PARAMETRIC EXCITATIONS OF A TRAVELING BEAM AND THE WASHBOARDING PROBLEM IN BANDSAWING

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ABSTRACT

This thesis investigates the effect of primary cutting parameters and simple bandsaw blade geometry on the occurrence and severity of Type I washboarding in bandsaws. The mechanisms causing Type I washboarding are not currently well understood. To achieve a better understanding of the problem, both an analytical approach dealing with parametric axial excitations of a traveling beam, and an experimental approach are undertaken and the results of both studies are compared.

The effect of in plane axial blade loading on a simple traveling beam model is examined. The governing differential equations of motion contain parametric stiffness terms that require the use of a perturbation method for an approximate solution. The perturbation method used in this work is the method of multiple scales. In one blade model, the effect of axial tension fluctuations at the tooth passing frequency due to fluctuations in the total axial cutting load is studied. In the second model, the effect of low frequency axial tension fluctuations in combination with a lateral excitation at the tooth passing frequency is studied. The second model gives response characteristics very similar to those seen in experimental cutting tests including a response at the tooth passing frequency and a response below the tooth passing frequency by an amount equal to the low frequency axial tension fluctuation.

Cutting test data is examined to determine the pertinent response characteristics during Type I washboarding. It is found that two main components of response lead to Type I washboarding; a response at the tooth passing frequency and one at a frequency slightly lower than the tooth passing frequency. The behavior of these two responses with changes in primary cutting parameters is very similar to what is predicted by the second traveling beam model.
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NOMENCLATURE

\[ \alpha \] natural frequencies of traveling beam with tension and centripetal effects

\[ \chi \] non-dimensional parameter for speed and tension effects in traveling beam

\[ \delta \] variational operator, Kronecker delta

\[ \delta_x \] non-dimensional parameter for speed, tension and axial cutting load effects in traveling beam

\[ \epsilon \] small parameter relating tension fluctuation to beam buckling load

\[ \phi_i \] mode shapes of stationary beam free vibration problem

\[ \gamma \] non-dimensional parameter incorporating coriolis effects, exponent determining stability of first order multiple scales solutions

\[ \eta \] non-dimensional lateral cutting coefficient

\[ \lambda_i \] natural frequencies of unperturbed damped traveling beam problem

\[ \zeta \] non-dimensional axial cutting force parameter

\[ \rho \] linear mass density of traveling beam

\[ \sigma \] detuning parameter used in multiple scales analysis

\[ \tau \] rescaled time, magnitude of axial tension fluctuation

\[ \omega \] axial tension fluctuation frequency used in analytical model

\[ \omega_0 \] natural frequency of unperturbed, undamped traveling beam

\[ \Omega \] lateral force frequency in analytical model

\[ \Gamma_i \] coefficient in multiple scales equations for \( A_i \)'s

\[ \Psi_{mr} \] coefficient in damped multiple scales equations

\[ a_i \] magnitude of solution for \( A_i \) in multiple scales solutions

\[ c \] blade speed

\[ c_{ii} \] non-dimensional Rayleigh damping coefficients

\[ cc \] complex conjugate terms

\[ f_b \] band passing frequency

\[ f_i \] gyroscopic natural frequency of traveling beam

\[ f_{ij} \] parametric coupling coefficients

\[ f_w \] wheel rotation frequency

\[ g_i \] coefficients of gyroscopic matrix

\[ i \] square root of (-1)
Nomenclature

i, j  integers
n, m  integers
t  time
ui  i\textsuperscript{th} modal coordinate of discretized traveling beam problem
u_{i0}  zero order multiple scales component of i\textsuperscript{th} modal coordinate
u_{i1}  first order multiple scales component of i\textsuperscript{th} modal coordinate
w  trial function for Galerkin finite element formulation
x  spatial dimension along blade/beam axial direction
y  deflection of beam out of plane
Ai  magnitude of i\textsuperscript{th} response component for multiple scales solution
Bi  magnitude of forced zero order multiple scales solution
Di  i\textsuperscript{th} order differential operator for multiple scales problem
El  beam bending rigidity
F  lateral harmonic force magnitude
F_3  tertiary response frequency
F_{ex}  excited response frequency
F_{tp}  tooth passing frequency
Hi  i\textsuperscript{th} Hermitian shape function for beam element
L  Lagrangian, length of cutting blade between guides
M  beam end bending moment
N  number of elements in finite element formulation
P_0  axial cutting load for single tooth
R_0  static bandsaw blade tension
T  beam total kinetic energy
T_0  multiple scales time scale
T_1  tension in top half of traveling beam, multiple scales time scale
T_2  tension in bottom half of traveling beam
V  beam end shear force
V_a  component of beam potential due to axial load effects
V_b  component of beam potential due to bending effects
V_{tot}  total beam potential energy
\Delta F  frequency difference
\Delta F_s  secondary frequency difference
\{d_e\}  vector of elemental displacements for traveling beam element
Nomenclature

[f] matrix of parametric coupling coefficients
[B] Jordan Canonical form
[G] assembled system gyroscopic matrix for traveling beam problem
[G°] elemental gyroscopic matrix for traveling beam element
[H] matrix of Hermitian shape functions
[H]^T transpose of matrix of Hermitian shape functions
[K°] elemental stiffness matrix for traveling beam element
[K°₁] elemental bending stiffness matrix for traveling beam element
[K°₂] elemental speed and tension stiffness matrix for traveling beam element
[K₃] time invariant portion of assembled stiffness matrix for traveling beam problem
[K₄] time dependent portion of assembled stiffness matrix for traveling beam problem
[M] assembled mass matrix for traveling beam problem
[M°] elemental mass matrix for traveling beam element
[P] modal matrix
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CHAPTER 1: INTRODUCTION

1 Background

Washboarding is a problem of major importance in the wood cutting industry in both circular saws and bandsaws. Washboarding is characterized by a sinusoidal pattern left on the sawn surface of wood due to high frequency transverse vibration of the sawblade. This pattern is removed subsequent to cutting and accounts for a significant amount of waste sawdust. In experimental cutting studies, two different types of washboarding patterns are observed. These two types of washboarding are labeled Type I and Type II and are shown in Figures 1.1 and 1.2 below:

![Figure 1.1 - Type I washboarding pattern](image1)

![Figure 1.2 - Type II washboarding pattern](image2)

It has been hypothesized that Type II washboarding is caused by lateral regenerative cutting forces and this model has been examined by Tian (1998 [1]) for unguided circular saws and Luo (2001 [2]) for bandsaws.

The cause of Type I washboarding in bandsaws is not yet fully understood but experimental observations have given some clues as to the underlying physics of the
problem. One such experimental observation is that Type I washboarding is strongly
dependent on the depth of wood being cut. In industry as well as in experimental cutting
tests, it is observed that washboarding typically only occurs above a certain depth of cut.
It is evident that the depth of cut determines how many teeth are engaged in the
workpiece at a given instant; with the larger the depth of cut, the more teeth in the wood.
One means of explaining this phenomenon is that as the number of teeth engaged in the
wood increases, the work done on the blade by the workpiece through interaction with
the teeth increases above the level of energy dissipation present in the system and
unstable or limit cycle oscillations occur.

Another important observation is that Type I washboarding behavior is also strongly
dependent upon the amount of material removed by each cutting tooth (bite). As bite
increases, the severity of the washboarding increases as well. It is well established that
increasing bite is associated with increasing the cutting forces acting on each tooth. The
largest component of the cutting force acts in the plane of the blade in the axial direction
and as a result, affects the in plane stress distribution in the sawblade.

Two other very important parameters affecting Type I washboarding are blade speed
and the level of blade tension. All four of these operating parameters will be investigated
in detail throughout this work.

In order to gain a better understanding of what causes Type I washboarding and how it
responds to changes in the operating parameters, two simple models will be developed
and compared to the characteristics observed during actual cutting tests.

2 Physics of Parametric Excitations

The basic problem considered in this work involves the modeling of a traveling beam
subjected to different types of loading. In order to investigate the behavior of such a
model, the continuous system described by a partial differential equation must be
discretized to give a system of ordinary differential equations. The class of ordinary
differential equations considered in this work contains parametric terms. Parametric
excitations occur whenever some physical property of the system involved in the
governing differential equation(s) of motion is time dependent. This property manifests
itself as non-autonomous, periodic coefficients in the equations of motion of the system.
Chapter 1: Introduction

For a regular linear gyroscopic mechanical system with speed dependent damping but without parametric excitation or any other external excitation, the discretized equations of motion may be given by:

\[ [M]\ddot{x} + [C + G]\dot{x} + [K]x = 0 \]  
\( \text{Eq. 1.1} \)

where the elements of \( M \) represent the mass and inertial characteristics of the system, the elements of \( C+G \) represent the damping and gyroscopic characteristics of the system and the elements of \( K \) represent the stiffness characteristics of the system. The \( x_i \) represent some generalized coordinates of the system that are functions of time. Note that in this formulation, all elements of the above-mentioned matrices are time independent.

In the bandsaw washboarding problem, it will be seen that the in plane stress field present in the bandsaw blade is an important contributor to the lateral stiffness of the blade. It will also be shown that the in plane stress field varies with time. In this manner, the blade lateral stiffness characteristics will also depend on time so that the discretized system equations of motion can be given by:

\[ [M][\dddot{x}] + [C + G][\ddot{x}] + [K(i)][\dot{x}] = 0 \]  
\( \text{Eq. 1.2} \)

where the elements of \( K(t) \) are time dependent and more specifically, periodic. Typically, exact analytical solutions are available for Eq. 1.1 and the stability behavior of the system can be studied by examining the nature of the eigenvalues of the characteristic equation for the system. Eq. 1.2 on the other hand, is not readily handled using analytic methods and approximate solutions must be investigated. The stability behavior is no longer simply analyzed by looking at system eigenvalues and a more rigorous approach using some sort of perturbation method must be taken. The perturbation method that will be used throughout this work is the method of multiple scales.

In addition to the homogeneous class of parametrically excited problems given by Eq. 1.2, the non-homogeneous case will also be considered where:
This problem corresponds to the case of a traveling beam (the bandsaw blade) with a fluctuating axial stress field as well as an applied harmonic lateral load. It is expected that both parametric and regular resonance phenomena will occur. The approximate solutions of the problem given in Eq. 1.3 have interesting features that may be related to the washboarding problem.

### 3 Previous Research

General research on the dynamic behavior of bandsaws has been pursued extensively over the past three decades. The bulk of the early work considered the low frequency behavior of the saws and dealt with the first handful of natural frequencies and mode shapes. A thorough review of this literature is given by Ulsoy and Mote (1978 [3]).

Parametric excitations in bandsawing are a more specific case of problem that have been investigated by a number of authors. The nature of the applied load leading to parametric terms in the equations of motion and the mathematical method used in their solution differs between authors but many similar results are found.

Wu and Mote (1986 [4]) investigated parametric instabilities in an axially moving thin beam (one dimensional model) due to harmonic distributed edge loading, \( p(x,t) \), in the plane of the band perpendicular to the direction of motion as shown in Figure 1.3. The equations of motion were analyzed using the method of multiple scales. It was found in this work that combination bending-torsion parametric instabilities and simple torsion parametric instabilities are the only parametric instabilities that can occur. In addition, the gyroscopic terms in the equations of motion are vital in determining the proper stability boundaries.
Lengoc and McCallion (1995 [5]), as the second part of a three part study, investigated separately the effect of periodic axial tension fluctuation and periodic tangential (axial) cutting force fluctuation on a two dimensional moving smooth blade using the method of harmonic balance. A schematic of their model is seen below in Figure 1.4. The instability behavior of the band was looked at in the range of the first four natural frequencies of the moving band. It was found that simple parametric resonances are possible for the case of periodic axial tension and combination parametric resonances dominate the behavior of a blade with periodic tangential edge loading.

The bulk of research conducted on bandsaw vibration problems deals with the low frequency behavior of the bandsaw and in particular, the first handful of natural frequencies and mode shapes. The washboarding phenomena is a high frequency problem and given the geometry of the toothed bandsaw blade, likely deals with complex mode shapes.
Chapter 1: Introduction

In order to investigate the high frequency behavior of washboarding in bandsaws, Luo (2001 [2]) studied the effect of regenerative moving cutting forces on both a smooth traveling band as well as a toothed traveling band in the washboarding frequency range. The regenerative lateral forces resulted in time delay, parametric stiffness terms in the discretized equations of motion. The approximate solution of this problem was found to agree well with the experimental results for Type II washboarding.

As mentioned in the previous section, the method of multiple scales will be used to find approximate solutions to a parametrically excited traveling beam problem. The method of multiple scales has many applications in the solution of systems of ordinary differential equations, particularly in the solution of non-linear equations and those with small non-autonomous coefficients as will be investigated later in this work. Nayfeh and Mook (1977 [6]) investigated the vibrations of a fixed free beam column subjected to a harmonically varying compressive follower end load. The solution of this problem is very similar to the problem of Chapter 3 as well as the homogeneous form of the model developed in Chapter 4.

4 Objectives and Scope

This thesis aims to give qualitative understanding of the mechanism(s) that might possibly cause Type I washboarding in bandsaws and how certain operating parameters affect the washboarding behavior. The two models presented in this thesis were conceived through consideration of the experimental observations of how Type I washboarding behavior is affected by the primary operating parameters. These physical models led to mathematical equations governing the behavior of the system. It is important to note that a number of simplifying assumptions have been made in the development of the mathematical models. These mathematical equations were then solved using approximate methods and the behavior of the models was compared to the results found experimentally.

In order to accomplish these tasks, the following sub-objectives were completed:
Chapter 1: Introduction

1. To describe the bandsaw system, the important operating parameters, the experimental testing program undertaken and the general response characteristics that lead to Type I washboarding.

2. To derive the equation of motion for a traveling beam subjected to time varying axial tension using Hamilton's Principle.

3. To investigate the effect of axial tension fluctuations due to cutting forces at the tooth passing frequency on the response characteristics of a traveling beam and to compare the predicted response to the experimental results.

4. To investigate the effect of axial tension fluctuations at low frequency in combination with lateral cutting forces at the tooth passing frequency and to compare the predicted response to the experimental results.

5. To investigate the effect of system damping on the response characteristics of the combined low frequency axial tension fluctuation and high frequency lateral excitation model.

6. To investigate the effect of bite, depth of cut and band strain level on the natural frequencies of a toothed blade.

7. To investigate the effect of bite, depth of cut, band strain level and blade speed on the experimental washboarding results and compare these results to those predicted by mathematical modeling.

8. To propose two possible mechanisms by which Type I washboarding occurs in wide bandsaws.

This thesis is organized in six chapters. Chapter 2 goes over basic bandsawing terminology, important process parameters, the experimental testing program conducted and some of the important blade response characteristics associated with Type I washboarding.

Chapters 3 and 4 present two different mathematical models that will attempt to uncover the underlying mechanism responsible to for Type I washboarding. Both models consider important parameters of the bandsawing process and predict certain response components as well as instability behavior.

Chapter 5 presents some pertinent experimental observations regarding Type I washboarding including the effect of important parameters on the washboarding effects.
Chapter 1: Introduction

response. The dependence of the blade natural frequencies and mode shapes on these same parameters will also be investigated and the analytical results of Chapters 3 and 4 will be compared with experiment.

Lastly, Chapter 6 gives conclusions regarding the findings of this study and also offers suggestions for work that could lend further clarification to the washboarding problem.
CHAPTER 2: FUNDAMENTAL TYPE I WASHBOARDING CHARACTERISTICS

1 Discussion of Process Variables

Bandsawing is a very complex process with a number of variables that must be considered when investigating washboarding behavior. A schematic diagram of an industrial bandsawing process is shown below in Figure 2.1.

As can be seen, the basic concept is that of a continuous steel band moving at speed \( c \), around two large wheels, one of which is driven, the other is not. On one side of the bandsaw is the cutting span of length \( L \), which is supported by two hydro-dynamically
Chapter 2: Fundamental Type I Washboarding Characteristics

lubricated guides that act to stabilize the blade and prevent unwanted transverse blade oscillations. The blade is also under a tensile load ($R_0$) created by the straining system attached to the top wheel. In addition to this tensile blade load, cutting forces act on one edge of the blade at each of the teeth located in the wood being cut. These cutting forces have a three dimensional nature and depend on a number of parameters including tooth geometry and the characteristics of the wood being cut. A number of variables are used to describe the shape of these teeth. Some of these variables are illustrated in Figure 2.2 below:

![Figure 2.2](image)

**Figure 2.2 – Geometric properties of bandsaw teeth**

The main geometric variables that will be dealt with in this work are the depth of gullet and the hook length.

In a typical bandsawing process, a number of operating variables can be controlled. These include blade speed, band tension, depth of cut, and bite. Blade speed is simply the speed at which the blade travels. Band tension is created by moving the top wheel away from the bottom via some straining mechanism thus inducing tension in the band. Depth of cut is defined as the height of the wood being cut and bite is the amount of wood removed per tooth. One other very important parameter is the tooth passing frequency ($F_{tp}$). The tooth passing frequency is dependent upon both the blade speed and the tooth pitch, $p$, and represents the frequency with which the cutting teeth pass a certain fixed point in space.
Chapter 2: Fundamental Type I Washboarding Characteristics

2 Experimental Washboarding Trials

Between February of 2001 and September of 2002, a series of cutting tests were conducted at Forintek Canada Corp. Vancouver using their 5' Cetec double-column bandmill. The object of these tests was to examine the effect of process parameters and tooth shape on the occurrence and severity of Type I washboarding in an industrial bandsaw. Three blades of different thicknesses (16, 17 and 18 gauges) were used in the cutting trials. All other aspects of the blade geometry were identical, including blade width and tooth profile. Each blade was tested for varying speeds, strains, bites and depths of cut. Once sufficient testing was performed with each blade and each tooth configuration, the tooth shape (depth of gullet and hook length) was systematically modified and the tests were repeated. In this manner, the effect of tooth shape was investigated. A detailed explanation of the apparatus, methods, analysis and results of this study were documented by Taylor et al. (2003 [7]).

In order to differentiate between individual cutting tests, a system was developed to describe the parameters used in each cut. Each cut is designated by a name such as 'b520112140a', where the leading 'b' refers to the main parameter being investigated, the '520' refers to a wheel speed of 520 rpm, the '112' refers to the feed speed in feet per minute, the '140' refers to a strain level of $R_0 = 14$ klbs and the final 'a' refers to the number of the test that was performed using the configuration described. Feed speed is the speed at which the wood being cut moves towards the blade and combined with the tooth passing frequency, determines the bite.

In order to describe the magnitude of the washboarding seen during each specific cutting test, a system was established to describe the area of the sawn surface covered with a washboarding pattern as well as the severity of the washboarding pattern. The extent of washboarding is described by a percentage value were 0% indicates no washboarding visible anywhere on the sawn surface and 100% indicates the entire sawn surface showing a washboarding pattern. The washboarding severity is indicated by a single letter, F, L, M or H. "F" indicates a very faint washboarding pattern not easily seen but noticeable to the touch. "L" indicates a light washboarding pattern that can be seen and is more noticeable to the touch. "M" indicates a medium washboarding pattern that
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is easily visible and noticeable to the touch. "H" indicates a heavy washboarding pattern that is easily seen with very significant raised ridges when felt.

Non-contact eddy current displacement probes were used to measure the blade lateral displacement. Four probes were used, three located immediately below the wood being cut and one above as seen in the schematic testing setup in Figure 2.3.

![Experimental setup for cutting trials](image)

Figure 2.3 – Experimental setup for cutting trials

The single probe located above the cut is used to track the tooth passing frequency while the three below the cut measure blade displacements.

As mentioned, over the course of the cutting tests, the tooth shape was periodically modified. A naming system was put in place to keep track of the blade and tooth
configurations. The blade designation consists of the gauge number (either 16, 17 or 18) followed by the gullet modification state and then the hook modification state. A sample blade designation is 17G1H0 where the 17 refers to the blade gauge, G1 refers to the first gullet modification and H0 refers to the original hook shape.

A modification to the gullet consists of increasing the depth of gullet by 1/16". An increment in hook modification number consists of increasing the tooth hook length by 1/16". A plot of the tooth shapes from G0H0 to G2H2 is shown below in Figure 2.4 for the 18 gauge blade:

![Tooth Profiles of the 18 Ga. Blade](image)

**Figure 2.4** – Tooth profiles for different modification states of 18 gauge blade

Using the non-contact eddy current probes, the lateral vibration behavior of the blade was recorded throughout each cut.

### 3 Tooth Passing Frequency, Excited Frequency, and Frequency Difference

Using the data acquired from the eddy current probes, a vibration spectrum history plot can be produced to show vibration power spectrum density (PSD) at the measured point as time evolves during the cut. A typical PSD plot of this kind is shown below in Figure 2.5.
The data in Figure 2.5 comes from the probe mounted closest to the tooth gullet. This probe location appears to be the most pertinent to the washboarding behavior of the sawblade as it is closest to the teeth which are responsible for the washboarding pattern left on the wood surface. The data shown in Figure 2.5 gives the entire frequency response of the blade at the measured point from 0 Hz all the way to approximately 1200 Hz (higher than the tooth passing frequency). A number of excitations occur over the range shown. Due to kinematic considerations developed by Lehmann and Hutton (1997 [8]) and Okai et al. (1996 [9]) and physical measurements of the washboarding pattern left on the wood, a much narrower frequency band can be investigated. In this case, the frequency range of interest is from 875 Hz to 950 Hz and is shown below in Figure 2.6.
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In this frequency window, three primary responses are seen. The first of these occurs at the tooth passing frequency, $F_{tp}$ (approximately 1030 Hz). The second occurs below the tooth passing frequency and is typically called the excited frequency, $F_{ex}$ (approximately 1010 Hz). The third response occurs above the tooth passing frequency at $F_3$ (approximately 1050 Hz). Experimental results show that this third response is not always visible and when visible, is always smaller in magnitude than either of the first two responses. It is thought that the response at the tooth passing frequency occurs due to some blade excitation at the tooth passing frequency. It is not yet understood what produces the excited response at $F_{ex}$ and the tertiary response at $F_3$ above $F_{tp}$. At present, it is thought that the response at $F_{ex}$ is either at a natural frequency or at some other frequency corresponding to some unknown blade excitation. Another fundamental feature of Figure 2.6 is the difference between $F_{tp}$ and $F_{ex}$ which will be called the frequency difference, $\Delta F$. The secondary frequency difference is given as $\Delta F_s$ and represents the difference between $F_3$ and $F_{tp}$. For Type I washboarding, at first glance, it appears that these frequency differences are constant throughout the cut and equal in magnitude. This phenomenon will be examined in Chapters 4 and 5. Two different
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models will be explored in this work to attempt to explain the nature of the responses seen in Figure 2.6.

It is also interesting to note that the relative magnitudes of the responses at $F_{ex}$ and $F_{tp}$ vary. For example, washboarding may occur when the response at $F_{ex}$ is larger than that at $F_{tp}$ or vice versa. In Figure 2.6, the response at $F_{tp}$ happens to be of the same magnitude as that at $F_{ex}$. It is the combination of the two responses that accounts for the washboarding pattern.

4 Low Frequency Response Characteristics

In order to describe the behaviour of the frequency difference, $\Delta F$, a parametric excitation model will be described and investigated in Chapter 4. One of the primary inputs of this model is the frequency of the parametric excitation. This frequency is of interest in possibly accounting for the frequency difference. Since this frequency difference is typically in the range of 10 - 30 Hz for Type I washboarding problems, the low frequency behavior of bandsaw blades is of interest. The aim of studying the low frequency behavior is to find a low frequency excitation component that corresponds with the measured frequency difference, $\Delta F$.

Two important low frequency excitations in bandsawing occur at the band passing frequency, $f_b$ and at the wheel rotation frequency, $f_w$. The band passing frequency corresponds to the time taken for the band to make one complete revolution. The wheel rotation frequency is simply the rate at which the wheels are turning. In addition to these two frequencies, their harmonics are also very important as can be seen in the typical low frequency waterfall plot for a washboarding blade in Figure 2.7 below.
As can be seen, excitations occur at a number of frequencies between 0 and 30 Hz. For the example chosen, the band passing frequency, $f_b$, is 4.36 Hz, and the wheel rotation frequency, $f_w$, is 9.35 Hz. The response for low frequency ($< 5$ Hz) is not clear due to large amplitude low frequency displacements. However, the $2f_b$, $3f_b$, $4f_b$, $5f_b$ and $6f_b$ responses show up clearly as marked on the Figure. It is not common to see lateral vibrations of the blade at the wheel rotation frequency, likely due to the influence of the guides in isolating the cutting span from lateral vibration caused by the wheels.

Another interesting feature of Figure 2.7 is the large amplitude response components after approximately $t = 3s$. This corresponds to the vibration of the blade immediately after the cut finishes. It is common for the blade to experience significant low frequency lateral displacement during the cut and once the cut is finished, the blade is no longer
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constrained by the wood being cut and springs back to its equilibrium position with large free vibrations of the blade as a result.

The main goal of the remainder of this work is to develop a mathematical model that predicts similar response characteristics to those seen through experiment. The model developed will then be compared with the results found from the experimental cutting tests.
CHAPTER 3: HOMOGENEOUS PARAMETRIC ONE-DIMENSIONAL BANDSAW MODEL

1 Background

In order to investigate the impact of primary operating parameters on Type I washboarding behavior, a simple mathematical model will be developed that exhibits similar response characteristics to those found in experimental cutting tests. The moving band will be modeled as a traveling beam with lateral displacement $y(x,t)$. Although this is a drastic geometric simplification, upon discretization of the equations of motion, the general behavior of the system under parametric loading will be investigated and qualitative comparisons will be made to the experimental results. The objective of this work is to gain a better understanding of the mechanisms involved in Type I washboarding rather than to accurately predict what combinations of operating parameters will lead to Type I washboarding.

Throughout this Chapter and Chapter 4, the symbol $\Omega$ will be used to represent a high frequency that corresponds to $F_{tp}$ from experiment. Using the model developed here and that in Chapter 4, the nature of the frequency difference, $\Delta F$ will be investigated. In this Chapter, the frequency difference will be compared to a natural frequency of the system given by $\omega_n$. The goal of the two models developed is to predict a response at both $\Omega$ and a frequency less than $\Omega$ by a value corresponding to the frequency difference.

Given a basic traveling beam model, it remains to determine the loads that will be applied to represent the interaction with the wood due to cutting forces as well as the tensioning load, $R_0$. As stated in Chapter 1, depth of cut and bite both play an important role in determining Type I washboarding behavior. Both of these parameters are closely related to the magnitude of the total axial cutting load applied to the blade. This Chapter will focus on a model that examines the effect of the main tangential cutting loads on a traveling beam model.
During cutting, tangential cutting loads applied at the tooth tips augment the overall in plane stress fields of the bandsaw blade which in turn will affect the blade stiffness characteristics. It should also be noted that since the blade is moving, these forces must move with the blade and combined with the fact that the depth of cut is not regularly equal to an integer number of tooth pitches, a varying number of teeth are engaged in cutting over the duration of the cut. To see this phenomenon more clearly, consider the situation in Figure 3.1 below:

At time $t_1$, five cutting teeth are engaged and the top tooth has just entered the wood. At time $t_2$, less than one tooth period later, five teeth are again engaged. At time $t_3$, the bottommost tooth shown has left the wood prior to a new tooth entering the wood from above leaving four teeth engaged in the wood. Some time later, $t_4$, at the top of the cut, another cutting tooth will enter giving five cutting teeth once again. The time taken between $t_1$ and $t_4$ corresponds to the tooth passing period. Given that each of the cutting teeth contributes to the total axial cutting load, this load will fluctuate at the tooth passing frequency. The fundamental period of this fluctuation will be equal to the tooth passing
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period. The total average cutting load will be determined by the number of cutting teeth which for a given tooth pitch, is dictated by the depth of cut.

In order to better understand how depth of cut and fluctuating blade axial loads affect the stability characteristics of industrial bandsaws, a simplified traveling beam model subjected to static axial tension and a superimposed fluctuating axial load applied at midspan will be investigated. The intent of studying this model is to simplify the complex cutting situation as a point load that varies at the tooth passing frequency about an average load proportional to the number of teeth engaged in the wood. The layout of the model is shown below:

Figure 3.2 – Traveling beam with central harmonic axial load

Given the discontinuity in axial load at the point of load application, the model domain has been split in half with the bottom part of the beam having lateral displacement \( y_1(x) \) and the top half of the beam having displacement \( y_2(x) \). Given the symmetry of the model about the \( x = L/2 \) axis and the relatively small magnitude of the total cutting load
compared to the static component of axial tension, the fluctuating axial load component will be assumed to be shared equally between the top and bottom halves of the beam. Thus, if the initial static tension in the beam is $R_0$, the total time dependent-tension in the bottom half of the beam is:

\[ T_1 = R_0 + \frac{P_0}{2} (n + \cos \Omega t) \quad \text{Eq. 3.1} \]

and that in the top half of the beam is:

\[ T_2 = R_0 - \frac{P_0}{2} (n + \cos \Omega t) \quad \text{Eq. 3.2} \]

where $P_0$ represents a force corresponding to the main cutting force on a single tooth, $n$ represents the average number of teeth engaged in the wood at a given time and $\Omega$ represents the tooth passing frequency. Note that in practice, the magnitude of $P_0$ is much smaller than that of the initial axial tension, $R_0$, so that no part of the beam will ever be in compression.

2 Derivation of Equation of Motion

The equation of motion for a traveling beam of the type illustrated above will be derived using Hamilton’s principle. In order to use this approach, the Lagrangian for the system must be developed which in turn requires expressions for the kinetic and potential energies of an infinitesimal element of the beam. For the case being considered here, there are no lateral external forces so no account will be made for work done on the system by external forces.

In order to determine the kinetic energy of an element of the beam, its velocity components must be described. Looking at Figure 3.3 below, it can be seen that the element of beam has two main velocities, the first being the blade speed in the direction of the blade, the second being the speed of the blade’s lateral displacement from the equilibrium state.
The first of these components can be resolved into two components, one being co-linear with the \( x \) direction, the other in the \( y \) direction. Thus the total velocity of the element in each of the \( x \) and \( y \) directions can be given as:

\[
\begin{align*}
    v_x &= c \cos \theta \\
    v_y &= c \sin \theta + \frac{\partial y}{\partial t}
\end{align*}
\]  

Eqs. 3.3a,b

respectively. The angle \( \theta \) represents the angle between the undeformed centerline of the blade and the displaced blade and assuming small deflections, the following holds:

\[
\theta \approx \frac{\partial y}{\partial x}
\]  

Eq. 3.4

so that Eqs. 3.3 can be rewritten as:
Chapter 3: Homogeneous Parametric One-Dimensional Bandsaw Model

\[ v_x = c \]  
\[ v_y = c \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} \]  

since \( \cos \theta \) is approximately one and \( \sin \theta \) is approximately \( \theta \). The magnitude of the absolute velocity of the blade can be given as:

\[ v_{tot} = \sqrt{c^2 + \left( c \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} \right)^2} \]  
Eq. 3.6

So that the kinetic energy of the beam element is given as:

\[ dT = \frac{1}{2} \rho \left[ c^2 + \left( c \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} \right)^2 \right] dx \]  
Eq. 3.7

Next, an expression for the potential energy of the beam must be developed. It is well known that the potential energy of a beam due to bending can be expressed as:

\[ dV_b = \frac{1}{2} EI \left( \frac{\partial^2 y}{\partial x^2} \right)^2 dx \]  
Eq. 3.8

The second component of beam potential energy is due to the axial load and is given by:

\[ dV_a = -\frac{1}{2} P \left( \frac{\partial y}{\partial x} \right)^2 dx \]  
Eq. 3.9

where \( P \) is assumed to be the axial compressive load in the beam. Now that both the kinetic and potential energies of the beam element are known, the Lagrangian for the beam may be written as:
Chapter 3: Homogeneous Parametric One-Dimensional Bandsaw Model

\[ L = T - V = \int_{0}^{L} \left[ \frac{1}{2} \rho \left( c^2 + (c \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t})^2 \right) - \frac{1}{2} \left( EI \frac{\partial^2 y}{\partial x^2} \right)^2 + \frac{1}{2} P \left( \frac{\partial y}{\partial x} \right)^2 \right] dx \]  \hspace{1cm} \text{Eq. 3.10}

Now, according to Hamilton's principle as described by Meirovitch (1970 [10]), the total variation of the above integral from arbitrary times \( t_0 \) to \( t_1 \) must be zero. This can be written as:

\[ \delta L = \delta \int_{t_0}^{t_1} L \, dx \, dt = 0 \]  \hspace{1cm} \text{Eq. 3.11}

Upon substitution of the expression for the Lagrangian (Eq. 3.10) into Eq. 3.11:

\[ \delta L = \delta \int_{t_0}^{t_1} \left[ \frac{1}{2} \rho \left( c^2 + (c \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t})^2 \right) - \frac{1}{2} \left( EI \frac{\partial^2 y}{\partial x^2} \right)^2 + \frac{1}{2} P \left( \frac{\partial y}{\partial x} \right)^2 \right] dx \, dt = 0 \]  \hspace{1cm} \text{Eq. 3.12}

The variation above can be expanded into the double integral to yield (Eq. 3.13):

\[ \delta L = \int_{t_0}^{t_1} \left[ \rho \left( c \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} \right) \delta \left( c \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} \right) - EI \frac{\partial^2 y}{\partial x^2} \delta \left( \frac{\partial^2 y}{\partial x^2} \right) + P \frac{\partial y}{\partial x} \delta \left( \frac{\partial y}{\partial x} \right) \right] dx \, dt = 0 \]

Expanding the variational operator through and collecting like terms in the variation gives Eq. 3.14:

\[ \delta L = \int_{t_0}^{t_1} \left[ \left( \rho \left( c^2 + P \frac{\partial y}{\partial x} + \rho c \frac{\partial y}{\partial t} \right) \right) \delta \left( \frac{\partial y}{\partial x} \right) - EI \frac{\partial^2 y}{\partial x^2} \delta \left( \frac{\partial^2 y}{\partial x^2} \right) + P \left( \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} \right) \delta \left( \frac{\partial y}{\partial t} \right) \right] dx \, dt = 0 \]

This variational integral involves three terms. The first contains a variation in the beam slope, the second contains a variation in the derivative of beam slope and the third contains a variation in the beam's lateral velocity. This expression can be simplified by integrating the first term by parts with respect to \( x \) once, integrating the second term by
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parts with respect to $x$ twice and integrating the third term by parts with respect to time once. The resulting expression is:

$$
\delta I = \int_0^L \left\{ \int_0^L \left[ EI \frac{\partial^4 y}{\partial x^4} + (\rho c^2 + P) \frac{\partial^2 y}{\partial x^2} + 2\rho c \frac{\partial^2 y}{\partial x \partial t} + \rho \frac{\partial^2 y}{\partial t^2} \right] dx dt \right\} \delta y dx
$$

Eq. 3.15

Once again, the entire variation above must be set to zero. The equation of motion is given in the first double integral:

$$
EI \frac{\partial^4 y}{\partial x^4} + (\rho c^2 + P) \frac{\partial^2 y}{\partial x^2} + 2\rho c \frac{\partial^2 y}{\partial x \partial t} + \rho \frac{\partial^2 y}{\partial t^2} = 0
$$

Eq. 3.16

Eq. 3.16 agrees with the equation of motion for a traveling beam given by Mote (1972 [11]). The boundary conditions can be determined from the second integral in Eq. 3.15. Knowing that the variation in displacement is zero at the boundaries (blade is constrained in displacement at the guides), $\delta y$ evaluated at $x = 0$ and $x = L$ is zero. At the same time, the variation in beam slope at the beam end points is not zero. Therefore, the natural boundary conditions are:

$$
\frac{\partial^2 y(0)}{\partial x^2} = \frac{\partial^2 y(L)}{\partial x^2} = 0
$$

Eq. 3.17

As stated, the kinematic boundary conditions are zero displacement at the guide supports or:

$$
y(0) = y(L) = 0
$$

Eq. 3.18

In the preceding derivation, the axial load, $P$, was considered a compressive load. In the case of bandsawing, the axial load, $P$, is tensile and for this particular model, can be approximated by Eqs. 3.1 and 3.2. Substituting Eqs. 3.1 and 3.2 into Eq. 3.16, the equation of motion for each of the two domains become:
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\[ EI \frac{\partial^4 y_1}{\partial x^4} + \left( \rho c^2 - R_0 - \frac{P_0}{2} (n + \cos \Omega t) \right) \frac{\partial^2 y_1}{\partial x^2} + 2 \rho c \frac{\partial^2 y_1}{\partial x \partial t} + \rho \frac{\partial^2 y_1}{\partial t^2} = 0 \]  
Eq. 3.19

\[ EI \frac{\partial^4 y_2}{\partial x^4} + \left( \rho c^2 - R_0 + \frac{P_0}{2} (n + \cos \Omega t) \right) \frac{\partial^2 y_2}{\partial x^2} + 2 \rho c \frac{\partial^2 y_2}{\partial x \partial t} + \rho \frac{\partial^2 y_2}{\partial t^2} = 0 \]  
Eq. 3.20

Eq. 3.19 is valid for \( 0 < x < L/2 \) and Eq. 3.20 for \( L/2 < x < L \). The first term in both of Eqs. 3.19 and 3.20 represents the restoring lateral force in the beam due to its bending stiffness. The second term represents the restoring lateral force due to axial load in the beam from centripetal effects, the static band strain and the applied axial cutting loads. The third term represents the lateral beam loads due to coriolis forces and the last term represents the inertial forces acting on the beam in the lateral direction.

In order to gain a better understanding of how the model variables affect the physics of the equation of motion, a non-dimensionalization process will be undertaken. To this end, suitable length and time scales must be chosen. The natural length scale to choose is the span of the blade, \( L \). Using this scale, the length variables \( x \) and \( y \) may be scaled as:

\[ x = \bar{x}L \quad \text{and} \quad y = \bar{y}L \]  
Eqs. 3.21

with an overbar indicating a dimensionless quantity. Using this scaling, derivatives with respect to \( x \) become:

\[ \frac{\partial}{\partial x} = \frac{1}{L} \frac{\partial}{\partial \bar{x}} \]  
Eq. 3.22

To non-dimensionalize time, an appropriate scale to use is the period of the first natural frequency of lateral vibration for the motionless beam:

\[ t = \bar{t}L^2 \left( \frac{\rho}{EI} \right)^{1/2} \]  
Eq. 3.23
so that derivatives with respect to time become:

\[
\frac{\partial}{\partial t} = \frac{1}{L^2 \sqrt{\rho}} \frac{\partial}{\partial t}
\]

Eq. 3.24

The non-dimensionalized tooth passing frequency now becomes:

\[
\Omega = \frac{1}{L^2} \sqrt{\frac{EI}{\rho}}
\]

Eq. 3.25

So that the PDE for \( y_1 \) becomes (where Eq. 3.26):

\[
\frac{EI}{L^3} \frac{\partial^4 y_1}{\partial x^4} - \frac{1}{L} \left( R_0 + \frac{P_0}{2} (n + \cos \Omega t) - \rho c^2 \right) \frac{\partial^2 y_1}{\partial x^2} + 2 \frac{c}{L^2} \sqrt{EI\rho} \frac{\partial^2 y_1}{\partial x \partial t} + \frac{EI}{L^3} \frac{\partial^2 y_1}{\partial t^2} = 0
\]

with the overbars omitted for simplicity. And dividing through by \( EI/L^3 \), we arrive at the non-dimensional form (Eq. 3.27):

\[
\frac{\partial^4 y_1}{\partial x^4} - \left( \frac{R_0 L^2}{EI} + \frac{P_0 L^2}{2EI} (n + \cos \Omega t) - \frac{\rho c^2 L^2}{EI} \right) \frac{\partial^2 y_1}{\partial x^2} + 2cL \sqrt{\frac{EI}{\rho}} \frac{\partial^2 y_1}{\partial x \partial t} + \frac{\partial^2 y_1}{\partial t^2} = 0
\]

where the following non-dimensional constants become apparent:

\[
\delta_\pm = \left( \frac{R_0 \pm \frac{nP_0}{2} - \rho c^2}{EI} \right) L^2, \quad \zeta = \frac{P_0 L^2}{2EI}, \quad \gamma = 2cL \sqrt{\frac{\rho}{EI}}
\]

Eqs. 3.28

The plus or minus depends upon whether the top half or bottom half of the beam is being considered. The first constant, \( \delta_\pm \), gives a measure of the lateral restoring force due to tension effects from the static band tension, the total axial cutting load and centripetal effects due to blade speed in comparison to the buckling characteristics of the beam. The second quantity, \( \zeta \), gives a measure of the component of fluctuating band...
tension in comparison to the buckling characteristics of the beam. The third non-dimensional parameter, $\gamma$, gives a measure of the time taken for a fixed point on the beam to traverse the span between guides compared to the period of the first lateral bending frequency of the beam.

In simplified form, the non-dimensionalized equation of motion for $y_1$ is:

$$\frac{\partial^4 y_1}{\partial x^4} - (\delta_1 + \zeta \cos \Omega t) \frac{\partial^2 y_1}{\partial x^2} + \gamma \frac{\partial^2 y_1}{\partial x \partial t} + \frac{\partial^2 y_1}{\partial t^2} = 0$$  \hspace{1cm} \text{Eq. 3.29}

Following the same procedure, the equation for $y_2$ becomes:

$$\frac{\partial^4 y_2}{\partial x^4} - (\delta_1 - \zeta \cos \Omega t) \frac{\partial^2 y_2}{\partial x^2} + \gamma \frac{\partial^2 y_2}{\partial x \partial t} + \frac{\partial^2 y_2}{\partial t^2} = 0$$  \hspace{1cm} \text{Eq. 3.30}

Upon non-dimensionalization, the boundary conditions (Eqs. 3.17 and 3.18) become:

$$y(0) = y(1) = 0$$  \hspace{1cm} \text{Eq. 3.31}

$$\frac{\partial^2 y(0)}{\partial x^2} = \frac{\partial^2 y(1)}{\partial x^2} = 0$$  \hspace{1cm} \text{Eq. 3.32}

From Eqs. 3.29 and 3.30, it can be seen that the behavior of the system is governed by three non-dimensional parameters, $\delta_1$, $\zeta$, and $\gamma$.

### 3 Discretization of Equation of Motion

In order to obtain an approximate solution to the equations of motion and investigate the stability behaviour of this system, a finite element discretization of the problem domain will be undertaken.

To begin, consider the possible equations of motion (Eqs. 3.29 and 3.30). Note that the equations differ depending upon the location in the beam domain. This will give rise to different element stiffness matrices depending upon the element location in the domain.
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At this stage, a Galerkin approach as outlined by Kwon and Bang (2000 [12]) will be applied to the beam differential equation of motion. To begin, an average weighted residual of Eqs. 29 and 30 will be taken:

\[
I = \int_0^l \left( \frac{\partial^4 y}{\partial x^4} - \left( \delta_z \pm \zeta \cos \Omega t \right) \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial^2 y}{\partial x \partial t} \right) w \, dx = 0 \quad \text{Eq. 3.33}
\]

where once again, w is an unknown trial function. Integrating the right hand side of Eq. 3.33 twice by parts and discretizing the domain into n elements, we arrive at (Eq. 3.34):

\[
I = \sum_{i=1}^n \left[ \int_0^{\frac{l_i}{l}} \left( \left( \delta_z \pm \zeta \cos \Omega t \right) \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial^2 y}{\partial x \partial t} \right) w \, dx + \int_0^{\frac{l_i}{l}} \frac{\partial^2 y}{\partial x^2} \frac{\partial^2 w}{\partial x \partial t} \, dx \right] + \left[ Vw - M \frac{\partial w}{\partial x} \right]_0 = 0
\]

where \( l \) represents the length of the element and V and M represent the shear and bending moments at each boundary. The beam elements to be used will have four degrees of freedom, lateral displacement at each end as well as slope at each end. Non-dimensional Hermitian shape functions as described in Appendix A, will be employed to describe the deformation of the beam, \( y(x) \) as well as the weight function, \( w(x) \). The result is given below.

\[
\sum_{i=1}^n \int_0^{\frac{l_i}{l}} \left[ H^T \left[ H \right] \{ \dot{d}^e \} \right] dx + \sum_{i=1}^n \int_0^{\frac{l_i}{l}} \gamma \left[ H^T \left[ H \right] \{ d^e \} \right] dx + \sum_{i=1}^n \int_0^{\frac{l_i}{l}} \left[ H^{-1} \right]^T \left[ H^{-1} \right] - \left( \delta_z \pm \zeta \cos \Omega t \right) \left[ H^{-1} \right]^T \left[ H \right] \{ d^e \} \right] dx \quad \text{Eq. 3.35}
\]

\[
+ \left[ Vw - M \frac{\partial w}{\partial x} \right]_0 = 0
\]

The matrices multiplying the vector of nodal coordinates, \( \{ d^e \} \), account for the stiffness characteristics of each element. Notice that there are two basic matrices that comprise each elemental stiffness matrix. Let these be denoted by \( K_1 \) and \( K_2 \).

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\[
\left[ K_1^e \right] = \int_0^L \left[ H^* \right]^T \left[ H^* \right] \cdot dx \quad \text{Eq. 3.36}
\]

\[
\left[ K_2^e \right] = \int_0^L \left[ H^* \right]^T \left[ H \right] \cdot dx \quad \text{Eq. 3.37}
\]

These matrices are also given in Appendix A and in combination with non-dimensional parameters are used to represent stiffness contributions from the bending stiffness, combined constant axial load and blade speed, and fluctuating component of axial load.

The combined elemental stiffness matrix is simply the combination of the above two basic matrices as follows:

\[
\left[ K^e \right] = \left[ K_1^e \right] - (\delta \pm \zeta \cos \Omega t) \left[ K_2^e \right] \quad \text{Eq. 3.38}
\]

where sign + or - depends upon the element location in the beam.

Next, the matrix multiplying the vector of nodal velocities, \( \{ \dot{v}^e \} \), is the elemental gyroscopic matrix \( [G] \) and can be expressed as:

\[
\left[ G^e \right] = \int_0^L \gamma \left[ H^* \right]^T \left[ H \right] \cdot dx \quad \text{Eq. 3.39}
\]

When this matrix is evaluated for a given element length, it is essentially skew symmetric as expected except for two non-zero diagonal terms. These non-zero diagonal terms will be resolved upon assembly of the system gyroscopic matrix \( [G] \).

The beam element mass matrix \( [M] \) multiplying the vector of nodal accelerations, \( \{ \ddot{v}^e \} \), can be given as:
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\[ M^e = \int_0^L [H]^T [H] \cdot dx \]

Eq. 3.40

The expanded form of the gyroscopic and mass matrices are given in Appendix A. Note that all of these elemental matrices are in non-dimensional form.

When all of the respective elemental matrices are assembled to represent the domain of the problem, the following linear matrix differential equation results:

\[ [M]{y} + [G]{y} + \{[K_3] + \zeta \cos \Omega t[K_4]\}{y} = 0 \]

Eq. 3.41

where \{y\} is a vector containing all of the model nodal coordinates, \([K_3]\) represents the assembled time invariant stiffness matrix and \(\eta \cos \Omega t[K_4]\) represents the assembled time dependent stiffness matrix. The key thing to note about this system of equations is that we have a linear system of non-autonomous equations due to the \(\cos \Omega t\) term multiplying \([K_4]\). As stated in Chapter 1, the stability characteristics of these equations are much different than for linear autonomous systems.

4 Solution Using Method of Multiple Scales

The method of multiple scales as described by Nayfeh (1973 [13]) is a useful uniformly valid expansion that recasts the governing equations as functions of more than one time scale such as \(T_0\) and \(T_1\) instead of just the single variable \(t\). It will be seen in Chapter 4 that the method of multiple scales has advantages when dealing with perturbed, damped equations.

In order to gain an understanding of the form of the approximate solutions that will result from applying the method of multiple scales to Eq. 3.41, the gyroscopic terms in Eq. 3.41 will be neglected at this time. Neglecting these gyroscopic terms will not affect the frequency components of the solution but will affect the stability behaviour of the results as will be seen at the end of this Chapter. A model containing these gyroscopic terms will be investigated in the next Chapter.
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At this stage, typical values of the dimensionless parameters of Eqs. 3.28 will be introduced. For the blade configuration used in the cutting tests, typical values of the blade variables are:

\[ EI = 5.63 \text{Nm}^2 \quad \rho = 1.18 \frac{\text{kg}}{\text{m}} \]

\[ L = 0.8m \quad R_0 = 31.2kN \]

\[ P_0 = 312N \quad n = 5.5 \]

Using these values, the non-dimensional parameters derived earlier in this Chapter may be evaluated.

\[ \delta_+ = \left( \frac{R_0 + \frac{nP_0}{2} - \rho c^2}{EI} \right) L^2 = 5694 \]

\[ \delta_- = \left( \frac{R_0 - \frac{nP_0}{2} - \rho c^2}{EI} \right) L^2 = 5389 \]

\[ \zeta = \frac{P_0 L^2}{2EI} = 27.7 \]

The preferred solution method for the governing system of second order ordinary differential equations is the method of multiple scales. Use of this method, however, requires that the parameter \( \zeta \) multiplying the non-autonomous term in the equations of motion be much less than 1. Clearly this is not the case with the current scaling. In order to avoid this problem, let time be rescaled as:

\[ \tau = \sqrt{\delta_+} t \quad \text{Eq. 3.42} \]

So that the rescaled tooth passing frequency becomes:

\[ \bar{\Omega} = \frac{\Omega}{\sqrt{\delta_+}} \quad \text{Eq. 3.43} \]
and any derivatives with respect to time become:

\[
\frac{\partial}{\partial t} = \sqrt{\delta} \frac{\partial}{\partial \tau} \quad \text{and} \quad \frac{\partial^2}{\partial t^2} = \delta \frac{\partial^2}{\partial \tau^2}
\]

Eqs. 3.44

Making these substitutions in the discretized equations of motion (Eq. 3.41):

\[
\delta_s[M_{i}y^*]+[K_3]+\zeta\cos(\Omega \tau)[K_4]y=0
\]

Eq. 3.45

where the overbar has been omitted for simplicity. Now dividing through by \( \delta_s \), the rescaled equations emerge:

\[
[M_{i}y^*]+\left(\frac{1}{\delta_s} [K_3]+\zeta\cos(\Omega \tau)[K_4]\right)y=0
\]

Eq. 3.46

where \( \zeta \) is given as:

\[
\zeta = \frac{\delta_s}{\delta_s} = 0.005
\]

Eq. 3.47

In Eq. 3.46, primes denote differentiation with respect to the rescaled time variable \( \tau \).

In order to find solutions of Eq. 3.46 using the method of multiple scales, they must be manipulated into a more usable form. Pre-multiplying Eq. 3.46 by the inverse of the mass matrix, results in the following:

\[
\{y^*\}+\left(\frac{1}{\delta_s} [M_{i}]^{-1} [K_3]+\zeta\cos(\Omega \tau)[M_{i}]^{-1} [K_4]\right)y=0
\]

Eq. 3.48

Next, the linear transformation \( \{y\} = [P]\{u\} \) is introduced such that the matrix \([B]\):

\[
[B]=[P]^{-1} [M_{i}]^{-1} [K_3] [P]
\]

Eq. 3.49
is in Jordan Canonical form. Given that the system in question is a one dimensional continuous system, the natural frequencies of the unperturbed system (eigenvalues of $[M]^{-1}[K]$) are all distinct and positive. In this case, the equations of motion reduce to the following set of coupled linear second order differential equations:

$$u''_n + \omega_n^2 u_n + 2\varepsilon \cos(\Omega \tau) \sum_{m=1}^{N} f_{nm} u_m = 0$$  \hspace{1cm} \text{Eq. 3.50}$$

where $N$ is the number of degrees of freedom of the system and the matrix $\mathbf{f}$ is related to $[K]$ as:

$$\mathbf{f} = \frac{1}{2} [P]^{-1} [M]^{-1} [K] \mathbf{I} [P]$$  \hspace{1cm} \text{Eq. 3.51}$$

The method of multiple scales as described in Nayfeh (1973 [13]) and Nayfeh and Mook (1979 [14]) will be used to find an approximate first order solution to Eq. 3.50. The stability behaviour of Eq. 3.50 will not be investigated here since the stability boundaries predicted by the method of multiple scales will be incorrect given the exclusion of gyroscopic terms in the analysis.

In order to use the method of multiple scales, the $u_n$ will be assumed to be functions of more than one time scale as follows:

$$u_n(\tau; \varepsilon) = u_n(0, T_0) + \varepsilon u_n(0, T_1) + ...$$  \hspace{1cm} \text{Eq. 3.52}$$

where the new time scales $T_0$ and $T_1$ are given by:

$$T_0 = \tau \quad \text{and} \quad T_1 = \varepsilon \tau = \varepsilon T_0$$  \hspace{1cm} \text{Eqs. 3.53}$$

With the different time scales, differentiation with respect to the scaled time variable $\tau$ becomes:
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\[
\frac{\partial}{\partial \tau} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} \quad \text{and} \quad \frac{\partial^2}{\partial \tau^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon \frac{\partial^2}{\partial T_0 \partial T_1}
\]
Eqs. 3.54

Introducing the \( D_i \) notation, Eqs. 3.54 can be written as:

\[
\frac{\partial}{\partial \tau} = D_0 + \varepsilon D_1 \quad \text{and} \quad \frac{\partial^2}{\partial \tau^2} = D_0^2 + 2\varepsilon D_0 D_1
\]
Eqs. 3.55

Now that the \( u_n \) and differentiation have been defined with respect to the new time scales, Eqs. 3.52 and 3.54 can be substituted into Eq. 3.50. When like powers of \( \varepsilon \) are collected, the zero and first order multiple scales equations result.

Zero order multiple scales equation (coefficients of \( \varepsilon^0 \)):

\[
D_0^2 u_{n0} + \omega_n^2 u_{n0} = 0
\]
Eq. 3.56

First order multiple scales equation (coefficients of \( \varepsilon^1 \)):

\[
D_0^2 u_{n1} + \omega_n^2 u_{n1} = -2D_0 D_1 u_{n0} - \sum_r f_m u_{r0} \left[ \exp(i\Omega T_0) + cc \right]
\]
Eqs. 3.57

where \( cc \) refers to the complex conjugate terms associated with \( \exp(i\Omega T_0) \). The first of these equations is simply the free vibration of the stationary beam in the absence of damping. The second equation incorporates the effect of the parametric axial excitation as a harmonic forcing term on the right hand side of the equation. Once solutions of the zero order equation are found, they may be substituted into the first order equation.

Eq. 3.56 represents a simple harmonic oscillator. Solutions for the \( u_{n0} \) can be given as:

\[
u_{n0} = A_n (T_1) \exp(i\omega_n T_0) + cc
\]
Eq. 3.58

This solution may then be substituted into the first order equation to yield (Eq. 3.59):
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\[ D_0^2 u_{nl} + \omega_n^2 u_{nl} = -2i\omega_n D_1 A_n \exp(i\omega_n T_0) - \sum_r f_{nr} A_r \{ \exp[i(\omega_r + \Omega)T_0] + \exp[i(\omega_r - \Omega)T_0] \} + cc \]

Examining Eq. 3.59, a number of interesting issues arise. The first issue is that the first term on the right hand side is a harmonic excitation at one of the natural frequencies of the zero order equation. This type of term is called a non-secular term and must be investigated. Notice that regardless the value of the axial excitation frequency, \( \Omega \), the non-secular terms will cause infinite resonant behaviour. For example, if the axial tension excitation is removed entirely, \( \Omega = 0 \), the system will have infinite response which is known not to be the case.

This term is not the only possible non-secular term on the right hand side of Eq. 3.59. There are a number of possible combinations of the tooth passing frequency, \( \Omega \), and the zero order natural frequencies that will lead to non-secular terms. These possible combinations are:

- \( \Omega = 2\omega_p \)
- \( \Omega = \omega_p + \omega_q \)
- \( \Omega = \omega_p - \omega_q \)

Each one of these combinations will result in one of the exponentials inside the summation sign on the right hand side of Eq. 3.59 being a secular term. These are all special cases corresponding to a particular value of the axial excitation frequency whose stability must be investigated separately as done in Nayfeh and Mook (1979 [14]). It has previously been stated that the stability behaviour of Eq. 3.59 will not accurately represent that of the complete gyroscopic system. The form of the solution will, however, exhibit the same form as if the gyroscopic terms were included. Assuming none of the three resonant conditions involving \( \Omega \) occur, elimination of the remaining non-secular term requires that:

\[ D_1 A_n = 0 \quad \text{Eq. 3.60} \]

which implies that \( A_n \) is a function of time scales higher than \( T_1 \). Given this condition, solutions to Eq. 3.59 are found to be:
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\[
u_{n1} = \sum_r f_n A_r \left[ \frac{\exp[i(\omega_r + \Omega)T_0]}{(\omega_r + \Omega)^2 - \omega_n^2} + \frac{\exp[i(\omega_r - \Omega)T_0]}{(\omega_r - \Omega)^2 - \omega_n^2} \right] + \text{cc} \quad \text{Eq. 3.61}
\]

Observing Eqs. 3.58 and 3.61, it is obvious that there is no component of the approximate response occurring at the tooth passing frequency, \(\Omega\). In addition, it has been found experimentally that the first natural frequency of the moving blade for the configuration used in the Forintek cutting tests is on the order of 100 Hz. The model of this Chapter was created in an attempt to predict components of response at exactly the tooth passing frequency, \(\Omega\), as well as at a frequency lower than the tooth passing frequency by a value in the range of 10 to 40 Hz. Obviously, \(\Omega - \omega_n\), where \(\omega_n\) is around 100 Hz does not very well reflect the experimental observations.

From these results, it can be concluded that the model developed here predicts responses that do not correspond with those found experimentally at \(F_{tp}\) and \(F_{ex}\). It can also be said that axial tension fluctuations at the tooth passing frequency are unlikely to be a cause of the Type I washboarding problem and so a new loading mechanism must be investigated for this problem. This new loading will be the focus of the next Chapter. It will be seen in the next Chapter that axial tension fluctuations at a much lower frequency combined with lateral excitations at the tooth passing frequency are a much more likely contributor to the Type I washboarding problem.

In order to justify the elimination of the gyroscopic terms in the multiple scales analysis, consider the following zero and first order equations with the gyroscopic terms included:

Zero order multiple scales equation including gyroscopic terms (coefficients of \(\varepsilon^0\)):

\[
D_0^2 u_{n0} + \sum_{m=1}^{N} g_{nm} D_0 u_{m0} + \omega_n^2 u_{n0} = 0 \quad \text{Eq. 3.62}
\]

First order multiple scales equation including gyroscopic terms (coefficients of \(\varepsilon^1\), Eqs. 3.63):
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\[ D_0^2 u_{n1} + \sum_{m=1}^{N} g_{nm} D_0 u_{nm} + \omega_n^2 u_{n1} = -2D_0 D_1 u_{n0} - \sum_{m=1}^{N} g_{nm} D_1 u_{m0} - \sum_{r} f_{rn} u_{r0} [\exp(i\Omega T_0) + cc] \]

The solutions of Eq. 3.62 can be given in the form:

\[ u_{n0} = \sum_{m=1}^{N} A_{nm} (T_1) \exp(i\omega_m' T_0) + cc \quad \text{Eq. 3.64} \]

where the \( \omega_m' \) represent the gyroscopic natural frequencies of the problem. When Eq. 3.64 is substituted into Eqs. 3.63, the resulting terms on the right hand side have frequency components \( \omega_m', \Omega + \omega_m', \Omega - \omega_m' \) and complex conjugates. These are exactly the same frequency components as for Eq. 3.59 save for the natural frequencies now being the gyroscopic natural frequencies rather than those for the stationary blade. For this reason, the general form of the frequency components of the multiple scales solution are unaffected by the gyroscopic effects. The second term on the right hand side of Eqs. 3.63 is very important in determining the stability behaviour of system as it is comprised of terms that contain the \( A_{nm}' \) which govern the stability of the zero and first order solutions.
CHAPTER 4: COMBINED PARAMETRIC AND LATERAL HARMONIC FORCE MODEL

1 Background

The previous Chapter showed that a fluctuating axial load applied at midspan of a traveling beam does not predict responses similar to those seen in washboarding experiments. For this reason, a new type of beam loading will be pursued consisting of the application of a lateral harmonic cutting force in conjunction with an axial tension fluctuation which has been shown to lead to parametric terms in the equations of motion. This type of problem has not received much attention in the literature as supported by Bolotin (1999 [15]) who states that one important area where significant work remains to be done is in the combination of parametric and forced excitations.

It will be shown in Section 2 that lateral vibration of the blade at the tooth passing frequency occurs regardless of whether the blade is cutting or not. Starting with an idling blade with a component of response at $F_{tp}$, when the blade begins to interact with the workpiece, it is foreseeable that lateral periodic forces on the blade exist at each tooth tip at the frequency of its vibration. A forcing function of this type will be used in the next blade model.

In conjunction with this lateral excitation, low frequency band tension variations occur. These fluctuations may have one or more frequency components corresponding to multiples of the band passing and wheel rotation frequencies. This tension variation will also be accounted for in the next blade model. A diagram of the proposed model is shown below in Figure 4.1.
In the previous chapter, it was found that a parametric excitation at the tooth passing frequency does not result in a matching component of response at the tooth passing frequency. For a regular non-homogeneous harmonic excitation of a continuous system, however, it is well established that a component of response will exist at the frequency of excitation. In this Chapter, a mathematical model of the system shown in Figure 4.1 will be developed to investigate how a lateral tooth passing frequency excitation and a low frequency axial parametric excitation will interact. The lateral force used in the model of Figure 4.1 has been simplified to act at midspan when in reality, the lateral cutting forces occur at discrete points through the depth of cut. In addition to there being a number of these lateral cutting forces, the forces move with the teeth. This effect will not be modeled in this work. A discussion of the two different system excitations follows.

1.1 Band Tension Fluctuation

During bandsaw operation, the tension in the bandsaw blade does not remain constant. A number of effects may occur to produce fluctuations in the band tension such as wheel
mounting eccentricities or out of round, influence of the band weld, dished areas interacting with the wheels or guides, and vibration of the machine structure. These effects will now be individually discussed.

1.1.1 Wheel Mounting Eccentricities and Out of Round

Consider the ideal situation of two perfectly circular wheels each rotating about their respective centers as shown below in Figure 4.2a:

![Figure 4.2](image)

Figure 4.2 – (a) Ideal wheel mounting arrangement and (b) Case of single wheel mounted eccentrically

In contrast, consider the situation in Figure 4.2b where the bottom wheel is mounted correctly while the top wheel is mounted eccentrically. Due to this eccentricity, the band will experience maximum extension once per top wheel revolution when the maximum radius, $r_{\text{max}}$, reaches the orientation shown and minimum extension once per wheel revolution when $r_{\text{max}}$ is located 180° to the location shown. This would translate to the band tension varying harmonically at the wheel rotation frequency, $f_w$. 

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The next case to consider again involves two perfectly circular wheels but now both wheels have a mounting eccentricity. A number of different situations can arise depending upon the phase between the two eccentricities. First consider the case where the two eccentricities are exactly in phase as depicted in Figure 4.3a:

![Figure 4.3](image)

Figure 4.3 – Three cases of both wheels mounted eccentrically (a) $r_{\text{max}}$ in phase (b) $r_{\text{max}}$ out of phase 180° (c) $r_{\text{max}}$ with general phase

For this configuration, there will be no variation in band tension since the distance between the top and bottom wheel extremities will remain constant. Figure 4.3b shows the situation where the wheel mounting eccentricities are 180° out of phase. For this situation, the band will once again experience maximum extension once per wheel revolution and minimum extension once per wheel revolution resulting in a band tension fluctuation at $f_w$. The third case depicted in Figure 4.3c shows the general case of two eccentrically mounted wheels where the eccentricities are neither exactly in, nor exactly 180° out of phase. For this situation, the band extension will reach a 'maximum' once per revolution of the bottom wheel and once per revolution of the top wheel. These maxima, however, do not occur simultaneously resulting in the fundamental period of the tension variation being at exactly twice the wheel rotation frequency.
In addition to eccentric wheel mounting, the bandsaw wheels themselves may not be perfectly circular. Depending on the variation in the wheel radius through the full 360° of the wheel, the band tension may fluctuate at any integer of the wheel rotation frequency. For example, consider the case of the top wheel having the shape of an ellipse and the bottom wheel being perfectly circular as shown in Figure 4.4 below:

![Diagram showing tension fluctuations on elliptical and circular wheels](image)

Figure 4.4 – Elliptical wheel causing tension fluctuations (a) Minimum tension orientation (b) Maximum tension orientation

Given that the two wheels are mounted without eccentricity, the band tension will experience maxima and minima twice per single revolution of the elliptical wheel leading to a variation in the band tension at twice the wheel rotation frequency. For wheels with more than two local maximum and minimum radii, the band tension will fluctuate with a fundamental frequency equal to some higher multiple of the wheel rotation frequency.

1.1.2 Band Imperfections

Another important low frequency in the bandsawing problem is the band passing frequency, $f_b$. It is believed that axial tension fluctuations also occur at integer multiples of the band passing frequency due to the band weld and dished areas in the band.
These are particularly important given the interaction between the blade and the guides and their proximity to the cutting span where washboarding occurs.

The band weld has different stiffness characteristics than the rest of the uniform band due to different local blade thickness at the weld and residual stress distributions. As this weld traverses the span between the guides, the stiffness characteristics of the cutting span change. This effect happens once per band revolution and leads to an axial tension fluctuation equal to the band passing frequency, \( f_b \).

Next consider a dished area in the blade as it interacts with the guide. Pressure between the guide and the blade combined with the curvature of the blade at the dished area will result in a change in stress distribution surrounding the dish. As the dished area leaves contact with the guide, it will take its original out of plane shape. This results in local motion of the blade out of its plane between the guides that may be picked up by the eddy current displacement probes. Given a single dished area of the band, there will be a displacement component at the band passing frequency. For every dished area on the band, \( n \), there will be a component of axial tension fluctuation at \( nf_b \).

### 1.1.3 Vibration of the Machine Structure

Any vibration of the machine structure that causes relative motion between the two wheel centers of the bandsaw will cause tension fluctuations in the band. Given that the eddy current probes measuring blade vibration were mounted on the machine structure itself, it is very unlikely that the lowest structural frequencies are recorded during running of the machine.

In the work by Zhan (1990 [16]), the lowest four natural frequencies of the five foot CanCar bandsaw in the Wood Machining Laboratory at the University of British Columbia were found to be 12Hz, 33 Hz, 45 Hz and 58 Hz. This bandsaw is the same basic size as that used in the cutting tests at Forintek Canada Corp. Given that the range of frequency differences observed during the cutting tests was between 10 and 40 Hz, the first two of these structural frequencies are potential candidates for the frequency difference. The dependence of these frequencies, however, on such parameters as wheel speed and strain level was not investigated.
1.1.4 Band Vibration

As the band vibrates, given that the wheel centers are for the most part, a fixed distance apart, the band tension will fluctuate. This is a non-linear effect that was not considered in the derivation of the beam equation of motion. It might, however, lead to low frequency tension fluctuations. The interesting thing to note is that for every cycle of band lateral vibration, the tension will experience maxima at both extremes of band motion and two minima as the band returns to the equilibrium position. For this reason, the band tension will fluctuate at twice the frequency of any lateral displacement.

1.2 Lateral Excitation

One common observation throughout the cutting trials conducted was that regardless of whether the blade is idling, cutting without washboarding, or cutting with washboarding, a component of response is always seen at $F_{tp}$. This indicates that the blade, near the teeth, is constantly vibrating at the tooth passing frequency. Since this behavior occurs both during idling as well as cutting, a blade interaction with something other than the workpiece must be causing this phenomena. PSD plots for three cases of operation are shown below in Figures 4.5 and 4.6 as well as in Figure 2.6. The first of these plots shows a cut occurring from time 0 to 3 seconds and an idling blade from 3 to 5 seconds for blade 18G0H0. Note the response at $F_{tp} = 1155$ to $1162$ Hz once the cut has finished even though the blade is no longer cutting. The steady increase in this tooth passing frequency from $t = 3$ seconds to $t = 5$ seconds is due to the saw speeding up as the cut ends. This response post cut shows that the response at the tooth passing frequency occurs even when the blade is not cutting. The second plot shows a cutting blade without washboarding with a response at $F_{tp}$ still evident. The last of these plots from Chapter 2, shows a washboarding condition with clear responses at $F_{ex}$, $F_3$, and $F_{tp}$. The model of Chapter 3 failed to predict this response at $F_{tp}$ and therefore, a new blade loading will be introduced.
Chapter 4: Combined Parametric and Lateral Harmonic Force Model

Post Cut Tooth Passing Frequency Response

Idling Blade with $F_{tp}$ response, $t = 3 - 5s$

Figure 4.5 – High frequency response showing idling condition at end of cut (18G0H0)

Typical High Frequency Response Without Washboarding

Figure 4.6 – High frequency response without washboarding showing response at $F_{tp}$
Chapter 4: Combined Parametric and Lateral Harmonic Force Model

The driving force of this response at $F_{tp}$ is not yet well understood but it has been proposed that blade interaction with the guides would produce a response at the tooth passing frequency. In a typical band saw setup, the guides are proud of the tangent line between the two wheels and as such, the blade between wheels will exhibit a static deformation of a shape shown in Figure 4.7 below:

![Figure 4.7 - Blade offset due to guide positioning](image)

It is apparent from the diagram that the cutting span of the blade is not in line with the dotted tangent line shown connecting the top and bottom wheels and as a result, the direction of the blade changes as it traverses either of the guides. In addition, the guide itself does not extend to support the whole width of the blade. The front edge of the blade is cantilevered from the edge of the guide as can be seen below in Figure 4.8.
Given the curvature of the blade going over the guide plus the fact that the front blade edge is unsupported and the presence of tension in the blade, the front edge of the blade will tend to deflect towards the center of curvature of the blade.

Next, with the addition of blade speed, this deflected front edge will be moving with the blade. Since the front edge is where the teeth are located, the mass and stiffness characteristics of the cantilevered front edge at the guide location vary. The degree of deflection of a cantilever subject to loading of any kind will depend upon its stiffness and mass characteristics. Therefore, the front edge deflection will vary with time at the frequency with which teeth pass the guide which is exactly $F_{tp}$. The effect of guide offset on the idling and cutting responses at $F_{tp}$ remains to be investigated.

Further testing remains to be done in order to examine the influence of guide-blade interaction on the response at $F_{tp}$ for the idling blade. Regardless how this excitation is created, its effect once the teeth enter the wood must be considered.
2 Equation of Motion

As was seen in Chapter 3, the equation of motion for an axially moving beam subjected to an axial tensile load with a small harmonically varying component was given by Eq. 3.16. For the problem described in Section 1, the equation of motion is the same as Eq. 3.16 except for the addition of a lateral harmonic cutting force at midspan instead of the axial harmonic cutting force at the same location. With this new blade loading, the equation of motion becomes:

\[
EI \frac{d^4 y}{dx^4} - \left( R_0 + \tau \cos \omega \cdot t - \rho c^2 \right) \frac{d^2 y}{dx^2} + 2 \rho c \frac{d^2 y}{dxdt} + \rho \frac{d^2 y}{dt^2} = F \cos \Omega t \cdot \delta \left( x - \frac{L}{2} \right)
\]

Eq. 4.1

For this problem, \( \omega \) represents a low frequency fluctuation of the band tension, \( \tau \) represents the magnitude of the band tension fluctuation, \( F \) represents the magnitude of the lateral midspan force, \( \Omega \) represents the tooth passing frequency and \( \delta \) is the Kronecker delta.

In comparison with Eqs. 3.19 and 3.20, the first term in Eq. 4.1 has an identical interpretation. The second term now represents lateral restoring forces due to the static and fluctuating strain components as well as the centripetal acceleration of the beam. The remaining terms on the left hand side are the same as for Eqs. 3.19 and 3.20. The major difference between Eqs. 3.19/3.20 and Eq. 4.1 is the harmonic lateral forcing term on the right hand side of Eq. 4.1.

As was done in Chapter 3, Eq. 4.1 will be non-dimensionalized. Using the same time and length scales as those introduced in Chapter 3, Eq. 4.1 becomes:

\[
\frac{d^4 y}{dx^4} - \left( \chi c \cos \omega \cdot t \right) \frac{d^2 y}{dx^2} + \gamma \frac{d^2 y}{dxdt} + \frac{d^2 y}{dt^2} = \eta \cos \Omega t \cdot \delta \left( x - \frac{1}{2} \right)
\]

Eq. 4.2

Where the same time scale has been used to non-dimensionalize \( \omega \) as was initially used for \( \Omega \) in Chapter 3. The two new non-dimensional constants \( \chi \) and \( \eta \) are given by:
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\[ X = \left( R_0 - \rho c^2 \right) L^2 \quad \eta = \frac{F L^3}{E I} \quad \text{Eqs. 4.3a,b} \]

Eq. 4.3a is exactly analogous to \( \zeta \) in the previous Chapter save the exclusion of the mean axial cutting load. The non-dimensional group \( \eta \) represents the magnitude of the lateral cutting force compared to the compressive buckling load of the unstrained band. The small parameter \( \varepsilon \) multiplying the parametric term in Eq. 4.2 is given by:

\[ \varepsilon = \frac{\varepsilon L^3}{E I} \quad \text{Eq. 4.4} \]

and represents the magnitude of the axial tension fluctuation compared to the compressive buckling load of the unstrained band.

The boundary conditions at the guides are identical to those for the model in Chapter 3 and can be represented as:

\[ y(0) = y(1) = 0 \quad \text{Eqs. 4.5a,b} \]

\[ \frac{\partial^2 y(0)}{\partial x^2} = \frac{\partial^2 y(1)}{\partial x^2} = 0 \]

The boundary conditions used represent a pinned condition at either end of the span with no axial constraint. These boundary conditions do not necessarily accurately reflect the true situation occurring in the bandsawing process but for analytical purposes, they lead to simple stationary mode shapes that ease the manipulation of the equations of motion.

Once again, the next step in finding an approximate solution is to discretize the equation of motion.
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3 Discretization of Equation of Motion

A major difference between the model developed in Chapter 3 and that proposed here is the symmetry of the model about the applied lateral load. Because of this symmetry, the band tension in the top and bottom halves of the cutting span is the same, unlike the model in Chapter 3. Therefore, the entire domain is governed by the same partial differential equation and the equation of motion may be discretized by expanding the solution in terms of the mode shapes of the free vibration of the stationary, homogeneous problem without the parametric axial excitation:

\[ y(x,t) = \sum_{m} u_m(t) \phi_m(x) \quad \text{Eq. 4.6} \]

The \( \phi_m \) are the mode shapes that solve the eigenvalue problem for the stationary, homogeneous free vibration problem:

\[ \phi_m''(x) - \kappa_m \phi_m''' - \kappa_m^4 \phi_m = 0 \]
\[ \phi_m(0) = \phi_m(1) = 0 \quad \text{Eqs. 4.7a,b,c} \]
\[ \phi_m''(0) = \phi_m''(1) = 0 \]

The solutions of the above eigenvalue problem are of the form:

\[ \phi_n(x) = \sin(n \pi x) \quad \text{Eq. 4.8} \]

If different boundary conditions than those given in Eqs. 4.5 had been used, the mode shapes that solve the corresponding eigenvalue problem would contain cos, cosh and sinh terms in addition to the sin term that results for the given boundary conditions. Such mode shapes are algebraically cumbersome. Substituting Eq. 4.8 into Eq. 4.6 and then into Eq. 4.2, the following results (Eq. 4.9):

\[ \sum_n \left[ (n \pi)^4 + (\chi + \epsilon \cos \omega \cdot t)(n \pi)^2 \right] u_n \sin(n \pi x) + \gamma \cdot n \pi \cdot \dot{u}_n \cos(n \pi x) + \ddot{u}_n \sin(n \pi x) \]
\[ = \eta \cos \Omega t \delta \left( x - \frac{1}{2} \right) \]

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The mode shapes found for the eigenvalue problem given by Eqs. 4.8 are orthogonal and therefore, if Eq. 4.9 is multiplied by $\phi_m(x) = \sin(m\pi x)$ and integrated over the domain of the problem (from $x = 0$ to $x = 1$), the following results (Eq. 4.10):

$$\ddot{u}_m + \sum_n g_{mn} \dot{u}_n + (m\pi)^2 \left[ \chi + (m\pi)^2 \right] u_m + (m\pi)^2 \varepsilon \cos(\omega_t t) u_m = 2\eta \cos(\Omega t) \sin \left( \frac{m\pi}{2} \right)$$

Eq. 4.10

This is an infinite set of coupled Mathieu equations. An important note however is that the coupling occurs due to the gyroscopic term rather than the parametric term. No coupling exists in the parametric term due to the selection of the boundary conditions that led to the simple sinusoidal stationary mode shapes. It is shown in Nayfeh and Mook (1979 [14]) that the parametric coupling terms affect the stability boundaries of the response. The potential frequency combinations leading to instabilities will, however, be the same as if Eqs. 4.10 exhibited parametric coupling. It will be seen later in this Chapter that the issue of instability is also strongly dependent upon damping. The forcing function on the right hand side of Eq. 4.10 is non-zero for odd $m$ and zero for even $m$ as is expected given the nature of the mode shapes and the location of the applied load. For the analysis that follows, Eq. 4.10 can be recast in a more standard form:

$$\ddot{u}_m + \sum_n g_{mn} \dot{u}_n + \alpha_m u_m + 2\varepsilon \cos(\omega_t t) f_{mm} u_m = 2\eta \cos(\Omega t) \sin \left( \frac{m\pi}{2} \right)$$

Eq. 4.11

Where the following substitutions have been made:

$$g_{mn} = 2\pi \gamma \left[ n \cos(n\pi x) \sin(m\pi x) \right]$$

$$\alpha_m = (m\pi)^2 \left[ \chi + (m\pi)^2 \right]$$

Eq. 4.12a, b, c

$$f_{mm} = \frac{(m\pi)^2}{2}$$
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It can be seen that in the absence of blade speed, Eq. 4.11 is completely decoupled. Therefore, for the homogeneous solution, similar results are expected for the stability and form of solution as are found for the single degree of freedom Mathieu Equation as described by Bolotin (1964 [17]) and Nayfeh and Mook (1979 [14]). In the next section, an approximate solution for Eq. 4.11 will be found using the method of multiple scales.

4 Solution Using Method of Multiple Scales

Similar to Chapter 3, the method of multiple scales will be used to investigate the approximate solutions and stability behavior of Eq. 4.11. The main differences between the analysis performed here and that of Chapter 3 is the inclusion of gyroscopic terms as well as a non-homogeneous forcing term on the right hand side of Eq. 4.11. In contrast to Eq 3.50 of Chapter 3, the off diagonal parametric coefficients, \( f_{mn} \) are all equal to zero for the present configuration.

4.1 Multiple Scales Equations

Once again, the solution for \( y \) will be expanded in powers of the small variable \( \varepsilon \):

\[
\begin{align*}
  u_m(t; \varepsilon) &= u_{m0}(T_0, T_1, T_2) + \varepsilon u_{m1}(T_0, T_1, T_2) + \varepsilon^2 u_{m2}(T_0, T_1, T_2) + \ldots \\
  \text{Eq. 4.13}
\end{align*}
\]

Substituting Eq. 4.13 into Eq. 4.11 and equating like powers of epsilon results in the following two equations (Eqs. 4.14 a,b):

\[
\begin{align*}
  D_0^2 u_{m0} + \sum_n g_{mn} D_0 u_{n0} + \alpha_m u_{m0} &= 2\eta \cos(\Omega t) \sin \left( \frac{m\pi}{2} \right) \\
  D_0^2 u_{m1} + \sum_n g_{mn} D_0 u_{n1} + \alpha_m u_{m1} &= -2D_0 D_1 u_{m0} - \sum_n g_{mn} D_1 u_{n0} - 2f_{mm} u_{m0} \cos \omega T_0 \\
\end{align*}
\]

where the first arises from equating terms containing \( \varepsilon^0 \) and the second arises from equating terms containing \( \varepsilon^1 \). It can be seen that the first equation represents a set of \( m \) undamped gyroscopic equations with a harmonic forcing term. If \( m \) is odd, the forcing term reduces to a harmonic excitation of magnitude \( 2\eta \) and if \( m \) is even, the forcing term is zero. This is due to the ability of the central lateral load to do work on the \( m^{th} \) mode.
shape. For the even stationary mode shapes, the center of the beam is a node and therefore, no work can be done on the even modes since no displacement occurs at the location of force application as seen in Figure 4.9 below. This is not true for the odd modes. This is of course an idealization given that the actual cutting zone occurs over a finite section of the span between guides.

![Odd Type Mode Shape](a) Odd Type Mode Shape  ![Even Type Mode Shape](b) Even Type Mode Shape

Figure 4.9 – Work done on (a) Odd mode shapes is non-zero, (b) Even mode shapes is zero

### 4.2 Two Mode Zero Order Solution

In order to simplify the following analysis and discussion, a two mode solution will be considered. This approach will not produce results that quantitatively compare to those found in experiment. Given the simple one-dimensional nature of the problem considered and the complex two dimensional mode shapes believed to be involved in the actual washboarding problem, including a large number of mode shapes in the analysis will simply serve to increase the computational complexity of the problem. The behavior of interest is in the relations between the two forcing frequencies, $\Omega$ and $\omega$, and two consecutive system natural frequencies. It will be seen that response components to first order contain frequency terms involving combinations of $\Omega$ or $\omega$ with each other or a
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single gyroscopic natural frequency of the system. Qualitative behavior of these scenarios can be investigated with a two-mode solution. For simplicity in numbering equations, the first and second modes will be used. In reducing the system of equations to two, Eq. 4.14a becomes:

\[ D_0^2 u_{10} + g_{12} D_0 u_{20} + \alpha_1 u_{10} = 2\eta \cos(\Omega t) \]

Eqs. 4.15

\[ D_0^2 u_{20} + g_{21} D_0 u_{10} + \alpha_2 u_{20} = 0 \]

Eqs. 4.15 describe the behavior of the traveling beam in absence of the axial tension fluctuation. The solution of Eqs. 4.15 will consist of two parts: the homogeneous solution and the particular solution. Solving first for the homogeneous solutions \( u_{10h} \) and \( u_{20h} \):

\[ u_{10h} = A_1(T_1) \exp(i\omega_1 T_0) + A_2(T_1) \exp(i\omega_2 T_0) + cc \]

Eqs. 4.16

\[ u_{20h} = \frac{i(\alpha_1 - \alpha_2)}{g_{12} \omega_1} A_1(T_1) \exp(i\omega_1 T_0) + \frac{i(\alpha_1 - \alpha_2)}{g_{12} \omega_2} A_2(T_1) \exp(i\omega_2 T_0) + cc \]

Where the speed dependent gyroscopic natural frequencies, \( \omega_1 \) and \( \omega_2 \) are the solutions of:

\[ \omega_n^4 - (\alpha_1 + \alpha_2 - g_{12} g_{21}) \omega_n^2 + \alpha_1 \alpha_2 = 0 \]

Eq. 4.17

Eqs. 4.15 also have a particular solution of the form:

\[ u_{10p} = B_1 \exp(i\Omega T_0) \]

Eqs. 4.18

\[ u_{20p} = B_2 \exp(i\Omega T_0) \]

where

\[
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} = \frac{2\eta}{(\alpha_1 - \Omega^2)(\alpha_2 - \Omega^2) + g_{12} g_{21} \Omega^2} \begin{bmatrix}
\alpha_2 - \Omega^2 \\
-i g_{21} \Omega
\end{bmatrix}
\]

Eq. 4.19
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The complete zero order solution is expressed as:

\[ u_{10} = u_{10h} + u_{10p} \]

\[ u_{20} = u_{20h} + u_{20p} \]

Eq. 4.20

It is evident that the system response to zero order contains three frequency terms, one at each of the two gyroscopic natural frequencies and the other at \( \Omega \). As with any standard undamped forced vibration problem, the particular solutions \( u_{10p} \) and \( u_{20p} \) will exhibit large magnitude vibration as \( \Omega \) approaches a gyroscopic natural frequency of the system since the denominator of Eq. 4.19 approaches zero.

4.3 Two Mode First Order Solution

Substituting Eqs. 4.20 into Eq. 4.14b, the first order equations become (Eqs. 4.21 a,b):

\[
D_0^2 u_{11} + g_{12} D_0 u_{21} + \alpha_1 u_{11} = \frac{i(\alpha_1 + \omega^2)}{\alpha_1} A'_1 e^{i\omega t_0} - \frac{i(\alpha_2 + \omega^2)}{\alpha_2} A'_2 e^{i\omega t_0} \\
- f_{11}[A_1 (e^{i(\omega_1 + \omega)\tau_0} + e^{i(\omega_1 - \omega)\tau_0})] \\
- f_{11}[A_2 (e^{i(\omega_2 + \omega)\tau_0} + e^{i(\omega_2 - \omega)\tau_0})] \\
- f_{11}[B_1 (e^{i(\Omega + \omega)\tau_0} + e^{i(\Omega - \omega)\tau_0})] + cc
\]

\[
D_0^2 u_{21} + g_{21} D_0 u_{11} + \alpha_2 u_{21} = \frac{2\alpha_2 - g_{12} g_{21}}{g_{12}} A'_1 e^{i\omega t_0} + \frac{2\alpha_1 - g_{12} g_{21}}{g_{12}} A'_2 e^{i\omega t_0} \\
- f_{22}\left[\frac{i(\alpha_1 - \omega^2)}{\alpha_1} A_1 (e^{i(\omega_1 + \omega)\tau_0} + e^{i(\omega_1 - \omega)\tau_0})\right] \\
- f_{22}\left[\frac{i(\alpha_2 - \omega^2)}{\alpha_2} A_2 (e^{i(\omega_2 + \omega)\tau_0} + e^{i(\omega_2 - \omega)\tau_0})\right] \\
- f_{22}\left[B_2 (e^{i(\Omega + \omega)\tau_0} + e^{i(\Omega - \omega)\tau_0})\right] + cc
\]

Notice that the differential operator comprising the form of the left hand side of Eqs. 4.21 will, for the corresponding homogeneous problem, yield gyroscopic natural frequencies equal to those found for the zero order problem. Therefore, any term on the right hand side of Eqs. 4.21 containing frequency components equal to a gyroscopic natural
frequency will cause a resonant condition. From the right hand sides of Eqs. 4.21, there are a number of these apparent resonant frequency combinations that can yield terms of this nature. These resonant frequencies include:

1. $\omega = 2\omega_1$
2. $\omega = 2\omega_2$
3. $\omega = \omega_1 + \omega_2$
4. $\omega = \omega_2 - \omega_1$
5. $\Omega + \omega = \omega_1$
6. $\Omega + \omega = \omega_2$
7. $\Omega - \omega = \omega_1$
8. $\Omega - \omega = \omega_2$

Given that the washboarding problem occurs in the vicinity of 1000 Hz with natural frequencies of the same order, and that $\Delta F$, which in this case is represented by $\omega$, is measured in the region of 10 – 40 Hz, the first three of these combinations cannot be associated with washboarding according to this model. As will be seen in Chapter 5, natural frequency spacing in the washboarding frequency range for a stationary toothed blade used in the experimental cutting trials is on the order of 20 Hz and therefore, resonant combination #4 is possible. Finally, the last four resonant combinations are all definitely possible and must be investigated. Resonant condition #4 represents a combination parametric resonance that may or may not lead to unstable behavior due to the solution containing an exponential term with a positive real exponent. Resonant conditions #5 - #8 represent “modified” resonances where the combination of $\Omega$ and $\omega$ is near a natural frequency.

It is important to discern between parametric resonances and regular forced resonances. Resonant conditions #1 through #4 represent parametric resonances while conditions #5 through #8 represents regular forced type resonances. The parametric resonances can be divided into simple parametric resonances involving a single natural frequency such as conditions #1 and #2 and combination parametric resonances involving the sum or difference of two natural frequencies given by conditions #3 and #4. It is possible that the solution for these conditions will exhibit unbounded exponential growth. The regular forced type resonances that occur for conditions #5 through #8 exhibit responses that contain a bounded harmonic component multiplied by a magnitude that becomes singular when the resonant condition is exactly satisfied. For given values of $\Omega$ and $\omega$, the magnitude is constant as compared to exhibiting exponential growth as for the case
of parametric resonance. To begin, the case of none of the above resonant conditions being satisfied will be investigated.

### 4.3.1 Non-resonant Case

Given the linearity of Eqs. 4.21, solutions can be found independently for each of the terms on the right hand sides of the equations. With this, and noting that the troublesome terms on the right hand sides of Eqs. 4.21 contain \( \exp(i\omega_0 T_0) \), Nayfeh and Mook (1979 [14]) show that:

\[
A_1' = 0 \quad \text{and} \quad A_2' = 0 \quad \text{Eqs. 4.22}
\]

Implying that \( A_1 \) and \( A_2 \) are functions of \( T_2 \) and higher order time scales only, for the non-resonant case. To examine the dependence of the \( A_n \) on \( T_2 \), the second order equations must be investigated. This will not be done here. Given the above restrictions on \( A_1 \) and \( A_2 \) for the non-resonant case, solutions to the remaining terms on the right hand side of Eqs. 4.21 may be found. Given the linear nature of the problem, each frequency component will be treated separately and then these solutions will be summed to obtain the total response.

For the response due to terms containing the frequencies \( \omega_1 \pm \omega \), Eqs. 4.21 reduce to:

\[
D_0^2 u_{11} + g_{12} D_0 u_{12} + \alpha_1 u_{11} = -f_{11} A_1 \exp[i(\omega_1 \pm \omega)T_0] 
\]

\[
D_0^2 u_{21} + g_{21} D_0 u_{11} + \alpha_2 u_{21} = -f_{22} \frac{i(\alpha_1 - \alpha_2^2)}{g_{12} \alpha_1} A_1 \exp[i(\omega_1 \pm \omega)T_0] 
\]

The solution of Eqs. 4.23 is given as (Eq. 4.24):

\[
\begin{bmatrix}
    u_{11} \\
    u_{21}
\end{bmatrix}_{1,2} = \frac{A_1}{\text{DET}(C_{1,2})} \begin{bmatrix}
    \alpha_2 - (\omega_1 \pm \omega)^2 & -i(\omega_1 \pm \omega)g_{12} \\
    -i(\omega_1 \pm \omega)g_{21} & \alpha_1 - (\omega_1 \pm \omega)^2
\end{bmatrix} \begin{bmatrix}
    -f_{11} \\
    if_{22}(\alpha_1 - \alpha_2^2) / g_{12} \alpha_1
\end{bmatrix} \exp[i(\omega_1 \pm \omega)T_0]
\]

Where \( \text{DET}(C_{1,2}) \) is given by:
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\[ \text{DET}(C_{1,2}) = (\omega_1 \pm \omega)^4 - (\alpha_1 + \alpha_2 - g_{12}g_{21})(\omega_1 \pm \omega)^2 + \alpha_1\alpha_2 \]  \hspace{1cm} \text{Eq. 4.25} \]

In the above Eqs. 4.24 and 4.25, the ‘1,2’ subscript refers to the ‘+’ and ‘-’ respectively in the ‘±’ for the frequency combination being considered.

Notice that the right hand side of Eq. 4.25 is of the same form as Eq. 4.17 so that if \( \omega_1 \pm \omega \) is near a gyroscopic natural frequency of the system, then this determinant is small. If \( \omega_1 \pm \omega \) is equal to a gyroscopic natural frequency of the system, then the determinant is equal to zero and the behavior must be examined more carefully to determine whether a parametric combination resonance occurs.

In a similar fashion, the responses to other frequency components on the right hand side of Eqs. 4.21 can be found.

For the response due to terms containing \( \omega_2 \pm \omega \), Eqs. 4.21 reduce to:

\[ D_0^2u_{11} + g_{12}D_0u_{21} + \alpha_1u_{11} = -f_{11}A_2 \exp[i(\omega_2 \pm \omega)t_0] \]  \hspace{1cm} \text{Eqs. 4.26} \]

\[ D_0^2u_{21} + g_{21}D_0u_{11} + \alpha_2u_{21} = -f_{22} \frac{i[\alpha_1 - \omega_2^2]}{g_{12}\omega_2} A_2 \exp[i(\omega_2 \pm \omega)t_0] \]

The solution of Eqs. 4.26 is given as (Eq. 4.27):

\[
\begin{bmatrix}
  u_{11} \\
  u_{21}
\end{bmatrix}
= \frac{A_2}{\text{DET}(C_{3,4})} \begin{bmatrix}
  \alpha_2 - (\omega_2 \pm \omega)^2 & -i(\omega_2 \pm \omega)g_{12} \\
  -i(\omega_2 \pm \omega)g_{21} & \alpha_1 - (\omega_2 \pm \omega)^2
\end{bmatrix}
\begin{bmatrix}
  -f_{11} \\
  -f_{22}(\alpha_1 - \omega_2^2)
\end{bmatrix}
\exp[i(\omega_2 \pm \omega)t_0]
\]

Where \( \text{DET}(C_{3,4}) \) is given by:

\[ \text{DET}(C_{3,4}) = (\omega_2 \pm \omega)^4 - (\alpha_1 + \alpha_2 - g_{12}g_{21})(\omega_2 \pm \omega)^2 + \alpha_1\alpha_2 \]  \hspace{1cm} \text{Eq. 4.28} \]

For the response due to terms containing \( \Omega \pm \omega \), Eqs. 4.21 reduce to:
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\[ D_0^2 u_{11} + g_{12} D_0 u_{21} + \alpha_1 u_{11} = -f_{11} B_1 \exp[i(\Omega \pm \omega)T_0] \]  
Eqs. 4.29

\[ D_0^2 u_{21} + g_{21} D_0 u_{11} + \alpha_2 u_{21} = -f_{22} B_2 \exp[i(\Omega \pm \omega)T_0] \]

The solution of Eqs. 4.29 is given by:

\[
\begin{bmatrix}
{u_{11}} \\
{u_{21}}
\end{bmatrix}_{5,6} = \frac{1}{\text{DET}(C_{5,6})}
\begin{bmatrix}
\alpha_2 - (\Omega \pm \omega)^2 & -i(\Omega \pm \omega)g_{12} \\
-i(\Omega \pm \omega)g_{21} & \alpha_1 - (\Omega \pm \omega)^2
\end{bmatrix}
\begin{bmatrix}
-f_{11} B_1 \\
-f_{22} B_2
\end{bmatrix} \exp[i(\Omega \pm \omega)T_0] \quad \text{Eq. 4.30}
\]

Where \( \text{DET}(C_{5,6}) \) is given by:

\[
\text{DET}(C_{5,6}) = (\Omega \pm \omega)^4 - (\alpha_1 + \alpha_2 - g_{12} g_{21})(\Omega \pm \omega)^2 + \alpha_1 \alpha_2 \quad \text{Eq. 4.31}
\]

Examining the form of solution for the frequency component \( \Omega \pm \omega \), it can be seen that in addition to \( \text{DET}(C_{5,6}) \) approaching zero as \( \Omega \pm \omega \) approaches a gyroscopic natural frequency of the unperturbed system, the \( B_n \) become large as \( \Omega \) approaches a natural frequency of the system. This would indicate that the response at \( \Omega \pm \omega \) can be significant if \( \Omega \) is near a natural frequency as well as if \( \Omega \pm \omega \) is near a natural frequency. Unlike the case for the frequency components \( \omega_n \pm \omega \), when \( \Omega \pm \omega \) is near or equal to a gyroscopic natural frequency, a forced resonance condition results. The important characteristic of this resonance is that it occurs when the combination \( \Omega \pm \omega \) is near a natural frequency even though the frequency of the lateral excitation is simply \( \Omega \).

The total response at first order is simply the sum of all 6 frequency components:

\[
\begin{bmatrix}
{u_{11}} \\
{u_{21}}
\end{bmatrix} = \sum_{k=1}^{6} \begin{bmatrix}
{u_{11}} \\
{u_{21}}
\end{bmatrix}_k \quad \text{Eq. 4.32}
\]

Therefore, a spectrum plot of the total first order solution given by Eq. 4.32 should show six distinct frequency spikes whose magnitudes depend on the system parameters such as blade speed and band tension as well as a number of frequency relations that will be discussed in the following sub-sections. These frequency relations dictate the proximity
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of different combinations of the two forcing frequencies, \( \Omega \) and \( \omega \), or combinations of the band tension fluctuation and system natural frequencies \( (\omega_n \pm \omega) \) to natural frequencies of the system.

4.3.2 \( \Omega \pm \omega \) Near \( \omega_i \)

To investigate the behavior of the system when \( \Omega \pm \omega \) is near \( \omega_i \), a detuning parameter will be introduced such that:

\[ \omega = \pm (\omega_i - \Omega) + \varepsilon \sigma \]  

Eq. 4.33

So that:

\[ \exp[(\Omega \pm \omega)T_0] = \exp(i\omega_i T_0) \exp(\pm i\sigma T_0) \]  

Eq. 4.34

In order to examine the effect of the detuning parameter on eliminating the troublesome terms from Eqs. 4.21, assume that the solutions for the \( u_{nl} \) are of the form:

\[ u_{11} = P_1(T_1) \exp(i\omega_i T_0) + Q_1(T_1) \exp(i\omega_2 T_0) \]  

Eqs. 4.35

\[ u_{21} = P_2(T_1) \exp(i\omega_i T_0) + Q_2(T_1) \exp(i\omega_2 T_0) \]

so that substituting Eqs. 4.35 into Eqs. 4.21 and equating the coefficients of \( \exp(i\omega_i T_0) \), the following system of equations for the \( P_n \) results (Eq. 4.36):

\[
\begin{bmatrix}
\alpha_i - \omega_i^2 & i \omega_i g_{12} \\
i \omega_i g_{21} & \alpha_2 - \omega_i^2
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
= \begin{bmatrix}
-\frac{i(\alpha_i + \omega_i^2)}{\omega_i} A_i' - f_{11} B_1 \exp(\pm i\sigma T_1) \\
-\frac{2\alpha_i - g_{12} g_{21} - 2\omega_i^2}{g_{12}} A_i' - f_{22} B_2 \exp(\pm i\sigma T_1)
\end{bmatrix}
\]

Note that the matrix multiplying \( P_n \) is singular according to Eq. 4.17 so that solutions do not exist unless:
Chapter 4: Combined Parametric and Lateral Harmonic Force Model

\[
\begin{vmatrix}
\alpha_1 - \omega_1^2 & - \frac{i(\alpha_1 + \omega_1^2)}{\alpha_1} A'_1 - f_{11} B_1 \exp(\pm i\sigma T) \\
\omega_1 g_{21} & - \frac{2\alpha_1 - g_{12} g_{21} - 2\omega_1^2}{g_{12}} A' - f_{22} B_2 \exp(\pm i\sigma T)
\end{vmatrix} = 0
\quad \text{Eq. 4.37}
\]

So that (Eq. 4.38):

\[
\left(\frac{(\alpha_1 - \omega_1^2)(2\alpha_1 - g_{12} g_{21} - 2\omega_1^2)}{g_{12}} + g_{21}(\alpha_1 + \omega_1^2)\right) \cdot A'_1 = \left[i\omega_1 g_{21} f_{11} B_1 - (\alpha_1 - \omega_1^2) f_{22} B_2\right] \exp(\pm i\sigma T)
\]

Solving Eq. 4.38 yields:

\[
A'_1(T) = \pm \frac{i\omega_1 g_{21} f_{11} B_1 - (\alpha_1 - \omega_1^2) f_{22} B_2}{i\sigma} \quad \text{Eq. 4.39}
\]

\[
\left(\frac{(\alpha_1 - \omega_1^2)(2\alpha_1 - g_{12} g_{21} - 2\omega_1^2)}{g_{12}} + g_{21}(\alpha_1 + \omega_1^2)\right)^{-1} \exp(\pm i\sigma T)
\]

By equating the coefficients of \(\exp(i\omega_2 T_0)\) to find the system governing the \(Q_n\), it can be shown that:

\[
A'_2 = 0
\quad \text{Eq. 4.40}
\]

which implies that \(A_2\) is a function of \(T_2\) and higher order time scales when \(\Omega \pm \omega\) is near \(\omega_1\) so that the zero order response at \(\omega_2\) is stable and of constant magnitude that depends upon the initial conditions.

Since \(\sigma\) is real, \(A_1\) is bounded for all time except for the case of \(\sigma\) equal to zero. In this case, \(\Omega - \omega\) is equal the first natural frequency of the system and a resonant condition occurs. Since \(A_1\) multiplies part of the zero order response, the component of this response at the first natural frequency tends to infinity as \(\sigma\) approaches zero. Given the certain presence of damping in the system and non-linear large amplitude effects, the response for this resonance condition will have a finite limit. A discussion of damping will be carried out in Section 6 of this Chapter. Note that it was shown in Section 4.3.1 that
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the component of the response at \( \Omega - \omega \) for the first order solution also becomes large when \( \Omega - \omega \) is near a natural frequency so that the zero order resonance and the first order resonance are compounded.

Unlike a simple or combination parametric resonance situation, when \( \Omega - \omega \) is near a natural frequency, a regular resonance phenomena occurs rather than an instability. There is only one value of the detuning parameter (\( \sigma = 0 \)) causing unbounded behavior rather than a range over which the amplitudes exhibit exponential growth.

4.3.3 \( \Omega \pm \omega \) Near \( \omega_2 \)

Using an analysis similar to the previous section, it can be shown that when \( \Omega \pm \omega \) is near \( \omega_2 \), \( A_2 \) is given by (Eq. 4.41):

\[
A_2(T_1) = \pm \frac{i \omega g_{21} f_{11} B_1 - (\alpha_1 - \omega^2) f_{22} B_2}{i \sigma} \left[ \frac{(\alpha_1 - \omega^2)(2 \alpha_1 + g_{12} g_{21} - 2 \omega^2)}{g_{12} + g_{21}(\alpha_1 + \omega^2)} \right]^{-1} e^{i \lambda t_1}
\]

So that \( A_2 \) is bounded for all time save for when \( \sigma \) is equal to zero. This is exactly analogous to the case of \( \Omega \pm \omega \) near \( \omega_1 \).

4.3.4 \( \omega \) Near \( \omega_2 - \omega_1 \)

This condition corresponds to a combination parametric resonance. For this condition, the lateral forcing frequency is not causing any resonant conditions and the stability behavior is governed by the homogeneous form of the first order equations (Eqs. 4.20). Once again, introducing the detuning parameter:

\[
\omega = \omega_2 - \omega_1 + \varepsilon \sigma \quad \text{Eq. 4.42}
\]

The terms involving frequencies \( \omega + \omega_1 \) and \( \omega_2 - \omega \) become:

\[
\exp[i(\omega + \omega_1)T_0] = \exp(i \omega_2 T_0 + i \sigma T_1) \quad \text{Eqs. 4.43}
\]

\[
\exp[i(\omega_2 - \omega)T_0] = \exp(i \omega_1 T_0 - i \sigma T_1)
\]
In order to investigate the stability behavior of the system in response to the resonant terms, particular solutions of the form in Eqs. 4.35 will be considered. When Eqs. 4.35 and 4.43 are substituted into Eqs. 4.21 and coefficients of \( \exp(i\omega_1 T_0) \) are equated, the following relations for the \( P_n \) and \( Q_n \) result (Eqs. 4.44 and 4.45):

\[
\begin{bmatrix}
\alpha_1 - \omega_1^2 & i\omega_1 g_{12} \\
i\omega_1 g_{21} & \alpha_2 - \omega_2^2
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
= \begin{bmatrix}
-i(\alpha_1 + \omega_1^2) \\
2\alpha_1 - g_{12} g_{21} - 2\omega_1^2
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
- \begin{bmatrix}
\frac{i(\alpha_1 + \omega_1^2)}{\omega_1} \\
\frac{2\alpha_1 - g_{12} g_{21} - 2\omega_1^2}{g_{12} \omega_2}
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
\frac{i(\alpha_1 - \omega_1^2)}{g_{12} \omega_2}
\end{bmatrix}
- \begin{bmatrix}
\frac{i(\alpha_1 - \omega_1^2)}{\omega_2} \\
\frac{2\alpha_1 - g_{12} g_{21} - 2\omega_1^2}{g_{12} \omega_2}
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
\frac{i(\alpha_1 - \omega_1^2)}{g_{12} \omega_2}
\end{bmatrix}
\end{bmatrix}
\]

From Eq. 4.17, it can be seen that the coefficient matrices on the left hand sides of Eqs. 4.44 and 4.45 are singular. As a result, in order for solutions for the \( P_n \) and \( Q_n \) to exist, the \( A_n \) must satisfy the following conditions:

\[
A'_1 = -\Gamma_1 A_2 \exp(-i\sigma T_1) \\
A'_2 = -\Gamma_2 A_2 \exp(i\sigma T_1)
\]

Eqs. 4.46

where:

\[
\Gamma_1 = \frac{1}{2} \left[ \frac{\alpha_1 - \omega_1^2}{g_{12}} - g_{21} \alpha_1 \right]^{-1} \left[ i\omega_1 g_{21} f_{11} - i \left( \frac{\alpha_1 - \omega_1^2}{g_{12} \omega_2} \right) \right]
\]

Eq. 4.47

\[
\Gamma_2 = \frac{1}{2} \left[ \frac{\alpha_1 - \omega_2^2}{g_{12}} - g_{21} \alpha_1 \right]^{-1} \left[ i\omega_1 g_{21} f_{11} - i \left( \frac{\alpha_1 - \omega_2^2}{g_{12} \omega_2} \right) \right]
\]

Eq. 4.48

It is important to note at this point that the \( \Gamma_n \) are purely imaginary. Solutions for the \( A_n \) can now be assumed in the form:

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\[ A_1 = a_1 \exp(\gamma T_1) \]  
Eqs. 4.49

\[ A_2 = a_2 \exp[(\gamma + i\sigma)T_1] \]

Nayfeh and Mook (1979 [14]) show that the exponent \( \gamma \) is given by:

\[ \gamma = -\frac{1}{2} i\sigma \pm \sqrt{\Gamma_1 \Gamma_2 - \frac{1}{4} \sigma^2} \]  
Eq. 4.50

Recalling that the \( \Gamma_n \) are both purely imaginary, \( \gamma \) can have a positive real part only for:

\[ \sigma > \pm 2\sqrt{\Gamma_1 \Gamma_2} \]  
Eq. 4.51

This leads to the stability boundaries in the \( \omega - \varepsilon \) parameter space given by:

\[ \omega = \omega_2 - \omega_1 \pm 2\varepsilon \sqrt{\Gamma_1 \Gamma_2} + O(\varepsilon^2) \]  
Eq. 4.52

Given that the \( \Gamma_n \) are dependent upon blade speed and blade tension, the stability boundaries will be dependent upon these as well as will be seen in Section 5.

5 Influence of Problem Parameters

In order to investigate the effect of various problem parameters on the physical behavior of the model being considered, typical numerical values are required. Given that the frequencies of interest in the bandsawing problem are on the order of 1000 Hz, the parameters will be chosen such that the two natural frequencies being considered will also be on the order of 1000 Hz. The values that will be used for the problem parameters are given in Table 4.1 below:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bending Stiffness</td>
<td>El</td>
<td>5.625 Nm²</td>
</tr>
<tr>
<td>Blade Span</td>
<td>L</td>
<td>0.8 m</td>
</tr>
<tr>
<td>Blade Linear Density</td>
<td>( \rho )</td>
<td>1.179 kg/m</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
<th>Blade Speed</th>
<th>C</th>
<th>42 m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tooth Pitch</td>
<td>P</td>
<td>0.0444 m (1.75&quot;)</td>
</tr>
<tr>
<td>Tooth Passing Frequency</td>
<td>Ω</td>
<td>946 Hz</td>
</tr>
<tr>
<td>Tension</td>
<td>T</td>
<td>31.2 kN (7000 lb)</td>
</tr>
<tr>
<td>Lateral Force Magnitude</td>
<td>F</td>
<td>1.10 N/m</td>
</tr>
</tbody>
</table>

Table 4.1 - Values of blade parameters used in response component calculations

Given these values, the typical values of the non-dimensional parameters may now be found as follows:

\[ \chi = \frac{(T - \rho c^2)E}{L^2} = 118.4 \quad \gamma = 2cL\sqrt{\frac{\rho}{EI}} = 30.8 \quad \eta = \frac{FL^3}{EI} = 0.1 \]

The fluctuations in axial tension were assumed small so that the parameter \( \varepsilon \ll 1 \). This assumption was used in the multiple scales analysis of the discretized equations. Given these parameters, the \( g_{mn} \), \( \alpha_m \) and \( f_{mn} \) may now be calculated according to Eqs. 4.12. In order to have the resulting natural frequencies on the order of 1000 Hz, it was found that the modes 9 and 10 should be considered. This implies that \( m = 9 \) and \( n = 10 \). Considering these two modes and the values defined above, the non-dimensional gyroscopic natural frequencies are:

\[ \omega_9 = 1653 \quad \omega_{10} = 2269 \]

which when dimensionalized, correspond to:

\[ f_9 = 898Hz \quad f_{10} = 1233Hz \]

Now that typical values for the majority of the parameters of interest have been established, the effect of varying each of these parameters independently will be examined. The response components of interest in the washboarding problem contain frequency terms \( \Omega \) and \( \Omega \pm \omega \). Looking at the zero order response, there are three frequency terms, one at each of the natural frequencies being considered and the third
at the tooth passing frequency. For reasons to be seen in Sections 6 and 7 once damping is incorporated into the model, the zero order terms containing the two natural frequencies and the first order terms containing the frequency combinations $\omega_n \pm \omega$ will not be considered at this time.

5.1 **Effect of Blade Speed**

Changing blade speed, $c$, in the problem affects a number of parameters beginning with the non-dimensional variables $\chi$ and $\gamma$. These parameters are important in determining the gyroscopic coefficients $g_{nm}$ and the equivalent frequency coefficients $\alpha_n$. Not only do these coefficients dictate the gyroscopic natural frequencies of the system, they play important roles in the magnitude of the system response as well as the determination of the stability boundaries for cases of parametric resonances and the amplitude magnification for the case of regular forced resonances involving responses at $\Omega$ and $\Omega \pm \omega$.

In addition to affecting the parameters mentioned above, blade speed directly determines the tooth passing frequency and therefore is very important in determining the frequency combinations $\Omega \pm \omega$. A waterfall frequency plot of the total response (zero plus first order terms) for $u_g (u_{90} + u_{91})$ with varying tooth passing frequency (blade speed) is shown below in Figure 4.10:
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Figure 4.10 - $u_9$ blade response with varying tooth passing frequency

For this plot, a non-dimensional band tension fluctuation frequency of $\omega = 30$ which corresponds to approximately 16 Hz was used. The response component at the lowest frequency showing peaks '1' and '2' corresponds to the first order $\Omega - \omega$ response. The central response component showing peak '3' corresponds to that at $\Omega$ from the zero order response. The highest frequency response component showing peaks '4' and '5' corresponds to that at $\Omega + \omega$ from the first order response.

A number of interesting features can be seen in Figure 4.10. The first observation is that as the tooth passing frequency changes, the response frequencies move with it as expected. The second observation is that the spacing between the response components remains constant and equal to $\omega$ (16 Hz) as the tooth passing frequency is varied. The most notable features of Figure 4.10 however, are the 5 peaks in the response components that are labeled 1 through 5 on the figure.
Beginning with the highest tooth passing frequency, 920 Hz, it can be seen that the responses at $\Omega - \omega$ and $\Omega$ are of the same order while that at $\Omega + \omega$ is quite small. As the tooth passing frequency is decreased to approximately 914 Hz, the response labeled “1” becomes large. This corresponds to the situation where $\Omega - \omega$ is approximately equal to $f_0$. Note that due to the lack of damping, the response should actually grow to infinity as $\Omega - \omega = f_0$ but given the frequency resolution used in the calculations, this does not occur in this diagram. At this value of $\Omega$, the responses at $\Omega$ and $\Omega + \omega$ are smaller than that at $\Omega - \omega$ and this scenario corresponds to what is seen during severe Type I washboarding.

As $\Omega$ is decreased further, the response at $\Omega - \omega$ decreases until $\Omega \sim 900$ Hz. This corresponds to the large amplitudes labeled “2”, “3” and “4” in Figure 4.10. In this scenario, $\Omega$ is very near $f_0$ so that a large response occurs at $\Omega$. It is interesting though, that large responses also occur at $\Omega \pm \omega$ for this case despite neither of these combinations being near $f_0$. The reason these responses are also large is that the first order response at $\Omega \pm \omega$ given by Eq. 4.30 contains the terms $B_1$ and $B_2$ which become large when $\Omega$ is near a natural frequency. This is in addition to the term $\text{DET}(C_{5,6})$ which causes a large response when $\Omega \pm \omega$ is near a gyroscopic natural frequency.

The last interesting feature of Figure 4.10 corresponds to the large response labeled “5”. This response corresponds to the case of $\Omega + \omega$ being near $f_0$. This response is analogous to that labeled “1”. This situation of a large response at $\Omega + \omega$ with significantly smaller responses at $\Omega$ and $\Omega - \omega$ is not seen in washboarding trials.

### 5.2 Effect of Blade Strain

Blade strain, $T$, has a direct effect on the non-dimensional variable $\chi$ and as such is very important in determining the natural frequencies of the system. It is well known that increasing the blade tension also increases the stationary bending natural frequencies of a stationary beam. The effect of tension on the gyroscopic natural frequencies of the system will be explored. Figure 4.11 shows the dependence of the non-dimensional 9th and 10th gyroscopic natural frequencies on blade strain:
As can be seen from Figure 4.11, increasing blade strain serves to increase the 9th and 10th gyroscopic natural frequencies of the beam as expected. Knowing the effect of strain on gyroscopic natural frequency, it is of interest to investigate the influence of strain on the response for $u_9$ for a given tooth passing frequency and bite. A plot of the $u_9$ response components with changing strain is shown below in Figure 4.12. The strain limits used in the plot correspond to 6500 and 7500 lbs which are adequate to cause $\omega_9$ to traverse the frequency range from $\Omega - \omega$ to $\Omega + \omega$. 

Figure 4.11 - Effect of blade strain on $\omega_9$ and $\omega_{10}$ ($F_{tp} = 900$ Hz)
Figure 4.12 - $u_9$ response components with varying strain ($F_{tp} = 900$ Hz)

Figure 4.12 was created using a tooth passing frequency of 900 Hz. Again, five large amplitude responses can be seen. As with Figure 4.10, the components of response from lowest frequency to highest frequency correspond to $\Omega - \omega$, $\Omega$, and $\Omega + \omega$ respectively. Given that the tooth passing frequency is constant, the response frequencies are constant as well. A line A-B, passing through peaks "1", "3" and "5" traces $f_9$ as the strain level increases.

Peak "1" corresponds to $\omega_y$ being equal to $\Omega - \omega$. Peaks "2", "3", and "4" correspond to $\omega_y$ being near $\Omega$ and the large magnitude of Peaks "2" and "4" is explained using the same argument found in Section 5.1. Peak "5" corresponds to $\Omega + \omega$ near $\omega_y$. Thus it is clear that changing strain affects the gyroscopic natural frequencies of the system and thus the magnitude of the response for all other variables held constant.
5.3 Effect of Bite

Echeverri (2003 [18]) has shown for single tooth cutting tests using wood, that increasing bite leads to increased axial cutting load on the tooth tip. This component of cutting force was considered in the model of the previous Chapter. Given the mechanism described in Section 1.2, it is also assumed that increasing bite will increase the magnitude of the direct lateral excitation, \( H \), which has a direct effect on the non-dimensional cutting force, \( \eta \). The non-dimensional cutting force in turn directly affects the magnitude of the forced response components at the frequencies \( \Omega - \omega \), \( \Omega \) and \( \Omega + \omega \).

Another observation concerning bite is that the chip thickness is increased leading to increased pressure on the rake face of the cutting tooth. This increased pressure translates to increased friction between the chip being removed and the tooth. It stands to reason therefore that the level of damping must also be affected by increasing bite. At this time, this is simply a qualitative observation. No attempt will be made here to relate bite to energy dissipation through lateral motion.

Given that bite affects the main axial cutting loads present on each tooth tip, each load serves to alter the local tension in the band. With the knowledge that band tension affects the natural frequencies of the blade, it is apparent that changing bite should have an effect on the blade natural frequencies. This effect has not been included in the present model and will be discussed further in the next Chapter.

Figure 4.13 below shows the impact of increasing bite on the zero and first order response components for \( u_9 \).
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Figure 4.13 - $u_9$ response components with varying $\eta$ (Bite) ($F_{ip} = 915$ Hz, $f_9 = 898$ Hz, $\Omega - \omega = 900$ Hz)

Responses "1", "2" and "3" in Figure 4.13 refer to the $\Omega - \omega$, $\Omega$ and $\Omega + \omega$ responses respectively. For this plot, the largest response is at $\Omega - \omega$ due to its proximity to $\omega_9$. It is clear from the plot that increasing bite serves to increase the magnitude of all three of the components of response.

5.4 Effect of Depth of Cut

The model considered in this Chapter to this point has made no allowance for the influence of depth of cut on the response. However, depth of cut can easily be incorporated into the model by including a number of direct lateral cutting forces acting at fixed locations along the blade instead of a single lateral cutting force at blade midspan. These cutting forces will be spaced at the tooth pitch and will be considered to have a phase relationship with each other. This situation is shown in Figure 4.14 below:

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Figure 4.14 – Schematic of multiple combined lateral excitation and axial tension fluctuation model

Considering the model represented above, the non-dimensionalized partial differential equation of motion can now be given as (Eq. 4.53):

\[
\frac{\partial^4 y}{\partial x^4} - (\chi + \epsilon \cos \omega \cdot t) \frac{\partial^2 y}{\partial x^2} + \gamma \frac{\partial^2 y}{\partial x \partial t} + \frac{\partial^2 y}{\partial t^2} = \sum_{k=1}^{N} \eta \cos(\Omega t + \phi_k) \delta \left[ x - \frac{h_0 + (k-1)p}{L} \right]
\]

which when discretized in the same manner as in Section 3, becomes (Eq. 4.54):

\[
\ddot{u}_m + \sum_n g_{mn} \dot{u}_n + \alpha_m u_m + 2\varepsilon \cos(\omega \cdot t) f_{mm} \dot{u}_m = 2\eta \sum_{k=1}^{N} \cos(\Omega t + \phi_k) \sin \left[ \frac{m\pi}{L} (h_0 + (k-1)p) \right]
\]

The most basic case to consider is that of a depth of cut involving a pair of direct lateral cutting forces. Let these forces be separated by one tooth pitch and centered between the guides. Given a tooth pitch of \( p = 1.75" \) and a length between guide centers of \( L = \)
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30° and once again considering the first and second modes (i.e. \( m = 1, 2 \)), Eq. 4.54 reduces to:

\[
\ddot{u}_m + \sum_n g_{mn} \dot{u}_n + \alpha_m u_m + 2\varepsilon \cos(\omega \cdot t) f_{mn} u_m = 2\left\{ \eta_{1m} \cos(\Omega t) + \eta_{2m} \cos(\Omega t + \phi) \right\}
\]

Eq. 4.55

where:

\[
\eta_{1m} = \eta \sin \left[ \frac{m\pi}{2} \left( 1 - \frac{p}{L} \right) \right] \quad \eta_{2m} = \eta \sin \left[ \frac{m\pi}{2} \left( 1 + \frac{p}{L} \right) \right]
\]

Eqs. 4.56

It is easily seen that the magnitude of the terms on the right hand side of Eq. 4.55 is dependent upon which mode is being considered. Combining terms on the right hand side of Eq. 4.55, it can be rewritten as:

\[
\ddot{u}_m + \sum_n g_{mn} \dot{u}_n + \alpha_m u_m + 2\varepsilon \cos(\omega \cdot t) f_{mn} u_m = 2\eta_{rm} \cos(\Omega t + \phi_{rm})
\]

Eq. 4.57

where:

\[
\eta_{rm} = \left( \eta_{1m}^2 + 2\eta_{1m} \eta_{2m} \cos \phi_2 + \eta_{2m}^2 \right)^{\frac{1}{2}} \quad \tan \phi_{rm} = \frac{\eta_{2m} \sin \phi_2}{\eta_{1m} + \eta_{2m} \cos \phi_2}
\]

Eqs. 4.58

Depending upon the phasing and magnitudes of the \( \eta_{km} \), the resultant forcing term on the right hand side of Eq. 4.57 may be larger or smaller than for the case of a single central lateral load. For example, if the phase of the second forcing term is \( \pi \) radians, and \( \eta_{1m} = \eta_{2m} \), the resultant \( \eta_{rm} \) is zero. On the other hand, if the second forcing term is exactly in phase with the first, and once again \( \eta_{1m} = \eta_{2m} \), the resultant \( \eta_{rm} \) will be twice \( \eta_{1m} \). These resultant magnitudes are important as they directly determine the magnitude of the forced responses at both the tooth passing frequency as well as the at the frequency combinations \( \Omega \pm \omega \) as seen in Section 4. This concept can be extended further to examine the general case of \( N \) teeth. Rewriting Eq. 4.54 as:
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\[ \ddot{u}_m + \sum_n g_{mn} \dot{u}_n + \alpha_m u_m + 2\varepsilon \cos(\omega \cdot t) f_{mn} u_m = 2 \sum_{k=1}^N \eta_{km} \cos(\Omega t + \phi_k) \]  
Eq. 4.59

where the \( \eta_{km} \) are given by:

\[ \eta_{km} = \eta \sin \left[ \frac{m\pi}{2} \left( 1 - p \frac{N - 2k + 1}{L} \right) \right] \]  
Eq. 4.60

It remains to consolidate the \( N \) terms on the right hand side of Eq. 4.59 into one single term given by:

\[ 2 \sum_{k=1}^N \eta_{km} \cos(\Omega t + \phi_k) = 2 \eta_{rm} \cos(\Omega t + \phi_{rm}) \]  
Eq. 4.61

where the resultant magnitude and phase are given by:

\[ \eta_{rm} = \sqrt{\left( \sum_{k=1}^N \eta_{km} \cos \phi_k \right)^2 + \left( \sum_{k=1}^N \eta_{km} \sin \phi_k \right)^2} \]  
Eq. 4.62

\[ \tan \phi_{rm} = \frac{\sum_{k=1}^N \eta_{km} \sin \phi_k}{\sum_{k=1}^N \eta_{km} \cos \phi_k} \]  
Eq. 4.63

In general, the magnitude of the resulting forcing term depends upon the phasing of the applied direct lateral loads as well as the mode shape of the \( m^{th} \) mode being considered. For some cases, depending upon the phases involved, increasing depth of cut will yield an increased value of \( \eta_{rm} \). This will have the same effect as increasing bite and lead to increased response magnitude as seen in Figure 4.13.
5.5 Instability When \( \omega \) is Near \( \omega_{10} - \omega_9 \)

The stability boundaries in the \( \omega-\varepsilon \) plane can be described by modifying Eq. 4.52 to represent the ninth and tenth modes as follows:

\[
\omega = \omega_{10} - \omega_9 \pm 2\varepsilon \sqrt{\Gamma_9 \Gamma_{10}} + O(\varepsilon^2)
\]  

Eq. 4.64

Using the parameters listed in Table 4.1, the values of \( \Gamma_9 \) and \( \Gamma_{10} \) can be calculated. As expected, both the \( \Gamma_m \) are purely imaginary. It also turns out that both of the \( \Gamma_m \) have positive imaginary parts so that when the product \( \Gamma_9 \Gamma_{10} \) is calculated, it is negative. For this reason, no unstable region exists for the combination resonance associated with \( \omega \) near \( \omega_{10} - \omega_9 \). In fact, when all the adjacent modes up to modes 100 and 101 are compared, it is found that the imaginary part of \( \Gamma_n \) and \( \Gamma_{n+1} \) are both positive as seen in Figure 4.15 below:

Figure 4.15 – Imaginary parts of \( \Gamma_m \) and \( \Gamma_{m+1} \) for increasing mode number
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This suggests that no combination parametric resonances involving adjacent modes can occur in the simple traveling beam model with axial tension fluctuation for two adjacent modes which agrees with the results found by Wu and Mote (1986 [4]) as well as Lengoc and McCallion (1995 [5]). It will be seen that the average natural frequency spacing of a stationary toothed blade like those used in the cutting trials in the washboarding frequency region is roughly 20 Hz. This combined with the measured frequency difference being between 10 and 40 Hz suggests that it is likely that a parametric combination resonance causing washboarding could only occur for \( \omega \) near the difference between two adjacent modes. However, it has just been shown that for the traveling beam model being considered, combination parametric resonances cannot occur for \( \omega \) near the difference of two adjacent modes. This does not necessarily discount parametric combination resonances from being a cause for Type I washboarding given the complex mode shapes of a real toothed blade. The analysis to this point has neglected the effect of damping. Once damping is considered in the next two sections, it will be seen that regular forced resonances of the types investigated in Sections 4.2, 4.3.2 and 4.3.3 are a more likely cause of the Type I washboarding problem. The next two sections will investigate the effect of small and large damping respectively.

6 Effect of Damping

Given the nature of the wood cutting problem, a thorough investigation of how damping is manifested in the cutting process is very difficult. In fact, damping may occur in a number of different ways including:

1. interaction between workpiece and blade
2. damping at the hydro-dynamically lubricated guides
3. internal damping in the blade
4. damping due to air resistance

In order to gain a very basic understanding how damping might play a role in the model considered throughout this chapter, Rayleigh damping, as described by Dimarogonas (1996 [19]), will be assumed where the damping matrix is a linear combination of the mass and stiffness matrices. As can be seen from the discretized equations of motion Eqs. 4.11, the mass and stiffness matrices are both diagonal and therefore, any linear
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combination of the two will also be diagonal. The elements of such a damping matrix will be designated $c_{nm}$, and the discretized equations of motion become (Eq. 4.65):

$$\ddot{u}_m + \alpha_{mm} \dot{u}_m + \sum_n g_{mn} \dot{u}_n + \alpha_m u_m + 2\epsilon \cos(\omega \cdot t) f_{mn} u_m = 2\eta \cos(\Omega t) \sin\left(\frac{m\pi}{2}\right)$$

It has been assumed that damping is small as it simplifies the multiple scales analysis to follow. Consideration of larger scale damping will be given in Section 7. At this stage, all elements of the diagonal damping matrix are considered real and positive definite. Notice that the gyroscopic terms remain the only terms that couple the equations of motion. Should non-diagonal damping be investigated, it would serve to alter the gyroscopic coefficients of the system and thus affect the stability characteristics for parametric resonances and the magnitude components for forced resonances.

6.1 Two Mode Zeroth Order Solution With Damping

As before, a two-mode model will be used to investigate the behavior of the system. Once again, using the method of multiple scales, the $\epsilon^0$ and $\epsilon^1$ order equations are:

Order $\epsilon^0$:

$$D_0^2 u_{10} + g_{12} D_0 u_{20} + \alpha_1 u_{10} = 2\eta \cos(\Omega t)$$

Eqs. 4.66

$$D_0^2 u_{20} + g_{21} D_0 u_{10} + \alpha_2 u_{20} = 0$$

Note that to this order, Eqs. 4.66 are identical to Eqs. 4.15 and will thus have identical solutions.

Order $\epsilon^1$ (Eqs. 4.67):

$$D_0^2 u_{11} + g_{12} D_0 u_{21} + \alpha_1 u_{11} = -2D_0 D_1 u_{10} - g_{12} D_1 u_{20} - c_{11} D_0 u_{10} - 2f_{11} u_{10} \cos \omega T_0$$

$$D_0^2 u_{21} + g_{12} D_0 u_{11} + \alpha_2 u_{21} = -2D_0 D_1 u_{20} - g_{21} D_1 u_{10} - c_{22} D_0 u_{20} - 2f_{22} u_{20} \cos \omega T_0$$

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As for the undamped case, the zero order total solution is given by (Eqs. 4.68):

\[ u_{10} = A_1(T_1) \exp(i\omega_1 T_0) + A_2(T_1) \exp(i\omega_2 T_0) + c + B_1 \exp(i\Omega T_0) \]

\[ u_{20} = \frac{i(\alpha_i - \alpha_2^2)}{g_{12}\omega} A_1(T_1) \exp(i\omega_1 T_0) + \frac{i(\alpha_i - \alpha_2^2)}{g_{12}\omega} A_2(T_1) \exp(i\omega_2 T_0) + c + B_2 \exp(i\Omega T_0) \]

Where \( B_1 \) and \( B_2 \) represent the amplitudes of the forced zero order response as given in Eq. 4.19. Given the assumption of small damping, the amplitudes of the zero order forced response can become singular as \( \Omega \) approaches a natural frequency of the system as was discovered in Section 4.2. It is understood at this point that prior to any amplitude experiencing a singularity, non-linear large amplitude effects not included in this model will govern the behavior of the system.

6.2 Two Mode First Order Solution With Damping

When the complete zero order solution is substituted into the right hand sides of Eqs. 4.67, the following results for the \( \varepsilon^1 \) system of equations (Eqs. 4.69a,b):

\[ D_0^2 u_{11} + g_{12} D_0 u_{21} + \alpha_1 u_{11} = -i \left[ \frac{(\alpha_1 + \omega_1^2)}{\omega_1} A_1 + c_1 \omega_1 A_1 \right] \exp(i\omega_1 T_0) \]

\[ -i \left[ \frac{(\alpha_1 + \omega_2^2)}{\omega_2} A_2 + c_1 \omega_2 A_2 \right] \exp(i\omega_2 T_0) \]

\[ - f_{11} \left[ A_1 \exp[i(\omega_1 + \omega) T_0] + A_1 \exp[i(\omega_1 - \omega) T_0] \right] \]

\[ - f_{11} A_2 \exp[i(\omega_2 + \omega) T_0] + A_2 \exp[i(\omega_2 - \omega) T_0] \] + c + \[ - f_{11} B_1 \left[ \exp[i(\Omega + \omega) T_0] + \exp[i(\Omega - \omega) T_0] \right] \]

\[ - i\Omega c_{11} B_1 \exp(i\Omega T_0) \]
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\[
D_0^2 u_{21} + g_{21} D_0 u_{11} + \alpha_2 u_{21} = \left[ \frac{2\alpha_1 - g_{12} g_{21} - 2\omega_1^2}{g_{12}} A_1' + \frac{c_{22}(\alpha_1 - \omega_1^2)}{g_{12}} A_1 \right] \exp(i\omega_1 T_0)
+ \left[ \frac{2\alpha_1 - g_{12} g_{21} - 2\omega_2^2}{g_{12}} A_2' + \frac{c_{22}(\alpha_1 - \omega_2^2)}{g_{12}} A_2 \right] \exp(i\omega_2 T_0)
- if_{22} \frac{\alpha_1 - \omega_1^2}{g_{12} \omega_1} A_1 \left\{ \exp[i(\omega_1 + \omega) T_0] + \exp[i(\omega_1 - \omega) T_0] \right\}
- if_{22} \frac{\alpha_1 - \omega_2^2}{g_{12} \omega_2} A_2 \left\{ \exp[i(\omega_2 + \omega) T_0] + \exp[i(\omega_2 - \omega) T_0] \right\} + cc
- f_{22} B_2 \left\{ \exp[i(\Omega + \omega) T_0] + \exp[i(\Omega - \omega) T_0] \right\}
- i\Omega c_{22} B_2 \exp(i\Omega T_0)
\]

As for the undamped case, from the right hand sides of Eqs. 4.69, there are a number of apparent resonant frequency combinations. These resonant frequencies include:

1. \( \omega = 2\omega_1 \)
2. \( \omega = 2\omega_2 \)
3. \( \omega = \omega_1 + \omega_2 \)
4. \( \omega = \omega_2 - \omega_1 \)
5. \( \Omega + \omega \approx \omega_1 \)
6. \( \Omega + \omega \approx \omega_2 \)
7. \( \Omega - \omega \approx \omega_1 \)
8. \( \Omega - \omega \approx \omega_2 \)
9. \( \Omega \approx \omega_1 \)
10. \( \Omega \approx \omega_2 \)

Once again, given the values of the tooth passing frequency and band tension fluctuation frequency, the first three of these cases do not occur during the washboarding process and will not be considered further. Two additional resonant possibilities (#9 and #10) occur at this order compared to the undamped case. Prior to investigating the remaining cases, the case where none of the above resonant conditions applies will be investigated and the general solution of the first order equations will be found. It should be noted that Eqs. 4.69 are identical to Eqs. 4.21 save for the addition of the final term on the right hand sides corresponding to an excitation at \( \Omega \). Therefore, the solution to the non-resonant terms is simply the particular solution of Eqs. 4.21 plus the solution of Eqs. 4.69 corresponding to the final term on the right hand side containing the term \( \exp(i\Omega T_0) \).
6.2.1 Non-resonant Case With Damping

Once again, in order to obtain valid solutions, the troublesome terms containing \( \exp(i \omega_0 T_0) \) must be dealt with. As done in Section 4.3.2, the first order solution will be assumed to have the form given in Eqs. 4.35. When Eqs. 4.35 are substituted into Eqs. 4.69 and the coefficients of \( \exp(i \omega_0 T_0) \) are equated, the following two systems of equations for the \( P_n, Q_n \) result:

\[
\begin{bmatrix}
\alpha_1 - \omega_1^2 & \omega_1 g_{12} \\
i \omega_1 g_{21} & \alpha_2 - \omega_1^2
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix} =
\begin{bmatrix}
R_1 \\
R_2
\end{bmatrix}
\]

Eq. 4.70

\[
\begin{bmatrix}
\alpha_2 - \omega_2^2 & i \omega_2 g_{12} \\
\omega_2 g_{21} & \alpha_2 - \omega_2^2
\end{bmatrix}
\begin{bmatrix}
Q_1 \\
Q_2
\end{bmatrix} =
\begin{bmatrix}
S_1 \\
S_2
\end{bmatrix}
\]

Eq. 4.71

where the \( R_n \) and \( S_n \) are given by:

\[
R_1 = -i \frac{\alpha_1 + \omega_1^2}{\omega_1} A_1' - ic_1 \omega_1 A_1
\]

Eqs. 4.72

\[
R_2 = \frac{2 \alpha_1 - g_{12} \omega_2 - 2 \omega_1^2}{g_{12}} A_1' + \frac{c_{22}(\alpha_1 - \omega_1^2)}{g_{12}} A_1
\]

\[
S_1 = -i \frac{\alpha_2 + \omega_2^2}{\omega_2} A_2' - ic_1 \omega_2 A_2
\]

Eqs. 4.73

\[
S_2 = \frac{2 \alpha_2 - g_{12} \omega_2 - 2 \omega_2^2}{g_{12}} A_2' + \frac{c_{22}(\alpha_2 - \omega_2^2)}{g_{12}} A_2
\]

Since the coefficient matrices multiplying the \( P_n \) and \( Q_n \) are singular according to Eq. 4.17, in order for solutions for the \( P_n, Q_n \) to exist, the following must hold:

\[
\begin{vmatrix}
\alpha_1 - \omega_1^2 & R_1 \\
i \omega_1 g_{21} & R_2
\end{vmatrix} = 0
\]

Eq. 4.74
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\[
\begin{vmatrix}
\alpha_1 - \omega_2^2 & S_1 \\ i\omega_2 g_{21} & S_2
\end{vmatrix} = 0
\]

Eq. 4.75

When these determinants are expanded, two independent first order differential equations result; one for \(A_1\), the other for \(A_2\):

\[
A_1' + \Psi_{1nr} A_1 = 0
\]

Eqs. 4.76

\[
A_2' + \Psi_{2nr} A_2 = 0
\]

where the prime indicates differentiation with respect to \(T\), and the 'nr' refers to the non-resonant case. The \(\Psi\)'s are given by:

\[
\Psi_{1nr} = \frac{1}{2} \left[ \frac{c_{22}(\alpha_1 - \omega_1^2)^2 - c_{11}g_{12}g_{21}\omega_1^2}{(\alpha_1 - \omega_1^2)^2 - g_{12}g_{21}\alpha_1} \right]
\]

Eqs. 4.77

\[
\Psi_{2nr} = \frac{1}{2} \left[ \frac{c_{22}(\alpha_1 - \omega_2^2)^2 - c_{11}g_{12}g_{21}\omega_2^2}{(\alpha_1 - \omega_2^2)^2 - g_{12}g_{21}\alpha_1} \right]
\]

Now solutions for the \(A_n\) can be given by:

\[
A_1 = C_1(T_2) \exp(-\Psi_{1nr} T_1)
\]

Eqs. 4.78

\[
A_2 = C_2(T_2) \exp(-\Psi_{2nr} T_1)
\]

This shows that to first order, the amplitudes \(A_n\) exhibit either exponential growth or exponential decay compared to the undamped case in which the \(A_n\) were constant \((D_0 A_n = 0)\). From Eqs. 4.77, it is evident that the \(\Psi_{knr}\) are real. In addition, since the product \(g_{12}g_{21}\) is negative given the opposite signs of \(g_{12}\) and \(g_{21}\), and given that \(\alpha_1\) and the \(c_{nn}\) are positive real quantities, the \(\Psi_{knr}\) must be positive as well. Therefore, the amplitudes \(A_n\) must decay exponentially indicating that for the non-resonant case, the only steady state zero order term in the solution will occur at the tooth passing frequency, \(\Omega\).
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With the troublesome terms eliminated, solutions for the remaining terms on the right hand sides of Eqs. 4.69 can be found. In exactly the same manner as for the undamped case, solutions to each of the non-resonant terms on the right hand sides of Eqs. 4.69 are given by Eqs. 4.24, 4.27 and 4.30. The solution to the final term on the right hand sides of Eqs. 4.69 can be found independently due to the linear nature of the problem.

For the response due to terms containing \( \Omega \), Eqs. 4.69 reduces to:

\[
D_0^2 u_{11} + g_{12} D_0 u_{21} + \alpha_1 u_{11} = -i \Omega c_{11} B_1 \exp(i \Omega T_0)
\]

\[
D_0^2 u_{21} + g_{21} D_0 u_{11} + \alpha_2 u_{21} = -i \Omega c_{22} B_2 \exp(i \Omega T_0)
\]

The solution of Eqs. 4.79 is given as:

\[
\begin{bmatrix}
  u_{11} \\
  u_{21}
\end{bmatrix}
= -\frac{i \Omega}{\text{DET}(C_{c7})} \begin{bmatrix}
  \alpha_2 - \Omega^2 & -i \Omega g_{12} \\
  -i \Omega g_{21} & \alpha_1 - \Omega^2
\end{bmatrix} \begin{bmatrix}
  c_1 B_1 \\
  c_2 B_2
\end{bmatrix} \exp(i \Omega T_0)
\]

Eq. 4.80

Where \( \text{DET}(C_{c7}) \) represents a modified frequency equation given by:

\[
\text{DET}(C_{c7}) = (\alpha_1 - \Omega^2)(\alpha_2 - \Omega^2) + g_{12} g_{21} \Omega^2
\]

Eq. 4.81

Notice that as \( \Omega \) approaches a natural frequency, \( \text{DET}(C_{c7}) \) becomes large and approaches infinity. This is another resonant case that will be investigated in the following section. This first order resonance will compound the resonance that can occur at zero order when \( \Omega \) approaches a natural frequency of the system.

It is important to note at this point that the responses corresponding to terms not involving \( \Omega \), as given by Eqs. 4.24 and 4.27, in the first order solution will all experience decaying behavior with time given the behavior of the \( A_n \) for the non-resonant condition. Therefore, in order to obtain large responses at the frequencies \( \omega_n \pm \omega \), the intensity of the instability must be great enough to overcome the energy dissipated by the damping inherent in the problem. This will in effect serve to narrow the frequency range over which unstable behavior, if any, will occur. At this point, it still remains to be seen if any
parametric combination resonances are possible. It is also apparent, from the single degree of freedom damped Mathieu equation discussed by Bolotin (1964 [17]) and Nayfeh and Mook (1979 [14]), that above a critical level of damping, the instability associated with \( \omega_n \pm \omega \) near a natural frequency will no longer occur. This is important as it would suggest that provided significant damping, the only responses of interest will be those either at the tooth passing frequency or those at a frequency combination containing the tooth passing frequency.

Now that the seven components of the first order response have been investigated for the non-resonant case, it remains to be seen what happens to the response components as any of the possible resonant frequency combinations occurs.

### 6.2.2 \( \Omega \) Near \( \omega_1 \)

In addition to the resonant conditions for the undamped case at first order, the damped case has an additional resonant case where the tooth passing frequency approaches a natural frequency of the system. In order to investigate this case, a detuning parameter will be introduced as before:

\[
\Omega = \omega_1 + \epsilon \sigma \quad \text{Eq. 4.82}
\]

Making use of Eq. 4.82 and assuming a solution for Eqs. 4.69 of the form in Eqs. 4.35, isolating terms containing \( \exp(i \omega_1 T) \) yields:

\[
\begin{bmatrix}
\alpha_1 - \omega_1^2 & i \omega_1 g_{12} \\
\omega_1 g_{21} & \alpha_2 - \omega_1^2
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
= \begin{bmatrix}
R_1 - i \Omega c_{11} B_1 \exp(i \sigma T) \\
R_2 - i \Omega c_{22} B_2 \exp(i \sigma T)
\end{bmatrix} \quad \text{Eq. 4.83}
\]

Once again, given that the coefficient matrix of the \( P_n \) in Eq. 4.83 is singular according to Eq. 4.17, the following must hold in order for solutions to exist for the \( P_n \):

\[
\begin{bmatrix}
\alpha_1 - \omega_1^2 & R_1 - i \Omega c_{11} B_1 \exp(i \sigma T) \\
i \omega_1 g_{21} & R_2 - i \Omega c_{22} B_2 \exp(i \sigma T)
\end{bmatrix}
= 0 
\quad \text{Eq. 4.84}
\]
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Evaluation of the determinant in Eq. 4.84 leads to the following first order differential equation for \( A_1 \):

\[
A_1' + \Psi_{1nr} A_1 = \Omega \left[ g_{21} \omega_1 c_{11} B_1 + i \left( \alpha_1 - \omega_2^2 \right) c_{22} B_2 \right] \exp(i \sigma T_1) \tag{4.85}
\]

The above equation is a linear, non-homogeneous first order ordinary differential equation with a homogeneous solution given by Eq. 4.78a and a particular solution given by:

\[
(A_1)_p = \frac{\Omega \left[ g_{21} \omega_1 c_{11} B_1 + i \left( \alpha_1 - \omega_2^2 \right) c_{22} B_2 \right]}{\Psi_{1nr} + i \sigma} \exp(i \sigma T_1) \tag{4.86}
\]

As discovered earlier, \( \Psi_{1nr} \) is a positive semi-definite real quantity. Given a non-zero \( \Psi_{1nr} \), the particular solution \((A_1)_p\) will have a finite magnitude regardless the proximity of \( \Omega \) to a gyroscopic natural frequency (\( \sigma = 0 \)). It was also found that for \( \Omega \) near a natural frequency, the first order solution component given by Eq. 4.80 becomes large and becomes singular for \( \Omega \) equal to a natural frequency. This serves to reinforce that the only steady state component of the total response \( u_{nm} \) dominated by the zero order solution is that at the tooth passing frequency.

It should also be noted that for \( \Omega = \omega_1 \), the exponential component of \((A_1)_p, \exp(i \sigma T_1)\) becomes equal to one as \( \sigma \) is equal to zero indicating that \((A_1)_p\) is a constant.

6.2.3 \( \Omega \) Near \( \omega_2 \)

As was done for the previous section, let:

\[
\Omega = \omega_2 + \epsilon \sigma \tag{4.87}
\]

and following the same procedure, the particular solution for \( A_2 \) is found to be:

\[
(A_2)_p = \frac{\Omega \left[ g_{21} \omega_2 c_{11} B_1 + i \left( \alpha_1 - \omega_2^2 \right) c_{22} B_2 \right]}{\Psi_{2nr} + i \sigma} \exp(i \sigma T_1) \tag{4.88}
\]
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As for the particular solution of $A_i$, the particular solution for $A_2$ will have a finite value given non-zero $\Psi_{2w}$.

6.2.4 $\Omega \pm \omega$ Near $\omega_1$

Once again assuming a solution of the form in Eqs. 4.35, using the detuning parameter $\sigma$:

$$\Omega \pm \omega = \omega_1 + c \sigma$$

Eq. 4.89

and equating terms containing $\exp(i\omega T_0)$, the following results for the $P_n$:

$$\begin{bmatrix} \alpha_1 - \omega_1^2 & i \omega_1 g_{12} \\ i \omega_1 g_{21} & \alpha_2 - \omega_2^2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} R_1 - f_{11} B_1 \exp(i \sigma T_1) \\ R_2 - f_{22} B_2 \exp(i \sigma T_1) \end{bmatrix}$$

Eq. 4.90

As seen before, in order for solutions for the $P_n$ to exist, the following must hold:

$$\begin{bmatrix} \alpha_1 - \omega_1^2 & R_1 - f_{11} B_1 \exp(i \sigma T_1) \\ i \omega_1 g_{21} & R_2 - f_{22} B_2 \exp(i \sigma T_1) \end{bmatrix} = 0$$

Eq. 4.91

which yields the following equation for $A_1$:

$$A_1' + \Psi_{1nr} A_1 = \left(\alpha_1 - \omega_1^2\right) f_{22} B_2 + i g_{21} \omega_1 f_{11} B_1 \exp(i \sigma T_1)$$

Eq. 4.92

Again, in addition to the homogeneous solution, which has been shown to lead to exponentially decaying solutions for $A_1$, the particular solution is given as:

$$\left( A_1 \right)_p = \frac{\left(\alpha_1 - \omega_1^2\right) f_{22} B_2 + i g_{21} \omega_1 f_{11} B_1}{\Psi_{1nr} + i \sigma} \exp(i \sigma T_1)$$

Eq. 4.93

As for the case $\Omega$ near $\omega_1$, the particular solution for $A_1$ does not blow up as $\Omega \pm \omega$ approaches $\omega_1$ given that $\Psi_{1nr}$ is nonzero. This indicates that the zero order solution at
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the frequency $\omega_1$ remains finite as both $\Omega$ and/or $\Omega \pm \omega$ approach the first natural frequency and only the zero order component of response at the tooth passing frequency blows up as $\Omega$ approaches a natural frequency.

6.2.5 $\omega$ Near $\omega_2 - \omega_1$

It was found in Section 5.5, that no combination parametric resonances of the type $\omega$ near $\omega_2 - \omega_1$ could occur. It remains to be seen what effect damping has on this possible resonant condition. For this case, the detuning parameter will be introduced as:

$$\omega = \omega_2 - \omega_1 + \varepsilon \sigma$$  \hspace{1cm} \text{Eq. 4.94}

so that:

$$\exp[i(\omega_2 - \omega)T_0] = \exp(i\omega_2 T_0) \exp(i\sigma T_1)$$  \hspace{1cm} \text{Eqs. 4.95}

$$\exp[i(\omega_2 - \omega)T_0] = \exp(i\omega_1 T_0) \exp(-i\sigma T_1)$$

Once again assuming the solution in Eqs. 4.35 and equating terms in Eqs. 4.69 containing $\exp(i\omega_n T_0)$ leads to two sets of equations for the $P_n$ and the $Q_n$ of the form:

$$\begin{bmatrix} \alpha_1 - \omega_1^2 & i\omega_1 g_{12} \\ i\omega_1 g_{21} & \alpha_2 - \omega_2^2 \end{bmatrix} \begin{bmatrix} P^1 \\ P^2 \end{bmatrix} = \begin{bmatrix} R_1 - f_{11} A_2 \exp(-i\sigma T_1) \\ R_2 - f_{22} \frac{\alpha_1 - \omega_1^2}{g_{12} \omega_2} A_2 \exp(-i\sigma T_1) \end{bmatrix}$$  \hspace{1cm} \text{Eq. 4.96}

$$\begin{bmatrix} \alpha_1 - \omega_1^2 & i\omega_2 g_{12} \\ i\omega_2 g_{21} & \alpha_2 - \omega_2^2 \end{bmatrix} \begin{bmatrix} Q^1 \\ Q^2 \end{bmatrix} = \begin{bmatrix} S_1 - f_{11} A_1 \exp(i\sigma T_1) \\ S_2 - f_{22} \frac{\alpha_1 - \omega_1^2}{g_{12} \omega_1} A_1 \exp(i\sigma T_1) \end{bmatrix}$$  \hspace{1cm} \text{Eq. 4.97}

As before, the coefficient matrices of the $P_n$ and $Q_n$ are singular so that in order for solutions to exist, the following determinants must be equal to zero:

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\[
\begin{vmatrix}
\alpha_1 - \omega_1^2 & R_1 - f_{11} A_1 \exp(-i\sigma T_1) \\
i\omega g_{21} & R_2 - f_{22} \frac{\alpha_1 - \omega_1^2}{g_{12}\omega_2} A_2 \exp(-i\sigma T_1)
\end{vmatrix} = 0
\]
Eq. 4.98

\[
\begin{vmatrix}
\alpha_1 - \omega_1^2 & S_1 - f_{11} A_1 \exp(i\sigma T_1) \\
i\omega g_{21} & S_2 - f_{22} \frac{\alpha_1 - \omega_1^2}{g_{12}\omega_1} A_1 \exp(i\sigma T_1)
\end{vmatrix} = 0
\]
Eq. 4.99

Satisfying these two conditions yields the following set of equations for the \( A_n \):

\[
A_1' + \Psi_{1nr} A_1 + \Gamma_1 A_2 \exp(-i\sigma T_1) = 0
\]
Eqs. 4.100

\[
A_2' + \Psi_{2nr} A_2 + \Gamma_2 A_1 \exp(i\sigma T_1) = 0
\]

where the \( \Gamma_n \) are given by Eqs. 4.47 and 4.48. Solutions of Eqs. 4.100 can be expressed in the form of Eqs. 4.49 as was done in Section 4.3.4:

\[
A_1 = a_1 \exp(\gamma T_1) \quad A_2 = a_2 \exp(\gamma + i\sigma T_1)
\]
Eqs. 4.101

provided the following holds:

\[
\gamma^2 + (i\sigma + \Psi_{1nr} + \Psi_{2nr}) \gamma - \Gamma_1 \Gamma_2 + \Psi_{1nr} \Psi_{2nr} = 0
\]
Eq. 4.102

This quadratic equation results in the following solutions for the \( \gamma \)’s:

\[
\gamma = -\left(\frac{\gamma + \Psi_{1nr} + \Psi_{2nr}}{2}\right) \pm \sqrt{\left(\frac{\gamma + \Psi_{1nr} + \Psi_{2nr}}{2}\right)^2 - 4\left(\Psi_{1nr} \Psi_{2nr} - \Gamma_1 \Gamma_2\right)}
\]
Eq. 4.103

In order to ascertain the nature of the roots of Eq. 4.103, it is important to recall that the \( \Psi \)’s are real and positive semi-definite. Also note that the product \( \Gamma_1 \Gamma_2 \) is real and negative given that each of the \( \Gamma_n \) are purely complex and the evidence in Figure 4.15. Since \( \Psi_{1nr} \Psi_{2nr} - \Gamma_1 \Gamma_2 \) is positive, when the square root is evaluated in Eq. 103, the real
part will be less than $\Psi_{1nr} + \Psi_{2nr}$ so that the real part of the exponent $\gamma$ will be negative regardless of the value of $\sigma$ indicating that similar to the undamped case, combination resonances where $\omega$ is near the difference of two adjacent gyroscopic natural frequencies cannot occur. This supports the idea that the only steady state response components to zero and first order are those with frequency components $\Omega$ and $\Omega \pm \omega$.

7 Effect of $\epsilon^0$ Order Damping

In the previous Section, damping was assumed to be small and as a result, the zero order multiple scales equations were identical to those without damping causing the solutions to have singularities as the tooth passing frequency approached a natural frequency of the system. In a real system with any level of damping such singularities do not occur. In this Section, the effect of zero order damping will be briefly investigated. Eq. 4.104 below is identical to Eq. 4.65 save for the order of damping.

$$u_m + c_{mm}u_m + \sum_n g_{mn}u_n + \alpha_m u_m + 2\epsilon \cos(\omega \cdot t)f_{mn}u_m = 2\eta \cos(\Omega t)sin\left(\frac{m\pi}{2}\right)$$

Once again using the method of multiple scales, Eq. 4.104 can be converted to the zero and first order multiple scales equations below:

Order $\epsilon^0$:

$$D_0^2u_{10} + D_0(c_{11}u_{10} + g_{12}u_{20}) + \alpha_{10}u_{10} = 2\eta \cos(\Omega t)$$

Eqs. 4.105

$$D_0^2u_{20} + D_0(g_{21}u_{10} + c_{22}u_{20}) + \alpha_{20}u_{20} = 0$$

Order $\epsilon^1$ (Eqs. 4.106):

$$D_0^2u_{11} + D_0(c_{11}u_{11} + g_{12}u_{21}) + \alpha_{11}u_{11} = -2D_0D_1u_{10} - g_{12}D_1u_{20} - c_{11}D_1u_{10} - 2f_{11}u_{10} \cos \omega T_0$$

$$D_0^2u_{21} + D_0(g_{21}u_{10} + c_{22}u_{20}) + \alpha_{21}u_{21} = -2D_0D_1u_{20} - g_{21}D_1u_{10} - c_{22}D_1u_{20} - 2f_{22}u_{20} \cos \omega T_0$$
Chapter 4: Combined Parametric and Lateral Harmonic Force Model

In contrast to the undamped case, the zero order total solution of Eq. 4.105 is given by (Eqs. 4.107):

$$u_{10h} = A_1(T_0) \exp(i\lambda_1 T_0) + A_2(T_0) \exp(i\lambda_2 T_0) + cc + B_{1c} \exp(i\Omega T_0)$$

$$u_{20h} = \frac{i(\alpha_i - \lambda_1^2 + i\lambda_1 \lambda_2)}{g_{12 \lambda_1}} A_1(T_0) \exp(i\lambda_1 T_0) + \frac{i(\alpha_i - \lambda_2^2 + i\lambda_1 \lambda_2)}{g_{12 \lambda_2}} A_2(T_0) \exp(i\lambda_2 T_0) + cc$$

$$+ B_{2c} \exp(i\Omega T_0)$$

where the $\lambda_n$'s come from the solution of:

$$\begin{align*}
(\alpha_i - \lambda_n^2 + i\lambda_n c_{ii})(\alpha_i - \lambda_n^2 + i\lambda_n c_{ii}) + \lambda_n^2 g_{12} g_{21} &= 0 \\
\text{Eq. 4.108}
\end{align*}$$

Given the presence of damping at zero order, the eigenvalues ($\lambda_n$'s) will have a positive imaginary part that when combined with $i$ in $\exp(i\lambda_n T_0)$ resulting in a negative real component of the exponent. This leads to exponential decay of the homogeneous zero order solutions.

The $B_{nc}$ are given by (Eq. 4.109):

$$\begin{align*}
\begin{bmatrix}
B_{1c} \\
B_{2c}
\end{bmatrix} &= \frac{2\eta}{(\alpha_i - \Omega^2 + i\Omega c_{ii})(\alpha_i - \Omega^2 + i\Omega c_{ii}) + g_{12} g_{21} \Omega^2} \begin{bmatrix}
\alpha_i - \Omega^2 + i\Omega c_{ii} \\
-ig_{21} \Omega
\end{bmatrix}
\end{align*}$$

The quantity in the denominator of Eq. 4.109 has the same form as the left hand side of Eq. 4.108. Knowing that Eq. 4.108 is only satisfied for complex values of the $\lambda_n$, and knowing that $\Omega$ is a purely real quantity, there is no value of $\Omega$ that leads to the $B_{nc}$'s having infinite magnitude. This is in contrast to the undamped and $\varepsilon$ order damping zero order particular solutions which both exhibit singularities as $\Omega$ approaches a gyroscopic natural frequency of the system.

Substituting Eqs. 4.107 into Eqs. 4.106, the following form of the $\varepsilon$ order multiple scales equations results (Eqs. 4.110a,b):
Chapter 4: Combined Parametric and Lateral Harmonic Force Model

\[ D_0^2 u_{11} + D_0 (c_{11} u_{11} + g_{12} u_{21}) + \alpha_1 u_{11} = -i \frac{\alpha_1 + \lambda_1^2}{\lambda_1} A' \exp(i \lambda_1 T_0) \]

\[-i \frac{\alpha_1 + \lambda_2^2}{\lambda_2} A' \exp(i \lambda_2 T_0) \]

\[-f_{11} \{A_1 \exp[i(\lambda_1 + \omega)T_0] + A_1 \exp[i(\lambda_1 - \omega)T_0]\} \]

\[-f_{11} \{A_2 \exp[i(\lambda_2 + \omega)T_0] + A_2 \exp[i(\lambda_2 - \omega)T_0]\} + cc \]

\[-f_{11} B_{1c} \{\exp[i(\Omega + \omega)T_0] + \exp[i(\Omega - \omega)T_0]\} \]

\[ D_0^2 u_{21} + D_0 (g_{21} u_{11} + c_{22} u_{21}) + \alpha_2 u_{21} = \]

\[ \begin{bmatrix}
2\alpha_1 - g_{12} g_{21} - 2\lambda_1^2 + 2ic_{11} \lambda_1 \\
-g_{12} \alpha_1 - \lambda_2^2 + ic_{11} \lambda_1 \\
b_{12} \alpha_1 - \lambda_2^2 + ic_{11} \lambda_2 \\
-g_{12} \alpha_1 - \lambda_1^2 + ic_{11} \lambda_2
\end{bmatrix}
\]

\[ A' \exp(i \lambda_1 T_0) \]

\[ A' \exp(i \lambda_2 T_0) \]

\[-if_{22} \frac{\alpha_1 - \lambda_1^2 + ic_{11} \lambda_1}{g_{12} \lambda_1} A_1 \{\exp[i(\lambda_1 + \omega)T_0] + \exp[i(\lambda_1 - \omega)T_0]\} \]

\[-if_{22} \frac{\alpha_1 - \lambda_2^2 + ic_{11} \lambda_2}{g_{12} \lambda_2} A_2 \{\exp[i(\lambda_2 + \omega)T_0] + \exp[i(\lambda_2 - \omega)T_0]\} + cc \]

\[-f_{22} B_{2c} \{\exp[i(\Omega + \omega)T_0] + \exp[i(\Omega - \omega)T_0]\} \]

As for the undamped and \( \varepsilon \) order damping cases, it is of interest to determine the possible resonant terms involving the parametric excitation frequency, \( \omega \), and the lateral excitation frequency, \( \Omega \), on the right hand sides of Eqs. 4.110. Given that the eigenvalues of the zero order problem, \( \lambda_n \)'s, are complex, and that \( \omega \) is real, the terms \( \exp[i(\lambda_n \pm \omega)] \) cannot cause resonance since the combination \( \lambda_n \pm \omega \) cannot be equal to \( \lambda_m \). This leaves only two remaining potentially troublesome terms containing \( \exp[i(\Omega \pm \omega)] \). Note however that the combination \( \Omega \pm \omega \) is purely real and therefore cannot be equal to any of the complex \( \lambda_n \)'s either. Therefore, with zero order damping, no parametric instabilities can occur in the solutions of the first order multiple scales equations. The particular solutions involving the frequency terms \( \exp[i(\Omega \pm \omega)] \), however, will have finite magnitude since the frequency combinations \( \Omega \pm \omega \) can never be equal to a damped gyroscopic natural frequency of the system. This is in contrast to the \( \varepsilon \) order
damping particular solutions for the \( \exp[i(\Omega \pm \omega)] \) terms which experience singularities as \( \Omega \pm \omega \) approaches any of the \( \omega_n \)'s. Given that combination parametric resonances have been shown not to occur for adjacent modes in both the undamped and \( \varepsilon \) order damping problems as well as the zero order damping problem, the zero order damping configuration yields realistic solutions as there are no singularities in either the zero or first order particular solutions.

The particular solutions for the terms on the right hand side of Eqs. 4.110 containing \( \exp[i(\Omega \pm \omega)] \) are given by (Eq. 4.101):

\[
\begin{bmatrix}
    u_{11} \\
    u_{21}
\end{bmatrix}
= \frac{\exp[i(\Omega \pm \omega)T_0]}{DET(C_c)} \begin{bmatrix}
    \alpha_2 - (\Omega \pm \omega)^2 + i(\Omega \pm \omega)c_{22} & -i(\Omega \pm \omega)g_{12} \\
    -i(\Omega \pm \omega)g_{21} & \alpha_1 - (\Omega \pm \omega)^2 + i(\Omega \pm \omega)c_{11}
\end{bmatrix}
\begin{bmatrix}
    -f_{11}B_{1c} \\
    -f_{22}B_{2c}
\end{bmatrix}
\]

where \( DET(C_c) \) is given by (Eq. 4.102):

\[
DET(C_c) = [\alpha_1 - (\Omega \pm \omega)^2 + ic_{11}(\Omega \pm \omega)][\alpha_2 - (\Omega \pm \omega)^2 + ic_{22}(\Omega \pm \omega)] + g_{12}g_{21}(\Omega \pm \omega)^2
\]

The determinant given by Eq. 102 cannot be equal to zero and the same was shown earlier for the denominator of the expressions for the \( B_{nc} \) from Eq. 4.109. Therefore, the solutions at the frequency \( \Omega \pm \omega \) are bounded for all real values of \( \Omega \pm \omega \). This is in contrast to the undamped and \( \varepsilon \) order damping cases which exhibit singularities in the zero and first order solutions as \( \Omega \) or \( \Omega \pm \omega \) approach a gyroscopic natural frequency.
CHAPTER 5: EXPLANATION OF TYPE I
WASHBOARDING BEHAVIOR

1 Experimental Behavior of Response Components

As was seen in Chapter 4, the combined parametric axial tension, lateral excitation model predicts, as desired, a response component at the frequency of the lateral excitation. In the washboarding problem being considered, this frequency corresponds to the tooth passing frequency. The magnitude of this response depends upon two main factors; the magnitude of the direct lateral excitation as well as the proximity of the tooth passing frequency to a gyroscopic natural frequency of the system. Similarly, the model predicts two other important responses at frequencies greater than and less than the tooth passing frequency by some value \( \omega \). These excited frequency responses also depend on the magnitude of the lateral excitation and the proximity of \( \Omega \pm \omega \) to a gyroscopic natural frequency of the system.

As seen in Chapter 2, experimental results show two main responses responsible for washboarding. These responses occur at the tooth passing frequency, \( F_{tp} \), and the excited frequency occurring at a frequency less than the tooth passing frequency, \( F_{ex} \), by some frequency difference \( \Delta F \). It was also seen that a tertiary response occurs above the tooth passing frequency at some frequency \( F_3 \) but is always smaller in magnitude than the tooth passing frequency response and the excited response. Given that the model of Chapter 4 predicts a large response at the excited frequency when \( \Omega - \omega \) is near a gyroscopic natural frequency of the system, the behaviour of the excited frequency will be investigated and compared to the natural frequencies of the blade as predicted by a finite element model of a toothed blade.

As has been observed in Chapter 4, the lateral excitation magnitude is believed to be largely influenced by both bite and depth of cut. The gyroscopic natural frequencies of the model were found to be dependent on blade speed (or tooth passing frequency), blade strain and blade geometry. During cutting trials, the effects of these primary operating parameters on the washboarding behavior of a band saw were tested. The
Chapter 5: Explanation of Type I Washboarding Behavior

general results of these tests are described next. The work by Taylor et al. (2003 [7]) contains a more complete discussion of the findings of the experimental cutting program.

1.1 Influence of Blade Speed on Type I Washboarding

Cutting trials for the three blades were run for a range of wheel speeds from 460 rpm to 700 rpm. Given the different dynamic behavior for each blade tested, the washboarding speed region for each blade and each configuration was essentially unique. For example, the thickest blade with the least amount of tooth modification (16G0H0, shallowest gullet) tended to have the highest washboarding speed region. Conversely, the thinnest blade with the most significant tooth modification (18G2H2) had the lowest washboarding speed region. This trend for the 18 gauge blade is seen in Figure 5.1 below.

As can be seen, washboarding was found to occur within a fairly broad range of wheel speeds for each blade and tooth configuration. At this point, it is unknown whether the washboarding response is associated with a natural frequency. It will be seen in Section 2 of this Chapter that a number of natural frequencies are found in the washboarding frequency range between 900 and 1200 Hz. Given the range of wheel speeds that lead to washboarding for a given blade configuration as shown in Figure 5.1 and assuming that washboarding is associated somehow with natural frequencies, the question arises as to whether a single natural frequency is responsible for the entire range or whether a number of natural frequencies are involved. Given the difficulty in experimentally determining high frequency mode shapes during cutting, this question is difficult to answer.

Another observation from Figure 5.1 is that with subsequent tooth shape modifications, the washboarding speed region decreases. This fact will be used when assessing the influence of tooth geometry on blade natural frequencies.
Chapter 5: Explanation of Type I Washboarding Behavior

W/B Extent for the 18 Ga. Blade (Bite= 0.024", D_c = 9.6")

1.2 Influence of Strain on Type I Washboarding

Three different strain levels were tested during the testing program, 10 klbs, 14 klbs and 18 klbs. The fundamental effect of changing strain during the testing program was to shift the speed region over which washboarding occurs. This may be seen in Figure 5.2 below:
The behaviour of blade natural frequencies with strain will be investigated in Section 2. As was seen in Chapter 4, increasing strain led to increased blade natural frequencies for an axially traveling beam. It would also appear that washboarding extent is reduced with increasing strain.

1.3 Influence of Bite on Type I Washboarding

The model considered in Chapter 5 predicted that the magnitude of the response components is directly proportional to bite. The model did not consider the effect of the main tangential cutting loads on the natural frequencies of the blade. This effect will be examined more closely in Section 2.3. Experimental results have shown that bite plays a very important role in determining blade washboarding severity and extent. Figure 5.3 below shows the dependence of washboarding extent on bite for blade 18G0H0 for constant speed, strain and depth of cut.
Chapter 5: Explanation of Type I Washboarding Behavior

Figure 5.3 - Effect bite on washboarding extent (18G0H0 blade)

The figure shows that for small bites, Type I washboarding does not occur (Type II does) and that as bite is increased, the washboarding extent does also. This agrees well with the proposed mathematical model.

1.4 Influence of Depth of Cut on Type I Washboarding

During the cutting tests, it was found that depth of cut had a strong impact on the washboarding behavior of the blades tested. In general, as depth of cut is increased, the severity of the washboarding increases as well. In addition, below a certain threshold depth of cut, no Type I washboarding occurs. Figure 5.4 below shows the influence of depth of cut on the extent of washboarding.
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It is evident that increased depth of cut does increase the extent of washboarding but it also increases the size of the speed region over which washboarding occurs in agreement with findings by Okai et al. (1997 [20]). From Eq. 4.62, increasing depth of cut will typically increase $\eta_m$ which from Eq. 4.19, will increase the magnitude of the responses at both $F_{tp}$ and $F_{ex}$. Assuming a minimum response magnitude exists in order to observe a washboarding pattern, as the $\eta_m$ increase, the range of tooth passing frequencies resulting in this minimum response magnitude will also increase according to the frequency determinant in the denominator of Eq. 4.19 and the frequency determinant $\text{DET}(C_{5,6})$ given by Eq. 4.31 that plays a prominent role in the response components given be Eq. 4.30. This result agrees with the above experimental findings.

From Figure 5.4, it would appear that washboarding severity also increases with depth of cut. From Eq. 4.62, it would appear that, depending upon the phasing between adjacent
cutting teeth, that the resulting lateral force magnitude could either increase or decrease. These two observations might suggest that the teeth are not completely out of phase during washboarding.

2 Investigation of Toothed Blade Natural Frequencies

In order to interpret the bulk of experimental data accumulated during the washboarding trials, it is helpful to have information on the natural frequencies and mode shapes of a toothed blade and how these frequencies and mode shapes are influenced by parameters such as speed, strain, depth of cut, bite and geometry. This information will be helpful when discussing the nature of the frequency difference later in this Chapter.

2.1 Stationary Mode Shapes and Natural Frequencies

In order to obtain the natural frequencies and mode shapes for a stationary toothed bandsaw blade, a finite element model was created with the same geometrical configuration as that used in the washboarding trials. Although the model created will not include the effects of blade speed, the general trends observed will be useful when related to observations from the physical data. Given that the blade 18G2H2 showed the greatest tendency to washboard, this blade was modeled first for a given strain (T = 7000 lbs), and no cutting forces.

2.1.1 Blade Modeling

The blade model created for the 18G2H2 blade consists of a section of blade supported on each end with a distributed tensile load applied at one end across the width of the band. The parameters used in creating this model are given below in Table 5.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blade Span</td>
<td>L</td>
<td>0.8 m</td>
</tr>
<tr>
<td>Blade Thickness</td>
<td>t</td>
<td>1.245 mm</td>
</tr>
<tr>
<td>Blade Width</td>
<td>W</td>
<td>175.31 mm</td>
</tr>
<tr>
<td>Tooth Shape</td>
<td>-</td>
<td>18G2H2</td>
</tr>
<tr>
<td>Total Blade Tension</td>
<td>(R_0)</td>
<td>31.2 kN (7000 lbs)</td>
</tr>
</tbody>
</table>

Table 5.1 – 18G2H2 finite element blade model parameters
Chapter 5: Explanation of Type I Washboarding Behavior

An important consideration in the blade model is the nature of the constraints at either end of the band. Figure 5.5 shows the base model for the 18G2H2 blade complete with constraints and applied axial load.

![Finite element model for blade 18G2H2 (R₀ = 14 klbs)](image)

**Figure 5.5 – Finite element model for blade 18G2H2 (R₀ = 14 klbs)**

The nodes on the left edge of the blade have all been constrained in the axial direction of the blade as well as in the direction across the width of the band. In addition, the nodes between the back edge and the edge of the guide have been constrained in the lateral direction while those nodes overhanging the guide are unconstrained in the lateral direction. At the other end of the blade, the nodes between the back edge of the blade and the edge of the guide are constrained in the lateral direction as well as the direction across the width of the band. These nodes are unconstrained in the axial direction in order to allow the applied tensile axial load to develop strain in the band.

Using MSC Nastran’s case control, each blade model was analyzed first to calculate the stress field due to the applied axial load and then this stress field was used to augment the blade stiffness matrix for use in calculating the blade natural frequencies and mode shapes.
2.1.2 Natural Frequencies and Mode Shapes

The range of frequencies of interest from a washboarding perspective is from approximately 900 Hz to 1200 Hz. In this range, 11 natural frequencies were discovered. All of these 11 natural mode shapes exhibited large tooth motion compared with other areas of the blade indicating that cutting forces might be able to perform significant amounts of work on the natural mode shape as the blade moves through the wood. It should be noted at this point that the analysis performed does not include the gyroscopic effects due to blade motion. Luo (2001 [2]) found that gyroscopic effects can either increase or decrease blade natural frequencies in the speed region of interest. However, the change in natural frequencies for a smooth band model over the speed region considered in this work is not large. In addition to the high frequency modes, when investigating the frequency difference, ΔF, the behaviour of the first few natural frequencies will be important.

In order to compare frequencies for different blade loadings, a consistent mode numbering system was used. Each mode was assigned a single number or sequence of numbers to describe the number of antinodes along the length of the blade followed by a comma and then a number indicating the number of antinodes across the width of the blade. For example, the 3,3 mode looks like:

Figure 5.6 - Mode 3,3fb for the 18G2H2 blade (R₀ = 14 klbs)
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For reasons that will become apparent, an additional tag of either 'b', 't' or 'fb' will be used after the value for the number of antinodes across the width of the blade. The 'b' tag indicates that the motion of the back edge of the blade is significantly larger than that of the toothed edge of the blade whereas an 'f' tag indicates the opposite situation. Occasionally, the motion of the back and toothed edges of the blade will be of the same order in which case the tag 'fb' will be used.

On occasion, the number of antinodes along the length of the blade differs from the front, toothed edge, to the back edge. In these instances, additional labels must be included. For example, the 7t5b,2fb mode looks like:

![Figure 5.7 - Mode 7t5b,2fb for the 18G2H2 blade (R₀ = 14 kls)](image)

This designation indicates that there are 7 antinodes along the 't' (toothed) edge of the blade and 5 antinodes along the 'b' (back) edge of the blade. For modes that have more than 2 antinodes across the width of the blade, additional tags must be used. For example the 10t4m₁2m₂2b,4f mode looks like:
Chapter 5: Explanation of Type I Washboarding Behavior

Figure 5.8 - Mode 10t4m12m22b,4f for the 18G2H2 blade (R₀ = 14 klbs)

This designation indicates 10 antinodes along the toothed edge, 4 antinodes along the first middle line of nodes, 2 antinodes along the second middle line of nodes and 2 antinodes along the back edge of the blade. Using this system, mode shapes for different tooth loads, strains and blade thicknesses are easily identified and frequencies compared. The first 5 natural frequencies and mode shapes for the 18G2H2 blade are listed below in Table 5.2. The 11 natural frequencies in the washboarding frequency range and their mode shapes are listed below in Table 5.3:

<table>
<thead>
<tr>
<th>Mode Designation</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,2f</td>
<td>123.7</td>
</tr>
<tr>
<td>1,2b</td>
<td>137.2</td>
</tr>
<tr>
<td>1,3fb</td>
<td>240.8</td>
</tr>
<tr>
<td>2,2f</td>
<td>243.4</td>
</tr>
<tr>
<td>2,2b</td>
<td>278.3</td>
</tr>
</tbody>
</table>

Table 5.2 – Low frequency mode shapes and natural frequencies for blade 18G2H2, R₀ = 14 klbs
Chapter 5: Explanation of Type I Washboarding Behavior

<table>
<thead>
<tr>
<th>Mode Designation</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4,4fb</td>
<td>905.3</td>
</tr>
<tr>
<td>8t6b,1fb</td>
<td>949.0</td>
</tr>
<tr>
<td>8t6b,2f</td>
<td>954.2</td>
</tr>
<tr>
<td>6,3fb</td>
<td>1038.7</td>
</tr>
<tr>
<td>9t5m5b,3fb</td>
<td>1053.0</td>
</tr>
<tr>
<td>5,4fb</td>
<td>1077.6</td>
</tr>
<tr>
<td>1,5fb</td>
<td>1092.7</td>
</tr>
<tr>
<td>10t4m2m2b,4f</td>
<td>1138.8</td>
</tr>
<tr>
<td>2,5fb</td>
<td>1153.1</td>
</tr>
<tr>
<td>7,2b</td>
<td>1163.7</td>
</tr>
<tr>
<td>11t7b,2f</td>
<td>1195.5</td>
</tr>
</tbody>
</table>

Table 5.3 – High frequency mode shapes and natural frequencies for blade 18G2H2, \( R_0 = 14 \) klbs

As can be seen, the frequency spacing of the high frequency modes listed ranges from 5 Hz to 84 Hz with an average spacing of 19 Hz. This gives a large number of modes potentially responsible for washboarding. By examining the behaviour of the natural frequencies and the frequency difference as the cutting conditions are changed, it might be possible to narrow the above list of possible washboarding modes.

2.2 Effect of Strain

Changing the blade strain has a very significant effect on the blade stress field in the main body of the blade. Given that the blade planar stress field plays an important role in determining the blade’s lateral stiffness characteristics, it is evident that strain is important in determining the system natural frequencies. It is well known from the model of Chapter 4 that increasing axial strain has a strong effect on the pure axial bending modes of a beam as seen in Figure 4.11. An important note, however, is that the local stress field at each of the teeth does not show the same dependence on blade strain as the stress field in the main portion of the blade. For this reason, it remains to be seen if the natural frequencies of the modes of interest that contain significant tooth motion are as strongly dependent upon strain as the main bending modes of the blade. The first five
natural frequencies of blade 18G2H2 for the three different values of blade strain used in the cutting trials are given below in Figure 5.9.

![Effect of Strain On Low Natural Frequencies (Blade 18G2H2)](image)

Figure 5.9 – Effect of strain on first five natural frequencies of blade 18G2H2

Notice that as expected, the first five natural frequencies increase with strain. It is interesting though, that for \( R_0 = 10 \) klbs, mode 1,3fb has a higher frequency than mode 2,2f in contrast with the case of the two higher strain levels. This would indicate that the higher the number of antinodes in the axial direction, the greater the increase in natural frequency for a given increase in axial strain. The effect of strain on the high frequency tooth dominated modes of interest is shown in Table 5.4. It should be noted that not all mode shapes occur for all cases of strain.

<table>
<thead>
<tr>
<th>Mode Designation</th>
<th>( R_0 = 10 ) klbs</th>
<th>( R_0 = 14 ) klbs</th>
<th>( R_0 = 18 ) klbs</th>
</tr>
</thead>
<tbody>
<tr>
<td>4,4fb</td>
<td>862.1</td>
<td>905.3</td>
<td>948.1</td>
</tr>
<tr>
<td>8t4m4b,3f</td>
<td>899.1</td>
<td>-</td>
<td>998.1</td>
</tr>
<tr>
<td>8t6b,1fb</td>
<td>-</td>
<td>949.0</td>
<td>-</td>
</tr>
<tr>
<td>8t6b,2f</td>
<td>-</td>
<td>954.2</td>
<td>-</td>
</tr>
<tr>
<td>5,4fb</td>
<td>-</td>
<td>-</td>
<td>1133.0</td>
</tr>
</tbody>
</table>
Chapter 5: Explanation of Type I Washboarding Behavior

<table>
<thead>
<tr>
<th>Mode Shape</th>
<th>Natural Frequency 1 (kHz)</th>
<th>Natural Frequency 2 (kHz)</th>
<th>Natural Frequency 3 (kHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6,2b</td>
<td>1048.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6,3fb</td>
<td>945.3</td>
<td>1038.7</td>
<td>1125.3</td>
</tr>
<tr>
<td>9t5m5b,3fb</td>
<td>1002.1</td>
<td>1053.0</td>
<td></td>
</tr>
<tr>
<td>9t5m5b,3fb*</td>
<td>1021.3</td>
<td>1077.6</td>
<td></td>
</tr>
<tr>
<td>7,2b</td>
<td>1040.2</td>
<td>1163.7</td>
<td></td>
</tr>
<tr>
<td>1,5fb</td>
<td>1090.5</td>
<td>1092.7</td>
<td></td>
</tr>
<tr>
<td>10t5m5b,3f</td>
<td>1098.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9t1m1m21b,4f</td>
<td>-</td>
<td></td>
<td>1092.9</td>
</tr>
<tr>
<td>9t1m1m21b,4f</td>
<td>-</td>
<td></td>
<td>1096.2</td>
</tr>
<tr>
<td>10l2m2m4b,4f</td>
<td>-</td>
<td></td>
<td>1174.9</td>
</tr>
<tr>
<td>10t4m2m2b,4f</td>
<td>-</td>
<td>1138.8</td>
<td></td>
</tr>
<tr>
<td>6t7m7b,3fb</td>
<td>1142.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,5fb</td>
<td>1144.7</td>
<td>1153.1</td>
<td>1160.7</td>
</tr>
<tr>
<td>11t4b,2f</td>
<td></td>
<td></td>
<td>1227.3</td>
</tr>
<tr>
<td>11t7b,2f</td>
<td></td>
<td>1195.5</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.4 – Effect of strain on high frequency modes of blade 18G2H2

It is apparent from Table 5.4 that only simple mode shapes such as '4,4fb', '6,3fb', and '2,5fb' exist for all three strain levels. It can be seen that changing strain not only has a drastic effect on the natural frequencies of the toothed blade, it also strongly affects the corresponding mode shapes often creating new modes for different values of strain.

Upon examination of the mode shapes 9t1m1m21b,4f for the case of $R_0 = 18$ klbs, it is apparent that the mode shape is very similar to the 1,5fb mode save for the fact that the teeth are not in phase as for the 1,5fb mode. Considering these modes to be analogous and looking at the frequency change with strain for the 1,5fb and 2,5fb modes, it is seen that the change in frequency is only 6 Hz for the 1,5fb modes and 16 Hz for the 2,5fb modes when the strain is increased from 10 klbs to 18 klbs. For all other modes, the frequency change is at least 50 Hz. This fact will be revisited later in this Chapter.

2.3 **Effect of Bite**

The main effect of increasing bite from the perspective of blade loading is to increase the main tangential cutting load on each tooth tip. Increasing the main tangential cutting
force on each tooth augments the blade stress field, thus affecting its lateral stiffness characteristics. This change in stiffness characteristics will result in altered natural frequencies.

In order to get an estimate of the maximum main tangential cutting load, the following calculation was made.

Mill drive power: 75 kW
Typical blade speed: 50 m/s
Maximum cutting load: \( \frac{75\text{kW}}{50\text{m/s}} = 1500 \text{ N} \)

This assumes that all the drive power is consumed in the main cutting force. This is obviously not the case and as such, the force estimated is an upper bound. Assuming a depth of cut that gives 6 cutting teeth, the main cutting force per tooth is maximally 250 N. In modeling the blade, a range of tooth loads from 0 N to 200 N was used (given that 250 N is an upper bounds estimate).

The 18G2H2 blade model was altered to include 4 increments of main tangential cutting force, 50 N per tooth, 100 N per tooth, 150 N per tooth and 200 N per tooth oriented in the axial direction on six of the eleven teeth in the cutting span.

Table 5.5 lists the same modes as given in Table 5.3 and also the natural frequencies for the different increments of main tangential cutting force (P).
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<table>
<thead>
<tr>
<th>Mode</th>
<th>P = 0 N</th>
<th>P = 50 N</th>
<th>P = 100 N</th>
<th>P = 150 N</th>
<th>P = 200 N</th>
<th>Change (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4,4fb</td>
<td>905.3</td>
<td>904.5</td>
<td>903.6</td>
<td>902.8</td>
<td>901.9</td>
<td>-3.4</td>
</tr>
<tr>
<td>8t6b,1fb</td>
<td>949.0</td>
<td>947.1</td>
<td>945.2</td>
<td>943.2</td>
<td>941.0</td>
<td>-8.0</td>
</tr>
<tr>
<td>8t6b,2f</td>
<td>954.2</td>
<td>951.9</td>
<td>949.4</td>
<td>946.9</td>
<td>944.4</td>
<td>-9.8</td>
</tr>
<tr>
<td>6,3fb</td>
<td>1038.7</td>
<td>1037.0</td>
<td>1035.4</td>
<td>1033.7</td>
<td>1032.0</td>
<td>-6.7</td>
</tr>
<tr>
<td>9t5m5b,3fb</td>
<td>1053.0</td>
<td>1051.0</td>
<td>1048.8</td>
<td>1046.5</td>
<td>1043.9</td>
<td>-9.1</td>
</tr>
<tr>
<td>5,4fb</td>
<td>1077.6</td>
<td>1076.4</td>
<td>1075.2</td>
<td>1074.0</td>
<td>1072.8</td>
<td>-4.8</td>
</tr>
<tr>
<td>1,5fb</td>
<td>1092.7</td>
<td>1092.5</td>
<td>1092.2</td>
<td>1092.0</td>
<td>1091.7</td>
<td>-1.0</td>
</tr>
<tr>
<td>10t4m,2m2b,4f</td>
<td>1138.8</td>
<td>1136.9</td>
<td>1134.7</td>
<td>1132.3</td>
<td>1129.7</td>
<td>-9.1</td>
</tr>
<tr>
<td>2,5fb</td>
<td>1153.1</td>
<td>1152.8</td>
<td>1152.5</td>
<td>1152.1</td>
<td>1151.8</td>
<td>-1.3</td>
</tr>
<tr>
<td>7,2b</td>
<td>1163.7</td>
<td>1161.6</td>
<td>1159.5</td>
<td>1157.5</td>
<td>1155.4</td>
<td>-8.3</td>
</tr>
<tr>
<td>11t7b,2f</td>
<td>1195.5</td>
<td>1192.9</td>
<td>1190.2</td>
<td>1187.6</td>
<td>1185.1</td>
<td>-10.4</td>
</tr>
</tbody>
</table>

Table 5.5 – Effect of main tangential cutting force (bite) on blade frequencies (18G2H2 blade, $R_0 = 14$ klbs)

The results show that for all modes, the natural frequencies decrease with increasing bite. This is not unexpected as the applied load was modeled as a conservative force without a follower nature. According to Timoshenko (1961 [21]), the natural frequency of a beam column will decrease with the application of a compressive conservative end load. In increasing the main tangential cutting load from zero to 200 N, the maximum change in any of the frequencies listed was 10.4 Hz. As will be seen when investigating the effect of bite on the frequency difference, this fact may prove significant.

2.4 Effect of Depth of Cut

In order to examine the effect of depth of cut on the blade model examined to this point, two factors must be considered. Increasing depth of cut leads to more teeth cutting wood which results in a greater total axial load on the blade as well as a change in the local tooth stress field at an increased number of teeth. The second factor deals with the interaction between the cutting teeth and the wood being cut. Given that the teeth are moving laterally while cutting, it is not unreasonable to expect that some lateral stiffness component exists between the teeth and the wood. This concept is illustrated in Figure 5.10 below:
The value of lateral stiffness, $k_t$, associated with each tooth will likely depend upon tooth geometry as well as bite. It is evident that including such stiffnesses in the blade model will cause an increase in the natural frequencies involving large tooth motion.

To analyze this effect, rod elements were added to six of the teeth for the 18G2H2 blade model. Three different stiffnesses were tested corresponding to cross sectional areas of $1.0 \times 10^{-9}$, $2.0 \times 10^{-9}$ and $3.0 \times 10^{-9}$ m$^2$. These stiffnesses were chosen to slightly alter the models natural frequencies without drastically altering the model mode shapes. All rods were one half meter in length and made of steel. The effect of the added stiffness on the five lowest natural frequencies is shown below in Figure 5.11:
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**Effect of Lateral Cutting Stiffness on Low Blade Frequencies**

![Graph showing the effect of lateral cutting stiffness on low blade frequencies.](image)

Figure 5.11 – Effect of lateral cutting stiffness on low blade frequencies

(Lateral Stiffnesses of 0, 1, 2 and 3 correspond to rods of areas 0, 1 e-9, 2e-9 and 3 e-9 m² respectively)

From the figure, the added stiffness does not drastically alter the natural frequencies. It can be seen however, that the natural frequencies with significant tooth edge motion, (1,2f), (1,3fb) and (2,2f) show noticeable increase (6.3, 3.2 and 3.9 Hz respectively) in going from no lateral stiffness to the third level of lateral stiffness. The back edge modes on the other hand increase by 0.5 Hz maximum for mode (1,2b). This is not unexpected as the added stiffness is placed at the toothed edge of the blade.

The effect of lateral stiffness on the high frequency modes is shown below in Table 5.6.

<table>
<thead>
<tr>
<th>Mode Designation</th>
<th>Natural Frequencies (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 Lateral Stiffness</td>
</tr>
<tr>
<td>4,4fb</td>
<td>905.3</td>
</tr>
<tr>
<td>8t6b,1fb</td>
<td>949.0</td>
</tr>
<tr>
<td>8t6b,2f</td>
<td>954.2</td>
</tr>
<tr>
<td>6,3fb</td>
<td>1038.7</td>
</tr>
<tr>
<td>9t5m5b,3fb</td>
<td>1053.0</td>
</tr>
<tr>
<td>5,4fb</td>
<td>1077.6</td>
</tr>
</tbody>
</table>
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Table 5.6 - Effect of lateral cutting stiffness on high blade frequencies

<table>
<thead>
<tr>
<th>Lateral stiffness</th>
<th>Frequency 1</th>
<th>Frequency 2</th>
<th>Frequency 3</th>
<th>Frequency 4</th>
<th>Frequency 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 1, 2 and 3</td>
<td>1092.7</td>
<td>1138.8</td>
<td>1153.1</td>
<td>1163.7</td>
<td>1195.5</td>
</tr>
<tr>
<td>x 10^9 m^2</td>
<td>1093.2</td>
<td>1139.9</td>
<td>1153.5</td>
<td>1163.7</td>
<td>1197.5</td>
</tr>
<tr>
<td></td>
<td>1093.8</td>
<td>1140.9</td>
<td>1153.9</td>
<td>1163.8</td>
<td>1199.2</td>
</tr>
<tr>
<td></td>
<td>1094.3</td>
<td>1141.9</td>
<td>1154.2</td>
<td>1163.8</td>
<td>1201.4</td>
</tr>
<tr>
<td></td>
<td>1.6</td>
<td>3.1</td>
<td>1.1</td>
<td>0.1</td>
<td>5.87</td>
</tr>
</tbody>
</table>

(Lateral stiffnesses of 0, 1, 2 and 3 correspond to rods of areas 0, 1 e-9, 2 e-9 and 3 e-9 m^2 respectively)

Once again, significant frequency changes only occur for those modes with significant tooth edge motion. It is apparent that the magnitude of the frequency change for the high frequencies is of the same order as that of the low frequencies (maximum of approximately 6 Hz).

The effect of the number of teeth under cutting load and with added lateral stiffness will not be examined here. It is assumed that for cases of less than six loaded or stiffened teeth, the frequency changes will be smaller than those found in Table 5.5, Figure 5.11 and Table 5.6 above.

2.5 Effect of Tooth Geometry and Blade Thickness

Given that the modification of the teeth conducted throughout the cutting tests did not drastically alter the basic dimensions (L x W) of the blade, it is expected that tooth modifications will not drastically affect the low frequencies and mode shapes of the blade. The same cannot, however, be said about the high frequency modes involving significant tooth motion. Increasing hook length or depth of gullet will result in significantly different localized stiffness characteristics. In order to investigate this effect, additional blade models were created for the 18G0H0 and 18G1H1 tooth configurations. A comparison of the first 5 natural frequencies and the 11 considered in Tables 5.2 and 5.3 are shown below in Figure 5.12 and Table 5.7.
As can be seen from Figure 5.12, the changes in natural frequency for the first five mode shapes with the change in tooth configuration from G0H0 to G2H2 are not very large (approximately 2.3% maximum change for mode 2,2f). In addition, it was observed that the mode shapes were identical for all three blade configurations.

<table>
<thead>
<tr>
<th>Mode Designation</th>
<th>18G0H0</th>
<th>18G1H1</th>
<th>18G2H2</th>
</tr>
</thead>
<tbody>
<tr>
<td>4,4fb</td>
<td>913.3</td>
<td>905.6</td>
<td>905.3</td>
</tr>
<tr>
<td>8t6b,1fb</td>
<td>944.8</td>
<td>949.0</td>
<td></td>
</tr>
<tr>
<td>8t6b,2f</td>
<td>979.1</td>
<td></td>
<td>954.2</td>
</tr>
<tr>
<td>6,3fb</td>
<td>1035.8</td>
<td>1037.8</td>
<td>1038.7</td>
</tr>
<tr>
<td>9t5m5b,3f</td>
<td>1200.2*</td>
<td>1070.8</td>
<td>1053.0</td>
</tr>
<tr>
<td>5,4fb</td>
<td>1084.4</td>
<td></td>
<td>1077.6</td>
</tr>
<tr>
<td>1,5fb</td>
<td>1103.7</td>
<td>1093.1*</td>
<td>1092.7</td>
</tr>
<tr>
<td>10t4m,2m,2b,4f</td>
<td>1325.6*</td>
<td>1174.6*</td>
<td>1138.8</td>
</tr>
<tr>
<td>2,5fb</td>
<td>1166.6</td>
<td></td>
<td>1153.1</td>
</tr>
<tr>
<td>7,2b</td>
<td>1156.5</td>
<td>1158.5</td>
<td>1163.7</td>
</tr>
<tr>
<td>11t7b,2f</td>
<td>1412.4</td>
<td>1232.2</td>
<td>1195.5</td>
</tr>
</tbody>
</table>

Table 5.7 – Effect of tooth modifications on high frequency modes ($R_0 = 14$ klbs)
Asterisks beside a natural frequency indicate a very similar but not identical mode shape to that listed in the first column. A number of interesting observations can be made about the behaviour of the natural frequencies listed in Table 5.7. First of all, it can be seen that not all mode shapes exist for each of the different tooth profiles. This would indicate that tooth profile plays a very important role in the high frequency mode shapes of the bandsaw blade. Secondly, it can be seen that in moving from tooth profile G0H0 to G2H2, the natural frequencies of a particular mode either increase or decrease depending upon whether the mode shape is dominated by tooth or back edge motion. It can be seen from modes (8t6b,2f), (9t5m5b,3f), (10t4m1,2m,2b,4f) and (11t7b,2f), that the natural frequencies decrease significantly with the tooth modification. For the modes containing both front and back edge motion (fb modes), the natural frequencies can either increase or decrease in moving from the G0H0 configuration to the G2H2 configuration but the magnitude of the change is quite small compared to that of the pure front edge motion modes. Lastly, it can be seen that the back edge dominated mode (7,2b) has an increasing frequency with the change in tooth profile. This agrees with the low frequency back edge modes (1,2b) and (2,2b) which also show an increasing frequency with tooth modification from G0H0 to G2H2.

The nature of the natural frequency change with tooth modification will become important when investigating the effect of tooth profile on the frequency difference. This will be discussed in the next section.

Blade thickness is another important parameter in determining blade natural frequencies. Increasing blade thickness both increases blade bending stiffness as well the mass per unit area of the blade. These two phenomena have opposite effects on the blade natural frequencies. Increasing blade lateral stiffness serves to increase natural frequencies while increasing blade mass density serves to decrease natural frequencies. For this reason, the impact of blade thickness on natural frequencies will be mode shape dependent. The variation in natural frequencies for the first five mode shapes with blade thickness is shown below in Figure 5.13.
Effect of Blade Thickness on Low Frequency Modes

As can be seen from Figure 5.13, the natural frequencies of the first four modes increase with decreasing blade thickness while for the fifth natural frequency, the opposite occurs. The influence of blade thickness on higher frequency modes is shown below in Figure 5.14.
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Effect of Blade Thickness on High Frequency Modes

Figure 5.14 - Effect of blade thickness on possible washboarding natural frequencies of G2H2 blades ($R_0 = 14$ klbs)

Figure 5.14 shows that the frequencies in the washboarding frequency range for the G2H2 blade configuration all exhibit decreasing natural frequencies with decreasing blade thickness. The rate at which natural frequencies decrease is mode dependent.

3 Investigation of Frequency Difference

The model presented in Chapter 5 included a harmonically varying axial tension that when combined with a lateral harmonic load at the tooth passing frequency, leads to response characteristics similar to those seen in washboarding experiments. One of the fundamental features of the model was the frequency with which the band tension fluctuated and how this is thought to correspond to the frequency difference observed in experiment. In order to get a better understanding of the true nature of the frequency difference, the experimental results were investigated to determine how $\Delta F$ changes with the fundamental operating parameters.
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3.1 Frequency Difference throughout Cut

Using data from a number of cutting tests performed with the 18 gauge blade, the frequency difference was measured at 0.125 s intervals throughout the cut. In order to obtain the frequency difference, the tooth passing frequency was obtained from the probe mounted above the cut and the excited frequency was obtained from the probe mounted closest to the toothed edge of the blade. For each cut, $\Delta F$ was measured from the start of cut through to the end. A table of the average frequency differences and their standard deviation for different cutting tests are shown in Table B.1 in Appendix B.

As can be seen from the table, the standard deviation in the frequency difference throughout the cut is less than one Hz for all cases listed except one. Given that the measurement of both the excited frequency and the tooth passing frequency is accurate to $\pm 0.5$ Hz, the difference between the two can be expected to be accurate to only $\pm 1.0$ Hz. Since the standard deviations are all within this $\pm 1.0$ Hz range, it would appear that the frequency difference remains constant throughout the cut.

This is an important observation since the analytical model proposed in Chapter 5 assumed a constant frequency difference related to some low frequency characteristic of the system. Over the course of a cut, the tooth passing frequency varies typically by up to 5 Hz which is roughly 0.5% of the tooth passing frequency. For the cutting tests considered, the frequency difference ranged from as low as 10 Hz up to 40 Hz. A 0.5% variation in these measured frequency differences correspond to 0.05 – 0.20 Hz. This would suggest that if the frequency difference is somehow linearly related to machine speed, the frequency difference should appear to be constant throughout each cut given the resolution of the frequency measurements. This was indeed the case.

Also included in Table B.1 is the secondary frequency difference, $\Delta F_s$. It was not possible to measure this secondary frequency difference for all cutting trials as the tertiary response is very weak for a number of these trials and not discernible from other noise. It is interesting to see that for all measurable cases, the secondary frequency difference is very close to $\Delta F$. In fact, for nearly all cases, the two differ by less than 0.5 Hz. This suggests that the two frequency differences are in fact the same within experimental error. This agrees nicely with the response components predicted in
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Chapter 4 where the frequency difference corresponds to the frequency of axial tension fluctuation.

3.2 Effect of Blade Speed on Frequency Difference

The analysis of Chapter 4 has shown that blade speed has an effect on the natural frequencies of an axially moving beam. Luo (2001 [2]) has also shown that blade speed affects the natural frequencies of an axially moving plate. However, blade speed does not affect every natural frequency in the same manner. Some natural frequencies increase with blade speed while others decrease with blade speed. With this in mind, Figures 5.15 and 5.16 below show the effect of blade speed on $\Delta F$.

![Effect of Blade Speed on Frequency Difference](image)

Figure 5.15 – Effect of speed on $\Delta F$ for 18G2H2, 18G1H1 and 18G1H0 Blades
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Effect of Blade Speed on Frequency Difference
(18G2H1 Blade, Bite = 24.4e-3 in, R₀ = 14 klbs, 7 Board Cant)

![Graph showing the effect of speed on frequency difference](image)

Figure 5.16 – Effect of speed on ΔF for 18G2H1 blade

It can be seen from Figures 5.15 and 5.16 that for the four blade configurations investigated, ΔF increases with blade speed. In fact, given the change in ΔF for the 18G1H0 blade of roughly 12 Hz going from 560 to 600 rpm, blade speed will be shown to have the most significant effect on ΔF for the range of parameters tested. It is very difficult to relate these changes in ΔF to the behaviour of natural frequencies given that as wheel speed changes in 20 rpm increments the tooth passing frequency changes by roughly 40 Hz. Given the spacing of the natural frequencies in the washboarding speed region, there is no evidence that the same natural frequency (if any) is being excited with each increment in wheel speed. This topic will be discussed further in Section 4.

3.3 Effect of Bite on Frequency Difference

From experiment, the frequency difference appears to increase with increasing bite. Therefore, for a given tooth passing frequency, Ω, the excited frequency, Ω - ω, decreases with increasing bite. As was seen in Section 2.3, finite element modeling has shown that the natural frequencies of a stationary blade decrease as the main axial cutting load (directly proportional to bite) is increased. The load applied in the finite element model was a conservative one (did not exhibit follower behavior) and frequencies were calculated for five values of cutting force ranging from no load up to

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200 N per tooth. As expected, only mode shapes showing significant tooth tip motion show marked decreases in natural frequency (up to 10 Hz). Figures 5.17 through 5.20 below show the dependence of $\Delta f$ on bite for the 18G2H2, 18G2H1, 18G1H1 and 18G1H0 blades.

Effect of Bite on Frequency Difference
(18G2H2 Blade, $n = 520$ rpm, $R_0 = 14$ klbs, 6 Board Cant)

Figure 5.17 – Effect of bite on $\Delta F$ for 18G2H2 blade

Effect of Bite on Frequency Difference
(18G2H1 Blade, $n = 540$ rpm, $R_0 = 14$ klbs, 7 Board Cant)

Figure 5.18 – Effect of bite on $\Delta F$ for 18G2H1 blade

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Effect of Bite on Frequency Difference
(I8G1H1 Blade, n = 580 rpm, R₀ = 14 klbs, 6 Board Cant)

Figure 5.19 – Effect of bite on ΔF for 18G1H1 blade

Effect of Bite on Frequency Difference
(I8G1H0 Blade, R₀ = 14 klbs, 6 Board Cant)

Figure 5.20 – Effect of bite on ΔF for 18G1H0 blade
For the 18G2H2 and 18G2H1 blades, the trend of increasing ΔF with bite is very noticeable. For the 18G1H1 blade, the trend is the same but the data is significantly more scattered. Finally, for the 18G1H0 blade, there were only four suitable cutting tests to compare the effect of bite on frequency difference. The trend however is still apparent and shows that for all cases investigated for the 18 gauge blade, increasing the bite leads to an increased frequency difference. It should be noted that over the range of bites from 0.020" to 0.035", the frequency difference increased by more than the ±1.0 Hz error associated with the measurement of ΔF.

Comparing the experimental frequency difference changes of approximately 3 or 4 Hz for the range of bites examined, it can be seen that it is of the same order as the change in natural frequency due to increased bite as calculated in Section 2.3. In fact, the change in natural frequency was calculated by comparing the natural frequencies for the unloaded blade, to those with a tooth tip load of 200 N. The experimental results show the effect of bite over a smaller range of bite. Therefore, the fact that increasing bite leads to an increased frequency difference could possibly be attributed to the natural frequency decreasing with bite.

3.4 Effect of Strain on Frequency Difference

Finite element results have shown that increasing blade strain also increases blade natural frequencies. Some natural frequencies show a greater dependence upon strain than others. Figures 5.21 and 5.22 below show the experimental dependence of ΔF on strain for the different configurations of the 18 gauge blade.
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Effect of Strain on Frequency Difference
(18G2H2 Blade, n = 500 rpm, Bite = 24.7e-3 in, 6 Board Cant)

Figure 5.21 – Effect of strain on ΔF for 18G2H2 blade

Effect of Strain on Frequency Difference
(18G2H1 Blade, n = 540 rpm, Bite = 24.4e-3 in, 7 Board Cant)

Figure 5.22 – Effect of strain on ΔF for 18G2H1 blade
Figure 5.21 and 5.22 clearly show that $\Delta F$ decreases with increasing strain for the 18G2H2 and 18G2H1 blades. Washboarding data for all three strain levels for the 18G1H1 and 18G1H0 blades was not available. If the excited frequency is indeed at or near a natural frequency, these results agree with the results found from the investigation of the dependence of blade natural frequencies on strain. However, observing that the change in $\Delta F$ in moving from a strain level of 10 klbs to 18 klbs is on the order of three hertz, only a certain number of mode shapes exhibit the same character. For others, the change in natural frequency with the change in strain is much larger than 3 Hz.

3.5 Effect of Depth of Cut on Frequency Difference

In examining the effect of depth of cut on $\Delta F$, it was found that $\Delta F$ decreased markedly with increased depth of cut. Once again, for a given blade speed and strain level, this would indicate that the excited frequency increases with depth of cut. The finite element model used in Section 2.4 to investigate the effect of depth of cut on the natural frequencies of the high frequency modes showed that provided increasing depth of cut increases the lateral stiffness of the system, the natural frequencies of the blade increase with depth of cut. This observation agrees with the concept of the excited frequency being at or near a natural frequency. Figure 5.23 below shows the dependence of $\Delta F$ on the number of boards in the cant being cut (depth of cut).
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3.6 Effect of Blade Thickness on Frequency Difference

It is difficult to get an accurate comparison of the frequency difference for the three blade thicknesses tested since the washboarding speed region for each of the three blades in a given tooth configuration are significantly different. The primary washboarding speed region for the 18 gauge blade was between 480 and 640 rpm. The primary washboarding speed region for the 17 gauge blade was between 640 and 780 rpm. Finally, the primary washboarding speed region for the 16 gauge blade was between 640 and 780 rpm. It must be pointed out that where these speed regions overlap, the tooth configurations do not coincide between the two blades. Given that no single speed is available at which to compare the frequency difference for the three different blade thicknesses, it will not be done here.

3.7 Effect of Tooth Profile on Frequency Difference

As was seen in Section 2.5, changing the tooth profile had a significant effect on the natural frequencies of modes dominated by large tooth edge motion. Using the PSD
plots for the 18 gauge blades, a comparison of the frequency difference for similar cutting conditions was conducted. It was found that the frequency difference for the 18G0H0 blade was less than that for the 18G1H1 blade which was less than that for the 18G2H2 blade for all comparable cutting tests. The behaviour of ΔF with the successive tooth modifications for the 18 Gauge blade can be seen in Figure 5.15 which shows the behaviour of the frequency difference with wheel speed for blades 18G2H2, 18G1H1 and 18G1H0. At a given speed, it would appear that the frequency difference is largest for the 18G2H2 blade and smallest for the 18G1H0 blade.

In order to understand this result, consider the natural frequencies and mode shapes for the 18G0H0, 18G1H1 and 18G2H2 blades as compared in Table 5.7. As outlined in Section 2.5, the change in natural frequency with tooth modification depended greatly upon the nature of the mode shape with those mode shapes dominated by front edge motion behaving differently than those dominated by back edge motion. When the frequency change was compared for all those modes exhibiting large relative tooth edge amplitude, the frequency for the 18G0H0 blade was always higher than for the 18G1H1 blade which in turn was always larger than that for the 18G2H2 blade. Conversely, for those modes with dominant back edge amplitude, the frequency for the 18G2H2 blade was always larger than that of the 18G1H1 blade which in turn was always larger than for the 18G0H0 blade. Lastly, for those modes exhibiting both large tooth and back edge amplitudes, the frequencies of the all three blades were very similar.

This lends strong evidence to the case of washboarding being caused by modes with large relative tooth motion since ΔF increased in moving from the G0H0 configuration through the G1H1 configuration to the G2H2 configuration while the natural frequencies decreased. It also lends strong evidence to the case of the excited frequency being at or very near a natural frequency of the system.

The argument that washboarding is caused only by modes with large relative tooth motion can also be justified from a work standpoint. In the model considered in Chapter 4, the only significant forces in the lateral direction are the cutting forces which are applied at the tooth tips. Work can only be done through the movement of these tooth tips while under load.
4 Possible Explanations for Observed Behaviour

So far, in this Chapter, the effect of changing basic operating parameters such as speed, bite, depth of cut and strain on washboarding behaviour, natural frequencies, mode shapes and the frequency difference has been investigated. The results from the analytical model of Chapter 4 showed that the response components occur at certain frequencies corresponding to $\Omega$ and $\Omega - \omega$. These frequencies do not necessarily correspond with natural frequencies of the system but the magnitudes of the responses depend on the proximity of either of these frequencies to a natural frequency of the system. In order to reconcile all the observations made in this Chapter, two mechanisms will be proposed and described.

4.1 $\Delta F$ Associated with $nf_b$ or $mf_w$

One of the proposed axial tension fluctuations was associated with either an integer multiple of the band passing frequency or an integer multiple of the wheel rotation frequency. From low frequency washboarding frequency plots, it was seen that lateral vibration at $f_b$ and its harmonics is present. It will be assumed here that these lateral vibrations result in fluctuations of the blade in plane stress field at $f_b$ and its harmonics. This is a non-linear effect that to this point has not been included in the mathematical models. It is believed that due to the presence of the guides, lateral vibration at $f_w$ and its harmonics is not measured but tension variations at these frequencies are still present.

The next observation is that high frequency modes with significant tooth motion have been shown to be on average roughly 20 Hz apart. Therefore, for tooth passing frequencies in the midst of these tooth modes, the likelihood of a natural frequency being within 10 to 30 Hz below the tooth passing frequency is likely. Given the multitude of frequencies at which the in plane stress field is fluctuating, there will be one single frequency which when subtracted from the tooth passing frequency is nearest to the natural frequency below the tooth passing frequency. For example, if the tooth passing frequency is 1000 Hz, the natural frequency closest to and below the tooth passing frequency is 980 Hz and $f_b$ is 4 Hz, the frequency difference corresponds to $5f_b$. This would suggest that the component of the fluctuating stress field at 20 Hz is responsible for the frequency difference.
Next consider what occurs with the manipulation of the basic operating parameters of the saw. As strain is increased, the natural frequency might also increase from 980 Hz to 985 Hz. The difference between the tooth passing frequency and the natural frequency is now 15 Hz so that the excited frequency will be 984 Hz due to a frequency difference of $4f_b$ (16 Hz). This agrees with the observation that frequency difference decreases with strain and that natural frequencies increase with strain.

As bite is increased, the natural frequency might decrease to 977 Hz. This would lead to the frequency difference corresponding to $6f_b$ (24 Hz) so that the excited response occurs at 986 Hz. This scenario also agrees with the experimental results and the finite element frequency results.

As depth of cut is increased, finite element analysis predicts that the natural frequencies will increase also. Let the new natural frequency be 983 Hz for an increased depth of cut so that once again, $4f_b$ is the multiple of $f_b$ that, when subtracted from the tooth passing frequency, comes closest to the natural frequency.

Lastly, consider the effect of speed. It has been shown experimentally that increasing speed leads to an increase in frequency difference. For the cutting tests conducted, wheel speed was set at 20 rpm increments. This corresponds to a change in tooth passing frequency of roughly 36 Hz. Consider a case where once again, the tooth passing frequency is 1000 Hz with a natural frequency, $f_1$, at 980 Hz and $\Delta F$ corresponds to $4f_b$. Now increasing the wheel speed by 20 rpm sets the new tooth passing frequency at 1036 Hz. The natural frequency $f_1$ will likely have changed slightly to $f_1'$. Let a second natural frequency, $f_2$ be at 1030 Hz. This second natural frequency is 4 Hz less than the tooth passing frequency giving $\Delta F = f_b$. This second frequency difference is much less than the first which disagrees with the observation that $\Delta F$ always increases with speed.

It can be seen that the only problem with the idea that the frequency difference is an integer multiple of $f_b$ is that the frequency seems to increase monotonically with wheel speed but there is no guarantee that as speed increases, the excited natural frequencies are progressively further from the tooth passing frequency. It turns out that considering the experimental evidence will help shed light on this proposed model.
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If the frequency difference does indeed correspond to either of these two frequencies or their harmonics, then the measured frequency difference should be an integer multiple of either \( f_b \) or \( f_w \). Recall, however, that the potential error in the measured frequency difference is ± 1 Hz. For this reason, if the integer relation does indeed hold, then the following must be true:

\[
\frac{\Delta F - 1}{f_b} < n < \frac{\Delta F + 1}{f_b} \quad \text{or} \quad \frac{\Delta F - 1}{f_w} < m < \frac{\Delta F + 1}{f_w}
\]

where \( n \) and \( m \) are integers. In investigating the frequency difference behaviour for the 18 gauge blade, 58 cutting tests were considered. Of these 58 tests, the first of Eqs. 5.1 was satisfied for only 26 of the cases and the second of Eqs. 5.1 was satisfied for only 15 of the cases. These results are shown in Tables B.2 and B.3 in Appendix B. This also is very strong evidence against the frequency difference being a multiple of either \( f_b \) or \( f_w \).

4.2 \( \Delta F \) Associated with Low Natural Frequency

Given that there is evidence against the frequency difference corresponding to integer multiples of \( f_b \) or \( f_w \), a suitable candidate for the low frequency axial tension fluctuation remains to be found. One possibility suggested in Chapter 4 was that the low frequency axial fluctuation corresponds to some low frequency machine structure vibration. This possibility can be discounted from the observation that frequency difference is dependent upon blade thickness, bite, depth of cut and tooth profile which do not have significant impacts on the machine structural frequencies.

A second possibility is that \( \Delta F \) corresponds to some low frequency blade mode that has yet to be identified. Lateral vibration of the blade leads, through non-linear effects, to a variation in the blade tension corresponding to twice the frequency of lateral vibration. The finite element model investigated earlier in this Chapter was setup mainly for investigation of high frequency modes and as a result, the sections of blade between the guides and wheels (outside the main cutting span) were not considered. In addition, the stiffness and damping characteristics of the hydrodynamically lubricated guides were also not considered. Obviously, bite, depth of cut, strain, blade speed and blade
thickness will all affect such a low frequency blade mode. The task remains to find such a mode and determine its dependence upon the aforementioned parameters.

Finite element analysis has shown that the low blade natural frequencies exhibit certain trends with the four main operating parameters, the tooth profile and the blade thickness. To start, the effect of speed on the low frequency blade modes was not investigated using the finite element model but it has been shown by Luo (2001 [2]) as well as Lengoc and McCallion (1995 [22]) that the first four natural frequencies of an axially travelling smooth plate all decrease with blade speed. This disagrees with the observation that the frequency difference increases with blade speed.

In addition to blade speed, it was shown that increasing bite (increasing main tangential cutting force) causes all natural frequencies to decrease. This does not agree with the observation that the frequency difference increases with bite. It should be noted that the applied tooth loads were conservative. If the loads were modeled as follower forces, some of the natural frequencies would likely increase while some would likely decrease as found by Timoshenko (1961 [21]) for a cantilever beam column with follower end load. Lengoc and McCallion (1995 [23]) also found that for a smooth traveling plate with follower edge load, the first natural frequency decreased with increasing edge load while the second increased.

Blade strain was also shown to have a significant impact on the frequency difference and low frequency modes. It was found that the frequency difference decreases with increasing strain in contrast with the low blade frequencies which all increase with increasing strain. Depth of cut was shown to cause an increase in the low blade frequencies while showing the opposite effect on the frequency difference.

Blade thickness was also shown to have an effect on the frequency difference as well as the low blade frequencies. The frequency difference was shown to increase with blade thickness while four of the first five calculated low frequency blade modes were shown to exhibit an increasing frequency with blade thickness. These two observations would support the proposition of the frequency difference being caused by a low frequency blade mode. Lastly, incremental tooth modifications were shown to decrease the
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frequency difference while no definitive trend was observed with respect to the low blade frequencies. This observation neither supports nor discredits the proposition at hand.

Given the evidence listed, it appears unlikely that the frequency difference is caused by vibration of the blade in a single low frequency mode. This conclusion will be supported by the low frequency behaviour to be discussed in Section 5.

4.3 Energy Transfer Between Widely Spaced Modes

Malatkar and Nayfeh (2003 [24]) discuss the nonlinear phenomenon of energy transfer from a directly excited high frequency mode to a lower frequency mode. The main features of the response consist of a response at the excitation frequency which could be compared to the response seen in Type I washboarding at $F_{tp}$ as well as a response at a the first natural frequency. Surrounding this excitation are a number of asymmetric sidebands with spacing equal to a Hopf bifurcation frequency that is very close to the first natural frequency of the system. Again, these asymmetric sidebands could correspond to the responses seen at $F_{ex}$ and $F_3$ which are definitely asymmetric considering that the magnitude of the response at $F_3$ is always small compared to that at $F_{ex}$. Given the presence of the first natural frequency response, the investigation of the low frequency behaviour during the washboarding trials becomes important. This will be discussed in Section 5.

4.4 General Thoughts on $\Delta F$

The analysis of Chapter 4 showed that the responses at $\Omega \pm \omega$ and $\Omega$ occur as the result of a regular resonance type phenomenon. The magnitude of the response depends largely on a frequency determinant in the coefficient multiplying the harmonic component of the response. This frequency determinant is very similar to the situation arising in a damped linear single degree of freedom system with a harmonic excitation. The general response of such a system is given by:
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\[
x(t) = \left( \frac{F}{k} \right) \frac{\cos(\omega t)}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2 \frac{c}{c_c} \frac{\omega}{\omega_n}\right)^2}}\ 
\]

where \( x(t) \) is the system response, \( F \) is the magnitude of the harmonic excitation, \( k \) is the system stiffness, \( c \) is the system damping, \( \omega \) is the forcing frequency, \( \omega_n \) is the system natural frequency and \( c_c \) is the critical level of damping. Plots of the magnitude of the response are shown below in Figure 5.24 for different levels of damping.

One DOF Forced Response With Damping

![Diagram of One DOF Forced Response With Damping](image)

**Figure 5.24** – Response of damped SDOF system for various levels of damping

The quantity \( x_0/x_{stat} \) gives the system response magnitude divided by the static deflection, \( F/k \). It can be seen that the maximum value of the response occurs when the forcing frequency is near the system natural frequency. This is exactly the case found in Chapter 4 where the largest responses are predicted for the cases when \( \Omega \pm \omega \) and \( \Omega \) are near natural frequencies. In addition, it can be seen that the magnitude of the response is significant compared to the static deflection for a large range of excitation frequencies. For example, for all of the damping levels shown in Figure 5.25, the system response is twice the static deflection for a range of \( \omega/\omega_n \) from about 0.8 to 1.2. For the
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washboarding problem where the natural frequencies likely being excited are on the order of 1000 Hz, this corresponds to significant resonant responses for excitation frequencies between 800 and 1200 Hz. This is a very large frequency range and suggests that the responses seen in washboarding trials at \( \Omega \pm \omega \) and \( \Omega \) may arise due to resonant excitation of a single or handful of natural frequencies. This is in contrast to the situation described in Section 4.1. The goal remains however, to find out if an axial tension fluctuation corresponding to the frequency difference exists and if it does, what causes it.

5 Low Frequency Blade Characteristics

In order to help answer the question of what causes the low frequency axial tension fluctuation, the low frequency behaviour of the band for the cutting tests already considered was investigated. As was seen in Section 4 of Chapter 2, the low frequency response of the band between the guides during cutting shows responses at all multiples of \( f_b \). For the cases where the first of Eqs. 5.1 is satisfied, there will be a low frequency response equal to an integer multiple of \( f_b \) that may also correspond to \( \Delta F \). This fact, however, is not very useful as the integer multiple relationship between \( \Delta F \) and \( n f_b \) or \( m f_w \) has been discounted. Instead, it will be more useful to examine the low frequency behaviour of those cuts where the first of Eqs. 5.1 is not satisfied. Any response, if present, corresponding to \( \Delta F \) for these cuts will not be associated with an integer multiple of \( f_b \) and must come from some other source.

A number of the washboarding power spectrum density plots were investigated for the 18 gauge blade, where \( \Delta F \) did not correspond to \( n f_b \), in the low frequency region from 0 to 50 Hz in an attempt to identify a low frequency blade response that could be related to \( \Delta F \). For all cases, the band passing frequency and its multiples were easily visible as well as a large steady state 0 Hz response as seen in Figure 2.7.

When these low frequency characteristics were compared with the frequency difference measured from the high frequency blade response, there was no definitive evidence relating a low frequency excitation to the frequency difference. In a small number of tests, the frequency difference corresponded to a peak in the low frequency response and while in most other tests, it did not. The results of this investigation are given in
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Table B.4 in Appendix B. Given the frequency resolution with which the measurements were taken, it is difficult to determine a definitive pattern. This would suggest that the mechanism discussed in Section 4.3 is not at work in the case of Type I washboarding. In addition, the first blade frequency of the experimental bandmill used in the cutting trials was much larger than any of the frequency differences measured during experiment. Therefore, nonlinear energy transfer between modes does not appear to be a contributor to Type I washboarding.

An important note regarding the low frequency response characteristics shown in Figure 2.7 is that they represent lateral vibrations of the blade. The frequency difference used in the model considered here corresponds to axial tension fluctuations. These fluctuations will not necessarily result in corresponding lateral displacement signals during testing. For this reason, direct measurement of the axial tension fluctuation during cutting would be useful.

Another important low frequency component identified in Chapter 4 was $\Delta F/2$. The low frequency behaviour of those plots where $\Delta F/2$ does not correspond to a multiple of $f_b$ was examined. The results are shown in Table B.5 in Appendix B and show that a response at $\Delta F/2$ rarely occurs. For this reason, it is unlikely that the frequency difference corresponds to the tension fluctuation arising from a low frequency lateral vibration of the band.
CHAPTER 6: CONCLUSIONS

Over the course of this work, a number of fundamental conclusions have been drawn about the possible nature of the Type I washboarding problem. These conclusions have come about through simple mathematical modeling of a number of mechanisms thought responsible for washboarding as well as through an intensive experimental cutting program. A number of important parameters that affect the washboarding behavior of bandsaw blades have been identified. The main points discussed in this work follow.

1. **Axial tension fluctuations at the tooth passing frequency do not, in isolation, appear to cause the response characteristics associated with Type I washboarding.**

Given the observations that Type I washboarding is strongly dependent upon bite and depth of cut and that a strong response occurs at the tooth passing frequency, a model was developed that considered the effect of axial cutting loads varying at the tooth passing frequency. The model consisted of a traveling beam with a central axial load superimposed on a much larger static tension. It was found that such a model does not predict responses at the tooth passing frequency as desired and therefore, the model was not considered further.

2. **Low frequency axial tension fluctuations coupled with lateral harmonic blade excitation at the tooth passing frequency predicts response characteristics similar to those seen experimentally.**

The traveling beam model was modified to incorporate the combination of a harmonically varying lateral load coupled with an axial tension fluctuation at some small frequency, $\omega$. The response predicted by this model had three main frequency components: $\Omega - \omega$, $\Omega$, and $\Omega + \omega$ that can be compared to the three responses seen experimentally at $F_{ex}$, $F_{tp}$, and $F_3$. The predicted behavior of these responses with changes in primary operating parameters such as blade speed, bite, depth of cut and axial strain matched well with that seen experimentally. The model does not predict however, that the
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magnitude of the response at $\Omega + \omega$ will always be smaller than those at the other two frequency components as is found experimentally.

3 **The frequency difference has been shown to vary with cutting parameters such as speed, strain, bite, depth of cut, tooth profile and blade thickness.**

An investigation of the frequency difference, $\Delta F$ was carried out to examine its dependence on parameters such as blade speed, bite, depth of cut and blade strain. The general trends were as follows:

- $\Delta F$ increases with increasing blade speed
- $\Delta F$ increases with increasing bite
- $\Delta F$ decreases with increasing depth of cut
- $\Delta F$ decreases with increasing strain
- $\Delta F$ increases with the level of tooth modification
- the dependence of $\Delta F$ on blade thickness is inconclusive as there was insufficient data to establish a trend

No satisfactory explanation has yet been found for the nature of the frequency difference and further experimental testing may be required.

4 **Stationary blade natural frequencies have been shown to vary with cutting parameters such as speed, strain, bite, depth of cut, tooth profile and blade thickness.**

Using a stationary blade finite element model of a toothed blade, the effect of strain, bite, depth of cut, blade thickness and tooth profile were investigated. The behavior of the natural frequencies with each of these parameters was different depending upon the the mode shape being considered. Specifically, different effects were noticed for modes involving significant tooth motion compared to those that don't involve significant tooth motion.
5. In disagreement with the mathematical model, the response at $F_3$ is not always present in experiment and is never of comparable magnitude to the responses at $F_{ex}$ and $F_{tp}$.

Although a response at $F_3$ is very commonly seen in the experimental washboarding plots, its magnitude is typically much smaller than those at $F_{ex}$ and $F_{tp}$. The analytical model of Chapter 4 predicts a large response at $\Omega + \omega$, which is the component corresponding to $F_3$, when $\Omega + \omega$ is near a gyroscopic natural frequency of the unperturbed system. It is suggested that by modeling the lateral cutting forces as moving lateral cutting forces, this situation may be rectified.

Using the models of Chapters 3 and 4, a fundamental qualitative understanding of the role of axial tension fluctuations as well as lateral excitations and their role in Type I washboarding has been acquired. Future improvements on the traveling beam models investigated might include the exploration of moving cutting forces as well as the effect of combining regular harmonic lateral forces with regenerative lateral forces such as those investigated by Luo (2001 [2]) and Tian (1998 [1]). The combination of these two types of loadings may more accurately reflect the true blade loading conditions during washboarding.

Experimentally, a number of items could use further clarification. These include a more thorough examination of the effect of blade speed on Type I washboarding by using a wheel speed increment much smaller than the 20 rpm used in the cutting trials. This would give a more accurate representation of the behavior of the responses at $F_{tp}$ and $F_{ex}$ as blade speed is increased.

In addition, an investigation of the axial strain fluctuation should be undertaken by instrumenting the main hydraulic straining cylinder and attempting to match the frequency difference seen on washboarding plots with some component of the axial strain fluctuation.

Another further area for study lies in attempting to measure the high frequency mode shapes of an idling blade as well as what mode shapes are excited during
Chapter 6: Conclusions

washboarding. This is a very difficult undertaking given the magnitude of the frequencies involved and the motion associated with the cutting teeth.

The effect of guide offset on the response at $F_p$ for both idling and cutting conditions and the corresponding effect on Type I washboarding behavior would also be useful.

Lastly, increased frequency resolution during cutting trials would be helpful in trying to associate the measured frequency difference with some low frequency response characteristic. Currently, the error of $\pm1.0$ Hz in the frequency difference measurement leads to difficulties in identifying equivalent low frequency responses.
REFERENCES


References


APPENDIX A: FINITE ELEMENT FORMULATION OF TRAVELING BEAM PROBLEM

In order to evaluate the element matrices for the finite element formulation of the traveling beam problem discussed in Chapter 3, Hermitian shape functions will be introduced. Each beam element used in the formulation had four degrees of freedom: one displacement and one slope at each end. The deflection of each beam element was assumed to be:

\[ y(x) = H_1(x)y_1 + H_2(x)\theta_1 + H_3(x)y_2 + H_4(x)\theta_2 \]  

Eq. A.1

where the \( H_i \) represent Hermitian shape functions given by:

\[
H_1(x) = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \quad \quad H_2(x) = x - \frac{2x^2}{l} + \frac{x^3}{l^2}
\]

Eqs. A.2

\[
H_3(x) = \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \quad \quad H_4(x) = \frac{x^2}{l} + \frac{x^3}{l^2}
\]

Eqs. A.3

As was done to the rest of the problem, the assumed displacement and Hermitian shape functions must be non-dimensionalized with the length scale \( l \). Using Eqs. 3.21, the non-dimensionalized displacement profile is given by:

\[ \bar{y}(\bar{x}) = \bar{H}_1(\bar{x})\bar{y}_1 + \bar{H}_2(\bar{x})\theta_1 + \bar{H}_3(\bar{x})\bar{y}_2 + \bar{H}_4(\bar{x})\theta_2 \]  

Eq. A.3

where the non-dimensionalized Hermitian shape functions are given by:

\[
H_1(x) = 1 - 3x^2N^2 + 2x^3N^3 \quad \quad H_2(x) = x - 2x^2N + x^3N^2
\]

Eqs. A.4

\[
H_3(x) = 3x^2N^2 - 2x^3N^3 \quad \quad H_4(x) = xN + x^3N^2
\]
Appendix A: Finite Element Formulation of Traveling Beam Problem

where the overbars have been removed for simplicity. In the above equations, \( N \) refers to the number of elements in the formulation and is given by:

\[
N = \frac{L}{l} \tag{Eq. A.5}
\]

Using Eqs. A.4 above, Eq. 3.36 may be evaluated to give the elemental bending stiffness matrix:

\[
[K]^e = \begin{bmatrix}
12N^3 & 6N^2 & -12N^3 & 6N^2 \\
4N & -6N^2 & 2N & \\
12N^3 & -6N^2 & 4N & \\
\end{bmatrix} \tag{Eq. A.6}
\]

The elemental stiffness due to speed and tension effects is given by:

\[
[K]^e = \begin{bmatrix}
\frac{6N}{5} & \frac{1}{10} & \frac{6N}{5} & \frac{1}{10} \\
\frac{2}{15N} & \frac{10}{6N} & \frac{30N}{6N} & \frac{2}{15N} \\
\end{bmatrix} \tag{Eq. A.7}
\]

The gyroscopic matrix defined in Eq. 3.39 is given by:

\[
[G]^e = \begin{bmatrix}
\frac{1}{2} & \frac{1}{10N} & \frac{1}{2} & \frac{1}{10N} \\
\frac{1}{10N} & 0 & \frac{1}{10N} & \frac{1}{60N^2} \\
\frac{2}{10N} & \frac{1}{10N} & \frac{2}{10N} & \frac{1}{10N} \\
\frac{1}{10N} & \frac{1}{60N^2} & \frac{1}{10N} & 0 \\
\end{bmatrix} \tag{Eq. A.8}
\]

The consistent mass matrix for the beam element is:
Appendix A: Finite Element Formulation of Traveling Beam Problem

\[
[M^*] = \frac{1}{420} \begin{bmatrix}
156 & 22 & 54 & -13 \\
N & N^2 & N & N^2 \\
4 & 13 & 3 \\
N^3 & N^2 & 156 & 22 \\
N & N^2 \\
SYM & 4 & N^3 \\
\end{bmatrix} \quad \text{Eq. A.9}
\]
APPENDIX B: FREQUENCY DIFFERENCE DATA

Table B.1 below shows the frequency difference, data accumulated for the 18 gauge blade. Both the frequency difference, $\Delta F$, and the secondary frequency difference, $\Delta F_s$, are listed together with the standard variation in their values as measured through the course of the cut. The notation NS stands for 'No Signal' where the tertiary response was not apparent on the washboarding plot.

| Blade   | Cutting Test | Average $\Delta F$ (Hz) | Standard Deviation (Hz) | Average $\Delta F_s$ (Hz) | Standard Deviation (Hz) | $|\Delta F - \Delta F_s|$ |
|---------|--------------|--------------------------|------------------------|---------------------------|------------------------|---------------------------|
| a480104100a.600 | 10.75          | 0.433                    | 11.00                  | 0.000                     | 0.25                   |
| a550110180a.600 | 11.67          | 0.596                    | NS                     | NS                        | NS                     |
| a501109100a.600 | 14.42          | 0.567                    | 14.55                  | 0.618                     | 0.13                   |
| a519111140a.600 | 15.67          | 0.745                    | NS                     | NS                        | NS                     |
| a521113180a.600 | 14.17          | 0.860                    | 14.65                  | 0.743                     | 0.48                   |
| a5401118180a.600| 16.90          | 0.661                    | 17.79                  | 0.247                     | 0.89                   |
| a541117140a.600 | 17.41          | 0.824                    | NS                     | NS                        | NS                     |
| a54122140a.600  | 17.65          | 0.969                    | NS                     | NS                        | NS                     |
| a561122180a.600 | 19.50          | 0.500                    | NS                     | NS                        | NS                     |
| b500110140a.600 | 13.08          | 0.615                    | 12.85                  | 0.502                     | 0.23                   |
| b52096140a.600  | 14.75          | 0.433                    | NS                     | NS                        | NS                     |
| b520109140a.600 | 15.92          | 0.675                    | NS                     | NS                        | NS                     |
| b520146140a.600 | 17.60          | 0.490                    | 18.50                  | 0.471                     | 0.90                   |
| b520162140a.600 | 17.76          | 0.807                    | 17.91                  | 0.706                     | 0.15                   |
| b540116140a.600 | 19.06          | 0.599                    | NS                     | NS                        | NS                     |
| c481103140a.600 | 10.15          | 0.963                    | NS                     | NS                        | NS                     |
| c500108140a.600 | 11.09          | 0.688                    | 11.05                  | 0.269                     | 0.04                   |
| c520112140a.600 | 13.15          | 0.502                    | 13.20                  | 0.367                     | 0.05                   |
| c540099140a.600 | 16.08          | 0.670                    | NS                     | NS                        | NS                     |
| c560102140a.600 | 18.50          | 0.469                    | NS                     | NS                        | NS                     |
| c560108140a.600 | 12.13          | 0.511                    | 12.13                  | 0.531                     | 0.00                   |
| c560122140a.600 | 16.88          | 0.705                    | 16.88                  | 0.484                     | 0.00                   |
| d520110140a.600 | 13.53          | 0.460                    | NS                     | NS                        | NS                     |
| d5401113140a.600| 17.09          | 0.424                    | 17.17                  | 0.589                     | 0.08                   |
| d500116180a.600 | 15.73          | 0.602                    | 15.80                  | 0.359                     | 0.07                   |
| d560121180a.600 | 19.32          | 0.692                    | 18.88                  | 0.599                     | 0.44                   |
| c520112100a.600 | 15.02          | 0.490                    | NS                     | NS                        | NS                     |
| c540116100a.600 | 18.42          | 0.534                    | NS                     | NS                        | NS                     |
| c540199140a.600 | 16.00          | 0.297                    | 15.75                  | 0.433                     | 0.25                   |
| c540123140a.600 | 17.86          | 0.431                    | 18.31                  | 0.609                     | 0.45                   |
| c540147140a.600 | 18.84          | 0.458                    | 18.53                  | 0.485                     | 0.31                   |
| c540171140a.600 | 20.24          | 0.366                    | 20.26                  | 0.593                     | 0.02                   |
| g520113140a.600 | 13.73          | 0.622                    | 13.77                  | 0.504                     | 0.04                   |
| g541116140a.600 | 17.33          | 0.653                    | 17.00                  | 0.650                     | 0.33                   |
| g558120140a.600 | 20.60          | 0.707                    | NS                     | NS                        | NS                     |
Table B.1 – Frequency difference data for 18 gauge blade

Table B.2 shows the relationship between the measured frequency difference and the band passing frequency, $f_b$. Given the possible measurement error in the frequency difference of ± 1.0 Hz, the frequency difference is compared to $f_b$ for both its lower and upper limits and then checked to see if an integer relation is possible.
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Appendix B: Frequency Difference Data

Table B.2 – Check if $\Delta F$ is a potential integer multiple of $f_b$ ($Y = yes, N = no$)

It can be seen that 32 of the 58 cutting tests do not confirm the hypothesis that the frequency difference is an integer multiple of $f_b$. This is strong evidence that the frequency difference is not related to the band passing frequency.

Table B.3 below shows the relationship between the wheel rotation frequency, $f_w$ and the frequency difference, once again, checking the feasibility of an integer relationship.

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Table B.3 - Check if $\Delta F$ is a potential integer multiple of $f_w$ ($Y = \text{yes}, N = \text{no}$)

Table B.3 shows that only 15 of the 58 cutting tests support the idea of the frequency difference corresponding to an integer multiple of $f_w$. This too is strong evidence that the frequency difference is not an integer multiple of $f_w$.

Table B.4 below indicates whether a low frequency response corresponding to the frequency difference was present for those cases where the frequency difference did not correspond to an integer multiple of $f_b$. 

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Table B.4 - Indication of low frequency response corresponding to ΔF (y = yes, n = no)

For the most part, there is no low frequency response corresponding to ΔF and for the entries marked “y”, the signals were sporadic and not entirely conclusive.
### Appendix B: Frequency Difference Data

#### Table B.5 - Indication of low frequency response corresponding to ∆F/2 (y = yes, n = no)

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Table B.5 also shows that a low frequency response corresponding to one half the frequency difference is not often discernible. Again, for the cases marked "y", the signals were sporadic and may have been due to noise or other sources.