A DENJOY-PERRON SECOND INTEGRAL

by

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Abstract

The Denjoy Perron Integral and the Perron Integral are known to be equivalent. It is shown that a Denjoy Perron Second Integral may be defined by extending the concept of Generalized Absolute Continuity. The Denjoy Perron Second Integral is shown to include the Perron Second Integral.
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INTRODUCTION

1. Definitions

1. The function of a real variable, \( F(x) \), is defined for all \( x \).

1.2 The second oscillation of a function \( F(x) \) over an interval \( I = (\alpha, \beta) \) denoted by \( O^2(F, I) \) is the maximum of the numbers \( m_1 \) and \( m_2 \) where

\[
m_1 = \alpha < t_1 < t_2 \leq \beta \quad \left( \frac{F(t_2) - F(\alpha)}{t_2 - \alpha} - \frac{F(t_1) - F(\alpha)}{t_1 - \alpha} \right) \right),
\]

\[
m_2 = \alpha \leq t_1 < t_2 < \beta \quad \left( \frac{F(\beta) - F(t_2)}{\beta - t_2} - \frac{F(\beta) - F(t_1)}{\beta - t_1} \right) .
\]

1.3 The function \( F(x) \) is said to be absolutely continuous in the second restricted sense on a bounded set \( E \), denoted by \( AC^2_x \), if to every \( \epsilon > 0 \) there corresponds an \( \eta > 0 \) such that, for every finite sequence of non-overlapping intervals \( I_k \) whose end points belong to \( E \), the inequality

\[
\sum |I_k| < \eta \quad \text{implies that} \quad \sum O^2(F, I_k) < \epsilon .
\]

1.4 The function \( F(x) \) is said to be generalized absolutely continuous in the second restricted sense, denoted by \( AC^2G_x \), on a bounded set \( E \), if \( F(x) \) is continuous on \( E \) and if \( E \)
is expressible as the sum of a sequence of bounded sets, on each of which \( F(x) \) is \( AC_x \).

1.5 The second restricted oscillation of a function \( F(x) \) over an interval \( I = (\alpha, \beta) \), denoted by \( W(F, I) \), defined if and only if \( F'(\alpha) \) and \( F'(\beta) \) exist and are finite, is the maximum of the numbers \( n_1 \) and \( n_2 \), where

\[
n_1 = \sup_{\alpha < t < \beta} \left| \frac{F(t) - F(\alpha)}{t - \alpha} - F'(\alpha) \right|
\]

\[
n_2 = \sup_{\alpha < t < \beta} \left| F'(\beta) - \frac{F(\beta) - F(t)}{\beta - t} \right|
\]

1.6 The upper (lower) restricted second oscillation of a function \( F(x) \) over an interval \( I = (\alpha, \beta) \) denoted by \( \bar{W}(F, I) \), \( (\bar{W}(F, I)) \), defined if and only if \( F'(\alpha) \) and \( F'(\beta) \) exist and are finite, is the maximum (minimum) of the numbers \( \bar{n}_1 \), \( (\bar{n}_1) \) and \( \bar{n}_2 \), \( (\bar{n}_2) \)

where

\[
\bar{n}_1, (\bar{n}_1) = \sup_{\alpha < t < \beta} (\inf) \left[ \frac{F(t) - F(\alpha)}{t - \alpha} - F'(\alpha) \right]
\]

\[
\bar{n}_2, (\bar{n}_2) = \sup_{\alpha < t < \beta} (\inf) \left[ F'(\beta) - \frac{F(\beta) - F(t)}{\beta - t} \right]
\]

1.7 The function \( F(x) \) is said to be upper absolutely continuous in the second restricted sense, if \( F'(x) \) exists and
is finite at every point of $E$, and if to every $\epsilon > 0$ there corresponds an $\eta > 0$ such that for every finite sequence of non overlapping intervals $I_k$ with end points belonging to $E$, the inequality $\sum |I_k| < \eta$ implies that $\sum W(F I_k) < \epsilon$.

1.8 The function $F(x)$ is said to be lower absolutely continuous in the second restricted sense, denoted by $LAC^2_x$, over a bounded set $E$ if $F'(x)$ exists and is finite at every point of $E$, and if to each $\epsilon > 0$ there corresponds an $\eta > 0$ such that for every finite sequence of non overlapping intervals $I_k$ whose end points belong to $E$, the inequality $\sum |I_k| < \eta$ implies that $\sum W(F I_k) > -\epsilon$.

1.9 The upper and lower generalized extreme second derivatives, denoted by $\Delta'' F(x)$ and $\delta'' F(x)$, respectively, are defined by the formulas

\[
\Delta'' F(x) = \limsup_{h \to 0} \liminf_{k \to 0} \frac{2}{h + k} \left[ \frac{F(x + h) - F(x)}{h} - \frac{F(x) - F(x - k)}{k} \right],
\]

\[
\delta'' F(x) = \liminf_{h \to 0} \limsup_{k \to 0} \frac{2}{h + k} \left[ \frac{F(x + h) - F(x)}{h} - \frac{F(x) - F(x - k)}{k} \right],
\]

where $h$ and $k$ may tend to zero in any manner.

1.10 If $\Delta'' F(x) = \delta'' F(x)$ the common value denoted by $D'' F(x)$ is equal to the generalized second derivative $D^2 F(x)$,
but the existence of $D^2 F(x)$ does not imply the existence of $D^n F(x)$.

2. Definition of the $P_x^2$ integral

This section is reproduced with minor modifications from a paper by James and Gage [2].

2.1 If $f(x)$ is defined in an interval $(a,c)$ the functions $M(x)$ and $m(x)$ are called major and minor functions, respectively, of $f(x)$ in $(a,c)$ if
   (a) $M(x)$ and $m(x)$ are continuous in $(a,c)$,
   (b) $M(a) = M(c) = m(a) = m(c) = 0$,
   (c) $\delta^"M(x) \geq f(x) \geq \Delta^"m(x)$,
   (d) $\delta^"M(x) \geq -\infty$, $\Delta^"m(x) < +\infty$,
for all $x$ in $(a,c)$ with the possible exception of an enumerable set.

2.2 A function $f(x)$ defined in an interval $(a,c)$ is said to be integrable over $(a,b,c)$, $a < b < c$ if for every $\epsilon > 0$ there exists a major function $M(x)$ and a minor function $m(x)$ such that
   $$0 \leq m(b) - M(b) < \epsilon$$

2.3 If $f(x)$ is integrable over $(a,x,c)$ then $F(x)$ exists such that $F(x)$ is the sup of all major functions $M(x)$ and $F(x)$ is the inf of all minor functions $m(x)$ and
\[ \int_{\text{axc}} f(x) \, dx = -F(x). \]

Furthermore

(a) \( M(x) - F(x) \) is convex in \((a, c)\),
(b) \( F(x) - m(x) \) is convex in \((a, c)\),
(c) \( M(x) - m(x) \) is convex in \((a, c)\).

2.4 If \( f(x) \) is integrable over \((a, b, c)\), \( a < b < c \) then there exists a major function \( M(x) \) and a minor function \( m(x) \) such that

\[ 0 \leq m(x) - M(x) < \text{Max.} \left[ \frac{c-x}{c-b} \epsilon, \frac{x-a}{b-a} \epsilon \right] \]

for all \( x \) in \((a, c)\).

Section 3  Functions \( AC^2_G_x, UA^2_G_x, LAC^2_x \) and \( AC^2_x \)

Denjoy [1] showed that if \(-\infty < \delta'' F(x) \leq \Delta'' F(x) < +\infty \) on a set \( E \) then \( F'(x) \) exists and is finite on \( E \). This result is used to find sufficient conditions that a function \( F(x) \) be \( AC^2_G_x, UA^2_G_x, LAC^2_x \), or \( AC^2_x \) on a set \( E \).

Theorem 3.1  If \( F(x) \) is a continuous function which fulfils the condition \(-\infty < \delta'' F(x) \leq \Delta'' F(x) < +\infty \) at all points of a set \( E \), except perhaps those of an enumerable subset, then \( F(x) \) is \( AC^2_G_x \) on \( E \).
Proof. For each positive integer \( n \), let \( A_n \) denote the set of points \( x \) of \( E \) such that \( 0 < h, 0 < k, 0 < h + k < \frac{1}{n} \) implies

\[-n \leq \frac{2}{n + k} \left[ \frac{F(x + h) - F(x)}{h} - \frac{F(x) - F(x - k)}{k} \right] \leq n,\]

and for every integer \( i \) let \( A_{n_i} \) denote the common part of \( A_n \) and of the interval \( \left[ \frac{i}{n}, \frac{i + 1}{n} \right] \).

\[E = \sum_{n=1}^{\infty} \sum_{i=-\infty}^{\infty} A_{n_i}.\]

If \( I \) is any interval \((a, \beta)\) whose end points belong to one of the sets \( A_{n_i} \), then for \( a \leq x - k \leq x \leq x + h \leq \beta \) where \( x \in A_{n_i} \) we have

\[-n \leq \frac{2}{h + k} \left[ \frac{F(x + h) - F(x)}{h} - \frac{F(x) - F(x - k)}{k} \right] \leq n.\]

If we let \( k \) tend to zero in this expression

\[-n \leq \frac{2}{h} \left[ \frac{F(x + h) - F(x)}{h} - F'(x) \right] \leq n.\]

Similarly if \( h \) tends to zero

\[-n \leq \frac{2}{h} \left[ \frac{F'(x) - F(x) - F(x - k)}{k} \right] \leq n.\]

Hence for \( a < t_1 < t_2 < \beta \)

\[-n \leq \frac{2}{t_2 - a} \left[ \frac{F(t_2) - F(a)}{t_2 - a} - F'(a) \right] \leq n.\]

Hence \(-\frac{n}{2}(t_2 - a) \leq \frac{F(t_2) - F(a)}{t_2 - a} - F'(a) \leq \frac{n}{2}(t_2 - a),\)

\[-\frac{n}{2}(t_1 - a) \leq \frac{F(t_1) - F(a)}{t_1 - a} - F'(a) \leq \frac{n}{2}(t_1 - a).\]
We obtain
\[-\frac{n}{2}(t-\alpha)-\frac{n}{2}(t-\alpha) \leq \left[ \frac{F(t_2)-F(\alpha)}{t_2-\alpha} - \frac{F(t_1)-F(\alpha)}{t_1-\alpha} \right] \leq \frac{n}{2}(t_2-\alpha)+\frac{n}{2}(t_1-\alpha),\]
and hence
\[\left| \frac{F(t_2)-F(\alpha)}{t_2-\alpha} - \frac{F(t_1)-F(\alpha)}{t_1-\alpha} \right| \leq \frac{n}{2} \left[ (t_2-\alpha)+(t_1-\alpha) \right] \leq n(\beta-\alpha).
\]
Similarly
\[\left| \frac{F(\beta)-F(t_2)}{\beta-t_2} - \frac{F(\beta)-F(t_1)}{\beta-t_1} \right| \leq n(\beta-\alpha),\]
and hence \(O^2(F, I) < n |I|\).

Then for any sequence of non-overlapping intervals belonging to one of the sets \(A_n^i\)

\[\sum O^2 (F, I_k) < n \sum |I_k| .\]

Therefore \(F(x)\) is AC\(_x\) on each of the sets \(A_n^i\) and hence AC\(^2\)G\(_x\) on \(E\).

**Theorem 3.2** If \(\Delta' F(x) < +\infty\) and \(F'(x)\) exists and is finite on a set \(E\), then \(E\) is the sum of an enumerable sequence of subsets \(E_n^i\) on each of which \(F(x)\) is UAC\(^2\)\(_x\).

**Proof** For each positive integer \(n\) let \(A_n^i\) denote the set for which the inequality \(0 < h + k < \frac{1}{n}\) implies
\[\frac{2}{h+k} \left[ \frac{F(x+h) - F(x)}{h} - \frac{F(x) - F(x-k)}{k} \right] \leq n \quad \text{and for every integer } i \text{ let } A_n^i \text{ denote the common part of the set } A_n \text{ and the interval } \left( \frac{i}{n}, \frac{i+1}{n} \right). \]

Then \(E = \sum_{n=1}^{\infty} \sum_{i=-\infty}^{\infty} A_n^i\).

For any interval \(I = (\alpha, \beta)\) with end points belonging to a set
where \( x \in A_n \)

\[
\frac{2}{h + k} \left[ \frac{F(x + h) - F(x)}{h} - \frac{F(x) - F(x - k)}{k} \right] \leq n .
\]

Letting \( k \) tend to zero in this expression we obtain

\[
\frac{F(x + h) - F(x)}{h} - F'(x) \leq \frac{n}{2} h ,
\]

and, letting \( h \) tend to zero

\[
F'(x) - \frac{F(x) - F(x - k)}{k} \leq \frac{n}{2} k .
\]

For \( \alpha < t < \beta \) we have

\[
\frac{F(t) - F(\alpha)}{t - \alpha} - F'(\alpha) \leq \frac{n}{2} (t - \alpha) ,
\]

and

\[
F'(\beta) - \frac{F(\beta) - F(t)}{\beta - t} \leq \frac{n}{2} (\beta - t) .
\]

Hence

\[
\overline{W}(F, I) < n |I| .
\]

Therefore for any finite sequence of non-overlapping intervals with end points belonging to a set \( A_n \)

\[
\sum \overline{W}(F, I_k) < n \sum |I_k| \leq \epsilon .\quad \text{For} \quad \sum |I_k| < \frac{\epsilon}{n} .
\]

Therefore \( F(x) \) is UAC\(^2_x\) on each of the sets \( A_n \).

**Theorem 3.2** If \( \delta'' F(x) > -\infty \) and \( F'(x) \) exists and is finite on a set \( E \), then \( E \) is the sum of an enumerable sequence of subsets \( S_m \) on each of which \( F(x) \) is LAC\(^2_x\).

**Proof** Similar to Theorem 3.2
Theorem 3.4  If \( F'(x) \) exists and is finite at all points of a set \( E \), and if \( F(x) \) is \( AC^2_x \) on \( E \), then \( F(x) \) is both \( UAC^2_x \) and \( LAC^2_x \) on \( E \). Moreover if \( F(x) \) is both \( UAC^2_x \) and \( LAC^2_x \) on \( E \) then \( F(x) \) is necessarily \( AC^2_x \) on \( E \).

**Proof:** If \( F'(x) \) exists and is finite on \( E \), we have for any interval \( I = (\alpha, \beta) \) with end points belonging to \( E \), for \( \alpha \leq t_1 \leq t_2 \leq \beta 
\[
\frac{F(t_2)-F(\alpha)}{t_2-\alpha} - \frac{F(t_1)-F(\alpha)}{t_1-\alpha} = \left[ \frac{F(t_2)-F(\alpha)}{t_2-\alpha} - F'(\alpha) \right] - \left[ \frac{F(t_1)-F(\alpha)}{t_1-\alpha} - F'(\alpha) \right]
\]

Hence \( \bar{n}_1 - \bar{n}_1 \leq \frac{F(t_2)-F(\alpha)}{t_2-\alpha} - \frac{F(t_1)-F(\alpha)}{t_1-\alpha} \leq \bar{n}_1 - \bar{n}_1 \).

Similarly \( n_2 - \bar{n}_1 \leq \frac{F(\beta)-F(t_2)}{\beta-t_2} - \frac{F(\beta)-F(t_1)}{\beta-t_1} \leq \bar{n}_2 - \bar{n}_2 \).

Therefore \( O^2 FI \leq \bar{W}(F,I) - W(F,I) \).

Therefore if \( F(x) \) is both \( UAC^2_x \) and \( LAC^2_x \) then \( F(x) \) is \( AC^2_x \) on \( E \).

However from the definitions
\[
\bar{n}_1 \leq \bar{m}_1, \\
\bar{n}_2 \leq \bar{m}_2, \\
\bar{n}_1 \geq \bar{m}_1, \\
\bar{n}_2 \geq \bar{m}_2.
\]

Therefore \( \bar{W}(F,I) \leq O^2(F,I) \), \( W(F,I) \geq -O^2(F,I) \).
Therefore if $F(x)$ is $AC^2_x$ on a set $E$ then $F(x)$ is both $UAC^2_x$ and $LAC^2_x$ on $E$.

4. **Properties of Convex Functions**

The results of this section are reproduced from the work of Hardy, Littlewood and Polya [5].

4.1 If $g(x)$ is convex in an interval $(a,c)$ and bounded above in some interval interior to $(a,c)$ then $g(x)$ has the following properties:

(a) $g(x)$ is continuous in $(a,c)$;

(b) The left and right hand derivatives $g^-(x)$ and $g^+(x)$ exist for all $x$ in $(a,c)$;

(c) $g^-(x) \leq g^+(x)$ for all $x$ in $(a,c)$;

(d) The derivative $g'(x)$ exists for all $x$ in $(a,c)$ with the possible exception of an enumerable set.

5. **Properties of major and minor functions, and of the function** $F(x) = -\int_{a}^{x} f(t) \, dt$.

Denjoy has shown that $\Delta'' F(x) < +\infty$ implies that $\bar{F}(x) \leq \tilde{F}(x)$ and that $\delta'' F(x) > -\infty$ implies that $\bar{F}(x)^- \leq \tilde{F}(x)^+ [1]$.

However $\bar{F}(x) \geq \tilde{F}(x)$ except on an enumerable set and $\bar{F}(x)^- \geq \tilde{F}(x)^+$ except on an enumerable set [3].

From these results it follows that

1. $\Delta'' m(x) < +\infty$ implies $\bar{m}(x) = \tilde{m}(x)$ except on an enumerable set.
2. \( \delta^\infty M(x) > -\infty \) implies \( \overline{M}(x) = \underline{M}(x) \) except on an enumerable set.

**Theorem 5.1** Let \( F(x) = - \int_a^x f(t) \, dt \), \( M(x) \) any major function, \( m(x) \) any minor function.

Then \( F'(x) \), \( M'(x) \) and \( m'(x) \) exist and are finite on the interval \((a, c)\) except perhaps on an enumerable set.

**Proof:** \( M(x) = m(x) + g(x) \) where \( g(x) \) is convex

Therefore \( \overline{M}(x) = \overline{m}(x) + g'(x) \),

and \( \underline{M}(x) = \underline{m}(x) + g'(x) \).

Therefore \( \overline{M}(x) - \underline{M}(x) = \overline{m}(x) - \underline{m}(x) \),

\[ 0 = \overline{m}(x) - \underline{m}(x) \]

and hence \( \overline{m}(x) = \underline{m}(x) \).

Therefore \( \overline{m}(x) = \overline{m}(x) \leq \overline{m}(x) = \overline{m}(x) \leq \overline{m}(x) \) except perhaps on an enumerable set.

Therefore \( m'(x) \) exists, except perhaps on an enumerable subset.

Furthermore \( \overline{M}(x) = \overline{m}(x) + g'(x) \),

\[ \underline{M}(x) = \underline{m}(x) + g'(x) \]

Therefore \( \overline{M}(x) - \underline{M}(x) = \overline{m}(x) - \underline{m}(x) = 0 \).

It follows that \( \overline{M}(x) = \overline{m}(x) \leq \overline{M}(x) = \overline{m}(x) \leq \overline{M}(x) \) except perhaps on an enumerable subset.

Therefore \( M'(x) \) exists, except perhaps on an enumerable subset.

Similarly since \( M(x) = F(x) + h(x) \) where \( h(x) \) is convex
F'(x) exists except perhaps on an enumerable subset.

**Theorem 5.2** If for a major function $M(x)$, the inequality $\delta'M(x) > -\infty$ is satisfied on an interval $(\alpha, \beta)$, then the interval $(\alpha, \beta)$ is the sum of at most an enumerable infinity of non-overlapping subsets on each of which $M(x)$ is $\text{LAC}_x$.

**Proof:** From Theorem 5.1 $M'(x)$ exists on $(\alpha, \beta)$ except perhaps on an enumerable set. Let $E$ be the subset of $(\alpha, \beta)$ on which $M'(x)$ exists. Then $(\alpha, \beta) = E + I_0$ where $I_0$ is an enumerable set. By Theorem 3.3 $E = \sum_{n=1}^{\infty} E_n$, at most an enumerable infinity of sets on each of which $M(x)$ is $\text{LAC}_x$. Now $I_0 = \sum_{n=1}^{\infty} a_n$ where each $a_n$ is a single point. Hence $M(x)$ is trivially $\text{LAC}_x$ on each $a_n$. Therefore $(\alpha, \beta) = \sum_{n=1}^{\infty} E_n + \sum_{n=1}^{\infty} a_n$, at most an enumerable infinity of non-overlapping subsets on each of which $M(x)$ is $\text{LAC}_x$.

**Theorem 5.3** If for a minor function $m(x)$, the inequality $\Delta m(x) < +\infty$ is satisfied on an interval $(\alpha, \beta)$, then the interval $(\alpha, \beta)$ is the sum of at most an enumerable infinity of non-overlapping subsets, on each of which $m(x)$ is $\text{UAC}_x$.

**Proof:** Similar to Theorem 5.2.

**Theorem 5.4** Let $F(x) = -\int_{\alpha}^{x} f(x) \, dx$ and let $(\alpha, \beta)$ be an interval interior to the interval $(\alpha, c)$. Then $F(x)$ is $\text{AC}_x^2$ on $(\alpha, \beta)$. 

Proof: By Theorem 5.1 we can write \((a, \beta) = E + \sum_{n=1}^{\infty} a_n\)
where \(F'(x), M'(x)\) and \(m'(x)\) exist and are finite on 
\(E\) and each \(a_n\) is a single point.

By Theorem 5.2 \(E = \sum_{n=1}^{\infty} E_n\) where \(M(x)\) is 
L.A.C. on each \(E_n\).

Now \(F(x) = M(x) - G(x)\) where \(G(x)\) is convex. Therefore 
for any sequence of non overlapping intervals \(I_k = (\alpha_k, \beta_k)\) 
with end points belonging to one of the \(E_n\), and such that 
\(\sum \|I_k\| < \infty\), we have for \(\alpha_k < t_k < \beta_k\)
\[
\frac{F(t_k) - F(\alpha_k)}{t_k - \alpha_k} - F'(\alpha_k) = \left[\frac{M(t_k) - M(\alpha_k)}{t_k - \alpha_k} - M'(\alpha_k)\right] - \left[\frac{G(t_k) - G(\alpha_k)}{t_k - \alpha_k} - G'(\alpha_k)\right],
\]
and
\[
F'(\beta_k) - \frac{F(\beta_k) - F(t_k)}{\beta_k - t_k} = \left[\frac{M'(\beta_k) - \frac{M(\beta_k) - M(t_k)}{\beta_k - t_k}}{\beta_k - t_k}\right] - \left[\frac{G'(\beta_k) - \frac{G(\beta_k) - G(t_k)}{\beta_k - t_k}}{\beta_k - t_k}\right].
\]

Since \(G(x)\) is convex
\[
G'(\alpha_k) \leq \frac{G(t_k) - G(\alpha_k)}{t_k - \alpha_k} \leq G'(\beta_k).
\]
\[
G'(\alpha_k) \leq \frac{G(\beta_k) - G(t_k)}{\beta_k - t_k} \leq G'(\beta_k).
\]

Whence
\[
\frac{F(t_k) - F(\alpha_k)}{t_k - \alpha_k} - F'(\alpha_k) \geq \left[\frac{M(t_k) - M(\alpha_k)}{t_k - \alpha_k} - M'(\alpha_k)\right] - \left[\frac{G'(\beta_k) - G'(\alpha_k)}{\beta_k - \alpha_k}\right].
\]
\[
F'(\beta_k) - \frac{F(\beta_k) - F(t_k)}{\beta_k - t_k} \geq \left[\frac{M'(\beta_k) - \frac{M(\beta_k) - M(t_k)}{\beta_k - t_k}}{\beta_k - t_k}\right] - \left[\frac{G'(\beta_k) - G'(\alpha_k)}{\beta_k - \alpha_k}\right].
\]

Hence
\[
\sum_{k=1}^{n} W(F, I_k) \geq \sum_{k=1}^{n} W(M, I_k) - \sum_{k=1}^{n} \left[\frac{G'(\beta_k) - G'(\alpha_k)}{\beta_k - \alpha_k}\right].
\]

Since \(g(x)\) is convex
\[
\sum_{k=1}^{n} \left[\frac{G'(\beta_k) - G'(\alpha_k)}{\beta_k - \alpha_k}\right] \leq G'(\beta) - G'(\alpha).
\]
Therefore \( \sum_{k=1}^{n} W(F, I_k) \geq \sum_{k=1}^{n} W(M, I_k) - \left[ G'(\beta) - G'(\alpha) \right] \).

However \( G'(\beta) \leq \frac{G(c) - G(\beta)}{c - \beta} \) and
\[
G'(\alpha) \geq \frac{G(\alpha) - G(c)}{\alpha - a} \quad \therefore \quad G'(\beta) - G'(\alpha) \leq \frac{G(c) - G(\beta)}{c - \beta} - \frac{G(\alpha) - G(c)}{\alpha - a}.
\]

But \( M(\alpha) = M(c) = F(\alpha) = F(c) = 0 \).

Therefore \( G(\alpha) = G(c) = 0 \)
and hence \( G'(\beta) - G'(\alpha) \leq \frac{-G(\beta)}{c - \beta} - \frac{G(\alpha)}{\alpha - a} \).

whence \( -\left[ G'(\beta) - G'(\alpha) \right] \geq \frac{G(\beta)}{c - \beta} + \frac{G(\alpha)}{\alpha - a} \).

However \( G(x) = -\left[ F(x) - M(x) \right] \).

Therefore \( -\left[ G'(\beta) - G'(\alpha) \right] \geq -\frac{F(\beta) - M(\beta)}{c - \beta} - \frac{F(\alpha) - M(\alpha)}{\alpha - a} \).

From Theorem 2.4
\[
F(x) - M(x) \leq m(x) - M(x) \leq \max_{\epsilon}(\frac{c-x}{c-b}, \frac{x-a}{b-a}) \epsilon
\]
for \( 0 \leq m(b) - M(b) < \epsilon \), \( a < b < c \).

We choose \( m(x) \) and \( M(x) \) such that
\[
0 < m(b) - M(b) < \min (c - b) \epsilon, (b - a) \epsilon.
\]

Therefore
\[
\frac{F(\beta) - M(\beta)}{c - \beta} < \epsilon \quad \text{and} \quad \frac{F(\alpha) - M(\alpha)}{\alpha - a} < \epsilon.
\]
It follows that \( \sum_{k=1}^{n} W(F, I_k) \geq \sum_{k=1}^{n} W(M, I_k) - 2 \epsilon \).

Therefore \( F(x) \) is \( LAC_x \) on each \( E_n \).

Similarly \( E = \sum S_m \) where \( m(x) \) is \( UAC_{\infty} \) on each \( S_m \)
and hence \( F(x) \) is \( UAC_{\infty} \) on each \( S_m \).

Now \( (\alpha, \beta) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_n S_m + \sum_{n=1}^{\infty} a_n \)
and \( F(x) \) is both \( LAC_x \) and \( UAC_x \) on each \( E_n S_m \)
and trivially \( AC_x \) on each of the points \( a_n \).
Therefore \( F(x) \) is \( AC_G x \) on \( (\alpha, \beta) \).

Section 6. The differentiability of the \( P^2 x \) integral.

This section is reproduced with minor changes from a paper
by James and Gage [2].

6.1 Let \( f(x) \) be integrable over \( (a, b, c) \) and let
\( F(x) = - \int_{a}^{c} f(x) \, dx \).

Then for almost all \( x \) in \( (a, c) \) the function \( D''F(x) \)
is finite and equal to \( f(x) \).

This result together with Denjoy's proof [1]
that \( D''F(x) = F''(x) \) (the second ordinary approximate
\( \text{derivative of } F(x) \) ) almost everywhere shows that if \( f(x) \)
is \( P^2 x \) integrable and \( F(x) = - \int_{a}^{c} f(x) \, dx \) then \( F(x) \)
is \( AC^2 G_x \) in any interval \( (\alpha, \beta) \) interior to \( (a, c) \) and that
\[ F_n(x) = f(x) \] for almost all \( x \) in \((\alpha, \beta)\).

Section 7. Properties of \( AC^2 G_x \) Functions.

Theorem 7.1 If \( F(x) \) is \( AC^2 G_x \) on a set \( E \), then \( F'(x) \) exists almost everywhere on \( E \).

Proof: Let \( I \) be an interval \((\alpha, \beta)\) with end points \( \alpha, \beta \) belonging to \( E \) and such that \( \alpha \) is not isolated on the right and \( \beta \) is not isolated on the left.

Let \( |I| < \eta \)

Since \( F(x) \) is \( AC^2 G_x \) on \( E \) we have for \( \alpha < t_1 < t_2 < \beta \)

\[
\left| \frac{F(t_2) - F(\alpha)}{t_2 - \alpha} - \frac{F(t_1) - F(\alpha)}{t_1 - \alpha} \right| < \epsilon
\]

Since \( \alpha \) is not isolated on the right let \( t_1 \) approach \( \alpha \) through points of \( E \), and let \( t_2 \) approach \( \beta \),

then \( -\epsilon < \frac{F(\beta) - F(\alpha)}{\beta - \alpha} - \overline{F}'(\alpha) < \frac{F(\beta) - F(\alpha)}{\beta - \alpha} - \underline{F}'(\alpha) < \epsilon \).

Hence \( \overline{F}'(\alpha) \) and \( \underline{F}'(\alpha) \) are finite.

Similarly

\( -\epsilon < \frac{F(\beta) - F(\alpha)}{\beta - \alpha} - \frac{F(\beta) - F(\alpha)}{\beta - \alpha} < \epsilon \).

\( \overline{F}'(\beta) \) and \( \underline{F}'(\beta) \) are finite.

Let \( E_1 \) denote the subset of \( E \) of points which are not isolated on both the left and the right.

Then for every point of \( E_1 \) the two Dini derivatives on the same non isolated side are finite.
Hence $F'(x)$ exists almost everywhere on $E_1$ [4].

Moreover the set of points isolated on both the left and the right is clearly enumerable. Therefore $F'(x)$ exists almost everywhere on $E$.

The Dini derivatives are known to be measurable [4] when $F(x)$ is defined over a measurable set $E$; and if in addition the Dini derivatives are VB over a set $E_1 \subset E$ then they are approximately derivable almost everywhere on $E$ [4]. We use these results to prove

**Theorem 7.2** If $F(x)$ is $AC^2_x$ on a bounded measurable set $E$, then $F''_{oa}(x)$ exists almost everywhere on $E$.

**Proof:** $F'(x)$ exists almost everywhere on $E$ (Theorem 7.1). Let $E_1$ be the set on which $F'(x)$ is defined.

$$E = E_1 + E_0$$ where $E_0$ is of measure zero.

Since $E$ is measurable so is $E_1$ and $|E| = |E_1|$.

Let $I = (\alpha, \beta)$ be any interval with end points belonging to $E_1$.

We have

$$0^2(F,I) < \frac{F(\beta) - F(\alpha)}{\beta - \alpha} - F'(\alpha) \leq 0^2(F,I),$$

$$0^2(F,I) < F'(\beta) \frac{F(\beta) - F(\alpha)}{\beta - \alpha} \leq 0^2(F,I).$$

Hence

$$-2 \ 0^2(F,I) < F'(\beta) - F'(\alpha) < 2 \ 0^2(F,I),$$

and hence $F'(x)$ is $AC$ on the set $E_1$ if $F(x)$ is $AC^2_x$ on $E_1$. Moreover $F'(x)$ being $AC$ on $E_1$ is necessarily VB on $E_1$ [4].
Now on $E_1$ $F'(x)$ coincides with the Dini derivatives. Therefore $F'(x)$ is approximately derivable almost everywhere on $E_1$ and hence almost everywhere on $E$.

**Section 8. Definition of the Integral.**

**Definition 8.1** $f(x)$ is said to be $D^2_x$ integrable over an interval $(a,c)$ if there exists a function $F(x)$ such that

1. $F(x)$ is AC $G_x$ on $(a,c)$.
2. $F''_{oa}(x) = f(x)$ almost everywhere.

The function $-F(x)$ is called the indefinite $D^2_x$ integral of $f(x)$.

**Definition 8.2** The definite $D^2_x$ integral of $f(x)$ over the interval $(a,b,c)$, denoted by $D^2_x \int_{a,b,c} f(x) \, dx$, $a < b < c$, is given by the formula

$$D^2_x \int_{a,b,c} f(x) \, dx = \frac{b-a}{c-a} F(a) + \frac{c-b}{c-a} F(c) - F(b).$$

**Theorem 8.3** If $f(x)$ is $P^2_x$ integrable in the interval $(a,b,c)$ then $f(x)$ is $D^2_x$ integrable in any interval interior to $(a,c)$.

**Proof:** By Theorem 6.1 if $F(x) = -P^2_x \int_{abc} f(x) \, dx$ then $F''_{oa} = f(x)$ almost everywhere on $(a,c)$, and from Theorem 5.4 $F(x)$ is $AC^2 G_x$ on any interval $(\alpha, \beta)$ interior to $(a,c)$, hence $f(x)$ is integrable in the $D^2_x$ sense.
Bibliography


