COUPLED-OSCILLATOR MODELS FOR VORTEX-INDUCED OSCILLATION OF A CIRCULAR CYLINDER

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ABSTRACT

The vortex-induced oscillation of a circular cylinder is modelled by a non-linear system with two degrees of freedom. The periodic lift acting on the cylinder due to the vortex-street wake is represented by a self-excited oscillator, which is coupled to the cylinder motion. Approximate solutions and stability criteria are presented which are valid over restricted intervals.

Changes to the form of the coupled-oscillator model and its approximate solution are examined in order to improve the comparison between predicted model and experimental results. The changes are motivated by the study of experimental evidence, and by comparison with the known properties of similar systems of non-linear equations.

Significant improvement in the coupled-oscillator model performance is obtained through the inclusion of an effective structural damping term which is dependent on wind speed and cylinder displacement.

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LIST OF SYMBOLS

A	Non-dimensional transverse cylinder displacement amplitude.
A_F	Component of A at ω_{V_F} (free component)
A_H	Component of A at ω_c (harmonic component)
C_L	Instantaneous lift coefficient
$\overline{C}_L$	Amplitude of lift coefficient
C_{L_o}	Amplitude of lift coefficient for stationary cylinder
C_H	Amplitude of the component of C_L at ω_c (harmonic component)
C_F	Amplitude of the component of C_L at ω_{V_F} (free component)
S	Strouhal number $= \dfrac{h\,\omega_{V_s}}{2\pi V}$
V	Free stream velocity
X_c	Instantaneous transverse cylinder displacement
X	Non-Dimensional transverse cylinder displacement $= \dfrac{X_c}{h}$
a	Mass parameter $= \dfrac{\rho\, h^2}{8\pi^2 S\, m}$
b	Coupling parameter
f	Damping parameter
h	Cylinder diameter
m	Cylinder mass per unit length
ω_c	Detuned frequency of cylinder oscillation (wind-on)
ω_n	Natural frequency of spring-cylinder system (still-air)
ω_V	Vortex formation frequency for the elastically mounted cylinder

ω_{V_s}	Vortex formation frequency for stationary cylinder
ω_{V_F}	Vortex formation frequency approximately at ω_{V_s} (elastically mounted cylinder)
Ω	$= \dfrac{\omega_c}{\omega_n}$
ω_o	$= \dfrac{\omega_{V_s}}{\omega_n}$
ω_F	$= \dfrac{\omega_{V_F}}{\omega_n}$
β	Critical damping ratio (wind-on)
β_o	Critical damping ratio (wind-off)
$\alpha, \gamma, \eta, \delta$	Coefficients of non-linear damping terms
ϕ	Phase angle by which C_L leads X
λ	Detuning parameter for cylinder oscillation frequency
ρ	Fluid density
τ	Non-dimensional time $= \omega_n t$

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1. INTRODUCTION

Dating from the early 1960's, there has been an active program in this department to study the effects on fixed or elastically supported bluff bodies of the wakes produced by them. In the Reynolds Number range which is of interest [$0(10^4)$], the wake is characterized by periodically shed vortices, the frequency of which is governed by the Strouhal relationship. This work is concerned with the interaction of an elastically mounted circular cylinder with its wake, for the case in which the Strouhal frequency is close to the resonance frequency of the cylinder-mounting system. Detailed experimental studies have been carried out by Ferguson (1) and Feng (2) to document the vortex-induced oscillation of just such a system.

As direct solution of the governing dynamic equations for the cylinder and its wake is not feasible at present, a variety of simplified mathematical models have been suggested to describe the interaction [a summary of the more promising suggestions is given by Parkinson (3)]. A proposal by Hartlen and Currie (4) seems to have particular merit. They consider the lift acting on the cylinder (due to its periodic wake) to be governed by a second order non-linear differential equation (of the type studied by van der Pol) which is coupled to the cylinder motion. Over a restricted interval, the results predicted by their model bear good resemblance to certain of the experimentally observed features. They fail to produce some important characteristics however.

Using the coupled-oscillator concept it is the intention of this work to suggest changes in the form of non-linear terms and examine the effects on the solution. The stimulus for this comes from the need to obtain better correlation between model predictions and experimental results.

2. PRELIMINARY

Figure I provides a summary of Feng's results for the vortex-induced oscillation of a circular cylinder (for given input conditions). As Feng determined only three values of lift coefficient amplitude, transient C'_p behaviour was used in establishing the location of the jumps in $\overline{C}_L$ [Parkinson (5)]. The results demonstrate that over a discrete range of flow speeds (the lock-in range), cylinder displacement and fluctuating lift are periodic in time, with the same frequency, which is close to that of the natural frequency of the spring-cylinder system. The amount by which the phase of the exciting force leads the cylinder displacement is measured as well. Important features to note are the hysteresis loops which exist for both amplitude (of displacement and lift) and phase. Also significant is the response for $\omega_o > 1.4$ (outside of lock-in), where cylinder oscillations persist at frequency close to ω_n while the frequency of the predominant excitation is considerably higher (ω_F).

Figure II describes the configuration and the important elements of the spring-cylinder system. With the effect of the vortex-street wake on the cylinder included as a forcing function, the differential equation for transverse displacement X_c is:

$$m\ddot{X}_c + 2\beta\omega_n m\dot{X}_c + m\omega_n^2 X_c = C_L(\frac{\rho}{2}V^2 h)$$

To nondimensionalize the equation, introduce

$$X \equiv \frac{X_c}{h}$$

$$\tau \equiv \omega_n t$$

$$V \equiv \frac{h\,\omega_{V_S}}{2\pi S} \qquad \text{(Strouhal Relationship)}$$

and obtain

$$X'' + 2\beta X' + X = a\omega_\circ^{\,2} C_L \qquad \ldots. \quad (2.1)$$

For modelling purposes, the problem now reduces to determining an expression for C_L.

Hartlen and Currie originally suggested that the lift coefficient be governed by the following differential equation

$$C_L'' - \alpha\omega_\circ C_L' + \frac{\gamma}{\omega_\circ} C_L'^{\,3} + \omega_\circ^{\,2}\, C_L = bX' \qquad \ldots. \quad (2.2)$$

This form was chosen because of its simplicity, and because away from resonance of the spring-cylinder system ($bX' \to 0$), self-excited oscillation of amplitude and frequency approximately equal to $\sqrt{\frac{4}{3}\frac{\alpha}{\gamma}}$ and $\omega_\circ$ respectively is predicted for C_L (provided α, γ are small). This behaviour is consistent with experimental observation if $\sqrt{\frac{4}{3}\frac{\alpha}{\gamma}}$ is set equal to the amplitude of the lift coefficient for a stationary cylinder ($C_{L_\circ}$).

The coupling term (bX') was included to provide a dependence of C_L on cylinder motion. Its presence leads to the prediction of interesting C_L behaviour for $\omega_\circ$ close to ω_n. Drawing a comparison between this system and the well-studied forced oscillation of the van der Pol equation [Stoker (6)], one would expect a range of $\omega_\circ$ for which C_L and X have the same oscillation frequency (lock-in), bounded by a range of $\omega_\circ$ for which C_L has components close to $\omega_\circ$ and ω_n (combination-oscillation). Figure III demonstrates that the postulated regions of characteristic response are consistent with experimental evidence - region A being associated with the typical forced response of an elastic system, region B with the transitional range in which frequency components close to ω_o and ω_n are present, and region C with the lock-in range. It is not possible to make further assumptions concerning the detailed nature of the response, as the forcing function is itself dependent on C_L through Equation (2.1).

Hartlen and Currie obtained an approximate solution to the system of coupled differential equations [Equations (2.1) and (2.2)] valid within the lock-in region, by assuming X and C_L to be given as follows (method of van der Pol)

$$X = A_H \sin \Omega\tau$$

$$C_L = C_H \sin (\Omega\tau + \phi_H) \qquad .... \ (2.3)$$

The actual analysis and a summary of results is included in Appendix A. Figure IV summarizes model predictions for the indicated input values. The results demonstrate the model's ability to generate certain of the features of vortex-induced oscillation.

The stability of the approximate solution is not given directly by the method of van der Pol. An alternate method which does provide such information is the K-B method [Minorsky (7)]. This analysis is introduced and developed in Appendix A. The results obtained allow one to confirm that the solutions summarized by Figure IV are stable, and that the two approximate methods of solution yield identical results provided that Ω, $\Omega^2 \cong 1$.

The results obtained are encouraging. The model fails to produce a double-amplitude response, however, and since the approximate solution is valid only within the lock-in region, the system behaviour for

$\omega_o > 1.4$ cannot be produced. The following work is concerned with an investigation of the form of model and solution used, with a view to improving the comparison between predicted and experimental results.

3. MODEL FORMULATION

3.1 HIGHER ORDER NON-LINEARITY

It was decided to investigate the effect of increasing the order of non-linearity in the governing equation for C_L. Following a suggestion by Landl (8), odd power terms to seventh order in C_L' were included. The equation for C_L then takes the form

$$C_L'' - \alpha\omega_o C_L' + \frac{\gamma}{\omega_o} (C_L')^3 - \frac{\eta}{\omega_o 3} (C_L')^5 + \frac{\delta}{\omega_o 5} (C_L')^7 + \omega_o^{\ 2} C_L = bX' \qquad \text{.... (3.1)}$$

where $\alpha, \gamma, \eta, \delta > 0$

The justification for including fifth and seventh powers of C_L' comes from examining the homogeneous form of Equation (3.1) ($bX' \to 0$). For $\alpha, \gamma, \eta, \delta$ small, then

$$C_L \cong C_F \sin \omega_o \tau$$

and C_F may have one or three positive real roots. In the latter case the middle root would be unstable, and the trivial solution $C_F = 0$ is unstable in either case. Considering the inhomogeneous form, it was hoped that the increase in non-linearity would result in the existence of two stable C_L amplitudes for a given ω_o within the lock-in region; a hysteresis effect possibly resulting from the manner of the dependence on ω_o.

Approximate solutions (by the methods of van der Pol and K-B) to the system of Equations (2.1) and (3.1) are included in Appendix B. Values for the non-linear coefficients $\alpha, \gamma, \eta, \delta$ are determined by requiring that three positive real roots C_{H_i} exist within lock-in (two of which are known from experiment), and that one real root C_{L_o} exist away from lock-in ($bX' \rightarrow 0$).

In order to match predicted with experimental values of lift coefficient amplitude within lock-in, the non-linear coefficients necessary were found to be of 0 (10). The effect of the magnitude of α, γ, η, δ on the approximate solution of Equation (3.1) has not been examined.

Figure V shows numerical results for the indicated input values. The stability analysis confirms that the middle amplitudes of C_H and A_H are unstable, and that the other amplitudes are stable.

The results demonstrate the system's ability to model the behaviour of C_L reasonably well within lock-in (as it was designed to). The frequency and phase variations remain a problem, however, as to a first order approximation they are independent of C_L and thus do not reflect jumps in amplitude which the system produces. The behaviour of the predicted cylinder amplitude is clearly a problem as well.

The predicted results indicate that an extension to seventh order non-linearity in C_L' results in only marginal improvement of the system behaviour, while introducing further complications in doing so.

3.2 COMBINATION-OSCILLATION SOLUTION

Currie and Oey (9) proposed that the double amplitude response could be accounted for by the existence of different solutions to the system of Equations (2.1) and (2.2) for harmonic, or combination-type forms of solution; that is, whether X and C_L are assumed to be of form given by Equation (2.3), or as shown below (combination-type)

$$X = A_H \sin \Omega\tau + A_F \sin \omega_F\tau \qquad \ldots. \ (3.2)$$

$$C_L = C_H \sin (\Omega\tau + \phi_H) + C_F \sin (\omega_F\tau + \phi_F)$$

They draw comparisons between the coupled-oscillator system and the forced oscillation of the van der Pol equation. Actual results of a detailed analysis have yet to be published.

Experimental evidence supports a combination-oscillation form of solution over a range of ω_o adjacent to the lock-in region (Figure III, region B). There is no evidence for a solution of this form within the lock-in region, however.

A study was carried out to see whether or not a solution of this form could realistically account for one of the amplitudes within lock-in, or the system behaviour outside of it. The actual analysis is included in Appendix C. A stability analysis was not carried out, as the approximations which are required in order to combine the K-B method with a combination-oscillation form of solution are not at all obvious.

Figure VI illustrates the important numerical results for the indicated input values. The phase and frequency variations for Ω and ϕ_H are identical to those for the harmonic case and thus have not been shown. Away from the neighbourhood of $\omega_o = 1$, the forced cylinder response at ω_F is negligible, thus A_F and ϕ_F have not been shown as well. The results demonstrate the possibility of the existence of a combination-type oscillation within lock-in. Unfortunately, the analysis predicts a solution valid only within lock-in, and a complicated C_L behaviour over this range - C_L is predicted to have components of approximately equal magnitude at frequencies of Ω and ω_F.

It would appear that the governing equations as formulated are not capable of accommodating a combination-type solution.

3.3 VARIABLE DAMPING

If one assumes the cylinder motion to be governed by Equation (2.1), and that within lock-in X and C_L may be approximated by Equation (2.3), then by substituting for X and C_L in Equation (2.1) and applying the principle of harmonic balance, the following result may be obtained:

$$2\beta = \frac{a\omega_\circ^2}{A_H \Omega} C_H \sin \phi_H$$

Since all the quantities on the right-hand-side of the equation are known or are measurable, the apparent structural damping during vortex-induced cylinder oscillation may be calculated. These calculated values are then to be compared with the value measured in still-air (which is the value given by Feng).

Table I summarizes the experimental results and the calculated ratio $\frac{2\beta}{(2\beta_\circ)}$, where $(2\beta_\circ)$ is the wind-off structural damping. The effective structural damping appears to depend on cylinder oscillation amplitude as well as wind speed.

$\omega_\circ$	A_H	C_H	ϕ_H	$\frac{2\beta}{2\beta_\circ}$
.98	.03	.45	2° ↔ 6° 4°	.57 ↔ 1.7 1.1
1.06	.11	.8	2 ↔ 14 9°	.3 ↔ 2.2 1.5
1.12	.21	1.5	10 ↔ 16 11°	1.7 ↔ 2.8 1.9
1.21	.48 .3	1.91 .5	37 ↔ 59 37 102	4 ↔ 5.6 4 2.7

$a = .0022$
$2\beta_\circ = .002$
$\Omega = .97$

TABLE I Effective Structural Damping During Vortex-Induced Oscillation.

It is clear that any model which fails to take this effect into account will have little chance of success in predicting experimental behaviour.

It is proposed that the effective structural damping be approximated by a relationship of form:

$$2\beta \doteq 2\beta_\circ \; (1 + f\omega_\circ^{\;2} \; A_H)$$

The $\omega_\circ^{\;2}$ and A_H provide a dependence of system damping on the wind force acting on the cylinder, and cylinder displacement respectively. One would expect the constant f to depend on the experimental configuration. An appropriate value can be calculated from the experimental results as follows:

$\omega_\circ$	A_H	$\frac{2\beta}{2\beta_\circ}$	f
.98	.03	1.1	3.5
1.06	.11	1.5	4.0
1.12	.21	1.9	3.4
1.21	.48 .3	4 2.7	4.3 3.9

TABLE II Damping Parameter Determination

A value of $f \cong 4$ would seem to be indicated.

The modified equation governing cylinder response then is

$$X'' + 2\beta_\circ \; (1 + f\omega_\circ^{\;2} \; A_H) \; X' + X = a\omega_\circ^{\;2} \; C_L \qquad \ldots . \; (3.3)$$

In order to assess the effect of the proposed variable damping term, the system of Equations (2.2) and (3.3) has been solved approximately, assuming harmonic and combination-type forms of solution for X and C_L. A stability analysis has been carried out for the harmonic solution and is included in Appendix D.

(i) Harmonic Solution

Within the lock-in range, assume X and C_L to be given by Equation (2.3). If one substitutes for X and C_L into Equations (2.2) and (3.3) and neglects terms in A_H', C_H', ϕ_H' and higher harmonics, the following system of equations can be obtained by applying the principle of harmonic balance:

$$a\omega_o^2 \, C_H \cos \phi_H = A_H \, (1 - \Omega^2)$$

$$a\omega_o^2 \, C_H \sin \phi_H = A_H \, \Omega B_o \, (1 + f\omega_o^2 \, A_H)$$

$$\frac{(\omega_o^2 - \Omega^2)}{\alpha \, \omega_o \, \Omega} \cos \phi_H + \sin \phi_H \, (1 - \frac{\Omega^2}{\omega_o^2} \, \rho_H) = 0 \qquad \ldots . \, (3.4)$$

$$\frac{(\omega_o^2 - \Omega^2)}{\alpha\omega_o\Omega} \sin \phi_H - \cos \phi_H \, (1 - \frac{\Omega^2}{\omega_o^2} \rho_H) = \frac{bA_H}{\alpha \, \omega_o C_H}$$

where $B_o \equiv 2\beta_o$

$$\rho_H \equiv (\frac{C_H}{C_{L_o}})^2$$

To proceed, it is necessary to make an assumption concerning the frequency behaviour $\Omega(\omega_o)$ (which is close to 1 throughout the lock-in region). Introduce

$$\Omega \equiv 1 - \frac{\lambda B_o}{2}$$

where $|\lambda| = 0(1)$

and make the assumption that

$$1 - \Omega^2 \cong \lambda B_\circ - \lambda^2 \frac{B_\circ^2}{4} \cong \lambda B_\circ$$

$$\Omega, \Omega^2 \cong 1$$

both of which are reasonable, since $B_\circ = 0(10^{-3})$. From Equation (3.4) then, one obtains

$$a\omega_\circ^2 C_H \cos \phi_H \cong A_H \lambda B_\circ$$

$$a\omega_\circ^2 C_H \sin \phi_H \cong A_H B_\circ (1 + f \omega_\circ^2 A_H)$$

$$\frac{\Delta}{\alpha\omega_\circ} \cos \phi_H + \sin \phi_H \left(1 - \frac{\rho_H}{\omega_\circ^2}\right) \cong 0 \qquad \ldots . (3.5)$$

$$\frac{\Delta}{\alpha\omega_\circ} \sin \phi_H - \cos \phi_H \left(1 - \frac{\rho_H}{\omega_\circ^2}\right) \cong \frac{bA_H}{\alpha\omega_\circ C_H}$$

$$\text{where } \Delta \equiv \omega_\circ^2 - 1$$

From Equations (3.5.1 and 2)

$$\tan \phi_H = \frac{1 + f\omega_\circ^2 A_H}{\lambda}$$

$$C_H^2 = A_H^2 \left(\frac{B_\circ}{a\omega_\circ^2}\right)^2 \left[\lambda^2 + (1 + f\omega_\circ^2 A_H)^2\right] \qquad \ldots . (3.6)$$

From Equations (3.5.3 and 4)

$$\lambda^2 = (1 + f\omega_\circ^2 A_H) \left[\frac{n\omega_\circ^2}{\Delta} - (1 + f\omega_\circ^2 A_H)\right] \ldots . (3.7)$$

$$\text{where } n \equiv \frac{ab}{B_\circ}$$

Substituting for λ^2 in Equation (3.6.2)

$$C_H^{\ 2} = A_H^{\ 2} \left(\frac{B_o}{a\omega_o^{\ 2}}\right)^2 (1 + f\omega_o^{\ 2} A_H) \frac{n\omega_o^{\ 2}}{\Delta} \qquad \ldots.(3.8)$$

Substituting for $\tan \phi_H$ in Equation (3.5.3)

$$\frac{\Delta}{\alpha\omega_o} + \left(\frac{1 + f\omega_o^{\ 2} A_H}{\lambda}\right) \left(1 - \frac{\rho_H}{\omega_o^{\ 2}}\right) = 0$$

then substituting for λ and ρ_H (from Equations (3.7 and 8)) one obtains

$$\left(\frac{\Delta}{\alpha\omega_o}\right)^2 \left(\frac{n\omega_o^{\ 2}}{\Delta} - (1 + f\omega_o^{\ 2} A_H)\right)$$

$$= (1 + f\omega_o^{\ 2} A_H) \left[1 - C_1 A_H^{\ 2} (1 + f\omega_o^{\ 2} A_H)\right]^2$$

$$\text{where } C_1 \equiv \left(\frac{b}{C_{L_o} \omega_o^{\ 2}}\right)^2 \frac{1}{n\Delta}$$

which can be expanded to yield

$$0 = g_1 A_H^{\ 7} + g_2 A_H^{\ 6} + \ldots.. + g_7 A_H^{\ 1} + g_8 A_H^{\ o} \qquad \ldots. (3.9)$$

where

$$g_1 \equiv C_1^{\ 2} (f\, \omega_o^{\ 2})^3$$

$$g_2 \equiv 3C_1^{\ 2} (f\omega_o^{\ 2})^2$$

$$g_3 \equiv 3C_1^{\ 2} f\, \omega_o^{\ 2}$$

$$g_4 \equiv C_1^{\ 2} - 2C_1 (f\omega_o^{\ 2})^2$$

$$g_5 \equiv - 4C_1 f\omega_o^{\ 2}$$

$$g_6 \equiv -2C_1$$

$$g_7 \equiv f\,\omega_o^2\left(1 + \left(\frac{\Delta}{\alpha\omega_o}\right)^2\right)$$

$$g_8 \equiv 1 + \frac{n\Delta}{\alpha^2}\left(\frac{\Delta}{n\omega_o^2} - 1\right)$$

The seventh order polynomial in A_H can be solved approximately as a function of ω_o and the input parameters (n, b, C_{Lo}, f). Once the roots A_{H_i} have been determined, values $C_{H_i}^2$ can be determined from Equation (3.8), and λ_i^2 from Equation (3.7). The sign of λ_i (and thus $\Omega_i \equiv 1 - \lambda_i \frac{B_o}{2}$) can be determined by substituting for $C_{H_i}^2$ and $\tan\phi_{H_i}$ in Equation (3.5.3).

Figure VII shows the results of such an analysis for the indicated input values. The results demonstrate the system's ability to generate multiple amplitudes in A_H, C_H, ϕ_H and Ω with varying ω_o. The possibility of producing a hysteresis effect exists as the upper branch of $A_H(\omega_o)$ is valid for $\Omega < 1$ only, and the two lower branches for $\Omega > 1$ only. The principle result of the stability analysis (Appendix D) is that the middle branch of $A_H(\omega_o)$ is unstable, while the upper and lower branches are stable. The arrows on Figure VII incorporate this information in describing possible behaviour for increasing or decreasing ω_o.

Although there are still remaining difficulties with the amplitudes of X and C_L, and with trends in the phase angle for $\Omega > 1$, the inclusion of the variable damping term has resulted in a significant improvement in model performance within the lock-in range.

(ii) Combination-Oscillation Solution

If one assumes X and C_L to be given by Equation (3.2), then substituting into Equations (2.2) and (3.3) and neglecting terms such as A_H', ϕ_F', higher harmonics and combination tones and finally applying the principle of harmonic balance, one obtains the following system of equations:

$$a\omega_o^2 \, C_F \cos \phi_F = A_F \, (1 - \omega_F^2)$$

$$a\omega_o^2 \, C_F \sin \phi_F = A_F \, \omega_F B_o \, (1 + f\omega_o^2 A_H)$$

$$a\omega_o^2 \, C_H \cos \phi_H = A_H \, (1 - \Omega^2)$$

$$a\omega_o^2 \, C_H \sin \phi_H = A_H \Omega B_o \, (1 + f\omega_o^2 A_H) \qquad \ldots. \ (3.10)$$

$$\frac{(\omega_o^2 - \omega_F^2)}{\alpha\omega_o \, \omega_F} \cos \phi_F + \sin \phi_F \left[1 - \left(\frac{\Omega}{\omega_o}\right)^2 \left(\rho_F \left(\frac{\omega_F}{\Omega}\right)^2 + 2\rho_H\right) \right] = 0$$

$$\frac{(\omega_o^2 - \omega_F^2)}{\alpha\omega_o\omega_F} \sin \phi_F - \cos \phi_F \left[1 - \left(\frac{\Omega}{\omega_o}\right)^2 \left(\rho_F \left(\frac{\omega_F}{\Omega}\right)^2 + 2\rho_H\right) \right] = \frac{bA_F}{\alpha\omega_o C_F}$$

$$\frac{(\omega_o^2 - \Omega^2)}{\alpha\omega_o\Omega} \cos \phi_H + \sin \phi_H \left[1 - \left(\frac{\Omega}{\omega_o}\right)^2 \left(\rho_H + 2\rho_F \left(\frac{\omega_F}{\Omega}\right)^2\right) \right] = 0$$

$$\frac{(\omega_o^2 - \Omega^2)}{\alpha\omega_o\Omega} \sin \phi_H - \cos \phi_H \left[1 - \left(\frac{\Omega}{\omega_o}\right)^2 \left(\rho_H + 2\rho_F \left(\frac{\omega_F}{\Omega}\right)^2\right) \right] = \frac{bA_H}{\alpha\omega_o C_H}$$

Next introduce

$$\sigma_F \equiv \frac{\omega_o^2 - \omega_F^2}{\alpha\omega_o\omega_F}$$

$$\sigma_H \equiv \frac{\omega_o^2 - \Omega^2}{\alpha\omega_o\,\Omega}$$

then from Equations (3.10.5 and 6)

$$\sigma_F\ (1 + \tan^2 \phi_F) = \frac{b\ a\ \omega_o\ \tan^2 \phi_F}{B_o\alpha\ \omega_F\ (1 + f\omega_o^2\ A_H)} \qquad \dots.\ (3.11)$$

which uses $\frac{bA_F}{\alpha\omega_o C_F} = \frac{ba\omega_o\ \sin\phi_F}{B_o\ \alpha\omega_F\ (1 + f\omega_o^2\ A_H)}$ from Equation (3.10.2). To proceed, it is necessary to make assumptions concerning the frequency behaviour $\Omega(\omega_o)$ (which is close to 1) and $\omega_F(\omega_o)$ (which is close to ω_o).

Assume that $\omega_F \cong \omega_o$, then from Equations (3.10.1 and 2)

$$\tan\phi_F \cong \frac{B_o\omega_o}{1 - \omega_o^2}\ (1 + f\omega_o^2\ A_H)$$

thus, $\tan^2 \phi_F << 1$ for ω_o^2 away from the immediate neighbourhood of $\omega_o = 1$. From Equation (3.11) then

$$\sigma_F \cong \frac{n}{\alpha}\left(\frac{B_o\omega_o}{\Delta}\right)^2\ (1 + f\ \omega_o^2 A_H)$$

where $\Delta \equiv \omega_o^2 - 1$

Substituting for σ_F and $\tan\phi_F$ in Equation (3.10.5) yields

$$\rho_F\left(\frac{\omega_o}{\Omega}\right)^2 + 2\rho_H \cong \left(1 - \frac{n\ B_o\,\omega_o}{\alpha\Delta}\right)\left(\frac{\omega_o}{\Omega}\right)^2 \qquad \dots.\ (3.12.1)$$

From Equation (3.10.7)

$$\rho_H + 2\rho_F\left(\frac{\omega_o}{\Omega}\right)^2 = \left(1 + \frac{\Delta}{\alpha\omega_o\ \tan\phi_H}\right)\left(\frac{\omega_o}{\Omega}\right)^2 \qquad \dots.\ (3.12.2)$$

then solving for $\rho_{F,H}$ from Equations (3.12.1 and 2)

$$\rho_H = \left(1 - 2\,\frac{n\ B_o\omega_o}{\alpha\ \Delta} - \frac{\Delta}{\alpha\omega_o\ \tan\phi_H}\right)\frac{\omega_o}{3}^2$$

$$\rho_F = (1 + \frac{n\ B_\circ \omega_\circ}{\alpha\ \Delta} + \frac{2\Delta}{\alpha\omega_\circ\ \tan\ \phi_H})/3 \qquad .... (3.13)$$

From Equations (3.10.7 and 8)

$$(\sigma_H - \frac{n\omega_\circ}{\alpha\Omega\ (1 + f\omega_\circ{}^2 A_H)})\tan^2\ \phi_H + \sigma_H = 0 \qquad .... (3.14)$$

introduce $\Omega \equiv 1 - \frac{\lambda B_\circ}{2}$ and assume that

$$1 - \Omega^2 = \lambda B_\circ - \lambda^2 \frac{B_\circ{}^2}{4} \cong \lambda B_\circ$$

$$\Omega,\ \Omega^2 \cong 1$$

then examine

$$\sigma_H \equiv \frac{\omega_\circ{}^2 - \Omega^2}{\alpha\omega_\circ\ \Omega} \cong \frac{\Delta}{\alpha\omega_\circ}$$

$$\tan\ \phi_H \equiv \frac{\Omega\ B_\circ}{1 - \Omega^2}\ (1 + f\ \omega_\circ{}^2\ A_H) \cong \frac{1 + f\ \omega_\circ{}^2\ A_H}{\lambda}$$

Substituting for σ_H and $\tan\ \phi_H$ in Equation (3.14), one obtains

$$\lambda^2 = (\frac{n\ \omega_\circ{}^2}{\Delta\ (1 + f\omega_\circ{}^2\ A_H)} - 1)\ (1 + f\omega_\circ{}^2\ A_H)^2 \qquad .... (3.15)$$

From Equations (3.10.1 and 2)

$$(a\omega_\circ{}^2)^2 C_H{}^2 = A_H{}^2\ B_\circ{}^2\ ((1 - \Omega^2)^2 + (1 + f\ \omega_\circ{}^2\ A_H)^2)$$

$$\cong A_H{}^2\ B_\circ{}^2\ (\lambda^2 + (1 + f\omega_\circ{}^2\ A_H)^2)$$

then substituting for λ^2:

$$C_H{}^2 = \frac{A_H{}^2\ b^2}{n\omega_\circ{}^2\Delta}\ (1 + f\omega_\circ{}^2\ A_H)$$

or

$$\rho_H = A_H^{\ 2} \ \frac{b^2}{n\omega_\circ^{\ 2}\ C_{L_\circ}^{\ \ 2}\ \Delta} \ \ (1 + f\omega_\circ^{\ 2} A_H) \qquad \ldots. \ (3.16)$$

Equating (3.13.1) and (3.16), and substituting for $\tan\phi_H$ and λ one obtains

$$\left(\frac{n\omega_\circ^{\ 2}}{\Delta(1 + f\omega_\circ^{\ 2}\ A_H)} - 1 \right) = \left[\frac{\alpha\omega_\circ}{\Delta} \ \left(\frac{3b^2}{n\Delta\ (C_{L_\circ}\omega_\circ^{\ 2})^2} A_H^{\ 2}\ (1 + f\omega_\circ^{\ 2}\ A_H) + \frac{2\ nB_\circ\omega_\circ}{\alpha\Delta} - 1 \right) \right]^2$$

which can be expanded to yield

$$0 = g_1\ A_H^{\ 7} + g_2\ A_H^{\ 6} + \ldots + g_8\ A_H^{\ \circ} \qquad \ldots. \ (3.17)$$

where

$$g_1 \equiv 9\ C_1^{\ 2}\ (f\omega_\circ^{\ 2})^3$$

$$g_2 \equiv 27\ C_1^{\ 2}\ (f\omega_\circ^{\ 2})^2$$

$$g_3 \equiv 27\ C_1^{\ 2}\ (f\omega_\circ^{\ 2})$$

$$g_4 \equiv 9\ C_1^{\ 2} + 6\ C_1\ C_2\ (f\omega_\circ^{\ 2})^2$$

$$g_5 \equiv 12\ C_1\ C_2\ f\omega_\circ^{\ 2}$$

$$g_6 \equiv 6\ C_1\ C_2$$

$$g_7 \equiv f\omega_\circ^{\ 2}\ \left(C_2^{\ 2} + \left(\frac{\Delta}{\alpha\omega_\circ}\right)^2\right)$$

$$g_8 \equiv C_2^{\ 2} + \left(\frac{\Delta}{\alpha\omega_\circ}\right)^2 \ \left(1 - \frac{n\omega_\circ^{\ 2}}{\Delta}\right)$$

where $$C_1 \equiv \left(\frac{b}{C_{L_o} \omega_o^{\,2}}\right)^2 \frac{1}{n\Delta}$$

$$C_2 \equiv \frac{2\, n\, B_o\, \omega_o}{\alpha\, \Delta} - 1$$

The seventh order polynomial in A_H can be solved in a manner similar to Equation (3.9). Once the roots A_{H_i} have been determined, values for $C_{H_i}{}^2$ can be determined from Equation (3.16), and $\lambda_i{}^2$ from Equation (3.15). The sign of λ_i (and thus $\Omega_i \equiv 1 - \frac{\lambda_i B_o}{2}$) can be determined by substituting for $C_{H_i}{}^2$ and $\tan \phi_{H_i}$ in Equation (3.13.1). $C_{F_i}{}^2$ is then given by Equation (3.13.2), and A_{F_i} and ϕ_{F_i} from Equations (3.10.1 and 2).

Figure VIII shows the results of such an analysis for the indicated input values. Since the forced cylinder response at ω_F is negligible away from the neighbourhood of $\omega_o = 1$, A_F and ϕ_F have not been shown. The results demonstrate the system's ability to generate a combination-type solution valid only at the extremes of the resonance region, and realistic behaviour of C_L for $\omega_o < 1.15$ or $\omega_o > 1.38$. These features are both characteristic of vortex-induced cylinder oscillation.

There is no solution for $1.15 < \omega_o < 1.28$ as C_{F_i} is imaginary over this range. There is no solution for $\omega_o < 1.05$ as the results are invalid in the neighbourhood of $\omega_o = 1$.

The inclusion of the variable damping term in the differential equation governing cylinder displacement appears to allow for the realistic accommodation of a combination-oscillation form of solution. This has the effect of extending the range of applicability of the coupled-oscillator model outside of the lock-in region.

4. DISCUSSION

Several changes in form of the governing equations of Hartlen and Currie's original coupled-oscillator model for vortex-induced oscillation have been suggested and examined. Various forms of solution to the modified equations and the question of their stability have been investigated as well. Predicted results have been compared with experimental information, in order to obtain a measure of their usefulness.

The results of this work show the application of a combination-oscillation form of solution to Hartlen and Currie's original model, and the extension to a seventh order non-linearity in C_L' to be unproductive. A positive contribution has been made, however, with the inclusion of an effective structural damping term dependent on wind speed and cylinder displacement. The modified governing equations then produce a hysteresis mechanism within the lock-in region (harmonic solution), and realistic system behaviour outside of lock-in (combination-oscillation form of solution).

The inclusion of a variable structural damping term (which is consistent with experimental evidence) has the effect of improving trends in the coupled-oscillator model performance, and extending its range of applicability. It is proposed that the results are encouraging enough to warrant further investigation of this form of non-linearity.

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APPENDIX A

Hartlen and Currie's original system of differential equation-solution by the methods of van der Pol and K-B

Governing System

$$X'' + 2\beta_\circ X' + X = a\omega_\circ^2 \, C_L$$

$$C_L'' - \alpha\omega_\circ C_L' + \frac{\gamma}{\omega_\circ} (C_L')^3 + \omega_\circ^2 \, C_L = bX' \qquad \ldots \text{A.1}$$

(i) Solution after van der Pol

Assume

$$X = A_H \sin \Omega\tau$$

$$C_L = C_H \sin (\Omega\tau + \phi_H) \qquad \ldots \text{A.2}$$

then substituting for X and C_L in Equation A.1 and neglecting terms such as A_H', ϕ_H', and higher harmonics, one obtains the following system of equations after applying the principle of harmonic balance:

$$a\omega_\circ^2 \, C_H \cos \phi_H = (1 - \Omega^2) \, A_H$$

$$a\omega_\circ^2 \, C_H \sin \phi_H = B_\circ\Omega \, A_H$$

$$\frac{(\omega_\circ^2 - \Omega^2)}{\alpha\omega_\circ \, \Omega} \cos \phi_H + \sin \phi_H \left(1 - \left(\frac{\Omega}{\omega_\circ}\right)^2 \rho_H\right) = 0 \qquad \ldots \text{A.3}$$

$$\frac{(\omega_\circ^2 - \Omega^2)}{\alpha\omega_\circ \, \Omega} \sin \phi_H - \cos \phi_H \left(1 - \left(\frac{\Omega}{\omega_\circ}\right)^2 \rho_H\right) = \frac{bA_H}{\alpha\omega_\circ \, C_H}$$

where

$$B_\circ \equiv 2\beta_\circ$$

$$C_{L_o} \equiv \sqrt{\frac{4}{3}\frac{\alpha}{\gamma}}$$

$$\rho_H \equiv \left(\frac{C_H}{C_{L_o}}\right)^2$$

From Equations A.3.1 and 2

$$\tan \phi_H = \frac{B_o \, \Omega}{1 - \Omega^2}$$

$$A_H^{\ 2} = \frac{C_H^{\ 2} \left(\frac{a\omega_o^{\ 2}}{B_o \, \Omega}\right)^2}{1 + C\,ot^2 \, \phi_H}$$

From Equations A.3.3 and 4 ... A.4

$$\omega_o^{\ 2} = \frac{\Omega^2}{1 - \frac{n}{1 + C\,ot^2 \, \phi_H}}$$

$$C_H^{\ 2} = \left(C_{L_o} \; \frac{\omega_o}{\Omega}\right)^2 \left(1 + \frac{(\omega_o^{\ 2} - \Omega^2)}{\alpha\omega_o \, \Omega \, \tan \phi_H}\right)$$

$$\text{where } n \equiv \frac{ab}{B_o}$$

(ii) Solution by the K-B method (to ascertain the stability of the approximate solutions to Equation A.1).

Rewriting Equation A.1

$$X'' + X = a\omega_o^{\ 2} \, C_L - \frac{B_o}{b} \left(C_L'' - \alpha\omega_o \, C_L' + \frac{\gamma}{\omega_o} C_L'^{\ 3} + \omega_o^{\ 2} \, C_L\right)$$

... A.5

$$\text{assuming} \quad X = A_H \, (\tau) \sin \left(\tau + \theta_H \, (\tau)\right)$$

... A.6

$$X' = A_H \, (\tau) \cos \left(\tau + \theta_H \, (\tau)\right)$$

which implies that

$$A_H'(\tau)\sin(\tau + \theta_H(\tau)) + A_H(\tau)\,\theta_H'(\tau)\cos(\tau + \theta_H(\tau)) = 0$$

or

$$\theta_H'(\tau) = -\frac{A_H'(\tau)}{A_H(\tau)}\,\frac{\sin(\tau + \theta_H(\tau))}{\cos(\tau + \theta_H(\tau))} \qquad \ldots \text{A.7}$$

then multiplying Equation A.5 by X', one can determine

$$A_H'(\tau) = \left(a\,\omega_\circ^2 C_L - \frac{B_\circ}{b}\left(C_L'' - \alpha\omega_\circ C_L + \frac{\gamma}{\omega_\circ} C_L'^3 + \omega_\circ^2 C_L\right)\right)\cos(\tau + \theta_H(\tau))$$

and from Equation A.7

$$\theta_H'(\tau) = -\left(a\,\omega_\circ^2 C_L - \frac{B_\circ}{b}\left(C_L'' - \alpha\omega_\circ C_L + \frac{\gamma}{\omega_\circ} C_L'^3 + \omega_\circ^2 C_L\right)\right)\frac{\sin(\tau + \theta_H(\tau))}{A_H(\tau)}$$

Since a, $B_\circ$ are 0 (10^{-3})

$$\overline{A_H'} \cong \frac{1}{2\pi}\int_0^{2\pi}\left(a\,\omega_\circ^2 C_L - \frac{B_\circ}{b}(\ldots)\right)\cos\psi\, d\psi$$

$$\overline{\theta_H'} \cong -\frac{1}{2\pi A_H}\int_0^{2\pi}\left(a\omega_\circ^2 C_L - \frac{B_\circ}{b}(\ldots)\right)\sin\psi\, d\psi \qquad \ldots \text{A.9}$$

where $\psi \equiv \tau + \theta_H$

If one assumes that $C_L = C_H(\tau)\sin(\tau + \phi_H(\tau))$

and that C_H', ϕ_H', are 0 (10^{-3}), then from Equation A.9

$$\overline{A_H'} \cong \frac{a\,C_H}{2}\left[-\left(\omega_\circ^2 + \frac{1}{n}(1 - \omega_\circ^2)\right)\sin\zeta + \frac{\alpha\omega_\circ}{n}\left(1 - \frac{\rho_H}{\omega_\circ^2}\right)\cos\zeta\right]$$

$$\overline{\theta_H'} = \frac{-a\ C_H}{2\ A_H}\left[(\omega_o{}^2 + \frac{1}{n}(1 - \omega_o{}^2))\ \cos\zeta + \frac{\alpha\omega_o}{n}\ (1 - \frac{\rho_H}{\omega_o{}^2})\ \sin\zeta\right] \qquad \text{... A.10}$$

$$\text{where} \quad \zeta \equiv \theta_H - \phi_H$$

$$\rho_H \equiv \frac{C_H{}^2}{\frac{4}{3}\ \frac{\alpha}{\gamma}} \equiv \frac{C_H{}^2}{C_{L_o}{}^2}$$

Stationary solutions to Equations A.10 exist for $\overline{A_H'} = 0$ in which case $A_H = A_{H_s}$, $C_H = C_{H_s}$ and $\overline{\theta_H'} = -\kappa_H$ which implies $\theta_{H_s} = -\kappa_H\tau$. To first order approximation ζ_s must not be a function of τ, thus $\phi_{H_s} = \theta_{H_s} - \zeta_s$ $= -\kappa_{H_s}\tau - \zeta_s$ where $\zeta_s \neq \zeta_s(\tau)$. In which case one obtains

$$-(\omega_o{}^2 + \frac{1}{n}(1 - \omega_o{}^2))\ \sin\zeta_s + \frac{\alpha\ \omega_o}{n}\ (1 - \frac{\rho_{H_s}}{\omega_o{}^2})\ \cos\zeta_s = 0$$

$$\text{... A.11}$$

$$(\omega_o{}^2 + \frac{1}{n}(1 - \omega_o{}^2))\ \cos\zeta_s + \frac{\alpha\omega_o}{n}\ (1 - \frac{\rho_{H_s}}{\omega_o{}^2})\ \sin\zeta_s = \frac{2\kappa_H\ A_{H_s}}{a\ C_{H_s}}$$

Two further equations are obtained by requiring that the stationary solutions satisfy Equation A.1. Substituting for X and C_L:

$$X = A_{H_s}\ \sin(\tau + \theta_{H_s}) \equiv A_{H_s}\ \sin(1 - \kappa_H)\tau \equiv A_{H_s}\ \sin\Omega\tau \qquad \text{... A.12}$$

$$C_L = C_{H_s}\ \sin(\tau + \phi_{H_s}) \equiv C_{H_s}\ \sin((1 - \kappa_H)\tau - \zeta_s) \equiv C_H\ \sin(\Omega\tau - \zeta_s)$$

$$\text{where} \quad \Omega \equiv 1 - \kappa_H$$

provides

$$a\omega_o^2 \, C_{H_s} \sin \zeta_s = - B_o \, \Omega \, A_{H_s}$$

... A.13

$$a\omega_o^2 \, C_{H_s} \cos \zeta_s = A_{H_s} \, (1 - \Omega^2)$$

Solving Equations A.11 and A.13 one obtains

$$\omega_o^2 = \frac{1}{1 - \dfrac{n}{1 + \cot^2\zeta_s}}$$

... A.14

$$A_{H_s}^{\ 2} = \left(\frac{a\omega_o^2}{B_o\Omega}\right)^2 \frac{C_{H_s}^{\ 2}}{(1 + \cot^2 \zeta_s)}$$

$$C_{H_s}^{\ 2} = (C_{L_o} \omega_o^2)^2 \left[1 - \frac{\tan_s}{\alpha\omega_o} \, (1 + \omega_o^2 \, (n - 1))\right]$$

$$\text{where} \quad \tan \zeta_s \equiv \frac{- B_o\Omega}{1 - \Omega^2}$$

which can be shown to be identical to the results obtained by the method of van der Pol (Equation A.4) provided that Ω, $\Omega^2 \cong 1$.

To determine the stability of a particular solution, one need examine the sign of $\dfrac{d\overline{A'_H}}{dA_H}$ only in the neighbourhood of the root A_{H_s}. From the expression for $\overline{A'_H}$ (Equation A.10.1) one can determine

$$\frac{d\overline{A'_H}}{dA_H} = \frac{dC_H}{dA_H} \left(\frac{\overline{A'_H}}{C_H} - \frac{a\alpha \, \rho_H}{n \, \omega_o} \cos \zeta \right)$$

thus

$$\left.\frac{d\,\overline{A_H'}}{d\,A_H}\right|_{A_H = A_{H_s}} = \frac{-a\alpha\,\rho_{H_s}}{n\,\omega_o}\cos\zeta_s\left.\frac{d\,C_H}{d\,A_H}\right|_{A_H = A_{H_s}} \qquad \ldots\ A.\ 15$$

From Equation A.14.2

$$\frac{d\,C_H}{d\,A_H} = \left(\frac{B_o\,\Omega}{a\omega_o{}^2}\right)^2 (1 + \cot^2\zeta)\ \frac{A_H}{C_H}$$

The stability criterion is

$$\left.\frac{d\,\overline{A_H'}}{d\,A_H}\right|_{A_H = A_{H_s}} = \begin{cases} < 0 & \text{Stable} \\ > 0 & \text{Unstable} \end{cases}$$

Examining Equation A.15, since $\frac{d\,C_H}{d\,A_H}$, a, α, ρ_{H_s}, n, ω_o, are all positive quantities, then the question of stability is decided by $-\cos\zeta$ or $-\cos(-\zeta)$. Thus

$$(-\zeta) < \pi/2 \rightarrow A_{H_s} \text{ will be stable}$$

$$(-\zeta) > \pi/2 \rightarrow A_{H_s} \text{ will be unstable}$$

APPENDIX B

Extension to 7th order non-linearity in C_L' - solution by the methods of van der Pol and K-B.

Governing System

$$X'' + B_o X' + X = a\omega_o^2 C_L$$

$$C_L'' - \alpha\omega_o C_L' + \frac{\gamma}{\omega_o}(C_L')^3 - \frac{\eta}{\omega_o^3}(C_L')^5 + \frac{\delta}{\omega_o^5}(C_L')^7 + \omega_o^2 C_L = bX' \qquad \text{... B.1}$$

(i) Solution after van der Pol

Assume

$$X = A_H \sin \Omega\tau$$

$$\text{... B.2}$$

$$C_L = C_H \sin(\Omega\tau + \phi_H)$$

Substituting for X and C_L into Equation B.1, and applying the principle of harmonic balance one obtains:

$$a\omega_o^2 C_H \cos\phi_H = A_H (1 - \Omega^2)$$

$$a\omega_o^2 C_H \sin\phi_H = A_H B_o\Omega \qquad \text{... B.3}$$

$$\frac{(\omega_o^2 - \Omega^2)}{\alpha \; \omega_o\Omega} \cos\phi_H +$$

$$\left[1 - \frac{3\gamma}{4\alpha}\frac{\Omega^2}{\omega_o^2} C_H^2 + \frac{5}{8}\frac{\eta}{\alpha}\left(\frac{\Omega}{\omega_o}\right)^4 C_H^4 - \frac{35}{64}\frac{\delta}{\alpha}\left(\frac{\Omega}{\omega_o}\right)^6 C_H^6\right] \sin\phi_H = 0$$

$$\frac{(\omega_o^2 - \Omega^2)}{\alpha\omega_o \; \Omega} \sin\phi_H - \left[1 - \frac{3\gamma}{4\alpha}\left(\frac{\Omega}{\omega}\right)^2 C_H^2 + \right.$$

$$\left.\frac{5}{8}\ \frac{\eta}{\alpha}\ \left(\frac{\Omega}{\omega_o}\right)^2 C_H^{\ 4} - \frac{35}{64}\ \frac{\delta}{\alpha}\ \left(\frac{\Omega}{\omega_o}\right)^6 C_H^{\ 6}\right]\cos\ \phi_H = \frac{b\ A_H}{\alpha\omega_o C_H}$$

Equations B.3.3 and 4 are obtained by assuming that

$$C_L' = C_H\ \Omega\ \cos\ (\Omega\tau + \phi_H)$$

$$(C_L')^3 \cong \frac{3}{4}\ C_H^{\ 3}\ \ \Omega^3 \cos\ (\Omega\tau + \phi_H)$$

$$(C_L')^5 \cong \frac{5}{8}\ \ C_H^{\ 5}\ \ \Omega^5\ \cos\ (\Omega\tau + \phi_H)$$

$$(C_L')^7 = \frac{35}{64}\ \ C_H^{\ 7}\ \Omega^7\ \cos\ (\Omega\tau + \phi_H)$$

From Equation B.3 one can determine:

$$\tan\ \phi_H = \frac{B_o\Omega}{1 - \Omega^2}$$

$$A_H^{\ 2} = \frac{C_H^{\ 2}\ \left(\frac{\alpha\omega_o}{B_o\Omega}\right)^2}{(1 + \cot^2\ \phi_H)} \qquad \text{... B.4}$$

$$\omega_o^{\ 2} = \frac{\Omega^2}{1 - \dfrac{\eta}{1 + \cot^2\ \phi_H}}$$

$$C_H^{\ 6} - \frac{8}{7}\,\frac{\eta}{\delta}\ \left(\frac{\omega_o}{\Omega}\right)^2 C_H^{\ 4} + \frac{48}{35}\ \ \frac{\gamma}{\delta}\ \ \left(\frac{\omega_o}{\Omega}\right)^6 \left[\ \frac{\alpha}{\delta}\ \ + \frac{(\omega_o^{\ 2} - \Omega^3)}{\delta\omega_o\ \ \Omega\tan\ \phi_H}\ \right] = 0$$

In order to provide for a double amplitude response within the lock-in region, three real roots of the cubic polynomial in $C_H^{\ 2}$ must exist. For a particular value of Ω (and thus ω_o) within the region, values of

$\overline{C}_{L_{max}}$ and $\overline{C}_{L_{min}}$ are available from experimental data and provide two equations for the determination of the non-linear coefficients. The choice of a third root ($\overline{C}_{L_{unstable}}$) is made in order to establish $\frac{\eta}{\delta}$, $\frac{\gamma}{\delta}$, $\frac{\alpha}{\delta}$ so that a single real root of predetermined amplitude (C_{L_o}) is predicted outside of the lock-in region. This requires the selection of an appropriate value of δ as well. Once the various parameters have been specified, Equation B.4.4 can be solved for $C_{H_i}^{\ 2}(\omega_o)$ by standard methods.

(ii) Solution by the K-B method

Rewrite Equations B.1

$$X'' + x = a\omega_o^{\,2}\, C_L - \frac{B_o}{b}\left[C_L'' - \alpha\omega_o\, C_L' + \frac{\gamma}{\omega_o}(C_L')^3 - \frac{\eta}{\omega_o^{\,3}}(C_L')^5 + \frac{\delta}{\omega_o^{\,5}}\, C_L'^{\,7} + \omega_o^{\,2}\, C_L \right] \qquad \text{.... B.5}$$

assume

$$X = A_H(\tau)\sin(\tau + \theta_H(\tau))$$

$$X' = A_H(\tau)\cos(\tau + \theta_H(\tau))$$

$$C_L = C_H(\tau)\sin(\tau + \phi_H(\tau))$$

then proceeding in a manner identical to that introduced in Appendix A (ii) one obtains

$$\overline{A_H'} \simeq \frac{a\, C_H}{2}\left[-\left(\omega_o^{\,2} + \frac{1}{n}(1 - \omega_o^{\,2})\right)\sin\zeta + \frac{\alpha\omega_o}{n}\, G(C_H)\cos\zeta \right]$$

$$\overline{\Theta_H'} \simeq \frac{-a\, C_H}{2\, A_H}\left[\left(\omega_o^{\,2} + \frac{1}{n}(1 - \omega_o^{\,2})\right)\cos\zeta + \frac{\alpha\omega_o}{n}\, G(C_H)\sin\zeta \right]$$

$$\text{... B.6}$$

where $\zeta \equiv \theta_H - \phi_H$

$$G\,(C_H) \equiv 1 - \frac{3\gamma}{4\alpha}\,\frac{C_H^{\;2}}{\omega_o^{\;2}} + \frac{5}{8}\,\frac{\eta}{\alpha}\,\frac{C_H^{\;4}}{\omega_o^{\;4}} - \frac{35}{64}\,\frac{\delta}{\alpha}\,\frac{C_H^{\;6}}{\omega_o^{\;6}}$$

Stationary solutions to Equation B.6 exist for

$$\overline{A_H{}'} = 0 \quad \text{in which case} \quad A_H = A_{H_s}$$

$$C_H = C_{H_s}$$

$$\overline{\Theta_H{}'} = -\,\kappa_H \quad \text{which implies} \quad \Theta_{H_s} = -\,\kappa_H \tau$$

To a first order approximation, ζ_s must not be a function of τ, thus

$$\phi_{H_s} = \Theta_{H_s} - \zeta_s = -\,\kappa_H \tau - \zeta_s \text{ where } \zeta_s \neq \zeta_s\,(\tau)$$

In which case one obtains

$$-\,(\omega_o^{\;2} + \frac{1}{n}\,(1 - \omega_o^{\;2}))\,\sin\zeta_s + \frac{\alpha\omega_o}{n}\,G\,(C_{H_s})\,\cos\zeta_s = 0$$

$$(\omega_o^{\;2} + \frac{1}{n}\,(1 - \omega_o^{\;2}))\,\cos\zeta_s + \frac{\alpha\omega_o}{n}\,G\,(C_{H_s})\,\sin\zeta_s = \frac{2\kappa_H}{\alpha}\,\frac{A_{H_s}}{C_{H_s}} \qquad \ldots \text{ B.7}$$

It is required as well that the stationary solution satisfy Equation B.1 for

$$X = A_{H_s}\,\sin\,(\tau + \Theta_{H_s}) \equiv A_{H_s}\,\sin\,(1 - \kappa_H)\tau \equiv A_{H_s}\,\sin\Omega\tau$$

$$C_L = C_{H_s}\,\sin\,(\tau + \phi_{H_s}) \equiv C_{H_s}\,\sin\,((1 - \kappa_H)\tau - \zeta_s) \equiv C_{H_s}\,\sin\,(\Omega\tau - \zeta_s)$$

which provides

$$a\omega_o^2 \, C_{H_s} \sin\zeta_s = -B_o \Omega A_{H_s}$$

$$a\omega_o^2 \, C_{H_s} \cos\zeta_s = A_{H_s} (1 - \Omega^2) \qquad \ldots \text{B.8}$$

From Equations B.7 and 8 one obtains:

$$\omega_o^2 = \frac{1}{1 - \dfrac{n}{1 + \cot^2\zeta_s}}$$

$$A_{H_s}^2 = \left(\frac{a\omega_o^2}{B\Omega}\right)^2 \frac{C_{H_s}^2}{(1 + \cot^2\zeta_s)} \qquad \ldots \text{B.9}$$

$$C_H^6 - \frac{8}{7} \omega_o^2 \frac{\eta}{\delta} C_H^4 + \frac{48}{35}\frac{\gamma}{\delta} \omega_o^4 C_H^2$$

$$- \frac{64}{35} \omega_o^6 \left[\frac{\alpha}{\delta} - \frac{n}{\omega_o \delta} \left(\omega_o^2 + \frac{1}{n}(1 - \omega_o^2)\right) \tan\zeta_s\right] = 0$$

$$\text{where } \tan\zeta_s = \frac{-\Omega B}{1 - \Omega^2}$$

which can be shown to be identical to the results obtained by the method of van der Pol (Equation B.4) provided that $\Omega, \Omega^2, \Omega^4, \Omega^6 \cong 1$.

To determine the stability of a root A_{H_s}, examine the sign of $\left.\dfrac{d\,\overline{A_H'}}{d\,A_H}\right|_{A_H = A_{H_s}}$. From Equation B.6.1 one has that $\overline{A_H'} = F(C_H)$, therefore

$$\frac{d\,\overline{A_H'}}{d\,A_H} = \frac{d\,F}{d\,C_H}\frac{d\,C_H}{d\,A_H}$$ which can be evaluated from Equations

B.6.1 and 9.2. The criterion of stability being

$$\left.\frac{d\,\overline{A_H'}}{d\,A_H}\right|_{A_H = A_{H_s}} = \begin{matrix} < 0 \text{ stable} \\ > 0 \text{ unstable} \end{matrix}$$

APPENDIX C

Combination-oscillation solution applied to Hartlen and Currie's original system (solution by the method of van der Pol).

Assume

$$X = A_H \sin \Omega\tau + A_F \sin \omega_F \tau$$

... C.1

$$C_L = C_H \sin (\Omega\tau + \phi_H) + C_F \sin (\omega_F \tau + \phi_F)$$

then substituting for X and C_L into Equation A.1 and neglecting terms such as A_H', ϕ_F', higher harmonics and combination tones, the following results are obtained after applying the principle of harmonic balance:

$$a\omega_\circ^2 \, C_F \cos \phi_F = A_F \, (1 - \omega_F^2)$$

$$a\omega_\circ^2 \, C_F \sin \phi_F = A_F \, B_\circ \, \omega_F$$

$$a\omega_\circ^2 \, C_H \cos \phi_H = A_H \, (1 - \Omega^2)$$

... C.2

$$a\omega_\circ^2 \, C_H \sin \phi_H = A_H \, B_\circ \, \Omega$$

$$\frac{(\omega_\circ^2 - \omega_F^2)}{\alpha\omega_\circ \, \omega_F} \cos \phi_F + \sin \phi_F \left[1 - \left(\frac{\Omega}{\omega_\circ}\right)^2 \left(\rho_F \left(\frac{\omega_F}{\Omega}\right)^2 + 2 \, \rho_H\right) \right] = 0$$

$$\frac{(\omega_\circ^2 - \omega_F^2)}{\alpha\omega_\circ \, \omega_F} \sin \phi_F - \cos \phi_F \left[1 - \left(\frac{\Omega}{\omega_\circ}\right)^2 \left(\rho_F \left(\frac{\omega_F}{\Omega}\right)^2 + 2 \, \rho_H\right) \right] = \frac{b \, A_F}{\alpha\omega_\circ \, C_F}$$

$$\frac{(\omega_\circ^2 - \Omega^2)}{\alpha\omega_\circ\,\Omega}\;\cos\phi_H + \sin\phi_H\left[1 - \left(\frac{\Omega}{\omega_\circ}\right)^2\left(\rho_H + 2\rho_F\left(\frac{\omega_F}{\Omega}\right)^2\right)\right] = 0$$

$$\left(\frac{\omega_\circ^2 - \Omega^2}{\alpha\omega_\circ\,\Omega}\right)\sin\phi_H - \cos\phi_H\left[1 - \left(\frac{\Omega}{\omega_\circ}\right)^2\left(\rho_H + 2\rho_F\left(\frac{\omega_F}{\Omega}\right)^2\right)\right] = \frac{b\,A_H}{\alpha\omega_\circ\,C_H}$$

$$\text{where}\quad \rho_H \equiv \frac{C_H^{\,2}}{\frac{4}{3}\,\frac{\alpha}{\gamma}} \equiv \left(\frac{C_H}{C_{L_\circ}}\right)^2$$

$$\rho_F \equiv \frac{C_F^{\,2}}{\frac{4}{3}\,\frac{\alpha}{\gamma}} \equiv \left(\frac{C_F}{C_{L_\circ}}\right)^2$$

Note that Equations C.2.5-7 assume that

$$(C_L')^3 \cong \frac{3}{4}\,\Omega^3\,C_F\,\frac{\omega_F}{\Omega}\cos(\omega_F\tau + \phi_F)\left[C_F^{\,2}\left(\frac{\omega_F}{\Omega}\right)^2 + 2\,C_H^{\,2}\right]$$

$$+ \frac{3}{4}\,\Omega^3\,C_H\cos(\Omega\tau + \phi_H)\left[C_H^{\,2} + 2\,C_F^{\,2}\left(\frac{\omega_F}{\Omega}\right)^2\right]$$

From Equations C.2.5 and 6 one obtains

$$\omega_\circ^2 = \frac{\omega_F^{\,2}}{1 - \dfrac{n}{1 + \cot^2\phi_F}}$$

$$2\rho_H + \rho_F\left(\frac{\omega_F}{\Omega}\right)^2 = \left(\frac{\omega_\circ}{\Omega}\right)^2\left[1 + \frac{(\omega_\circ^{\,2} - \omega_F^{\,2})}{\alpha\omega_\circ\,\omega_F\tan\phi_F}\right] \qquad \ldots\ \text{C.3}$$

From Equations C.2.7 and 8

$$\omega_\circ^2 = \frac{\Omega^2}{1 - \dfrac{n}{1 + \cot^2\phi_H}}$$

$$\rho_H + 2\rho_F \left(\frac{\omega_F}{\Omega}\right)^2 = \left(\frac{\omega_\circ}{\Omega}\right)^2 \left[1 + \frac{(\omega_\circ^2 - \Omega^2)}{\alpha\omega_\circ\Omega \tan\phi_H}\right]$$

From Equations C.2.1-4

$$\tan\phi_F = \frac{\omega_F B_\circ}{1 - \omega_F^2}$$

$$A_F^2 = \left(\frac{a\ \omega_\circ^2}{B_\circ\ \omega_F}\right)^2 \frac{C_F^2}{(1 + \cot^2\phi_F)}$$

$$\tan\phi_H = \frac{\Omega B_\circ}{1 - \Omega^2}$$

$$A_H^2 = \left(\frac{a\ \omega_\circ^2}{B_\circ\ \Omega}\right)^2 \frac{C_H^2}{(1 + \cot^2\phi_H)}$$

If one assumes that $\Omega \cong 1$ and $\omega_F \cong \omega_\circ$ then Equations C.3 can be solved for $\rho_{H,F}$, $A_{H,F}$, $\phi_{H,F}$ as functions of $\omega_\circ$, α,γ and η. $\Omega(\omega_\circ)$ and ω_F $(\omega_\circ)$ are given by the appropriate roots of Equations C. 3.3 and 1.

APPENDIX D

Variable structural damping - solution by the K-B method.

Governing system

$$X'' + B_\circ\ (1 + f\omega_\circ^2\ A_H)\ X' + X = a\omega_\circ^2\ C_L$$

$$C_L'' - \alpha\omega_\circ\ C_L' + \frac{\gamma}{\omega_\circ}\ (C_L')^3 + \omega_\circ^2\ C_L = b\ X' \qquad \ldots\ D.1$$

Assume

$$X = A_H\ (\tau)\ \sin(\tau + \theta_H\ (\tau))$$

$$X' = A_H\ (\tau)\ \cos\ (\tau + \theta_H\ (\tau))$$

$$C_L = C_H\ \sin\ (\tau + \phi_H\ (\tau))$$

Then proceeding in a manner identical to that introduced in Appendix A(ii) one obtains

$$\overline{A_H'} \cong \frac{-C_H\ a}{2}\left[\sin\zeta\ (\omega_\circ^2 - \frac{1}{n}\ (1 + f\omega_\circ^2\ A_H)\ (\omega_\circ^2 - 1)) - \cos\zeta\quad (1 + f\ \omega_\circ^2\ A_H)\ \frac{\alpha\omega_\circ}{n}\quad \frac{(1 - \rho_H)}{\omega_\circ^2}\right] \qquad \ldots\ D.2$$

$$\overline{\theta_H'} \cong \frac{-C_H\ a}{2\ A_H}\left[\cos\zeta\ (\ \omega_\circ^2 - \frac{1}{n}\ (1 + f\omega_\circ\ A_H)\ (\omega_\circ^2 - 1)) + \sin\zeta\quad (1 + f\omega_\circ^2\ A_H)\ \frac{\alpha\omega_\circ}{n}\quad \frac{(1 - \rho_H)}{\omega_\circ^2}\right]$$

Stationary solutions to Equation D.2 exist for

$$\overline{A_H{}'} = 0 \quad \text{in which case } A_H = A_{H_s}$$

$$C_H = C_{H_s}$$

and $\overline{\theta_H{}'} = -\kappa_H$ which implies $\theta_{H_s} = -\kappa_H\tau$. To a first order approximation, ζ_s must not be a function of τ, thus $\phi_{H_s} = \theta_{H_s} - \zeta_s = -\kappa_H\tau - \zeta_s$ where $\zeta_s \neq \zeta_s(\tau)$. Two further equations are obtained by requiring that the stationary solution satisfy Equation D.1.1 for

$$X = A_{H_s} \sin(1 - \kappa_H)\tau \equiv A_{H_s} \sin \Omega\tau$$

$$C_L = C_{H_s} \sin((1 - \kappa_H)\tau - \zeta_s) = C_{H_s} \sin(\Omega\tau - \zeta_s)$$

The expressions which are derived for A_{H_s}, C_{H_s}, Ω and ζ_s are identical to those obtained by the method of van der Pol (Equations (3.6-9)), where

$$\zeta_s \equiv -\phi$$

$$-\kappa_H \equiv \frac{-\lambda B_o}{2}$$

To determine the stability of a root A_{H_s}, examine the sign of $\left.\frac{d\,\overline{A_H{}'}}{d\,A_H}\right|_{A_H = A_{H_s}}$. From Equation D.2.1 one has that $\overline{A_H{}'} = F\,(A_H, C_H, \zeta)$, therefore

$$\frac{d\,\overline{A_H{}'}}{d\,A_H} = \frac{\partial F}{\partial A_H} + \frac{\partial F}{\partial C_H}\frac{d\,C_H}{d\,A_H} + \frac{\partial F}{\partial \zeta}\frac{d\zeta}{d\,A_H} \qquad \ldots \text{D.3}$$

The partial derivatives can be obtained from Equation D.2.1, and the exact differentials from Equations (3.5.2) and (3.8). The criterion for stability being

$$\left.\frac{d\,\overline{A_H'}}{d\,A_H}\right|_{A_H = A_{H_S}} = \begin{cases} < 0 & \text{stable} \\ > 0 & \text{unstable} \end{cases}$$

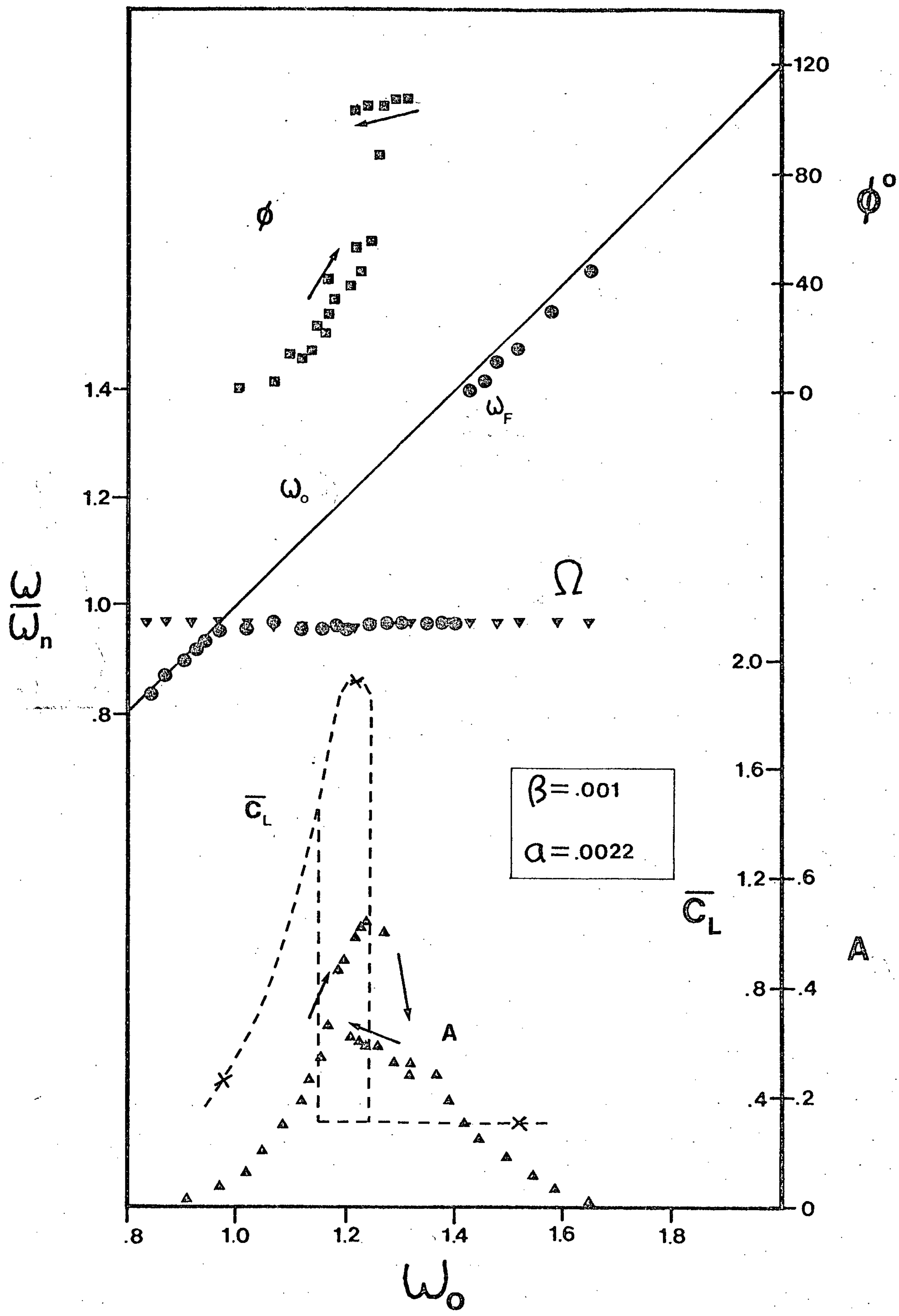

FIGURE 1: Experimental Results For Vortex-Induced Oscillation of a Circular Cylinder (Feng)

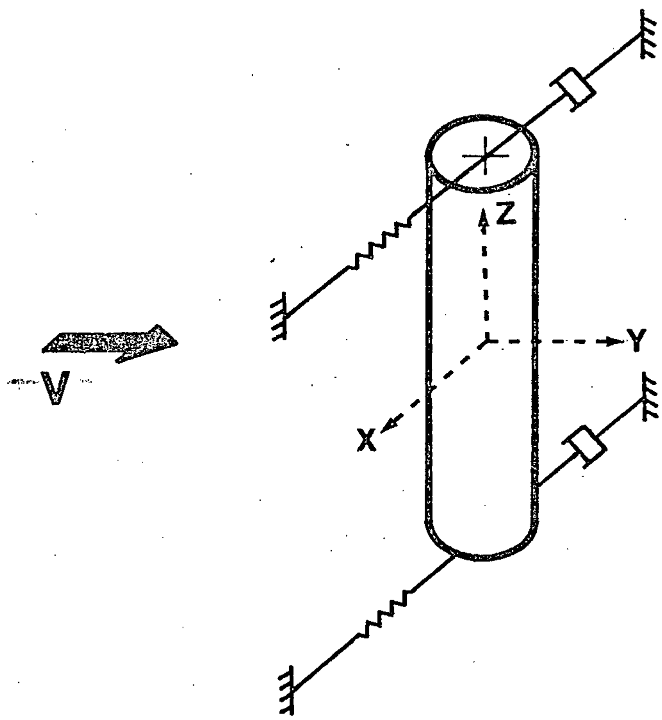

FIGURE II: Schematic Diagram of Experimental Configuration

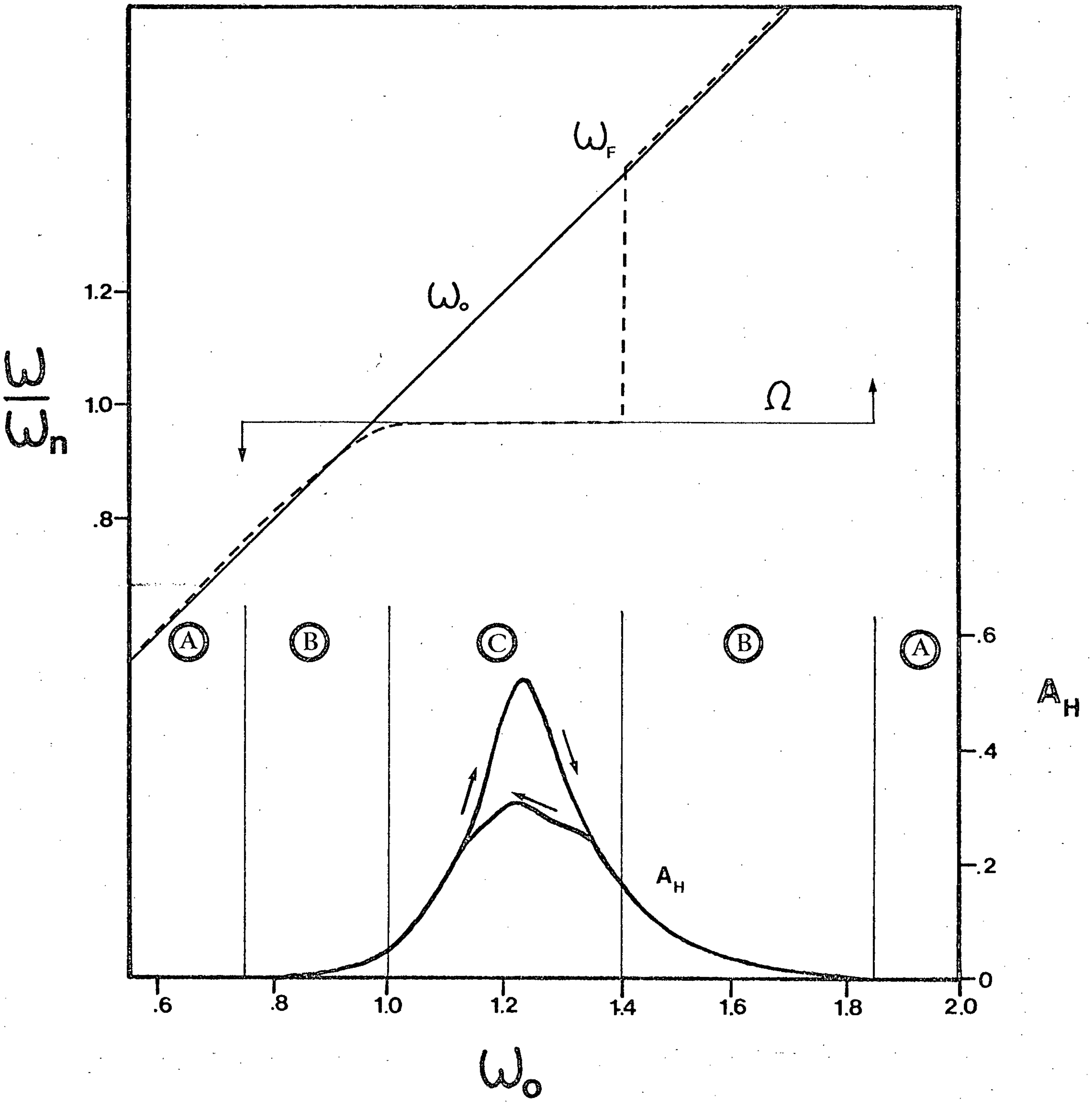

FIGURE III: Characteristic Domains of Vortex-Induced Oscillation

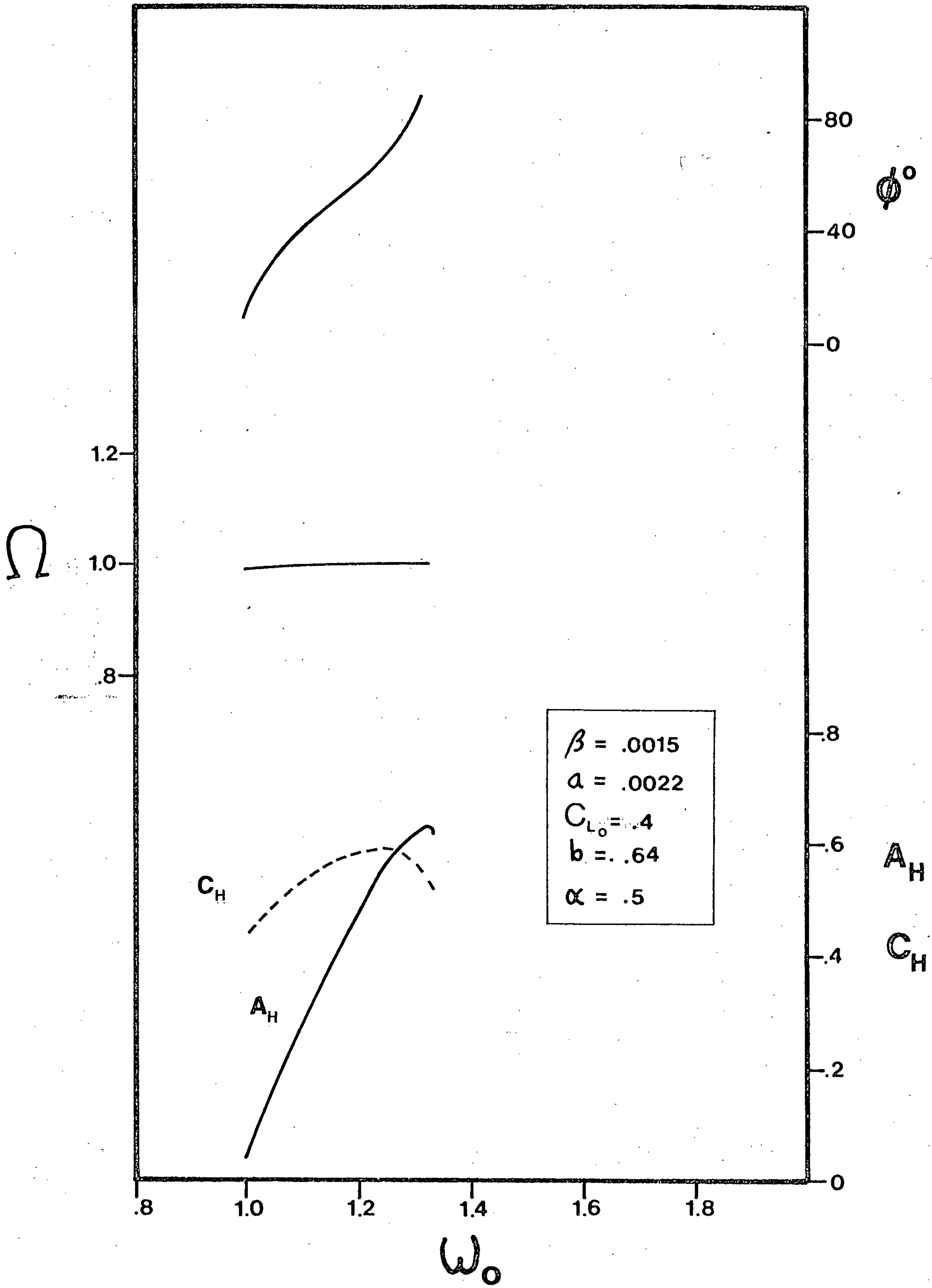

FIGURE IV: Theoretical Predictions from Hartlen and Currie's Original Model

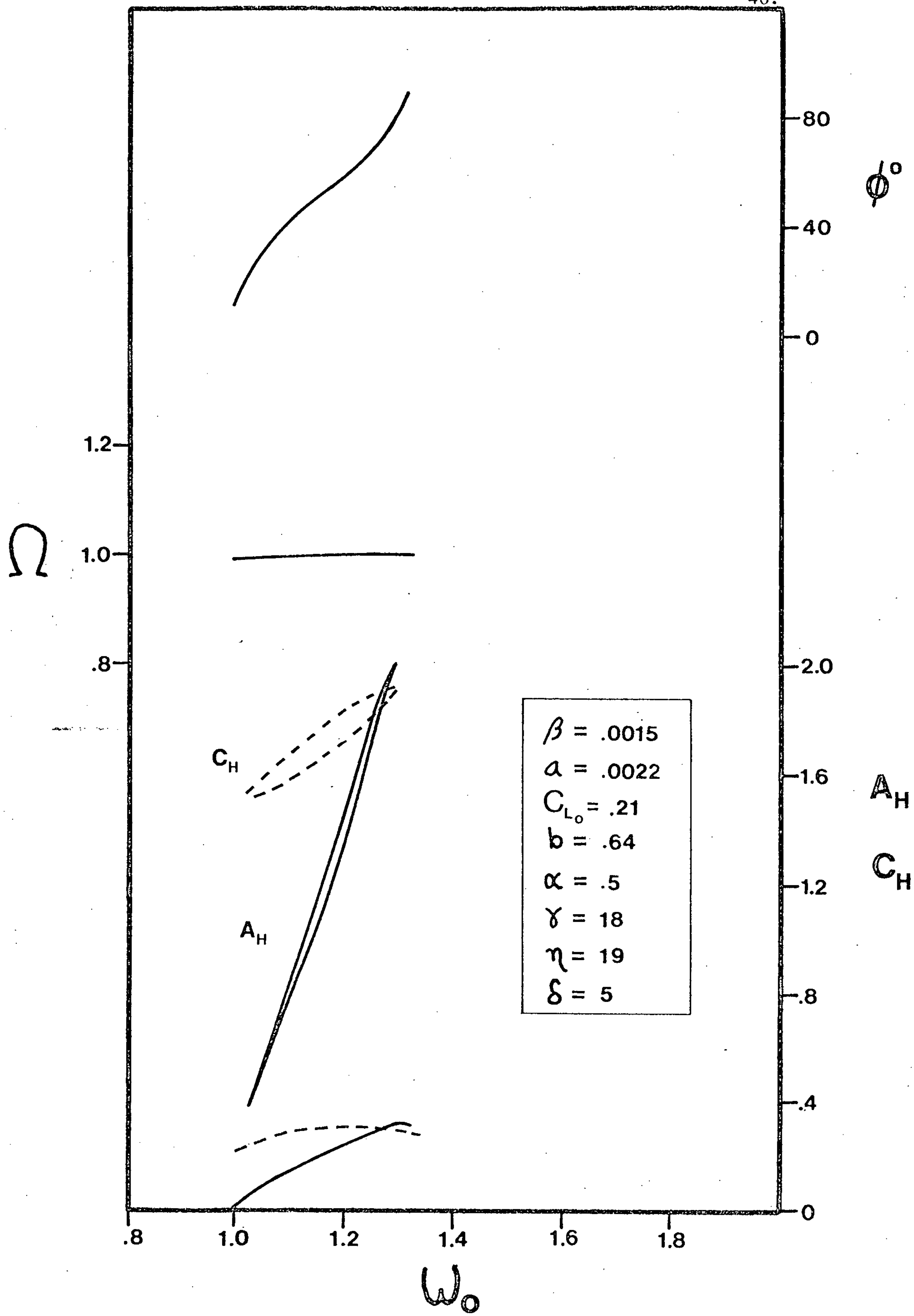

FIGURE V: Theoretical Predictions for a Higher Order Nonlinearity in C_L'

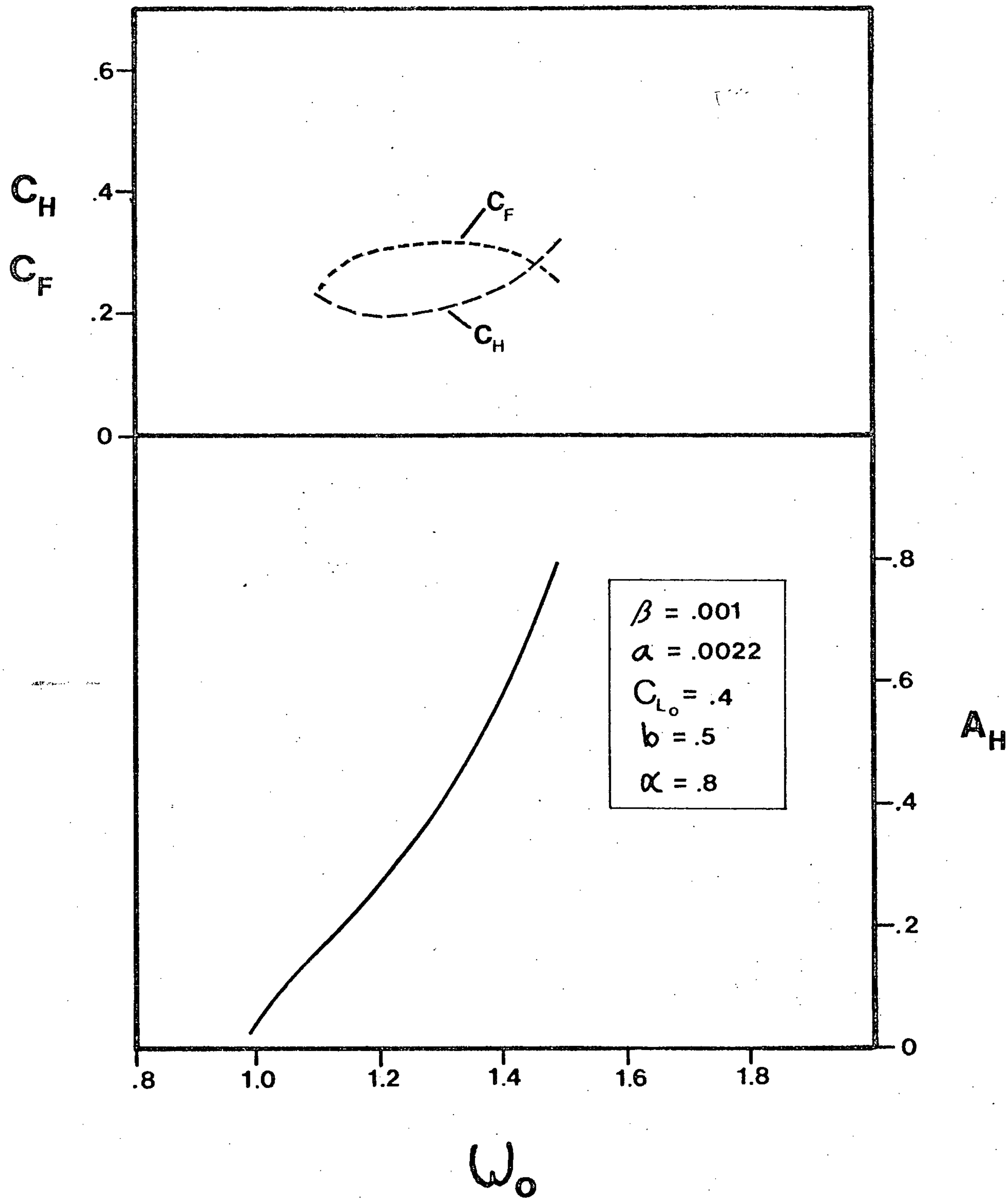

FIGURE VI: Theoretical Predictions for Combination-Oscillation Solution Applied to Hartlen and Currie's Original Model

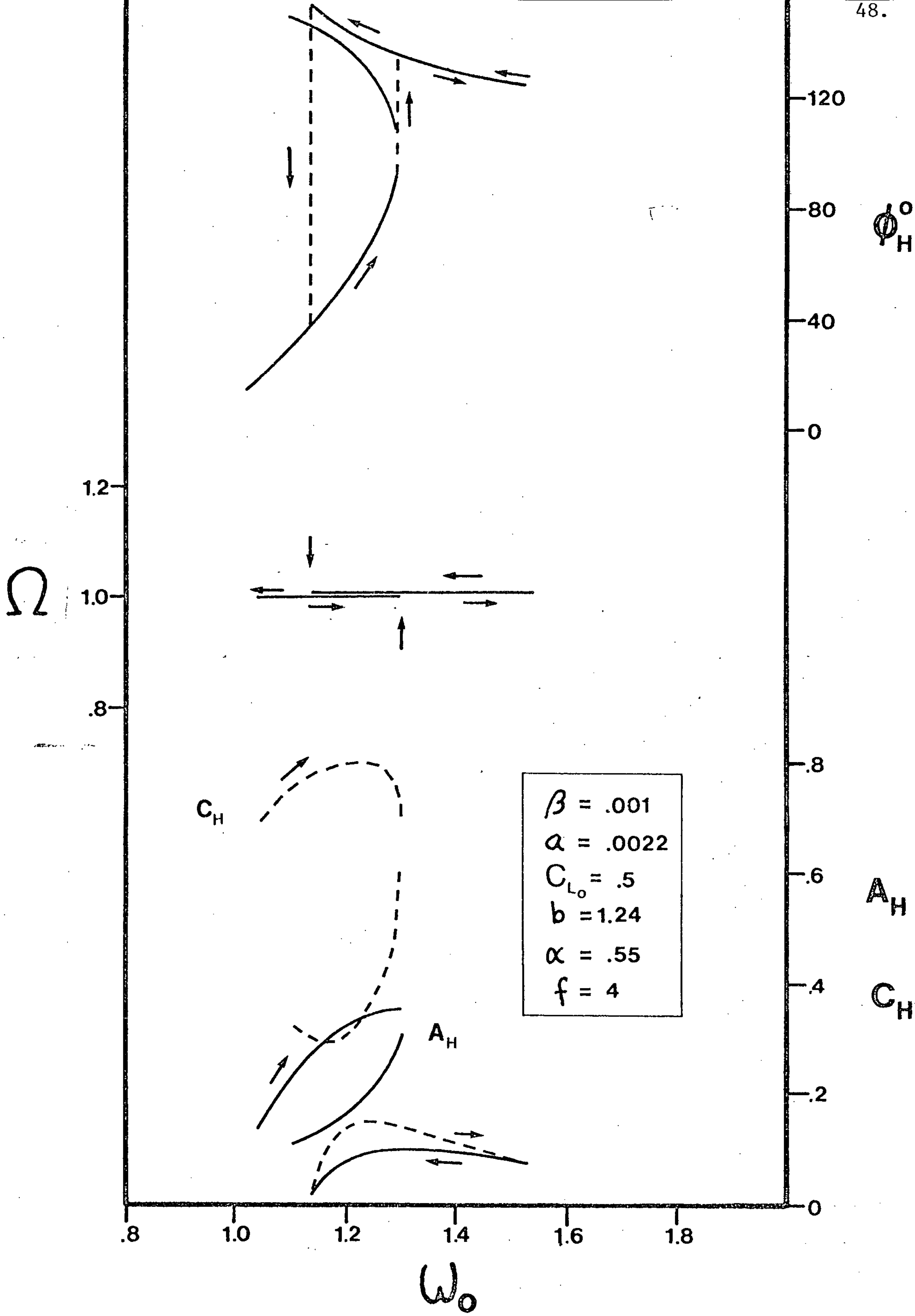

FIGURE VII: Theoretical Predictions of the Effect of Variable Structural Damping — Harmonic Solution

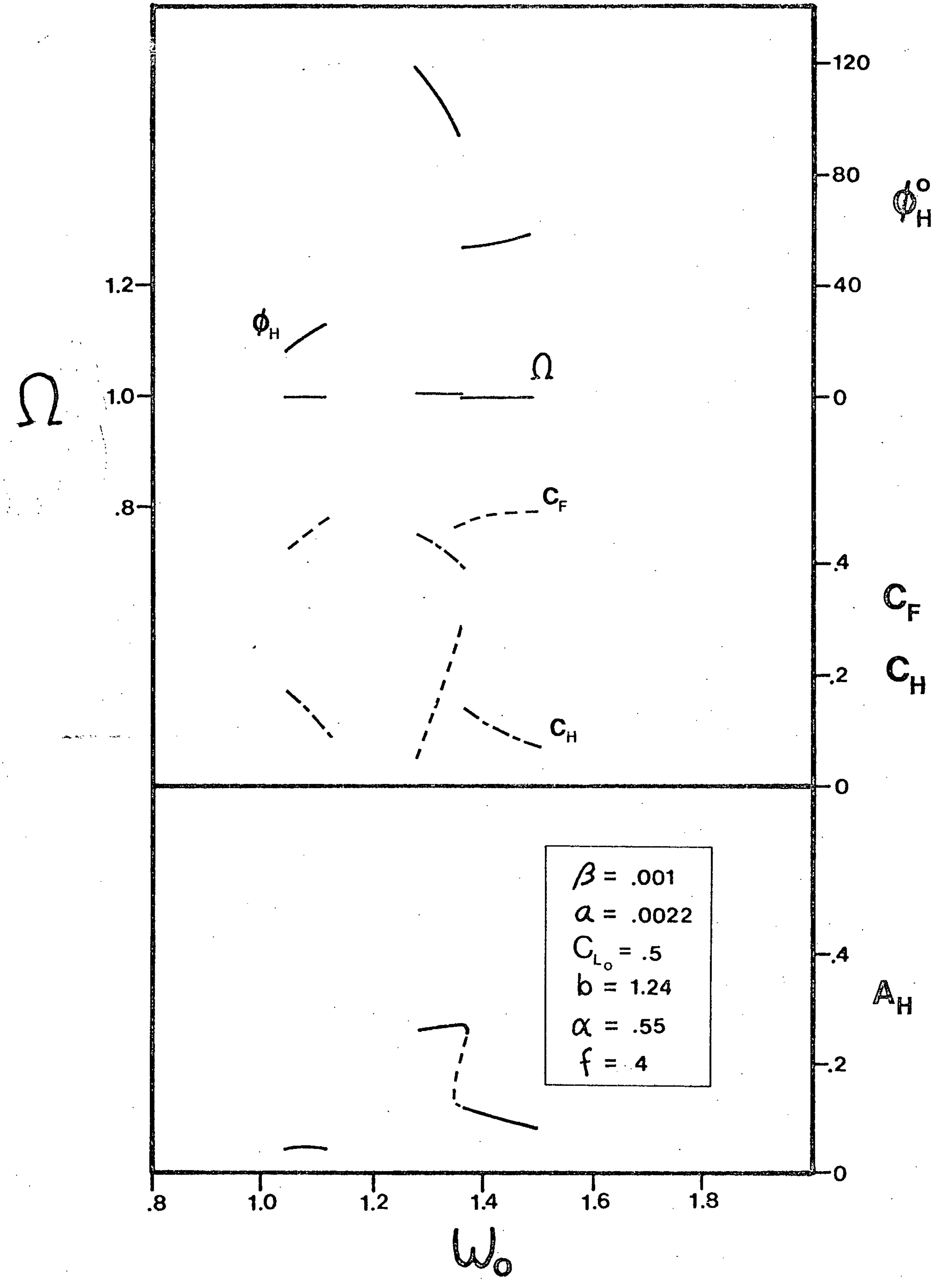

FIGURE VIII: Theoretical Predictions of the Effect of Variable Structural Damping—Combination-Oscillation