EFFECTS OF VISCOUS DISSIPATION ON COMBINED FREE AND FORCED CONVECTION THROUGH VERTICAL DUCTS AND PASSAGES

by

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ABSTRACT

The effects of viscous dissipation on the flow phenomena and heat transfer rate for fully developed laminar flow through vertical ducts and passages has been analysed under the condition of combined free and forced convection. The fluid properties are considered to be constant except for the variation of density in the buoyancy term of the momentum equation. The thermal boundary condition of uniform heat flux per unit length in the flow direction has been considered. The investigation is carried out for two geometries; (a) Circular ducts and (b) Concentric annuli. The governing momentum and non-linear energy equations are solved for the circular duct by three methods; (i) Power Series Method (ii) Galerkin's Method and (iii) Numerical Integration Method. The solutions for the concentric annuli are obtained by Numerical Integration Method. Results for the velocity and temperature distribution in the flow field are obtained, and information of engineering interest like Nusselt numbers have been evaluated.

For combined free and forced convection, the momentum and energy equations are coupled, and hence viscous dissipation affects both the velocity and temperature fields. The effect of viscous dissipation on the velocity field is to reduce the flow velocity near the heated wall(s) and thus it counteracts the effect of free convection on the velocity field for the present study of heating in upflow. The effect of viscous dissipation on the temperature field is to act as a heat source in the fluid and

reduce the temperature differences in the system. Viscous dissipation opposes the externally impressed heating and reduces the heat transfer rate when the surface transfers heat to the fluid. Consequently, lower Nusselt number values are obtained when viscous dissipation is taken into consideration. The quantitative effect of viscous dissipation on Nusselt number is found to be small for the case of circular ducts. However, for flow through annular passages and for the corresponding values of the same parameters, the effect of viscous dissipation on the heat transfer rate may not be ignored.

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NOMENCLATURE

A = Area of cross-section

c_p = Specific heat of the fluid at constant pressure

C = $\frac{\partial T}{\partial Z}$, temperature gradient in the flow direction

D_h = $\frac{4 \times \text{area of cross-section}}{\text{heated perimeter}}$, equivalent diameter

Eck = $\frac{U^2}{C_p \Delta T}$, Eckert number, dimensionless

g = Gravitational acceleration

L = $\frac{-(\frac{dp}{dZ} + \rho_w g) D_h^2}{4 \mu U}$, pressure drop parameter dimensionless

M = $\frac{Eck}{Re}$, viscous dissipation parameter, dimensionless

Nu = $\frac{h D_h}{\kappa}$, Nusselt number, dimensionless

q = Wall heat flux

r = Radial coordinate

R = $\frac{2r}{D_h}$, for circular ducts, dimensionless

= $\frac{r}{r_o}$, for concentric annuli, dimensionless

Ra = $\frac{\rho^2 \beta g c_p C D_h^4}{16 \mu \kappa}$, Rayleigh number, dimensionless

Re = $\frac{U D_h \rho}{\mu}$, Reynolds number, dimensionless

T = Temperature

U = Average axial velocity

v_z = Axial velocity

V = $\frac{v_z}{U}$, dimensionless axial velocity

$\overline{V}$ = $\frac{V}{L}$, dimensionless

Z = Axial coordinate in flow direction

β = Coefficient of volumetric expansion

η = $(Ra)^{1/4}$

κ = Thermal conductivity of the fluid

λ = $\frac{r_i}{r_o}$, radius ratio, dimensionless

μ = Dynamic viscosity of the fluid

ρ = Density of the fluid

ϕ = $\frac{(T - T_w)}{\rho U c_p C D_h^2/4\kappa}$, temperature function, dimensionless

$\overline{\phi}$ = $\frac{\phi}{L}$, dimensionless

∇^2 = $\frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR}$

Subscripts

i. . . . . . . inside

o. . . . . . . outside

w. . . . . . . wall

1. INTRODUCTION

In the flow of all real fluids, viscosity plays an important role and when viscous fluids flow on solid surfaces by and large velocity gradients exist. These velocity gradients give rise to shear stresses which results in the dissipation of frictional energy into heat. Consequently in a heat transfer process for the flow of a real fluid, the omission of viscous dissipation in the thermal energy balance of a moving fluid element would be unrealistic from the physics of fluids.

Hallman [7]* and Morton [14] have investigated the effect of free convection on forced convection and have shown that the effect of free convection on the forced velocity field is to increase the velocity gradients near the walls of the duct in up flow when heat is transferred from the surface to the fluid. From the results of these investigations it seems that the study of the effects of viscous dissipation which is associated with velocity gradients could be quite interesting in the field of combined free and forced convection.

The study of the effects of viscous dissipation can be divided into two broad categories, (i) External flows and (ii) Internal flows. A brief survey of the available literature under these two categories is presented below.

External Flows

For external flows, the effect of viscous dissipation is found to be quite significant because of the energy generated in

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*Numbers in brackets designate references at the end of the thesis.

the boundary layer, and the skin temperatures that are attained at very high velocities [8]. Several studies have been made in this regard because the phenomena of 'Aerodynamic Heating' at high Mach numbers can cause severe problems due to the temperature limitations of structural materials commonly used in the manufacture of aircraft parts and missiles. Studies in the area of Aerodynamic Heating have been reported by Schlichting [22], Shapiro [23] and Truitt [24] among others.

The study of the effects of viscous dissipation in natural convection was carried out by Gebhart [5] for flow over a semi-infinite plate parallel to the body force direction. He used the perturbation method and has calculated the first temperature perturbation function for Prandtl numbers from 10^{-2} to 10^{4}. He has shown that the magnitude of the viscous dissipation effect depends upon the dissipation parameter which is small for most engineering devices with common fluids for the gravitational field strength of the earth.

<u>Internal Flows</u>

The study of the effects of viscous dissipation in internal laminar flows can be divided into three parts, (i) Forced convection, (ii) Free convection and (iii) Combined free and forced convection.

(i) <u>Forced Convection</u>

Tyagi [25, 26, 27, 28] in a series of papers has studied the effect of viscous dissipation in forced convection through non-circular channels. He has used the method of complex variables and has obtained solutions for both Neumann and Dirichlet type thermal boundary conditions showing that viscous dissipation has significant effect on the Nusselt number.

Cheng [3] has studied the effects of viscous dissipation for flow through regular polygonal ducts using the method of point-matching. Exact solutions were obtained for the governing partial differential equations and the boundary conditions were satisfied only at selected points. He has also obtained results for a circular duct and has shown that the effect of viscous dissipation is greater for circular ducts than for non-circular ducts.

(ii) Free Convection

Ostrach [6, 15, 16, 17, 18] has investigated the effects of viscous dissipation in natural convection flows through channels formed by two parallel long plane surfaces and has shown that the flow and heat transfer are not only functions of Prandtl and Grashof numbers but also depend on the dimensionless frictional heating parameter which may appreciably affect the mode of heat transfer.

(iii) Combined Free and Forced Convection

The only available work in the field of combined free and forced convection is that of Ostrach [19, 20]. He has used the method of successive approximations to analyse the problem of taking into account the effects of frictional heating in flow between vertical parallel plane surfaces and has obtained results similar to his free convection analysis.

No work seems to have been done to study the effects of viscous dissipation for flow through circular ducts and annular passages and is the subject of the present thesis. In the next section, the formulation of the problem and the methods of solution for the circular duct are presented.

2. SECTION I

CIRCULAR DUCTS

2.1 Formulation Of The Problem

Consider a vertical straight circular duct of constant cross-section as shown in Fig. 1. The flow is considered to be laminar and fully developed both hydrodynamically and thermally, and is in the vertical upward direction along the positive Z-axis. The thermal boundary condition of uniform heat flux per unit length in the direction of flow is considered. The fluid properties are considered to be constant except for the variation of density in the buoyancy term of the equation of motion. The pressure work term in the energy equation has been neglected.

Under the above mentioned conditions, the differential form of the continuity equation is identically equal to zero. The governing momentum and energy equations can be written as [1]*

$$0 = -\frac{dp}{dZ} + \mu\left(\frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{dv_z}{dr}\right) - \rho g, \qquad (1)$$

$$\rho c_p v_z \frac{\partial T}{\partial Z} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \mu\left(\frac{dv_z}{dr}\right)^2. \qquad (2)$$

For the condition of uniform heat input in the flow direction and constant fluid properties, the axial temperature gradient at the wall and for the fluid are constant and equal. Thus $\frac{\partial T}{\partial Z} = C$, where C is a constant.

In the above equations density is to be considered variable only in the buoyancy term of the momentum equation (1). This assumption is known to be valid as long as the density variations in the flow field are small [9]. Under this condition the equation of state in the linear form can be written as,

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* For details see Appendix C

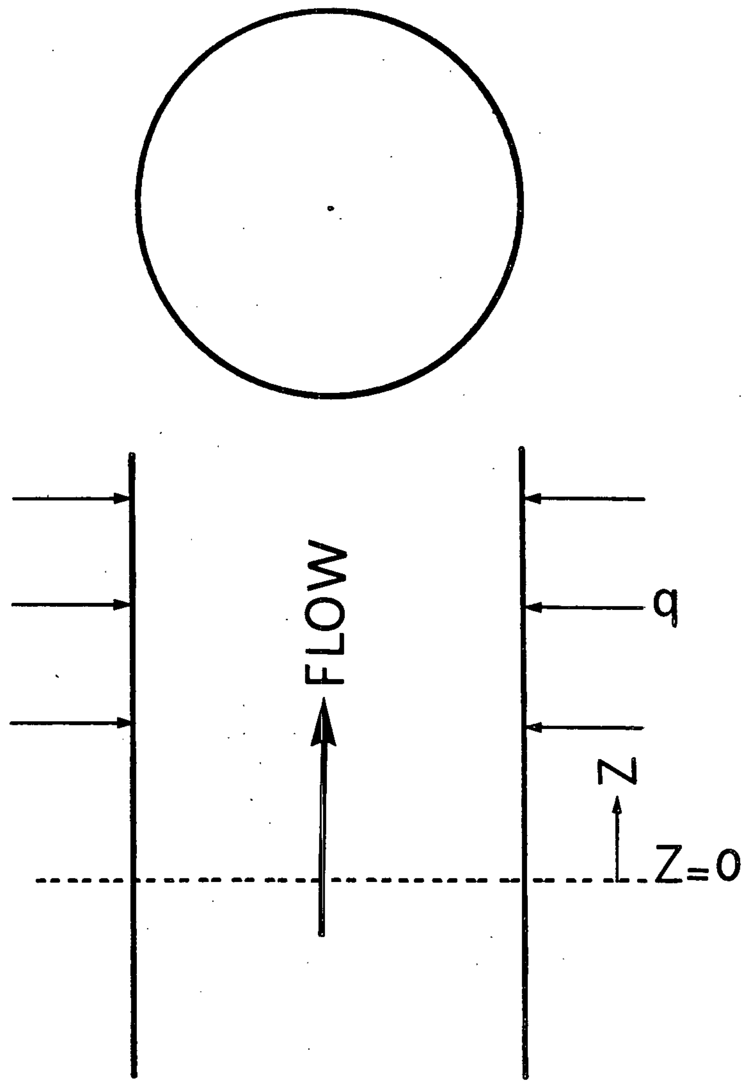

FIGURE 1 Coordinate System for Flow Through a Vertical Circular Duct

$$\rho = \rho_w \left[1 - \beta \left(T - T_w \right) \right] , \qquad (3)$$

where ρ_w denotes the density of the fluid at the corresponding axial point on the duct wall. The wall temperature is defined by,

$$T_w = T_o + Z \frac{\partial T}{\partial Z} ,$$

where T_o is the reference temperature at $Z = 0$.

By choosing the following non-dimensional parameters,

$$R = 2r/D_h , \quad V = v_z/U ,$$
$$\phi = (T - T_w) / (\rho U c_p C D_h^2 / 4k) ,$$

and inserting equation (3) in equation (1), the following non-dimensional forms of the momentum and energy equations are obtained,

$$\nabla^2 V + Ra\,\phi + L = 0 , \qquad (4)$$

$$\nabla^2 \phi - V + 4M \left(\frac{dV}{dR} \right)^2 = 0 , \qquad (5)$$

where $$\nabla^2 = \frac{d^2}{dR^2} + \frac{1}{R} \frac{d}{dR} .$$

In equations (4) and (5), Rayleigh number Ra and the viscous dissipation parameter M are prescribed quantities while V, ϕ and L are the three unknown quantities to be determined.

From the principle of continuity, for constant fluid properties, the integral form of the continuity equation can be written as,

$$\iint v_z \, dA = \iint U \, dA ,$$

or $$\iint V\,dA = \iint dA \;. \tag{6}$$

In the present analysis for the case of circular duct, equations (4), (5) and (6) have been solved for the following boundary conditions:

Boundary Conditions

$$\text{At } R = 1,\; V = \phi = 0 \;. \tag{7}$$

In order to compare the results with viscous dissipation effects to those without it, the available solution for the latter case [7] is first presented here briefly.

3. SOLUTIONS

3.1 Exact Solution Without Viscous Dissipation Term

When the viscous dissipation term is neglected from the energy equation (2), the problem does not remain non-linear any more, and an exact solution is available [7]. This exact solution in the form of Kelvin functions is presented in a more simplified manner below.

By neglecting the viscous dissipation term, equations (4) and (5) can be rewritten as,

$$\nabla^2 V + Ra\,\phi + L = 0\,, \tag{4}$$

$$\nabla^2 \phi - V = 0\,. \tag{8}$$

Since the pressure drop parameter L is independent of the coordinate system, equations (4) and (8) can be divided by L to give the following equations:

$$\nabla^2 \bar{V} + Ra\,\bar{\Phi} + 1 = 0\,, \tag{9}$$

$$\nabla^2 \bar{\Phi} - \bar{V} = 0\,, \tag{10}$$

where $\bar{V} = V/L\,,\quad \bar{\Phi} = \phi/L\,.$

Equations (9) and (10) can be combined together to give,

$$\nabla^4 \bar{V} + \eta^4 \bar{V} = 0\,, \tag{11}$$

A general solution of equation (11) can be written [13] as,

$$\bar{V} = A_1 ber_0(\eta R) + A_2 bei_0(\eta R) + A_3 ker_0(\eta R) + A_4 kei_0(\eta R) \,. \tag{12}$$

The non-dimensional temperature function can be obtained from equation (9) as,

$$\bar{\Phi} = -\frac{1}{Ra}\left[1 + \nabla^2 \bar{V}\right] , \tag{13}$$

where $\nabla^2 \bar{V} = \eta^2 \left[-A_1 bei_0(\eta R) + A_2 ber_0(\eta R) - A_3 kei_0(\eta R) + A_4 ker_0(\eta R)\right].$

In the present case of flow through a circular duct, the ker and kei terms drop out from equations (12) and (13). The remaining constants A_1 and A_2 are obtained by applying the boundary conditions $\bar{V} = \bar{\phi} = 0$ at the wall. Once $\bar{V}$ is known, the pressure drop parameter L is obtained from the continuity equation,

$$L = \frac{\iint dA}{\iint \bar{V} dA} \,.$$

The non-dimensional velocity and temperature functions are then determined from,

$$V = \bar{V} . L,$$

$$\phi = \bar{\phi} . L.$$

Having obtained the velocity and temperature functions, the Nusselt numbers can be evaluated from the following expression, M being zero for this case.

Nusselt Number*

$$Nu = \frac{\left[-1 + 8M\int_0^1 \left(\frac{dV}{dR}\right)^2 R\,dR\right]}{\int_0^1 \phi V R\,dR \Big/ \int_0^1 V R\,dR} \qquad (14)$$

3.2 Solutions With Viscous Dissipation Term

Now we will deal with the methods of solution of the problem when the viscous dissipation term is included in the energy equation. Since the problem is non-linear, an exact solution does not seem possible at present. Therefore the solution for the present problem was obtained by three approximate but fairly accurate methods. The three methods used were,

1. Power Series Method
2. Galerkin's Method
3. Numerical Integration Method

3.2.1 Power Series Method

In the theory of bending of circular plates with large deflection, equations somewhat similar to equation (5) occur and Way [29] has used the power series method to solve such a problem. The essence of this method is that an infinite series is assumed

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* For details see Appendix A

for the function, and after substituting this series expression in the differential equation, the unknown coefficients are lumped together in the form of a recursion expression. Now assigning a numerical value to the first coefficient, all the remaining coefficients of the series can be determined from this recursion expression. The values of these coefficients are then improved upon by iteration to satisfy the boundary conditions.

The above method was used to obtain solutions for V and ϕ. Since V and ϕ are symmetrical functions of R, they can be expanded in series of even powers of R.

Let the dimensionless velocity and temperature functions V and ϕ be expressed in the form of infinite power series with unknown coefficients as,

$$V = C_o + C_1 R^2 + C_2 R^4 + C_3 R^6 + \cdots\cdots, \tag{15}$$

$$\phi = D_o + D_1 R^2 + D_2 R^4 + D_3 R^6 + \cdots\cdots, \tag{16}$$

where C_o, C_1, C_2 .... C_n and D_o, D_1, D_2 .... D_n are the unknown coefficients.

Substituting the power series expressions (15) and (16) in equation (4) and performing the required differentiations the following expression is obtained,

$$\left(4C_1 + 16C_2R^2 + 36C_3R^4 + 64C_4R^6 + \cdots\cdots\right) + Ra\left(D_0 + D_1R^2 + D_2R^4 + D_3R^6 + \cdots\cdots\right) + L = 0. \quad (17)$$

Now equating the coefficients of terms of like powers of R, the following expressions result,

$$4C_1 + RaD_0 + L = 0 \quad \text{for } R^0, \quad (18)$$

$$16C_2 + RaD_1 = 0 \quad \text{for } R^2, \quad (19)$$

$$36C_3 + RaD_2 = 0 \quad \text{for } R^4, \quad (20)$$

$$64C_4 + RaD_3 = 0 \quad \text{for } R^6, \quad (21)$$

$$\vdots$$

From the above expressions it can be seen that except for the coefficients of R^0, the coefficients of the remaining powers of R can be written as,

$$4n^2C_n + RaD_{n-1} = 0 \quad \text{for } n = 2,3,4 \cdots\cdots \infty. \quad (22)$$

Now substituting the power series expressions (15) and (16) in the energy equation (5) and performing the required differentiations the following expression is obtained,

$$\left(4D_1 + 16D_2R^2 + 36D_3R^4 + 64D_4R^6 + \cdots\cdots\right)$$
$$-\left(C_0 + C_1R^2 + C_2R^4 + C_3R^6 + \cdots\cdots\right)$$
$$+ 4M\left(2C_1R + 4C_2R^3 + 6C_3R^5 + \cdots\cdots\right)^2 = 0. \qquad (23)$$

In equation (23), the last term within the parenthesis can be written as,

$$\left(2C_1R + 4C_2R^3 + 6C_3R^5 + \cdots\cdots\right)^2$$
$$= \sum_{n=1}^{\infty}\left(2n\,C_n\,R^{2n-1}\right)^2$$
$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} 2n\cdot 2k\cdot C_n\,C_k\cdot R^{2n-1}\cdot R^{2k-1}$$
$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} 4\,n\,k\,C_n\,C_k\,R^{(2n+2k-2)} \qquad (24)$$

Let $2n + 2k - 2 = 2S$

Therefore, $n + k - 1 = S$

or $n = S + 1 - k$

Since $n \geqslant 1$, therefore, $k \leqslant S$.

Thus expression (24) becomes

$$\sum_{S=1}^{\infty}\left[\sum_{k=1}^{\infty} 4k\,(S+1-k)\,C_{S+1-k}\,C_k\right]R^{2S}. \qquad (25)$$

Now substituting expression (25) in equation (23), the following expression is obtained,

$$\left(4D_1 + 16D_2R^2 + 36D_3R^4 + 64D^4R^6 + \cdots\cdots\right)$$
$$-\left(C_0 + C_1R^2 + C_2R^4 + C_3R^6 + \cdots\cdots\right)$$
$$+4M\sum_{S=1}^{\infty}\left[\sum_{k=1}^{\infty} 4k(s+1-k)C_{S+1-K}\,C_K\right]R^{2S} = 0. \quad (26)$$

The coefficients of terms of like powers of R are now equated to give the following set of equations,

$$4D_1 - C_0 = 0 \qquad \text{for } R^0, \quad (27)$$

$$16D_2 - C_1 + 4M\sum_{k=1}^{S=1} 4k(s+1-k)C_{s+1-k}C_k = 0 \quad \text{for } R^2,$$

or $$4(S+1)^2D_{S+1} - C_S + 4M\sum_{K=1}^{S} 4K(S+1-K)C_{S+1-K}C_K = 0 \quad (28)$$

for $S = 1, 2, 3 \cdots\cdots \infty$
$K \leq S.$

Collecting equations (18), (22), (27) and (28) together, we have,

$$4C_1 + Ra\,D_0 + L = 0, \quad (18)$$

$$4n^2C_n + Ra\,D_{n-1} = 0 \quad \text{for } n = 2,3,4\cdots\infty, \quad (22)$$

$$4D_1 - C_0 = 0 , \tag{27}$$

$$4(S+1)^2 D_{S+1} - C_S + 4M \sum_{k=1}^{S} 4k(S+1-k) C_{S+1-k} C_k = 0 \tag{28}$$

for $S = 1, 2, 3 \ldots\ldots \infty$

$K \leq S.$

From equations (18), (22), (27) and (28), it can be seen that knowing the values of C_0, D_0 and L, all the successive coefficients C_n and D_n can be calculated for any prescribed values of Rayleigh number Ra and viscous dissipation parameter M.

Applying the boundary conditions (7) on equations (15) and (16), the following expressions are obtained,

At R = 1,

$$\sum_{n=0}^{\infty} C_n = 0 , \tag{29}$$

$$\sum_{n=0}^{\infty} D_n = 0 . \tag{30}$$

Substituting the power series expression (15) in the integral form of the continuity equation (6) and performing the required integration, the following expression is obtained,

$$C_0 + \sum_{n=1}^{\infty} \frac{C_n}{n+1} = 1 . \tag{31}$$

In order to evaluate these coefficients, the initial estimates of C_0 and D_0 were made from the results of the exact solution as C_0 and D_0 are the velocity and temperature difference at the centre of the duct. These values were then improved by iteration so that the coefficients obtained from equations (18), (22), (27) and (28) satisfy the boundary conditions (29) and (30) and equation (31).

Determination of the required coefficients gives the solution for the velocity and temperature field. Knowing the velocity and temperature functions, Nusselt numbers were then evaluated from equation (14).

3.2.2 Galerkin's Method

The second method used for the solution of the problem is the Galerkin's Method [2, 10]. By this method an approximate solution of a differential equation can be obtained by choosing an expression with a certain system of functions for the unknown quantity satisfying the boundary conditions and using the optimization technique, the resulting equations are solved simultaneously to determine the unknown coefficients of the expression.

Let the dimensionless velocity and temperature functions be expressed as,

$$V = (1-R^2)(C_0 + C_1R^2 + C_2R^4), \tag{32}$$

$$\Phi = (1-R^2)(D_0 + D_1R^2 + D_2R^4), \tag{33}$$

where C_0, C_1, C_2 and D_0, D_1, D_2 are the unknown coefficients. The factor $(1-R^2)$ in expressions (32) and (33) ensures satisfaction of the boundary conditions (7).

Expressions (32) and (33) are not the exact solutions for V and ϕ and substituting these expressions in equations (4) and (5), we obtain the following expressions which are a measure of the accuracy of the approximations,

$$Y_1 = 4C_1 + 16C_2R^2 - 4C_0 - 16C_1R^2 - 36C_2R^4 + Ra\,(D_0 + D_1R^2 + D_2R^4 - D_0R^2 - D_1R^4 - D_2R^6) + L\,, \tag{34}$$

$$Y_2 = 4D_1 + 16D_2R^2 - 4D_0 - 16D_1R^2 - 36D_2R^4 - (C_0 + C_1R^2 + C_2R^4 - C_0R^2 - C_1R^4 - C_2R^6) + 4M\,(2C_1R + 4C_2R^3 - 2C_0R - 4C_1R^3 - 6C_2R^5)^2. \tag{35}$$

If expressions (32) and (33) were exact solutions for V and ϕ respectively, then Y_1 and Y_2 would be identically equal to zero.

Now multiplying Y_1 with the first, second and third term respectively of (32), and integrating over the duct cross-section, the following equations are obtained,

$$\int_0^1 Y_1\,(1-R^2)\,R\,dR = 0\,, \tag{36}$$

$$\int_0^1 Y_1 (1-R^2) R^3 dR = 0 \quad , \tag{37}$$

$$\int_0^1 Y_1 (1-R^2) R^5 dR = 0 \quad . \tag{38}$$

Proceeding in a similar manner and using the expression for Y_2 and equation (33) we obtain,

$$\int_0^1 Y_2 (1-R^2) R\, dR = 0 \quad , \tag{39}$$

$$\int_0^1 Y_2 (1-R^2) R^3 dR = 0 \quad , \tag{40}$$

$$\int_0^1 Y_2 (1-R^2) R^5 dR = 0 \quad . \tag{41}$$

After performing the required integrations, the following combination of linear and non-linear algebraic equations are obtained,

$$-C_0 - \frac{C_1}{3} - \frac{C_2}{6} + \frac{L}{4} + \frac{Ra}{6}\left(D_0 + \frac{D_1}{4} + \frac{D_2}{10}\right) = 0 \ , \tag{42}$$

$$-C_0 - C_1 - \frac{7}{10}C_2 + \frac{L}{4} + \frac{Ra}{4}\left(\frac{D_0}{2} + \frac{D_1}{5} + \frac{D_2}{10}\right) = 0, \tag{43}$$

$$-\frac{C_0}{6} - \frac{7}{30}C_1 - \frac{C_2}{5} + \frac{L}{24} + \frac{Ra}{30}\left(\frac{D_0}{2} + \frac{D_1}{4} + \frac{D_2}{7}\right) = 0, \tag{44}$$

$$-\frac{C_0}{6} - \frac{C_1}{24} - \frac{C_2}{60} - D_0 - \frac{D_1}{3} - \frac{D_2}{6} + 4M\left(\frac{C_0^2}{3} + \frac{C_1^2}{15} + \frac{C_2^2}{35} - \frac{C_0C_2}{15} + \frac{C_1C_2}{15}\right) = 0, \tag{45}$$

$$-\frac{C_0}{24} - \frac{C_1}{60} - \frac{C_2}{120} - \frac{D_0}{3} - \frac{D_1}{3} - \frac{7}{30}D_2 + 4M\left(\frac{C_0^2}{6} + \frac{C_1^2}{30} + \frac{C_2^2}{60} + \frac{C_0C_1}{15} + \frac{4}{105}C_1C_2\right) = 0, \tag{46}$$

$$-\frac{C_0}{60} - \frac{C_1}{120} - \frac{C_2}{210} - \frac{D_0}{6} - \frac{7}{30}D_1 - \frac{D_2}{5} + 4M\left(\frac{C_0^2}{10} + \frac{C_1^2}{42} + \frac{C_2^2}{84} + \frac{C_0C_1}{15} + \frac{2}{105}C_0C_2 + \frac{C_1C_2}{35}\right) = 0. \tag{47}$$

In these six equations (42) to (47) there are seven unknowns C_0, C_1, C_2, D_0, D_1, D_2 and L to be determined. Therefore an additional equation is required which is obtained by substituting equation (32) in the continuity equation (6) to give,

$$C_0 + \frac{C_1}{3} + \frac{C_2}{6} = 2 \quad . \tag{48}$$

Equations (42) to (48) are solved simultaneously by Powell's method to obtain the values of the unknown coefficients and the pressure drop parameter L. Once the values of the unknown coefficients are determined they can be substituted in expressions (32) and (33) to give the values of the velocity and temperature functions and Nusselt numbers can then be evaluated.

3.2.3 Numerical Integration

The numerical integration method of Runge-Kutta of order four was used to obtain the solutions for the governing differential equations (4) and (5). The method requires the complete set of functional values V, ϕ and their gradients at the starting boundary point and the estimates of the missing initial boundary conditions were made from the exact solution results. The resulting solutions were then improved by iteration to obtain the desired solutions satisfying the boundary conditions (7) and the continuity equation (6) simultaneously.

The error involved in the fourth order R-K method is of the order of h^5 where h is the step size. A step size of 0.01 was taken in this case and the solutions obtained were checked by reducing the step size to 0.001. It was seen that the two solutions did not differ up to six significant figures.

This completes the methods of solutions for the circular duct. In the next section, the formulation and solutions for the concentric annulus are presented.

4. SECTION II

CONCENTRIC ANNULI

4.1 Formulation Of The Problem

Consider the fully developed laminar flow of a fluid in the vertical upward direction through the annular passage as shown in Figure 2.

The assumptions made in the formulation of the problem, and the governing equations in the dimensional form for the concentric annulus remain the same as for the circular duct (Section I), and are not repeated here. In addition, the boundary condition of no slip at the walls will still apply. However, the thermal boundary condition will depend on the following three situations,

Case I: Outer wall heated and inner wall perfectly insulated.

Case II: Inner wall heated and outer wall perfectly insulated.

Case III: Both walls heated with equal wall temperatures at a given axial position.

Choosing the following non-dimensional parameters,

$$R = r/r_o \ ,$$

$$V = v_z/U \ ,$$

$$\phi = \frac{(T - T_w)}{(\rho U C_p C D_h^2/4K)}$$

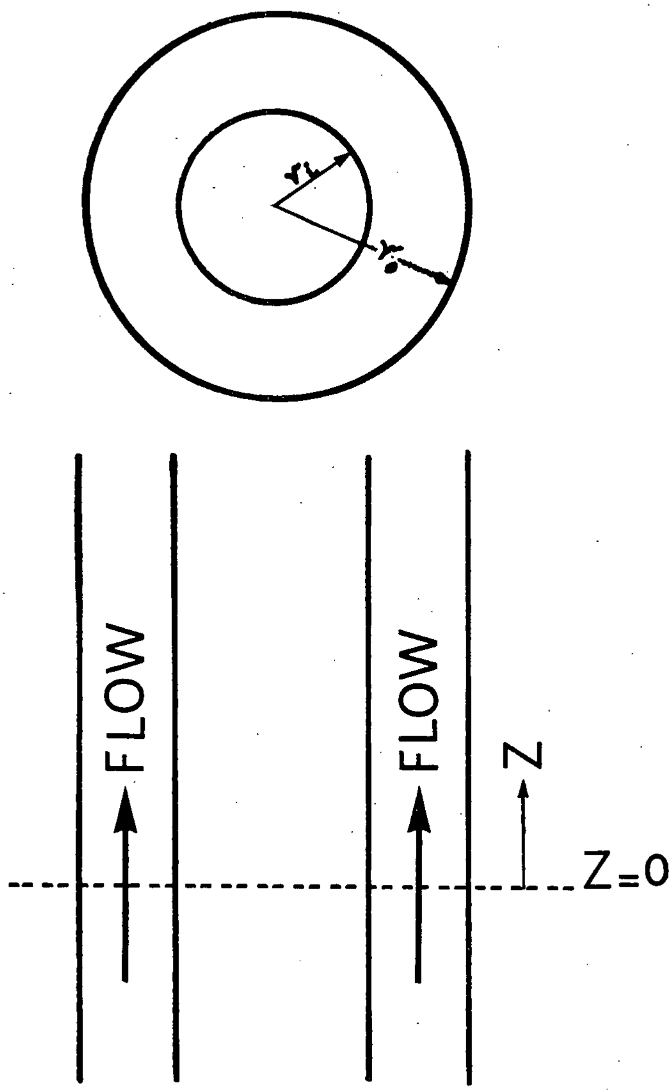

FIGURE 2 Coordinate System for Flow Through a Vertical Concentric Annulus

the non-dimensional form of the governing equations and the boundary conditions for Case I, II and III respectively are as follows:

Case I: Outer Wall Heated, Inner Wall Insulated.

First of all we redefine the equivalent diameter and evaluate it.

Equivalent Diameter

For this case, the equivalent diameter is given by,

$$D_h = \frac{4 \text{ x Area of cross-section}}{\text{Heated perimeter}}$$

$$= 2r_o\left(1 - \lambda^2\right), \tag{49}$$

where r_o = radius of the outer tube

λ = radius of inner tube/radius of outer tube.

Substituting the non-dimensional parameters and equation (49) in the momentum and energy equations (1) and (2) respectively, the following non-dimensional equations are obtained,

$$(1 - \lambda^2)^2 \nabla^2 V + Ra\,\phi + L = 0\,, \tag{50}$$

$$(1-\lambda^2)^2\,\nabla^2\phi - V + 4(1-\lambda^2)^2 M\left(\frac{dV}{dR}\right)^2 = 0. \qquad (51)$$

Equations (50) and (51) along with the continuity equation (6) are to be solved for the following boundary conditions:

Boundary Conditions

$$\text{At } R=\lambda\,,\quad V = \frac{d\phi}{dR} = 0\,, \qquad (52)$$

$$\text{At } R=1\,,\quad V = \phi = 0\,. \qquad (53)$$

Nusselt Number

The Nusselt number is given by*

$$Nu = \frac{\left[-1 + 8(1-\lambda^2)M\int_\lambda^1\left(\frac{dV}{dR}\right)^2 R\,dR\right]}{\int_\lambda^1 \phi V R\,dR \Big/ \int_\lambda^1 V R\,dR} \qquad (54)$$

Case II: Inner Wall Heated, Outer Wall Insulated

Equivalent Diameter

For this case the equivalent diameter is given by,

$$D_h = 2r_0(1-\lambda^2)/\lambda\,. \qquad (55)$$

Using this value for D_h in equations (1) and (2), the non-dimensional momentum and energy equations are obtained as,

$$\frac{(1-\lambda^2)^2}{\lambda^2}\nabla^2 V + Ra\,\phi + L = 0, \qquad (56)$$

$$\frac{(1-\lambda^2)^2}{\lambda^2}\nabla^2\phi - V + \frac{4(1-\lambda^2)^2}{\lambda^2}M\left(\frac{dV}{dR}\right)^2 = 0. \qquad (57)$$

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* For details see Appendix B

For case II equations (56) and (57) along with the continuity equation (6) are to be solved for the following boundary conditions:

Boundary Conditions

$$\text{At } R = \lambda, \quad V = \phi = 0, \tag{58}$$

$$\text{At } R = 1, \quad V = \frac{d\phi}{dR} = 0. \tag{59}$$

Nusselt Number

The Nusselt number expression is given by*

$$Nu = \frac{\left[-1 + \frac{8(1-\lambda^2)}{\lambda^2} M \int_\lambda^1 \left(\frac{dV}{dR}\right)^2 R\,dR\right]}{\int_\lambda^1 \phi V R\,dR \Big/ \int_\lambda^1 V R\,dR}. \tag{60}$$

Case III: Both Walls Heated

Equivalent Diameter

For this case the equivalent diameter is obtained as,

$$D_h = 2r_o(1-\lambda). \tag{61}$$

Using this value for D_h, the non-dimensional momentum and energy equations are obtained as,

$$(1-\lambda)^2 \nabla^2 V + Ra\,\phi + L = 0, \tag{62}$$

$$(1-\lambda)^2 \nabla^2 \phi - V + 4(1-\lambda)^2 M \left(\frac{dV}{dR}\right)^2 = 0. \tag{63}$$

Equations (62) and (63) along with the continuity equation (6) are to be solved for the following boundary conditions:

Boundary Conditions

$$\text{At } R = \lambda, \quad V = \Phi = 0, \tag{64}$$

$$\text{At } R = 1, \quad V = \Phi = 0. \tag{65}$$

Nusselt Number

The Nusselt number expression is obtained as*,

$$Nu = \frac{\left[-1 + 8\left(\frac{1-\lambda}{1+\lambda}\right) M \int_{\lambda}^{1} \left(\frac{dV}{dR}\right)^2 R\,dR\right]}{\int_{\lambda}^{1} \Phi V R\,dR \Big/ \int_{\lambda}^{1} V R\,dR} . \tag{66}$$

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* For details see Appendix B

5. SOLUTIONS

5.1 Exact Solution Without Viscous Dissipation

A general form of the exact solution without viscous dissipation [11] for the concentric annulus in the form of Kelvin functions is presented in a more simplified manner for case I only, the approach for the other two cases being similar.

For case I, outer wall heated, inner wall insulated equations (50) and (51) reduce to,

$$\nabla^2 V + \frac{Ra}{(1-\lambda^2)^2}\phi + \frac{L}{(1-\lambda^2)^2} = 0, \tag{67}$$

$$\nabla^2 \phi - \frac{V}{(1-\lambda^2)^2} = 0. \tag{68}$$

Let $$\frac{1}{(1-\lambda^2)^2} = B. \tag{69}$$

Substituting equation (69) in (67) and (68) and dividing equations (67) and (68) by the pressure drop parameter L we obtain the following equations,

$$\nabla^2 \bar{V} + BRa\,\bar{\phi} + B = 0, \tag{70}$$

$$\nabla^2 \bar{\phi} - B\bar{V} = 0, \tag{71}$$

where $$\bar{V} = V/L, \quad \bar{\phi} = \phi/L.$$

Combining (70) and (71) we obtain,

$$\nabla^4 \bar{V} + \eta^4 \bar{V} = 0, \tag{72}$$

where $$\eta^4 = B^2 Ra \tag{73}$$

Equation (72) is identical to (11) and the solution is given by (12) in Section I and is repeated here.

$$\overline{V} = C_1 \,\mathrm{ber}_0(\eta R) + C_2 \,\mathrm{bei}_0(\eta R) + C_3 \,\mathrm{ker}_0(\eta R) + C_4 \,\mathrm{kei}_0(\eta R) . \quad (74)$$

from equation (70) we obtain,

$$\overline{\Phi} = \frac{-1}{BRa}\left[B + \nabla^2 \overline{V}\right], \quad (75)$$

where

$$\nabla^2 \overline{V} = \eta^2\left[-C_1 \,\mathrm{bei}_0(\eta R) + C_2 \,\mathrm{ber}_0(\eta R) - C_3 \,\mathrm{kei}_0(\eta R) + C_4 \,\mathrm{ker}_0(\eta R)\right] . \quad (76)$$

The unknowns C_1, C_2, C_3 and C_4 in (74) and (75) are obtained by applying the boundary conditions (52) and (53). This results in the following four equations:

$$0 = C_1 \,\mathrm{ber}_0(\eta\lambda) + C_2 \,\mathrm{bei}_0(\eta\lambda) + C_3 \,\mathrm{ker}_0(\eta\lambda) + C_4 \,\mathrm{kei}_0(\eta\lambda) , \quad (77)$$

$$0 = -\frac{\eta^3}{BRa}\left[-C_1 \,\mathrm{bei}_0'(\eta\lambda) + C_2 \,\mathrm{ber}_0'(\eta\lambda) - C_3 \,\mathrm{kei}_0'(\eta\lambda) + C_4 \,\mathrm{ker}_0'(\eta\lambda)\right], \quad (78)$$

$$0 = C_1 \,\mathrm{ber}_0(\eta) + C_2 \,\mathrm{bei}_0(\eta) + C_3 \,\mathrm{ker}_0(\eta) + C_4 \,\mathrm{kei}_0(\eta) , \quad (79)$$

$$0 = -\frac{1}{Ra} - \frac{\eta^2}{BRa}\left[-C_1 \,\mathrm{bei}_0(\eta) + C_2 \,\mathrm{ber}_0(\eta) - C_3 \,\mathrm{kei}_0(\eta) + C_4 \,\mathrm{ker}_0(\eta)\right] . \quad (80)$$

Equations (77), (78), (79) and (80) are solved simultaneously to determine the values of the unknown coefficients C_1, C_2, C_3 and C_4.

Thus $\overline{V}$ and $\overline{\phi}$ can be evaluated and the pressure drop parameter L is obtained from the continuity equation, The non-dimensional velocity and temperature functions are then determined from

$$V = \bar{V} \cdot L \ , \quad \Phi = \bar{\Phi} \cdot L \ ,$$

and Nusselt numbers can be evaluated from equation (54).

The solutions for case II, inner wall heated and outer wall insulated and for case III, both walls heated were obtained in a similar manner.

5.2 Solutions With Viscous Dissipation Term

Now we will deal with the problem taking into account the viscous dissipation term in the energy equation. A power series method similar to that for circular duct was attempted without success. It appears that Galerkin's method could be applied for case III where it is easier to set up the temperature function to satisfy the wall conditions. However, for cases I and II where one of the walls is insulated, it is difficult to set up suitable expressions for the temperature function. Thus the numerical integration method of Runge-Kutta of order four was used to obtain the solutions. The general procedure for the Runge-Kutta method is given in section (3.2.3). The step size taken for this problem was h = 0.01 $(1 - \lambda)$ where h is the step size and λ is the radius ratio (r_i/r_o). Solutions were also obtained by reducing the step size but no difference was observed in the solutions up to six significant figures.

6. DISCUSSION OF RESULTS

The effects of viscous dissipation on the flow phenomena and heat transfer rate as studied from the results obtained are discussed under two sections, (i) Circular ducts and (ii) Concentric annuli.

6.1 Circular Ducts

For the circular ducts, we will first discuss briefly the solution details and then present the results for the velocity and the temperature fields and the Nusselt numbers.

6.1.1 Solution Details

All calculations were made on an IBM digital computer. For the exact solution without viscous dissipation effects (M=0), the Kelvin function terms ber and bei were evaluated in Double Precision Arithmetic giving an accuracy up to fourteen significant figures. These functions were evaluated from the expressions in the form of infinite series given in McLachlan [13]. In the evaluation of the functions, the convergence was very rapid for the value of the argument up to eight.

In the power series method, seven sets of initial estimates very close to the values obtained from the exact solution results were used and employing the minimizing and iteration procedure the final coefficients were obtained. The coefficients of the series were very fast converging and the maximum number of terms in the series to be calculated did not exceed more than thirty five.

The Galerkin's method involved the solution of simultaneous non-linear algebraic equations and close enough initial guesses of the solution were very essential for rapid convergence. These 'educated' guesses were estimated from the results obtained with the exact solution for M=0.

In the Runge-Kutta's fourth order method, estimates of the initial guesses of the missing boundary conditions and the pressure drop parameter L were made from the results of the exact solution, and were then iterated upon to obtain the desired solution satisfying the boundary conditions at the end point. The error involved in the Runge-Kutta's fourth order method is of the order of h^5 where h is the step size for integration. Results for the present case were obtained by taking a step size of 0.01 and it was noted that the reduction in step size to 0.001 did not alter the solution up to six significant figures.

Coming to the accuracy of the methods used, a first check on the accuracy was carried out by calculating results for M=0 (no viscous dissipation effects) by the three methods and comparing them with the exact solution results as shown in Tables 1 and 2. Table 1 is for velocity and temperature functions and Table 2 shows the Nusselt number values. From these tables it can be seen that the results obtained by the three methods are in good agreement with the exact solution. These tables also show that for non-zero finite values of the dissipation parameter M, the agreement between the three methods is very good.

In upflow heating of a fluid the effect of free convection is to accelerate the velocity near the wall [12]. To satisfy continuity, the velocity near the tube centre is reduced. If the buoyancy rate is increased sufficiently, then it is theoretically possible to create flow reversal at the centre of the duct. However, it is known [7, 21] that just before negative velocity could occur, the flow becomes unstable and eventually turbulent. We, therefore, need to limit our attention only up to that value of Rayleigh number which creates flow reversal. Rayleigh number as defined in the nomenclature for the present study should not exceed 625 to maintain laminar flow. Thus for the present analysis the maximum value of Rayleigh number used was 625 for the case of circular duct.

The viscous dissipation parameter is defined as M=Eckert number/Reynolds number. The maximum value of this parameter used in the present analysis was 5×10^{-4}.

Now the effect of viscous dissipation on the velocity and temperature fields and the Nusselt numbers will be discussed.

6.1.2 Velocity Field

For the case of pure forced convection (Ra=0), the velocity field is independent of the temperature field and hence viscous dissipation has no effect on the velocity field. However, for the case of combined free and forced convection, the momentum and energy equations (4) and (5) respectively are coupled and hence viscous dissipation not only affects the temperature field but also the velocity field. The measure of free convection is the non-dimensional parameter Rayleigh number and as Rayleigh number increases the coupling becomes more and more strong and hence the dissipation effect becomes more pronounced.

From the results obtained it is seen that the effect of viscous dissipation on the velocity field is to reduce the flow velocity near the duct walls and consequently increase it near the centre. Table 1 shows the increase in velocity at the centre of the duct under the influence of viscous dissipation for various values of Rayleigh number. This trend becomes more pronounced with the increase in Rayleigh number. As stated, the reduction in the velocity near the duct walls has been observed, however, this data is not presented here for brevity. From Hallman's [7] investigation it is known that for the case of upflow heating, the effect of free convection on the velocity field is to increase the flow velocity near the duct walls and to reduce it near the centre. Thus viscous dissipation acts contrary to the free convection (buoyancy) effect on the flow field. From this it therefore follows, that the effect of free convection is to increase the shear stress at the wall whereas the effect of viscous dissipation is to reduce the same.

6.1.3 Temperature Field

The effect of viscous dissipation is to convert frictional energy into heat and hence it reduces the temperature differences in the system when the transfer of heat takes place from the surface to the fluid. Table 1 shows the temperature differences at the centre of the duct for various values of Rayleigh number taking into account viscous dissipation effect, and it can be seen that the temperature differences are reduced. This trend is also observed at all points along the tube radius, however, this data is not presented here.

6.1.4 Nusselt Numbers

One of the main parameters of engineering interest is the Nusselt number which is a measure of the heat transfer rate, and the effect of viscous dissipation on Nusselt number is an important aspect of the present analysis. As mentioned earlier, due to the conversion of frictional energy into heat the impressed external heating is opposed and the heat transfer rate is reduced. Consequently because of viscous dissipation effect, lower Nusselt number values are obtained. Table 2 shows the values of Nusselt numbers for different Rayleigh numbers taking into account viscous dissipation effects. From this table it can be seen that the Nusselt number values are reduced and the reduction becomes more pronounced at higher Rayleigh numbers.

6.2 Concentric Annuli

Now we will discuss the solutions obtained and the viscous dissipation effects for the three cases of the annular flow.

6.2.1 Solution Details

The exact solutions with M=0 for the concentric annulus also involved the derivatives of Kelvin functions because of the thermal boundary condition of one wall being insulated. These functions were evaluated in Double Precision from McLachlan [13].

The number of terms required for convergence was of the order of 20. The non-linear problem (M>0) was solved by Runge-Kutta fourth order method in Double Precision and the accuracy of R-K method was judged by obtaining results for M=0 and comparing them with the exact solution results. Table 3 shows the Nusselt number values as obtained by the exact solution and Runge-Kutta method for different values of Rayleigh number. From this table it can be seen that the results obtained by the two methods are in good agreement. A further check on the accuracy was made by comparing the results obtained by the two methods for Ra=1 which approximates to forced convection flow with the results of Cheng [4] since no results seem to be available in published literature for combined free and forced convection through annular passages. These results were also found to be in very close agreement.

Now we will discuss the velocity and temperature fields and the effect of viscous dissipation, for the three cases studied.

6.2.2 Velocity Field

First of all we will discuss the velocity profiles for M=0 as shown in figures 3 to 8.

Figure 3 shows the velocity profiles for the case of outer wall heated, inner wall insulated (case I) for λ=0.25. From this figure it can be seen that as Rayleigh number increases, the velocity gradients near the outer wall (heated wall) increase. This increase in velocity near the outer wall reduces the same near the inner wall and eventually flow reversal takes place at Ra≃2000. In Figure 4 are shown the velocity profiles for λ=0.5.

A similar trend is observed here by increasing Ra with flow reversal taking place now at Ra≈4000.

The velocity profiles for the case of inner wall heated, outer wall insulated (case II) are shown in Figure 5 for λ=0.25. From this figure it can be seen that by increasing Rayleigh number, the velocity gradients near the inner wall (heated wall) are increased with flow reversal occuring at Ra≈28×10^4. Figure 6 shows the velocity profiles for λ=0.5 and for this case flow reversal occurs at Ra≈5×10^4.

For the case of both walls heated (case III), the velocity profiles for λ=0.25 are shown in Figure 7. This figure shows that as Ra increases, the velocity gradients near both the walls increase. This increase in velocity near both the walls reduces the same near the central region and eventually a reversal of flow occurs at Ra≈6500. Figure 8 shows the velocity profiles for λ=0.5. The same effect of Rayleigh number is observed here on the velocity field with flow reversal now occuring at Ra≈7000.

6.2.3 Temperature Field

Now we will discuss the temperature profiles for M=0. Figures 9 and 10 show the temperature profiles for outer wall heated, inner wall insulated for λ=0.25 and 0.5 respectively. From these figures it can be seen that the temperature differences are reduced by increasing the Rayleigh number.

Figures 11 and 12 show the temperature profiles for the case of inner wall heated and outer wall insulated with λ=0.25 and 0.5 respectively. For this case also it can be seen that with the increase in Ra, the temperature differences are reduced.

In Figures 13 and 14 are shown the temperature profiles for the case of both walls heated for λ=0.25 and 0.5 respectively. For this case too, the temperature differences are reduced with increasing Rayleigh number.

Since the effects of viscous dissipation on the velocity and temperature field is found to be very small, it is not convenient to present the results graphically and, therefore, a general trend is represented by the following tables.

Tables 4 and 5 show the effect of viscous dissipation on the velocity and temperature fields for the case of outer wall heated, inner wall insulated with λ=0.75 for Ra=1 and 1000 respectively. Table 4 for Ra=1 is almost a pure forced convection case and it can be seen that there is no significant effect of viscous dissipation on the velocity field as the velocity field is independent of the temperature field. However, it can be seen that the temperature differences are reduced. Table 5 shows that as Ra has increased, the effect of viscous dissipation on the velocity field becomes more pronounced. The velocity near the outer wall (heated wall) is reduced while it is increased near the inner wall. The temperature differences are reduced throughout.

Table 6 and 7 show the dissipation effects for inner wall heated and outer wall insulated. From Table 6 it can be seen that for Ra=1, dissipation parameter M has no effect on the velocity field but the temperature differences are reduced. As Ra increases, it can be seen from Table 7 for Ra=1000, that viscous dissipation reduces the flow velocity near the inner wall (heated wall). The temperature differences are also reduced with increasing M.

For the case of both walls heated the effect of M is shown in Tables 8 and 9. From Table 8 for Ra=1, it can be seen that there is no significant effect of M on the velocity field though the temperature differences are reduced. Table 9 for Ra=2000 shows that for higher values of Ra, viscous dissipation reduces the flow velocity near both the walls and the temperature differences.

6.2.4 Nusselt Number

As mentioned earlier viscous dissipation opposes the impressed external heating and reduces the heat transfer rate resulting in lower values of Nusselt numbers. Figure 15 shows the effect of viscous dissipation on Nusselt numbers for outer wall heated, inner wall insulated with λ=0.25 and 0.5. From this figure it can be seen that Nusselt numbers decrease with increase in the dissipation parameter M. The reduction in Nusselt numbers becomes more pronounced at higher Rayleigh numbers.

Figure 16 shows the effect of M on Nusselt numbers for inner wall heated, outer wall insulated. For this case too, it can be seen that lower values of Nusselt numbers are obtained when viscous dissipation is taken into account.

The effect of M on Nusselt numbers for the case of both walls heated is shown in Figure 17. As anticipated the Nusselt numbers are again reduced with increasing M and this reduction becomes more pronounced at higher Rayleigh numbers.

6.2.5 Radius Ratio*

The effect of radius ratio λ on the Nusselt numbers can be seen from Figures 15, 16 and 17. Figures 15 and 17 show that for outer wall heated and inner wall insulated or for both walls heated, high values of Nusselt numbers are obtained by increasing λ whereas from Figure 16 it can be seen that for inner wall heated, outer wall insulated the Nusselt number values are reduced.

A comparison of the reduction in Nusselt numbers for the same value of the dissipation parameter M has also been studied. It is found that the maximum reduction occurs for the case of inner wall heated, outer wall insulated and the minimum reduction occurs for the case of both walls heated.

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* For details see Appendix C

7. CONCLUSIONS

The effects of viscous dissipation on the flow phenomena and heat transfer rate for combined free and forced convection through vertical circular ducts and concentric annuli has been studied.

From the results obtained it is concluded that the effects of viscous dissipation on the flow field is to reduce the velocity near the heated wall(s) thereby counteracting the effect of free convection on the velocity field in upflow when the transfer of heat takes place from the surface to the fluid. Thus it follows that due to viscous dissipation effects, the shear stress at the wall(s) is reduced. Viscous dissipation reduces the temperature differences in the system and hence the effect of buoyancy is decreased. The dissipation of frictional energy into heat reduces the heat transfer rate when heat is transferred from the surface to the fluid and results in lower Nusselt number values.

TABLE I

Velocities and Temperature Differences at the Centre of a Vertical Circular Duct due to Viscous Dissipation Effects.

		Exact Solution		Power Series Method		Galerkin's Method		Runge-Kutta Method	
Rayleigh Number Ra	Viscous Dissipation Parameter M	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ
1	0	1.9913	-0.3742	1.9912	-0.3742	1.9913	-0.3742	1.9924	-0.3744
	0.0001			1.9913	-0.3739	1.9913	-0.3739	1.9924	-0.3740
	0.0005			1.9913	-0.3723	1.9913	-0.3723	1.9924	-0.3725
10	0	1.9152	-0.3681	1.9152	-0.3681	1.9152	-0.3681	1.9163	-0.3682
	0.0001			1.9153	-0.3677	1.9153	-0.3677	1.9164	-0.3679
	0.0005			1.9155	-0.3663	1.9154	-0.3663	1.9165	-0.3665
50	0	1.6139	-0.3432	1.6139	-0.3432	1.6131	-0.3433	1.6149	-0.3434
	0.0001			1.6139	-0.3432	1.6132	-0.3430	1.6150	-0.3432
	0.0005			1.6143	-0.3420	1.6135	-0.3420	1.6154	-0.3422
100	0	1.3061	-0.3173	1.3061	-0.3173	1.3035	-0.3173	1.3069	-0.3175
	0.0001			1.3061	-0.3173	1.3035	-0.3171	1.3070	-0.3173
	0.0005			1.3065	-0.3163	1.3038	-0.3163	1.3073	-0.3165
500	0	0.1564	-0.2091	0.1564	-0.2091	0.1346	-0.2079	0.1563	-0.2093
	0.0001			0.1583	-0.2087	0.1370	-0.2073	0.1583	-0.2089
	0.0005			0.1660	-0.2069	0.1465	-0.2052	0.1660	-0.2071
625	0	0.0123	-0.1921	0.0123	-0.1921	0.0123	-0.1904	0.0121	-0.1923
	0.0001			0.0149	-0.1915	-0.0090	-0.1898	0.0146	-0.1918
	0.0005			0.0248	-0.1895	0.0035	-0.1873	0.0246	-0.1897

TABLE II

Effect of Viscous Dissipation Parameter on Nusselt Numbers for a Vertical Circular Duct

		Nusselt Number Nu			
Rayleigh Number Ra	Viscous Dissipation Parameter M	Exact Solution	Power Series Method	Galerkin's Method	Runge-Kutta Method
1	0	4.3743	4.3734	4.3742	4.3713
	0.0001		4.3653	4.3665	4.3633
	0.0005		4.3329	4.3354	4.3308
10	0	4.4688	4.4679	4.4689	4.4658
	0.0001		4.4591	4.4604	4.4568
	0.0005		4.4237	4.4267	4.4214
50	0	4.8735	4.8721	4.8735	4.8694
	0.0001		4.8542	4.8619	4.8572
	0.0005		4.8105	4.8156	4.8079
100	0	5.3429	5.3407	5.3428	5.3375
	0.0001		5.3181	5.3270	5.3204
	0.0005		5.2552	5.2633	5.2519
500	0	7.9516	7.9445	7.9518	7.9369
	0.0001		7.8782	7.8925	7.8705
	0.0005		7.6122	7.6538	7.6040
625	0	8.4911	8.4827	8.4934	8.4739
	0.0001		8.3998	8.4196	8.3908
	0.0005		8.0665	8.1228	8.0579

TABLE III

Nusselt Number Values for M=0
Obtained by Exact Solution and Runge-Kutta
Method for Concentric Annulus with Radius Ratio 0.5

	Nusselt Number Nu					
Rayleigh Number	Case I: Outer Wall Heated, Inner Wall Insulated.		Case II: Inner Wall Heated, Outer Wall Insulated.		Case III: Both Walls Heated.	
Ra	Exact Solution	Runge-Kutta Method	Exact Solution	Runge-Kutta Method	Exact Solution	Runge-Kutta Method
1	7.556	7.557	18.546	18.545	8.117	8.117
500	8.923	8.927	18.747	18.750	9.318	9.334
1000	10.078	10.086	18.949	18.953	10.362	10.396

TABLE IV

Velocity Distribution and Temperature Differences due to Viscous Dissipation Effects for Concentric Annulus with Outer Wall Heated, Inner Wall Insulated for Ra=1, λ=0.75

	Dissipation Parameter					
	M = 0.0		M = 0.0003		M = 0.0005	
Radius R	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ
0.75	0.0	-0.1517	0.0	-0.1451	0.0	-0.1406
0.77	0.5625	-0.1514	0.5625	-0.1450	0.5625	-0.1406
0.79	0.9893	-0.1493	0.9894	-0.1434	0.9893	-0.1394
0.82	1.2852	-0.1442	1.2852	-0.1388	1.2852	-0.1352
0.84	1.4543	-0.1350	1.4543	-0.1304	1.4542	-0.1273
0.87	1.4873	-0.1215	1.4873	-0.1175	1.4872	-0.1149
0.89	1.4271	-0.1036	1.4271	-0.1003	1.4270	-0.0981
0.92	1.2376	-0.0816	1.2376	-0.0790	1.2375	-0.0772
0.94	0.9348	-0.0563	0.9348	-0.0543	0.9348	-0.0530
0.97	0.5214	-0.0286	0.5214	-0.0275	0.5214	-0.0268
1.0	0.0	0.0	0.0	0.0	0.0	0.0

TABLE V

Velocity Distribution and Temperature Differences due to Viscous Dissipation Effects for Concentric Annulus with Outer Wall Heated, Inner Wall Insulated for Ra=1000, λ=0.75

	Dissipation Parameter M					
	M=0.0		M=0.0003		M=0.0005	
Radius R	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ
0.75	0.0	-0.1327	0.0	-0.1287	0.0	-0.1260
0.77	0.3472	-0.1325	0.3540	-0.1286	0.3586	-0.1260
0.79	0.6693	-0.1312	0.6788	-0.1275	0.6852	-0.1250
0.82	0.9630	-0.1278	0.9718	-0.1243	0.9777	-0.1220
0.84	1.2182	-0.1214	1.2238	-0.1183	1.2276	-0.1162
0.87	1.3055	-0.1113	1.3094	-0.1085	1.3120	-0.1066
0.89	1.5237	-0.0969	1.5201	-0.0945	1.5176	-0.0929
0.92	1.4993	-0.0781	1.4919	-0.0760	1.4869	-0.0747
0.94	1.2851	-0.0550	1.2761	-0.0533	1.2701	-0.0522
0.97	0.8117	-0.0284	0.8045	-0.0273	0.7997	-0.0266
1.0	0.0	0.0	0.0	0.0	0.0	0.0

TABLE VI

VELOCITY DISTRIBUTION AND TEMPERATURE DIFFERENCES DUE TO VISCOUS DISSIPATION EFFECTS FOR CONCENTRIC ANNULUS WITH INNER WALL HEATED, OUTER WALL INSULATED FOR Ra=1, λ=0.75

	Dissipation Parameter M					
	M=0.0		M=0.0003		M=0.0005	
Radius R	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ
0.75	0.0	0.0	0.0	0.0	0.0	0.0
0.77	0.5629	-0.0208	0.5630	-0.0194	0.5627	-0.0184
0.79	0.9899	-0.0401	0.9900	-0.0375	0.9896	-0.0358
0.82	1.2857	-0.0570	1.2859	-0.0535	1.2854	-0.0512
0.84	1.4546	-0.0710	1.4548	-0.0667	1.4542	-0.0638
0.87	1.4875	-0.0820	1.4877	-0.0770	1.4871	-0.0736
0.89	1.4269	-0.0901	1.4271	-0.0843	1.4265	-0.0803
0.92	1.2372	-0.0953	1.2374	-0.0888	1.2369	-0.0844
0.94	0.9343	-0.0982	0.9344	-0.0910	0.9341	-0.0862
0.97	0.5210	-0.0993	0.5211	-0.0917	0.5209	-0.0866
1.0	0.0	-0.0995	0.0	-0.0917	0.0	-0.0865

TABLE VII

VELOCITY DISTRIBUTION AND TEMPERATURE DIFFERENCES DUE TO VISCOUS DISSIPATION EFFECTS FOR CONCENTRIC ANNULUS WITH INNER WALL HEATED, OUTER WALL INSULATED FOR Ra=1000, λ=0.75

	Dissipation Parameter M					
	M=0.0		M=0.0003		M=0.0005	
Radius R	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ
0.75	0.0	0.0	0.0	0.0	0.0	0.0
0.77	0.6896	-0.0208	0.6820	-0.0194	0.6768	-0.0185
0.79	1.1438	-0.0398	1.1341	-0.0375	1.1274	-0.0359
0.82	1.4046	-0.0561	1.3964	-0.0531	1.3908	-0.0510
0.84	1.5076	-0.0695	1.5031	-0.0657	1.5000	-0.0632
0.87	1.5113	-0.0797	1.5086	-0.0754	1.5067	-0.0724
0.89	1.3490	-0.0870	1.3534	-0.0821	1.3564	-0.0787
0.92	1.1257	-0.0917	1.1332	-0.0862	1.1382	-0.0824
0.94	0.8226	-0.0942	0.8308	-0.0882	0.8364	-0.0842
0.97	0.4462	-0.0951	0.4521	-0.0888	0.4561	-0.0846
1.0	0.0	-0.0953	0.0	-0.0888	0.0	-0.0845

TABLE VIII

VELOCITY DISTRIBUTION AND TEMPERATURE DIFFERENCES DUE TO VISCOUS DISSIPATION EFFECTS FOR CONCENTRIC ANNULUS WITH BOTH WALLS HEATED FOR Ra=1, λ=0.75

	Dissipation Parameter M					
	M=0.0		M=0.0003		M=0.0005	
Radius R	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ
0.75	0.0	0.0	0.0	0.0	0.0	0.0
0.77	0.5632	-0.0520	0.5632	-0.0515	0.5630	-0.0511
0.79	0.9900	-0.0969	0.9900	-0.0961	0.9896	-0.0955
0.82	1.2853	-0.1308	1.2853	-0.1299	1.2849	-0.1293
0.84	1.4539	-0.1511	1.4539	-0.1502	1.4534	-0.1495
0.87	1.4868	-0.1566	1.4867	-0.1557	1.4863	-0.1550
0.89	1.4265	-0.1472	1.4265	-0.1463	1.4269	-0.1456
0.92	1.2373	-0.1241	1.2373	-0.1232	1.2369	-0.1226
0.94	0.9349	-0.0895	0.9349	-0.0888	0.9346	-0.0882
0.97	0.5217	-0.0467	0.5217	-0.0462	0.5215	-0.0458
1.0	0.0	0.0	0.0	0.0	0.0	0.0

TABLE IX

VELOCITY DISTRIBUTION AND TEMPERATURE DIFFERENCES DUE TO VISCOUS DISSIPATION EFFECTS FOR CONCENTRIC ANNULUS WITH BOTH WALLS HEATED FOR Ra=2000, λ=0.75

	Dissipation Parameter M					
	M=0.0		M=0.0003		M=0.0005	
Radius R	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ	Velocity V	Temperature Difference ϕ
0.75	0.0	0.0	0.0	0.0	0.0	0.0
0.77	1.2565	-0.0483	1.2510	-0.0472	1.2475	-0.0466
0.79	1.3567	-0.0835	1.3537	-0.0822	1.3517	-0.0814
0.82	1.0294	-0.1047	1.0306	-0.1032	1.0313	-0.1023
0.84	0.7093	-0.1151	0.7135	-0.1136	0.7162	-0.1127
0.87	0.6360	-0.1180	0.6409	-0.1165	0.6440	-0.1156
0.89	0.7721	-0.1146	0.7769	-0.1131	0.7799	-0.1122
0.92	1.1313	-0.1036	1.1333	-0.1021	1.1346	-0.1011
0.94	1.4545	-0.0817	1.4523	-0.0803	1.4509	-0.0795
0.97	1.3050	-0.0463	1.3002	-0.0453	1.2972	-0.0446
1.0	0.0	0.0	0.0	0.0	0.0	0.0

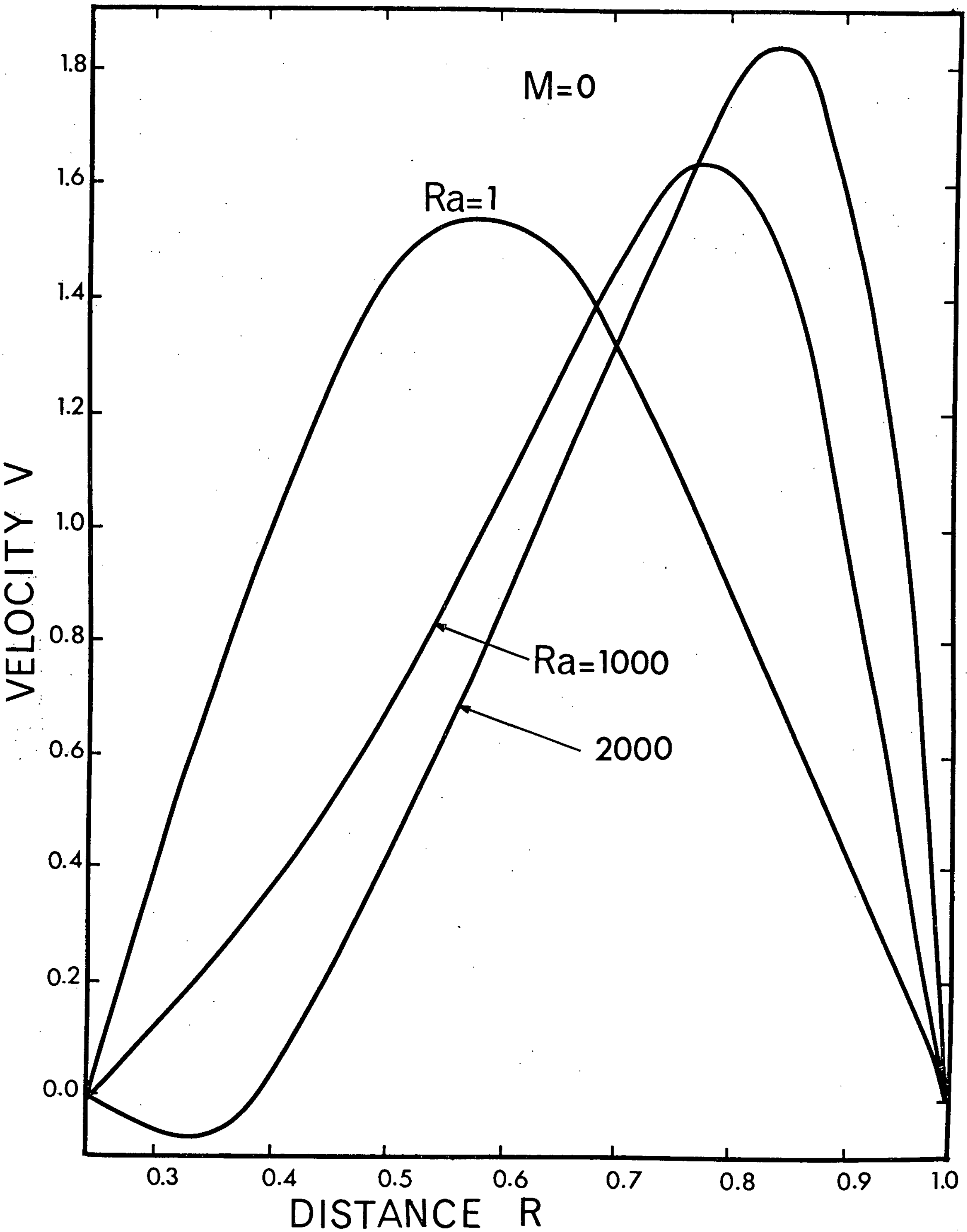

FIGURE 3 Velocity Profiles for Concentric Annulus with Outer Wall Heated, Inner Wall Insulated for Radius Ratio 0.25

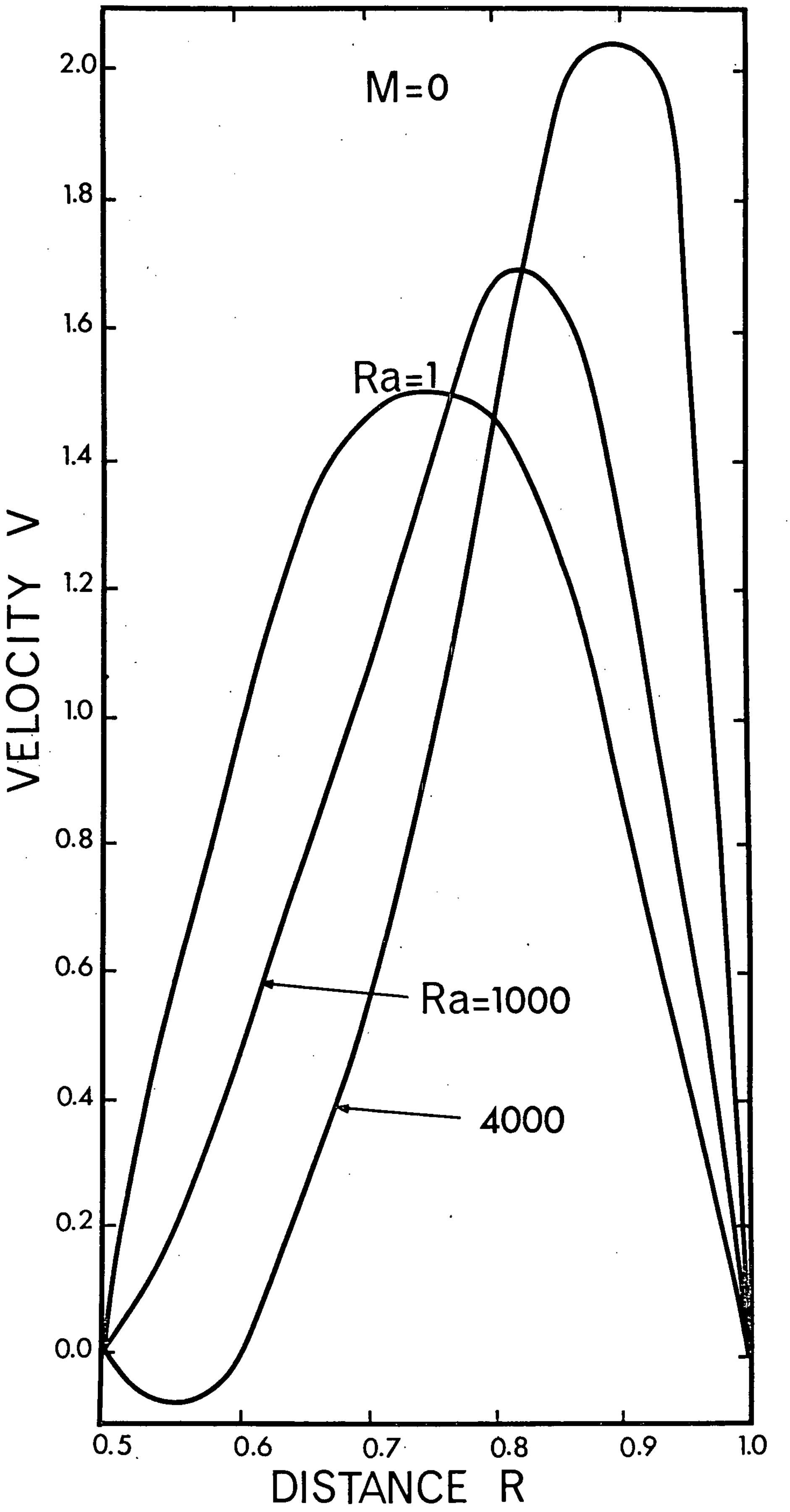

FIGURE 4 Velocity Profiles for Concentric Annulus with Outer Wall Heated, Inner Wall Insulated for Radius Ratio 0.5

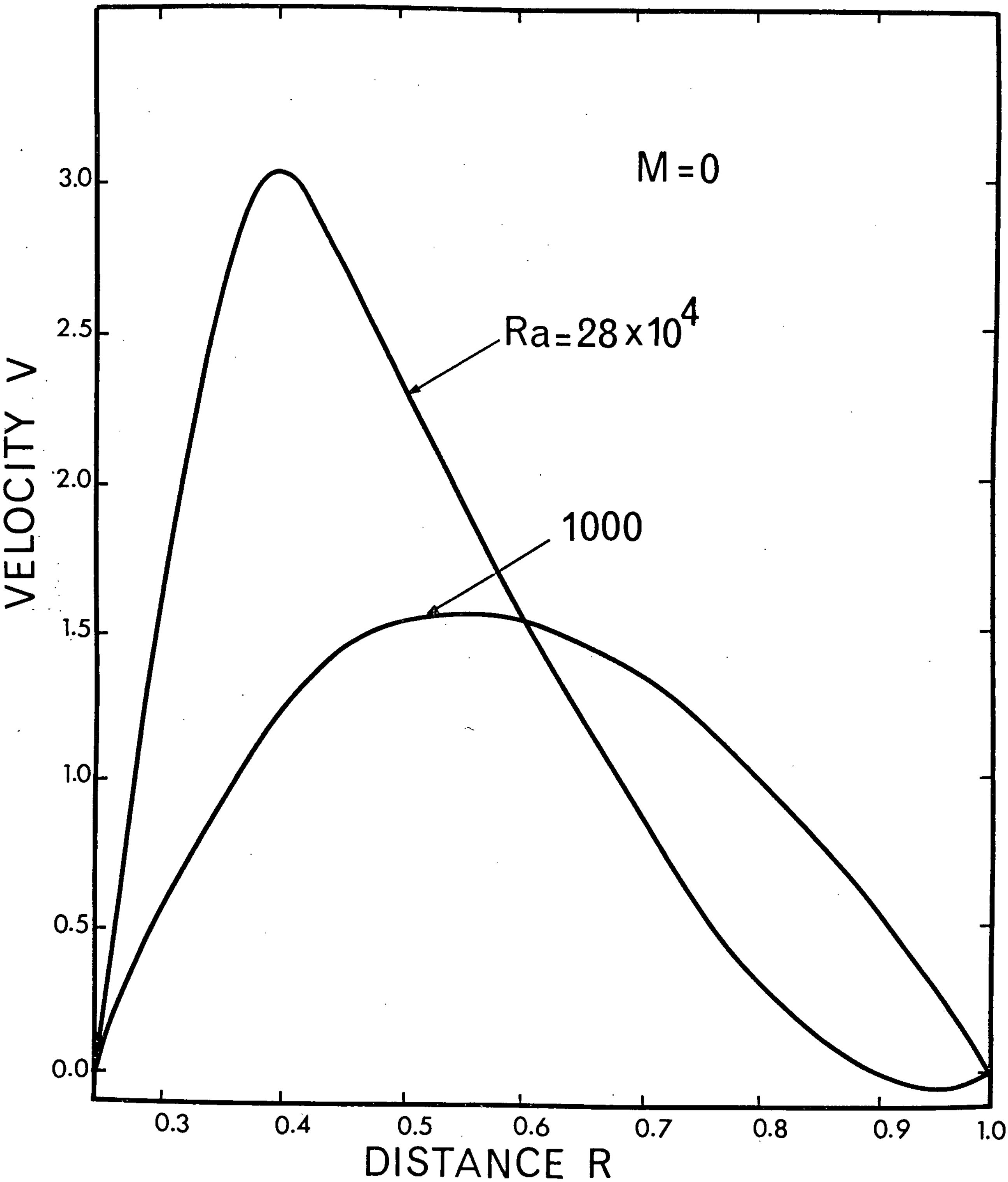

FIGURE 5 Velocity Profiles for Concentric Annulus with Inner Wall Heated, Outer Wall Insulated for Radius Ratio 0.25

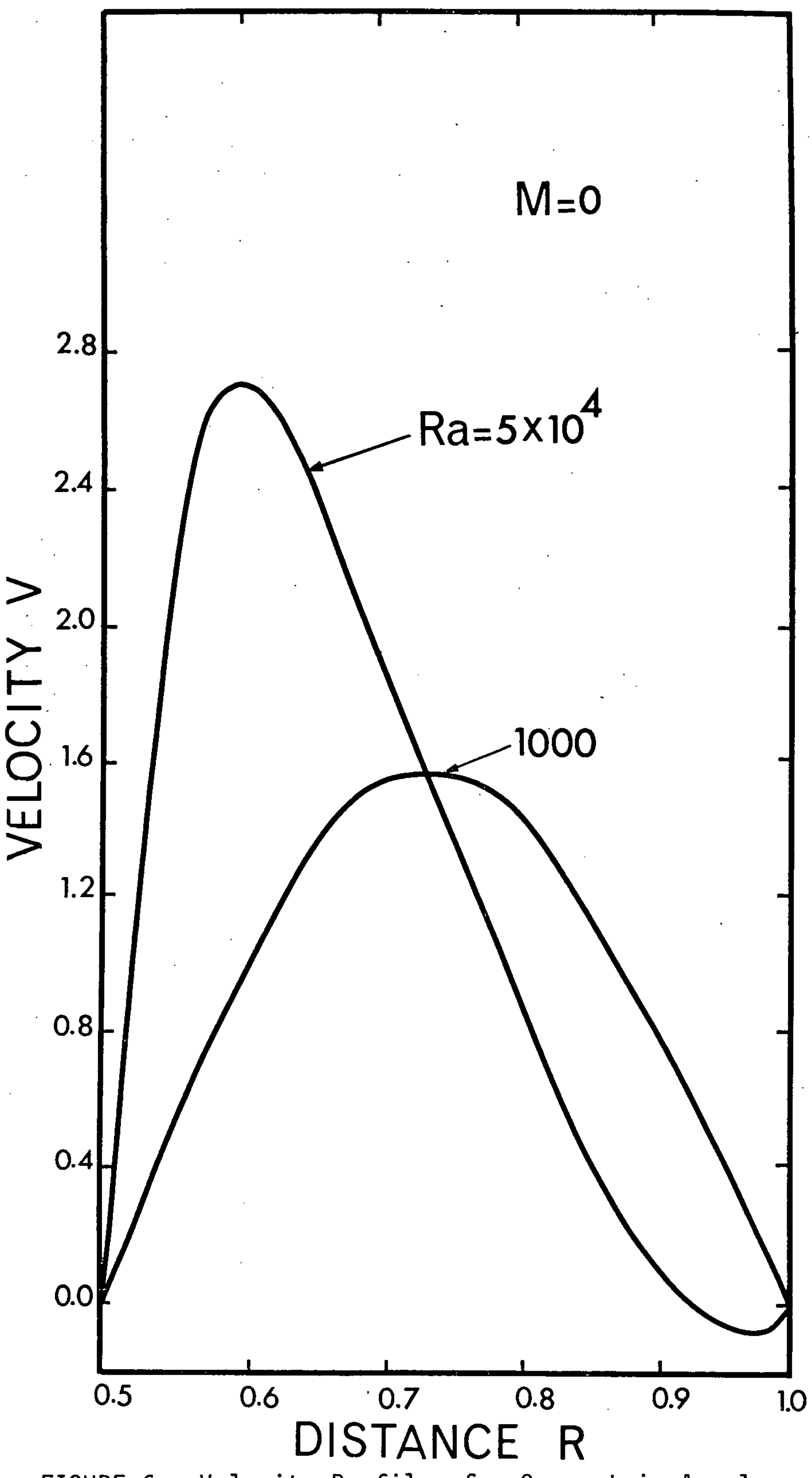

FIGURE 6 Velocity Profiles for Concentric Annulus with Inner Wall Heated, Outer Wall Insulated for Radius Ratio 0.5

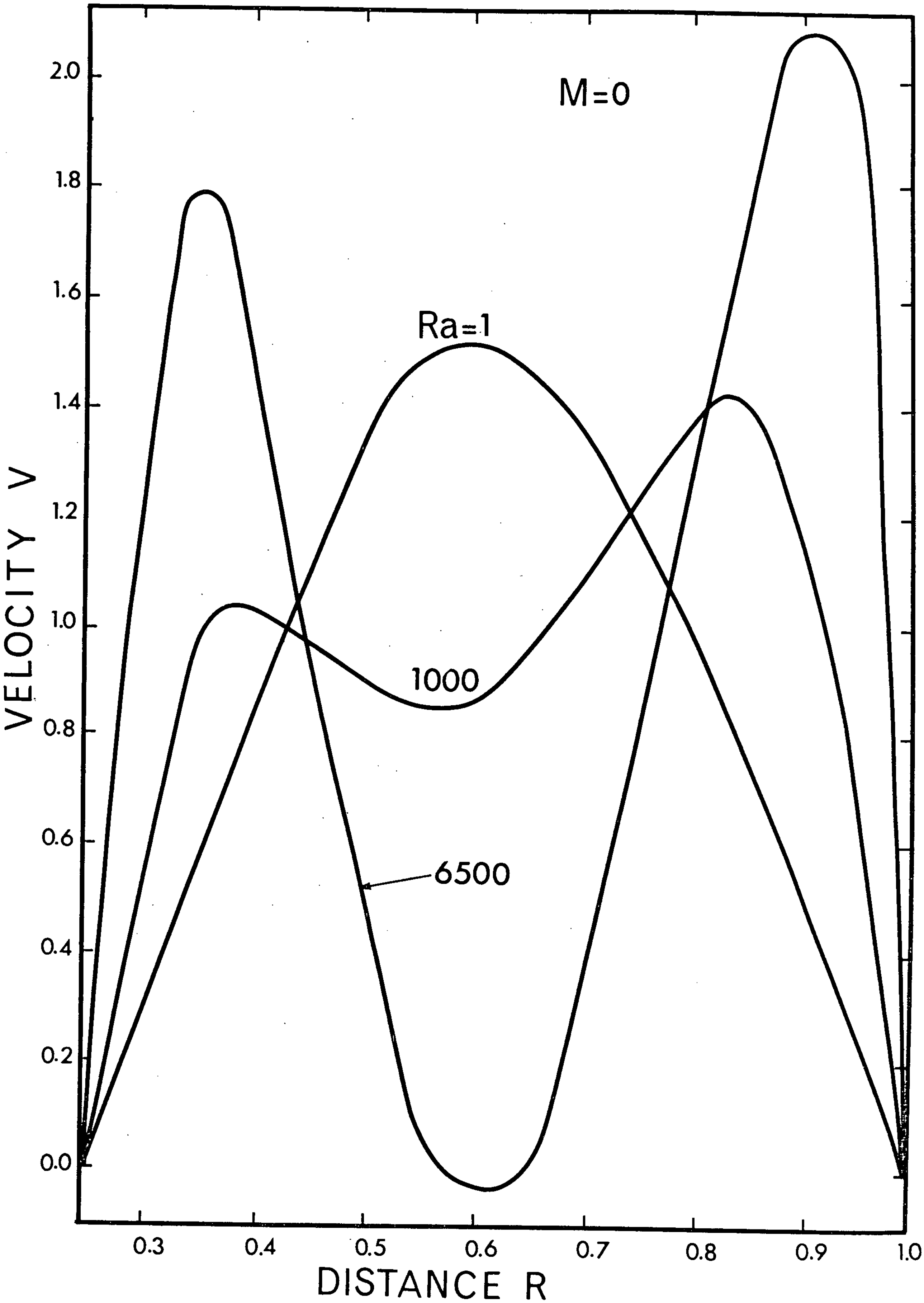

FIGURE 7 Velocity Profiles for Concentric Annulus with Both Walls Heated for Radius Ratio 0.25

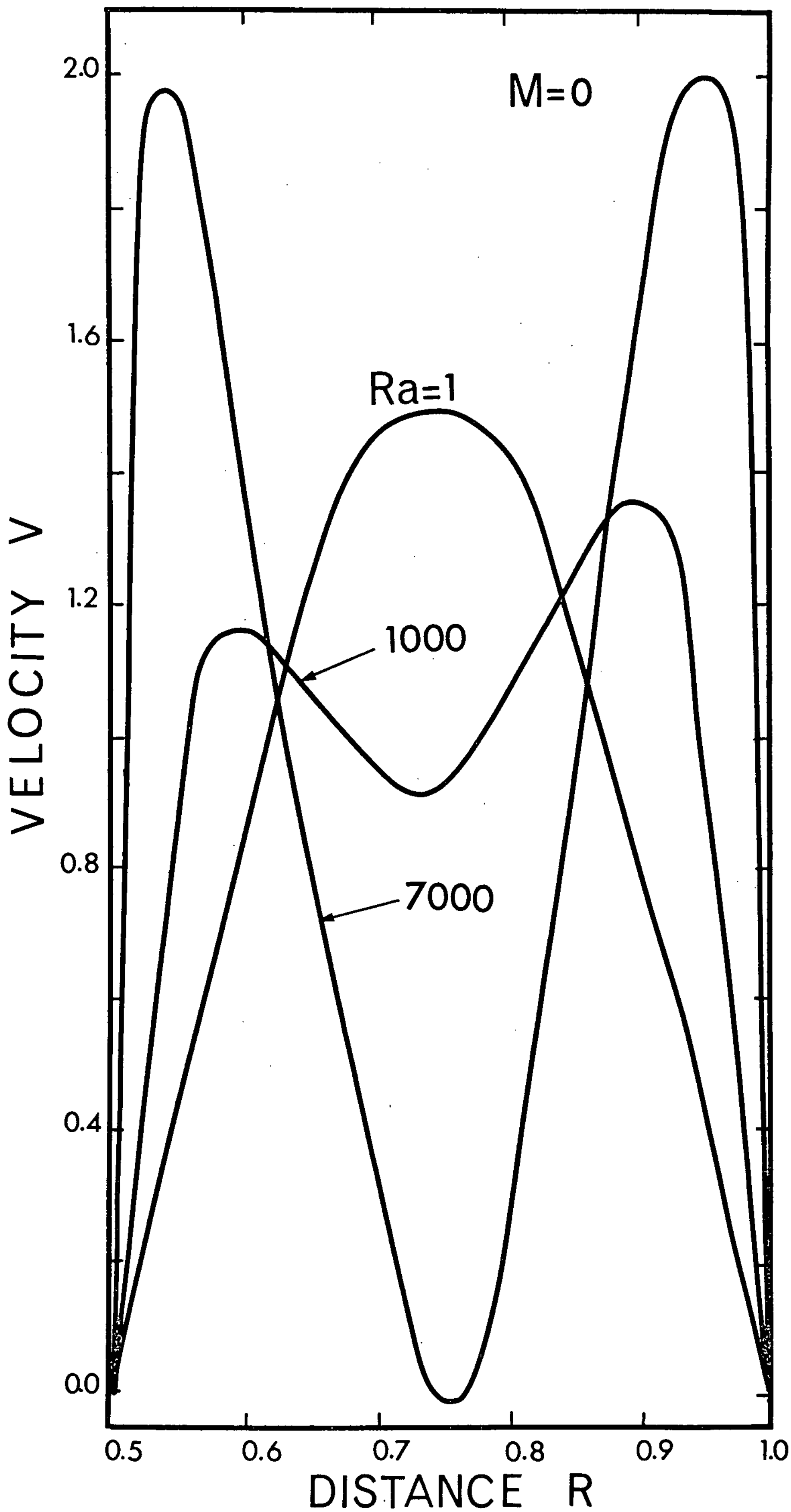

FIGURE 8 Velocity Profiles for Concentric Annulus with Both Walls Heated for Radius Ratio 0.5

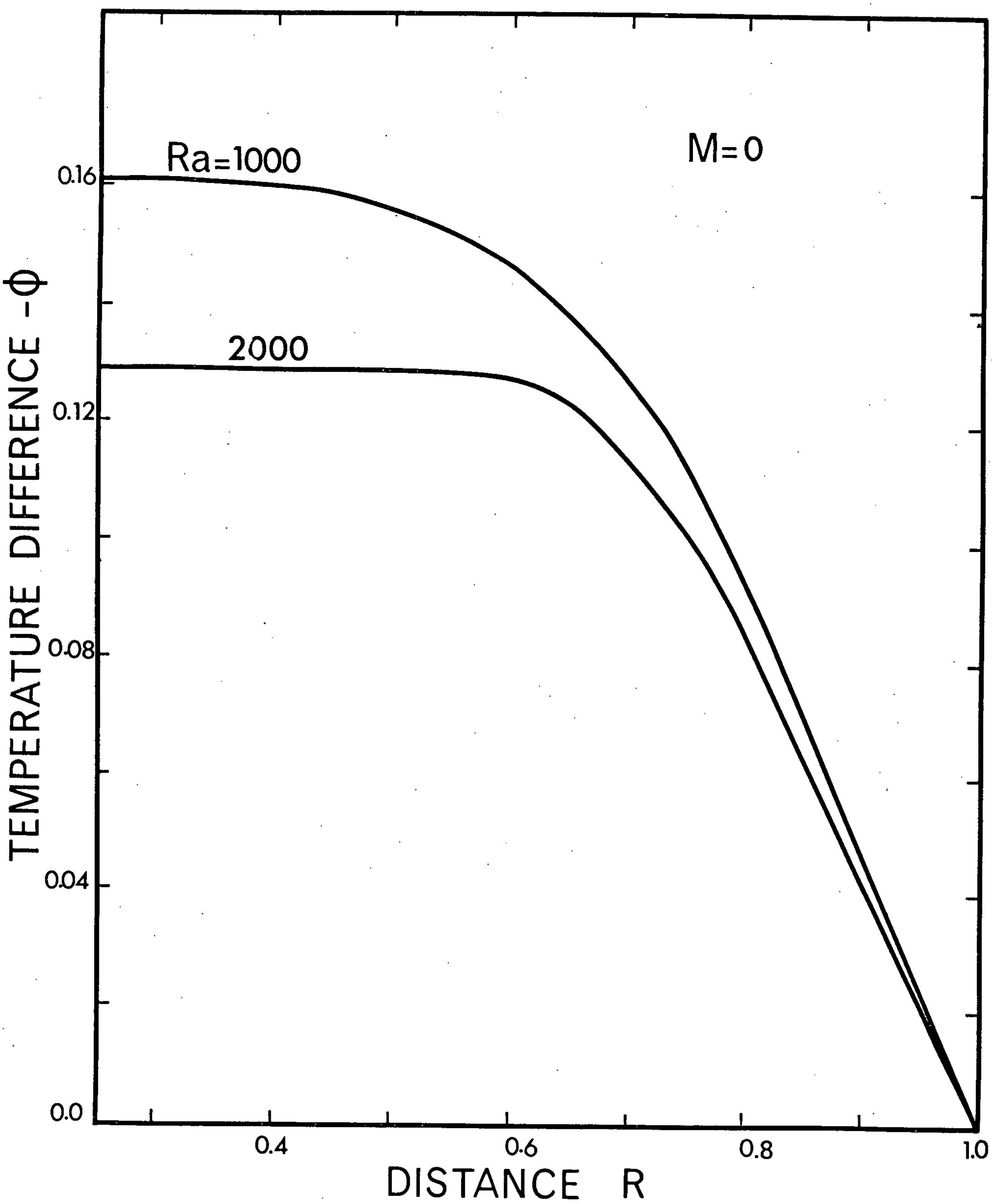

FIGURE 9 Temperature Profiles for Concentric Annulus with Outer Wall Heated, Inner Wall Insulated for Radius Ratio 0.25

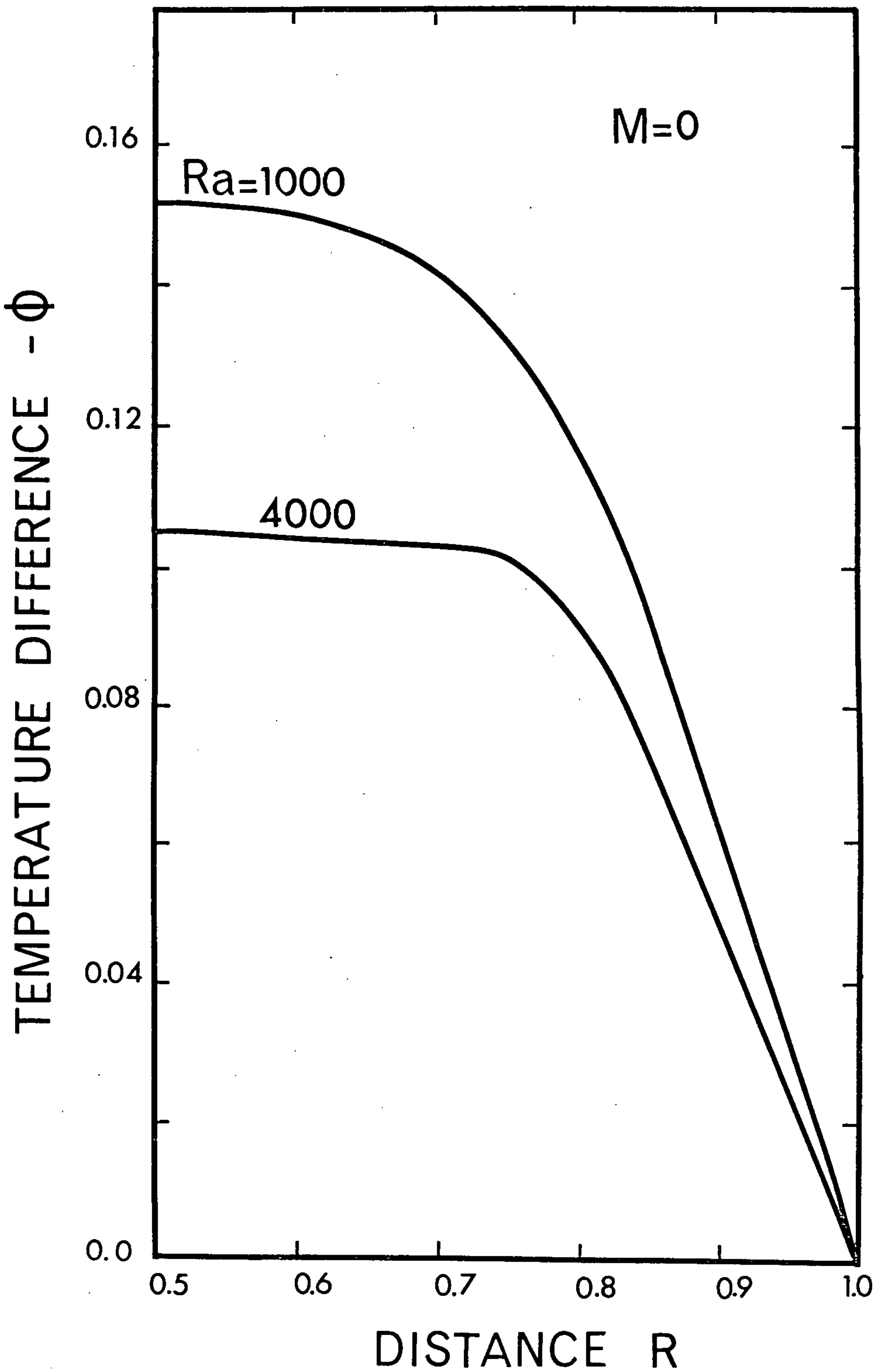

FIGURE 10 Temperature Profiles for Concentric Annulus with Outer Wall Heated, Inner Wall Insulated for Radius Ratio 0.5

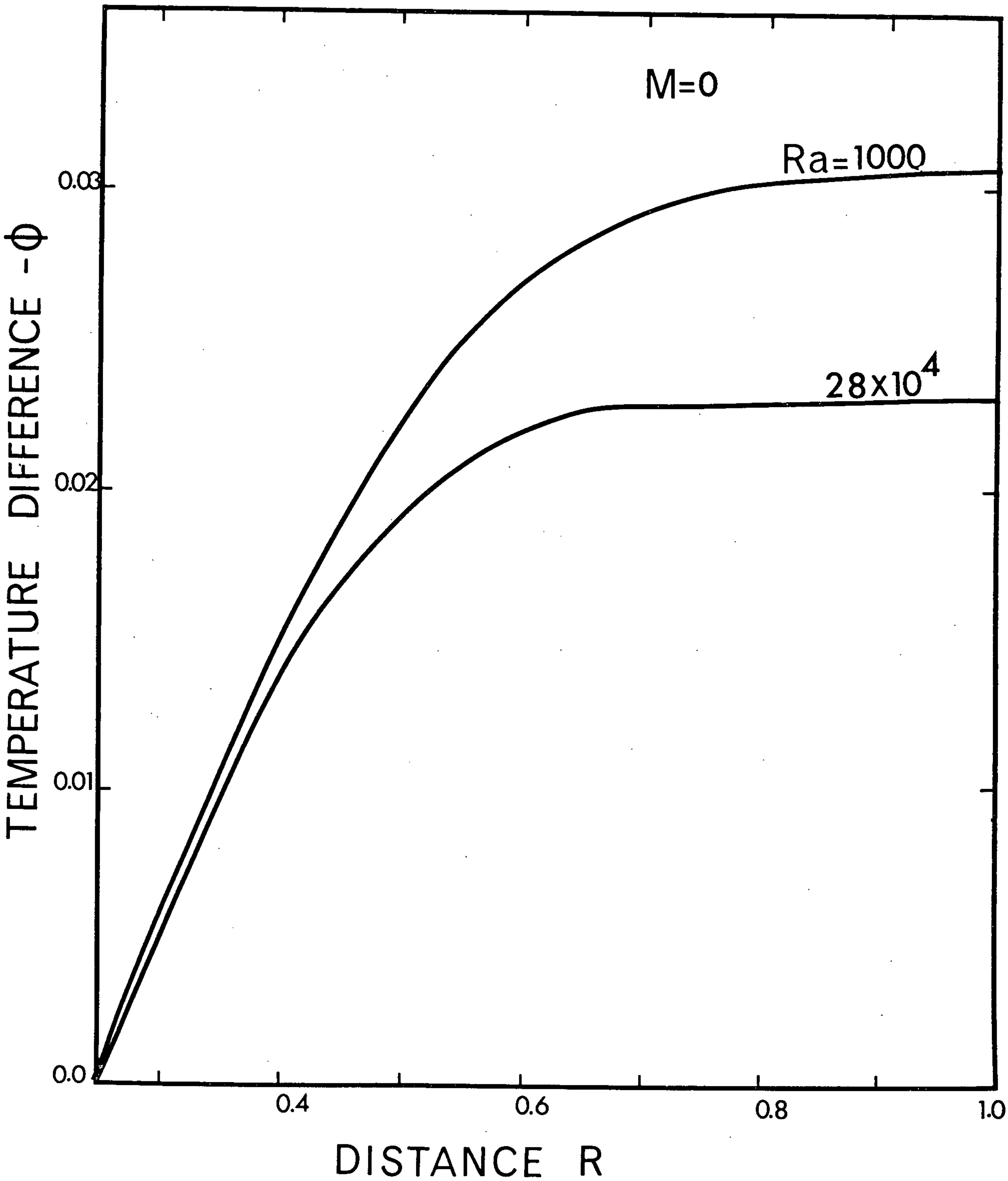

FIGURE 11 Temperature Profiles for Concentric Annulus with Inner Wall Heated, Outer Wall Insulated for Radius Ratio 0.25

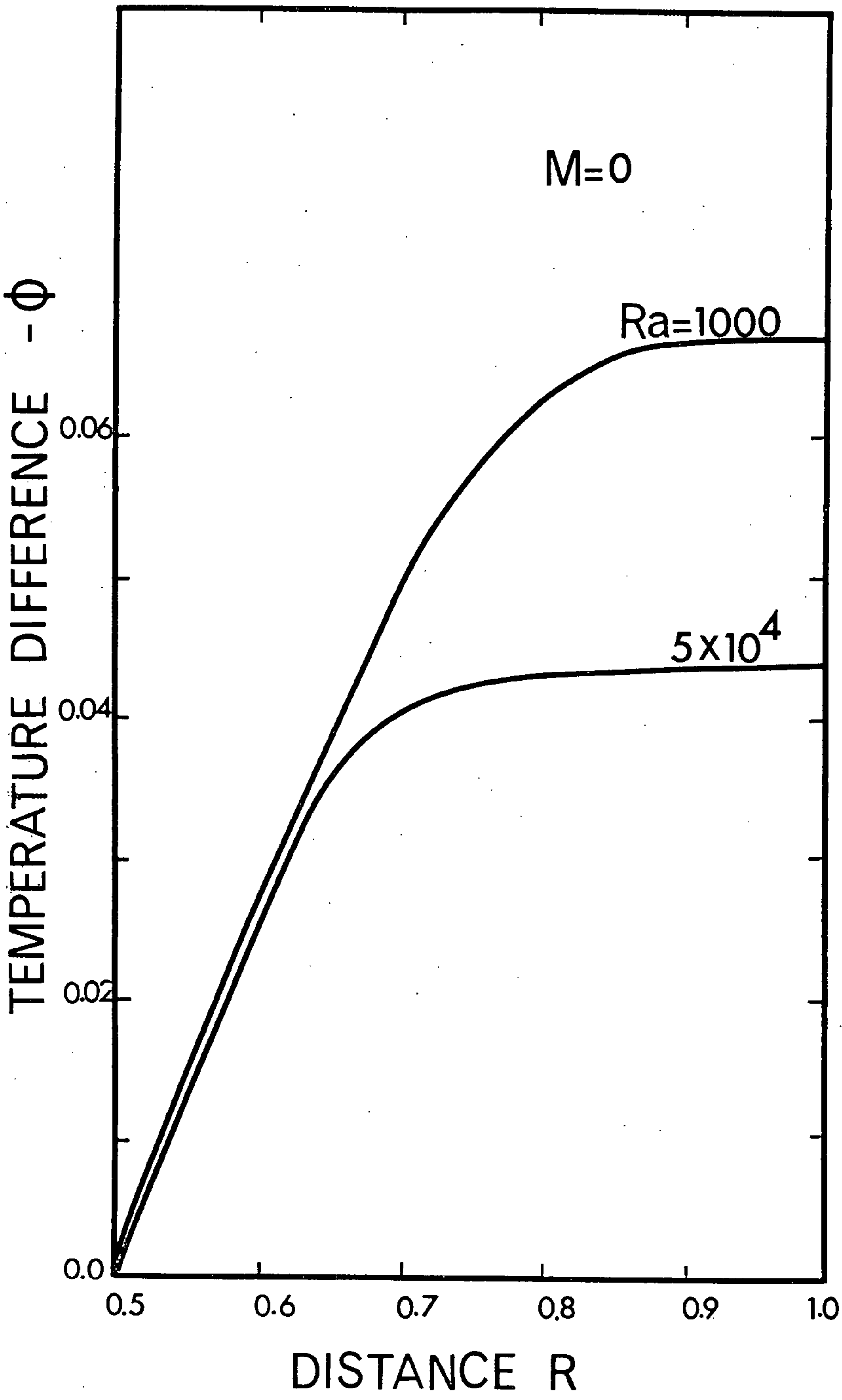

FIGURE 12 Temperature Profiles for Concentric Annulus with Inner Wall Heated, Outer Wall Insulated for Radius Ratio 0.5

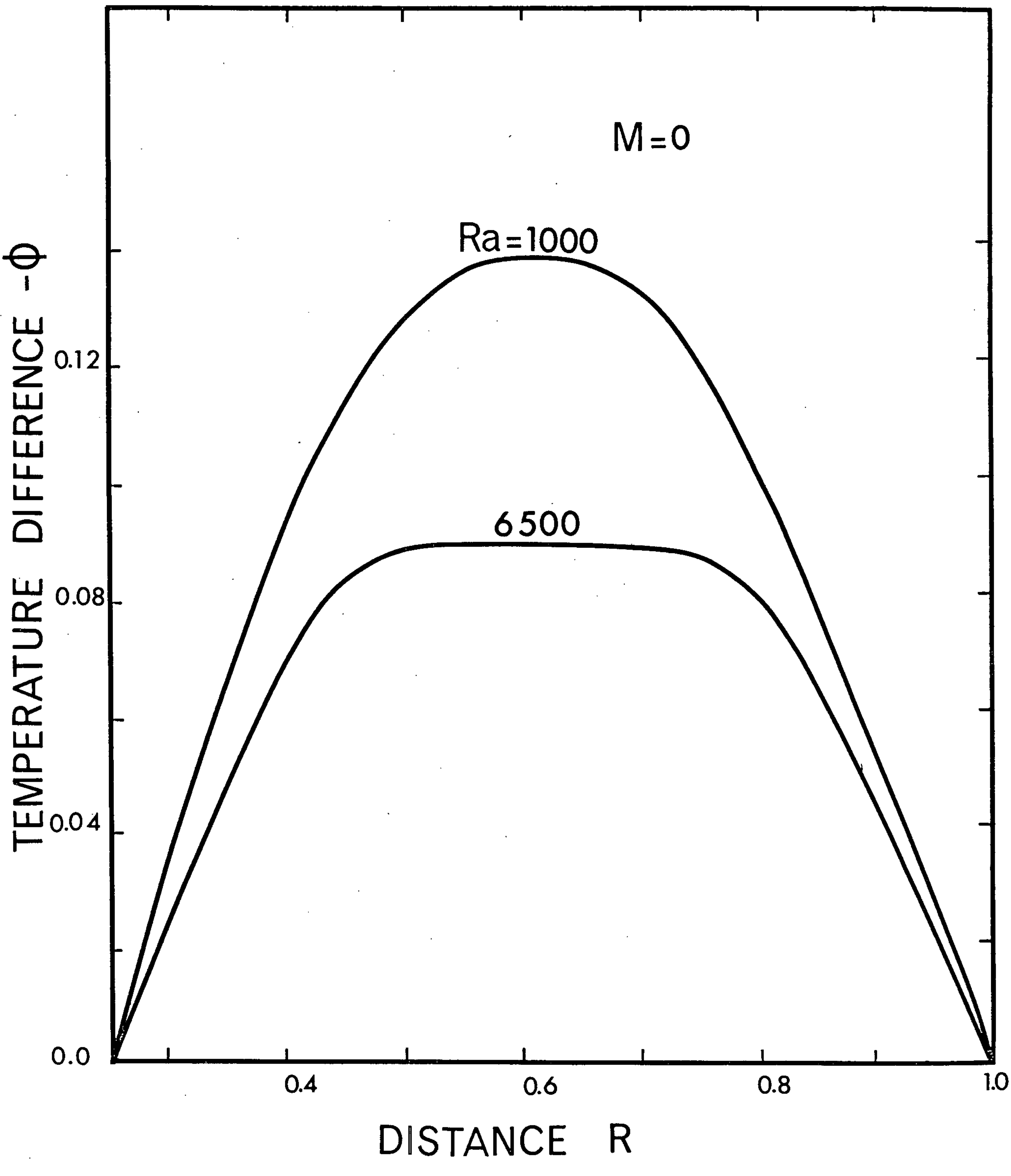

FIGURE 13 Temperature Profiles for Concentric Annulus with Both Walls Heated for Radius Ratio 0.25

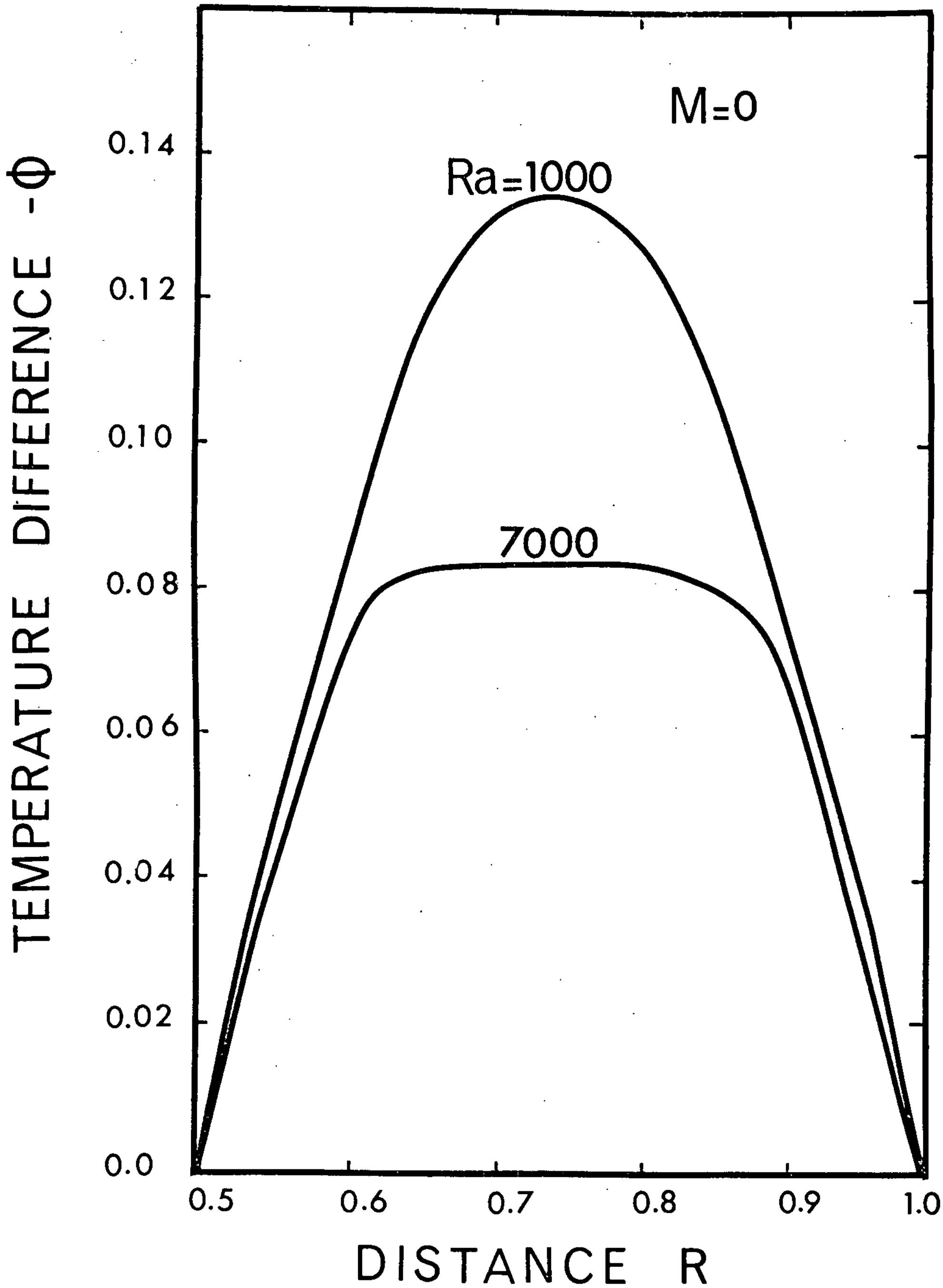

FIGURE 14 Temperature Profiles for Concentric Annulus with Both Walls Heated for Radius Ratio 0.5

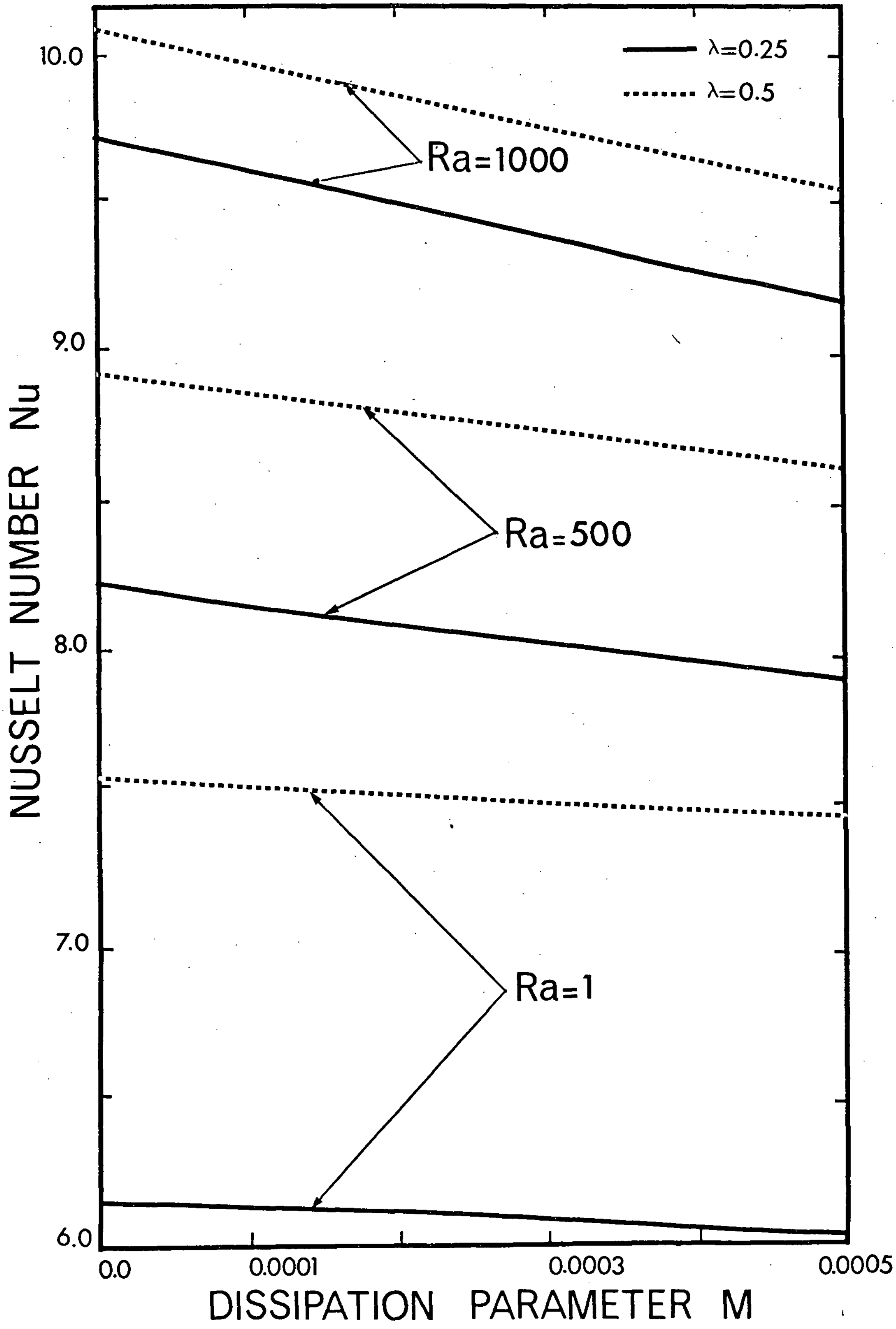

FIGURE 15 Effect of Viscous Dissipation Parameter on Nusselt Number for Concentric Annulus with Outer Wall Heated, Inner Wall Insulated

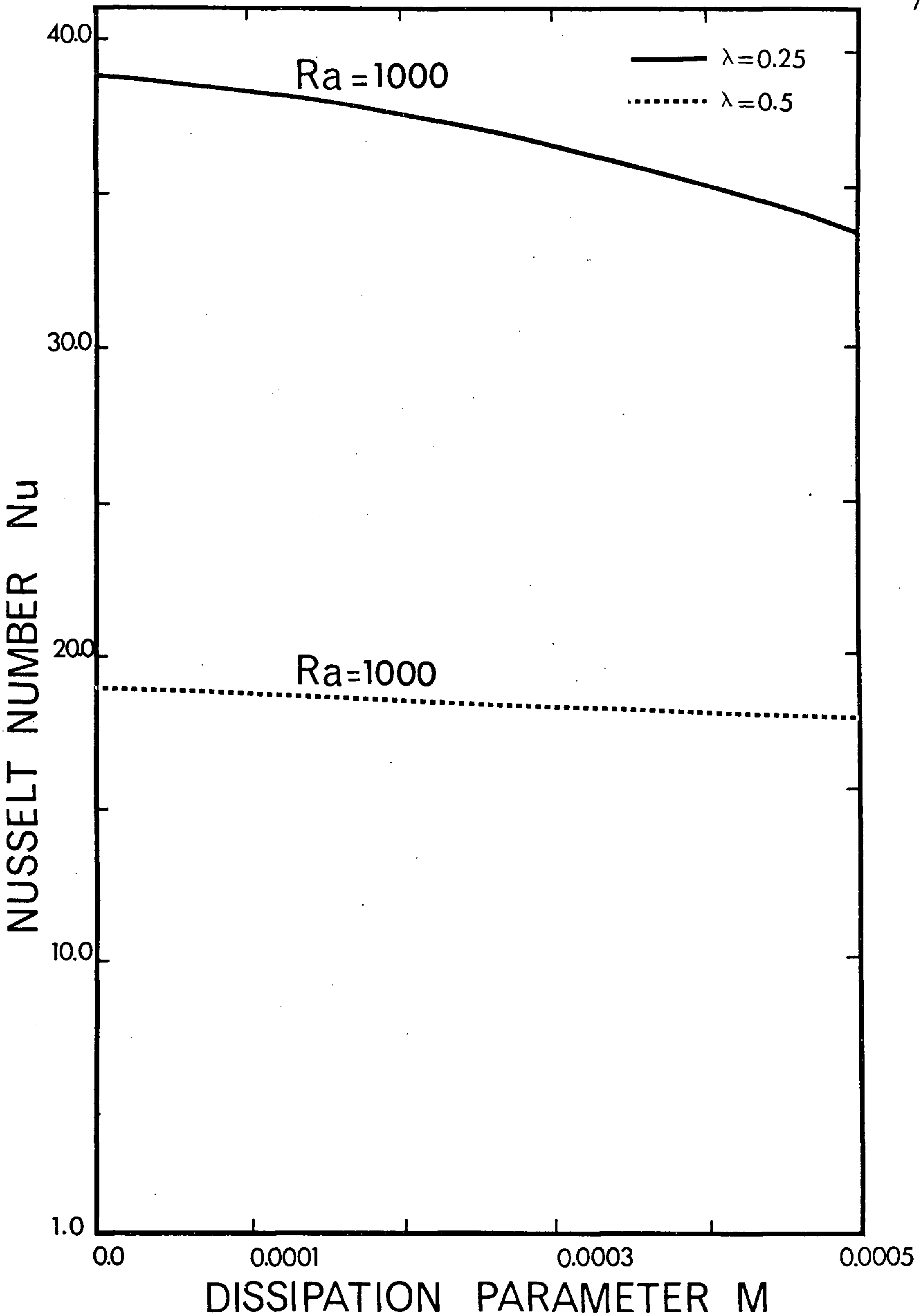

FIGURE 16 Effect of Viscous Dissipation Parameter on Nusselt Number for Concentric Annulus with Inner Wall Heated, Outer Wall Insulated

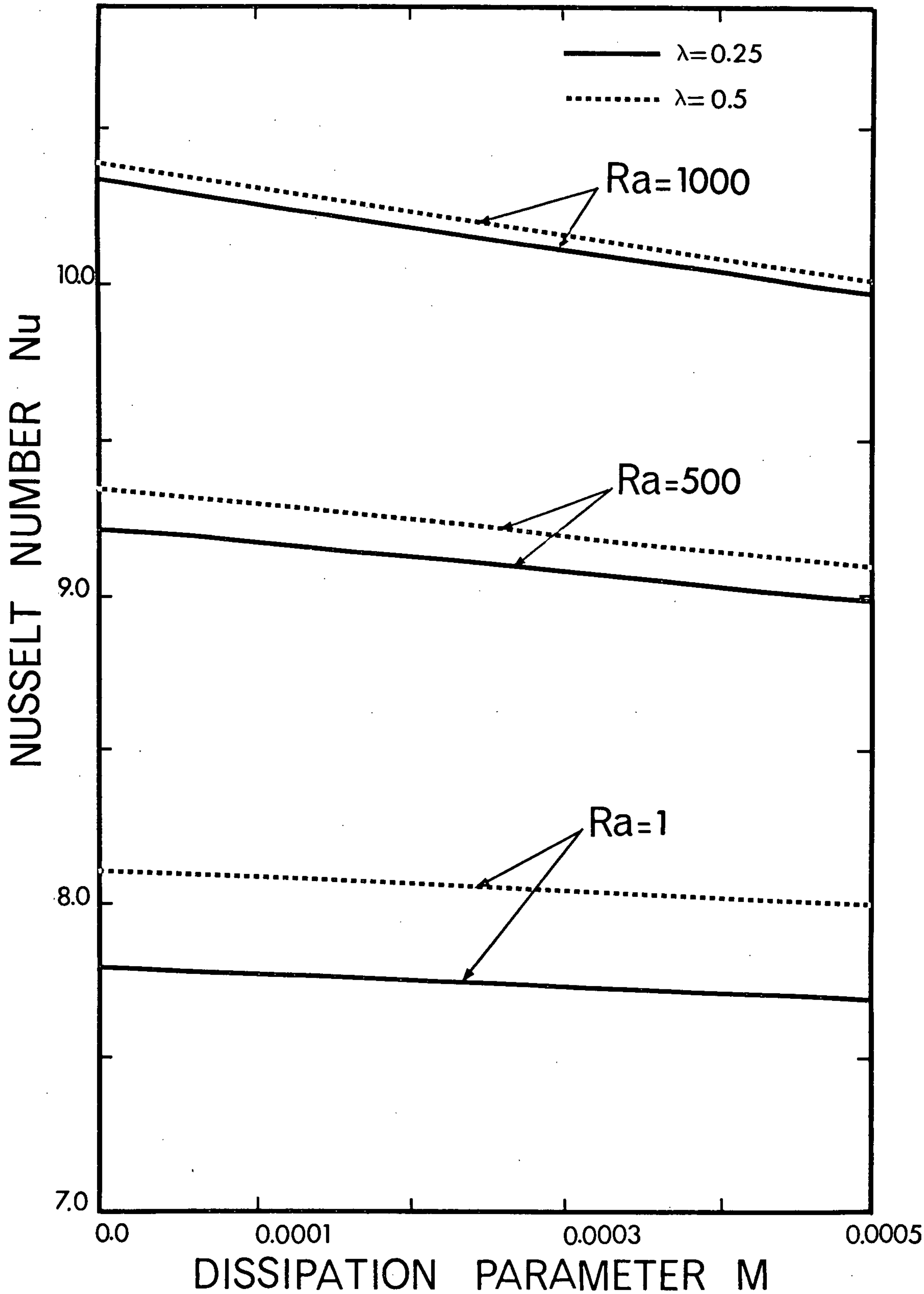

FIGURE 17 Effect of Viscous Dissipation Parameter on Nusselt Number for Concentric Annulus with Both Walls Heated

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APPENDICES

APPENDIX A

DERIVATION OF NUSSELT NUMBER EXPRESSION FOR CIRCULAR DUCTS

The Nusselt number expression for circular ducts in terms of the dimensionless variables is obtained as shown below,

$$Nu = \frac{h D_h}{k} = \frac{D_h}{k} \cdot \frac{q}{T_w - T_b} , \qquad \text{(A-1)}$$

where q = average heat flux

T_w = temperature of the wall

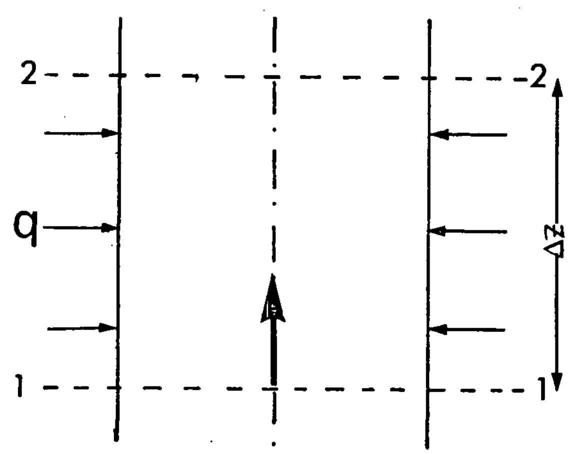
T_b = bulk temperature of the fluid.

The bulk temperature can be written as,

$$T_b = \iint T v_z \, dA \Big/ \iint v_z \, dA . \qquad \text{(A-2)}$$

Substituting (A-2) in (A-1), we obtain,

$$Nu = \frac{D_h}{k} \cdot \frac{q}{T_w - \left[\iint T v_z \, dA \big/ \iint v_z \, dA\right]} \qquad \text{(A-3)}$$



Energy Balance Between Sections 1&2

Now consider a fluid flowing between sections 1 and 2 of a circular duct as shown in the figure. By making an energy balance, we obtain,

$$\rho U c_p A \left(T_2 - T_1\right) = q P \Delta Z + \mu\left[\iint\left(\frac{dv_z}{dr}\right)^2 dA\right]\Delta Z, \qquad \text{(A-4)}$$

where

T_1 and T_2 are the bulk temperatures at sections 1 and 2 respectively and P is the heated perimeter of the duct.

Substituting $\frac{\partial T}{\partial Z}$ = C in equation (A-4), we obtain,

$$q = \rho c_p U \frac{D_h}{4} C - \frac{2\mu U^2}{D_h}\int_0^1\left(\frac{dV}{dR}\right)^2 R\,dR. \qquad \text{(A-5)}$$

Now substituting (A-5) in (A-3), we obtain,

$$Nu = \frac{\frac{D_h}{K}\left[\rho c_p U \frac{D_h}{4} C - \frac{2\mu U^2}{D_h}\int_0^1\left(\frac{dV}{dR}\right)^2 R\,dR\right]}{T_w - \left[\iint T v_z\,dA \Big/ \iint v_z\,dA\right]}, \qquad \text{(A-6)}$$

$$= \frac{\frac{D_h}{K}\left[\rho c_p U \frac{D_h}{4} C - \frac{2\mu U^2}{D_h}\int_0^1\left(\frac{dV}{dR}\right)^2 R\,dR\right]}{-\rho U c_p C D_h^2 \int_0^1 \phi V R\,dR \Big/ 4K \int_0^1 V R\,dR}, \qquad \text{(A-7)}$$

$$= \frac{-1 + 8M\int_0^1\left(\frac{dV}{dR}\right)^2 R\,dR}{\int_0^1 \Phi V R\,dR \Big/ \int_0^1 V R\,dR} \qquad \text{(A-8)}$$

APPENDIX B

DERIVATION OF NUSSELT NUMBER EXPRESSIONS FOR CONCENTRIC ANNULI

The Nusselt number expressions for the concentric annuli are obtained as shown below,

Case I: Outer Wall Heated, Inner Wall Insulated

Nusselt number is given by the expression

$$Nu = \frac{h D_h}{k} = \frac{D_h}{k} \cdot \frac{q}{T_w - T_b} \cdot \qquad \text{(B-1)}$$

The equivalent diameter for this case is given by,

$$D_h = 2 r_o (1 - \lambda^2) , \qquad \text{(B-2)}$$

where r_o is the radius of the outer tube and λ is the radius ratio r_i/r_o.

By making an energy balance as shown for the circular duct we obtain the following expression,

$$\rho c_p U A (T_2 - T_1) = q P \Delta Z + \mu \left[\iint \left(\frac{dv_z}{dr} \right)^2 dA \right] \Delta Z , \qquad \text{(B-3)}$$

where P is the heated perimeter.

From equation (B-3) we obtain,

$$q = \rho c_p U \frac{D_h}{4} C - \frac{\mu U^2}{r_o} \int_{\lambda}^{1} \left(\frac{dV}{dR} \right)^2 R dR . \qquad \text{(B-4)}$$

Using the value of D_h from (B-2) and substituting (B-4) in (B-1) we obtain,

$$Nu = \frac{\frac{D_h}{k}\left[\rho C_p U \frac{D_h}{4} C - \frac{\mu U^2}{r_o}\int_\lambda^1 \left(\frac{dV}{dR}\right)^2 R\,dR\right]}{T_w - T_b} \tag{B-5}$$

$$= \frac{\frac{D_h}{k}\left[\rho C_p U \frac{D_h}{4} C - \frac{\mu U^2}{r_o}\int_\lambda^1 \left(\frac{dV}{dR}\right)^2 R\,dR\right]}{-\rho U C_p C D_h^2 \int_\lambda^1 \phi V R\,dR / 4k \int_\lambda^1 V R\,dR} \tag{B-6}$$

$$= \frac{-1 + 8(1-\lambda^2) M \int_\lambda^1 \left(\frac{dV}{dR}\right)^2 R\,dR}{\int_\lambda^1 \phi V R\,dR / \int_\lambda^1 V R\,dR} . \tag{B-7}$$

<u>Case II: Inner Wall Heated, Outer Wall Insulated</u>

Nusselt Number $Nu = \frac{hD_h}{k} = \frac{D_h}{k} \cdot \frac{q}{T_W - T_b}$. (B-1)

The equivalent diameter for this case is given by,

$$D_h = 2r_o(1-\lambda^2)/\lambda . \tag{B-8}$$

By making an energy balance the following expression is obtained,

$$\rho C_p U A (T_2 - T_1) = q P \Delta Z + \mu\left[\iint \left(\frac{dv_z}{dr}\right)^2 dA\right] \Delta Z . \tag{B-9}$$

From equation (B-9), substituting (B-8) for D_h we obtain,

$$q = \rho C_p U \frac{D_h}{4} C - \frac{\mu U^2}{r_i}\int_\lambda^1 \left(\frac{dV}{dR}\right)^2 R\,dR . \tag{B-10}$$

Substituting (B-10) in (B-1), we obtain,

$$Nu = \frac{\frac{D_h}{k}\left[\rho c_p U \frac{D_h}{4} C - \frac{\mu U^2}{r_i}\int_\lambda^1 \left(\frac{dV}{dR}\right)^2 R\,dR\right]}{T_w - T_b} \tag{B-11}$$

$$= \frac{\frac{D_h}{k}\left[\rho c_p U \frac{D_h}{4} C - \frac{\mu U^2}{r_i}\int_\lambda^1 \left(\frac{dV}{dR}\right)^2 R\,dR\right]}{-\rho U c_p C D_h^2 \int_\lambda^1 \phi V R\,dR \Big/ 4k \int_\lambda^1 V R\,dR} \tag{B-12}$$

$$= \frac{-1 + \frac{8(1-\lambda^2)}{\lambda^2} M \int_\lambda^1 \left(\frac{dV}{dR}\right)^2 R\,dR}{\int_\lambda^1 \phi V R\,dR \Big/ \int_\lambda^1 V R\,dR} . \tag{B-13}$$

Case III: Both Walls Heated

$$Nu = \frac{h D_h}{k} = \frac{D_h}{k} \frac{q}{T_w - T_b} . \tag{B-1}$$

The equivalent diameter for this case is given by,

$D_h = 2r_o\ (1-\lambda)$

By making an energy balance we obtain,

$$\rho c_p U A (T_2 - T_1) = (q_i P_i + q_o P_o)\Delta Z + \mu\left[\iint \left(\frac{dv_z}{dr}\right)^2 dA\right]\Delta Z, \tag{B-14}$$

where q_i and q_o are the average heat flux at inner and outer wall respectively.

From equation (B-14) we obtain,

(B-15)

where $q = q_{average}$

Substituting (B-15) in (B-1) the following equation is obtained,

$$Nu = \frac{\frac{D_h}{k}\left[\rho c_p U \frac{D_h}{4} C - \frac{\mu U^2}{(r_o + r_i)}\int_\lambda^1 \left(\frac{dV}{dR}\right)^2 R\,dR\right]}{T_w - T_b} \qquad \text{(B-16)}$$

$$= \frac{\frac{D_h}{k}\left[\rho c_p U \frac{D_h}{4} C - \frac{\mu U^2}{(r_o + r_i)}\int_\lambda^1 \left(\frac{dV}{dR}\right)^2 R\,dR\right]}{-\rho U c_p C D_h^2 \int_\lambda^1 \phi V R\,dR \Big/ 4k \int_\lambda^1 V R\,dR} \qquad \text{(B-17)}$$

$$= \frac{-1 + 8\left(\frac{1-\lambda}{1+\lambda}\right) M \int_\lambda^1 \left(\frac{dV}{dR}\right)^2 R\,dR}{\int_\lambda^1 \phi V R\,dR \Big/ \int_\lambda^1 V R\,dR} \; . \qquad \text{(B-18)}$$

APPENDIX C

DETAILS OF GOVERNING EQUATIONS AND LIMITATIONS

The final form of the governing equations as given by (1) and (2) were obtained in the following manner:

On the basis of the assumptions on page 7, the equations of motion in r and Θ directions can be ignored. The basic momentum equation in Z-direction for constant ρ and μ is given by [1],

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\partial^2 v_z/\partial\theta^2 + \partial^2 v_z/\partial z^2\right] \pm \rho g_z \cdot \quad \text{(C-1)}$$

For steady flow $\frac{\partial v_z}{\partial t} = 0$ and because of symmetry the component of velocity in Θ-direction vanishes. For fully developed laminar flow, $v_r = \frac{\partial v_z}{\partial z} = 0$ and since pressure is only a function of Z, equation (C-1) reduces to

$$0 = -\frac{dp}{dz} + \mu\left(\frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{dv_z}{dr}\right) \pm \rho g_z \cdot \quad \text{(C-2)}$$

Since Z is measured positive in the upward direction, the negative sign before g_z is taken.

Thus we have,

$$0 = -\frac{dp}{dz} + \mu\left(\frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{dv_z}{dr}\right) - \rho g_z \cdot \quad \text{(C-3)}$$

The basic differential energy equation for constant κ, μ

and ρ can be written as,

$$\rho C_p \frac{DT}{Dt} = k\nabla^2 T + Q_i + \mu\Phi + T\beta\frac{Dp}{Dt}, \quad \text{(C-4)}$$

where Q_i = Internal heat generation source energy

Φ = Viscous dissipation function

Φ is given by [1],

$$\Phi = 2\left\{\left(\frac{\partial v_r}{\partial r}\right)^2 + \left[\frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\right]^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right\} + \left\{\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)^2 + \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)^2 + \left[\frac{1}{r}\frac{\partial v_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]^2\right\}. \quad \text{(C-5)}$$

Re-writing (C-4) in an expanded form we have,

$$\rho C_p\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right) = K\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right] + Q_i + \mu\Phi + T\beta\left(\frac{\partial p}{\partial t} + v_r\frac{\partial p}{\partial r} + \frac{v_\theta}{r}\frac{\partial p}{\partial \theta} + v_z\frac{\partial p}{\partial z}\right). \quad \text{(C-6)}$$

Eliminating the terms which are equal to zero for conditions mentioned earlier, and for no internal heat generation source, equation (C-6) reduces to

$$\rho C_p v_z\frac{\partial T}{\partial z} = K\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] + \mu\left(\frac{dv_z}{dr}\right)^2 + T\beta v_z\frac{dp}{dz}. \quad \text{(C-7)}$$

The relative significance of compression work to that of viscous dissipation can be seen by comparing the last two terms on the right hand side of equation (C-7). Equation (C-7) in the non-dimensional form can be written as,

$$\nabla^2\phi - V + \frac{4\mu U^2}{D_h^2}\left(\frac{dV}{dR}\right)^2 + \frac{C^2 U^4 C_p C D_h \beta}{4k}\phi V\frac{dp}{dz}. \quad \text{(C-8)}$$

Dividing the coefficient of compression work term by that of viscous dissipation, we obtain the factor (1/16) Pe Re β C D_h. This factor shows that for small values of Peclet number, Reynolds number, β the coefficient of volumetric expansion and the temperature rise in the flow direction, the compression work term can be neglected.

Thus equation (C-7) reduces to

$$\rho C_p v_z \frac{\partial T}{\partial z} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] + \mu\left(\frac{dv_z}{dr}\right)^2. \qquad \text{(C-9)}$$

A discussion on the inclusion of compression work term has also been given by Tyagi [27].

The variability of the physical properties with temperature makes the problem highly non-linear and thus extremely difficult to solve. Hence for this reason, the present study deals with constant properties except for the variation of density in the buoyancy term of the momentum equation. To ensure this, the temperature differences in the system should be small since all the physical properties are a function of temperature. Moreover, the duct length has to be small to avoid variation of properties along the duct length.

LIMITATIONS OF THE RADIUS RATIO FOR ANNULUS

For the concentric annulus, the range of radius ratio λ is from 0 to 1. For λ = 0, the annulus reduces to a circular duct with a wire in the centre parallel to the axis, whereas for λ = 1, the configuration of parallel plates is obtained. The range of λ used for the present analysis is from 0.25 to 0.75. If the annular gap is too small, the physical properties may not remain constant due to large viscous heating effects. On the other hand, if the annular gap is too large, the system approximates almost to flow along a single vertical cylinder in which case a fully developed flow is obtained only beyond a very large entrance length and is not of interest for the present study.