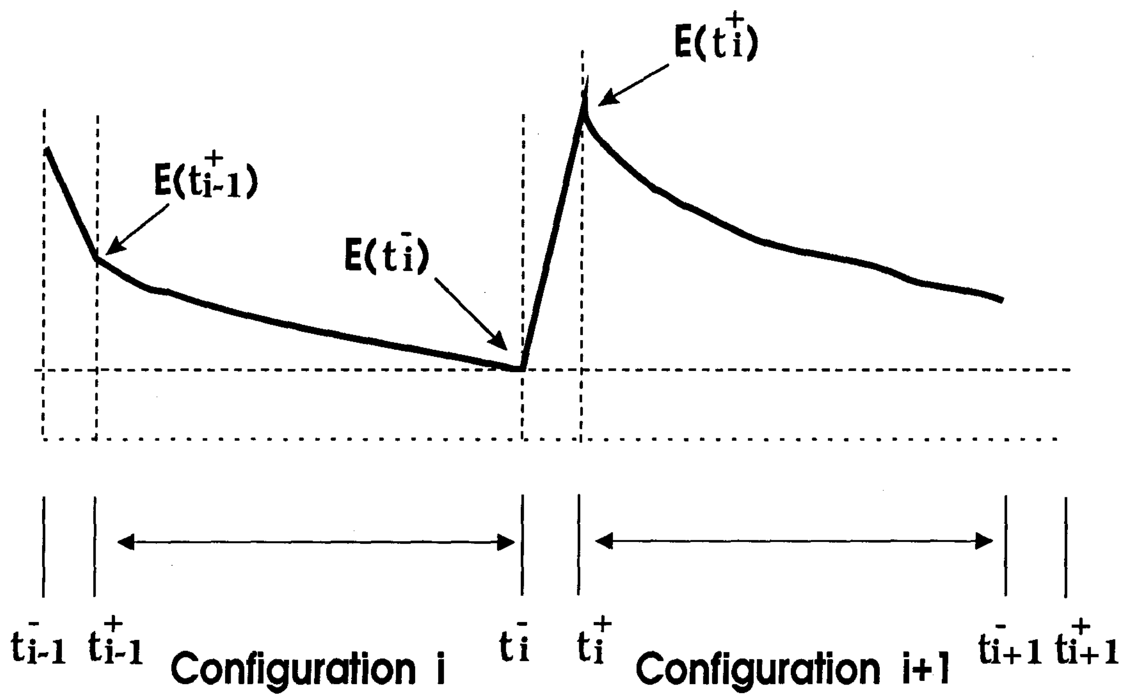


Figure 4.1: Energy Function

## 4.3 Modeling of Structural Perturbations

In the analysis of the dynamic stability of an **SVS**, the key issue is to look at how the energy function would change, if it does at all, with the variations of the system structure. It is known that the order of the overall system will change if the constraint condition between subsystems varies. In the situation when the system order is varying, we no longer have $dim\{\mathbf{d}_i(t)\} = dim\{\mathbf{d}_{i+1}(t)\}$. The parameter matrices of different system configurations cannot be directly compared to each other due to the fact that $dim\{\mathrm{M}_{i+1}\} \neq dim\{\mathrm{M}_i\}$. Hence, the theorems for stability evaluation of an **SVS** as derived in Chapter 3 become invalid.

In order to study the stability of the order-varying **SVS**, the structural perturbation has to be modeled first. Considering the fact that the orders of two consecutive system configurations ($ith$ and $i+1th$ for instance) are not equal, we assume

$$\begin{aligned} dim\{\mathbf{d}_i\} &= n_i \\ dim\{\mathbf{d}_{i+1}\} &= n_{i+1} \end{aligned}$$

and

$$n_{i+1} = n_i + \Delta n_{i+1}$$

$\Delta n_{i+1}$ more (or less) degrees of freedom are added to (or subtracted from) the previous system configuration $i$ after the system structure varies. Hence, $\Delta n_{i+1}$ more coordinates are required in order to fully define the dynamics of the new system configuration $i+1$. We select

$$\mathbf{d}_{i+1} = \begin{bmatrix} \tilde{\mathbf{d}}_i(t) \\ \mathbf{d}_{\Delta i+1}(t) \end{bmatrix} \tag{4.47}$$

where the symbol "$\sim$" is used to indicate that this part of $\mathbf{d}_{i+1}$ is *inherited* from the coordinate $\mathbf{d}_i(t)$ of the previous configuration. $\mathbf{d}_{\Delta i+1}(t)$ is the new coordinates added to the new system configuration. Then, we partition the parameter matrices accordingly

$$\begin{aligned} \mathbf{M}_{i+1} &= \begin{bmatrix} \mathbf{M}_i^p & \mathbf{m}_{i+1}^c \\ (\mathbf{m}_{i+1}^c)^T & \mathbf{M}_{\Delta i+1} \end{bmatrix} \\ \mathbf{K}_{i+1} &= \begin{bmatrix} \mathbf{K}_i^p & \mathbf{k}_{i+1}^c \\ (\mathbf{k}_{i+1}^c)^T & \mathbf{K}_{\Delta i+1} \end{bmatrix} \end{aligned} \tag{4.48}$$

where $\mathbf{M}_i^p$ and $\mathbf{K}_i^p$ are of the same dimensions as $\mathbf{M}_i$ and $\mathbf{K}_i$. $\mathbf{M}_{\Delta i+1}$ and $\mathbf{K}_{\Delta i+1}$ can be considered as parameter matrices describing the *newly-created* part of the **SVS** due to the structural perturbation. $\mathbf{m}_{i+1}^c$ and $\mathbf{k}_{i+1}^c$ can be considered as parameter matrices of the connection between the *original* part and the newly-created part of the **SVS**.

Since $\mathbf{M}_i^p$, $\mathbf{K}_i^p$ have the same dimension as $\mathbf{M}_i$ and $\mathbf{K}_i$, we can define the perturbation of parameter matrices on the previous system configuration as

$$\Delta\mathbf{M}_i^p = \mathbf{M}_i^p - \mathbf{M}_i \tag{4.49}$$

$$\Delta\mathbf{K}_i^p = \mathbf{K}_i^p - \mathbf{K}_i \tag{4.50}$$

where $\Delta\mathbf{M}_i^p$ and $\Delta\mathbf{K}_i^p$ are the perturbations of parameter matrices on the previous configuration. On the other hand, we can define

$$\begin{aligned} \hat{\mathbf{M}}_i &= \begin{bmatrix} \mathbf{M}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \hat{\mathbf{K}}_i &= \begin{bmatrix} \mathbf{K}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \end{aligned} \tag{4.51}$$

where the dimensions of the $\hat{\mathbf{M}}_i$ and $\hat{\mathbf{K}}_i$ are compatible with those of $\mathbf{M}_{i+1}$ and $\mathbf{K}_{i+1}$. Therefore, we have

$$\begin{aligned} \Delta\mathbf{M}_{i+1} &= \mathbf{M}_{i+1} - \hat{\mathbf{M}}_i \\ \Delta\mathbf{K}_{i+1} &= \mathbf{K}_{i+1} - \hat{\mathbf{K}}_i \end{aligned} \tag{4.52}$$

Substituting equation (4.48) and (4.51) into equation (4.52), we obtain

$$\begin{aligned} \Delta\mathbf{M}_{i+1} &= \begin{bmatrix} \Delta\mathbf{M}_i^p & \mathbf{m}_{i+1}^c \\ (\mathbf{m}_{i+1}^c)^T & \mathbf{m}_{\Delta i+1} \end{bmatrix} \\ \Delta\mathbf{K}_{i+1} &= \begin{bmatrix} \Delta\mathbf{K}_i^p & \mathbf{k}_{i+1}^c \\ (\mathbf{k}_{i+1}^c)^T & \mathbf{k}_{\Delta i+1} \end{bmatrix} \end{aligned} \tag{4.53}$$

$\Delta\mathbf{M}_{i+1}$ and $\Delta\mathbf{K}_{i+1}$ can be considered as *generalized parameter perturbation matrices* for configuration *i+1*. They describe the change of the system model due to the structural variation occurring at time instant $t = t_i$.

- **Example:** Consider the system described in figure 4.2. We assume that initially the two switches are both on.

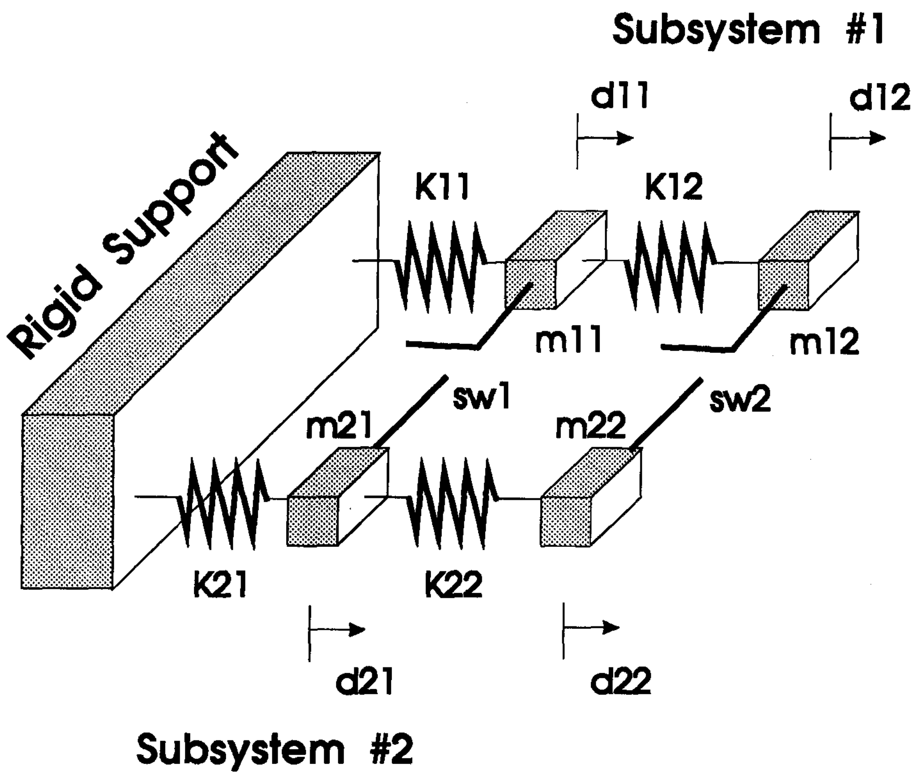


Figure 4.2: Example of A Varying-Order SVS

The system is actually the same as the one shown in (a) of figure 4.3. At time $t = t_1$, switch #2 is turned off so that the system takes a new configuration, as is shown in (b) of figure 4.3. The parameter matrices for system configuration #1 are

$$\mathbf{M}_1 = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad \mathbf{K}_1 = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

At $t = t_1$, the system configuration changes. Its order increases by 1.

The parameter matrices for the system configuration #2 are

$$\mathbf{M}_2 = \begin{bmatrix} m_1' & 0 & 0 \\ 0 & m_2' & 0 \\ 0 & 0 & m_3' \end{bmatrix} \qquad \mathbf{K}_2 = \begin{bmatrix} k_1' & -k_2' & -k_3' \\ -k_2' & k_2' & 0 \\ -k_3' & 0 & k_3' \end{bmatrix}$$

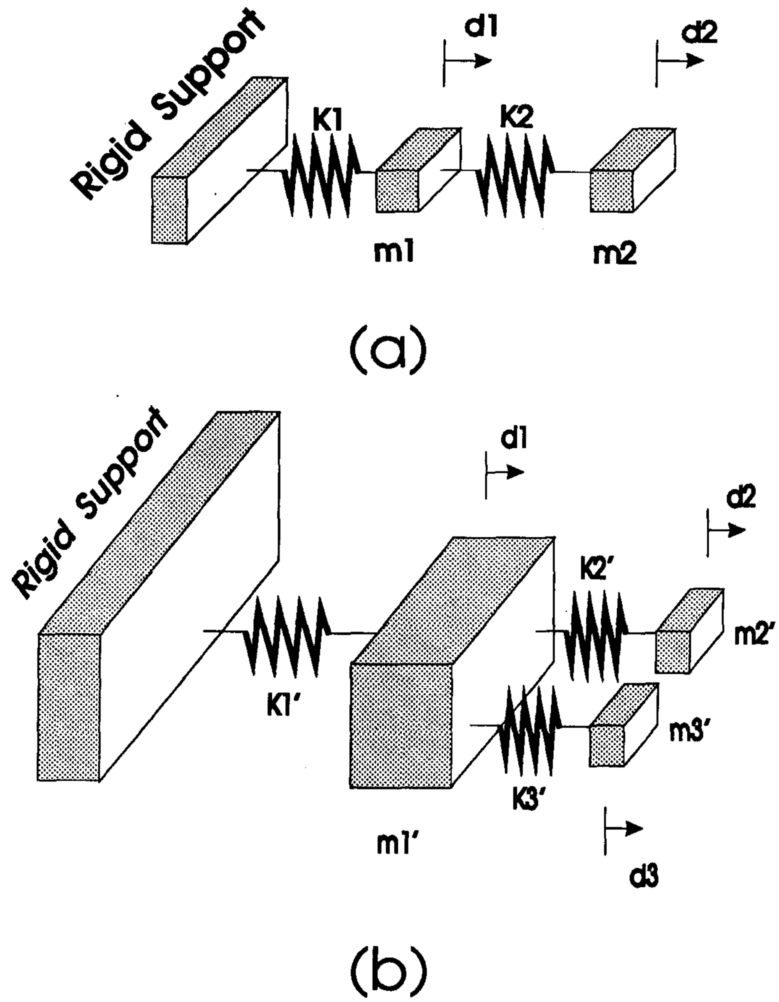


Figure 4.3: Structural Variation of the Example System

We partition the parameter matrices of $\mathbf{M}_2$ and $\mathbf{K}_2$ as

$$\mathbf{M}_2 = \begin{bmatrix} \mathbf{M}_1^p & \mathbf{0} \\ \mathbf{0} & m_3' \end{bmatrix} \qquad \mathbf{K}_2 = \begin{bmatrix} \mathbf{K}_1^p & \mathbf{K}_2^c \\ \mathbf{K}_2^c & k_{\Delta 2} \end{bmatrix}$$

where

$$\mathbf{M}_1^p = \begin{bmatrix} m_1' & 0 \\ 0 & m_2' \end{bmatrix} \qquad \mathbf{K}_1^p = \begin{bmatrix} k_1' & -k_2' \\ -k_2' & k_2' \end{bmatrix}$$

$$\mathbf{K}_2^c = \begin{bmatrix} -k_3' \\ 0 \end{bmatrix} \qquad k_{\Delta 2} = k_3'$$

Hence, the perturbation of the parameter matrices on configuration #1 can be determined,

$$\begin{aligned} \Delta\mathbf{M}_1^p = \mathbf{M}_1^p - \mathbf{M}_1 &= \begin{bmatrix} m_1' & 0 \\ 0 & m_2' \end{bmatrix} - \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & m_2' - m_2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \Delta\mathbf{K}_1^p = \mathbf{K}_1^p - \mathbf{K}_1 &= \begin{bmatrix} k_1' & -k_2' \\ -k_2' & k_2' \end{bmatrix} - \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \\ &= \begin{bmatrix} -k_2 & -(k_2' - k_2) \\ -(k_2' - k_2) & k_2' - k_2 \end{bmatrix} \end{aligned}$$

where $m_1' = m_1$ and $k_1' = k_1$ have been used. Also by equation (4.51), we have

$$\hat{\mathbf{M}}_1 = \begin{bmatrix} \mathbf{M}_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\mathbf{K}}_1 = \begin{bmatrix} \mathbf{K}_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence,

$$\begin{aligned} \Delta \mathbf{M}_2 = \mathbf{M}_2 - \hat{\mathbf{M}}_1 &= \begin{bmatrix} m_1' & 0 & 0 \\ 0 & m_2' & 0 \\ 0 & 0 & m_3' \end{bmatrix} - \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & m_2' - m_2 & 0 \\ 0 & 0 & m_3' \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \Delta \mathbf{K}_2 = \mathbf{K}_2 - \hat{\mathbf{K}}_1 &= \begin{bmatrix} k_1' & -k_2' & -k_3' \\ -k_2' & k_2' & 0 \\ -k_3' & 0 & k_3' \end{bmatrix} - \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -k_2 & -(k_2' - k_2) & k_3' \\ -(k_2' - k_2) & k_2' - k_2 & 0 \\ k_3' & 0 & k_3' \end{bmatrix} \end{aligned}$$

which are the generalized parameter perturbation matrices of configuration #2.

## 4.4 Modeling of Switching Instants

As has been discussed in Chapter 2, any **SVS** can been modeled by a series of configurations and switching instants over a period of time. Each configuration of the **SVS** can be treated as a time-invariant system and the structural change of the **SVS** occurs at the switching instant. In this section, the switching instant will be analyzed and its dynamic model will be developed. It has to be pointed out that since the model of the switching instant is developed for the analysis of the stability of the **SVS**, the emphasis is placed on the *dynamic characteristics* rather than the *physical characteristics* of the switching instant. More detailed analysis of the physical characteristics is being carried out in our Industrial Automation Laboratory in another project.

### 4.4.1 Modeling of Switching Instants by Process Compatibility

In order to illustrate the idea of process compatibility, we start with a simple system which initially consists of one mass node with one degree of freedom (d.o.f). The schematic diagram of the system is shown in Figure 4.4. An external impulsive force is applied on the mass node at the time t. As a result, the mass is broken into two smaller mass nodes and each of these smaller masses will have one d.o.f.

At the instant of the break-up, the system momentum may change due to the application of the external impulsive force. After the separation, the system total energy may also change. If it goes up, we say the variation of the structure makes the system unstable. In other words, the system has the *trend* of increasing kinetic energy and therefore is said to be dynamically unstable. On the other hand, if the kinetic energy level remains or even decreases, the system is said to be dynamically stable. The key issue here is to determine the change of the energy of the system and find its varying trend.

In order to calculate the energy of the system, the velocities of the mass nodes have to be determined first. It is known that

$$F = p_{post} - p_{pre} = m_1 v_1 + m_2 v_2 - mv \tag{4.54}$$

where $p_{post}$ and $p_{pre}$ are the momenta of the system before and after the break-up. $F$ is the external impulsive force applied on the initial mass node. $v_1$ and $v_2$ are the velocities of the two mass nodes after the separation. $v$ is the velocity of the mass node before

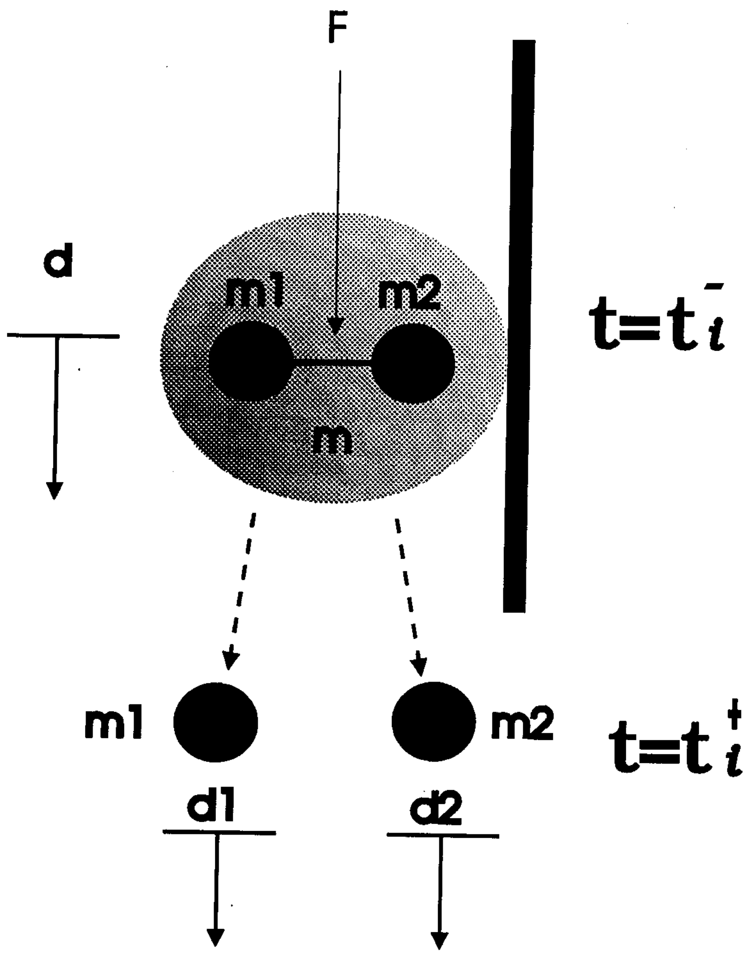


Figure 4.4: Illustration of A Breaking Instant (1)

separation. For simplicity, we assume that the mass node is broken into two equal parts, i.e., $m_1 = m_2 = m_p = m/2$, which is called an *equal split.* The equal split will lead to $v_1 = v_2 = v_p$. Therefore

$$\begin{aligned} F &= p_{post} - p_{pre} = 2\, m_p v_p - mv \\ &= mv_p - mv = m(v_p - v) \end{aligned} \tag{4.55}$$

i.e.,

$$\Delta v = v_p - v = \frac{F}{m} \tag{4.56}$$

or

$$v_p = v + \frac{F}{m} \tag{4.57}$$

It can be seen that the velocity of post-separation $v_p$ is determined by the velocity of pre-separation $v$ and the external impulsive force $F$ which is applied on the mass node at the switching instant. Equation (4.56) determines the change of the velocity of the example system. Since the displacement of the system cannot change instantly, we have

$$d = d_1 = d_2 \tag{4.58}$$

where the definitions of $d$, $d_1$ and $d_2$ can be found in Figure 4.4. If the switching instant of the system can be modeled by process compatibility relations such as equation (4.57) and (4.58), we say that the system is in *process compatibility* at the switching instant.

For the system shown, the kinetic energy function for the initial configuration is

$$E_{pre} = \frac{1}{2} mv^2 \tag{4.59}$$

where $m$ is the mass and $v$ is the velocity of the mass node. After the system is broken up, the kinetic energy function becomes

$$E_{post} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$
\begin{aligned}
&= \frac{1}{2}m_p v_p^2 + \frac{1}{2}m_p v_p^2 \\
&= m_p v_p^2 = \frac{1}{2}m v_p^2 \qquad (4.60)
\end{aligned}
$$

where $m_p$ is the mass of two smaller mass nodes and $v_p$ is the velocity of two mass nodes after equal split. Based on two energy functions, we can find the varying trend of the kinetic energy due to the structural variation.

$$
\begin{aligned}
\Delta E &= E_{post} - E_{pre} \\
&= \frac{1}{2}m\, v_p^2 - \frac{1}{2}m\, v^2 \\
&= \frac{1}{2}m\, (v_p^2 - v^2) \qquad (4.61)
\end{aligned}
$$

It can be seen that the sign of $\Delta E$ is dependent on the difference of $v_p$ and $v$. If

$$v_p^2 > v^2$$

then

$$\Delta E > 0$$

The system would be said to be dynamically unstable after the break-up. On the other hand, if

$$v_p^2 \leq v^2$$

then

$$\Delta E \leq 0$$

and the system would be said to be dynamically stable after the break-up.

Substituting equation (4.57) into equation (4.61), we have

$$
\begin{aligned}
\Delta E &= \frac{1}{2}m\, [(v + \frac{F}{m})^2 - v^2] \\
&= \frac{1}{2}m\, [v^2 + 2\frac{F}{m}v + (\frac{F}{m})^2 - v^2] \\
&= \frac{1}{2}\, [2Fv + \frac{F^2}{m}] \qquad (4.62)
\end{aligned}
$$

It can be seen that the stability condition of the system can then be stated in terms of the direction of the externally applied impulsive force $F$. As long as $F$ is applied in the same direction as $v$, we would have

$$\Delta E > 0$$

Therefore we know the system energy is going to increase after the break-up and consequently the system is dynamically unstable. For the system to be dynamically stable, we must have $\Delta E \leq 0$, which means

$$2Fv + \frac{F^2}{m} \leq 0$$

Since $F$ has to be applied in the opposite direction to the $v$, we have

$$-2 \mid F \mid \cdot \mid v \mid + \frac{\mid F \mid^2}{m} \leq 0$$

or

$$-2 \mid v \mid + \frac{\mid F \mid}{m} \leq 0$$

Finally

$$\mid F \mid \leq 2 \mid v \mid m$$

This result indicates that the external impulsive force has to be applied in the opposite direction of the velocity of the mass node and its magnitude must lie *within* certain range if we don't want to increase the energy of the system. In other words, if the mass node is pushed *forward*, or pushed *backward* too hard, the system would be dynamically unstable. $F$ has to be applied in a certain direction and stay within a certain range in order for the system to be dynamically stable.

In order to incorporate this break-up model into the analysis of stability of general order-varying **SVS**, we study the *multi-boundary-node* breaking process. Looking at the

Figure 4.5, we can see there are $\beta$ boundary nodes which are to be split into two separate smaller mass nodes at time $t = t_i$. After the break-up, there would be $2\beta$ mass nodes produced from the original $\beta$ boundary nodes.

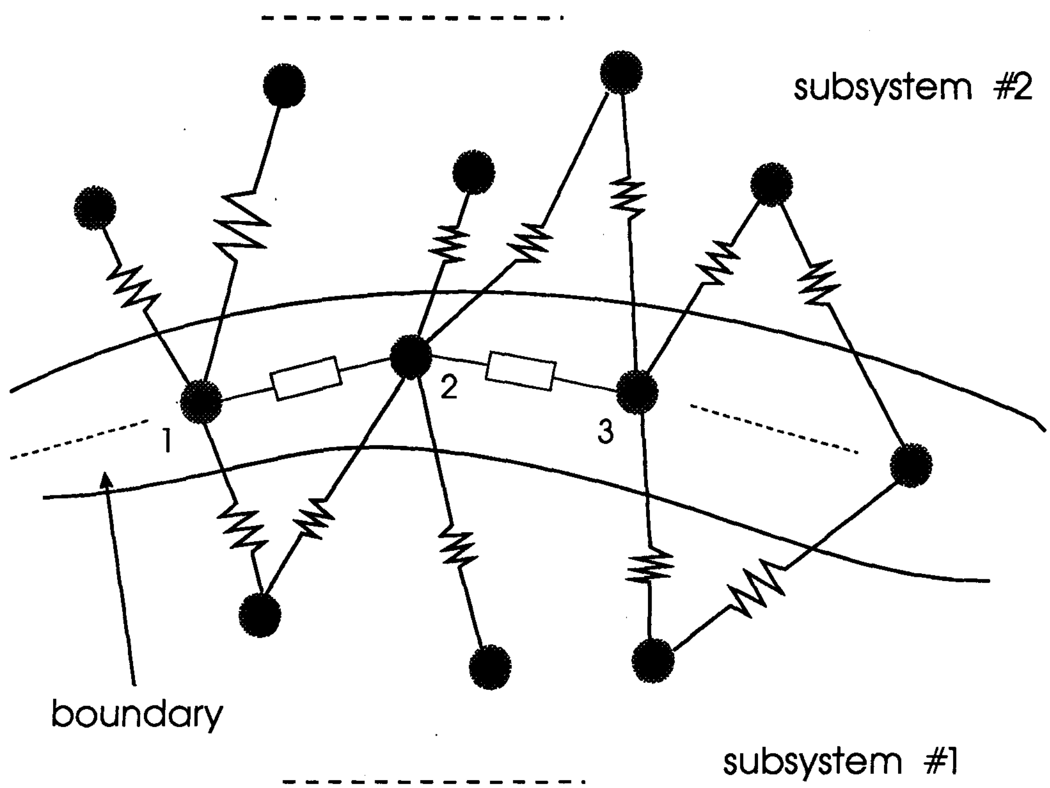


Figure 4.5: Schematic Diagram of the Order-Varying **SVS**

Although these $\beta$ boundary nodes are connected to both subsystems, we assume that at the instance of structural variation, the forces applied on boundary nodes from other *internal* parts of the system are negligible compared to the *externally* applied impulsive force **F**. Therefore, at the switching instant, the system boundary can be considered as a group of *isolated* mass nodes as is shown in Figure 4.6.

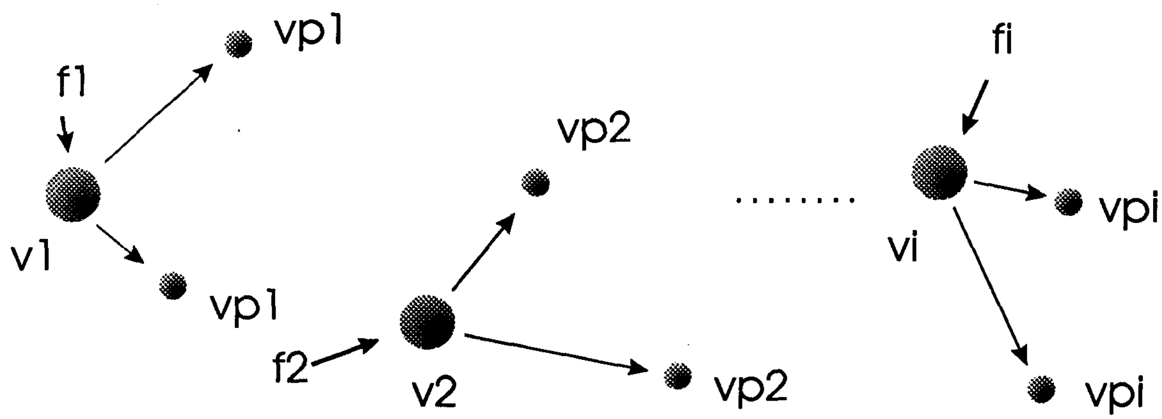


Figure 4.6: Schematics of the **SVS** at the Breaking Instance

Based on previous analysis, it is not difficult to obtain

$$v_{pi} = v_i + \frac{f_i}{m_i} \tag{4.63}$$

for each of the $\beta$ boundary nodes. Writing them in vector form, we have

$$\mathbf{v}_p = \mathbf{v} + \mathbf{M}_{bnd}^{-1}\mathbf{F} \tag{4.64}$$

or

$$\mathbf{v}_p = \mathbf{v} + \Delta\mathbf{v} \tag{4.65}$$

where $\mathbf{v}_p = \{v_{p1}, v_{p2}, \cdots, v_{p\beta}\}^T$, $\mathbf{v} = \{v_1, v_2, \cdots, v_\beta\}^T$, $\mathbf{M}_{bnd} = diag\{m_1, m_2, \cdots, m_\beta\}$, $\mathbf{F} = \{f_1, f_2, \cdots, f_\beta\}^T$, and $\Delta\mathbf{v} = \mathbf{M}_{bnd}^{-1}\mathbf{F}$. The physical meaning of these variables is clearly shown in figure (4.6). $\mathbf{v}$ is the *pre-breaking* velocity vector of the $\beta$ boundary nodes. $\mathbf{v}_p$ is the *post-breaking* velocity vector of the $2\beta$ newly-created mass nodes. $\mathbf{M}_{bnd}$ is the mass matrix of the $\beta$ boundary nodes before break-up. $\mathbf{F}$ is the vector of the externally applied impulsive force at the breaking instance. From equation (4.65), the velocities of the pre-breaking boundary nodes and post-breaking mass nodes are related through the impulsive force $\mathbf{F}$.

### 4.4.2 Modeling of Switching Instants by Motion Compatibility

In the previous section, the process compatibility has been used in the analysis of the switching instants of the **SVS**. It has been seen that the velocities of the boundary nodes of the **SVS** change instantly at the switching instant due to the externally applied impulsive forces. The important point is that the external impulsive force is applied in the same d.o.f. of the boundary nodes. Hence, the velocities of the post-breaking mass nodes change after the switching instant. If the impulsive force which causes the breaking of the boundary mass nodes is applied in a slightly different way, the pre-breaking velocity and post-breaking velocity of the boundary mass nodes will be exactly the same. In this situation, *motion compatibility* occurs, and will be used to analyze the switching instants.

As has been discussed before, the boundary mass nodes can be modeled as a group of isolated mass nodes without any connection to any subsystems at the switching instants. To demonstrate the idea of motion compatibility, we present the model for each of the boundary nodes in Figure 4.7. The boundary node can be thought of being composed of two equal smaller mass nodes. The external impulsive force is applied on the boundary node at time $t$, which will be broken into two mass nodes and each of them has one d.o.f.. In this case, the external impulsive force is applied in the d.o.f. in which the mass is constrained.

If the equal split is assumed, we will have $m_1 = m_2 = m/2$. Hence,

$$p_{post} = m_1 v_1 + m_2 v_2 = \frac{1}{2} m (v_1 + v_2) \tag{4.66}$$

$$p_{pre} = mv \tag{4.67}$$

where $p_{post}$ and $p_{pre}$ are the momenta of the system before and after the break-up. $v$ is the velocity of the mass node before separation. $v_1$ and $v_2$ are the velocities of the two mass nodes after the separation. It is not difficult to obtain $v_1 = v_2 = v_p$ if $m_1 = m_2$.

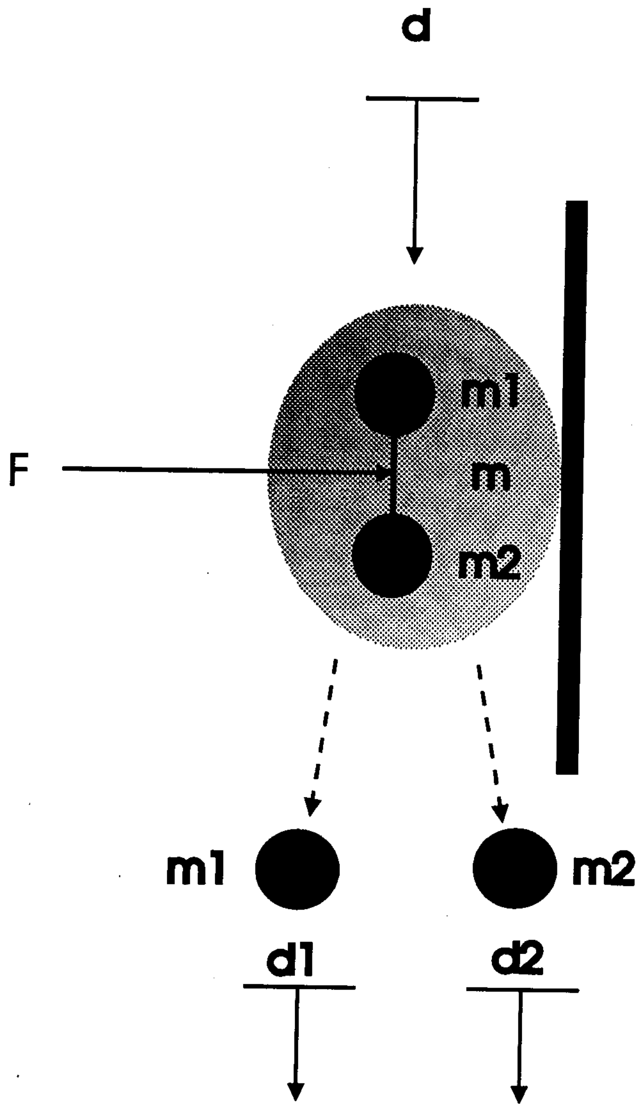


Figure 4.7: Illustration of A Breaking Instant (2)

According the law of conservation of momentum, we have

$$p_{pre} = p_{post} \tag{4.68}$$

i.e.,

$$mv = \frac{1}{2}m(v_p + v_p) = mv_p \tag{4.69}$$

Therefore,

$$v = v_p \tag{4.70}$$

It can be seen that the velocity of post-separation $v_p$ is the same as that of pre-separation $v$. Since the displacement of the system cannot change instantly, we have

$$d = d_1 = d_2 \tag{4.71}$$

where the definitions of $d$, $d_1$ and $d_2$ can be found in figure (4.7). If the switching instant of the system can be modeled by equation (4.70) and (4.71), we say that the system is of *motion compatibility* at the switching instant.

The kinetic energy function of the system for the initial configuration is

$$E_{pre} = \frac{1}{2}mv^2 \tag{4.72}$$

After the system is broken up, the kinetic energy function becomes

$$\begin{aligned} E_{post} &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \\ &= \frac{1}{2}(\frac{1}{2}m)(v_p^2 + v_p^2) = \frac{1}{2}mv_p^2 \\ &= \frac{1}{2}mv^2 \end{aligned} \tag{4.73}$$

Therefore, we have

$$E_{pre} = E_{post} \tag{4.74}$$

It can be seen that the energy of the boundary nodes remains unchanged at the switching instant.

By incorporating this break-up process into the stability analysis of multi-boundary-node order-varying **SVS**, we have

$$v_{pi} = v_i \tag{4.75}$$

for each of the $\beta$ boundary nodes. Writing them in the vector form, we have

$$\mathbf{v}_p = \mathbf{v} \tag{4.76}$$

where $\mathbf{v}_p = \{v_{p1}, v_{p2}, \cdots, v_{p\beta}\}^T$ is the the *post-breaking* velocity vector of the $2\beta$ newly-created mass nodes. $\mathbf{v} = \{v_1, v_2, \cdots, v_\beta\}^T$ is the *pre-breaking* velocity vector of the $\beta$ boundary nodes.

Comparing equations (4.57), (4.58) with equations (4.70), (4.71), we observe that the motion compatibility is actually a special case of the process compatibility with $\mathbf{F} = 0$. When the external impulsive force which causes the break-up of the boundary node is applied in the d.o.f. in which the boundary node is constrained, we will have $\mathbf{F} = 0$. There is no instantaneous change in the velocity of the boundary node. Hence, the switching instant of the **SVS** can be analyzed by using motion compatibility. On the other hand, when the external impulsive force is applied in the same d.o.f. as the boundary node, $\mathbf{F} \neq 0$. There is an instantaneous change in the velocity of the boundary node. Therefore, the switching instant of the **SVS** has to be analyzed by using process compatibility.

## 4.5 Analysis of Dynamic Stability Using Process Compatibility

To evaluate the dynamic stability, the energy functions for two consecutive system configurations have to be calculated

$$\begin{aligned} E_i(t) &= \frac{1}{2}\dot{\mathbf{d}}_i^T \mathbf{M}_i \dot{\mathbf{d}}_i + \frac{1}{2}\mathbf{d}_i^T \mathbf{K}_i \mathbf{d}_i && (4.77)\\ E_{i+1}(t) &= \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T \mathbf{M}_{i+1} \dot{\mathbf{d}}_{i+1} + \frac{1}{2}\mathbf{d}_{i+1}^T \mathbf{K}_{i+1} \mathbf{d}_{i+1} && (4.78)\end{aligned}$$

Substituting parameter matrix equation (4.52) into equation (4.78) yields

$$\begin{aligned} E_{i+1}(t_i^+) &= \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T(t_i^+)\, \mathbf{M}_{i+1}\dot{\mathbf{d}}_{i+1}(t_i^+) + \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\, \mathbf{K}_{i+1}\mathbf{d}_{i+1}(t_i^+)\\ &= \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T(t_i^+)\, (\hat{\mathbf{M}}_i + \Delta\mathbf{M}_{i+1})\dot{\mathbf{d}}_{i+1}(t_i^+) + \frac{1}{2}\mathbf{d}_{i+1}^T\, (\hat{\mathbf{K}}_i + \Delta\mathbf{K}_{i+1})\mathbf{d}_{i+1}\\ &= \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T(t_i^+)\, \hat{\mathbf{M}}_i\dot{\mathbf{d}}_{i+1}(t_i^+) + \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\, \hat{\mathbf{K}}_i\mathbf{d}_{i+1}(t_i^+)\\ &\quad + \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T(t_i^+)\, \Delta\mathbf{M}_{i+1}\dot{\mathbf{d}}_{i+1}(t_i^+) + \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\, \Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1}(t_i^+)\\ &= e_i(t_i^+) + \Delta e_{i+1}(t_i^+) && (4.79)\end{aligned}$$

where

$$\begin{aligned} e_i(t_i^+) &= \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T(t_i^+)\, \hat{\mathbf{M}}_i\dot{\mathbf{d}}_{i+1}(t_i^+) + \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\, \hat{\mathbf{K}}_i\mathbf{d}_{i+1} && (4.80)\\ \Delta e_{i+1}(t_i^+) &= \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T(t_i^+)\, \Delta\mathbf{M}_{i+1}\dot{\mathbf{d}}_{i+1}(t_i^+) + \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\, \Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1}(t_i^+) && (4.81)\end{aligned}$$

Substituting equation (4.47) and equation (4.51) into equation (4.80), we have

$$\begin{aligned} e_i(t_i^+) &= \frac{1}{2}\left( \begin{bmatrix} \dot{\tilde{\mathbf{d}}}_i(t_i^+) \\ \dot{\mathbf{d}}_{\Delta i+1}(t_i^+) \end{bmatrix}^T \begin{bmatrix} \mathbf{M}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\tilde{\mathbf{d}}}_i(t_i^+) \\ \dot{\mathbf{d}}_{\Delta i+1}(t_i^+) \end{bmatrix} \right.\\ &\quad \left. + \begin{bmatrix} \tilde{\mathbf{d}}_i(t_i^+) \\ \mathbf{d}_{\Delta i+1}(t_i^+) \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_i & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{d}}_i(t_i^+) \\ \mathbf{d}_{\Delta i+1}(t_i^+) \end{bmatrix} \right)\\ &= \frac{1}{2}[\dot{\tilde{\mathbf{d}}}_i^T(t_i^+)\, \mathbf{M}_i\, \dot{\tilde{\mathbf{d}}}_i(t_i^+) + \tilde{\mathbf{d}}_i^T(t_i^+)\mathbf{K}_i\, \tilde{\mathbf{d}}_i(t_i^+)] && (4.82)\end{aligned}$$

After the energy function is determined, the stability of the order-varying **SVS** can be studied. From equation (4.65), the initial velocity of a configuration can be established in terms of the final velocity of the last configuration at a structural switching instant,

$$\dot{\mathbf{d}}_{i+1}(t_i^+) = \dot{\tilde{\mathbf{d}}}_i(t_i^-) + \Delta\dot{\mathbf{d}}_{i+1}(t_i) \tag{4.83}$$

where $\dot{\mathbf{d}}_{i+1}(t_i^+)$ is the initial velocity of the system configuration $i+1$, $\dot{\tilde{\mathbf{d}}}_i(t_i^-)$ is the final velocity of the system configuration $i$ with the dimension adjusted to the $\dot{\mathbf{d}}_{i+1}(t_i^+)$ and $\Delta\dot{\mathbf{d}}_{i+1}(t_i)$ is the velocity perturbation introduced by process compatibility. $\dot{\mathbf{d}}_{i+1}(t_i^+)$ can be divided into three elements,

$$\dot{\mathbf{d}}_{i+1}(t_i^+) = \begin{bmatrix} \dot{\tilde{\mathbf{d}}}_i^{int}(t_i^+) \\ \dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+) \\ \dot{\mathbf{d}}_{i+1}^{new}(t_i^+) \end{bmatrix} \tag{4.84}$$

where superscript *int* denotes *internal nodes*, superscript *bnd* denotes the *boundary nodes* and superscript *new* denotes *newly-created nodes* in the new system configuration. As before, the symbol "$\sim$" denotes inherited from the coordinate of the previous system configuration. The physical meaning of every element of the coordinate can be seen in Figure 4.8.

Hence, we know that the first element $\dot{\tilde{\mathbf{d}}}_i^{int}(t_i^+)$ is inherited from the coordinates of the internal nodes of the previous system configuration $i$. Since these internal nodes are not changed at the instant of the structural variation, we have

$$\dot{\tilde{\mathbf{d}}}_i^{int}(t_i^+) = \dot{\mathbf{d}}_i^{int}(t_i^-) \tag{4.85}$$

where $\dot{\mathbf{d}}_i^{int}(t_i^-)$ is the *real* coordinate of the internal nodes of the system configuration $i$.

For the second element $\dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+)$, it is inherited from the coordinates of the boundary nodes of the previous system configuration and the mass nodes they correspond to are

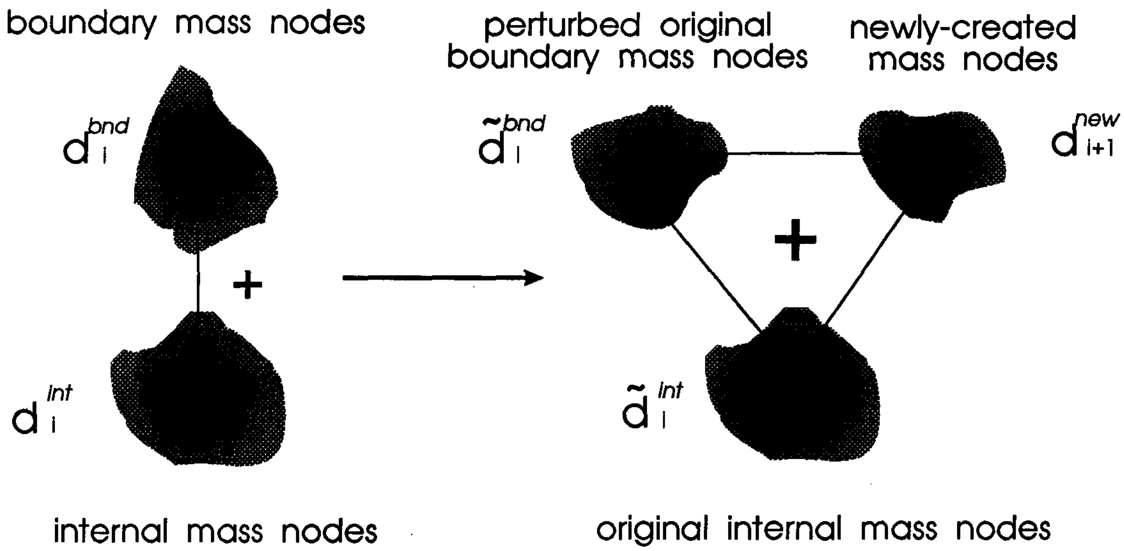


Figure 4.8: Definition of the Coordinates

split at the instance of structural variation. Therefore, it can be related to $\dot{\mathbf{d}}_i^{bnd}(t_i^-)$ by equation (4.65), i.e.,

$$\dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+) = \dot{\mathbf{d}}_i^{bnd}(t_i^-) + \Delta\dot{\mathbf{d}}_i^{bnd}(t_i) \tag{4.86}$$

where $\dot{\mathbf{d}}_i^{bnd}(t_i^-)$ is the *real* coordinate of the boundary nodes of the system configuration $i$ and $\Delta\dot{\mathbf{d}}_i^{bnd}(t_i)$ is the velocity perturbation on the $\dot{\mathbf{d}}_i^{bnd}(t_i)$.

The third element $\dot{\mathbf{d}}_{i+1}^{new}(t_i^+)$ corresponds to the newly-created mass nodes, which do not exist in the previous system configuration. However, since they are produced from the boundary nodes of the previous system configuration, the following relation applies to it if the equal split is assumed,

$$\begin{aligned}\dot{\mathbf{d}}_{i+1}^{new}(t_i^+) &= \dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+) \\ &= \dot{\mathbf{d}}_i^{bnd}(t_i^-) + \Delta\dot{\mathbf{d}}_i^{bnd}(t_i)\end{aligned} \tag{4.87}$$

It has been shown from the previous analysis that the velocity perturbation at the time $t = t_i$ is the function of external impulsive force $\mathbf{F}$, i.e.,

$$\Delta\dot{\mathbf{d}}_i^{bnd}(t_i) = (\mathbf{M}_i^{bnd})^{-1}\,\mathbf{F} \tag{4.88}$$

where $\mathbf{M}_i^{bnd} = diag\{m_1^{bnd}, m_2^{bnd}, \cdots, m_\beta^{bnd}\}$, $\mathbf{F} = \{f_1, f_2, \cdots, f_\beta\}^T$ and $\beta$ is the number of the boundary nodes of the previous system configuration $i$. Substituting equations (4.85), (4.86), (4.87) and (4.88) into equation (4.84) yields

$$\begin{aligned}
\dot{\mathbf{d}}_{i+1}(t_i^+) &= \begin{bmatrix} \dot{\mathbf{d}}_i^{int}(t_i^+) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^+) \\ \dot{\mathbf{d}}_{i+1}^{new}(t_i^+) \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{d}}_i^{int}(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) + \Delta\dot{\mathbf{d}}_i^{bnd}(t_i) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) + \Delta\dot{\mathbf{d}}_i^{bnd}(t_i) \end{bmatrix} \\
&= \begin{bmatrix} \dot{\mathbf{d}}_i^{int}(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \Delta\dot{\mathbf{d}}_i^{bnd}(t_i) \\ \Delta\dot{\mathbf{d}}_i^{bnd}(t_i) \end{bmatrix} \\
&= \begin{bmatrix} \dot{\mathbf{d}}_i^{int}(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ (\mathbf{M}^{bnd})_i^{-1}\,\mathbf{F} \\ (\mathbf{M}^{bnd})_i^{-1}\,\mathbf{F} \end{bmatrix} \qquad (4.89)
\end{aligned}$$

Comparing to equations (4.83), we have

$$\dot{\mathbf{d}}_i(t_i^-) = \begin{bmatrix} \dot{\mathbf{d}}_i^{int}(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) \end{bmatrix} \qquad (4.90)$$

$$\Delta\dot{\mathbf{d}}_{i+1}(t_i) = \begin{bmatrix} \mathbf{0} \\ (\mathbf{M}_i^{bnd})^{-1}\,\mathbf{F} \\ (\mathbf{M}_i^{bnd})^{-1}\,\mathbf{F} \end{bmatrix} \qquad (4.91)$$

At the switching instant, there is a *sudden* change in velocity, which is determined by the externally applied impulsive force $\mathbf{F}$.

Using this model of break-up, the dynamic stability of the order-varying **SVS** can be analyzed by process compatibility. As previously derived in equation (4.79), we have

$$E_{i+1}(t_i^+) = e_i(t_i^+) + \Delta e_{i+1}(t_i^+)$$

with

$$\begin{aligned} e_i(t_i^+) &= \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T(t_i^+)\,\hat{\mathbf{M}}_i\dot{\mathbf{d}}_{i+1}(t_i^+) + \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\,\hat{\mathbf{K}}_i\mathbf{d}_{i+1} \\ \Delta e_{i+1}(t_i^+) &= \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T(t_i^+)\,\Delta\mathbf{M}_{i+1}\dot{\mathbf{d}}_{i+1}(t_i^+) + \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\,\Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1}(t_i^+) \end{aligned}$$

We further assume

$$e_i(t_i^+) = E_i(t_i^-) + \Delta e_F(t_i) \tag{4.92}$$

where $\Delta e_F(t_i)$ is a small perturbation term which is caused by the velocity perturbation, or more directly by the externally applied impulsive force $\mathbf{F}$. Substituting equation (4.92) into equation (4.79) yields

$$\begin{aligned} E_{i+1}(t_i^+) &= E_i(t_i^-) + \Delta E_i(t_i) \\ &= E_i(t_i^-) + \Delta e_F(t_i) + \Delta e_{i+1}(t_i^+) \end{aligned} \tag{4.93}$$

where $E_i(t_i^-)$ is the energy value just before the structural variation. $\Delta E_i(t_i)$ is the energy perturbation occurring at the instant of structural variation. This perturbation term consists of two terms, one is the structural perturbation term $\Delta e_{i+1}(t_i^+)$ and the other is the *state variable* perturbation term $\Delta e_F(t_i)$. In order for a system to be dynamically stable, we must have

$$\Delta E_i(t_i) = E_{i+1}(t_i^+) - E_i(t_i^-) \leq 0$$

i.e.,

$$\Delta e_F(t_i) + \Delta e_{i+1}(t_i^+) \leq 0$$

Since the condition of $\Delta e_{i+1}(t_i^+) \leq 0$ is easy to find, the condition of $\Delta E_i(t_i) \leq 0$ can be determined if we can find the condition of $\Delta e_F(t_i) \leq 0$.

In order to determine $\Delta e_F(t_i)$, we substitute equation (4.83) into equation (4.80),

$$\begin{aligned}
e_i(t_i^+) &= \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T(t_i^+)\,\hat{\mathbf{M}}_i\dot{\mathbf{d}}_{i+1}(t_i^+) + \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\,\hat{\mathbf{K}}_i\mathbf{d}_{i+1} \\
&= \frac{1}{2}[(\dot{\tilde{\mathbf{d}}}_i(t_i^-) + \Delta\dot{\mathbf{d}}_{i+1}(t_i))^T\,\hat{\mathbf{M}}_i(\dot{\tilde{\mathbf{d}}}_i(t_i^-) + \Delta\dot{\mathbf{d}}_{i+1}(t_i))] + \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\,\hat{\mathbf{K}}_i\mathbf{d}_{i+1} \\
&= \frac{1}{2}[\dot{\tilde{\mathbf{d}}}_i(t_i^-)^T\,\hat{\mathbf{M}}_i\dot{\tilde{\mathbf{d}}}_i(t_i^-) + 2\Delta\dot{\mathbf{d}}_{i+1}(t_i)^T\,\hat{\mathbf{M}}_i\dot{\tilde{\mathbf{d}}}_i(t_i^-) \\
&\quad + \Delta\dot{\mathbf{d}}_{i+1}(t_i)^T\,\hat{\mathbf{M}}_i\Delta\dot{\mathbf{d}}_{i+1}(t_i)] + \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\,\hat{\mathbf{K}}_i\mathbf{d}_{i+1}
\end{aligned} \tag{4.94}$$

Since

$$\dot{\tilde{\mathbf{d}}}_i(t_i^-) = \begin{bmatrix} \dot{\mathbf{d}}_i^{int}(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{d}}_i(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) \end{bmatrix} \tag{4.95}$$

and

$$\tilde{\mathbf{d}}_i(t_i^-) = \begin{bmatrix} \mathbf{d}_i^{int}(t_i^-) \\ \mathbf{d}_i^{bnd}(t_i^-) \\ \mathbf{d}_i^{bnd}(t_i^-) \end{bmatrix} = \begin{bmatrix} \mathbf{d}_i(t_i^-) \\ \mathbf{d}_i^{bnd}(t_i^-) \end{bmatrix} \tag{4.96}$$

we have

$$\dot{\tilde{\mathbf{d}}}_i^T(t_i^-)\,\hat{\mathbf{M}}_i\dot{\tilde{\mathbf{d}}}_i(t_i^-) = \begin{bmatrix} \dot{\mathbf{d}}_i(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) \end{bmatrix}^T \begin{bmatrix} \mathbf{M}_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{d}}_i(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) \end{bmatrix} \tag{4.97}$$

$$= \dot{\mathbf{d}}_i^T(t_i^-)\mathbf{M}_i\dot{\mathbf{d}}_i(t_i^-) \tag{4.98}$$

and

$$\mathbf{d}_{i+1}^T(t_i^+)\,\hat{\mathbf{K}}_i\mathbf{d}_{i+1} = \begin{bmatrix} \mathbf{d}_i(t_i^-) \\ \mathbf{d}_i^{bnd}(t_i^-) \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{d}_i(t_i^-) \\ \mathbf{d}_i^{bnd}(t_i^-) \end{bmatrix} \tag{4.99}$$

$$= \mathbf{d}_i^T(t_i^-)\mathbf{K}_i\mathbf{d}_i(t_i^-) \tag{4.100}$$

Hence, we can rearrange equation (4.94) as

$$e_i(t_i^+) = \frac{1}{2}[\dot{\tilde{\mathbf{d}}}_i^T(t_i^-)\,\hat{\mathbf{M}}_i\dot{\tilde{\mathbf{d}}}_i(t_i^-) + 2\Delta\dot{\mathbf{d}}_{i+1}^T(t_i)\,\hat{\mathbf{M}}_i\dot{\tilde{\mathbf{d}}}_i(t_i^-)$$

$$
\begin{aligned}
& +\Delta\dot{\mathbf{d}}_{i+1}^T(t_i)\,\hat{\mathbf{M}}_i\Delta\dot{\mathbf{d}}_{i+1}(t_i)] + \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\,\hat{\mathbf{K}}_i\mathbf{d}_{i+1} \\
= \; & \frac{1}{2}\dot{\mathbf{d}}_i^T(t_i^-)\mathbf{M}_i\dot{\mathbf{d}}_i(t_i^-) + \frac{1}{2}\mathbf{d}_i^T(t_i^-)\mathbf{K}_i\mathbf{d}_i(t_i^-) \\
& +\Delta\dot{\mathbf{d}}_{i+1}^T(t_i)\,\hat{\mathbf{M}}_i\dot{\hat{\mathbf{d}}}_i(t_i^-) + \frac{1}{2}\Delta\dot{\mathbf{d}}_{i+1}^T(t_i)\,\hat{\mathbf{M}}_i\Delta\dot{\mathbf{d}}_{i+1}(t_i) \\
= \; & E_i(t_i^-) + \Delta e_F(t_i)
\end{aligned} \tag{4.101}
$$

where

$$
E_i(t_i^-) = \frac{1}{2}\dot{\mathbf{d}}_i^T(t_i^-)\mathbf{M}_i\dot{\mathbf{d}}_i(t_i^-) + \frac{1}{2}\mathbf{d}_i^T(t_i^-)\mathbf{K}_i\mathbf{d}_i(t_i^-) \tag{4.102}
$$

$$
\Delta e_F(t_i) = \Delta\dot{\mathbf{d}}_{i+1}^T(t_i)\,\hat{\mathbf{M}}_i\dot{\hat{\mathbf{d}}}_i(t_i^-) + \frac{1}{2}\Delta\dot{\mathbf{d}}_{i+1}^T(t_i)\,\hat{\mathbf{M}}_i\Delta\dot{\mathbf{d}}_{i+1}(t_i) \tag{4.103}
$$

The energy perturbation $\Delta e_F(t_i)$ due to the velocity perturbation is then determined. We define

$$
\alpha_i[\mathbf{F}, \mathbf{d}_i(t_i^-)] = \Delta\dot{\mathbf{d}}_{i+1}^T(t_i)\,\hat{\mathbf{M}}_i\dot{\hat{\mathbf{d}}}_i(t_i^-) \tag{4.104}
$$

then,

$$
\Delta e_F(t_i) = \alpha_i[\mathbf{F}, \mathbf{d}_i(t_i^-)] + \frac{1}{2}\Delta\dot{\mathbf{d}}_{i+1}^T(t_i)\,\hat{\mathbf{M}}_i\Delta\dot{\mathbf{d}}_{i+1}(t_i) \tag{4.105}
$$

$\alpha_i(\mathbf{F}, \mathbf{d}_i(t_i^-))$ can be considered as a *control variable* that provides the *constraint condition* on the externally applied impulsive force $\mathbf{F}$. If $\mathbf{F}$ satisfies the condition, the system would be dynamically stable. Based on equation (4.105), the criteria for evaluating the dynamic stability of the order-varying **SVS** using process compatibility can be derived.

**Theorem 4.1** Assume that dynamics of the structural variation is dominated by the process compatibility. An order-varying **SVS** would be dynamically stable iff $\Delta e_F(t_i) + \Delta e_{i+1}(t_i^+) \leq 0$.

**Theorem 4.2** Assume that dynamics of the structural variation is dominated by the process compatibility. An order-varying **SVS** would be dynamically stable if

1. Both $\Delta\mathbf{M}_{i+1}$ and $\Delta\mathbf{K}_{i+1}$ are negative semi-definite,

2. $2\alpha_i[\mathbf{F}, \mathbf{d}_i(t_i^-)] + \Delta\dot{\mathbf{d}}_{i+1}^T(t_i)\,\hat{\mathbf{M}}_i\Delta\dot{\mathbf{d}}_{i+1}(t_i) \leq 0$.

**Proof**: Since both $\Delta\mathbf{M}_{i+1}$ and $\Delta\mathbf{K}_{i+1}$ are negative semi-definite,

$$\dot{\mathbf{d}}_{i+1}^T(t_i^+)\,\Delta\mathbf{M}_{i+1}\dot{\mathbf{d}}_{i+1}(t_i^+) + \mathbf{d}_{i+1}^T(t_i^+)\,\Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1}(t_i^+) \leq 0$$

Since $2\alpha_i[\mathbf{F}, \mathbf{d}_i(t_i^-)] + \mathbf{d}_{i+1}^T(t_i^+)\,\Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1}(t_i^+) \leq 0$, we have

$$\begin{aligned} & \alpha_i[\mathbf{F}, \mathbf{d}_i(t_i^-)] + \Delta\dot{\mathbf{d}}_{i+1}^T(t_i)\,\hat{\mathbf{M}}_i\Delta\dot{\mathbf{d}}_{i+1}(t_i) \\ + \;& \dot{\mathbf{d}}_{i+1}^T(t_i^+)\,\Delta\mathbf{M}_{i+1}\dot{\mathbf{d}}_{i+1}(t_i^+) + \mathbf{d}_{i+1}^T(t_i^+)\,\Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1}(t_i^+) \leq 0 \end{aligned}$$

i.e., $\Delta e_F(t_i) + \Delta e_{i+1}(t_i^+) \leq 0$. According to the definition, we know that the configuration $i+1$ of the **SVS** is dynamically stable.

**Theorem 4.3** Assume that dynamics of the structural variation is dominated by the process compatibility. A conservative order-varying **SVS** would be dynamically unstable iff $\Delta e_F(t_i) + \Delta e_{i+1}(t_i^+) > 0$.

More specifically, we have

**Theorem 4.4** Assume that dynamics of the structural variation is dominated by the process compatibility. A conservative order-varying **SVS** would be dynamically unstable if

1. Both $\Delta\mathbf{M}_{i+1}$ and $\Delta\mathbf{K}_{i+1}$ are positive semi-definite,

2. $\alpha_i[\mathbf{F}, \mathbf{d}_i(t_i^-)] > 0$

**Proof**: Since both $\Delta\mathbf{M}_{i+1}$ and $\Delta\mathbf{K}_{i+1}$ are positive semi-definite,

$$\dot{\mathbf{d}}_{i+1}^T(t_i^+)\,\Delta\mathbf{M}_{i+1}\dot{\mathbf{d}}_{i+1}(t_i^+) + \mathbf{d}_{i+1}^T(t_i^+)\,\Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1}(t_i^+) \geq 0$$

Considering $\alpha_i[\mathbf{F}, \mathbf{d}_i(t_i^-)] > 0$ and $\hat{\mathbf{M}}_i$ is positive semi-definite, we have

$$\begin{aligned} & \alpha_i[\mathbf{F}, \mathbf{d}_i(t_i^-)] + \frac{1}{2}\Delta\dot{\mathbf{d}}_{i+1}^T(t_i)\,\hat{\mathbf{M}}_i\Delta\dot{\mathbf{d}}_{i+1}(t_i) \\ + \;& \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T(t_i^+)\,\Delta\mathbf{M}_{i+1}\dot{\mathbf{d}}_{i+1}(t_i^+) + \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\,\Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1}(t_i^+) > 0 \end{aligned}$$

i.e., $\Delta e_F(t_i)+\Delta e_{i+1}(t_i^+) > 0$. According to the definition, we know that the configuration $i+1$ of the **SVS** is dynamically unstable.

### 4.5.1 Perturbation on Kinetic Energy Function

It has been shown that the energy function changes suddenly at the structural switching instant. At the structural switching instant, the forces applied on mass nodes from other internal parts of the system can be considered relatively small compared to the externally applied impulsive force and therefore are neglected. The mass nodes can then be treated as a group of isolated ones and the mass matrix of the system at the switching instant can be expressed by two diagonal matrices,

$$\mathbf{M}_{swt}(t_i^-) = \begin{bmatrix} \mathbf{M}_{int} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{bnd} \end{bmatrix} \tag{4.106}$$

$$\mathbf{M}_{swt}(t_i^+) = \begin{bmatrix} \mathbf{M}_{int} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{bnd}^p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{new} \end{bmatrix} \tag{4.107}$$

where $\mathbf{M}_{swt}(t_i^-)$ is the mass matrix pre-structural switching and $\mathbf{M}_{swt}(t_i^+)$ is the mass matrix post-structural switching. $\mathbf{M}_{int}$, $\mathbf{M}_{bnd}$, $\mathbf{M}_{bnd}^p$ and $\mathbf{M}_{new}$ represent system internal, boundary, perturbed boundary and newly-created mass nodes. They are all diagonal. Since no mass is added to or removed from the system, the system overall mass is a constant., $\mathbf{M}_{new}$ is separated from $\mathbf{M}_{bnd}$. Therefore,

$$\mathbf{M}_{bnd}^p = \mathbf{M}_{bnd} - \mathbf{M}_{new} \tag{4.108}$$

Using the velocity relation given in equation (4.89), we can write the kinetic energy function at the switching instant,

$$E_{pre}^{ke}(t_i^-) \;=\; \frac{1}{2}\dot{\mathbf{d}}_i^T(t_i^-)\,\mathbf{M}_{swt}(t_i^-)\,\dot{\mathbf{d}}_i(t_i^-)$$

$$
\begin{aligned}
&= \frac{1}{2}\begin{bmatrix} \dot{\mathbf{d}}_i^{int}(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) \end{bmatrix}^T \begin{bmatrix} \mathbf{M}_{int} & 0 \\ 0 & \mathbf{M}_{bnd} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{d}}_i^{int}(t_i^-) \\ \dot{\mathbf{d}}_i^{bnd}(t_i^-) \end{bmatrix} \\
&= \frac{1}{2}[\dot{\mathbf{d}}_i^{int}(t_i^-)]^T \, \mathbf{M}_{int} \, \dot{\mathbf{d}}_i^{int}(t_i^-) + \frac{1}{2}[\dot{\mathbf{d}}_i^{bnd}(t_i^-)]^T \, \mathbf{M}_{bnd} \, \dot{\mathbf{d}}_i^{bnd}(t_i^-)
\end{aligned}
$$

and

$$
\begin{aligned}
E_{post}^{ke}(t_i^+) &= \frac{1}{2}\dot{\mathbf{d}}_{i+1}^T(t_i^+) \, \mathbf{M}_{swt}(t_i^+) \, \dot{\mathbf{d}}_{i+1}(t_i^+) \\
&= \begin{bmatrix} \dot{\tilde{\mathbf{d}}}_i^{int}(t_i^+) \\ \dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+) \\ \dot{\mathbf{d}}_{i+1}^{new}(t_i^+) \end{bmatrix}^T \begin{bmatrix} \mathbf{M}_{int} & 0 & 0 \\ 0 & \mathbf{M}_{bnd}^p & 0 \\ 0 & 0 & \mathbf{M}_{new} \end{bmatrix} \begin{bmatrix} \dot{\tilde{\mathbf{d}}}_i^{int}(t_i^+) \\ \dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+) \\ \dot{\mathbf{d}}_{i+1}^{new}(t_i^+) \end{bmatrix} \\
&= \frac{1}{2}[\dot{\tilde{\mathbf{d}}}_i^{int}(t_i^+)]^T \, \mathbf{M}_{int} \, \dot{\tilde{\mathbf{d}}}_i^{int}(t_i^+) + \frac{1}{2}[\dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+)]^T \, \mathbf{M}_{bnd}^p \, \dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+) \\
&\quad + \frac{1}{2}[\dot{\mathbf{d}}_{i+1}^{new}(t_i^+)]^T \, \mathbf{M}_{new} \, \dot{\mathbf{d}}_{i+1}^{new}(t_i^+)
\end{aligned}
$$

where $E_{pre}^{ke}(t_i^-)$ is the kinetic energy function just before the structural variation and $E_{post}^{ke}(t_i^+)$ is the kinetic energy function just after the structural variation. The change of the kinetic energy function at the switching instant can then be expressed as

$$
\begin{aligned}
\Delta E^{ke}(t_i) &= E_{post}^{ke}(t_i^+) - E_{pre}^{ke}(t_i^-) \\
&= \frac{1}{2}[\dot{\tilde{\mathbf{d}}}_i^{int}(t_i^+)]^T \, \mathbf{M}_{int} \, \dot{\tilde{\mathbf{d}}}_i^{int}(t_i^+) - \frac{1}{2}[\dot{\mathbf{d}}_i^{int}(t_i^-)]^T \, \mathbf{M}_{int} \, \dot{\mathbf{d}}_i^{int}(t_i^-) \\
&\quad + \frac{1}{2}[\dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+)]^T \, \mathbf{M}_{bnd}^p \, \dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+) - \frac{1}{2}[\dot{\mathbf{d}}_i^{bnd}(t_i^-)]^T \, \mathbf{M}_{bnd} \, \dot{\mathbf{d}}_i^{bnd}(t_i^-) \\
&\quad + \frac{1}{2}[\dot{\mathbf{d}}_{i+1}^{new}(t_i^+)]^T \, \mathbf{M}_{new} \, \dot{\mathbf{d}}_{i+1}^{new}(t_i^+) \qquad (4.109)
\end{aligned}
$$

Substituting equation (4.85) and equation (4.108) into equation (4.109) yields

$$
\begin{aligned}
\Delta E^{ke}(t_i) &= \frac{1}{2}[\dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+)]^T \, (\mathbf{M}_{bnd} - \mathbf{M}_{new}) \, \dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+) \\
&\quad - \frac{1}{2}[\dot{\mathbf{d}}_i^{bnd}(t_i^-)]^T \, \mathbf{M}_{bnd} \, \dot{\mathbf{d}}_i^{bnd}(t_i^-) + \frac{1}{2}[\dot{\mathbf{d}}_{i+1}^{new}(t_i^+)]^T \, \mathbf{M}_{new} \, \dot{\mathbf{d}}_{i+1}^{new}(t_i^+) \\
&= \frac{1}{2}[\dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+)]^T \, \mathbf{M}_{bnd} \, \dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+) - \frac{1}{2}[\dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+)]^T \, \mathbf{M}_{new} \, \dot{\tilde{\mathbf{d}}}_i^{bnd}(t_i^+)
\end{aligned}
$$

$$-\frac{1}{2}[\dot{\mathbf{d}}_i^{bnd}(t_i^-)]^T \, \mathbf{M}_{bnd} \, \dot{\mathbf{d}}_i^{bnd}(t_i^-)$$
$$+\frac{1}{2}[\dot{\mathbf{d}}_{i+1}^{new}(t_i^+)]^T \, \mathbf{M}_{new} \, \dot{\mathbf{d}}_{i+1}^{new}(t_i^+) \tag{4.110}$$

Also substituting equation (4.86) and equation (4.87) into equation (4.110), we can have

$$\begin{aligned}
\Delta E^{ke}(t_i) &= \frac{1}{2}[\dot{\mathbf{d}}_i^{bnd}(t_i^-) + \Delta\dot{\mathbf{d}}_i^{bnd}(t_i)]^T \, \mathbf{M}_{bnd} \, [\dot{\mathbf{d}}_i^{bnd}(t_i^-) + \Delta\dot{\mathbf{d}}_i^{bnd}(t_i)] \\
&\quad -\frac{1}{2}[\dot{\mathbf{d}}_i^{bnd}(t_i^-)]^T \, \mathbf{M}_{bnd} \, \dot{\mathbf{d}}_i^{bnd}(t_i^-) \\
&= \frac{1}{2}[\Delta\dot{\mathbf{d}}_i^{bnd}(t_i)]^T \, \mathbf{M}_{bnd} \, \dot{\mathbf{d}}_i^{bnd}(t_i^-) + \frac{1}{2}[\Delta\dot{\mathbf{d}}_i^{bnd}(t_i)]^T \, \mathbf{M}_{bnd} \, \Delta\dot{\mathbf{d}}_i^{bnd}(t_i) \\
&\quad +\frac{1}{2}[\dot{\mathbf{d}}_i^{bnd}(t_i^-)]^T \, \mathbf{M}_{bnd} \, \Delta\dot{\mathbf{d}}_i^{bnd}(t_i)
\end{aligned}$$

Since $\mathbf{M}_{bnd}$ is diagonal,

$$\begin{aligned}
\Delta E^{ke}(t_i) &= [\Delta\dot{\mathbf{d}}_i^{bnd}(t_i)]^T \, \mathbf{M}_{bnd} \, \dot{\mathbf{d}}_i^{bnd}(t_i^-) \\
&\quad +\frac{1}{2}[\Delta\dot{\mathbf{d}}_i^{bnd}(t_i)]^T \, \mathbf{M}_{bnd} \, \Delta\dot{\mathbf{d}}_i^{bnd}(t_i)
\end{aligned} \tag{4.111}$$

It can be seen that the perturbation on kinetic energy is determined by (1) the parameter matrix of the boundary mass, (2) the velocity perturbation due to the externally applied impulsive force which causes the system structural variation and (3) the time the impulsive force is applied.

### 4.5.2 Perturbation on Potential Energy Function

The perturbation on the potential energy function of an **SVS** can be derived as follows

$$\begin{aligned}
E_{pre}^{pot}(t_i^-) &= \frac{1}{2}\mathbf{d}_i^T(t_i^-) \, \mathbf{K}_{swt}(t_i^-) \, \mathbf{d}_i(t_i^-) \\
&= \frac{1}{2}\mathbf{d}_i^T(t_i^-) \, \mathbf{K}_i \, \mathbf{d}_i(t_i^-)
\end{aligned}$$

and

$$\begin{aligned}
E_{post}^{pot}(t_i^+) &= \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+) \, \mathbf{K}_{swt}(t_i^+) \, \mathbf{d}_{i+1}(t_i^+) \\
&= \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+) \, \mathbf{K}_{i+1} \, \mathbf{d}_{i+1}(t_i^+)
\end{aligned}$$

where $\mathbf{K}_{swt}(t_i^-)$ is actually the stiffness matrix of configuration $i$ and $\mathbf{K}_{swt}(t_i^+)$ is the stiffness matrix of configuration $i+1$. $E^{pot}_{pre}(t_i^-)$ is the potential energy function just before the structural variation and $E^{pot}_{post}(t_i^+)$ is the potential energy function just after the structural variation. The change of the potential energy function at the switching instant can then be expressed as

$$\begin{aligned}
\Delta E^{pot}(t_i) &= E^{pot}_{post}(t_i^+) - E^{pot}_{pre}(t_i^-) \\
&= \frac{1}{2}\mathbf{d}^T_{i+1}(t_i^+)\,\mathbf{K}_{i+1}\,\mathbf{d}_{i+1}(t_i^+) - \frac{1}{2}\mathbf{d}^T_i(t_i^-)\,\mathbf{K}_i\,\mathbf{d}_i(t_i^-) \\
&= \frac{1}{2}\mathbf{d}^T_{i+1}(t_i^+)\,(\hat{\mathbf{K}}_i + \Delta\mathbf{K}_{i+1})\,\mathbf{d}_{i+1}(t_i^+) - \frac{1}{2}\mathbf{d}^T_i(t_i^-)\,\mathbf{K}_i\,\mathbf{d}_i(t_i^-)
\end{aligned}$$

Considering equation (4.99), we have

$$\Delta E^{pot}(t_i) = \frac{1}{2}\mathbf{d}^T_{i+1}(t_i^+)\,\Delta\mathbf{K}_{i+1}\,\mathbf{d}_{i+1}(t_i^+) \tag{4.112}$$

The perturbation on the system potential energy is determined by the generalized stiffness perturbation matrix.

Therefore, the perturbation on the energy function at the switching instant is

$$\begin{aligned}
\Delta E(t_i) &= \Delta E^{ke}(t_i) + \Delta E^{pot}(t_i) \\
&= [\Delta\dot{\mathbf{d}}^{bnd}_i(t_i)]^T\,\mathbf{M}_{bnd}\,\dot{\mathbf{d}}^{bnd}_i(t_i^-) + \frac{1}{2}[\Delta\dot{\mathbf{d}}^{bnd}_i(t_i)]^T\,\mathbf{M}_{bnd}\,\Delta\dot{\mathbf{d}}^{bnd}_i(t_i) \\
&\quad + \frac{1}{2}\mathbf{d}^T_{i+1}(t_i^+)\,\Delta\mathbf{K}_{i+1}\,\mathbf{d}_{i+1}(t_i^+) \\
&= \alpha^{bnd}[\mathbf{F}, \dot{\mathbf{d}}^{bnd}_i(t_i^-)] + \alpha^{bnd}[\mathbf{F}] + \alpha[\Delta\mathbf{K}_{i+1}]
\end{aligned} \tag{4.113}$$

where

$$\begin{aligned}
\alpha^{bnd}[\mathbf{F}, \dot{\mathbf{d}}^{bnd}_i(t_i^-)] &= [\Delta\dot{\mathbf{d}}^{bnd}_i(t_i)]^T\,\mathbf{M}_{bnd}\,\dot{\mathbf{d}}^{bnd}_i(t_i^-) \\
\alpha^{bnd}[\mathbf{F}] &= \frac{1}{2}[\Delta\dot{\mathbf{d}}^{bnd}_i(t_i)]^T\,\mathbf{M}_{bnd}\,\Delta\dot{\mathbf{d}}^{bnd}_i(t_i) \\
\alpha[\Delta\mathbf{K}_{i+1}] &= \frac{1}{2}\mathbf{d}^T_{i+1}(t_i^+)\,\Delta\mathbf{K}_{i+1}\,\mathbf{d}_{i+1}(t_i^+)
\end{aligned}$$

The dynamic stability of the **SVS** can then be restated as follows:

**Theorem 4.5** Assume that dynamics of the structural variation is dominated by the process compatibility. An order-varying **SVS** would be dynamically stable if

1. $\Delta\mathbf{K}_{i+1}$ is negative semi-definite,
2. $\alpha^{bnd}[\mathbf{F}, \dot{\mathbf{d}}_i^{bnd}(t_i^-)] \leq -\alpha^{bnd}[\mathbf{F}]$.

**Theorem 4.6** Assume that dynamics of the structural variation is dominated by the process compatibility. A conservative order-varying **SVS** would be dynamically unstable if

1. $\Delta\mathbf{K}_{i+1}$ is positive semi-definite,
2. $\alpha^{bnd}[\mathbf{F}, \dot{\mathbf{d}}_i^{bnd}(t_i^-)] > -\alpha^{bnd}[\mathbf{F}]$.

The proof of **Theorem 4.5** and **Theorem 4.6** is straightforward. The negative semidefiniteness of the $\Delta\mathbf{K}_{i+1}$ would lead to

$$\alpha[\Delta\mathbf{K}_{i+1}] = \mathbf{d}_{i+1}^T(t_i^+)\,\Delta\mathbf{K}_{i+1}\,\mathbf{d}_{i+1}(t_i^+) \leq 0$$

If $\alpha^{bnd}[\mathbf{F}, \dot{\mathbf{d}}_i^{bnd}(t_i^-)] \leq -\alpha^{bnd}[\mathbf{F}]$,

$$\Delta E(t_i) \leq 0$$

and the **SVS** will be dynamically stable. Condition #1 is determined by system structural variation and condition #2 is determined by the way the externally applied impulsive force is applied on the boundary mass nodes. In general, if the impulsive force is applied oppositely to the moving direction of the boundary mass nodes within a certain range, the system dynamic stability condition would be satisfied. The proof of **Theorem 4.6** can be carried out in exactly the same way as that of **Theorem 4.5**.

In the following example, the application of **Theorem 4.5** and **Theorem 4.6** will be demonstrated.

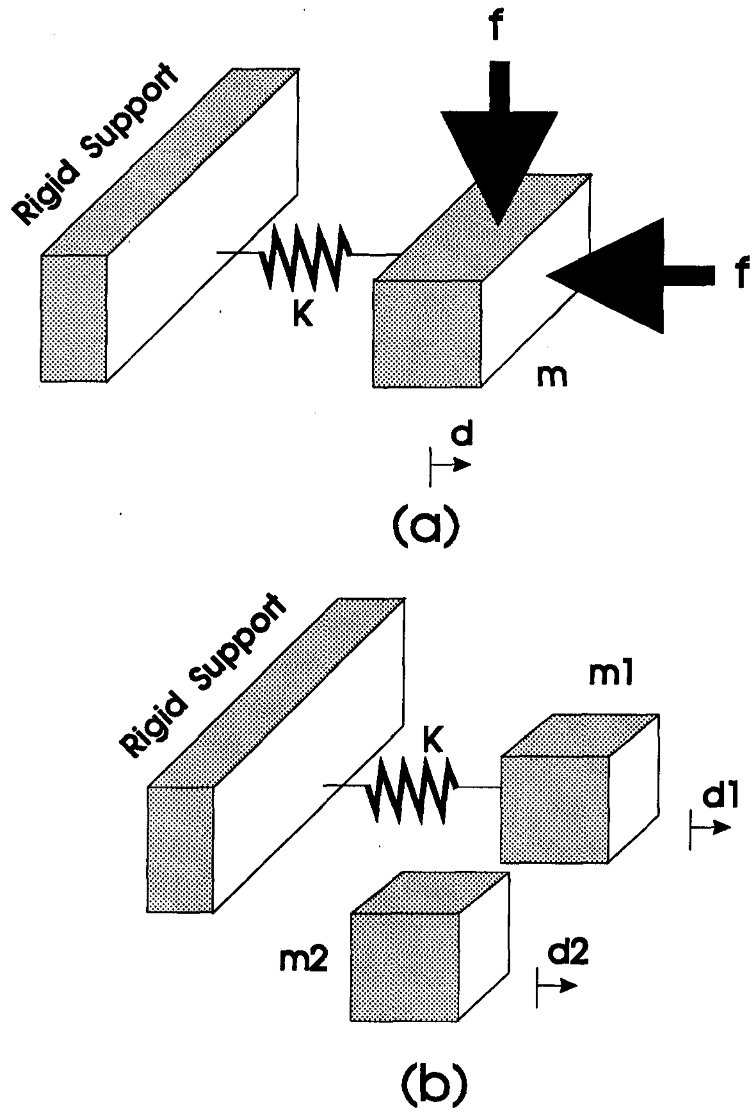


Figure 4.9: Example System

- **Example:** Consider the system described in (a) of Figure 4.9.

The equation of motion is given by

$$m\ddot{d} + kd = 0, \quad d(0) = d_0$$

and

$$\mathbf{M}_1 = \mathbf{M}_{bnd} = m, \quad \mathbf{K}_1 = k$$

At time $t = t_1$, an external impulsive force $\mathbf{f}$ (the horizontal one) is applied on the mass node and the mass is divided into two pieces instantly. The system takes a new configuration, which is shown in (b) of the Figure 4.9. The equations of motion are

$$\begin{aligned} m_1\ddot{d}_1 + kd_1 &= 0 \\ m_2\ddot{d}_2 &= 0 \end{aligned}$$

We have

$$\mathbf{M}_2 = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \mathbf{K}_2 = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}$$

which gives

$$\Delta\mathbf{K}_2 = \mathbf{K}_2 - \hat{\mathbf{K}}_1 = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} = 0$$

The velocity perturbation $\Delta\dot{d}$ due to $\mathbf{f}$ can be calculated as follows

$$\Delta\dot{d}(t_1) = \frac{\mathbf{f}}{m}$$

Hence,

$$\alpha^{bnd}[\mathbf{F}] = \frac{1}{2}\, m\, (\frac{\mathbf{f}}{m})^2 = \frac{f^2}{2m}$$

$$\alpha^{bnd}[\mathbf{F}, \dot{\mathbf{d}}_i^{bnd}(t_i^-)] = \frac{\mathbf{f}}{m}\, m\, \dot{d}(t_1^-) = \mathbf{f}\, \dot{d}(t_1^-)$$

Since $\Delta\mathbf{K}_2 = 0$, the system dynamic stability is determined by the relation between $\alpha^{bnd}[\mathbf{F}]$ and $\alpha^{bnd}[\mathbf{F}, \dot{\mathbf{d}}_i^{bnd}(t_i^-)]$ only. If we assume $\alpha^{bnd}[\mathbf{F}, \dot{\mathbf{d}}_i^{bnd}(t_i^-)] \leq -\alpha^{bnd}[\mathbf{F}]$, we have

$$\alpha^{bnd}[\mathbf{F}, \dot{\mathbf{d}}_i^{bnd}(t_i^-)] = \mathbf{f}\, \dot{d}(t_1) \leq -\alpha^{bnd}[\mathbf{F}] = -\frac{f^2}{2m}$$

It is not difficult to see that the external impulsive force $\mathbf{f}$ has to be applied in the opposite direction of the $\dot{d}(t)$, in order for the system to be dynamically stable, i.e.,

$$\alpha^{bnd}[\mathbf{F}, \dot{\mathbf{d}}_i^{bnd}(t_i^-)] = -f\, \dot{d}(t_1^-) \leq -\alpha^{bnd}[\mathbf{F}] = -\frac{f^2}{2m}$$

Finally

$$f \leq 2m\dot{d}(t_1^-)$$

According to **Theorem 4.5**, the system will be dynamically stable if $\mathbf{f}$ is applied in the opposite direction of the $\dot{d}(t)$ and $f \leq 2m\dot{d}(t_1^-)$.

On the other hand, the system would be dynamically unstable if

$$\alpha^{bnd}[\mathbf{F}, \dot{\mathbf{d}}_i^{bnd}(t_i^-)] = \mathbf{f}\, \dot{d}(t_1^-) > -\alpha^{bnd}[\mathbf{F}] = -\frac{f^2}{2m}$$

If $\mathbf{f}$ is applied in the same direction as $\dot{\mathbf{d}}(t_1^-)$, the above equation would be satisfied. According to **Theorem 4.6**, the system would be dynamically unstable.

The energy function before the structural variation is

$$E_1(t) = E(t_1^-) = \frac{1}{2}\, m\, \dot{d}^2(t_1^-) + \frac{1}{2}\, k\, d^2(t_1^-) = \frac{1}{2}\, k\, d^2(0)$$

The energy function after the structural variation is

$$E_2(t) = E(t_1^+) = \frac{1}{2}\, m_1\, \dot{d}_1^2(t_1^+) + \frac{1}{2}\, m_2\, \dot{d}_2^2(t_1^+) + \frac{1}{2}\, k\, d_1^2(t_1^+)$$

Since

$$\begin{aligned}
\dot{d}_1(t_1^+) &= \dot{d}_1(t_1^-) + \Delta \dot{d}_1(t_1) = \dot{d}_1(t_1^-) + \frac{\mathbf{f}}{m} \\
\dot{d}_2(t_1^+) &= \dot{d}_1(t_1^+) = \dot{d}_1(t_1^-) + \frac{\mathbf{f}}{m} \\
d_1(t_1^+) &= d_1(t_1^-)
\end{aligned}$$

we have

$$\begin{aligned}
E_2 &= \frac{1}{2}\, m_1\, [\dot{d}_1(t_1^-) + \frac{\mathbf{f}}{m}]^2 + \frac{1}{2}\, m_2\, [\dot{d}_1(t_1^-) + \frac{\mathbf{f}}{m}]^2 + \frac{1}{2}\, k\, d_1(t_1^-) \\
&= \frac{1}{2}\, m\, [\dot{d}_1(t_1^-) + \frac{\mathbf{f}}{m}]^2 + \frac{1}{2}\, k\, d_1(t_1^-) \\
&= \frac{1}{2}\, m\, [\dot{d}_1^2(t_1^-) + (\frac{\mathbf{f}}{m})^2 + 2\frac{\mathbf{f}}{m}\dot{d}_1(t_1^-)] + \frac{1}{2}\, k\, d_1(t_1^-) \\
&= \frac{1}{2}\, m\, \dot{d}_1^2(t_1^-) + \frac{1}{2}\, k\, d_1(t_1^-) + \frac{1}{2}\, (\frac{\mathbf{f}^2}{m}) + \mathbf{f}\dot{d}_1(t_1^-) \\
&= E_1 + \frac{1}{2}\, (\frac{\mathbf{f}^2}{m}) + \mathbf{f}\dot{d}_1(t_1^-)
\end{aligned}$$

Therefore,

$$\Delta E_2 = E_2 - E_1 = \frac{1}{2}\, (\frac{\mathbf{f}^2}{m}) + \mathbf{f}\dot{d}_1(t_1^-)$$

i.e.,

$$\Delta E_2 = E_2 - E_1 = \frac{1}{2}\, (\frac{\mathbf{f}^2}{m}) + \mathbf{f}\dot{d}(t_1^-)$$

If **f** is applied in the opposite direction to $\dot{d}(t_1^-)$, we have

$$\Delta E_2 = \frac{1}{2}\left(\frac{f^2}{m}\right) - f\,\dot{d}(t_1^-)$$

If $f \leq 2m\dot{d}(t_1^-)$ (i.e., $f/2m - \dot{d}(t_1^-) \leq 0$), we will have

$$\frac{1}{2}\left(\frac{f^2}{m}\right) - f\,\dot{d}_1(t_1^-) = f\,\left[\frac{f}{2m} - \dot{d}_1(t_1^-)\right] \leq 0$$

i.e.,

$$\Delta E_2 \leq 0$$

which proves that the system is dynamically stable.

On the other hand, if **f** is applied in the same direction as $\dot{d}(t_1^-)$, it is not difficult to see that

$$\begin{aligned}\Delta E_2 &= \frac{1}{2}\left(\frac{\mathbf{f}^2}{m}\right) + \mathbf{f}\dot{d}(t_1^-) \\ &= \frac{1}{2}\left(\frac{f^2}{m}\right) + f\dot{d}(t_1^-) > 0\end{aligned}$$

Therefore, the system is dynamically unstable. It can be seen that the previously derived analytical results have been verified.

The numerical simulation results are shown in Figure 4.10 and Figure 4.11 for dynamically stable and dynamically unstable cases respectively. The initial conditions of the system $d(0) = 10$, $\dot{d}(0) = 0$ are assumed. The following parameters are used in the numerical simulation,

$$m = 0.05kg, \qquad m_1 = m_2 = 0.025kg, \qquad k = 6N/m$$

The impulsive forces $f = 1.0N$ and $f = -1.5N$ are used in stable and unstable cases respectively. The symbols are defined as follows,

- $d(t)$: position of the mass node before structural variation; $d_1(t)$, $d_2(t)$, positions of the mass nodes after structural variation.
- $v(t)$: velocity of the mass node before structural variation; $v_1(t)$, $v_2(t)$, velocities of the mass nodes after structural variation.
- $E(t)$: energy of the mass node before structural variation; $E_1(t)$, $E_2(t)$, energy of the mass nodes after structural variation.

It can be seen from (a), (b) of Figure 4.10 that although the change of the system position is continuous, there is a sudden change in the system velocity. The change is caused by the external impulsive force f. In this case, f is applied in the same direction as $\dot{d}(t_1^-)$ and also $f \leq 2m\dot{d}(t_1^-)$. The system energy level decreases after the structural variation, which can be observed from (c) of Figure 4.10. Hence, the system is dynamically stable. In another case shown in Figure 4.11, f is applied in the same direction as $\dot{d}(t_1^-)$. There is also a sudden jump in the velocity, which can be observed from (b) of Figure 4.11. From (c) of Figure 4.11, we can see that the system energy level increases after the structural variation. Hence, the system is dynamically unstable.

### 4.5.3 Experimental Study of Dynamic Stability Using Process Compatibility

In order to further illustrate the application of analytical results developed in this research and verify the numerical simulation results, an experiment is carried out.

The experimental setup consists of three parts, a mechanical moving device, an EKTA 1000 motion analyser (i.e., high-speed camera) and an image acquisition and processing system. The mechanical moving device, which is shown in Figure 4.12, is composed of four SPB 8 super pillow blocks with linear bearings and a tubular solenoid. The

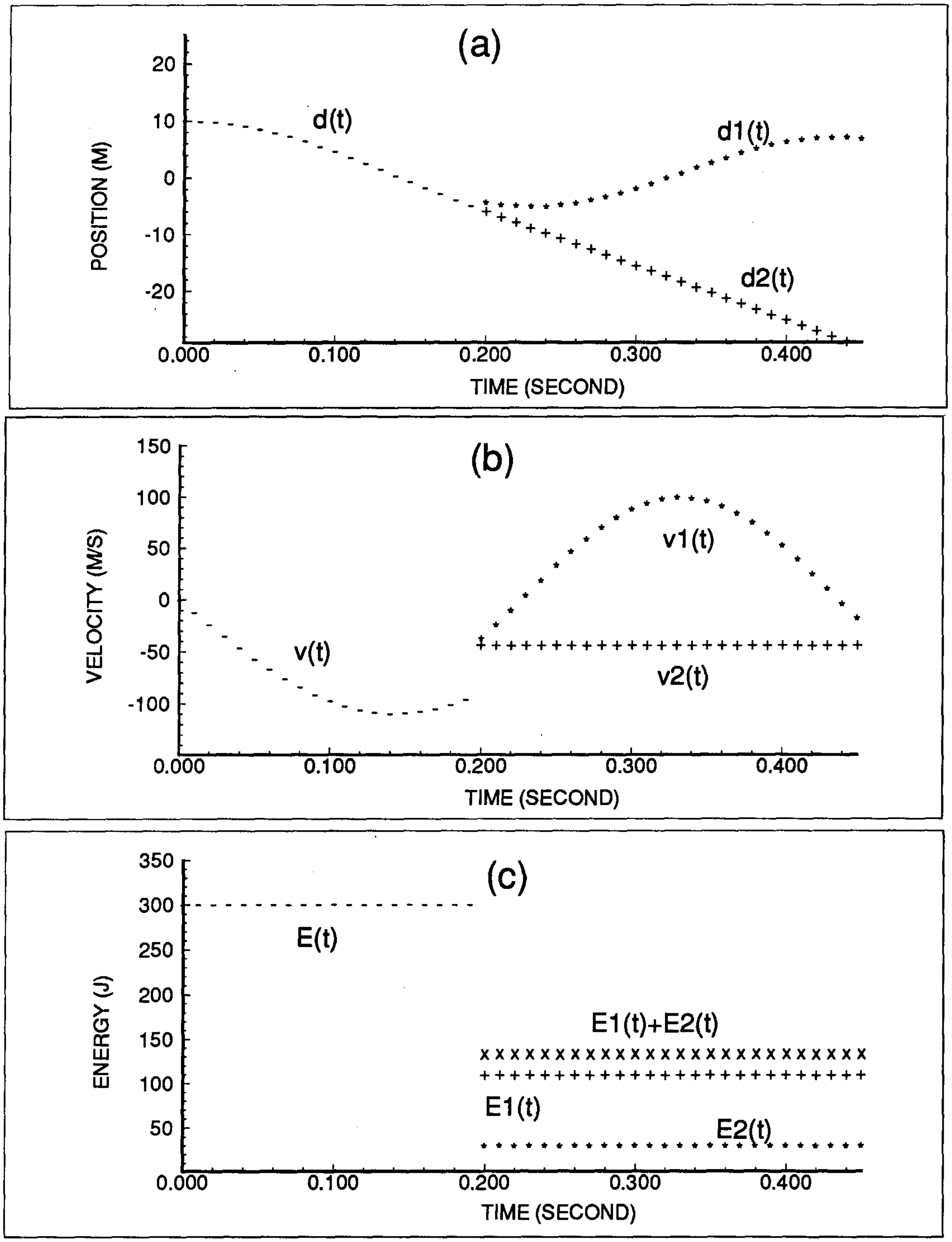


Figure 4.10: Result of Numerical Simulation: A Dynamic Stable Case Using Process Compatibility

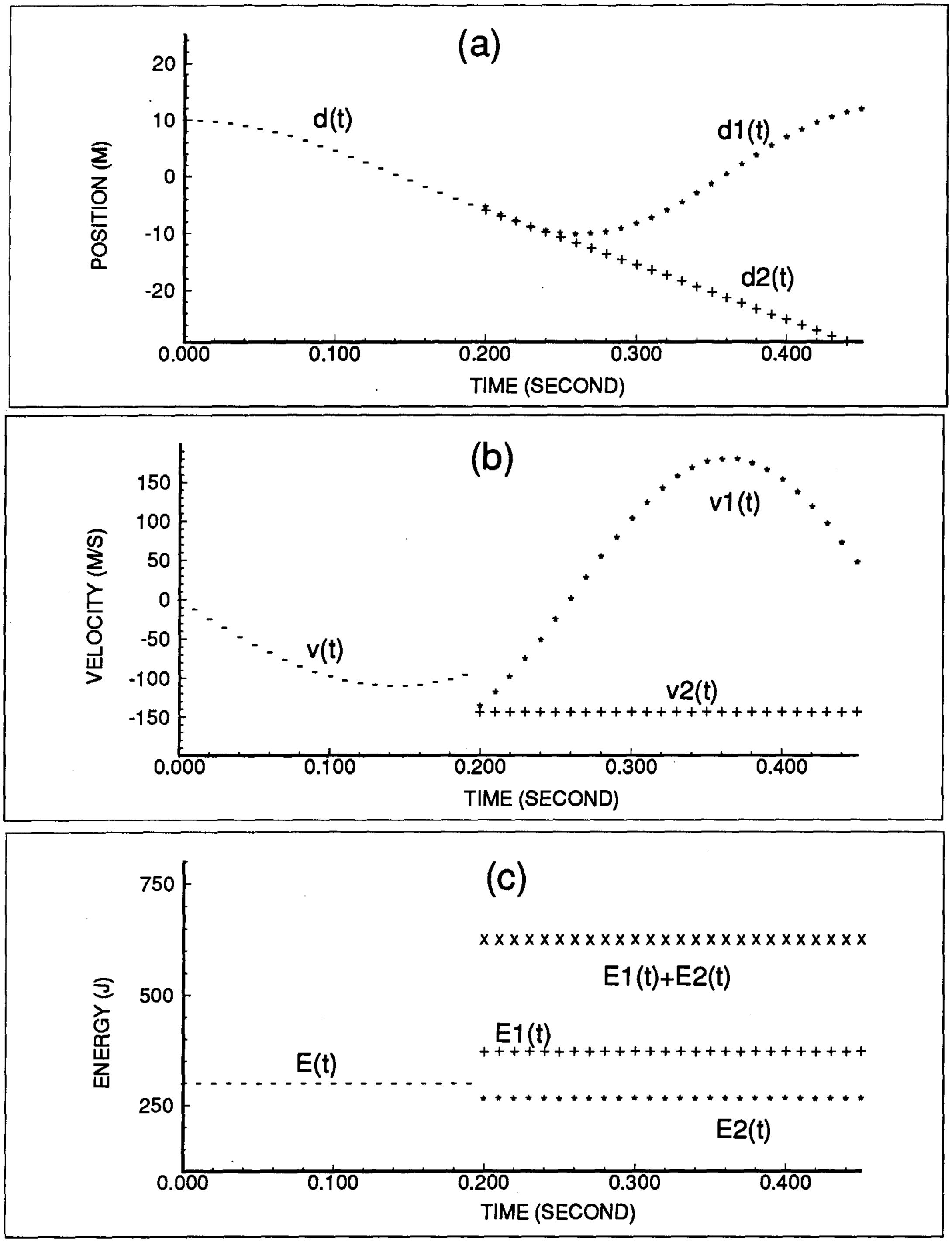


Figure 4.11: Result of Numerical Simulation: A Dynamic Unstable Case Using Process Compatibility

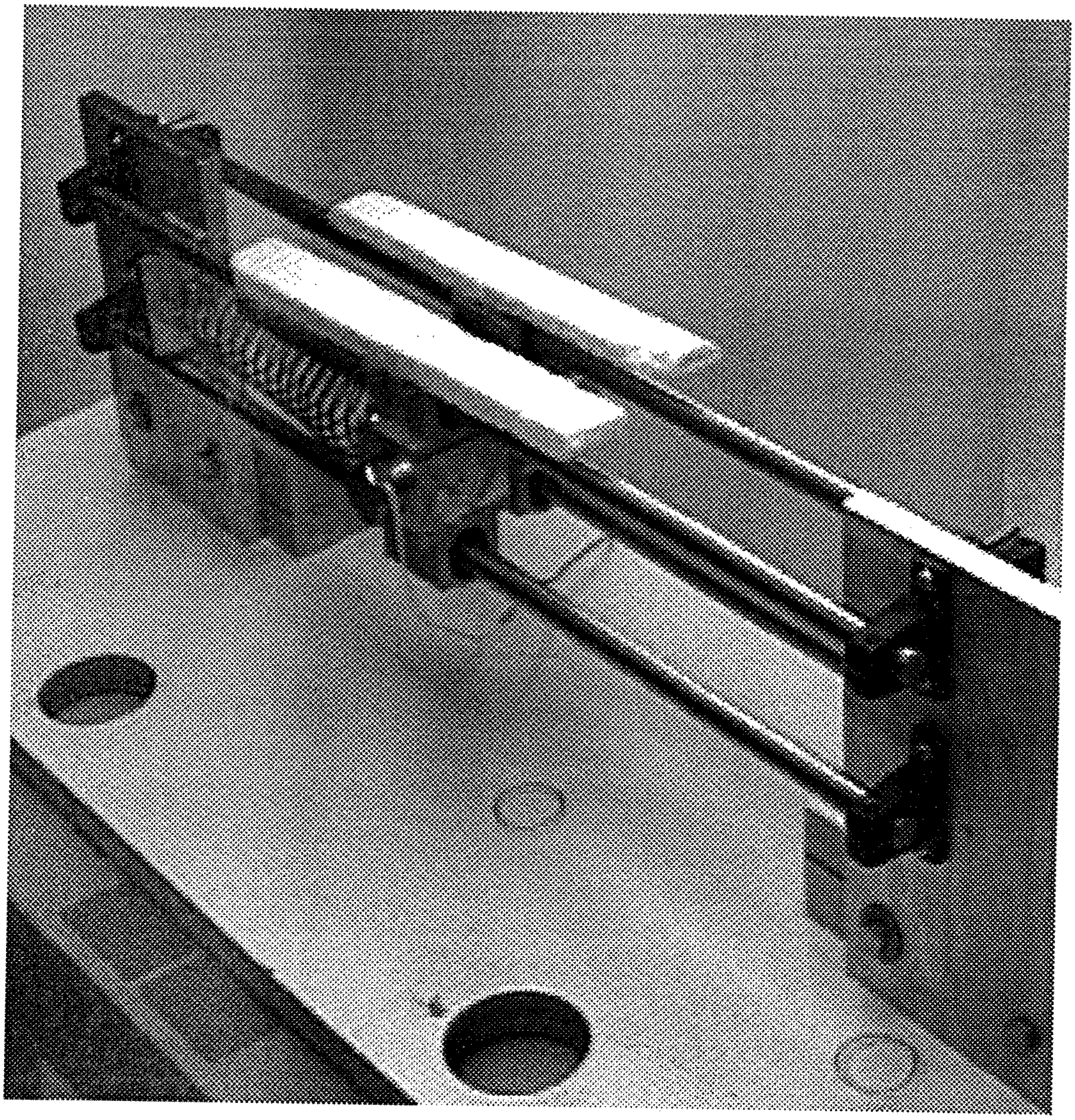

Figure 4.12: A Moving Mechanical Device

Figure 4.13: Image Processing System

blocks are used to emulate two mass nodes and the tubular solenoid is used to emulate a switchable connection between the mass nodes. The tubular solenoid is controlled by an electrical signal. The mass of each of the two mass nodes is $0.52kg$. One of the mass nodes is connected to a spring. The stiffness of the spring is $70N/m$. The other end of the spring is fixed to the supporting structure. The external impulsive force **f** is generated by a $100psi$ air jet, which can be turned on and off by a switch. The EKTA 1000 motion analyser is used to collect and store the experimental data. The sampling rate is set at 500 frames/second. The position is measured by using the EKTA 1000 motion analyser. The velocity and energy are calculated from the measured position data. After the position data is collected, it is then sent to a PC-based image processing system, which is shown in Figure 4.13, and the data is then processed there.

Initially, two mass nodes are connected through the tubular solenoid. An initial position is given to the mass nodes. When they are released, they start to move. At a point, an air jet is applied to the mass nodes in the same direction as the velocity of the mass nodes and the tubular solenoid is activated so that the two mass nodes are separated. From previous analysis, we know that the air jet will cause the system to be dynamically unstable.

The experimental results are shown in Figure 4.14 through Figure 4.16. Figure 4.14 shows the position profile of the system. Two mass nodes are separated and go in different ways after the solenoid is activated. Figure 4.15 shows the velocity profiles. The velocity data is calculated from the experimental position data. It can be seen that it is very noisy. In order to eliminate the nosie, a low-pass filter with a cutoff frequency of 8 hz is designed using MATLAB. The velocity signal is filtered and the true velocity signal can then be obtained. The change of the velocity at switching instant can be seen in the Figure 4.15. The sudden change of the velocity is caused by the air jet. Since the system energy is increased, which can be seen in Figure 4.16, we know that the system

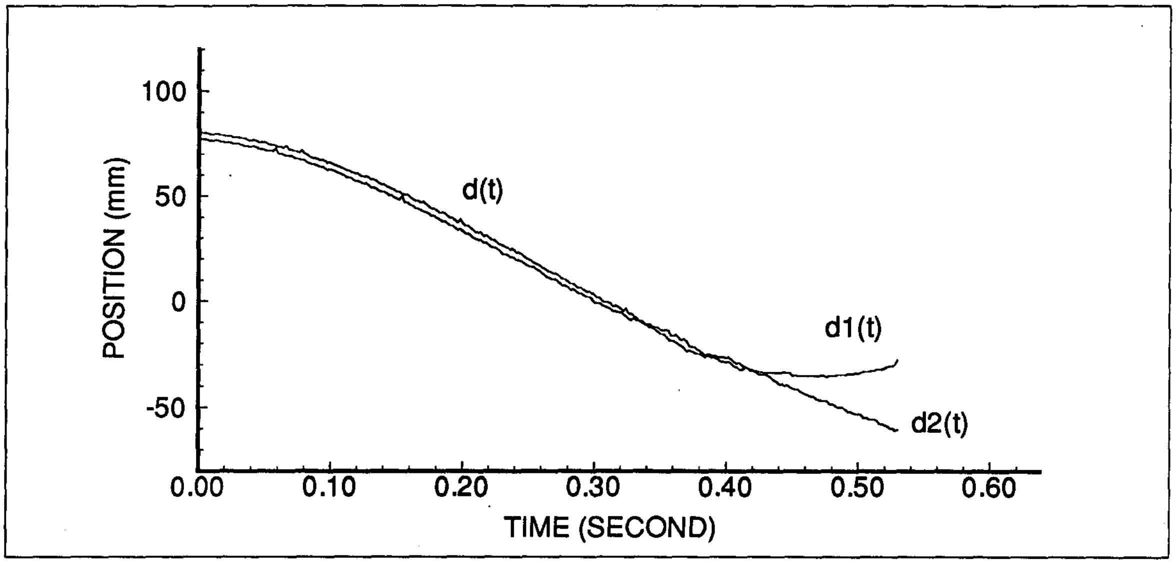


Figure 4.14: Experimental Result: Position Profile

is unstable after the change of the structure takes place.

This experiment further illustrates the concept of dynamic stability of the **SVS** (and dynamic instability of the **SVS**) and shows the application of previously derived theoretical results.

## 4.6 Analysis of Dynamic Stability Using Motion Compatibility

In the previous section, the stability of the order-varying **SVS** has been studied using the process compatibility. In this section, the stability of the order-varying **SVS** is analyzed using the motion compatibility. By motion compatibility, we have at the instant of structural variation,

$$\begin{aligned} \mathbf{d}_i(t_i^-) &= \tilde{\mathbf{d}}_i(t_i^+) \\ \dot{\mathbf{d}}_i(t_i^-) &= \dot{\tilde{\mathbf{d}}}_i(t_i^+) \end{aligned} \tag{4.114}$$

According to equation (4.111), we have

$$\Delta E^{ke}(t_i) \quad = \quad [\Delta \dot{\mathbf{d}}_i^{bnd}(t_i)]^T \, \mathbf{M}_{bnd} \, \dot{\mathbf{d}}_i^{bnd}(t_i^-) + \frac{1}{2}[\Delta \dot{\mathbf{d}}_i^{bnd}(t_i)]^T \, \mathbf{M}_{bnd} \, \Delta \dot{\mathbf{d}}_i^{bnd}(t_i)$$

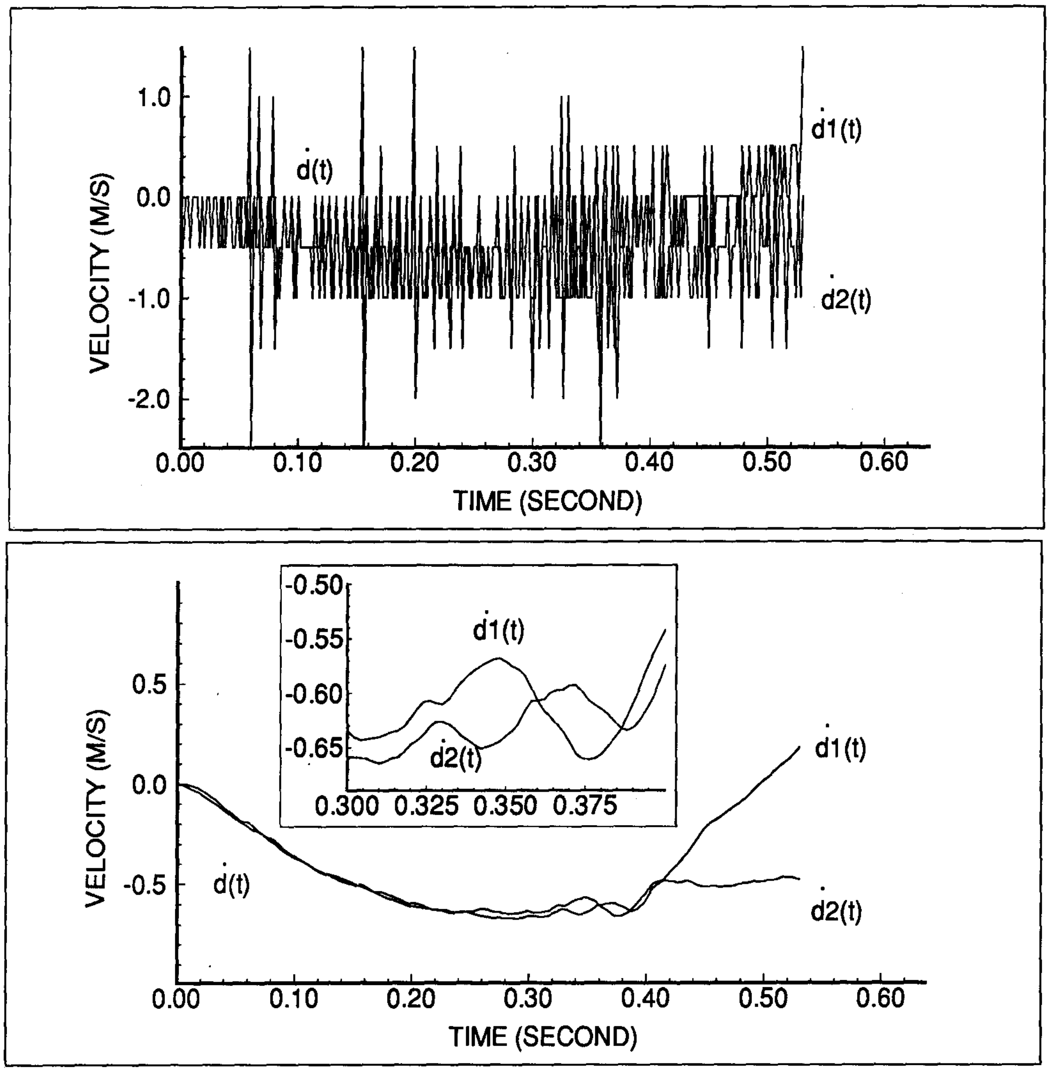


Figure 4.15: Experimental Result: Velocity Profile

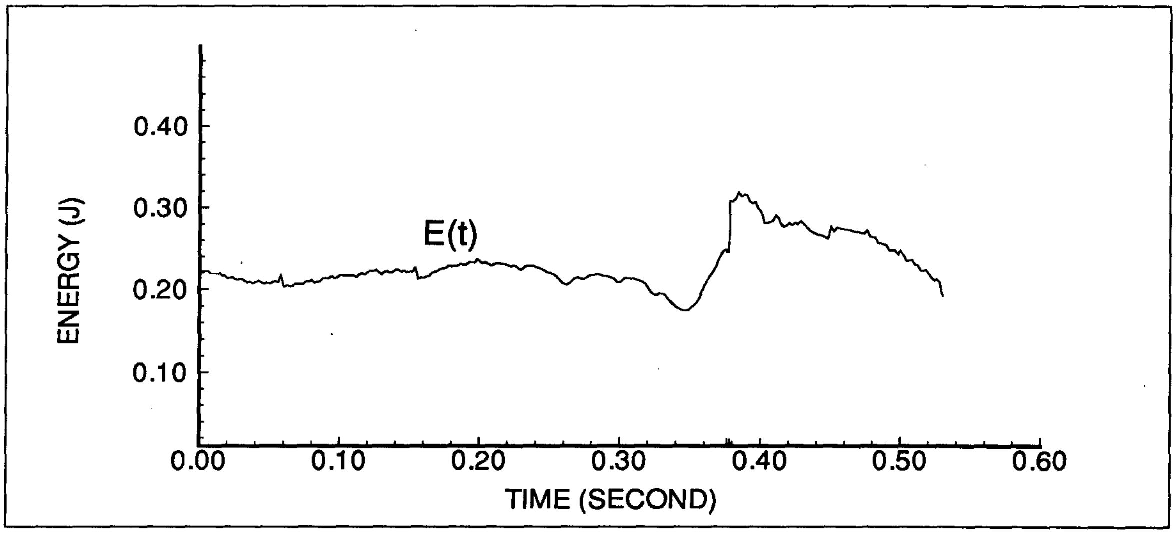


Figure 4.16: Experimental Result: Energy Profile

The motion compatibility (described in equation (4.114)) leads to $\Delta\dot{\mathbf{d}}_i^{bnd}(t_i) = 0$ at the switching instant. Hence,

$$\Delta E^{ke}(t_i) = 0 \tag{4.115}$$

which means that there is no change of kinetic energy at the instant of structural variation. It is not difficult to see that motion compatibility is actually a special case of process compatibility. The perturbation on the energy function is solely determined by the potential energy function, i.e.,

$$\Delta E(t_i) = \Delta E^{pot}(t_i) = \frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\;\Delta\mathbf{K}_{i+1}\;\mathbf{d}_{i+1}(t_i^+) \tag{4.116}$$

The stability theorem can then be stated according to the generalized stiffness perturbation matrix $\Delta\mathbf{K}_{i+1}$.

**Theorem 4.7:** Assume that dynamics of the structural variation is dominated by motion compatibility. If $\Delta\mathbf{K}_{i+1}$ is negative semi-definite, the configuration *i+1* will be dynamically stable.

**Proof:** As is shown in equation (4.116), $\Delta E(t_i)$ is of quadratic form. If $\Delta\mathbf{K}_{i+1}$ is negative semi-definite, we will have

$$\frac{1}{2}\mathbf{d}_{i+1}^T(t_i^+)\,\Delta\mathbf{K}_{i+1}\mathbf{d}_{i+1}(t_i^+) \leq 0$$

no matter what the values the $\mathbf{d}_{i+1}(t_i^+)$ takes. Therefore, $\Delta E(t_i) \leq 0$. According to the definition, we know the configuration *i+1* of the **SVS** is dynamically stable, which concludes the proof.

**Theorem 4.8:** Assume that dynamics of the structural variation is dominated by motion compatibility. For any conservative **SVS**, if $\Delta\mathbf{K}_{i+1}$ is positive definite, the configuration *i+1* will be dynamically unstable.

**Proof:** Since the **SVS** is a conservative system, we have

$$\begin{aligned} \mathbf{max}\{E_i(t)\} &= E_{ic} \\ \mathbf{max}\{E_{i+1}(t)\} &= E_{i+1c} \end{aligned}$$

where $E_{ic}$ and $E_{i+1c}$ are constant over the time period $[t_{i-1}, t_i]$ and $[t_i, t_{i+1}]$ respectively. If $\Delta\mathbf{K}_{i+1}$ is positive definite, $\Delta E(t_i) = E_{i+1c} - E_{ic} > 0$, i.e., $E_{i+1c} > E_{ic}$. There is a sudden jump of energy at the instant $t = t_i$, which makes the configuration *i+1* of the system dynamically unstable.

**Theorem 4.7** and **Theorem 4.8** provide criteria for evaluation of the dynamic stability and instability of the order-varying **SVS** for the motion compatibility case. Using these two theorems, the dynamic stability of the order-varying **SVS** can be predicted based on given structural perturbation, which is described by $\Delta\mathbf{K}_{i+1}$.

- **Example:** Consider the system described in (a) of Figure 4.9 again and assume the $\mathbf{f}$ is applied perpendicularly to the direction of motion of the mass node, which will lead to the process of the structural variation dominated by motion compatibility.

Then the velocity perturbation is $\Delta\dot{d} = 0$. Therefore,

$$\begin{aligned} \alpha^{bnd}[\mathbf{F}, \dot{\mathbf{d}}_i^{bnd}(t_i^-)] &= 0 \\ \alpha^{bnd}[\mathbf{F}] &= 0 \end{aligned}$$

It is known that $\Delta\mathbf{K_2} = 0$, i.e., $\alpha[\Delta\mathbf{K}_{i+1}] = 0$. According to **Theorem 4.7**, we know that the system is dynamically stable. In fact, the system energy remains unchanged in this case although the system structure has changed.

The numerical simulation results are shown in figure (4.17). Parameters used are

$$m = 0.05kg, \qquad m_1 = m_2 = 0.025kg, \qquad k = 6N/m$$

The symbols are defined as follows,

- $d(t)$: position of the mass node before structural variation; $d_1(t)$, $d_2(t)$, positions of the mass nodes after structural variation.
- $v(t)$: velocity of the mass node before structural variation; $v_1(t)$, $v_2(t)$, velocities of the mass nodes after structural variation.
- $E(t)$: energy of the mass node before structural variation; $E_1(t)$, $E_2(t)$, energy of the mass nodes after structural variation.

It can be seen from (a) of Figure 4.17 that after the two mass nodes separate, they take different trajectories. There is no sudden change in either the position or velocity profiles, which can be observed from (a) and (b) of Figure (4.17) . The energy remains unchanged after the structural variation. Therefore, the system is dynamically stable.

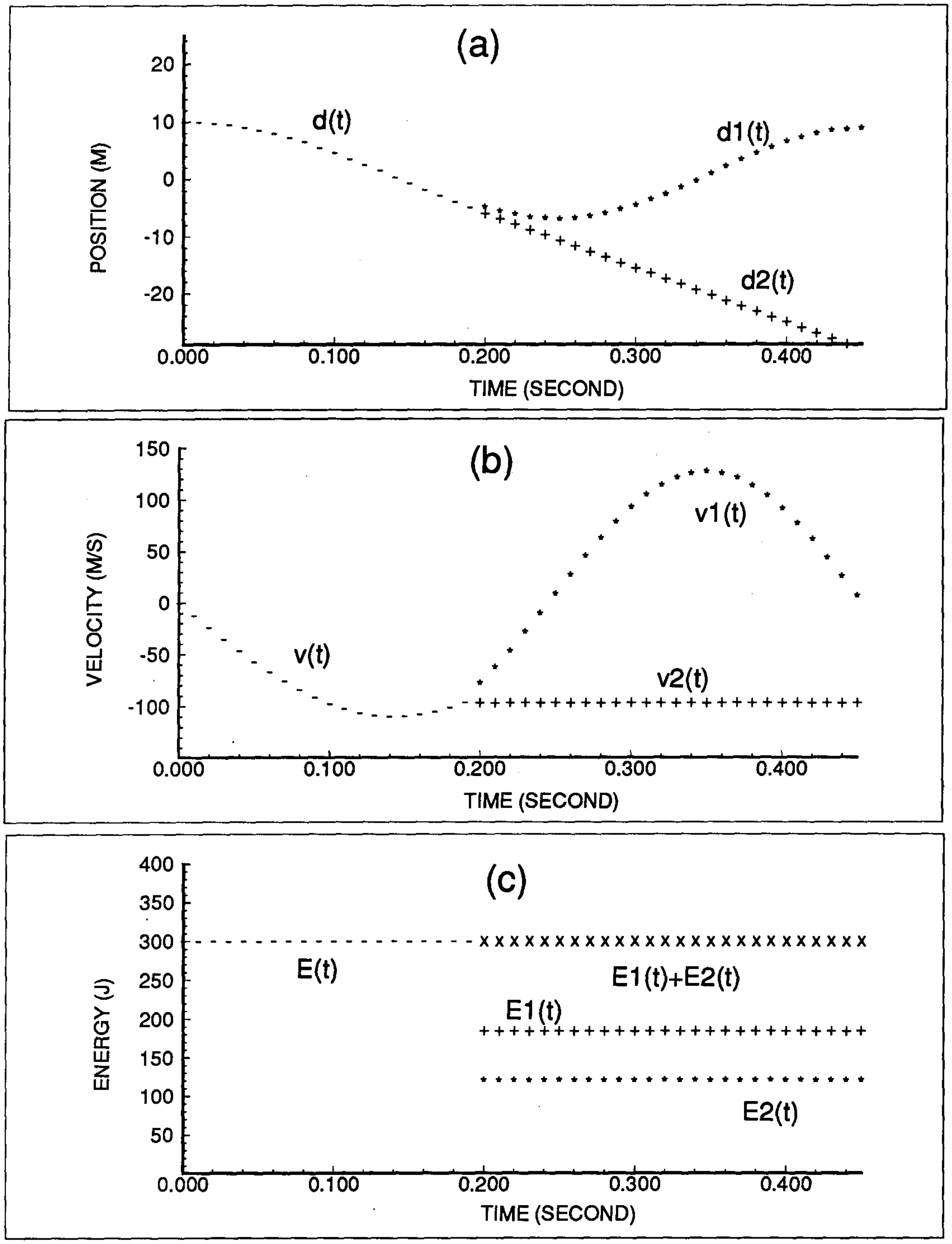


Figure 4.17: Result of Numerical Simulation: A Dynamic Stable Case using Motion Compatibility

## 4.7 Issues of Dynamic Control of the SVS

In this section, the control strategies and their implementation will be discussed. It has to be pointed out that since this research project originated from interest in the analysis of **SVS** dynamics and its stability, the previously developed analysis approach is mainly for that purpose. It is not our attempt here to present an exhaustive or even the representative picture of the general control theory of time-varying dynamic systems. The objective instead is to demonstrate what control strategies are applicable in the stabilization and control of **SVS** and how these control strategies can be implemented. Since the systems we have investigated are of matrix-second-order form, the control problem will be discussed in "mechanical" or physical coordinates.

The systems considered have the form

$$\mathbf{M}(t)\,\ddot{\mathbf{d}} + \mathbf{C}(t)\,\dot{\mathbf{d}} + \mathbf{K}(t)\,\mathbf{d} = \mathbf{f} \tag{4.117}$$

where $\mathbf{M}(t)$, $\mathbf{C}(t)$ and $\mathbf{K}(t)$ are the mass matrix, damping and stiffness matrices for the overall system, $\mathbf{d}$ is the displacement vector for the overall system, and $\mathbf{f}$ is the control force generated from $n(t)$ force actuators which can be further represented by

$$\mathbf{f} = \mathbf{B}(t)\mathbf{u} \tag{4.118}$$

where $\mathbf{f}$ denotes the control force, $\mathbf{u}$ denotes the $n(t)$ control inputs, one for each control device (actuator), and $\mathbf{B}(t)$ can be considered as actuator gain matrix. In general, its dimensions are determined by (1) the dimension of the system mass matrix $\mathbf{M}(t)$ and (2) the dimension of the control input $\mathbf{u}$, which is $n(t) \times 1$. It can be observed that since the dimension of the system mass matrix is not constant, the dimension of the control force could also vary over the time period of operation.

In order to feedback the position and velocity signals, we have to have sensor output

$$\mathbf{y} = \mathbf{C}_p(t)\mathbf{d} + \mathbf{C}_v(t)\dot{\mathbf{d}} \tag{4.119}$$

where $\mathbf{y}$ is the sensor output and $\mathbf{C}_p(t)$ and $\mathbf{C}_v(t)$ are position and velocity sensor gain matrices, which could include the mathematical model of the transducers and possibly the signal processing units used in measuring of the system response. If we assume the control strategy is modeled by $\mathbf{G}_c(t)$, the control input can be represented by

$$\mathbf{u} = -\mathbf{G}_c(t)\mathbf{y} = -\mathbf{G}_c(t)\mathbf{C}_p(t)\mathbf{d} - \mathbf{G}_c(t)\mathbf{C}_v(t)\dot{\mathbf{d}} \tag{4.120}$$

Hence, the control force can be written as

$$\begin{aligned} \mathbf{f} &= -\mathbf{B}(t)\mathbf{G}_c(t)\mathbf{y} \\ &= -\mathbf{B}(t)\mathbf{G}_c(t)\mathbf{C}_p(t)\mathbf{d} - \mathbf{B}(t)\mathbf{G}_c(t)\mathbf{C}_v(t)\dot{\mathbf{d}} \end{aligned} \tag{4.121}$$

Substituting equation (4.121) into equation (4.117) yields

$$\mathbf{M}(t)\,\ddot{\mathbf{d}} + \mathbf{C}(t)\,\dot{\mathbf{d}} + \mathbf{K}(t)\,\mathbf{d} = -\mathbf{B}(t)\mathbf{G}_c(t)\mathbf{C}_v(t)\dot{\mathbf{d}} - \mathbf{B}(t)\mathbf{G}_c(t)\mathbf{C}_p(t)\mathbf{d}$$

or simply

$$\mathbf{M}(t)\,\ddot{\mathbf{d}} + \mathbf{C}(t)\,\dot{\mathbf{d}} + \mathbf{K}(t)\,\mathbf{d} = -\Delta\mathbf{C}_{ctrl}(t)\dot{\mathbf{d}} - \Delta\mathbf{K}_{ctrl}(t)\mathbf{d} \tag{4.122}$$

where

$$\begin{aligned} \Delta\mathbf{K}_{ctrl}(t) &= \mathbf{B}(t)\mathbf{G}_c(t)\mathbf{C}_p(t) \\ \Delta\mathbf{C}_{ctrl}(t) &= \mathbf{B}(t)\mathbf{G}_c(t)\mathbf{C}_v(t) \end{aligned}$$

Usually, $\mathbf{B}(t)$ and $\mathbf{C}_p(t)$, $\mathbf{C}_v(t)$ are constant matrices for a particular configuration of the **SVS**. $\mathbf{G}_c(t)$ can then be designed to make $\Delta\mathbf{K}_{ctrl}(t)$ and $\Delta\mathbf{C}_{ctrl}(t)$ satisfy the stability requirement. $\Delta\mathbf{K}_{ctrl}(t)$ and $\Delta\mathbf{C}_{ctrl}(t)$ can be considered as *special perturbations* of the system. Rearranging equation (4.122), we have

$$\mathbf{M}(t)\,\ddot{\mathbf{d}} + [\mathbf{C}(t) + \Delta\mathbf{C}_{ctrl}(t)]\,\dot{\mathbf{d}} + [\mathbf{K}(t) + \Delta\mathbf{K}_{ctrl}(t)]\,\mathbf{d} = 0 \tag{4.123}$$

or

$$\mathbf{M}(t)\,\ddot{\mathbf{d}} + \mathbf{C}'(t)\,\dot{\mathbf{d}} + \mathbf{K}'(t)\,\mathbf{d} = 0 \tag{4.124}$$

where

$$\begin{aligned} \mathbf{C}'(t) &= \mathbf{C}(t) + \Delta\mathbf{C}_{ctrl}(t) \\ \mathbf{K}'(t) &= \mathbf{K}(t) + \Delta\mathbf{K}_{ctrl}(t) \end{aligned}$$

The implementation of control strategies can be discussed respectively in two cases.

- **Original System Contains No Controller**

  If the system expressed in equation (4.117) does not contain a controller, the control strategy can be designed and implemeted by going through the procedure decribed in this section from equation (4.118) to equation (4.124). Besides general engineering concerns, such as the place and the number of sensors, the place and the number of actuators, the symmetry of the system parameter matrices has to be carefully maintained in order to apply the previously developed approaches for the stability analysis of the system, which implies that $\Delta\mathbf{C}_{ctrl}(t)$ and $\Delta\mathbf{K}_{ctrl}(t)$ have to be symmetric. Since in general the **SVS** is a time-varying system, $\Delta\mathbf{C}_{ctrl}(t)$ and $\Delta\mathbf{K}_{ctrl}(t)$ should accommodate this feature, which means the controller for the **SVS** should have the ability to adapt to the ever-changing system dynamics. In particular, if the system order changes, it may be necessary to add (or remove) some of the sensors or actuators to (or from) the system. In general, not only the parameters of $\Delta\mathbf{C}_{ctrl}(t)$ and $\Delta\mathbf{K}_{ctrl}(t)$ have to adapt to the different system configurations, but also the orders of the $\Delta\mathbf{C}_{ctrl}(t)$ and $\Delta\mathbf{K}_{ctrl}(t)$ have to adapt as well. From the perturbation point of view, $\Delta\mathbf{C}_{ctrl}(t)$ and $\Delta\mathbf{K}_{ctrl}(t)$ can be considered as special perturbations superimposed on the structural perturbations so that the system dynamics are properly controlled.

- **Original System Contains A Controller**

If the system expressed in equation (4.117) contains a controller, the implementation of control strategies would be slightly different. It is known from chapter 3 that the parameter matrices of equation (4.117) can be further expressed as

$$\begin{aligned} \mathbf{M}(t) &= \mathbf{M}^o + \mathbf{M}^c(t) \\ \mathbf{C}(t) &= \mathbf{C}^o + \mathbf{C}^c(t) \\ \mathbf{K}(t) &= \mathbf{K}^o + \mathbf{K}^c(t) \end{aligned}$$

where $\mathbf{M}^o$, $\mathbf{C}^o$ and $\mathbf{K}^o$ are determined by subsystem parameter matrices. $\mathbf{M}^c(t)$, $\mathbf{C}^c(t)$ and $\mathbf{K}^c(t)$ are connection parameter matrices. Since it is assumed that the subsystems are time-invariant, only connection matrices can be modified to change the dynamics of the overall system. Therefore, the control strategies can only be implemented through connection matrices.

It is not difficult to obtain

$$\mathbf{M}^o\,\ddot{\mathbf{d}} + \mathbf{C}^o\,\dot{\mathbf{d}} + \mathbf{K}^o\,\mathbf{d} = -\mathbf{M}^c(t)\,\ddot{\mathbf{d}} - \mathbf{C}^c(t)\,\dot{\mathbf{d}} - \mathbf{K}^c(t)\,\mathbf{d} \tag{4.125}$$

For simplicity of illustration, we assume $\mathbf{M}^c(t) = 0$. Then

$$\mathbf{M}^o\,\ddot{\mathbf{d}} + \mathbf{C}^o\,\dot{\mathbf{d}} + \mathbf{K}^o\,\mathbf{d} = -\mathbf{C}^c(t)\,\dot{\mathbf{d}} - \mathbf{K}^c(t)\,\mathbf{d} \tag{4.126}$$

If we separate $\mathbf{C}^c(t)$ and $\mathbf{K}^c(t)$ as

$$\begin{aligned} \mathbf{C}^c(t) &= \mathbf{C}_{cs}(t) + \Delta\mathbf{C}_{ctrl}(t) \\ \mathbf{K}^c(t) &= \mathbf{K}_{cs}(t) + \Delta\mathbf{K}_{ctrl}(t) \end{aligned}$$

where $\mathbf{C}_{cs}(t)$, $\mathbf{K}_{cs}(t)$ model the system structural perturbations, and $\Delta\mathbf{C}_{ctrl}(t)$, $\Delta\mathbf{K}_{ctrl}(t)$ model the dynamics of the controller. Then, equation (4.126) can be rewritten as

$$\begin{aligned} & \mathbf{M}^o\,\ddot{\mathbf{d}} + [\mathbf{C}^o + \mathbf{C}_{cs}(t)]\,\dot{\mathbf{d}} + [\mathbf{K}^o + \mathbf{K}_{cs}(t)]\,\mathbf{d} \\ = \;& -\Delta\mathbf{C}_{ctrl}(t)\,\dot{\mathbf{d}} - \Delta\mathbf{K}_{ctrl}(t)\,\mathbf{d} \end{aligned}$$

or

$$\mathbf{M}^o\,\ddot{\mathbf{d}} + \mathbf{C}(t)\,\dot{\mathbf{d}} + \mathbf{K}(t)\,\mathbf{d} = -\Delta\mathbf{C}_{ctrl}(t)\,\dot{\mathbf{d}} - \Delta\mathbf{K}_{ctrl}(t)\,\mathbf{d} \tag{4.127}$$

Comparing equation (4.127) and equation (4.122), we see that both equations essentially have the same form. The design and implementation of $\Delta\mathbf{C}_{ctrl}(t)$ and $\Delta\mathbf{K}_{ctrl}(t)$ have been discussed before. It is not difficult to realize them through proper selection of sensors, actuators and control parameters by using the procedures decribed previously. However, the symmetry of $\Delta\mathbf{C}_{ctrl}(t)$ and $\Delta\mathbf{K}_{ctrl}(t)$ has to be maintained in order to use the previously developed approach to analyze the stability of the system.

## 4.8 Summary

The stability of order-varying **SVS** has been studied in this chapter by using the energy function as the evaluation function. Both process compatibility and motion compatibility have been studied. A number of criteria for evaluation of the dynamic stability of the order-varying **SVS** have been derived. The control strategy and the implementation of the control strategy have also be discussed for the **SVS**.

# Chapter 5

## Concluding Remarks

The stability theory of dynamic systems has been constantly evolving during the past two centuries. Various approaches have been developed for the stability analysis of different dynamic systems. These approaches are loosely related to each other and are usually applicable only in the stability analysis of particular kinds of dynamic systems. Although there has long been an effort to unify the stability theory for all branches of mechanics, significant results have rarely been achieved.

The proposed research is on the modeling and stability analysis of a special subset of time-varying dynamic systems, called structurally-varying systems or **SVS**. The main feature of the **SVS** is that it consists of a number of subsystems which are connected together through a group of time-varying constraints. The dynamic model and the stability condition of the system will usually change if the constraint condition changes. In real applications, it is always desirable to predict the change of the stability condition due to the variation of the constraint condition, or in other words, the structural perturbation. The real-time application sometimes even demands speed in the algorithm for the evaluation of the change of the stability condition.

In order to meet these requirements, new concepts of the stability have been designed and new approaches have been developed to analyze the stability of the **SVS**. We have used both the state response and the energy function as the evaluation function in the stability analysis of the **SVS**. The static and dynamic stability of the **SVS** have been thoroughly studied. The major contributions of the work can be summarized as follows:

- Based on the comprehensive study of various stability theories, new concepts of stability have been proposed for the special dynamic systems, which are called structurally-varying systems. The new concepts lay the foundation for the stability analysis of the **SVS**.

- The recursive state space model of the **SVS** has been developed so that the dynamic model for any system configuration can be derived in two ways, (1) using a model of its previous configuration and the current structural perturbation, (2) using the *unconstrained* subsystem models and constraint matrix which provides the most current system constraint information.

- A *qualitative* measure of the stability for the **SVS** has been established. A recursive estimation algorithm ($\gamma$-approach) has been developed for the evaluation of the stability of the **SVS**. By the $\gamma$ function, the stability of the fixed order **SVS** can be evaluated. In particular, the algorithm has the features of simplicity and recursiveness. Therefore it is appropriate for real-time applications.

- Thorough analysis has been carried out on the process of structural switching instants. Motion compatibility and process compatibility have been proposed and applied to the stability analysis of the **SVS**. Different dynamic performances have been revealed and their influences on the stability of the **SVS** have been investigated. A number of criteria for evaluating the stability of the **SVS** have been derived. The applications of the analytical results have been illustrated computationally and experimentally.

- Using both state response and system energy functions as the evaluation function, a new method for analyzing the stability of the **SVS** has been developed. Criteria based on two evaluation functions for predicting the static and dynamic stability of the **SVS** have been derived.

## Nomenclature

| Symbol | description |
|---|---|
| SVS | structurally-varying system |
| $m$, $\mathbf{M}$ | mass and mass matrix |
| $\mathbf{C}$ | damping matrix |
| $k$, $\mathbf{K}$ | stiffness and stiffness matrix |
| $\mathbf{d}(t)$ | displacement vector |
| $\mathbf{x}$ | state vector |
| $\mathbf{A}, \mathbf{B}$ | system matrix and input matrix |
| $\mathbf{\Phi}$ | state transition matrix |
| $\mu$ | matrix measure |
| $exp$ | exponential |
| $o$ | original |
| $c$ | constrained |
| $\Delta$ | change |
| $E, e$ | energy |
| $dim$ | dimension |
| $n$ | degree of freedom |
| $p$ | perturbed |
| $T$ | transposed |
| $v$ | velocity |
| $\mathbf{F}$, $\mathbf{f}$ | impulsive force vector |
| $int$ | internal |

| | |
|---|---|
| *bnd* | boundary |
| *new* | newly-created |

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# Appendix A

## Norms and Inner Product

Some mathematical definitions of norm and inner product are reviewed in this section. The materials come mainly from [Chen, 1988, DeCarlo, 1989]

**Definition 1:** Let $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in \mathbf{R}^n$. The norm of $\mathbf{x}$, which is a function: $R^n \rightarrow R$, can be defined by one of the following expressions:

$$\| \mathbf{x} \|_1 \triangleq \sum_{i=1}^{n} | \mathbf{x}_i |$$

$$\| \mathbf{x} \|_2 \triangleq (\sum_{i=1}^{n} | \mathbf{x}_i |^2)^{1/2}$$

$$\| \mathbf{x} \|_\infty \triangleq max | \mathbf{x}_i |$$

or in general,

$$\| \mathbf{x} \|_p \triangleq (\sum_{i=1}^{n} | \mathbf{x}_i |^p)^{1/p}$$

where $p$ ranges between 1 and $\infty$. In particular, the norm $\| \mathbf{x} \|_2$ is called the *Euclidean norm* or $l_2$ norm on $R^n$.

Each of the norms defined here has the following properties:

1. $\| \mathbf{x} \| \geq 0$ and $\| \mathbf{x} \| = 0$ iff $\mathbf{x} = \mathbf{0}$.

2. $\| \alpha \mathbf{x} \| = | \alpha | \| \mathbf{x} \|$ for all $\alpha \in \mathbf{R}$.

3. $\| \mathbf{x}_1 + \mathbf{x}_2 \| \leq \| \mathbf{x}_1 \| + \| \mathbf{x}_2 \|$.

These properties are easy to verify from the definition.

**Definition 2:** Let $\mathbf{A} = (a_{ij}) \in \mathbf{R}^{n\times n}$. The norm of $\mathbf{A}$ is defined as:

$$\begin{aligned} \| \mathbf{A} \| &\triangleq sup\{\frac{\| \mathbf{Ax} \|}{\| \mathbf{x} \|} : \ \| \mathbf{x} \| \neq 0\} \\ &= sup\{\| \mathbf{Ax} \| : \ \| \mathbf{x} \| = 1\} \end{aligned}$$

An immediate consequence of the definition of $\| \mathbf{A} \|$ is that for any $\mathbf{x} \in \mathbf{R}^n$,

$$\| \mathbf{Ax} \| \leq \| \mathbf{A} \| \cdot \| \mathbf{x} \|$$

Since the norm of $\| \mathbf{A} \|$ is defined through the norm of $\| \mathbf{x} \|$, it is called an *induced norm*. For different $\| \mathbf{x} \|$, we have different $\| \mathbf{A} \|$.

1. For $\| \mathbf{x} \|_1$: $\quad \| \mathbf{A} \|_1 = max(\sum_{i=1}^{n} | a_{ij} |), \quad j = 1, 2, \cdots.$

2. For $\| \mathbf{x} \|_2$: $\quad \| \mathbf{A} \|_2 = \{\lambda_{max}(\mathbf{A}^T\mathbf{A})\}^{1/2}.$

3. For $\| \mathbf{x} \|_\infty$: $\quad \| \mathbf{A} \|_\infty = max(\sum_{j=1}^{n} | a_{ij} |), \quad i = 1, 2, \cdots.$

The norm of a matrix has the following properties:

1. $\| \mathbf{A} \| = 0$ iff $\mathbf{A} = 0$.

2. $\| \alpha\mathbf{A} \| = | \alpha | \| \mathbf{A} \|$ for all $\alpha \in \mathbf{R}$.

3. $\| \mathbf{A}_1 + \mathbf{A}_2 \| \leq \| \mathbf{A}_1 \| + \| \mathbf{A}_2 \|$.

# Appendix B

## The Solution of $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$

For the state-space model of a system,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

we have the solution

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau) \cdot \mathbf{B}(\tau) \cdot \mathbf{u}(\tau) d\tau$$

where

1. $\mathbf{\Phi}(t, t_0)\mathbf{x}_0$ is the zero-input state response,
2. $\int_{t_0}^{t} \mathbf{\Phi}(t, \tau) \cdot \mathbf{B}(\tau) \cdot \mathbf{u}(\tau) d\tau$ is the zero-state response.

and in particular, in time-invariant cases,

$$\mathbf{\Phi}(t, t_0) = \mathbf{\Phi}(t - t_0, 0) = exp[\mathbf{A}(t - t_0)]$$

Hence, the complete solution for a time-invariant syetem is

$$\begin{aligned} \mathbf{x}(t) &= exp[\mathbf{A}(t - t_0)]\mathbf{x}_0 + \int_{t_0}^{t} exp[\mathbf{A}(t - \tau)] \cdot \mathbf{B} \cdot \mathbf{u}(\tau) d\tau \\ &= exp[\mathbf{A}(t - t_0)]\mathbf{x}_0 + exp(\mathbf{A}t) \int_{t_0}^{t} exp(-\mathbf{A}\tau) \cdot \mathbf{B} \cdot \mathbf{u}(\tau) d\tau \end{aligned}$$

# Appendix C

## Matrix Measure

**Definition**: Let $\|\cdot\|$ be an induced matrix norm on $\mathbf{R^{n\times n}}$. The matrix measure is defined as a function $\mu$: $\mathbf{R^{n\times n}} \rightarrow \mathbf{R}$

$$\mu[\mathbf{A}] = \lim_{\varepsilon\rightarrow 0^+} \frac{\| \mathbf{I} + \varepsilon\mathbf{A} \| - 1}{\varepsilon}$$

From a purely mathematical point of view, the measure $\mu[\mathbf{A}]$ of a matrix $\mathbf{A}$ can be thought of as the directional derivative of the norm function $\|\cdot\|$, as evaluated at $\mathbf{I}$ in the direction $\mathbf{A}$. The matrix measure has some useful properties, which are provided here.

1. $-\| \mathbf{A} \| \leq -\mu[-\mathbf{A}] \leq \mu[\mathbf{A}] \leq \| \mathbf{A} \|, \qquad \forall\, \mathbf{A} \in \mathbf{R}^{n\times n}$.

2. $\mu[\alpha\mathbf{A}] = \alpha\mu[\mathbf{A}], \qquad \forall\, \alpha \geq 0$ and $\forall\, \mathbf{A} \in \mathbf{R}^{n\times n}$.

3. $\max\, \{\mu[\mathbf{A}] - \mu[-\mathbf{B}], -\mu[-\mathbf{A}] + \mu[\mathbf{B}]\} \leq \mu[\mathbf{A}+\mathbf{B}] \leq \mu[\mathbf{A}] + \mu[\mathbf{B}]$.

4. $-\mu[-\mathbf{A}] \leq Re(\lambda_j) \leq \mu[\mathbf{A}]$ whenever $\lambda_j$ is an eigenvalue of $\mathbf{A}$.

The proof of the properties can be found in [Vidyasagar, 1978]. Using the matrix measure, we present a useful theorem.

**Theorem C1**: Consider the differential equation $\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x}$, $t \geq 0$, where $\mathbf{x} \in \mathbf{R^n}$, $\mathbf{A}(t) \in \mathbf{R^{n\times n}}$, $\mathbf{A}(\cdot)$ is piecewise-continuous. Let $\|\cdot\|$ be an norm on $\mathbf{R^n}$, and $\mu[\mathbf{A}]$ denote the the matrix measure on $\mathbf{R^{n\times n}}$. Then, whenever $t \geq t_0 \geq 0$, we have

$$\| \mathbf{x}(t_0) \|\; exp\{\int_{t_0}^{t} -\mu[-\mathbf{A}(\tau)]d\tau\} \leq \| \mathbf{x}(t) \| \leq \| \mathbf{x}(t_0) \|\; exp\{\int_{t_0}^{t} \mu[\mathbf{A}(\tau)]d\tau\}$$

and

$$exp\{\int_{t_0}^{t} -\mu[-\mathbf{A}(\tau)]d\tau\} \leq \| \mathbf{\Phi}(t,t_0) \| \leq exp\{\int_{t_0}^{t} \mu[\mathbf{A}(\tau)]d\tau\}$$

This theorem gives both upper and lower bounds of state variables $\mathbf{x}$ and state transition matrix $\mathbf{\Phi}(t,t_0)$. Proof of this theorem can be found in [Vidyasagar, 1978]. Using this theorem, the stability of structurally-varying systems can be studied based on the concept of matrix measure.

The calculation of the matrix measure on different norm is provided.

1. For $\| \mathbf{x} \|_\infty = \max | x_i |$, $\mu_\infty[\mathbf{A}] = max_i\{a_{ii} + \sum_{j \neq i} | a_{ij} |\}$.

2. For $\| \mathbf{x} \|_1 = \sum_{i=1}^{n} | x_i |$, $\mu_1[\mathbf{A}] = max_j\{a_{jj} + \sum_{i \neq j} | a_{ij} |\}$.

3. For $\| \mathbf{x} \|_2 = (\sum_{i=1}^{n} | x_i |^2)^{1/2}$, $\mu_2[\mathbf{A}] = \lambda_{max}\{(\mathbf{A}^* + \mathbf{A})/2\}$.

# Appendix D

## Properties of Symmetric Matrices

**Theorem D.1:** If an $n \times n$ matrix $\mathbf{A}$ is real and symmetric, its eigenvalues are all real.

**Theorem D.2:** If an $n \times n$ matrix $\mathbf{A}$ is real and symmetric, then all its eigenvalues are

1. positive if $\mathbf{A}$ is positive definite.
2. nonnegative if $\mathbf{A}$ is positive semidefinite.
3. negative if $\mathbf{A}$ is negative definite.
4. nonpositive if $\mathbf{A}$ is negative semidefinite.

The proof of these theorem can be found in [Orteg, 1987].