ON OPTIMIZATION
IN
PROBABILISTIC DESIGN
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ABSTRACT

In classical design the inputs to the design process, namely the relevant material property M and the load L are taken as deterministic values. The probabilistic design approach recognizes the variation in design inputs, and the consequent random behavior of these variables. The material property varies from one lot of production to another and within the same production lot. The applied load to a particular specimen varies widely within some range. The design output, a dimensional parameter A, also varies within the given range of tolerances, and therefore is randomly distributed among specimens. Failure occurs at the first instance the load value L is larger than the resisting strength of the material, S. Among n load applications the first exceedance of S by L results in failure. The relevant load variable is therefore the extreme value of a number n of loads that correspond to a given design mission time. Within this framework, design reliability and mission time emerge as the appropriate design inputs.

In probabilistic design the designer has a wide range of choice for both input parameters, reliability R and mission time n. In this Thesis, the optimal combination of n and R is determined in a logical way. A cost-benefit analysis is made, resulting in an optimal combination using the benefit and cost to the decision maker.
Since benefit is relative, the information required to determine the benefit function is acquired from the decision maker through several questions and a "reference gamble". Cost is analyzed and a cost function is constructed. Using the benefit and cost functions, indifference and constant cost functions are derived, resulting in suboptimal combinations of R and n. The optimal combination is chosen among the suboptimal combinations by minimizing the cost-benefit ratio. An example is presented to illustrate this decision model.
# TABLE OF CONTENTS

## I. INTRODUCTION

<table>
<thead>
<tr>
<th>Section</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1. General</td>
<td>1</td>
</tr>
<tr>
<td>2. Load L, and Material Property M</td>
<td>2</td>
</tr>
<tr>
<td>3. Probabilistic Design</td>
<td>7</td>
</tr>
<tr>
<td>B. Literature Survey</td>
<td>11</td>
</tr>
<tr>
<td>C. Present Research Problem</td>
<td>14</td>
</tr>
</tbody>
</table>

## II. BENEFIT

<table>
<thead>
<tr>
<th>Section</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Definition of Benefit</td>
<td>15</td>
</tr>
<tr>
<td>1. Definition</td>
<td>15</td>
</tr>
<tr>
<td>2. Considerations in Assessing Benefit</td>
<td>16</td>
</tr>
<tr>
<td>3. Preference Functions</td>
<td>18</td>
</tr>
<tr>
<td>B. Benefit of Mission Time at a Given Value of Reliability</td>
<td>20</td>
</tr>
<tr>
<td>1. General</td>
<td>20</td>
</tr>
<tr>
<td>2. Non-decreasing Preference Functions</td>
<td>22</td>
</tr>
<tr>
<td>3. Non-increasing Slope of Preference Function</td>
<td>25</td>
</tr>
</tbody>
</table>
C. Benefit of Reliability at a Given Value of Mission Time. .......................... 30
1. General ............................................. 30
2. Non-decreasing Preference Function ...... 32
3. Non-increasing Slope of Preference Function ........................................ 35

D. Benefit of Both Reliability and Mission Time. 41
1. Interdependence of Preference Functions.. 41
2. Benefit of Reliability and Mission Time, Indifference Functions ............... 42

E. Derivation of Preference Functions ............ 44
1. Information Required to Construct Preference Functions ....................... 44
2. Questions to the Decision Maker .................. 45
3. Reference Gamble .................................. 52
4. Construction of Preference Functions ...... 57
5. Construction of Indifference Functions ... 60

III. COST ................................................. 63

A. Material Cost ............................................. 65
1. Dependence on Parameters of Material p.d.f.............................................. 65
2. Dependence on Size Parameter .................. 69
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>A Typical Probability Distribution Function for Material Property M</td>
<td>4</td>
</tr>
<tr>
<td>1.2</td>
<td>A Typical Probability Distribution Function for Load L</td>
<td>6</td>
</tr>
<tr>
<td>1.3</td>
<td>P.d.f. of Load, Extreme Value of Load, and Material Strength</td>
<td>9</td>
</tr>
<tr>
<td>2.1</td>
<td>The Preference Function for Mission Time at Constant Reliability R' u(nR')</td>
<td>29</td>
</tr>
<tr>
<td>2.2</td>
<td>The Preference Function for Reliability at Constant Mission Time n u(Rn)</td>
<td>40</td>
</tr>
<tr>
<td>2.3</td>
<td>Indifference Functions for Reliability and Mission Time n(Rn=C)</td>
<td>43</td>
</tr>
<tr>
<td>2.4</td>
<td>Location of Initial and End Points of Preference Function for Reliability at Minimum Mission Time</td>
<td>47</td>
</tr>
<tr>
<td>2.5</td>
<td>Location of Initial Points of Preference Functions for Reliability at n_2</td>
<td>50</td>
</tr>
<tr>
<td>2.6</td>
<td>Location of Initial and End Points of Preference Function for Reliability at Mission Time Values, n_2</td>
<td>51</td>
</tr>
<tr>
<td>2.7</td>
<td>Location of these Points for Each Preference Function for Reliability at n_1</td>
<td>55</td>
</tr>
<tr>
<td>2.8</td>
<td>Preference Function for Different Values of R'_1</td>
<td>59</td>
</tr>
<tr>
<td>2.9</td>
<td>Preference Functions u(Rn_1) for Reliability at Constant Mission Time</td>
<td>62</td>
</tr>
<tr>
<td>2.10</td>
<td>Indifference Functions n(R</td>
<td>u=C)</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>3.1</td>
<td>Unit-Material Cost as a Function of $u_1$</td>
<td>67</td>
</tr>
<tr>
<td>3.2</td>
<td>Unit-Material Cost as a Function of $u_2$</td>
<td>68</td>
</tr>
<tr>
<td>3.3</td>
<td>Cost of Failure as Function of R for Specimens with Catastrophic Failure</td>
<td>72</td>
</tr>
<tr>
<td>3.4</td>
<td>Benefit-Cost Analysis for Mission Time</td>
<td>77</td>
</tr>
<tr>
<td>3.5</td>
<td>Cost of Mission Time at Constant Reliability $C(R, n_R = n)$</td>
<td>82</td>
</tr>
<tr>
<td>3.6</td>
<td>Constant Cost Functions</td>
<td>84</td>
</tr>
<tr>
<td>4.1</td>
<td>Locus of Sub-optimal Points</td>
<td>87</td>
</tr>
<tr>
<td>A.1</td>
<td>Preference Functions, $u(R</td>
<td>n_i)$, for R at constant $n_i$</td>
</tr>
<tr>
<td>A.2</td>
<td>Indifference Functions, $n(R</td>
<td>u)$</td>
</tr>
<tr>
<td>A.3</td>
<td>Constant Material Cost Functions</td>
<td>107</td>
</tr>
<tr>
<td>A.4</td>
<td>Locus of Suboptimal Points</td>
<td>110</td>
</tr>
</tbody>
</table>
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NOMENCLATURE

A : Design Parameter
C : Cost
C_f : Cost of Failure
C_m : Material Cost
C_r : Operation Cost
F : Probability Distribution Function
F_{\alpha, \text{max}} : Extreme Value Asymptote
L : Load
M : Material Property
n : Number of Load Applications, Mission Time
n(R|C) : Constant Cost Function
n(R/\mu) : Indifference Function
P : Price
P(F) : Probability of Failure
\Gamma : Net Resources at Completion
R : Reliability
R' : Certainty Equivalent Reliability
S : Material Strength
t : Thickness
T : Mission Length
u : Utility
W : Net Resources at Start
\Phi_{(n)} : Exact Extreme Value Distribution
\sigma^2 : Variance
\[ \mu \] : Benefit
\[ \mu_1 \] : Expected Value, Mean
\[ \mu_2 \] : Variance
\[ 2 \] : Safety Factor
<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
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</tr>
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<td>An Indifference Function is a trade-off function between reliability and mission time in terms of benefit. It is a locus of all combinations of R and n that have the same given constant benefit.</td>
</tr>
</tbody>
</table>
Mission Time : Mission Time, \( n \), of the design specimen is the time length, during which specimen is subjected to applied loads. \( n \) may be expressed in terms of time units on number of applied loads.

Reliability : Reliability is the probability of survival of design specimens by the end of the mission time. It is complimentary to the probability of failure.
1A. INTRODUCTION

1. General

In a design process, the physical details of a specimen are determined such that a given function is performed by the specimen under given conditions. The inputs to a design process are a) the relevant material property of the material, b) the load which is applied to design specimen and c) the chosen failure criterion.

Under the conventional design approach the inputs to the design process are taken as fixed values. A single representative value $L'$ is chosen for the load. The handbook value of the material property, $M'$, is used to represent the material property. A safety factor $v$ is introduced to take into account unforeseen contingencies. The specimen designed is meant to sustain all future loads, although even designs with a very high safety factor $v$ do fail occasionally. Failure of the design means that the load value $L$ was larger than the relevant strength value on the occasion the failure. The conventional design procedure of a specimen is based on the relation.

$$S = vL$$  \hspace{1cm} (1.1)

Where $S = \text{strength of the material}$

$L = \text{load value}$.

$v = \text{safety factor}$.
S is a function of the material property M, and relevant design dimension, A. A particular value of v is selected on the basis of engineering experience, judgement and knowledge of similar designs. By choosing a sufficiently high safety factor, the probability of failure is assumed to be eliminated. However, in practice even the most conservative designs do fail. Furthermore, the safety factor does not predict the performance of the design specimen; that is, the safety factor does not provide any information as to the likelihood of failure and the likely time length the design would operate prior to failure.

2. Load L, and Material Property M

If several similar specimens of a particular material are tested for a specific material property, it is likely that different property values are obtained. Material properties of a given material vary from one production lot to another, and within a particular production lot. The handbook value is usually a central value that represents the whole range of values. Material properties of a sample of specimens are randomly distributed. That is, material property is a random variable among specimens.

If measurements are taken of the load on similar design specimens under given operating conditions, the results are likely to fall in a wide range of values. A histogram of applied loads on a particular specimen often shows that load values occur randomly, with larger load
values occurring less frequently than smaller load values. But these large load values are the ones that cause failure and are therefore of importance.

As both the material property $M$ and load $L$ should be recognized as random variables, due to inherent variabilities, these variables are represented by some mathematical function, called the probability density function, $(p.d.f.)$. (1)

The probability density function, $f(x; \theta)$ represents the magnitude of $x$ in terms of its frequency of occurrence, where $\theta$ is the parameter value of the statistical model (2).

Each property of each specific material gives rise to a specific family of density functions. The control over the production process, (control of temperature, homogeneity, and percentage of alloying elements, etc.) determines the observed variation in the material property and thus determines the function $f$ and its parameter value, $\theta$. For load values, the nature of the p.d.f. and parameter value $\theta$ are determined by the operating conditions imposed on the specimen.

The handbook value of a material property $M$, is usually the mean value of the associated p.d.f. Data on material property show that the p.d.f. of $M$ is usually skewed to the right, since quality control results in elimination of variance in low values of $M$, while high value of $M$ often go unchecked. The lognormal, Gamma, and Weibull distributions are commonly used to represent the p.d.f. of $M$. (3) A typical p.d.f. for $M$ is shown in Fig. 1.1.
Fig. 1.1. A Typical Probability Density Function for Material Property M.
Most load values occur around a nominal value L', which is used for conventional design procedures while relatively high load values occur less frequently. Hence the p.d.f. of load L, is usually skewed to the right. A typical p.d.f. for L is shown in fig. 1.2. Failure will occur when a load value exceeds the resisting strength of the design specimen. Since the failure occurs at the first exceedance of S by L, the interest lies not in what portion of load values exceeds S, but in the number of load applications prior to the first exceedance of S. The probability that the strength S will not be exceeded in n consecutive load applications is given by the exact extreme value p.d.f. of the initial load distribution,

\[ f(n)(S) = \left[ F(L)|_{L=S} \right]^n \] (1.2)

To represent the p.d.f. of L, f(L), Lognormal, Gamma and Weibull models are employed. The extreme value asymptote, \( F_{I,\text{max}}(L) \) is constructed from the initial model of L, for a given number of contemplated loads, n(2).

The extreme value p.d.f. \( F_I(L) \) is shown in fig. 1.3.

The mission length, T, of the design specimen is the time length, during which the specimen is subjected to applied loads. T is expressed in time units. For many design specimens, the number of loads applied per unit time is a constant. If \( \Delta n \) is the number of loads applied per unit time,

\[ n = T \cdot \Delta n \] (1.3)
Fig. 1.2. A Typical Probability Density Function for Load $L$. 
Hence \( n \) is the number of applied loads to which the design is subjected during its mission time. In the rest of the discussion, \( n \) is used to represent the mission time and it may be easily converted to time units by dividing by \( \Delta n \).

3. **Probabilistic Design**

When the random behavior of a relevant material property \( M \) and of the load \( L \) are recognized, the conventional approach of designing with deterministic values of \( M' \) and \( L' \) is replaced by designing with the p.d.f's of the random variables \( M \) and \( L \).

The resisting strength of the material, \( S \), is a function of \( M \) and a critical design dimension, \( A \). In the case of a simple tension link for which the failure criterion is rupture,

\[ S = MA, \quad (1.4) \]

Where \( A \) is the area of the critical crosssection.

Failure occurs when \( S \) is exceeded by the load:

\[ L > S \]

The probability of failure, during the mission time \( n \) is

\[ P(n) = P(L > S) \quad (1.5) \]

The reliability of the design specimen is defined as the complimentary probability.

\[ R(n) = 1 - P(L > S) \quad (1.6) \]
The actual dimensions of a design element vary among specimens. Quality control results in variations being between given limits on the average. But as long as each element is not inspected, the design dimension, \( A \), is an unknown and must be considered a random variable. \(^{(2)}\)

Hence \( S \) is a product of two random variables and is therefore itself a random variable. The p.d.f. of the r.v. \( S \) is shown in fig. 1.3.

Under the probabilistic design approach, the reliability \( R \) and mission time \( n \) are chosen by the decision maker. Using eq. \((1.6)\),

\[
R(n) = 1 - P(L > S),
\]

the dimension of the design element is calculated, see ref (2).

The safety factor \( v \),

\[
v = \frac{S}{L},
\]

is the quotient of two random variables, and is therefore itself a random variable.

The reliability, \( R(n) \), may be defined in terms of \( v \); failure occurs when \( v < 1 \), and hence the probability that failure occurs during a mission time \( n \) is:

\[
P(n) = P(v < 1),
\]

\[
R(n) = 1 - P(v < 1).
\]

In fig. 1.3. a typical p.d.f. of \( L \), \( f(L) \) is shown. A corresponding extreme value model \( f_\text{L}(L) = P(f(L), n) \) derived from the initial model \( f(L) \) is also illustrated. A strength p.d.f. \( f(S) \) is presented in fig. 1.3, and the
Fig. 1.3. P.d.f. of Load, Extreme Value of Load, and Material Strength.
probability of failure is indicated qualitatively.

An example of obtaining p.d.f.'s for M and L and calculating R(n) from these p.d.f.'s is illustrated in Appendix I, within the framework of illustrating the decision analysis.
I. B. LITERATURE SURVEY

A study published by Freudenthal (5) in 1947 introduced the random behavior of material strength and load, and the concept of reliability to the design process. This work was followed by numerous discussions and studies in the literature by Mittenbergs, (6),(7), Weibull (8), and many others. An extensive survey of the literature pertaining to random behavior of material strength and the concept of probabilistic design is presented by Agrawal (9) in his thesis.

On the optimization of the probabilistic design process, that is on searching for optimal values of the design parameter, reliability R and mission time n, however, there is very little available literature. Freund(10) introduces risk into a programming model. Utility is expressed as

\[ u(R) = 1 - e^{-aR} \]

where

- \( R \) = net revenue of project
- \( a \) = constant cost factor.

Freund shows that if \( R \) is normally distributed, utility is optimized by maximizing

\[ E(R) - a\sigma^2/2, \]

where \( E(R) \) is the expected value of \( R \) and \( \sigma^2 \) is its variance.

This approach may prove to be useful when utility can be
expressed as above. But the condition that net revenue, \( R \), is to be of normal distribution makes the approach very unpractical, since it is very unlikely that \( R \), which is a function of non-normally distributed variables, will be a normally distributed variable.

Weisman & Holzman (11) argue that the utility function can be considered concave everywhere and using Freund's function, they define

\[
    r = W + P - C,
\]

where \( r \) = net resources at completion
\( W \) = net resources at start
\( P \) = Price or Revenue
\( C \) = Cost.

Hence the utility function becomes

\[
    \mu(r) = 1 - e^{-ar}.
\]

Considering price as deterministic,

\[
    \mu(r) = 1 - e^{-a(w+P)} \int e^{ac} f(c) \, dc.
\]

They show that if the cost model is unimodal and symmetric, the utility is maximized by

minimizing

\[
    E(c) + a\sigma^2 / 2
\]

When the cost function is not unimodal they give an upper bound for the objective function.

Singh and Kumar (12) discuss system reliability with pay-offs. Pay-offs are introduced to weigh the risk. The formulation given takes into account systems of two or more components. A loss matrix is assumed and used, but no study of getting the loss matrix is indicated.
In the field of optimization there is a number of excellent studies in the literature. Fishburn (13) gives a general theory of subjective probabilities and expected utilities. Debreu (14) presents a representation of preference ordering by a numerical function. Rader (15) discusses the existence of a utility function to represent preferences. Suppes (16) has an excellent study on the role of subjective probability and utility function in decision making.

Utility functions for multi-attributed consequences and independence of these functions is presented in articles by Keeney (17, 18). A methodical approach of estimating additive utilities is presented by Fishburn (19).

The statistical decision models are presented thoroughly by Schlaiffer (20 to 22) and by Pratt, Raifa, and Schlaiffer (23, 24). Other works in the area include those by Weiss (25), DeGroot (26), and Myron (27).
I. C. PRESENT RESEARCH PROBLEM

In probabilistic design, the designer can choose the reliability $R$ and mission time $n$. He has a wide range of choice for both input parameters, and the values of $R$ and $n$ determine the value of the dimensional parameter $A$.

The purpose of this study is to construct a rational decision process that provides the most optimal combination of $R$ and $n$ for the decision maker. A rational decision maker would want to maximize the benefit he derives from the design, while minimizing the cost of the design. So the problem is defined as

Minimize the cost-benefit ratio of the design subject to

- Minimum reliability,
- Maximum reliability,
- Minimum mission time,

It is therefore seen that in the problem as stated, the design parameters, $R$ and $n$, become the relevant decision variables.
II. BENEFIT

IIA. DEFINITION OF BENEFIT.

1. Definition

Benefit is the amount of satisfaction derived by a decision maker, resulting from a state of values of a variable (or variables), relative to some standard amount of satisfaction. Satisfaction results from a number of factors affected by the variable considered; these factors are discussed in section IIA2.

Benefit is an overall term that incorporates the satisfaction of output values without any reference to input values. In a design process, benefit is the satisfaction derived from the state of values of mission time and of the design without reference to the cost of design and other inputs to the design process. Cost factors are considered separately in chapter III.

Benefit is a relative term. Some factors of benefit can be expressed in absolute terms. For example, total sales revenue can be expressed in terms of monetary units. Benefit as a whole composed of all the factors discussed in section IIA2, cannot be expressed in absolute terms. Hence, benefit is relative to some norm and expressed as a ratio. The norm, or standard, is usually an easily defined state of output factors.
Benefit derived from a state is subjective. That is, it is different for different decision makers and at different states of outside factors. It is not necessarily the same for all decision makers but general trends can be analyzed. Sections IIB and IIC analyse these general trends for benefit derived from design reliability and design mission time.

2. Considerations in Assessing Benefit

A decision maker, in assessing the benefit derived from a certain state of an output, must take the factors affected by that output into consideration. The satisfaction derived from the output depends on the relative effect of the output on these factors and relative states and importance of these factors.

The considerations in assessing the benefit derived from a probabilistic design process in which the outputs are reliability \( R \) and mission time \( n \) can be listed as follows:

   a) **Sales revenue**

   Increased reliability and/or mission time may allow an increase in sales price or sales volume. The increase in sales revenue resulting from the ability to ask higher prices or the ability to sell a larger volume is obviously of benefit to the decision maker and his organization.
b) **Reputation**

The reputation of the decision maker and his organization is affected by the reliability and the mission time of the design. The higher these values are, the higher is the reputation for the decision maker. This implies present and future benefits for the decision maker, such as stronger market position, higher prices, and favourable consumer bias.

c) **Objectives**

The objectives of the decision maker are affected by the output values. A longer mission time is of more benefit to a decision maker who wants the design perform for a long time, while it is of less benefit to a decision maker who wants the design replaced in the near future.

d) **Marketing Considerations**

The marketing strategy of the decision maker should be considered in assessing the benefit. The market, in which the decision maker desires to operate (for example, high quality, intermediate quality, or low quality markets,) the desired market share, side product markets, the
maintanence market, all affect the marketing strategy. The benefit depends on the marketing strategy formulated by the decision maker.

e) Obsolescence

Development in technology, changes in customer tastes, and other outside factors may bring the design into obsolescence, these factors should be considered in assessing the benefit. A design designated to operate after it is obsolete, is over-designed.

In considering the above factors to assess the relative benefits, the decision maker makes use of all pertinent information and utilizes his experience.

How these considerations are utilized when determining a decision maker's benefit is illustrated in Sections IIB and IIC.

3. Preference functions

A decision maker will prefer a state of output variable with a higher benefit to a state of output variables with a lower benefit. In the case where benefit increases as the value of an output variable increases within an interval of values for the variable, in that interval the decision maker will prefer a higher value of the variable to a lower value of the variable. Benefit is a function of the output
variable and this function is called a preference function. In probabilistic design, the output is the reliability and mission time, and benefit is a function of both. (see Sections II.B. and II.C.) Benefit as a function of reliability, when mission time is fixed, forms the preference function for reliability at the given mission time. Benefit as a function of mission time at a fixed reliability value, forms the preference function for mission time at the given reliability value.
II.B. Benefit of Mission Time at a Given Value of Reliability

1. General

The preference function for mission time of a constant reliability, \( \mu(n|R) \), is analyzed in the interval between minimum and maximum mission time values. Minimum mission time, \( n_{\text{min}} \), is defined as the lowest beneficial value assessed by the decision maker from experience and pertinent information. Maximum mission time, \( n_{\text{max}} \), is defined as that value after which an increase in mission time does not result in an increase in the benefit or results in a decrease in the benefit.

In the interval between minimum and maximum mission times, \( (n_{\text{min}}, n_{\text{max}}) \), the preference function for mission time at given reliability, \( \mu(n|R) \), is a non-decreasing function. Benefit increases as the mission time increases, and maximum mission time, by definition, is the point where benefit is maximum. For discussion of the above statement, see Section II.B:2.

As mission time increases the marginal increase in the preference function, \( \mu(n) \), for unit increase in mission time declines. That is, the marginal benefit derived by the decision maker declines as mission time increases. The preference function has a non-increasing slope reaching zero at maximum mission time.
The above statements are discussed and illustrated with an example in the following sections.

The purpose of the decision process is to minimize the cost-benefit ratio, and for this reason mission time values larger than maximum mission time, $n_{\text{max}}$, which result in increased cost but decreased benefit are not considered in the analysis.
2. **Non-decreasing Preference Function.**

As stated in the previous section, the preference function for mission time at given reliability, \( \mu(n|\bar{R}) \), is non-decreasing. This is due to following factors:

a) The longer is the mission time, the higher is the price that can be charged and the larger is the revenue from the design. In the case the decision maker is the designer-user of the design, longer usage of the design implies higher revenue per specimen, since there are fewer replacements of the design. In the case the decision maker is the designer-seller, larger prices and/or more sales result in higher revenue. Since an increase occurs in revenue this is an increase in the benefit. For example, if the design is a machine tool, longer usage or higher prices increases the revenue of the decision maker. A machine tool used by the decision maker is in use a longer time if the mission time is longer and one sold by the decision maker yields higher prices and/or larger sales.
b) A design with a longer mission time has a higher reputation value for the decision maker. It enjoys the reputation of durability. This is of benefit to the decision maker, because it may prompt buyers to regard all designs by the decision maker as durable, including the design in question. For example, a decision maker who designs a machine tool with a longer mission time has the reputation of durable design and this will be of benefit in present and in future sales. New designs by that decision maker will be easier to introduce to the market, and are likely able to command higher prices.

c) Market share for the design may increase as it is longer-lasting. Among the designs for the same purpose, one with a longer mission time may be preferred by the buyer and the market share may increase, giving the decision maker the opportunity to expand and/or operate at more beneficial price levels. This is an increase in the benefit for the decision maker. For example, a machine tool with a higher mission time will be chosen more often by the buyers and its share in the market is likely
to increase. This increase in market share will enable the decision maker to consider expanding its production line, and enable him to increase his volume to a more optimal value.

These factors cause the benefit to increase as the mission time increases, but their effects are not constant. That is, benefit does not increase at a constant rate as mission time increases. This is discussed in the next section.
3. **Non-increasing Slope of Preference Function.**

As stated in section II.B.1, as the mission time increases the marginal increase in the preference function for mission time at given reliability, \( \mu(n/R) \), for unit increase in mission time \( n \) declines and approaches zero at the maximum mission time \( n_{\text{max}} \). That is, it has a non-increasing slope.

The marginal increase in benefit declines because of the factors listed below:

a) Higher mission time values bring the design closer to potential obsolescence near the end of its life. That is, the probability that the design will be obsolete prior to completion of its mission time increases. The advancement of technology and change in consumer-tastes and industry-needs and other outside factors cause designs to become obsolete. When a design becomes obsolete, the remaining life **time** is of little value. Buyers do not want to commit themselves to a design that has a high probability of becoming obsolete before the completion of its mission time. This factor negates the second factor in Section II.B.2, that longer mission time results in higher reputation since obsolete designs will result in reducing the positive
affect that longer life time has on reputation. In the case a machine tool is the design considered, advancement in technology (better design concepts), change in industry needs (shift in demand), and other expected and unexpected outside factors (such as energy shortage) may cause a machine tool to become obsolete within a certain life span (product life). A machine tool built to last longer has less marginal benefit as the mission time increases.

b) As the mission time increases the total market for the design during some extended time period shrinks and the demand for new items declines because the items already in use last longer. When designs have a high mission time, failure occurs less frequently and the necessity of replacement decreases. Even though, as stated in Factor C in section II B2., the market share for the design increases, the total market shrinks and the actual market volume for the design does not increase as much. Hence, the beneficial affect of increasing market share decreases as the mission time increases.

In the machine tool industry, for example, suppose that a manufacturer designs for a higher mission time than others do. The market
share increases, but since the items in use last longer, the total demand in the long run decreases. Hence, the total volume does not increase as much as the increase in the market share.

c) Longer mission times and associated longer revenue may tend to price the design out of its market. The buyer, even if it may be more beneficial in the long run, may not be willing or able to give the higher price for the design with the higher mission time. Hence, as mission time increases and price increases in accordance, the buyer group of the design gets smaller. Although the revenue for each design increases, as less items are sold, the total revenue will not increase as much, and if mission time is very high total revenue may, in fact, decline.

This factor lessens the effect of the factor $A$ in Section IIB2, that the higher mission time results in higher revenue for the decision maker.

In the machine tool industry, for example, the designs with higher mission time may have smaller buyer groups. The decision maker designing for a higher mission time may not receive as much revenue as those designing for lower mission time. Even though the
unit revenue (price per design specimen) is higher, sales volume may be lower, and may result in a lower total revenue.

In conclusion, the preference function for mission time at constant reliability, $\mu(n|R)$, is a non-decreasing function with a non-increasing slope. When the slope reaches zero, the corresponding mission time is defined as the maximum mission time, $n_{\text{max}}$. The function $\mu(n|R)$ is as shown in Fig. 2.1 in the interval between minimum and maximum mission time values, $(n_{\text{min}}, n_{\text{max}})$. 
Fig. 2.1. The Preference Function for Mission
Time at Constant Reliability \( R', \mu(n|R') \)
II. C. Benefit of Reliability at a Given Value of Mission Time

1. General

The preference function for reliability at a given value of mission time, $\mu(R|n)$, is a non-decreasing function in the range between minimum and maximum reliability, $(R_{\min}, R_{\max})$. Minimum reliability, $R_{\min}$, is assessed by the decision maker as the lowest value acceptable in the industry. A reliability value of one is not attainable in practice. Maximum reliability is defined as that value (less than one) that is technologically and/or economically attainable.

As the value of reliability increases, the benefit to the decision maker increases. In the subsequent analysis, the reliability values between minimum and maximum reliability are considered only, since the values below the former are not acceptable, and values above the latter are not attainable.

As the reliability increases, the marginal increase in the preference function for reliability at constant mission time, $\mu(R|n)$, for a unit increase in reliability declines. Hence, $\mu(R|n)$ features a non-increasing slope. Since the maximum reliability $R_{\max}$ is defined as the largest reliability value which is technologically and/or economically feasible, the slope of the preference function for constant mission time, must be zero at $R_{\max}$, since no further input of technological and other resources could possibly
result in a reliability increase.

Therefore, the preference function for reliability at a constant mission time, $\mu(R|n)$ is a concave function in the range between minimum and maximum reliability. The reason for the above statements are presented in Section II.C.2 and II.C.3.
2. **Non-decreasing Preference Function.**

The preference function for reliability at a given mission time, \( u(R|n) \), derived by the decision maker is a non-decreasing function. That is, the higher the reliability is, larger is the benefit the design provides to the decision maker.

The following factors comprise the reasoning behind the above statement.

a) The higher the probability that a design does not fail before the completion of the mission time, the higher is the revenue derived from the design. Suppose there exist two samples of 100 design items with reliability \( 1-F' \) and \( 1-2F' \), respectively. Before the completion of mission time, on the average \( F' \) items fail in the former case while \( 2F' \) items fail in the latter case. In the case of items with higher reliability, \( (1-F') \), the designer-user has more items in operation and needs fewer replacements, so that he derives higher revenue than with the items with lower reliability, \( (1-2F') \). A designer-seller is able to ask higher prices (revenue per item) for the items with higher reliability, due to obvious higher revenue for the
user, than he is able to ask for the lower reliability items.
For example, if the design item is a machine tool, the higher reliability items will return higher revenues through higher prices or through a higher percentage completing the mission time resulting in fewer replacements and/or repairs. The revenue of the decision maker will increase as the reliability of the item increases.

b) An increase in the reliability of the design results in a higher market share for the design item. A buyer typically prefers a more reliable item to one with lower reliability at the same mission time. All other things being equal, a higher reliability item sells better and therefore obtains a higher share of the market. This trend may reverse after a certain value of reliability. This point is discussed in Section II.C.3.
For example, a machine tool with a higher reliability will be chosen more often by buyers than a machine tool with a lower reliability. A customer will be inclined to prefer a machine tool that has a lower probability to fail (higher reliability) in its given mission time to one that has a higher probability to fail (lower reliability).
c) A design with higher reliability has a safer operation, and for critical designs, this feature is of considerable benefit. An increase in reliability decreases the probability of an undesired shutdown, work accidents, and unplanned and/or unwanted delays. An increase in reliability makes the operation of design items safer and more dependable, and so increases the benefit to a decision maker. A safer operation results from a machine with high reliability, as an example. Since there is a lower probability of failure during its mission time, the machine tool with higher reliability has lower expected loss in revenue resulting from delays and shutdowns.

The above statements demonstrate that an increase in reliability causes the benefit of the design to decision maker to increase. Since there is an increase in benefit with an increase in reliability, at a given mission time, a higher reliability design is preferred to a lower reliability item by the decision maker. Hence, the preference function of reliability at a constant mission time is a non-decreasing function.

The increase in reliability does not imply an increase in benefit at a constant rate. As the reliability approaches its maximum value, the increase in benefit declines. This point is discussed in Section II.C.3.
3. **Non-increasing Slope of Preference Function**

With increasing reliability values, the marginal increase in preference function for reliability at a given mission time, \( \mu(R|n) \), per unit increase in reliability \( R \), decreases. That is, \( \mu(R|n) \) has a non-increasing slope. The factors that cause a decline in the marginal increase of \( \mu(R|n) \) are discussed below.

a) A higher reliability implies that more of the design items will not have failed at the completion of the mission time. This results in less frequent replacements and therefore a shrinkage for the demand of the design item. Although the design item with higher reliability may cause more buyers to prefer it to a lower reliability item (causing the market share to increase for the higher reliability item), the total market shrinks due to a decrease in demand caused by fewer failures. Hence the increase in revenue may not be as much as anticipated otherwise. As the reliability gets closer to its maximum value the increase in market share is offset to some extent by shrinkage in the market demand.
When a machine tool is considered as an example, a higher reliability tool may capture a higher share of the market, but as the reliability increases, the total market demand decreases, because there is less need to replace the existing machine tools. The decrease in market demand causes the increase in revenue to decline.

b) A higher reliability value, with higher price (revenue per item) may tend to price the design item out of the market. This may cause a decline in the market share, and an accompanying decline in the total revenue. Increasing reliability increases the market share, but the price of the item also increases. When the price goes beyond a certain limit for each buyer, the buyer chooses a lower reliability item because the required initial capital outlay surpasses the advantages of higher reliability for the buyer. Hence, as the reliability becomes higher the rate of increase in revenue declines.

For example, a buyer for a machine tool chooses a higher reliability tool, all other things being equal. But, after the price of the design item goes beyond the amount the buyer can or is willing to allocate, he may
choose a lower reliability item. The limiting amount differs for each buyer, but the price increase, as reliability becomes very high, may cause the price to be beyond the amount which most buyers can allocate.

c) A decision maker may also be interested in spare parts and maintenance markets for the design item. As the reliability gets higher, the decision maker's volume in these side markets will decrease. Hence, as the reliability gets higher, the increase in revenue through more sales in the primary market is negated by the decrease in volume in parts and maintenance market. For example, a machine tool designer is able to obtain less revenue in his spare parts and maintenance enterprise as the reliability of the design item increases. The loss in benefit in side markets negates the increase in benefit by the increasing reliability for the decision maker.

d) Higher reliability typically results in bulkier and heavier designs which is not desirable for many designs. Increase in size and weight may go beyond the desirable limits for many decision makers after a certain value of reliability.
e) Higher reliability implies that a larger number of design items complete the mission time without failure. This increases the probability that a design item will become obsolete while still in use. For designs subject to rapid technological change, high reliability often implies overdesign. For such designs, the marginal increase in benefit clearly declines with increasing reliability. As an example, a power tool (especially a portable one) with excessive dimensions and/or weight may be impractical in usage and even though it is more reliable may not be desirable for many decision makers.

In conclusion, the above factors cause the marginal increase in benefit per unit increase in reliability to decline as the reliability increases. In Section II. B.2., it was seen that the preference function for reliability at a constant mission time, \( R_n \), is a non-decreasing function. In this section it is indicated that the slope of \( R_n \) is non-increasing. Since maximum reliability \( R_{\text{max}} \) is defined as the largest reliability value which is technologically and/or economically feasible, the slope of the preference function for constant mission time, \( R_n \), must be zero at maximum reliability, since no further input of technological and other resource could possibly result in an increase in reliability.
The function $u(R|n)$ is therefore a non-decreasing function, with a non-increasing slope, and the slope is zero at the maximum value $R_{\text{max}}$. In the interval between minimum and maximum reliability values $(R_{\text{min}}, R_{\text{max}})$, the preference function for reliability at a given mission time, $u(R|n)$, is as shown in Fig.2.2.
Fig. 2.2. The Preference Function for Reliability at Constant Mission Time $n'$, $\mu(R|n')$
II. D. Benefit of Both Reliability and Mission Time.

1. Interdependence of Preference Functions.

As stated in Section II.C., reliability is a function of the mission time, \( R(n) \). In that section, a preference function is developed for reliability at constant mission time, \( \mu(R|n) \) and in Section II. B. a preference function for mission time at constant reliability, \( \mu(n|R) \), is developed.

Since reliability and mission time are dependent on each other, (see Section II.A.) the benefit of reliability is only meaningful when it is defined at a given mission time and vice versa. The benefit resulting from reliability and mission time is a three dimensional preference function, \( \mu(R,n) \). \( \mu(R,n) \) increases as reliability and/or mission time increases and the slope of \( \mu(R,n) \) decreases as reliability and/or mission time increases (see Sections II.B. and II.C.), thus forming a concave surface within the intervals \( (n_{\min}, n_{\max}) \) and \( (R_{\min}, R_{\max}) \).

In order to express benefit as a function of both reliability and mission time, indifference (constant benefit) functions are introduced in Section II. D. 2.
II. D. 2. Benefit of Reliability and Mission Time

Indifference Function

A decision maker is expected to be indifferent to two choices of reliability and mission time combinations which yield the same benefit to the decision maker. That is, the decision maker is indifferent between \((R_1, n_1')\) and \((R_2, n_2'')\) when \(y(R_1, n_1') = y(R_2, n_2'')\). Since the concern of the decision maker is to increase his benefit, two sets of the parameters reliability and mission time would be of same value to him as long as the combined benefit derived by the decision maker is the same.

An indifference function \(I(R, n)\) is the locus of all sets \((R, n)\) with an equal value of benefit, \(y(R, n)\). The indifference function, \(I(R, n)\) is therefore a trade-off function between the reliability and mission time.

When the preference function for reliability at constant mission time, \(y(R | n)\), and preference function for mission time at constant reliability, \(y(n | R)\), are both concave functions (as is the case in this analysis, see Sections II. B. and II. C.), the indifference functions are convex functions, as shown in Fig. 2.3.

The construction and analysis of indifference functions are discussed in Section II. E. 5.
Fig. 2.3. Indifference Functions for Reliability and Mission Time, \( I(R|\mu=\text{cons.}) \).
II. E. Derivation of Preference Functions

1. Information Required to Construct Preference Functions

In order to construct the preference functions, certain information is required from the decision maker. The decision maker is to assess the relative benefit he derives from a certain combination of reliability and mission time compared to certain other combinations. This information is acquired by having the decision maker respond to several questions, and from a "reference gamble," which is explained in following sections.

This information is utilized to construct preference functions for reliability at several constant values of mission time. These functions, then, are used to construct the indifference functions, introduced in II.D.3.
2. **Questions to Decision Maker.**

The decision maker is to respond to the following questions, in order to assess his preference function for reliability at a given mission time. The purpose of the questions in this section is to locate the initial and end points of the preference function for reliability at given mission time, $\mu(R|n)$, for several different mission time values $n_i$ using a common scale. That is, the information obtained here is utilized to establish the relative benefits derived by the decision maker for the design at minimum reliability, $R_{\text{min}}$, and maximum reliability, $R_{\text{max}}$, at mission time values $n_i$.

As stated in Section II.B. & II.C., values for the minimum mission time $n_{\text{min}}$, minimum reliability $R_{\text{min}}$, and maximum reliability $R_{\text{max}}$, are defined and therefore taken as given.

**a)** The first question relates the benefit derived from minimum and maximum reliability values at minimum mission time. The information is used to locate the initial and end points of the preference function for reliability at minimum mission time, $\mu(R|n_{\text{min}})$. These points are also used as the basis from which to determine the initial and end points of all other preference function, $\mu(R|n_i)$, by using information obtained from question 2, in this section.
Question 1. Given the minimum mission time \( n = n_{\text{min}} \), what would be the ratio of the benefit derived at maximum reliability \( R_{\text{max}} \), to the benefit derived at minimum reliability \( R_{\text{min}} \):

\[
\frac{m_1}{m_2} = \frac{\mu(R_{\text{max}}, n_{\text{min}})}{\mu(R_{\text{min}}, n_{\text{min}})}
\]

If we let \( \mu(R_{\text{min}}, n_{\text{min}}) = \mu_0 \) and since our concern is in the relative benefit, let us assume some constant value for \( \mu_0 \).

Therefore:

\[
\mu(R_{\text{max}}, n_{\text{min}}) = m_1 \mu_0 \quad (2.1)
\]

A value for \( \mu(R_{\text{max}}, n_{\text{min}}) \), (benefit at maximum reliability and minimum mission time) is acquired utilizing the information obtained from question 1., and assuming a constant for \( \mu_0 \). The initial and end points for the benefit-reliability function at minimum mission time are located using the above information as shown in Fig.2.4 (marked by circles).

b) The second question relates the benefits derived at different values of mission time, \( n_i \), given the minimum reliability, \( R_{\text{min}} \). The ratio of benefit derived at these values of mission time, \( n_i \), given \( R_{\text{min}} \), to the benefit derived at the minimum reliability and minimum mission time, \( \mu(R_{\text{min}}, n_{\text{min}}) = \mu_0 \), locates the initial points of the preference function for reliability at these values of mission time.
Fig. 2.4. Location of Initial and End Points of Preference Function for Reliability at Minimum Mission Time. (Note that the preference function itself -dashed line- has not been obtained at this point.)
The maximum mission time is defined in Section II.B.1 as that value of mission time, after which further increases in mission time do not result in further increases in benefit. The minimum mission time is defined in the same section as the lowest useful value. In order to obtain several values of mission time \( n_i \), denote the maximum mission time given by \( n_{\text{max}} \) and divide \((n_{\text{max}}-n_{\text{min}})\), i.e. the difference between estimated maximum mission time and given minimum mission time, by an integer \( N \) to obtain a value, say \( n_n \), define \( n_i \) such that:

\[
n_i = n_{\text{min}} + i \cdot n_n ; \quad i=0,1,2,\ldots,N
\]

Since an increase of \( n_i \) from \( n_N \) to \( n_{N+1} \) in mission time does not result in an increase in benefit, \( n_N \) is by definition maximum mission time, \( n_{\text{max}} \).

Question 2. Given that the reliability is equal to the minimum reliability, \( R_{\text{min}} \), what would be the ratio of benefit derived at \( n_i \) to benefit derived at \( n_{\text{min}} \):

\[
\frac{\mu(R_{\text{min}}, n_i)}{\mu_0} ; \quad i=1,2,\ldots,N
\]

Let the above ratio be donated by \( P_i \). Then initial points of preference functions for reliability at mission time \( n_i \) are

\[
\mu(R_{\text{min}}, n_i) = P_i \mu_0 \quad (2.2)
\]
Since a value was assumed for \( \mu_0 \) these initial points are now located, using the information gained in Question 2, as shown in Fig. 2.5. (marked by squares).

c) The third question seeks the ratio of the benefit derived at different values of the mission time \( n_i \), given the maximum reliability \( R_{\text{max}} \), to the benefit derived at the minimum mission time \( n_{\text{min}} \), and the maximum reliability \( R_{\text{max}} \). The benefit derived at \( R_{\text{max}} \) and \( n_{\text{min}} \), as shown in Section II.E.2b above, is \( \mu(R_{\text{max}}, n_{\text{min}}) = m_1 \mu_0 \).

The information thus obtained is utilized to locate the end points of preference functions for reliability at \( n_i \).

Question 3. Given that the reliability is equal to the maximum reliability \( R_{\text{max}} \), what is the ratio of benefit derived at \( n_i \) to benefit derived at \( n_{\text{min}} \):

\[
\frac{\mu(R_{\text{max}}, n_i)}{M_1 \mu_0}.
\]

Let the above ratio be denoted by \( Q_i \). Then end points of the preference functions for reliability at mission time \( n_i \), are

\[
\mu(R_{\text{max}}, n_i) = Q_i m_1 \mu_0.
\]

Since these points are also expressed in terms of \( \mu_0 \) they are located on the same scale as the initial points. Utilizing the information obtained in Question 3, the end points are located as shown in Fig. 2.6. (marked by crosses).
Fig. 2.5. Location of Initial Points of Preference Function for Reliability at $n_i$. 
Fig. 2.6. Location of Initial and End Points of Preference Function for Reliability at Mission Time Values $n_i$. (Note that the preference functions -dashed lines- have not been obtained at this point.)
3. Reference Gamble

In the previous section the information obtained enabled the location of the initial and end points for the preference functions for reliability at mission time values \( n_i: \mu(R, n_i) \). A reference gamble is employed to locate a third point between initial and end points. Having three points, a suitable concave curve is fitted using the general criteria established in Section II.C, see Section II.E.4. This section explains what a reference gamble is as applied to the case in hand.

Suppose that at a given mission time \( n \), the reliability of the design is uncertain. Suppose also that there is a probability, \( q \), that the reliability is equal to the maximum value \( R_{\max} \), and that there is a probability \( (1-q) \), that the reliability is equal to the minimum value \( R_{\min} \). The decision maker derives a certain benefit from the given mission time and minimum reliability, or from the given mission time and maximum reliability. Now, there is not the certainty of minimum and maximum reliability, but a probability \( 1-q \) or \( q \) of getting either one or the other. The benefit derived from this uncertain situation by the decision maker is likely to be higher than the benefit derived if \( R_{\min} \) pertained for certain and lower than the benefit derived if \( R_{\max} \) pertained for certain.

That is; \( \mu(R_{\min}, n) < \mu(n, q) < \mu(R_{\max}, n) \),

where \( \mu(n, q) \) is the benefit derived at \( n \) with \( q \) and \( (1-q) \)
probabilities of $R_{\text{max}}$ and $R_{\text{min}}$, respectively.

Since $\mu(R,n)$ is a non-decreasing function, there is a value of $R$ such that

$$\mu(R|n) = \mu(n,q).$$

$R'$ is called the certainty equivalent of the uncertain state of $q$ and $(1-q)$ probabilities of $R_{\text{max}}$ and $R_{\text{min}}$. $R'$ is a function of $q$ and depends on the given mission time $n$: $R' = R'(q|n)$. The value of $R'(q)$ is found as follows.

The decision maker establishes what certain benefit he derives from the state at which there are $q$ and $(1-q)$ probabilities of $R_{\text{max}}$ and $R_{\text{min}}$, respectively, and decides at which value of $R'$, he is indifferent between the above state and the prospect of obtaining $R'$ for certain.

For clarity and convenience the probability value $q=1-q=.5$ is chosen. It is thus supposed that there is a fifty percent chance that the reliability is $R_{\text{max}}$ and fifty percent chance that the reliability is $R_{\text{min}}$. Using the reference gamble, a certainty equivalent $R'(0.5)$ is obtained. At $R=R'(0.5)$, the decision maker derives the same benefit as having a fifty percent chance of each $R_{\text{max}}$ and $R_{\text{min}}$.

Therefore, the benefit derived at $R'(0.5)$ is its expected value

$$E\{\mu(R'(0.5)|n)\} = 0.5 \cdot \mu(R_{\text{max}}|n) + 0.5 \cdot \mu(R_{\text{min}}|n)$$

$$= 0.5 \cdot (\mu(R_{\text{max}}|n) + \mu(R_{\text{min}}|n))$$
This benefit value at the reliability value $R_i$ provides a third point for the preference function for reliability at a given mission time $n$.

The required information, then, is the certainty equivalent $R'(0.5|n)$ at which the decision maker would be indifferent to a 50-50 gamble on $R_{\min}$ and $R_{\max}$. This information is obtained from the decision maker by the following question.

Question 4. (Reference Gamble) Given that the mission time is $n_i$, Proposition 1 is that there is a 50% chance of obtaining $R_{\min}$ and a 50% chance of obtaining $R_{\max}$. Proposition 2 is that a reliability value $R_i$ can be obtained for certain. At which value of $R_i$ would you be indifferent between the two propositions.

where $n_i=n_{\min}, n_1, n_2, \ldots, n_N$

The resulting intermediate points for the preference function for reliability are

$$\mu\{R_i'(0.5)|n_i\} = \{\mu(R_{\max}|n_i) + \mu(R_{\min}|n_i)\}/2.$$  

$$= (Q_i m_1 \mu_0 + P_i \mu_0)/2.$$  

$$= (\mu_0/2.) (Q_i m_1 + P_i) \quad (2.3)$$

Since these points are also expressed in terms of $\mu_0$, they are located on the same scale as the initial and end points. The location of these points is shown in Fig.2.7 (marked by triangles).
Fig. 2.7. Location of three Points for Each Preference Function for Reliability at \( n_i \). (Note that the preference functions -dashed lines- have not been obtained at this point.)
In Section II.C., it is observed that $\mu(R|n_i)$ is a concave function, and in this section we showed that

$$\mu(R^*_i|n_i) = 0.5\{\mu(R_{\max}|n_i) + \mu(R_{\min}|n_i)\} \quad (2.4)$$

For a concave function $F(x)$,

if $F(x_2) = 0.5\{F(x_1) + F(x_3)\}$

then $x_2 \leq 0.5(x_1 + x_3)$.

Hence, $R^*_i \leq 0.5(R_{\max} + R_{\min})$. 
4. **Construction of Preference Functions.**

In Sections II. E. 2 and II. E.3, an initial, an intermediate, and an end point were found for the preference function for reliability at a given mission time, $\mu(R)$. A suitable curve is to be fitted to these data that meets the criteria established in Section II. C.

As stated in Section II. C, the preference function for reliability:

a. is a non-decreasing function, i.e. $\mu'(R,n_i) \geq 0$

b. has a non-increasing slope, i.e. $\mu''(R,n_i) \leq 0$

c. has a zero slope at $R=R_{\text{max}}$, i.e. $\mu'(R_{\text{max}},n_i)=0$

In Sections II. E.2 & II. E.3, we established three points for each mission time value $n_i$. Let

- a. $\mu(R_{\text{min}},n_i)=\mu_{1i}$
- b. $\mu(R_i',n_i)=\mu_{2i}$
- c. $\mu(R_{\text{max}},n_i)=\mu_{3i}$

A function $\mu(R,n_i)$ is now suggested that fits the above three points suitably and confirms the restriction stated above:

$$
\mu(R,n_i) = A - B \left\{ (C-R) e^{-(C-R)} \right\}^D
$$

Where

- $A = \mu_{3i}$
- $C = R_{\text{max}}$
- $D = \frac{\ln (\mu_{3i} - \mu_{1i}) - \ln (\mu_{3i} - \mu_{2i})}{R_{\text{min}} - R_i' + \ln (R_{\text{max}} - R_{\text{min}}) - \ln (R_{\text{max}} - R_i')}$
- $B = \frac{(\mu_{3i} - \mu_{1i})}{[(R_{\text{max}} - R_{\text{min}}) e^{\exp (R_{\text{min}} - R_{\text{max}})}]^D}$
This function is illustrated in Fig. 2.8 for different values of $R_i$ at given values $R_{\text{min}}$ and $R_{\text{max}}$. A value of zero is taken for $\mu_{1i}$ and a value of one for $\mu_{3i}$. Values of .900 and .999 are used for $R_{\text{min}}$ and $R_{\text{max}}$ respectively. The value of $R_i$ is varied from .905 to .949.
Fig. 2.8. Preference Function for Different Values of $R_1'$. 
5. **Construction of Indifference Functions.**

In Section II.C. we developed preference function for reliability at constant mission time. In Section II.E.4 a function was suggested for $\mu(R|n)$. This function is utilized in this section to construct the indifference function for reliability and mission time.

An indifference function, introduced in Section II.D.2 is a trade-off function between reliability and mission time in terms of benefit. It is a locus of all combinations of reliability and mission time that have the same given constant benefit. If we let

$$\mu(R,n_i) = \mu_j$$

where $\mu_j$ is a constant, and solve for $R$ at all $n_i$ for which $\mu_j$ is in the range of $\mu(R,n_i)$, we obtain several combinations of $(R;n'_i)$ which result in the benefit $\mu_j$. Since all these combinations $(R;n'_i)$ are of equal benefit to the decision maker, he would be indifferent among these combinations.

In Fig. 2.9. curves of $\mu(R,n_i)$ are presented at equal intervals of $n_i$. As discussed in Section II.C., these functions are concave functions, since the increase in benefit at constant reliability, per unit increase in mission time, decreases as $n$ increases (see Section II.B.3).

The difference between $\mu(R|n_{i+1})$ and $\mu(R|n_i)$ decreases as the mission time $n$ increases. Hence, the difference
between the intersections of \( \mu(R|n_{i+1}) \) and \( \mu(R|n_i) \) with a vertical (\( R=\text{constant} \)) line decreases as \( n \) increases. This is illustrated in Fig. 2.9.

In Fig. 2.10, indifference functions \( n(R|\mu=C) \) are illustrated for the preference functions \( \mu(R|n_i) \) shown in Fig. 2.9.

Indifference functions derived from the preference functions for reliability and mission time, \( \mu(R|n) \) and \( \mu(n|R) \), are convex functions with negative slope and positive second derivative. The reasoning for the shape of indifference functions is given in Appendix 4.

Through the points \( (n'_i, R') \) obtained by letting

\[
\mu(R|n) = \mu_1 \text{ a constant}
\]

a smooth convex function is fitted. This is illustrated within the framework of the example given in Appendix I.
Fig. 2.9. Preference Functions $\mu(R|n_i)$, for Reliability at Constant Mission Time.

Fig. 2.10 Indifference Functions $n(R|\mu)$
III. **COST**

The cost of a probabilistic design is a function of reliability and the mission time of the design. The cost of a design may be analyzed in three categories:

a) Material Cost, b) Operation Cost, c) Production Cost.

a) Material Cost is, among other things, a function of the parameters of the p.d.f. of the material properties, see Section III.A. If a choice of material is specified, this cost will not vary and is omitted from the analysis. However, when a choice of material (e.g., aluminum, steel, or different grades of steel) is in question, this cost will be of importance. The analysis presented in this report would then be repeated for different materials, and the results compared in order to choose the optimum material.

Material Cost is also a function of the size of the specimen. The size of the specimen, in terms of some critical dimension $A$, determines the amount of material used for the production of the specimen. Reliability and mission time are both functions of the parameters of the p.d.f. of material property and the size of the specimen, see Section I.C. Hence, the cost of material can be expressed as a function of reliability and the mission time. Most reliable and longer lasting designs usually call for more exotic materials, resulting in increased material costs. See Section III.A.
b) Total operation cost decreases as the reliability of the design increases due to a smaller probability of failure and fewer shutdowns, etc., see Section III.B.1. Total operation cost increases as the mission time increases because of longer useful life of the specimen. (Note that benefit too, increases as mission time increases.) This effect is analyzed in Section III.B.2.

c) Production Cost for the specimen depends on the production method chosen for the design. Production cost is a major factor of the cost of the design. Production methods are affected by the properties of the material.

Even though it may be less costly to use a given material compared to a second material on the basis of its material properties, if the production cost associated with the first material is high compared to the second material, it offsets the cost advantage of using the first material. The second material becomes the better choice. Therefore production cost should always be considered within the context of the cost of the material which the type of production is associated with.

The changes in cost by changes in material properties is discussed in Section III.A.1.
III. A. MATERIAL COST

1. Dependence on Parameters of Material p.d.f.

Material cost per unit weight increases as the expected value \( \mu_1 \) and the variance \( \mu_2 \) of the relevant material property become more favourable; that is as \( \mu_1 \) increases and/or \( \mu_2 \) decreases. \( \mu_1 \) and \( \mu_2 \) are properties of the material p.d.f. and they are functions of the parameters of the p.d.f.

a) The cost of material increases as the expected value, \( \mu_1 \) increases, since a higher \( \mu_1 \) implies more exotic raw materials and more expensive production methods for the material.

If \( C_{m,\mu} \) is defined as the cost of material per unit weight; then

\[
C_{m,\mu} \uparrow \text{ as } \mu_1 \uparrow
\]

Hence,

\[
\frac{\partial C_{m,\mu}}{\partial \mu_1} > 0. \tag{3.1}
\]

and since additional incremental increases in \( \mu_1 \) become more costly as \( \mu_1 \) increases,

\[
(\frac{\partial C_{m,\mu}}{\partial \mu_1}) \uparrow \text{ as } \mu_1 \uparrow.
\]

Hence,

\[
\frac{\delta^2 C_{m,\mu}}{\delta (\mu_1)^2} > 0. \tag{3.2}
\]
This analysis determines the general shape of the function $C_{m,\mu}(\mu_1^i)$ as shown in Fig. 3.1. The exact shape differs for each material.

b) The cost of material increases as the variance $\mu_2$ decreases because a decrease in $\mu_2$ implies tighter production controls. Hence,

$$\frac{\partial c_{m,\mu}}{\partial \mu_2} < 0.$$  \hspace{1cm} (3.3)

Additional incremental decreases in the value of $\mu_2$ will be increasingly expensive as $\mu_2$ decreases, hence,

$$\frac{\partial c_{m,\mu}}{\partial \mu_2} \uparrow \text{ as } \mu_2 \downarrow.$$  

Hence,

$$\frac{\partial^2 c_{m,\mu}}{\partial (\mu_2)^2} > 0.$$  \hspace{1cm} (3.4)

The above analysis gives the general shape of the function $C_{m,\mu}(\mu_2)$ as shown in Fig. 3.2.
Fig. 3.1. Unit Material Cost as a Function of $\mu_i'$. 
Fig. 3.2. Unit Material Cost as a function of $\mu_2$. 
III. A. 2. **Dependence on Size Parameter**

The design configuration is a result of design engineering analysis. It is not taken as a variable in this study. The analysis presented is applicable to designing for a given configuration. The size parameter of the design affects both the reliability and the mission time as explained in Section I.A.3. The cost is also a function of the size parameter.

The material cost increases as the amount of material used for the design increases. The volume of material used for the design is a direct function of the size parameter. If the size parameter is a cross-sectional area, for example, the volume increases directly with the area. The material cost is therefore a direct function of the size parameter.

If the choice of material is desired to be taken as a variable in the analysis, the cost of material per unit weight, \( c_{m,\mu} \), varies. Since the specific weight of the material is constant, the cost of the material per unit volume, \( c_v \), is a direct function of \( c_{m,\mu} \).

Hence, the cost of material per design specimen is

\[
c_m = c_v \cdot V,
\]

where \( V \) is the volume of material used. Since the volume is a direct function of the size parameter \( A \),

\[
c_m = c_v \cdot K \cdot A,
\]

(3.5)
where $K = V/A$ is a constant relating the size parameter and volume.
III. B. COST OF OPERATION

1. Dependence on Reliability

The probability of failure of the design is related to the reliability of the design: (Section 1)

\[ P(n) = 1 - R(n). \]

That is, the probability that failure will occur during its intended mission time \( n \), is the complementary probability of reliability.

For critical designs, for which failure is catastrophic, the cost of failure maybe excessively high.

For such designs, the minimum reliability will be high, and in the range between minimum and maximum reliability, cost will decrease as the reliability increases. For non-repairable and for non-replaceable specimens the expected reliability-dependent operation cost is

\[ C_R = (1-R(n))C_F \]

where \( P(F) = (1-R(n)) \) = probability of failure is the complementary probability of reliability and \( C_F \) is the assessed cost of failure. See Fig. 3.3.

For replaceable and/or repairable design specimens, for which failure is not catastrophic, the reliability-dependent cost of failure decreases as reliability increases. In the hypothetical case when reliability is equal to one, \( C_R \)
Fig. 3.3. Cost of Failure as function of $R$ for Specimens with Catastrophic Failure.
is equal to zero. As reliability increases, the probability of failure decreases; hence among \( K \) specimens (large \( K \)), the expected number of specimens failing during the mission time, \( n \), decreases. Among \( K \) specimens the expected number of failures is equal to

\[
P(\text{F}) \cdot K = (1 - R(n)) \cdot K.
\]

If we assume a failing specimen is replaced or repaired immediately, the new specimen is to serve mission time \( n' \), the remainder of the mission time \( n \) of the failed specimen. Hence,

\[n' < n.\]

Since the new specimen has a smaller mission time, while the design parameter is the same,

\[R(n') > R(n).\]

Hence the probability of failure for the replaced or repaired specimen is lower than the probability of failure of the original specimens, and it is less likely to fail during the mission time \( n \).

The relation between \( R(n) \) and \( R(n') \), \( g(S,L,A) \), depends on the nature of the p.d.f. of material strength, \( S \), load \( L \), and the design parameter, \( A \):

\[R(n') = g(S,L,A) \cdot R(n).\]
Since \( R(n') > R(n) \) the relating function \( g \) becomes larger as \( R(n) \) increases.

If we assume \( K \) (large \( K \)) specimens were originally put into operation, \((1-R(n))K\) specimens will likely fail and be replaced and/or repaired. Among these \((1-R(n))K\) new specimens,

\[
P(F)(1-R(n))K = (1-R(n'))(1-R(n))K
\]

will likely fail and be replaced and/or repaired. The new replacement, (third generation), has a small mission time, \( n'' \), left to complete since

\[
n'' \ll n
\]

we can assume

\[
R(n'') = 1.0.
\]

That is, \( R(n'') \) is very close to one, and failure during \( n'' \) is very unlikely and therefore its effect may be neglected.

Since, starting with \( K \), and replacements of two generations, to have \( K \) specimens complete the mission time, \( K' \) specimens are needed where

\[
K' = K + (1-R(n))K + g(s,L,\Lambda)(1-R(n))^2K.
\]

So \((K'-K)\) repairs and/or replacements are needed

\[
K' - K = \left\{(1-R(n)) \left[1 + g(s,L,\Lambda)(1-R(n))^2\right]\right\}^2 K.
\]

If we let \( C_F \) be the cost of failure, i.e. the cost of repairing and/or replacing the failed specimens, the cost for \( K \) specimen is

\[
C_R(K) = \left\{(1-R(n)) \left[1 + g(s,L,\Lambda)(1-R(n))^2\right]\right\}^2 K \cdot C_F.
\]
the cost per specimen is

\[ C_R = \left\{ (1 - R(n)) \left[ 1 + g(s, L/A)(1 - R(n)) \right] \right\} \frac{1}{3} C_F. \]
2. Dependence on Mission Time

The operation of a specimen involves a cost. This cost increases as the specimen ages, due to more frequent shutdowns and failures. The costs of failure and shutdowns are taken into consideration in previous sections in connection with reliability, because their frequency is dependent on the reliability of the specimen.

Hence, the fixed part of the operation cost is taken into consideration. The fixed cost is directly proportional to the length of the mission time.

Let $C_{o,s} = \text{Cost of operation per specimen}$

$C_{o,R} = \text{Cost of operation per time unit per specimen.}$

$C_{o,s} = n C_{o,n}$ in monetary units/ specimens.

Here, as the mission time is increased, $C_{o,s}$ increases, and it may seem as a penalty cost. That is, the longer mission time appears to be less desirable. But when the benefit of the mission time was considered, the benefit was seen to be increasing with $n$. Therefore an increase in the mission time is associated both with an increase in cost and in benefit.

If the cost and benefit of the mission time were the only variables considered in decision making at a constant reliability, $R$, then the relation graphed in Fig 3.4 would have resulted.
Fig. 3.4. Benefit-Cost Analysis for Mission Time.
As seen from Fig. 3.4, if only the benefit of the mission time (omitting reliability), is considered, the smaller of the maximum mission time or the mission time value \( n' \) (at which the benefit is equal to the cost) would be chosen, since that would be the value which provides the maximum benefit. In case \( n_{\text{max}}(1) \) is the maximum mission time, \( n_{\text{max}}(1) \) would be chosen; if \( n_{\text{max}}(2) \) is the maximum mission time, \( n' \) would be chosen since \( n' < n_{\text{max}}(2) \).
III. C. COST FUNCTION

1. Dependence of Cost on Reliability and Mission Time

In Section I, it was shown that the reliability of a design may be expressed as (1.8)

\[ R(n) = 1 - P(\nu < 1) \]  

3.10

where

\[ \nu = \frac{S}{L} \]  

3.11

see eq. (1.7)

The resisting strength of the material, \( S \), is a function of the material property, \( M \), and the size parameter, \( A \). Hence the reliability can be expressed as

\[ R(n) = 1 - P(MxA \leq L|n) \]  

3.12

From this relation the size parameter \( A \), can be expressed in terms of reliability \( R \) and mission time \( n \). When the maximum-load distribution and strength distribution are approximated as log-normal models (see App. III):

\[ R(n) = F_N\left(\frac{\mu_v}{\sigma_v}; 0, 1\right) \]  

3.13

where from A.1.11 and A.1.12 in App. I,

\[ \mu_v = \mu_s - \mu_v \]  

3.14

and

\[ \sigma_v = \left(\sigma_s^2 + \sigma_v^2\right)^{0.5}. \]

But

\[ \mu_s = \mu_m + \ln A. \]

Hence

\[ \ln A = \mu_v - \mu_m + \mu_v. \]  

3.15
Hence, $\frac{\mu}{\sigma}$ is calculated from relation 3.13 and therefore is a function of the reliability $R$ and $\mu_L$ is a function of mission time $n$. (see Section I). Hence we have expressed the size parameter $A$ as a function of $n$, and $R$. See Appendix I, for an example.

In Section III.A. 2., (3.5), it was shown that the material cost is a direct function of the size parameter $A(R,n)$.

Hence, the cost of material is

$$C_m = C_v \cdot K \cdot A(R,n). \quad \text{(3.16)}$$

As shown in Section III. B. 2, the cost of operation per specimen as a function of mission time is

$$C_{o,s} = n \cdot C_{o,n} \quad \text{(3.17)}$$

Hence, the total cost function is the sum of three cost functions, namely the cost of material $C_m(R,n)$, the cost of operation (dependent on mission time) $C_{o,s}(n)$, and the cost of operation (dependent of reliability) $C_R(R)$:

$$C(R,n) = C_m(R,n) + C_{o,s}(n) + C_R(R) \quad \text{(3.19)}$$

When the material choice is also a variable in the decision process, the cost of material $C_m(R,n)$ is found separately for all choices of material. The cost function for each material is, hence, given by 3.19 using appropriate $C_m(R,n)$. 

IV. C. 2. **Constant Cost Curves**

Using relation 3.12,

\[ R(n) = 1 - P(S < L) \]

several values of \( n \) and \( R \) can be found for several fixed values of \( A \). Since \( A \) is a direct function of cost of material, using this information a function is developed of the cost of material as a function of mission time at constant reliability: \( C_m(R, n|R) \)

This is illustrated by the dashed line in Fig. 3.5.

Since the cost of operation dependent on mission time is a direct function of \( n \), this function \( C_{o,s}(n) \) (illustrated in Fig. 3.5 by the dotted line) and is added to \( C_m(R, n|R) \). Since the value of cost of reliability at constant reliability is a constant, \( C_R(R, n|R) \) (illustrated in Fig. 3.5 by dash-dot-dash line) is a horizontal line. Adding also \( C_R(R|R=C) \), we get the total cost function for mission time at constant reliability, \( C(R, n|R=C) \). See Fig. 3.5. (See Appendix I for numerical example.)

This procedure is repeated for \( N \) constant reliability values, where \( N \) is the number of constant mission time values chosen in Section 2.

Using these cost functions \( C(R, n|R=C) \), and taking fixed cost values, we get a set of values of \( (R,n) \) at the same fixed cost. Using these sets of \( (R,n) \), with the
Fig. 3.5. Cost of Mission Time at Constant Reliability, $C(R,n|R=c.)$. 
method employed to construct indifference functions in Section 2, constant cost functions are constructed, see Fig. 3.6.

Since total cost increases as mission time increases, and cost of material increases as reliability increases, the constant cost functions are concave functions, see Appendix IV.
Fig. 3.6. Constant Cost Functions
IV. DECISION PROCESS

A. Intersection of Indifference and Constant Cost Functions.

In Section II.E.5. it is shown that indifference functions are convex functions, and are expressed in terms of R and n. In Section IV.C.2., it is shown that constant cost functions are concave functions, and also are expressed in terms of R and n. Hence some of the indifference functions and some of the constant cost functions intersect at two points, and one or both of these points may be in the acceptable range \((R_{\min}, R_{\max})\) and \((n_{\min}, n_{\max})\). An indifference function will be tangent to a constant cost function, if many such functions are developed.

At the point \((R'; n')\) where an indifference function with benefit \(\mu'\) is tangent to a constant cost function with cost \(C'\), \(\mu'\) is realized with lowest cost and \(C'\) is realized with highest benefit.

Hence \((R'; n')\) is called a suboptimal point, since the cost-benefit ratio is lowest when cost \(C'\) and benefit \(\mu'\) are considered.

In the actual decision process, when most of the indifference functions and constant cost functions chosen are likely not to be tangent, the locus of likely tangency points is approximated graphically. Fig. 4.1. shows the result.
Joining these points gives the locus of suboptimal points.

B. The Optimal Combination of R and n.

Once the suboptimal points are found, each point is associated with a benefit value $\mu$, and cost value $C$.

The ratio of cost to benefit,

$$X = \frac{C}{\mu},$$

is found and a smooth function is fit to $X$ versus $R$. The minimum point of $X(R)$ gives the value of optimal reliability $R^*$, for which the cost-benefit ratio is a minimum. The corresponding optimal value for the mission time $n^*$ is located from the suboptimality line of Fig.4.1. at $R^*$.

An extensive example of the decision process is presented in Appendix I.
Fig. 4.1. Locus of Suboptimal Points.
V. CONCLUSION AND RECOMMENDATIONS

In the classical design approach, a safety factor is chosen based on past engineering experience. This factor does not contain information on the mission length of the design nor its reliability. Even the most conservative designs do fail occasionally. A probabilistic design approach conveys the information on expected mission time to failure, and the fraction of designs expected to fulfill this mission time, by the values of the decision parameters $n$ and $R$.

In the present work, a method is developed to obtain $R$ and $n$ values by a rational decision analysis. The decision criterion used in the analysis, is to minimize the cost-benefit ratio for the decision maker. Preference functions for the decision maker, for both reliability and mission time, are developed. Indifference functions are derived from these preference functions. Indifference functions are trade-off functions between reliability and mission time in terms of benefit. Constant cost functions, developed from cost functions for $R$ and $n$, are trade-off functions between reliability and mission time in terms of cost.

Indifference and constant cost functions are utilized to obtain the sub-optimal line which defines the locus of points $(n, R)$ that have the highest benefit at given cost and lowest cost at given benefit. Among these points, the one with the minimum cost-benefit ratio is chosen. This combination of $R^*$ and $n^*$ gives the values of $R$ and $n$ at which the decision maker minimizes his cost-benefit ratio. The
probabilistic design parameter $A^*$ is obtained directly from $R^*$ and $n^*$.

This decision model may be adopted for use when weight is the critical decision variable rather than cost. Constant weight functions may be developed using the same method as for constant cost functions. The final analysis may be utilized to find minimum weight-benefit ratio in the same manner.

It is recommended that the present work be extended by making material choice a decision parameter. In that case, the constant cost functions are developed for each material considered. A set of constant cost functions, derived for each material, and indifference functions result in a sub-optimal line for each material. The minimum cost-benefit points on each sub-optimal line are compared, and the one with the lowest cost benefit ratio gives the optimal reliability $R^*$, optimal mission time $n^*$, optimal design parameter $A^*$, and optimal material choice $M^*$. 
BIBLIOGRAPHY

(A) List of Journals Surveyed:

List of Articles and Books Referred:


List of Related Articles Perused:


I. The Problem

The decision process to find an optimal combination of mission time \( n \) and reliability \( R \), is illustrated with an example of a shear panel for an airplane.

**Problem:** To find the optimum combination of reliability \( R^* \) and mission time \( n^* \), such that the ratio of cost to benefit is minimized, for a shear panel of an airplane, probabilistically designed, and to find the corresponding design specification, \( t^* \) ins, thickness of the panel.

**Given Information:**

1. Preference function for reliability at constant mission time assessed by the decision maker.

2. Distribution of shear load, \( F(L) \): Weibull distribution with mean load, \( L_{me} = 1 \) Kip, and shape parameter, \( \lambda = 1.5 \).

3. Choice of material: Aluminum 24S with handbook values of Young's modulus,
\[ E = 10.6 \times 10^6 \text{ psi and Poisson's ratio } \nu = 0.33. \text{ Material distribution is assumed to be a log-normal distribution, } F_{LN}(E) \text{ with } \mu'(E) = \bar{E}, \text{ and } \gamma(E) = 0.1. \]

4. a. Minimum mission time, \( n_{\text{min}} = 100,000 \) applications.

b. Minimum reliability, \( R_{\text{min}} = 0.90. \)

5. Maximum reliability, \( R_{\text{max}} = 0.99. \)

5. Configuration: all edges clamped, rectangular side panel, with dimensions; \( a = 6\text{ft} \) and \( b = 3\text{ft}. \)

II. The Failure Mode

The mode is assumed to be yielding in shear. From Handbook of Engineering Mechanics by Flügge, the critical yield stress in shear, \( \tau_{cr} \), is related to design parameters as

\[ \tau_{cr} \frac{b^2 t}{\pi^2 k} = 10.34 \quad (A.1.1) \]

where \( k = \frac{E t^3}{12(1-\nu^2)} \)

Hence;

\[ \tau_{cr} = 10.34 \frac{\pi^2 E t^2}{12(1-\nu^2)} b^2 \quad (A.1.2) \]

The shear strength is

\[ S = b t \tau_{cr} \]

\[ = 10.34 \frac{\pi^2 t^3 E}{12(1-\nu^2)} b \quad (A.1.3) \]
III. Probabilistic Design

Failure occurs at the first occurrence of the load, L, which is greater than the strength S:

\[ L > S \]

\[ L > 10.34 \pi^2 t^3 E / 12 (1-v^2) b \]  \hspace{1cm} (A.1.4.)

A. P.d.f. of Material Property E.

P.d.f. of E is assumed as a log-normal distribution, with

\[ \mu'_E(E) = \bar{E} \]

\[ \gamma(E) = 0.1 \]

For a log-normal distribution,

\[ \mu'_E(E) = \exp(\mu_E + 0.5\sigma_E^2) \]

\[ \gamma(E) = (\exp(\sigma_E^2) - 1)^{0.5} \]  \hspace{1cm} (A.1.5.)

Hence, using given \( \bar{E} \) and \( \gamma(E) \) values,

\[ \mu_E = 14.849 \]

\[ \sigma_E = 0.0324 \]

Since from eq. (A.1.3),

\[ S = [10.34 \pi^2 t^3 / 12 (1-v^2)] E \]  \hspace{1cm} (A.1.6.)

S is log-normally distributed with

\[ \mu_S = \mu_E + \ln [10.34 \pi^2 t^3 / 12 (1-v^2) b] \]

\[ \sigma_S = \sigma_E \]  \hspace{1cm} (A.1.7.)

B. P.d.f. of Load L.

Load is assumed Weibull distributed with

\[ L_{NE} = 1.0 \]  \hspace{1cm} (Kip),

\[ \lambda_w = 1.5 \]
Since for a Weibull distribution

\[ \sigma_W = \frac{L_{0.6} \ln 2}{\ln \lambda W}, \]
\[ \sigma_W = 1.28 \text{ kips}. \]

The extreme value distribution of the Weibull model is estimated by a log-normal model, this is shown in Appendix III.

The log-normal model that estimates the extreme value distribution of the Weibull model is

\[ F_{LN}(L, \mu_L, \sigma_L) \]

Where, from A. 3.15 and A.3.16 in Appendix III,

\[ \mu_L = \ln \left[ 0.873 (\ln n)^{-0.33} \frac{(1.5 \ln n + 1.82)^{2.72}}{(1.5 \ln n + 3.00)^{4.72}} \right] \] (A.1.8.)

and
\[ \sigma_L = 1.65 \ln \left[ (1.5 \ln n + 3.00) / (1.5 \ln n + 1.82) \right] \] (A.1.9.)

Here, as shown in Appendix III, the values for \( \lambda_W \) and \( \sigma_W \) are substituted and the only independent variable is \( n \).

C. Reliability

Failure occurs when \( L \geq S \).

Define a variable \( V \) such that

\[ V = \frac{S}{L}. \] (A.1.10)

Hence failure occurs when

\[ V \leq 1. \]

The variable \( V \) is log-normally distributed with

\[ \mu_V = \mu_S - \mu_L \]

and
\[ \sigma_V = (\sigma_S^2 + \sigma_L^2)^{0.5} \] (A.1.11.)
The probability of failure is
\[ P = P(\nu \leq x), \]
and the reliability is
\[ R = F_N\left( \frac{\mu - x}{\sigma} ; 0, 1 \right), \]
where \( F_N \) is the standardized normal distribution function. See Chapter 13 in ref. (2)

Equation (Al.12) is a relation among the design parameter \( R \) and \( n \), and the corresponding (probabilistic) design thickness \( t \).

IV. The Preference Function for Reliability at a Given Mission Time

A. Answers to Questions:

Question 1; (see Section II.E.2)
Given the minimum mission time of 100,000 applications, the ratio of benefit derived at \( R = .99 \) to benefit derived at \( R = .90 \) is
\[ m_l = 2.0 \]

Question 2; (see Section II.E.2)
Estimate maximum mission time
\[ n_{\text{max}}' = 200,000 \text{ applications} \]
divide \( (n_{\text{max}}' - n_{\text{min}}) \) by \( N = 5 \)
hence \( n_n = 20,000 \) applications.
\[ n_i = n_{\text{min}} + i n_n ; \quad i=1,2,\ldots,N. \]
Given that the reliability is $R_{\text{min}} = 0.90$, the ratio of benefit derived at $n_i$ to benefit derived at $n_{\text{min}} = 100,000$ is

$$P_i = \frac{\mu(n_i | R_{\text{min}})}{\mu_0}$$

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<thead>
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<th>$n_i$</th>
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<tbody>
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<td>220,000</td>
<td>3.0</td>
</tr>
</tbody>
</table>

Since $p_i$ does not increase beyond 200,000, the maximum mission time $n_{\text{max}}$ is 200,000 applications, as was estimated previously.

Question 3; (see Section II.E.2.C.)

Given that the reliability is $R_{\text{max}} = 0.99$, the ratio of benefit derived at $n_i$ to benefit derived at $n_{\text{min}} = 100,000$ is

$$Q_i = \frac{\mu(n_i | R_{\text{max}})}{\mu_0}$$

<table>
<thead>
<tr>
<th>$n_i$</th>
<th>$Q_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>120,000</td>
<td>1.30</td>
</tr>
<tr>
<td>140,000</td>
<td>1.50</td>
</tr>
<tr>
<td>160,000</td>
<td>1.65</td>
</tr>
<tr>
<td>180,000</td>
<td>1.75</td>
</tr>
<tr>
<td>200,000</td>
<td>1.80</td>
</tr>
<tr>
<td>220,000</td>
<td>1.80</td>
</tr>
</tbody>
</table>

Hence $n_{\text{max}} = 200,000$ applications, as before.
B. Reference Gamble (see Section II.B.3):

Given the mission time $n_i$

Proposition 1: there is a 50% chance of obtaining $R_{\text{min}} = 0.90$ and 50% chance of obtaining $R_{\text{max}} = 0.99$.

Proposition 2: A reliability value $R_i'$ can be obtained for certain.

At what value of $R_i'$ would you be indifferent between the two propositions?

<table>
<thead>
<tr>
<th>$n_i$</th>
<th>$R_i'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100,000</td>
<td>0.930</td>
</tr>
<tr>
<td>120,000</td>
<td>0.927</td>
</tr>
<tr>
<td>140,000</td>
<td>0.924</td>
</tr>
<tr>
<td>160,000</td>
<td>0.922</td>
</tr>
<tr>
<td>180,000</td>
<td>0.921</td>
</tr>
<tr>
<td>200,000</td>
<td>0.920</td>
</tr>
</tbody>
</table>

C. Preference Function for Reliability at Constant Mission Time

As explained in Section II.E.4, the above information is utilized to construct $\mu(R|n_i)$.

Let $\mu_0$ (benefit at minimum mission time $n_{\text{min}}$ and minimum reliability $R_{\text{min}}$) be 1:

$$\mu_0 = 1.0.$$
From Section III.E.2,

\[ \mu(R_{\max}^i | h_{\min}^i) = \mu_0^{\prime} \]
\[ \mu(R_{\max}^i | h_i^i) = \mu_0^{\prime} \]
\[ \mu(R_{\max}^i | h_{\min}^i) = \Omega_i \mu_0 \]

From Section II.E.3,

\[ \mu(R_i^i (.50), n_i) = (\mu_0^i/2)(\Omega_i m_i + p_i) \]

The preference function for reliability at constant mission time \( \mu(R_i^i) \) suggested in Section II.E.4, has been fitted to these points:

\[ \mu(R_i^i) = A - B \left\{ (C-R)^{-d} \right\} \]

The values of the constants for \( n_i \) are as follows:

<table>
<thead>
<tr>
<th>( n_i )</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>100,000</td>
<td>2.00</td>
<td>100.63</td>
<td>0.99</td>
<td>1.846</td>
</tr>
<tr>
<td>120,000</td>
<td>2.60</td>
<td>171.85</td>
<td>0.99</td>
<td>2.103</td>
</tr>
<tr>
<td>140,000</td>
<td>3.00</td>
<td>339.52</td>
<td>0.99</td>
<td>2.422</td>
</tr>
<tr>
<td>160,000</td>
<td>3.30</td>
<td>651.99</td>
<td>0.99</td>
<td>2.683</td>
</tr>
<tr>
<td>180,000</td>
<td>3.50</td>
<td>859.71</td>
<td>0.99</td>
<td>2.833</td>
</tr>
<tr>
<td>200,000</td>
<td>3.60</td>
<td>1068.90</td>
<td>0.99</td>
<td>2.997</td>
</tr>
</tbody>
</table>

These functions are illustrated in Fig. (A.1)
D. Indifference Functions

For several Constant values of benefit, the preference functions for reliability at constant mission time are solved to get values for \( R \) and \( n_i \). That is, for given values of \( n \) and \( \mu \), the value of \( R \) is obtained from equation (A.1.13.).

The results are:

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( n_i )</th>
<th>( R )</th>
<th>( n_i )</th>
<th>( R )</th>
<th>( n_i )</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.83</td>
<td>100,000</td>
<td>0.972</td>
<td>120,000</td>
<td>0.912</td>
<td>130,000</td>
<td>0.900</td>
</tr>
<tr>
<td>2.33</td>
<td>110,000</td>
<td>0.990</td>
<td>120,000</td>
<td>0.943</td>
<td>140,000</td>
<td>0.906</td>
</tr>
<tr>
<td>2.60</td>
<td>120,000</td>
<td>0.990</td>
<td>140,000</td>
<td>0.925</td>
<td>160,000</td>
<td>0.900</td>
</tr>
<tr>
<td>2.97</td>
<td>140,000</td>
<td>0.980</td>
<td>160,000</td>
<td>0.920</td>
<td>180,000</td>
<td>0.903</td>
</tr>
<tr>
<td>3.17</td>
<td>160,000</td>
<td>0.940</td>
<td>180,000</td>
<td>0.918</td>
<td>200,000</td>
<td>0.908</td>
</tr>
<tr>
<td>3.33</td>
<td>160,000</td>
<td>0.970</td>
<td>180,000</td>
<td>0.936</td>
<td>200,000</td>
<td>0.925</td>
</tr>
</tbody>
</table>

A convex function, \( n(R|\mu) \), is fitted to the points \((n_i, R)\) corresponding to each fixed value, \( \mu \):

\[
n(R|\mu) = a_2 R^2 + a_1 R + a_0 \quad (A.1.14.)
\]

The resulting constants are:
These functions are graphed in Fig. A.2.

This concludes the analysis of the benefit to the decision maker of various combinations of n and R.
Fig. A.1. Preference Functions, $u(R|n_i)$, for $R$ at cons. $n_i$.

Fig. A.2. Indifference Functions, $n(R|\mu)$
Constant Cost Functions

There are failure cost, production cost, and material cost. Production cost is fixed and proportional to panel thickness, hence this cost component does not influence the location of the optimum cost-benefit ratio. Failure cost is essentially part of the operational cost. Since realistic data for this cost component are difficult to obtain, this component is excluded in the analysis. This omission does not affect the nature of the analysis, but certainly alters the location of the true optimum cost-benefit ratio. The following cost analysis deals with material cost only and is therefore only a first approximation to the true optimum cost-benefit ratio.

Since cost of the material is equal to cost per weight, \( C_w \), times the weight \( W \) of the material, and since the specific weight \( \nu \) of the material is constant:

\[
C = C_w \cdot \nu W.
\]

The volume \( V \) is

\[
V = a \cdot b \cdot t.
\]

Hence the cost is proportional to \( t \):

\[
C = Gt, \quad \text{where} \quad G = C_w a \cdot b \cdot \nu, \quad \text{is a constant}.
\]

It follows that a constant value of \( t \) implies constant value of material cost. Previously, it was shown that

\[
R = F_N \left( \frac{\mu}{\sigma \nu} ; 0,1 \right).
\]
where \( \mu_V = \mu_S - \mu_L \)

and \( \sigma_V = (\sigma_S^2 + \sigma_L^2). \)

But \( \mu_S = \mu_E + \ln \left[10.34 n^2 t^3 / 12(1-v^2) b^2 \right] \)

is a function of only \( t \), since other values are constants, given the material choice.

Furthermore,

\[
\mu_L = \ln \left[ 855 (\ln n)^{-0.33} \frac{(1.5 \ln n + 1.82)^{2.72}}{(1.5 \ln n + 3.02)^{1.72}} \right]
\]

and \( \sigma_L = 1.65 \frac{(1.5 \ln n + 3.02)}{(1.5 \ln n + 1.82)} \)

are functions of \( n \) only.

Since

\[
R = F_N \left( \frac{\mu_V}{\sigma_V}; 0, 1 \right) \tag{A.1.14}
\]

gives \( R \) as a function of \( n \) and \( t \), and since the cost of material is a direct function of thickness \( t \), we have a function relating the cost of material to the mission time and the reliability.

The constant material cost functions (see Chap. III) are, therefore, obtained from equation (A.1.14) by putting the thickness, \( t \), equal to a series of constant values in that equation. These functions of \( n \) in terms of \( R \) are obtained for various constant thickness (constant cost) values. Figure A.3. shows the resulting curves.
Fig. A.3. Constant material cost functions; each value of \( t \) implies a constant value of material cost.
VI  Decision Analysis

The next step is to obtain the locus of sub-optimal points of the cost-benefit ratios. Fig. A.4. shows the super-position of Fig. A.2. (indifference functions) and Fig. A.3. (constant cost functions). The line of tangency between these two sets of curves is the required locus of suboptimal points, see Fig. A.4.

Assuming that aluminium panels in thickness-multiples of 0.010 ins., the cost-benefit ratio of these thickness is obtained from Fig. A.4. by interpolating between indifference curves by the appropriate values of benefit, \( \mu \). The following table shows the resulting suboptimal cost-benefit ratios.

<table>
<thead>
<tr>
<th>Material cost</th>
<th>Benefit</th>
<th>Cost-benefit Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>((G\cdot t))</td>
<td>((\mu))</td>
<td>((G \cdot t/\mu))</td>
</tr>
<tr>
<td>.11G</td>
<td>2.48</td>
<td>.0443G</td>
</tr>
<tr>
<td>.12G</td>
<td>3.00</td>
<td>.0400G</td>
</tr>
<tr>
<td>.13G</td>
<td>3.24</td>
<td>.0402G</td>
</tr>
<tr>
<td>.14G</td>
<td>3.38</td>
<td>.0415G</td>
</tr>
<tr>
<td>.15G</td>
<td>3.47</td>
<td>.0422G</td>
</tr>
<tr>
<td>.16G</td>
<td>3.56</td>
<td>.0450G</td>
</tr>
</tbody>
</table>

From the above table, the optimum cost-benefit ratio is seen to be 0.0400, so that the optimum design thickness is \( t^* = 0.120 \) ins. The corresponding optimum values of the decision parameters are \( R^* = 0.918 \) and \( n^* = 164,000 \). The
interpretation of this optimum solution (based on material cost only) is that the decision maker minimizes cost-benefit; furthermore, for a panel mission time corresponding to n=164,000 load applications, the proportion of surviving shear panels is 0.918.
Fig. A.4. Locus of Suboptimal Points
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APPENDIX II

Analysis of Preference Function.

In Section III.E.4 a function \( \mu(R|n_i) \), was suggested

\[
\mu(R|n_i) = A - B \left[ (c-R) \exp(c-R) \right]^D
\]

(A.2.1)

where

A = \( M_{si} \)

C = R_{\text{max}}

D = \frac{\ln(M_{zi} - M_{si}) - \ln(M_{zi} - M_{ii})}{\ln(R_{\text{max}} - R_{\text{min}})}

B = \frac{(M_{zi} - M_{ii})}{\ln(R_{\text{max}} - R_{\text{min}})} \exp(R_{\text{min}} - R_{\text{max}})

This function is to meet the conditions, established in Section III-C, that

a) \( \mu'(R|n_i) \geq 0 \)

b) \( \mu''(R|n_i) \leq 0 \)

c) \( \mu'(R_{\text{max}}|n_i) = 0 \)

The purpose of this appendix is to show that \( \mu(R|n_i) \) meets the conditions listed above.

\[
\mu(R|n_i) = A - B \left[ (c-R) \exp(c-R) \right]^D
\]

(A.2.1)

\[
\mu'(R|n_i) = BD \exp[(c-R)D] (c-R)^D-1 (c-R+1)
\]

(A.2.2)

\[
\mu''(R|n_i) = -BD \left[ \exp(c-R) \right]^2 (c-R)^D-2 \frac{[D (c-R+1)^2 - 1]}{(c-R+1)^2 - 1}
\]

(A.2.3)
Substitute \( R = R_{\text{max}} \) in \( \mu'(R|n_i) \)

\[
\mu' = B \cdot D \cdot e^{(C-R_{\text{max}})} \cdot (C-R_{\text{max}})^{D-1} (C-R_{\text{max}} + 1)
\]

Since \( C = R_{\text{max}} \),

\[
(C-R_{\text{max}}) = 0,
\]

\[
\mu'(R_{\text{max}}|n_i) = B \cdot D \cdot e^0 \cdot 0 \cdot 1
\]

\[
\mu'(R_{\text{max}}|n_i) = 0.
\]

The third condition is met.

In order to analyze the first two conditions, first the constants are analyzed.
The characteristics of inputs of the function are:

a. \( 0 < R_{\text{min}} < R_i < R_{\text{max}} < 1.0 \) \hspace{1cm} (A.2.4)

b. from Section II.E.3
\[ R_i' \leq 0.5 (R_{\text{max}} + R_{\text{min}}) \]
\[ \therefore R_i - R_{\text{min}} \leq R_{\text{max}} - R_i' \] \hspace{1cm} (A.2.5)

c. since \( 0 < R_{\text{min}} < R_i < R_{\text{max}} < 1. \)
\[ 0 < R_{\text{max}} - R_{\text{min}} < 1. \] \hspace{1cm} (A.2.6)
\[ 0 < R_{\text{max}} - R_i' < 1. \] \hspace{1cm} (A.2.7)
\[ 0 < R_i' - R_{\text{min}} < 1. \] \hspace{1cm} (A.2.8)
To analyze constant \( D \),

Let \( X = R_i - R_{\text{min}} \)

\[ Y = R_{\text{max}} - R_i \]

\[ X + Y = R_{\text{max}} - R_{\text{min}} \]

From eg. A.2.6. thru eg. A.2.8.

\[ 0 < x < 1 \] (A.2.9)

\[ 0 < y < 1 \] (A.2.10)

\[ 0 < x + y < 1 \] (A.2.11)

We have

\[ e^x = 1 + x + \frac{x^2}{2!} + \cdots \]

and

\[ (1-x)^{-1} = 1 + x + x^2 + \cdots \]

Looking into above expansions of \( e^x \) and \((1-x)^{-1}\)

\[ \frac{x^2}{2!} < x^2 , \quad \frac{x^3}{3!} < x^3, \cdots \]

Therefore

\[ e^x < (1-x)^{-1} \]

\[ e^x < 1 + \frac{x}{1-x} \] (A.2.12)

since \( X > 0 \)

\[ \frac{(e^x-1)}{x} < \frac{1}{(1-x)} \] (A.2.13)

from eg. A.2.11.

\[ y < (1-x) \]

and since \( 0 < y < 1 \)

\[ \frac{1}{y} > \frac{1}{1-x} \]
using eq. A.2.13

\[
\frac{e^x - 1}{x} < \frac{1}{1-x} < \frac{1}{y} \quad (A.2.14)
\]

\[
e^x < \frac{y+x}{y}
\]

Resubstituting values for x and y.

\[
e^{\exp (R'_c - R_{\min})} < \frac{(R_{\max} - R_{\min})}{R_{\max} - R'_c} \quad (A.2.15)
\]

taking natural logarithm of both sides

\[
R'_c - R_{\min} < \ln (R_{\max} - R_{\min}) - \ln (R_{\max} - R'_c)
\]

\[
\ln (R_{\max} - R_{\min}) - \ln (R_{\max} - R'_c) > R_{\min} - R'_c > 0
\]

This is the denominator of constant D, therefore we have shown that denominator of D is positive.

The numerator of constant D is

\[
\ln (\mu_{3i} - \mu_{li}) - \ln (\mu_{3i} - \mu_{2i})
\]

From Section II.E.3.

\[
\mu_{2i} = 0.5(\mu_{3i} + \mu_{li})
\]

\[
\mu_{3i} - \mu_{li} = 2(\mu_{2i} - \mu_{li}) \quad (A.2.17)
\]

Hence

\[
\ln (\mu_{3i} - \mu_{2i}) - \ln (\mu_{2i} - \mu_{li}) = \ln 2 \quad > 0
\]

Since both numerator and denominator of D are positive

\[
D > 0. \quad (A.2.18)
\]
Looking into constant $B$.

$$ B = \frac{\mu_{3i} - \mu_{1i}}{[(R_{\text{max}} - R_{\text{min}}) \exp(R_{\text{min}} - R_{\text{max}})]^D} $$

\[ \mu_{3i} > \mu_{1i} \]

\[ \mu_{3i} - \mu_{1i} > 0 \]

\[ R_{\text{max}} > R_{\text{min}} \]

\[ R_{\text{max}} - R_{\text{min}} > 0 \]

\[ : \quad B > 0 \] \hspace{1cm} (A.2.19)

Constant $A$.

$$ A = \mu_{3i} > 0 $$ \hspace{1cm} (A.2.20)

Constant $C$.

$$ C = R_{\text{max}} > 0 $$ \hspace{1cm} (A.2.21)

Looking into condition (a)

$$ \mu' (R | n_i) > 0 $$

eg. A.2.2. gives

$$ \mu' (R | n_i) = BD e^{(c-R)^D} (c-R)^{D-1} (c-R+1) $$

Since

$$ c-R+1 > 0 $$

$$ B > 0 $$

and $$ D > 0 $$

$$ \mu' (R | n_i) > 0 $$

So, condition (a) is satisfied.
From A.2.8.

\[ R'_i > R_{\text{min}} \]
\[ R'_i - R_{\text{min}} > 0 \]
\[-(E'_i - R_{\text{min}}) < 0 \]

Adding expression \[ \ln \left( \frac{R_{\text{max}} - R_{\text{min}}}{R_{\text{max}} - R'_i} \right) \] to both sides.

\[ \ln \left( \frac{R_{\text{max}} - R_{\text{min}}}{R_{\text{max}} - R'_i} \right) > \ln \left( \frac{R_{\text{max}} - R_{\text{min}}}{R_{\text{max}} - R'_i} \right) - (E'_i - R_{\text{min}}) \quad (A.2.22) \]

From eg. A.2.5.

\[ R'_i - R_{\text{min}} \leq R_{\text{max}} - R'_i \]

Adding \( (R_{\text{max}} - R'_i) \) to both sides

\[ R_{\text{max}} - R_{\text{min}} \leq 2(R_{\text{max}} - R'_i) \]

\[ \ln \left( \frac{R_{\text{max}} - R_{\text{min}}}{R_{\text{max}} - R'_i} \right) \leq \ln 2. \quad (A.2.23) \]

Comparing with eg. A.2.22.

\[ \ln \left( \frac{R_{\text{max}} - R_{\text{min}}}{R_{\text{max}} - R'_i} \right) - (E'_i - R_{\text{min}}) \leq \ln 2 \]

From eg. A.2.16

\[ \ln \left( \frac{R_{\text{max}} - R_{\text{min}}}{R_{\text{max}} - R'_i} \right) - (E'_i - R_{\text{min}}) > 0. \]

Then dividing both sides of A.2.24, by the above expression.

\[ \frac{\ln 2}{\ln \left( \frac{R_{\text{max}} - R_{\text{min}}}{R_{\text{max}} - R'_i} \right) - (E'_i - R_{\text{min}})} > 1 \]
Since left hand side is equal to \( D \) (A.2.1c),

\[ D > 1. \]  

(A.2.25)

By definition of \( R_{\text{max}}' \)

\[
\begin{align*}
R_{\text{max}}' & \geq R \\
R_{\text{max}}' - R & \geq 0
\end{align*}
\]

Squaring both sides

\[
(R_{\text{max}}' - R + 1)^2 \geq 1
\]  

(A.2.26)

Since the expressions on left sides of A.2.25. and A.2.26. are both equal to or greater than one, their product is also equal to or greater than one.

\[
D (R_{\text{max}}' - R + 1)^2 \geq 1
\]

Since \( C = R_{\text{max}}' \)

\[
D (C - R + 1)^2 \geq 1
\]

\[
D (C - R + 1)^2 - 1 \geq 0
\]  

(A.2.27)

e.g. A.2.3. gives

\[
\mu'' = -1 \frac{5}{2} BD e^{(C-R)D} (C-R)^{D-2} \left[ D(C-R+1)^2 - 1 \right]^{-1}
\]

From Eg. A.2.19.

\( B > 0 \)

From eg. A.2.18.

\( D > 0 \)

Since \( C = R_{\text{max}}' \)

\( C-R > 0 \) by definition of \( R_{\text{max}}' \)

From eg. A.2.27

\[
[D(C-R+1)^2 - 1] \geq 0
\]
The expression in the large bracket on the right side of eg. A.2.28., is a product of all positive expressions, hence it is equal to or larger than zero. The whole bracket is multiplied by (-1), therefore

$$\mu''(R \setminus n_i) \leq 0$$

The constraint (b) is, therefore, met.
APPENDIX XIII

Estimation of Weibull Extreme by Log-normal

In Appendix I the load, a random variable, is modeled by a Weibull model. In this Section it will be shown how the type I asymptote of a largest observation of a Weibull model is estimated by a Log-normal model.

The load is modeled by a Weibull distribution with parameters $\sigma_w$ and $\lambda_w$. The density and distribution function of load are

$$f_w(L; \sigma_w, \lambda_w) = \frac{\lambda_w}{\sigma_w} \cdot L^{\lambda_w - 1} e^{-\left(\frac{L}{\sigma_w}\right)^\lambda_w} \quad (A.3.1)$$

$$F_w(L; \sigma_w, \lambda_w) = 1 - e^{\frac{-L}{\sigma_w} e^{\lambda_w}} \quad (A.3.2)$$

The type I asymptote of a largest observation from a Weibull model has parameters

$$\mu_e = \sigma_w \left(\ln n\right)^\lambda_w \quad (A.3.3)$$

$$\sigma_e = \frac{\sigma_w}{\lambda_w} \left(\ln n\right)^{1 - \lambda_w} \quad (A.3.4)$$

see reference (2).

In order to estimate the type I asymptote by a log-normal model, "estimation by quantiles" is employed. Since the right tail of the maximum-load distribution is of high importance, two high-order quantiles, $L_{0.85}$ and $L_{0.95}$ are chosen for the estimation process.

The quantile of order $L_q$ for an extreme value variable is

$$L_q = \mu_e - \left(\ln \ln \frac{1}{q}\right) \sigma_e \quad (A.3.5)$$
For a log-normal model the parameters $\mu_L$ and $\sigma_L$ in terms of quantiles $L_{q_1}$ and $L_{q_2}$ are

$$\sigma_L = \frac{\ln L_{q_2} - \ln L_{q_1}}{z_{q_2} - z_{q_1}}$$  \hspace{1cm} (A.3.6)$$

$$\mu_L = \ln L_{q_1} - z_{q_1} \cdot \sigma_L$$  \hspace{1cm} (A.3.7)$$

Substituting A.3.6. for $L_{q_1}$ and $L_{q_2}$ into the above equations:

$$\sigma_L = \frac{\ln\left[\mu_e - (\ln \ln \frac{q_2}{q_1}) \sigma_e\right] - \ln\left[\mu_e - (\ln \ln \frac{q_2}{q_1}) \sigma_e\right]}{z_{q_2} - z_{q_1}}$$  \hspace{1cm} (A.3.8)$$

$$\mu_L = \ln\left[\mu_e - (\ln \ln \frac{q_2}{q_1}) \sigma_e\right] - z_{q_1} \cdot \sigma_L$$  \hspace{1cm} (A.3.9)$$

Substituting A.3.4. and A.3.5. for $\mu_e$ and $\sigma_e$ in the above equations:

$$\sigma_L = \sigma_L (\sigma_w, \lambda w, \eta)$$  \hspace{1cm} (A.3.10)$$

$$\mu_L = \mu_L (\sigma_w, \lambda w, \eta)$$  \hspace{1cm} (A.3.11)$$

In equations A.3.11. and A.3.12., $\sigma_L$ and $\mu_L$, the parameters of log-normal approximation of the type I asymptote model of the original Weibull model for load, are expressed in terms of the parameters of the original Weibull model for load, $\sigma_w$, and $\lambda w$.

Substituting $q_1=0.85$ and $q_2=0.95$ in eq. A.3.11. and rearranging

$$\sigma_L = 1.65 \ln \left[ \frac{\lambda w \ln \eta + 3.00}{\lambda w \ln \eta + 1.82} \right]$$  \hspace{1cm} (A.3.12)$$
Substituting $q_1=0.85$, $q_2=0.95$, and A.3.13 for $\sigma_L$ in eg. A.3.12., and rearranging gives

$$
\mu = \ln \left[ \frac{\sigma_{\omega}}{\lambda \omega} \left( \ln \frac{n}{\lambda} \right)^{\frac{1}{\lambda \omega} - 1} \frac{(\lambda \omega \ln n + 1.82)^{2.72}}{(\lambda \omega \ln n + 3.00)^{1.72}} \right]
$$

(A3.14)

In Appendix I the parameter values for the original Weibull model for load are

$$
\sigma_{\omega} = 1.28 \text{ kips}
$$

and

$$
\lambda \omega = 1.5
$$

Substituting these values in equation A.3.13 and A.3.14, we have

$$
\mu = \ln \left[ 0.873 (\ln n)^{-0.53} \frac{(1.5 \ln n + 1.82)^{2.72}}{(1.5 \ln n + 3.00)^{1.72}} \right]
$$

(A.3.15)

$$
\sigma_L = 1.65 \ln \left[ \frac{1.5 \ln n + 3.00}{1.5 \ln n + 1.82} \right]
$$

(A.3.16)
APPENDIX IV

Analysis of Indifference Function.

In Section II. E. 5., indifference functions are developed. An indifference function is a collection of points at which the benefit has the same value. Thus, any specified level of benefit defines an implicit function between the mission time and reliability.

Since the benefit \( \mu \) is a function of both \( R \) and \( n \), the indifference function for benefit level \( \mu_0 \) may be obtained from

\[
\mu_0 = f(n, R), \tag{A.4.1}
\]

which is an implicit function in two variables. Hence there exists a function \( g \) such that

\[
n = g(R, \mu_0). \tag{A.4.2}
\]

This is the indifference function for benefit level \( \mu_0 \).

In fact, function \( g \) defines one indifference function for each given level of benefit, \( \mu \).

To investigate the slope of \( g \), substitute for \( n \) in

(A.4.1) from (A.4.2):

\[
\mu_0 = f\left[g(R, \mu_0), R\right]. \tag{A.4.3}
\]

differentiating with respect to \( R \):

\[
o = f_n g' + f_R \tag{A.4.4}
\]

where

\[
g' = \frac{dg}{dR},
\]

\[
f_n = \frac{df}{dn},
\]

\[
f_R = \frac{df}{dR};
\]
and so
\[ -g' = \frac{f_R}{f_n} \]  \hspace{1cm} (A.4.5)

But from Sections II.B.2 and II.C.2 we know that the slopes of preference function for reliability $f_R$, and preference function for mission time $f_n$, are both positive. Hence
\[ -g' = \frac{f_R}{f_n} > 0 \]  \hspace{1cm} (A.4.6)

This shows that when one variable is decreased, and benefit is to remain constant, the other variable must be increased.

Differentiating $g'$ with respect to $R$, we get
\[ g'' = -(f_n)^{-3} \left[ f_{nn} f_R^2 - 2 f_{n} f_{nR} f_R + f_{RR} (f_R)^2 \right] \]  \hspace{1cm} (A.4.7)

From Sections II.B.3 and II.C.3 we know that the second derivatives of $\mu(R/n)$ with respect to $R$, $f_{RR}$, and $\mu(n/R0)$ with respect to $n$ are both negative. Since both $f_n$ and $f_R$ are positive, $f_{nR}$ is also positive. Hence the terms within the bracket in (A.4.7) are all negative terms, and so
\[ g'' > 0 \]  \hspace{1cm} (A.4.8)

Hence indifference function at a constant benefit value is a convex function with negative slope and positive second derivative.