BIFURCATION OF A SQUARE PLATE
TWISTED BY CORNER FORCES

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B.A.Sc. (Mechanical Engineering), University of Waterloo, 1992

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF APPLIED SCIENCE

in
THE FACULTY OF GRADUATE STUDIES
DEPARTMENT OF MECHANICAL ENGINEERING

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA
April 1994
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Abstract

A square plate twisted by corner forces is described by classical linear theory as a saddle surface. In an experiment, as the plate deforms to any noticeable deflection, it appears not as a saddle surface, but as a cylindrical surface. The transformation in mode shapes presents problems in determining material behaviour by shear in a plate twisting experiment. The two mode shapes can be described by either displacement or curvature of the surface. The purpose of this work is to investigate the buckling of a square plate twisted by corner forces by determining the bifurcation point and comparing the present FEA work with the experimental results of Howell and other results found in literature. The problem is examined using nonlinear finite element buckling analysis. The bifurcation point is determined by load-displacement plots. The critical value of Gaussian curvature at the centre of the plate is determined by the Southwell plot method. The critical value of Gaussian curvature is found to occur before the bifurcation point. Gaussian curvature is found to vary by an order of magnitude over the plate at bifurcation.
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\( a \)  
plate length

\( C \)  
coefficient for the critical value of twist

\( C_a, C_b, C_c, C_d \)  
plate corners

\( D \)  
plate flexural rigidity

\( E \)  
Young's modulus of elasticity

\( h \)  
plate thickness

\( P \)  
corner force

\( \bar{P} \)  
nondimensionalised corner force

\( P_{\sigma} \)  
corner force at bifurcation

\( P_{K\sigma} \)  
corner force at critical value of Gaussian curvature (from Gaussian-mean curvature plot)

\( P_{Sc\sigma} \)  
corner force at critical value of Gaussian curvature (from Southwell plot)

\( u, v, w \)  
displacement

\( \delta_c \)  
deflection at corner \( C_c \) of plate

\( \delta_o \)  
deflection at centre of plate
\bar{\delta} \quad \text{nondimensionalised deflection}

\epsilon_x, \epsilon_y, \gamma_{xy} \quad \text{strain}

\kappa_x, \kappa_y \quad \text{curvature of the surface}

\kappa, \kappa_{xy} \quad \text{twist of the surface}

K \quad \text{Gaussian curvature}

K_{cr} \quad \text{critical value of Gaussian curvature}

\mu \quad \text{mean curvature}

\nu \quad \text{Poisson's ratio}
Acknowledgement

The author would like to thank Professor Hilton Ramsey for his guidance during the researching and writing of this thesis.

The author would also like to thank the Natural Sciences and Engineering Research Council of Canada for their financial support.
Chapter 1

Introduction

1.1 Purpose

The purpose of this work is to investigate the buckling of a linear elastic, isotropic square plate twisted by corner forces, by determining the bifurcation point. The problem is examined by a nonlinear finite element buckling analysis using the commercially available software package ANSYS Revision 5.0. A square plate twisted by corner forces is described by classical linear theory as a saddle surface. In an experiment, as the plate deforms to any noticeable deflection, it appears not as a saddle surface, but as a cylindrical surface. The transformation in mode shapes presents problems in determining material behaviour in shear by experiment. The findings of the present FEA work can be used in future study to develop a nonlinear relationship to account for this transformation and/or an upper bound to the application of a plate twist experiment. The two mode shapes can be described by either displacement or curvature of the surface. The critical value of corner force at bifurcation is determined from load-displacement plots. The critical value of Gaussian curvature at the centre of the plate is determined from the Southwell plot method. There is a discrepancy in the results found in literature using different methods of analysis and assumptions. The present FEA work is compared to the experimental results of Howell and other results in literature.
1.2 Literature Review

In 1890, Kelvin and Tait noted a transition in deformation surfaces of a square plate twisted by corner forces but did not attempt to find the point of instability.

In 1971, Lee and Hsu[2] investigated the buckling problem numerically, using finite difference methods and the nonlinear von Kármán equations for plates. The critical value of corner force at bifurcation was determined by displacement-load plots.

In 1975, Miyagawa, Hirata, and Shibuya[3] investigated the buckling problem experimentally and numerically, using deflection measurements in the experimental approach, and using a polynomial deformed configuration, von Kármán theory, and stress functions in the numerical approach. The critical value of corner force at bifurcation was determined by load-deflection plots.

In 1985, Ramsey[5] investigated the buckling problem analytically, using the kinematic results of Green and Naghdi for small deformations superposed on a large deformation of an elastic Cosserat surface, and the restricted form of the general nonlinear theory of shells and plates of Naghdi. The critical value of twist at bifurcation was determined from a Rayleigh quotient.

In 1991, Howell[1] investigated the buckling problem experimentally, using strain measurements and Kirckhoff theory to determine curvatures. The critical value of Gaussian curvature at bifurcation was determined by the Southwell plot.
2.1 Physics of the Problem

Classical linear theory of flat plates describes deflection $w$ of a square plate twisted by corner forces $P$ (figure 2.1):

$$w = \frac{P}{2(1-\nu)D} xy$$

in terms of the surface coordinates of the plate $x, y$. Flexural rigidity of the plate $D$:

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

is a function of Young’s modulus of elasticity $E$, Poisson’s ratio $\nu$, and plate thickness $h$.

Deflection can also be expressed in terms of twist $\kappa$ of the surface:

$$w = \kappa xy$$
These results (equations 2.1, 2.3) are well known in fundamental classical linear plate theory. The plate appears as a saddle surface (figure 2.2).

However, in an experiment, as the plate deforms to any noticeable deflection, it appears not as a saddle surface, but as a cylindrical surface with generators parallel to a plate diagonal (figure 2.3).

The mode of the plate can be determined by the surface characteristics with either displacement or curvature attributes. The saddle surface has equal magnitude deflections in the four corners relative to a fixed centre. The cylindrical surface has equal magnitude deflections in two opposite corners and zero deflection in the other two corners relative to a fixed centre.

Curvature can be viewed on a Mohr’s circle for curvature (figure 2.4). The abscissa
Chapter 2. Theory

Figure 2.4: Mohr’s Circle for Curvature

represents curvature $\kappa_n$ and the ordinate represents twist $\kappa_{nt}$ of the surface. Principal curvatures are $\kappa_1$, $\kappa_2$. The saddle surface is anticlastic—the two principal curvatures have opposite signs (figure 2.4a). Principal directions are parallel to the plate diagonals. Principal curvatures are equal and opposite, resulting in zero mean curvature and negative Gaussian curvature. The cylindrical surface is synclastic—curvatures in all orientations have like signs (figure 2.4b). Principal directions are parallel to the plate diagonals. One principal curvature is zero and the other non-zero, resulting in a non-zero mean curvature and a zero Gaussian curvature.

Classical linear theory of flat plates neglects all quadratic terms in the Green-Lagrange strain:

$$\varepsilon_x = u_x + \frac{1}{2}(u_x^2 + v_x^2 + w_x^2) \quad (2.4)$$
$$\varepsilon_y = v_y + \frac{1}{2}(u_y^2 + v_y^2 + w_y^2) \quad (2.5)$$
$$\varepsilon_z = w_z + \frac{1}{2}(u_z^2 + v_z^2 + w_z^2) \quad (2.6)$$
$$\gamma_{xy} = v_x + u_y + (u_xu_y + v_xv_y + w_xw_y) \quad (2.7)$$
$$\gamma_{yz} = w_y + v_z + (u_yu_z + v_yv_z + w_yw_z) \quad (2.8)$$
$$\gamma_{zx} = u_z + w_x + (u_zu_x + v_zv_x + w_zw_x) \quad (2.9)$$

The approximation of neglecting the nonlinear terms fails to account for the deformation in the middle plane of the plate due to bending. Midsurface strains can only
be neglected if the deflections of the plate are small in comparison with its thickness in non-developable surfaces (non-zero Gaussian curvature, such as saddle shapes, spheres) or the deflections are of the order of its thickness in developable surfaces (zero Gaussian curvature, such as cylinders, cones)[9]. Because of this approximation, classical linear plate theory cannot predict buckling.

2.2 Finite Element Theory

The finite element used in the analysis is an 8 node isoparametric quadrilateral shell element. It is labeled SHELL93 in the ANSYS Revision 5.0 element library. There are 5 degrees of freedom per node: 3 translations and 2 rotations. This element includes features of Green-Lagrangian strains and Mindlin plate theory. Green-Lagrangian strains (equations 2.4—2.9) take into account mid-surface strains of the plate. Mindlin plate theory allows for transverse shear deformation. This means that a line that is straight and normal to the mid-surface before loading, is assumed to remain straight but not necessarily normal to the mid-surface after loading. Displacements \( u, v \) of a point in the plate a distance \( z \) from the mid-surface are:

\[
\begin{align*}
    u & = \bar{u} - z\alpha \\
v & = \bar{v} - z\beta
\end{align*}
\]

where \( \alpha, \beta \) are small angles of rotation of a line that was normal to the mid-surface before loading and \( \bar{u}, \bar{v} \) are the displacements at the plate mid-surface. Strains \( \varepsilon_x, \varepsilon_y \) and shear strains \( \gamma_{xy}, \gamma_{yz}, \gamma_{xz} \) are:

\[
\begin{align*}
    \varepsilon_x & = \bar{u}_x + \frac{1}{2}(\bar{u}_x^2 + \bar{v}_x^2 + w_x^2) - z\alpha_x \\
    \varepsilon_y & = \bar{v}_y + \frac{1}{2}(\bar{u}_y^2 + \bar{v}_y^2 + w_y^2) - z\beta_y \\
    \gamma_{xy} & = \frac{1}{2}(\bar{u}_x \bar{v}_y + \bar{v}_x \bar{u}_y + w_x w_y) - z(\beta_x + \alpha_y)
\end{align*}
\]
\begin{align*}
\gamma_{yz} &= w_y + \overline{v}_y + (\overline{u}_y \overline{u}_y + \overline{u}_y \overline{v}_y + w_y w_y) - \beta \\
\gamma_{zx} &= \overline{u}_z + w_z + (\overline{u}_z \overline{u}_z + \overline{u}_z \overline{v}_z + w_z w_z) - \alpha \\
\end{align*}

Strains $\varepsilon_z$, $\varepsilon_y$ and shear strain $\gamma_{zy}$ are assumed to vary linearly through the plate thickness. Transverse shear strains $\gamma_{yz}$, $\gamma_{zx}$ are assumed to be constant through the plate thickness.

In the stress-strain relationship:

$$\{\sigma\} = [D] \{\varepsilon\} \quad (2.17)$$

the stress vector $\{\sigma\}$, the strain vector $\{\varepsilon\}$, and the material property matrix for the element $[D]$ are defined as:

$$\{\sigma\} = \begin{bmatrix} \sigma_z & \sigma_y & \tau_{zy} & \tau_{yz} & \tau_{zx} \end{bmatrix}^T \quad (2.18)$$

$$\{\varepsilon\} = \begin{bmatrix} \varepsilon_z & \varepsilon_y & \gamma_{zy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix}^T \quad (2.19)$$

$$[D] = \frac{E}{1 - \nu^2} \begin{bmatrix}
1 & \nu & 0 & 0 & 0 \\
\nu & 1 & 0 & 0 & 0 \\
0 & 0 & \frac{1 - \nu}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{1 - \nu}{2f} & 0 \\
0 & 0 & 0 & 0 & \frac{1 - \nu}{2f} \\
\end{bmatrix} \quad (2.20)$$

where $f$ is the shear factor:

$$f = \begin{cases} 
1.2, & A/h^2 \leq 25 \\
1.0 + 0.2 \frac{A}{25h^2}, & A/h^2 > 25 
\end{cases} \quad (2.21)$$

where $A$ is the area of the element and $h$ is the plate thickness. The shear factor is designed to avoid shear locking. As the element becomes thin, the $A/h^2$ ratio becomes large. The shear factor $f$ is thus increased and the stiffness associated with the transverse shears is reduced. The correct method to avoid shear locking is through selective integration, but ANSYS does not accommodate this. The SHELL93 element uses a $2 \times 2$ reduced quadrature rule.
2.3 Plate Theory

Kirkhoff theory is used to calculate curvatures from strain output of the finite element software. This is to be consistent with Howell’s experimental analysis so results can be compared. Kirkhoff theory neglects transverse shear deformations. This means that a line that is straight and normal to the midsurface before loading, is assumed to remain straight and normal to the midsurface after loading.

Extensional strain \( \varepsilon_s \) at an arbitrary point a distance \( z \) from the plate midsurface is:

\[
\varepsilon_s = \varepsilon_m + z\kappa \quad (2.22)
\]

where the membrane strain \( \varepsilon_m \) appears along the plate midsurface, and the curvature \( \kappa \) is associated with bending strain.

Solving the above equation for the top and bottom of the plate and equating midsurface strains gives the curvatures \( \kappa_x, \kappa_y \) and the twist \( \kappa_{xy} \) of the midsurface:

\[
\begin{align*}
\kappa_x &= \frac{\varepsilon^t - \varepsilon^b}{h} \\
\kappa_y &= \frac{\varepsilon^t - \varepsilon^b}{h} \\
\kappa_{xy} &= \frac{\gamma_{xy}^t - \gamma_{xy}^b}{2h}
\end{align*}
\quad (2.23, 2.24, 2.25)
\]

where \( \varepsilon^t, \varepsilon^b \) are the top and bottom surface strains of the plate respectively, and \( h \) is the plate thickness.

Principal curvatures \( \kappa_1, \kappa_2 \) from Mohr’s circle of curvatures are:

\[
\kappa_1, \kappa_2 = \frac{\kappa_x + \kappa_y}{2} \pm \sqrt{\left(\frac{\kappa_x - \kappa_y}{2}\right)^2 + \kappa_{xy}^2} 
\quad (2.26)
\]

Mean curvature \( \mu \) is the average of the two principal curvatures:

\[
\mu = \frac{1}{2}(\kappa_x + \kappa_y) = \frac{1}{2}(\kappa_1 + \kappa_2) 
\quad (2.27)
\]
Chapter 2. Theory

Gaussian curvature $K$ is product of the two principal curvatures:

$$K = \kappa_x \kappa_y - \kappa_{xy}^2 = \kappa_1 \kappa_2$$  \hspace{1cm} (2.28)

2.4 Southwell Plot

The Southwell plot is a common method to determine the elastic buckling load of a structural system. In experiments, there exists some imperfection in the undeformed shape and/or applied loading. As the compressive load increases, the lowest critical load buckling mode dominates. A linear function can be expressed in terms of applied load and deflection by neglecting contributions from higher modes.

In 1932, Southwell considered a simply supported column with an initial imperfection subjected to a compressive load $P[6]$. He expressed a linear relationship:

$$\frac{\delta}{P} = \frac{1}{P_{cr}} \delta + \frac{1}{P_{cr}} a$$  \hspace{1cm} (2.29)

in terms of the incremental deflection $\delta$, the Euler load $P_{cr}$, and coefficient $a$. The Southwell plot of $\delta/P$ versus $\delta$ gives a straight line whose slope is equal to the inverse of the buckling load.

The Southwell plot method claims accuracy only as $P \rightarrow P_{cr}$. Spencer[7] states that constructing Southwell plots using Kármán's strut data with loads up to $0.91P_{cr}$, to $0.88P_{cr}$, and to $0.82P_{cr}$ ($P_{cr}$ being defined as the critical load which Southwell obtained by plotting Kármán's data to $0.98P_{cr}$) gives errors of 3, 5, and 25 percent respectively.

The critical load $P_{cr}$ is a theoretical concept and should be independent of initial deflection. Spencer[7] showed that in buckling of a uniaxially compressed simply supported plate, the Southwell plot begins to underestimate the critical load when:

$$w_o/h > 0.5$$  \hspace{1cm} (2.30)

where $w_o$ is the initial deflection at the plate centre and $h$ is the plate thickness.
Chapter 3

Analysis

The analysis was performed on a SUN SPARC workstation. The preprocessing and the solution utilized ANSYS Revision 5.0, and the postprocessing utilized FORTRAN77 and TECPLIT Revision 5.0.

3.1 Preprocessing

The plate is modelled with the SHELL93 8 node isoparametric shell element.

The plate material is modelled as T6061-T6 Aluminium (table 3.1) for comparison with the experimental results of Howell[1]. Material nonlinearity, such as plasticity, is not considered in the analysis.

The plate geometry is square with plate length to thickness ratios \(a/h\) (table 3.2) for comparison with the experimental results of Howell[1].

The plate is meshed with square elements \(N\) per side, where \(N\) is even to provide a node at the centre of the plate to take displacement and strain measurements—the same location as Howell's strain gauges[1]. There are a total of \(N^2\) elements and \((3N^2 + 4N + 1)\) nodes for the model.

<table>
<thead>
<tr>
<th>Table 3.1: Material Constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E) (69 \times 10^9) Pa</td>
</tr>
<tr>
<td>(v) (0.33)</td>
</tr>
</tbody>
</table>

10
Table 3.2: Plate Geometry

<table>
<thead>
<tr>
<th>$a/h$ ratio</th>
<th>$a$ (m)</th>
<th>$h$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>49.2</td>
<td>0.1524</td>
<td>0.003099</td>
</tr>
<tr>
<td>63.2</td>
<td>0.2032</td>
<td>0.003216</td>
</tr>
<tr>
<td>80.3</td>
<td>0.2540</td>
<td>0.003162</td>
</tr>
<tr>
<td>96.0</td>
<td>0.3048</td>
<td>0.003175</td>
</tr>
<tr>
<td>196.7</td>
<td>0.6096</td>
<td>0.003099</td>
</tr>
</tbody>
</table>

Figure 3.1: Constraints

3.2 Solution

The plate is constrained at corners $C_a$, $C_b$, $C_d$ (figure 3.1) to zero displacement in the $z$ direction, to simulate the self equilibrating corner forces associated with the applied force at corner $C_c$. These are the same constraints in the experiment by Howell[1]. To prevent rigid body motion, additional DOF constraints are specified. The plate is constrained at corner $C_a$ to zero displacement in the $x$ and $y$ directions to prevent translation, and constrained at corner $C_b$ to zero displacement in the $y$ direction to prevent rotation. These constraints satisfy the kinematic, but not the static boundary conditions of a plate with free edges.

When applying the Southwell plot method to find critical values, an initial hydrostatic
Chapter 3. Analysis

pressure is applied in the positive $z$ direction. A nominal value of hydrostatic loading is used which produces deflections small compared to the plate thickness.

The applied force $P$ at corner $C_r$ is in the positive $z$ direction.

These boundary conditions provide a stable post buckling response. The applied corner force $P$ can exceed the value at the bifurcation point $P_{cr}$ without the instability of ill conditioned matrices, such as a negative main diagonal in the stiffness matrix.

Body forces, such as gravity loads, are not included in the analysis.

3.3 Postprocessing

Displacements $\delta_o$, $\delta_c$ are calculated at the centre of the plate and at corner $C_c$. The critical value of corner force at bifurcation is determined from load-deflection plot.

Strains $\epsilon_x$, $\epsilon_y$, $\gamma_{xy}$ are calculated at the top and bottom surfaces at the node at the centre of the plate using nodal point averaging in ANSYS. These values are exported to a FORTRAN code which calculates curvatures using Kirchhoff plate theory. The critical value of Gaussian curvature $K_{cr}$ is determined from the Southwell plot. The Southwell plot uses $\mu$ as the abscissa and $\mu/K$ as the ordinate. The asymptotic behaviour of the curve determines $K_{cr}$ as $K/K_{cr} \rightarrow 1$. 
Chapter 4

Results

4.1 Deflection

The deflection for the plate with 3 pinned corners is zero at the pinned corners $C_a$, $C_b$, $C_d$ and a maximum at the corner with the applied force $C_c$ (figure 4.1).

The load-deflection curve of corner with the applied force $C_c$ is smooth and shows no indication of buckling (figure 4.2). The load-deflection curve of the centre of the plate has an abrupt change in the slope at the bifurcation point $P_{cr}$.

The finite element analysis deflection of the plate centre agrees well with linear theory (equation 2.1) for deflections less than a plate thickness (figure 4.2). The FEA deflection of the plate corner $C_c$ agrees well with linear theory for deflections less than 4 plate thicknesses.

The finite element results of the plate with 3 pinned corners can be rotated to show the characteristic surface. The plate can be rotated so the deflections of corners $C_a$ and $C_c$ are equal, and translated so the deflection of the centre of the plate is zero. The deflections become $\delta_c/2 - \delta_o$ at $C_a$ and $C_c$, and $-\delta_o$ at $C_b$ and $C_d$ (figure 4.3), where $\delta_o$ and $\delta_c$ are the deflections of the unrotated results for the centre of the plate and corner $C_c$ respectively.

The bifurcation point is where the magnitude of rotated corners $C_a$, $C_c$ and rotated corners $C_b$, $C_d$ significantly diverge. The critical value of corner load varies slightly with Howell's[1] $a/h$ ratios (figure 4.4 and table 4.1). The mesh density of 144 elements is
Chapter 4. Results

Deflection Contour Plot
3 pinned corners at $P_{cr}$
96 a/h ratio, 144 elements

Figure 4.1: Deflection Contour Plot for 3 pinned corners at $P_{cr}$

Load-Deflection Plot
3 pinned corners
96 a/h ratio, 144 elements

Figure 4.2: Load-Deflection Plot for 3 pinned corners
sufficient to provide displacement convergence (figure 4.5 and table 4.2).

Below the bifurcation point, the magnitudes of deflection for rotated corners \( C_a, C_c \) and rotated corners \( C_b, C_d \) are almost equal (figure 4.4). The plate is bending to a saddle surface (figure 4.6).

Above the bifurcation point, the magnitude of deflection for rotated corners \( C_a, C_c \) is decreasing, and the magnitude of deflection for rotated corners \( C_b, C_d \) is increasing (figure 4.4). The plate is bending to a cylindrical surface (figure 4.7).

Nondimensionalized corner force \( \bar{P} \) is defined as[3]:

\[
\bar{P} = \frac{Pa^2}{2Dh} \tag{4.1}
\]

Nondimensionalized deflection \( \bar{\delta} \) is defined as[3]:

\[
\bar{\delta} = \frac{\delta}{h} \tag{4.2}
\]
Figure 4.4: Load-Deflection Plot for 3 pinned corners (rotated) with varying $a/h$ ratios

Table 4.1: Critical Values from load-deflection plot

<table>
<thead>
<tr>
<th>$a/h$ ratio</th>
<th>$P_{cr}$ (N)</th>
<th>$P_{cr}$</th>
<th>$\delta_a$, $\delta_c$</th>
<th>$\delta_b$, $\delta_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>49.2</td>
<td>1380</td>
<td>26.9</td>
<td>4.17</td>
<td>4.48</td>
</tr>
<tr>
<td>63.2</td>
<td>870</td>
<td>26.0</td>
<td>4.32</td>
<td>4.52</td>
</tr>
<tr>
<td>80.3</td>
<td>510</td>
<td>25.5</td>
<td>4.41</td>
<td>4.53</td>
</tr>
<tr>
<td>96.0</td>
<td>355</td>
<td>25.1</td>
<td>4.45</td>
<td>4.55</td>
</tr>
<tr>
<td>196.7</td>
<td>80</td>
<td>24.9</td>
<td>4.54</td>
<td>4.58</td>
</tr>
</tbody>
</table>
Chapter 4. Results

Figure 4.5: Load-Deflection Plot for 3 pinned corners (rotated) with varying mesh density

Table 4.2: Critical Values from load-deflection plot

<table>
<thead>
<tr>
<th>96 a/h ratio</th>
<th>elements</th>
<th>$P_{cr}$ (N)</th>
<th>$P_{cr}$</th>
<th>$\delta_a$, $\delta_c$</th>
<th>$\delta_b$, $\delta_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>352</td>
<td>25.0</td>
<td>4.45</td>
<td>4.55</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>355</td>
<td>25.1</td>
<td>4.45</td>
<td>4.55</td>
<td></td>
</tr>
<tr>
<td>144</td>
<td>355</td>
<td>25.1</td>
<td>4.45</td>
<td>4.55</td>
<td></td>
</tr>
<tr>
<td>256</td>
<td>355</td>
<td>25.1</td>
<td>4.45</td>
<td>4.55</td>
<td></td>
</tr>
</tbody>
</table>
Chapter 4. Results

Figure 4.6: Deflection Contour Plot for 3 pinned corners (rotated) at $P_{cr}/2$

Figure 4.7: Deflection Contour Plot for 3 pinned corners (rotated) at $2P_{cr}$
4.2 Southwell Plot

Howell[1] determined the critical value of Gaussian curvature using the Southwell plot method. The Southwell plot method requires an initial curvature in the structure.

The FEA Southwell plot is constructed from strains at the centre of the plate with 3 pinned corners and initial hydrostatic pressure. The Southwell plot produces parallel lines for varying intensity of initial hydrostatic pressure (figure 4.8).

The initial hydrostatic pressure creates an initial deflection of the centre of the plate $\delta_0$. The critical value of Gaussian curvature $K_{\text{Gr}}$ and corner force is $P_{\text{Gr}}$ determined by the Southwell plot method are not affected by deflections $\delta_0$ less than one tenth of a plate thickness (table 4.3).

The coefficient $C$ is defined as[5]:

$$ C = \kappa \frac{a^2}{h} \quad (4.3) $$
Table 4.3: Critical Values from Southwell plot

<table>
<thead>
<tr>
<th>pressure (Pa)</th>
<th>$\bar{\delta}_0$</th>
<th>$P_{Scr}$ (N)</th>
<th>$P_{Scr}$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0000961</td>
<td>284</td>
<td>20.1</td>
<td>9.04</td>
</tr>
<tr>
<td>1</td>
<td>0.000961</td>
<td>284</td>
<td>20.1</td>
<td>9.04</td>
</tr>
<tr>
<td>10</td>
<td>0.00961</td>
<td>284</td>
<td>20.1</td>
<td>9.03</td>
</tr>
<tr>
<td>100</td>
<td>0.0961</td>
<td>274</td>
<td>19.4</td>
<td>9.01</td>
</tr>
<tr>
<td>500</td>
<td>0.478</td>
<td>226</td>
<td>16.0</td>
<td>8.65</td>
</tr>
</tbody>
</table>

for the critical value of twist $\kappa$ at the centre of the plate.

For load levels less than $P_{Scr}$, the Southwell plot method over or under predicts $K_{Scr}$ (figure 4.9) and $C$ (figure 4.10) depending on the magnitude of the initial deflection.

4.3 Gaussian Curvature and Mean Curvature

The Southwell plot determines the critical value of Gaussian curvature where the slope on the Gaussian-mean curvature plot (figure 4.11) is zero[7]. The Gaussian-mean curvature plot is constructed from strain calculations at the centre of the plate with 3 pinned corners and no initial hydrostatic pressure.

Gaussian curvature is zero for the undeformed plate (no initial curvature), increases in magnitude as the plate deforms to a saddle surface, reaches a maximum value, begins to decrease in magnitude, and after bifurcation decreases in magnitude as the plate deforms to a cylindrical surface (figure 4.12).

Mean curvature is zero for the undeformed plate (no initial curvature) remains zero as the plate deforms to a saddle surface, and after bifurcation increases in magnitude as the plate deforms to a cylindrical surface (figure 4.13).

The corner load at the critical value of Gaussian curvature $P_{Kcr}$ (tables 4.4–4.5) is less than the corner load at bifurcation $P_{cr}$ (figure 4.12).
Figure 4.9: Critical Value of Gaussian Curvature from Southwell plot

Figure 4.10: Coefficient from Southwell plot
Chapter 4. Results

Gaussian-Mean Curvature Plot

Gaussian curvature $K_a h^2$

at centre of plate

96 a/h ratio, 144 elements

Figure 4.11: Gaussian-Mean Curvature Plot

Load-Gaussian Curvature Plot

load $P a^2/2Dh$

96 a/h ratio, 144 elements

Figure 4.12: Load-Gaussian Curvature Plot
Chapter 4. Results

Load-Mean Curvature Plot

at centre of plate
96 a/h ratio, 144 elements

0.000 0.050 0.100 0.150 0.200 0.250 0.300 0.350
mean curvature $\mu$ [m$^2$]

0 5 10 15 20 25 30
load $Pa^22Dh$

Table 4.4: Critical Values from Gaussian-mean curvature plot

<table>
<thead>
<tr>
<th>$a/h$ ratio</th>
<th>$P_{Kcr}$ (N)</th>
<th>$P_{Kcr}$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>49.2</td>
<td>1112</td>
<td>21.7</td>
<td>9.01</td>
</tr>
<tr>
<td>63.2</td>
<td>696</td>
<td>20.8</td>
<td>9.01</td>
</tr>
<tr>
<td>80.3</td>
<td>406</td>
<td>20.3</td>
<td>9.02</td>
</tr>
<tr>
<td>96.0</td>
<td>284</td>
<td>20.1</td>
<td>9.03</td>
</tr>
<tr>
<td>196.7</td>
<td>63</td>
<td>19.7</td>
<td>9.06</td>
</tr>
</tbody>
</table>

Figure 4.13: Load-Mean Curvature Plot
Table 4.5: Critical Values from Gaussian-mean curvature plot

<table>
<thead>
<tr>
<th>Elements</th>
<th>( P_{K_{cr}}) (N)</th>
<th>( \bar{P}<em>{K</em>{cr}})</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>278</td>
<td>19.7</td>
<td>9.09</td>
</tr>
<tr>
<td>64</td>
<td>276</td>
<td>19.6</td>
<td>9.03</td>
</tr>
<tr>
<td>144</td>
<td>284</td>
<td>20.1</td>
<td>9.03</td>
</tr>
<tr>
<td>256</td>
<td>286</td>
<td>20.3</td>
<td>9.01</td>
</tr>
</tbody>
</table>

Gaussian curvature is a minimum absolute value at the centre of the plate, a maximum absolute value near the corners of the plate, and varies over the plate by an order of magnitude at \( P_{K_{cr}} \) (figure 4.14) and \( P_{cr} \) (figure 4.15).

Mean curvature is a zero at the centre of the plate and varies positive and negative values over the plate at \( P_{K_{cr}} \) (figure 4.16) and \( P_{cr} \) (figure 4.17).

4.4 Curvature

The load-curvature plot is constructed from strains at the centre of the plate with 3 pinned corners and no initial hydrostatic pressure. Curvatures \( \kappa_x, \kappa_y \) are zero for the undeformed plate (no initial curvature), remain zero as the plate deforms to a saddle surface, and after bifurcation increase in magnitude as the plate deforms to a cylindrical surface (figure 4.18).

Twist \( \kappa_{xy} \) is zero for the undeformed plate (no initial curvature), increases in magnitude as the plate deforms to a saddle surface, reaches a maximum value, begins to decrease in magnitude, and after bifurcation continues to increase in magnitude as the plate deforms to a cylindrical surface.

The FEA twist \( \kappa_{xy} \) agrees well with linear theory (equations 2.1–2.3) for deflections less than half the plate thickness, and agrees well with membrane stress theory ([4] and...
Chapter 4. Results

Gaussian Curvature Contour Plot

at $P_{cr}$
96 a/h ratio, 144 elements

Figure 4.14: Gaussian Curvature Contour Plot at $P_{cr}$

Gaussian Curvature Contour Plot

at $P_{cr}$
96 a/h ratio, 144 elements

Figure 4.15: Gaussian Curvature Contour Plot at $P_{cr}$
Chapter 4. Results

Mean Curvature Contour Plot
at $P_{K_{cr}}$
96 a/h ratio, 144 elements

Figure 4.16: Mean Curvature Contour Plot at $P_{K_{cr}}$

Mean Curvature Contour Plot
at $P_{cr}$
96 a/h ratio, 144 elements

Figure 4.17: Mean Curvature Contour Plot at $P_{cr}$
Figure 4.18: Load-Curvature Plot

equation 2.22) for deflections less than a plate thickness (figure 4.18).

Curvature $\kappa_c$ is a zero at the centre of the plate and varies positive and negative values over the plate at $P_{K\sigma}$ (figure 4.19) and $P_{\sigma}$ (figure 4.20).

Twist $\kappa_{xy}$ is a minimum absolute value at the centre of the plate, a maximum absolute value near the corners of the plate, and varies over the plate by an order of magnitude at $P_{K\sigma}$ (figure 4.21) and $P_{\sigma}$ (figure 4.22).

4.5 Midsurface Strain

The load-midsurface strain plot is constructed from strains at the centre of the plate with 3 pinned corners and no initial hydrostatic pressure. Midsurface strains $\varepsilon_x$, $\varepsilon_y$ are zero for the undeformed plate (no initial curvature), are equal and compressive as the plate deforms to a saddle surface, and after bifurcation decrease in magnitude as the plate
Chapter 4. Results

Figure 4.19: Curvature $\kappa_z$ Contour Plot at $P_{Kcr}$

Figure 4.20: Curvature $\kappa_z$ Contour Plot at $P_{cr}$
Chapter 4. Results

Twist $\kappa_{xy}$ Contour Plot
at $P_{cr}$
96 a/h ratio, 144 elements

Figure 4.21: Twist $\kappa_{xy}$ Contour Plot at $P_{cr}$

Twist $\kappa_{xy}$ Contour Plot
at $P_{cr}$
96 a/h ratio, 144 elements

Figure 4.22: Twist $\kappa_{xy}$ Contour Plot at $P_{cr}$
deforms to a cylindrical surface (figure 4.23).

Midsurface shear strain $\gamma_{xy}$ is zero for the undeformed plate (no initial curvature), remains zero as the plate deforms to a saddle surface, and after bifurcation increases in magnitude as the plate deforms to a cylindrical surface.

The midsurface of the plate is in maximum compression at the centre of the plate and maximum tension at the edge of the plate at $P_{K\sigma}$ (figure 4.24) and $P_{\sigma}$ (figure 4.25).

Midsurface shear strain $\gamma_{xy}$ is zero at the centre of the plate and varies positive and negative values over the plate at $P_{K\sigma}$ (figure 4.26) and $P_{\sigma}$ (figure 4.27).

4.6 Fixed Plate Centre

The FEA plate buckles without an initial perturbation because of the 3 pinned corner constraints—the same used by Howell[1]. As the plate deforms only corner $C_c$ is free to
Chapter 4. Results

Figure 4.24: Midsurface Strain $\varepsilon_x$ Contour Plot at $P_{Kcr}$

Figure 4.25: Midsurface Strain $\varepsilon_x$ Contour Plot at $P_{cr}$
Chapter 4. Results

Figure 4.26: Midsurface Strain $\gamma_{xy}$ Contour Plot at $P_{Kc}$

Figure 4.27: Midsurface Strain $\gamma_{xy}$ Contour Plot at $P_{cr}$
Figure 4.28: Load-Deflection Plot for fixed plate centre

deflect. The corner forces remain parallel to the z axis and are no longer normal to the tangent plane of the centre of the plate.

Constraints which do not initiate buckling are created by fixing the centre of the plate in 5 degrees of freedom: displacements $u, v, w$ and rotations about the $x$ and $y$ axis. The drilling rotation about the $z$ axis is fixed by a constraint on $C_a$ in the direction of the $y = -x$ diagonal. The applied corners loads are $P$ at $C_a$ and $C_c$ and $-P$ at $C_b$ and $C_d$.

The orientation of the corner forces in the fixed plate centre loading case remain normal to the tangent plane of the centre of the plate. There is no perturbation, and the plate is loaded beyond the bifurcation point without experiencing buckling (figure 4.28).

The critical value of Gaussian curvature remains the same for the 3 pinned corners and the fixed plate centre loading cases (figure 4.29).

The 3 pinned corners and the fixed plate centre loading cases create membrane tension for large deflections due to the corner forces remaining in their original orientation and
Chapter 4. Results

Figure 4.29: Load-Gaussian Curvature Plot for fixed plate centre

stretching the plate. Further study involving “follower forces” which remain normal to
the plate surface is recommended to study the effects of the added membrane tension.

4.7 Alternate Finite Element

The finite element analysis was also performed modelling the plate with the SHELL43
4 node shell element. The SHELL43 element is claimed by ANSYS to be well suited
to model nonlinear thin to moderately-thick shell structures[8]. The SHELL43 element
accommodates rotational degrees of freedom and shear deformations but since the prob-
lem under consideration is highly nonlinear, the bilinear SHELL43 element would not
be expected to model the plate as well as the quadratic SHELL93 element. The plate
modelled with SHELL43 elements did not buckle for the loading case of 3 pinned corners
and no initial hydrostatic pressure. The SHELL43 element model only buckled with
Table 4.6: SHELL43 Element Critical Values from Southwell plot

96 $a/h$ ratio, 400 elements

<table>
<thead>
<tr>
<th>pressure (Pa)</th>
<th>$\delta_o$</th>
<th>$P_{Scr}$ (N)</th>
<th>$\bar{P}_{Scr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00094</td>
<td>did not buckle</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.0047</td>
<td>did not buckle</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.0094</td>
<td>did not buckle</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.018</td>
<td>did not buckle</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>0.028</td>
<td>295</td>
<td>20.9</td>
</tr>
<tr>
<td>40</td>
<td>0.037</td>
<td>292</td>
<td>20.7</td>
</tr>
<tr>
<td>50</td>
<td>0.047</td>
<td>285</td>
<td>20.2</td>
</tr>
<tr>
<td>100</td>
<td>0.094</td>
<td>285</td>
<td>20.2</td>
</tr>
<tr>
<td>500</td>
<td>0.47</td>
<td>225</td>
<td>15.9</td>
</tr>
</tbody>
</table>

Table 4.7: SHELL43 Element Critical Values from Southwell plot

96 $a/h$ ratio, 50 Pa hydrostatic pressure

<table>
<thead>
<tr>
<th>elements</th>
<th>$P_{Scr}$ (N)</th>
<th>$\bar{P}_{Scr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>did not buckle</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>did not buckle</td>
<td></td>
</tr>
<tr>
<td>144</td>
<td>285</td>
<td>20.2</td>
</tr>
<tr>
<td>256</td>
<td>285</td>
<td>20.2</td>
</tr>
<tr>
<td>400</td>
<td>285</td>
<td>20.2</td>
</tr>
</tbody>
</table>

an initial deflection greater than 0.3 plate thicknesses (table 4.6) and greater than 144 elements (table 4.7).

For the reasons of problems in buckling, the SHELL43 element was not used in the analysis.

4.8 Non-convergence

The post buckling response for the finite element analysis of the 196.7 $a/h$ ratio plate with 3 pinned corners does not converge. The plate bends to a saddle surface up to
the bifurcation point without numerical difficulties. Shear locking does not seem to be a factor, since increasing the mesh density—decreasing the element $a/h$ ratio—does not rectify the problem.
Chapter 5

Discussions

5.1 FEA Comparison of $P_{cr}$ and $P_{Kcr}$

Gaussian curvature at the centre of the plate reaches a maximum absolute value $K_{cr}$ at corner force $P_{Kcr}$. The absolute value of $K$ then begins to decrease in magnitude before the bifurcation point $P_{cr}$ (figure 5.1).

Attempting to determine the bifurcation point by a critical Gaussian curvature criterion, such as the Southwell plot method, will underestimate the bifurcation point (figure 5.2).

The critical Gaussian curvature point $P_{Kcr}$ is 80 percent of bifurcation point $P_{cr}$ (table 5.1).

The values of $K_{cr}$ and $P_{Kcr}$ remain the same for the 3 pinned corners and the fixed plate centre loading cases (figure 4.29). The present work uses corner forces which maintain their original vertical direction parallel to $z$ axis. For large deflections, the orientation

<table>
<thead>
<tr>
<th>$a/h$ ratio</th>
<th>$P_{cr}$</th>
<th>$P_{Kcr}$</th>
<th>$P_{Kcr}/P_{cr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>49.2</td>
<td>26.9</td>
<td>21.7</td>
<td>0.81</td>
</tr>
<tr>
<td>63.2</td>
<td>26.0</td>
<td>20.8</td>
<td>0.80</td>
</tr>
<tr>
<td>80.3</td>
<td>25.5</td>
<td>20.3</td>
<td>0.80</td>
</tr>
<tr>
<td>96.0</td>
<td>25.1</td>
<td>20.1</td>
<td>0.80</td>
</tr>
<tr>
<td>196.7</td>
<td>24.9</td>
<td>19.7</td>
<td>0.79</td>
</tr>
</tbody>
</table>

Table 5.1: Comparison of $P_{Kcr}$ and $P_{cr}$
Figure 5.1: Load-Gaussian Curvature Plot

Figure 5.2: Load-Deflection Plot
of the forces causes tensile membrane stresses which may have an effect on the value of $P_{K\sigma}$. Further investigation using “follower forces” which remain normal to the plate surface is recommended.

5.2 Comparisons with Bifurcation Points in Literature

5.2.1 Howell

Howell determined the bifurcation point by experiment[1]. The constraints on the plate in the experiment were three corners pinned and the loaded corner free to deflect with the load applied by a constant direction tensile cable. The 3 pinned corners loading case in the present FEA models this experimental setup. Strain gauges measured strains on the top and bottom surfaces of the plate, and Kirchhoff theory was used to calculate the curvatures from the strains. The critical value of Gaussian curvature at bifurcation was determined using the Southwell plot method.

Howell gives the bifurcation point:

$$\kappa a = 10.8h/a \quad (5.1)$$

for the critical value of twist $\kappa$ at the centre of the plate.

The present FEA work using the Southwell plot method gives:

$$\kappa a = 9.0h/a \quad (5.2)$$

The difference between the result of Howell and the FEA is mainly due to Howell’s limit of applied corner force. Howell limited the maximum applied corner force $P_{max}^{Howell}$ to avoid plastic yielding of the material. This corresponds to corner loads less than half of $P_{K\sigma}$ (table 5.2). The Southwell Plot method only claims accuracy as the load approaches the critical load $P \to P_{\sigma}$ (section 2.4).
Table 5.2: Comparison of Coefficient with Howell

<table>
<thead>
<tr>
<th>$a/h$ ratio</th>
<th>$C^{\text{Howell}}$</th>
<th>$C$</th>
<th>$P_{\text{max}}^{\text{Howell}}$</th>
<th>$P_{\text{max}}^{\text{Howell}}/P_{\text{cr}}^{\text{Howell}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>49.2</td>
<td>17.85</td>
<td>9.01</td>
<td>4.8</td>
<td>0.18</td>
</tr>
<tr>
<td>80.3</td>
<td>11.07</td>
<td>9.02</td>
<td>10.0</td>
<td>0.39</td>
</tr>
<tr>
<td>96.0</td>
<td>10.61</td>
<td>9.03</td>
<td>9.9</td>
<td>0.39</td>
</tr>
<tr>
<td>196.7</td>
<td>10.25</td>
<td>9.06</td>
<td>10.9</td>
<td>0.43</td>
</tr>
</tbody>
</table>

Table 5.3: Modified Coefficient

<table>
<thead>
<tr>
<th>pressure (N)</th>
<th>$C'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>8.9</td>
</tr>
<tr>
<td>1</td>
<td>9.5</td>
</tr>
<tr>
<td>10</td>
<td>10.0</td>
</tr>
<tr>
<td>100</td>
<td>9.9</td>
</tr>
<tr>
<td>300</td>
<td>9.4</td>
</tr>
<tr>
<td>500</td>
<td>8.9</td>
</tr>
</tbody>
</table>

The magnitude of initial deflection affects the Southwell's plot prediction of the critical value. The initial deflection for the experiment of Howell is unknown (figure 5.3).

The Southwell plot method using Gaussian and mean curvatures only up to $P_{\text{max}}^{\text{Howell}}$ gives modified coefficient $C'$ values closer to Howell's results (table 5.3).

5.2.2 Ramsey

Ramsey determined the bifurcation point by analytical methods[5]. The kinematic results of Green and Naghdif for small deformations superposed on a large deformation of an elastic Cosserat surface, and the restricted form of the general nonlinear theory of shells and plates of Naghdif were used. The critical value of twist $\kappa$ at bifurcation was determined from a Rayleigh quotient.
Ramsey gives the bifurcation point:

$$\kappa a = 3.29 h/a$$

(5.3)

for the critical value of twist $\kappa$ at bifurcation at the centre of the plate.

The present FEA work using the Southwell plot method gives results in equation 5.2.

The difference between the result of Ramsey and the FEA is mainly due to Ramsey's assumption of the Gaussian curvature behaviour. Ramsey assumed the Gaussian curvature at bifurcation to be uniform over the plate. The present FEA work shows the Gaussian curvature varies by an order of magnitude over the plate (figure 4.14–4.15).

5.2.3 Miyagawa, Hirata, and Shibuya

Miyagawa, Hirata, and Shibuya determined the bifurcation point by experimental and numerical methods[3].
In the experiment of Miyagawa et al., the critical value of corner force at bifurcation was determined from the load-deflection plot. The plate dimension ratio varied $40 < a/h < 120$. Bifurcation occurred only at ratios $a/h > 80$.

Miyagawa et al. give the bifurcation point experimentally:

$$\overline{P}_{cr} = \frac{Pa^2}{2Dt} \approx 21$$  \hspace{1cm} (5.4)

for the dimensionless corner force $\overline{P}_{cr}$.

In the numerical work of Miyagawa et al., the deformed configuration of the plate was approximated as a polynomial. Stresses in the middle of the plate were approximated by combining von Kármán theory, an assumed stress function, and experimental results. The relation between load and deflection was determined by minimizing the total energy of: strain energy due to bending and twisting, strain energy in the middle of the plate due to membrane stretching, and work done by the loads.

Miyagawa et al. give the bifurcation point numerically:

$$\overline{P}_{cr} = 22.8$$  \hspace{1cm} (5.5)

The present FEA work using load-deflection plot gives:

$$\overline{P}_{cr} = 25$$  \hspace{1cm} (5.6)

In the experiment of Miyagawa et al., the four loading points were applied by flat roller bearings which simulated "follower loads" to reduce the stretching forces along the plate. The plate material experienced plastic yielding resulting in the experimental bifurcation point of Miyagawa et al. lower than the numerical bifurcation point of Miyagawa et al.[3].

5.2.4 Lee and Hsu

Lee and Hsu determined the bifurcation point by finite difference methods[2]. The critical value of corner force at bifurcation was determined by the displacement-load plot.
Lee and Hsu give the bifurcation point:

\[
\bar{M}_{cr} = \left[ \frac{12(1 - \nu^2)}{4(1 - \nu)} \frac{a^2}{Dh} P \right]_{cr} = 21
\]  \hspace{1cm} (5.7)

for the dimensionless corner force \( \bar{M}_{cr} \).

The present FEA work using load-deflection plot gives:

\[
\bar{M}_{cr} = 61
\]  \hspace{1cm} (5.8)

The difference between the result of Lee and Hsu and the FEA is mainly due to the limited model of Lee and Hsu. The mesh used by Lee and Hsu in the finite difference scheme was not dense enough to provide convergence of \( M_{cr} \). No attempt was made to calculate \( M_{cr} \) more precisely.
Describing the surface of a square plate twisted by corner forces based on either displacement or curvature values gives different results for the critical point. The load-displacement plot determines the bifurcation point $P_{cr}$. The present FEA work gives $P_{cr} = 25$. The Southwell plot based on curvature determines the critical Gaussian curvature point $P_{Kcr}$. The present FEA work gives $P_{Kcr} = 20$.

The present FEA work gives the coefficient for the critical value of twist at the centre of the plate $C = 9.0$ from the Southwell plot. This result compares well with the experiment of Howell taking into account the low load levels Howell used to avoid plastic yielding of the material. Southwell plots constructed from curvature data of load levels less than $P_{Kcr}$, will overpredict the calculated value of $P_{Kcr}$ for initial deflections of the plate centre between $0.001 < \delta_c/h < 0.5$.

The result of the present FEA work does not compare well with the analytical work of Ramsey. Ramsey assumed Gaussian curvature to be uniform over the plate at bifurcation. The present FEA work shows that the problem is highly nonlinear and Gaussian curvature varies over the plate by an order of magnitude at $P_{Kcr}$ and $P_{cr}$.

The applied forces in the present FEA work maintain their original orientation even for large deflections. This will create significant tensile membrane stresses in the plate for deflections much larger than the plate thickness. Further FEA investigation involving "follower forces" which remain normal to the plate surface, and inclusion of nonlinear material properties is recommended.


