EXISTENCE OF PERIODIC SOLUTIONS OF CERTAIN NON-LINEAR DIFFERENTIAL EQUATIONS

by

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ABSTRACT

The theory of Poincaré and Bendixson is applied to establish the existence of periodic solutions of the differential equation

\[ \ddot{x} + f(x, \dot{x})\dot{x} + g(x) x = 0 \]

One part of the work is concerned with those equations which can be considered as arising from small perturbations of other equations of the same type, already possessing periodic solutions. Two existence theorems are demonstrated and the stability and uniqueness of the periodic solutions is also discussed.

The other part contains several theorems stating sufficient conditions for existence of periodic solutions which cannot be treated by perturbation methods.
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INTRODUCTION

The investigation carried out in the present work is concerned with the existence of periodic solutions of the differential equation

\[ \frac{d^2x}{dt^2} + f(x, \frac{dx}{dt}) \cdot \frac{dx}{dt} + q(x) \cdot x = 0 \]

where both variables \( t \) and \( x \) are assumed to be real.

In the particular case when the functions \( f(x, v) \) and \( g(x) \) are constant, the above equation becomes a linear equation with constant coefficients and is readily integrable. The question of periodicity of a solution is then most readily settled by examining its analytical expression obtained by integration.

A wide class of physical problems leads, however, to differential equations of type (A) which are not linear. The majority of such non-linear equations can be integrated only by approximation methods which have to be applied to each particular equation separately and which may require excessive time or labour. On the other hand the question of major physical interest is often just the eventual existence (and stability) of a periodic solution, while the actual shape of the resulting oscillation is of secondary importance. For this reason the question of the existence of periodic solutions has been investigated by a number of authors over the past thirty years.
One of the first investigators of the problem was B. van der Pol [11] who considered the equation

\[ \frac{d^2 \alpha}{dt^2} + \mu (\alpha^2 - 1) \frac{d\alpha}{dt} + \alpha = 0 \quad \mu > 0 \]

governing the oscillations in a circuit containing a vacuum triode. It was shown by Lienard [8] that for all values of \( \mu \) equation (B) possesses a periodic solution such that all non-periodic solutions approach, as \( t \to \infty \), the periodic one in an "asymptotic" fashion.

Similar results were later established for wider classes of non-linear equations of type (A) by N. Levinson and O. Smith [7], Filippor [5], A.V. Dragilev [4], A. de Castro [2] and others. All of these authors used in their investigations the Poincare-Bendixson theory of second order autonomous differential systems developed in the works of H. Poincare' [10] and I. Bendixson [1]. This theory forms also the basis of the present work. The particular techniques in demonstrating some of the theorems are, however, slightly different from those used by previous investigators.

Chapter I gives a brief account of the well-known general properties of solutions of equation (A) and some basic theorems used later in the text. Chapters II and III contain several cases of sufficient conditions for the existence of non-trivial periodic solutions. Some of the results appear to be original.
CHAPTER ONE. Preliminary Theorems

1.1. The Phase Plane

Let us consider the differential equation

\[ \frac{d^2x}{dt^2} + f(x, \frac{dx}{dt}) \cdot \frac{dx}{dt} + g(x) \cdot x = 0 \]  

(1.1.1)

This equation is equivalent to the following second order differential system

\[ \frac{dx}{dt} = v \]

(1.1.2)

\[ \frac{dv}{dt} = - f(x, v) \cdot v - g(x) \cdot x \]

which, in turn, can be reduced to the first order differential equation

\[ \frac{dv}{dx} = - \frac{f(x, v)}{v} - \frac{g(x)}{v} \cdot \frac{x}{v} \]

(1.1.3)

If a solution \( v = v(x) \) of (1.1.3) is known then the corresponding solution of (1.1.2) or (1.1.1) can be obtained by straightforward integration. Consequently, the problem is reduced to the investigation of solutions of (1.1.3).

Let us now consider the \((x, v)\) - plane, known as the phase-plane.

The solutions \( v = v(x) \) of (1.1.3) may be represented in the phase-plane by certain curves, called the trajectories (or characteristics) of the system (1.1.2) (or (1.1.1)). For a given point \((x, v)\) the vector \( (\frac{dx}{dt}, \frac{dv}{dt}) \), as given by
(1.1.2) and called the field-vector, has the direction of the tangent to the trajectory through this point. It is clear, therefore, that the direction of a trajectory through a given point is uniquely defined for all points in the phase plane except those where both $\frac{dx}{dt}$ and $\frac{dv}{dt}$ vanish, which are called the singular points (or singularities). One such singularity is the origin. The others may occur on the x-axis at the points where $g(x) = 0$.

Points of the phase-plane which are not singular are called regular points.

1.2. Existence and General Behavior of the Solutions.

It is well known (see, for instance [6]) that the system (1.1.2) possesses a unique solution $x = x(t)$ satisfying given initial conditions $x(t_0) = x_0$, $v(t_0) = v_0$ provided the functions $f(x, v)$ and $g(x)$ are locally Lipschitzian; that is, they satisfy a Lipschitz condition in every bounded region in the phase-plane. This property of functions $f(x, v)$ and $g(x)$ will, consequently, be tacitly assumed throughout the present work.

Trajectories of (1.1.2) in the phase plane possess some important properties which are constantly used in proofs of the theorems given in the subsequent two chapters. One such property is that two trajectories cannot intersect at a regular point. A rigorous proof of this fact can be found
elsewhere. It is also clear from (1.1.2) that the trajectories are directed curves, the positive direction being that of increase in \( t \). In particular, a trajectory passing through a (regular) point \( M \) not on the \( x \)-axis must necessarily be crossing the vertical line through \( M \) from left to right if \( M \) is above the \( x \)-axis and from right to left if \( M \) is below the \( x \)-axis. At the regular points on the \( v \)-axis the slope of trajectories is vertical and these are the only points with this property.

It is well known that periodic solutions of (1.1.2) correspond to closed trajectories (with no singularities on them) and that each such closed trajectory, if it exists, must contain at least one singularity in its interior. Without loss of generality we may assume this singularity to be the origin because if it were a point \( S = (a, 0) \) \((a \neq 0)\), the transformation \( x_1 = x - a \) will reduce the equation (1.1.1) to an equation of the same type for \( x_1 \) with the relevant singularity at the origin.

From (1.1.2) it is clear that the origin itself (and, therefore, every other singularity) may be considered as a degenerated (closed) trajectory, corresponding to the trivial solution

\[
\varpi(t) = 0 \quad \nu(t) = 0
\]

Since this solution is ever-present it is understood that any investigation of existence of periodic solutions actually involves non-trivial solutions only.
1.3. Conservative Systems.

As mentioned in the introduction, the system (1.1.2) is generally not integrable explicitly. Apart from the linear and other particular cases, there exists an important exception to this rule, namely when $f(x, v) = 0$. In this case the equation (1.1.3) assumes the form

$$\frac{dv}{dx} = -q(x) \cdot \frac{x}{v}$$

and is readily integrable, yielding

$$\frac{1}{2} v^2 + G(x) = \lambda = \text{Const}.$$

where

$$G(x) = \int_0^x q(\gamma) \cdot \gamma \, d\gamma$$

and $\lambda$ represents a parameter depending upon the initial conditions, namely

$$\lambda = \frac{1}{2} \left[ v(t_0) \right]^2 + G\left[ x(t_0) \right].$$

Solving (1.3.2) for $v = v(x)$ and substituting into the first of equations (1.1.2) we obtain the solution of (1.1.1). The systems which have the property that $f(x, v) = 0$ are usually called conservative.

It is to be noted that the question of existence of closed trajectories of (1.3.1) may be usually settled without actually solving the equation. For instance if $g(x)$ is positive at $(0, 0)$ and, consequently, in a neighbourhood
R of the origin, then it is easy to see that there exist a neighbourhood $R_1$ of the origin such that all trajectories passing through $R_1$ are closed so that there is an infinity of periodic solutions. On the other hand, if $g(0) < 0$ and there are no other singular points it can be shown that no periodic solutions are possible.

Similar conclusions are also possible if $f(x, v)$ is an odd function of $x$, viz.

$$f(-\infty, v) = -f(\infty, v)$$

at least in a neighbourhood of the origin. It was shown by E. McHarg [9] that if this is the case and if $g(0) > 0$ then there exists a neighbourhood of the origin containing an infinity of closed trajectories, just as in the case of a conservative system.

1.4. The Theory of Poincaré and Bendixson.

Investigation of systems which do not fall into the categories described in the preceding paragraph is considerably more difficult. The main tool used by practically all authors in this field is the so-called Poincaré-Bendixson theory of plane trajectories. In the particular question of existence of a periodic solution the following theorem is of utmost importance:

Theorem 1.1 (Poincaré-Bendixson)

"If the arc of a trajectory of (1.1.2), corresponding to values of $t$ such that $t > t_0$ (for some fixed $t_0$) lies
entirely within a bounded closed region $R$ which is free of singularities, then the trajectory is either a closed curve itself, or it approaches spirally a closed trajectory of (1.1.2) lying within $R$.

Proof of this theorem may be found, for instance, in [1] and will not be reproduced here. It is to be remarked that all existence theorems in this thesis are demonstrated by the application of Theorem 1.1.

In addition to this we shall quote another helpful theorem.

Theorem 1.2. (Dragilev)

"Suppose the D.E.

$$\frac{d^2x}{dt^2} + f_1(x, \frac{dx}{dt}) \cdot \frac{dx}{dt} + q(x) \cdot x = 0$$

(1.4.1)

(where the functions $f_1(x, v)$ and $g(x)$ are locally Lipschitzian) possesses a non-trivial periodic solution. Then the D.E.

$$\frac{d^2x}{dt^2} + f_2(x, \frac{dx}{dt}) \cdot \frac{dx}{dt} + q(x) \cdot x = 0$$

(1.4.2)

will also possess a non-trivial periodic solution provided

(a) $f_2(x, v)$ is locally Lipschitzian

(b) $f_2(0, 0) < 0$

(c) $f_2(x, v) \geq f_1(x, v)$ for all $(x, v)$.

Moreover, if $L_1$ and $L_2$ are closed curves in the phase-plane representing the periodic solutions of (1.4.1)
and (1.4.2), respectively, and if $R$ is the interior of $L_1$ then $L_2$ lies in $\overline{R}$ (closure of $R$).

Proof of this theorem is given in Dragilev's paper [4], as well as in a paper by Yelshin [12] who obtains the theorem as a special case of his "principle of comparison".

1.5. Non-conservative systems and their properties.

Let us now consider the system (1.1.2) in the most general case when $f(x, v)$ is an arbitrary (locally Lipschitzian) function of $x$ and $v$. By putting $f(x, v) = 0$ we obtain an associated conservative system

\[
\frac{dx}{dt} = v,
\]

(1.5.1)

\[
\frac{dv}{dt} = -g(x) \cdot \frac{x}{v}.
\]

Although the general picture of the trajectories in the phase-plane may be quite different for (1.1.2) and (1.5.1) it is clear, for instance, that both systems have the same singular points.

Moreover, it was already shown by Poincare that, unless $g(0) = 0$, the most important properties of these singularities also remain the same. In particular, it can be easily shown that if $g(0) < 0$ and if there are no singularities other than the origin then there can exist no non-trivial periodic solution of (1.1.2), just as is the case with (1.5.1).
If we define the function \( \lambda(x, v) \) by means of

\[
\lambda(x, v) = \frac{1}{2} v^2 + G(x)
\]

(1.5.2)

where

\[
G(x) = \int_0^x q(y) \gamma \, dy
\]

then \( \lambda(x, v) \) will no longer remain constant along the trajectories of (1.1.2) but, instead of this, will possess the following property:

Lemma 1.3.

"If \((x_0, v_0)\) and \((x_1, v_1)\) are two points on a trajectory \( T \) of (1.1.2) then

\[
\lambda(x_1, v_1) - \lambda(x_0, v_0) = - \int_{(x_0, v_0)}^{(x_1, v_1)} q(x, v) \cdot v \cdot dx
\]

(1.5.3)

where the integral on the r.h.s. is a curvilinear integral taken from \((x_0, v_0)\) to \((x_1, v_1)\) along the curve \( T \)."

Proof. Upon multiplication by \( v dx \) (1.1.3) becomes:

\[
v \cdot dv + q(x) \cdot x \cdot dx = - f(x, v) \cdot v \cdot dx
\]

Integrating from \((x_0, v_0)\) to \((x_1, v_1)\) along \( T \) we obtain

\[
\frac{1}{2} (v_1^2 - v_0^2) + \int_{x_0}^{x_1} q(x) \cdot x \cdot dx = - \int_{(x_0, v_0)}^{(x_1, v_1)} f(x, v) \cdot v \cdot dx
\]

From here (1.5.3) follows immediately by applying the definition (1.5.2).

In conclusion, we shall state another lemma which will be often used in later text:
Lemma 1.4.

"If \( g(0) > 0 \) and \( f(0, 0) < 0 \) then no trajectory of \((1.1.2)\) passing through a regular point will approach the origin as \( t \to \infty \)."

Proof. Consider the system of curves given by:

\[
\frac{1}{2} v^2 + G(x) = \lambda \quad \lambda > 0
\]

Because of the continuity of \( f(x, v) \) and \( g(x) \) there will exist a neighbourhood \( R' \) of the origin such that \( f(x, v) < 0 \) and \( g(x) > 0 \) everywhere in \( R' \). Then \( G(x) \) is non-negative in \( R' \) and increasing as \( |x| \) increases. From this, in turn, easily follows that \((1.5.4)\) represents, for small values of \( \lambda_1 \) a family of closed curves \( C_\lambda \) encircling the origin such that if \( \lambda_1 < \lambda_2 \) then \( C_{\lambda_1} \) lies in the interior of \( C_{\lambda_2} \).

Consider now a trajectory of \((1.1.2)\) passing through a regular point of \( R' \). From \((1.5.2)\) and \((1.1.2)\) it follows that along such a trajectory

\[
\frac{d\lambda}{dt} = \frac{\partial \lambda}{\partial v} \frac{dv}{dt} + \frac{\partial \lambda}{\partial x} \frac{dx}{dt} = v [ -f(x, v) \cdot v - g(x) \cdot x ] +
\]

\[
+ g(x) \cdot x \cdot v = -f(x, v) \cdot v^2
\]

It is seen that \( \frac{d\lambda}{dt} \geq 0 \) within \( R' \) so that the trajectories of \((1.1.2)\) recede from the origin and the lemma is proven.
2.1. Perturbations of conservative systems.

In this section we shall consider the systems which differ but very little from the conservative systems. It is convenient to represent such systems in the form

\[ \frac{dx}{dt} = \nu \]
\[ \frac{d\nu}{dt} = -\mu \cdot f(x, \nu) \cdot \nu - q(x) \cdot \infty \]

where the parameter \( \mu \) is assumed to be small.

Putting \( \mu = 0 \) in equations (2.1.1) we obtain a conservative system which is usually called the generating system corresponding to (2.1.1):

\[ \frac{dx}{dt} = \nu \]
\[ \frac{d\nu}{dt} = -q(x) \cdot \infty \]

We shall investigate the possibility of existence of a periodic solution of (2.1.1) around the singularity at the origin. For this purpose it is natural to start with the consideration of system (2.1.2). We shall restrict ourselves to the case when \( g(0) \neq 0 \).

It was already mentioned before that if \( g(0) < 0 \) and if the origin is the only singularity of the system then no (non-trivial) periodic solutions are possible. This will
be also true for the system (2.1.1), even if \( \mu \) is not small. Therefore we shall henceforward assume that \( g(0) > 0 \).

In this case there will exist a neighbourhood \( R_1 \) around the origin such that all trajectories of (2.1.2) passing through it will be closed (see Fig. 1). These trajectories will be given by

\[
\frac{1}{2} v^2 + G(\infty) = \lambda
\]

where

\[
G(\infty) = \int q(y) y \, dy
\]

and \( \lambda \) - a variable non-negative parameter. If \( R_1 \) is restricted in size the range of \( \lambda \) will also be restricted. This can actually happen if our system possesses other singularities beside the origin.

To maintain full generality we shall restrict ourselves to an open region \( R \) around the origin defined as the interior of a closed trajectory of the type (2.1.3) corresponding to a particular value \( \lambda = \lambda' \).

Suppose now that \( \mu \) is no longer zero so that we are dealing with the system (2.1.1). If \( \mu \) is small enough it is natural, however, to expect that the trajectories of (2.1.1) will differ but very little from the trajectories of (2.1.2).

This is indeed true and can be expressed as follows:
Lemma II.1.

Let \([x_1(t), v_1(t)]\) and \([x_2(t), v_2(t)]\) be the solutions of (2.1.1) and (2.1.2), respectively such that

\[
\| [x_1(t_0), v_1(t_0)] - [x_2(t_0), v_2(t_0)] \| < \varepsilon
\]

where \(\| (x, v) \| = \sqrt{x^2 + v^2}\) and both points \([x_1(t_0), v_1(t_0)]\) and \([x_2(t_0), v_2(t_0)]\) belong to an open neighbourhood \(R\) of the origin.

Then, as long as the points \([x_1(t), v_1(t)]\) and \([x_2(t), v_2(t)]\) remain within \(R\) the following inequality will hold for \(t \geq t_0\)

\[
\| [x_1(t), v_1(t)] - [x_2(t), v_2(t)] \| \leq \varepsilon \cdot \exp [L(t-t_0)] + \frac{M \cdot M}{L} \left\{ \exp [L(t-t_0)] - 1 \right\}
\]

where \(M = \sup_{(x,v) \in R} |f(x,v) \cdot v|\) and \(L\) is a constant \((x,v) \in R\).

Proof. The systems (2.1.2) and (2.1.1) can be respectively written in the vector notation as follows:

\[
\begin{align*}
\frac{dx}{dt} &= A(x) \\
\frac{dv}{dt} &= A(x) + B(x)
\end{align*}
\]

where \(Z = Z(t) = [x(t), v(t)]\) \(A(x) = [v, -q(x) \cdot x]\)

\(B(x) = [0, -\mu ; f(x, v) \cdot v]\)

Let \(\phi = \phi(t)\) and \(\psi = \psi(t)\) be the solution-vectors satisfying (2.1.5) and (2.1.6) respectively, such that
\[ \varphi(t_0) = [x_1(t_0), v_1(t_0)] \quad \psi(t_0) = [x_2(t_0), v_2(t_0)] \]

Then from (2.1.5) and (2.1.6) follows that

\[ \varphi(t) = \varphi(t_0) + \int_{t_0}^{t} A[\varphi(s)] \cdot ds \]

\[ \psi(t) = \psi(t_0) + \int_{t_0}^{t} \left\{ A[\varphi(s)] + B[\varphi(s)] \right\} ds \]

so that

\[ (2.1.7) \quad \|\psi(t) - \varphi(t)\| \leq \|\psi(t_0) - \varphi(t_0)\| + \int_{t_0}^{t} \|A(\varphi) - A(\psi)\| \cdot ds + \int_{t_0}^{t} \|B(\psi)\| \cdot ds \]

Let

\[ \Theta(t) = \|\psi(t) - \varphi(t)\| \]

Since \( g(x) \) is Lipschitzian and \( f(x, v)v \) is bounded by \( M \) we can write

\[ \|A[\psi(t)] - A[\varphi(t)]\| \leq L \cdot \Theta(t) \quad L = \text{Const.} \]

\[ \|B[\psi(t)]\| \leq \mu \cdot M \]

Consequently (2.1.7) becomes

\[ (2.1.8) \quad \Theta(t) \leq \epsilon + L \cdot \int_{t_0}^{t} \Theta(s) \cdot ds + \mu \cdot M \cdot (t - t_0) \]

If we define

\[ \Theta(t) = \int_{t_0}^{t} \Theta(s) \cdot ds \]

then the above inequality becomes

\[ \Theta'(t) - L \cdot \Theta(t) \leq \epsilon + \mu \cdot M \cdot (t - t_0) \]
Multiplying by \( \exp \left[ -L(t-t_0) \right] \) and integrating from \( t_0 \) to \( t \) we obtain
\[
\Theta(t) \cdot \exp \left[ -L(t-t_0) \right] \leq \frac{\varepsilon}{L} \left\{ 1 - \exp \left[ -L(t-t_0) \right] \right\} + \\
+ \frac{\mu \cdot M}{L^2} \left\{ 1 - [1 + \mu \cdot M(t-t_0) \exp \left[ -L(t-t_0) \right]] \right\}
\]
so that
\[
\Theta(t) \leq \frac{\varepsilon}{L} \left\{ \exp \left[ L(t-t_0) \right] - 1 \right\} + \frac{\mu \cdot M}{L^2} \left\{ \exp \left[ L(t-t_0) \right] - 1 - \mu \cdot M(t-t_0) \right\}.
\]
Substituting this expression into (2.1.8) we get
\[
\Theta(t) \leq \varepsilon \cdot \exp \left[ L(t-t_0) \right] + \frac{\mu \cdot M}{L} \left\{ \exp \left[ L(t-t_0) \right] - 1 \right\}
\]
and the lemma is proven.

Let us now choose a point \( A(x_0, v_0) \) in \( \mathbb{R} \) and consider a trajectory of (2.1.1) passing through it. If \( \mu = 0 \) this trajectory is closed and given by
\[
(2.1.9) \quad v = v_i(\infty, \lambda_0) = \sqrt{2\lambda_0 - 2G(\infty)}
\]
where
\[
\lambda_0 = \frac{1}{2} v_0^2 + G(\infty)
\]
In view of Lemma II.1 it is obvious that for \( \mu \neq 0 \) but sufficiently small our trajectory will have, in general, a "spiral shape". By this we mean that it will encircle the origin and cross the radiusvector \( AO \) through \( A \) at some point \( C \) lying close to \( A \). We are, of course, interested whether \( C \) will be closer to the origin than \( A \) or not. To answer this question we shall consider the following function:
\[
W(\lambda) = \int [x, v, (x, \lambda)] \cdot v, (x, \lambda) \, dx
\]
where the integral should be taken in clockwise direction around the closed curve defined by (2.1.9). Function \( W(\lambda) \) can be evaluated for all values of \( \lambda \) such that \( 0 \leq \lambda < \lambda_0 \). Its importance is seen from the following theorem.

Theorem II.2.

Let \( A = (x_0, v_0) \in R \) and let \( W(\lambda_0) > 0 \) where \( \lambda_0 = \frac{1}{2} v_0^2 + G(x_0) \). Then a number \( \mu_0 > 0 \) can be found such that for all values of \( \mu \) satisfying \( 0 < \mu \leq \mu_0 \) the trajectory \( T \) of (2.1.1) passing through \( A \) will make its successive crossing of radiusvector \( AO \) at a point \( C = (x_1, v_1) \) such that

\[
\sqrt{x_1^2 + v_1^2} < \sqrt{x_0^2 + v_0^2}
\]

Proof. Let \( \psi(t) \) and \( \varphi(t) \) be the vector-functions representing the solutions of (2.1.1) and (2.1.2), respectively, which pass through point \( A \) at \( t = t_0 \), viz.

\[
\psi(t) = [x_1(t), v_1(t)] \quad \psi(t_0) = [x_0, v_0]
\]

\[
\varphi(t) = [x_2(t), v_2(t)] \quad \varphi(t_0) = [x_0, v_0]
\]

We know that \( \varphi(t) \) is periodic. Let its period be \( T \). Then from Lemma II.1 follows that for \( t \in [t_0, t_0 + T] \) we shall have

\[
(2.1.11) \quad ||\psi(t) - \varphi(t)|| \leq \frac{\mu M}{L} \left( \exp LT - 1 \right) = \delta
\]

provided \( \psi(t) \) remains within \( R \). This inequality means that the point \( B \) defined by \( \psi(t_0 + T) \) will be within a
circle of radius \( \delta \) centered at the point A (see Fig. 2). By choosing \( \mu \) sufficiently small we can make \( \delta \) arbitrarily small. This will also ensure that \( \psi(t) \) remains within \( R \) for \( t \in [t_0, t_0 + T] \) so that (2.1.11) will be actually applicable.

Let us continue (if necessary) the trajectory defined by \( \psi(t) \) beyond the point B until we intersect the radius-vector \( AO \) and call the point of intersection \( C \).

We proceed to show that

\[ \lambda(C) < \lambda(A) \]

From Lemma 1.3 we know that

\[ Q = \lambda(A) - \lambda(B) = \]

\[ = \int_{A}^{B} \mu \cdot \mathbf{f}[x(t), \mathbf{v}(t)] \cdot \mathbf{v}(t) \cdot dx \]

Transforming this line-integral into an ordinary integral with respect to \( t \) we get

\[
Q = \int_{t_0}^{t_0 + T} \mu \cdot \mathbf{f}[x(t), \mathbf{v}(t)] \cdot [\mathbf{v}(t)]^2 \cdot dt
\]

Similarly, from (2.1.10) we obtain

\[
\mu \cdot W(\lambda_c) = \int_{t_0}^{t_0 + T} \mu \cdot \mathbf{f}[x_2(t), \mathbf{v}_2(t)] \cdot [\mathbf{v}_2(t)]^2 \cdot dt
\]

so that
Since $f(x, v)$ is Lipschitzian in $R$ we have

$$|f(x_1, v_1) - f(x_2, v_2)| \leq K \cdot \|v_1 - v_2\|$$

Therefore

$$|Q - \mu W| \leq \mu K \int_t^T \|v_1 - v_2\|^2 dt$$

Using the inequality (2.1.11) and also recalling that $|f(x, v)| \leq M$ and

$$|v_1 - v_2| \leq \|v_1 - v_2\|$$

we obtain

$$|Q - \mu W| \leq \mu K \delta \int_t^T v_1^2 dt + 2 \mu M \delta \int_t^T dt$$

Finally, $v_1$ is bounded in $R_1$ so $|v_1| \leq b$ and we obtain

$$|Q - \mu W| \leq \mu K \delta b^2 T + 2 \mu M \delta T = \mu^2 D = \Theta(\mu^2)$$

where

$$D = \frac{M \cdot T}{L} (K b^2 + 2 M) [\exp LT - 1] = \text{Const}.$$ 

It follows that if $\mu$ is chosen sufficiently small $Q$ will have the same sign as $\mu W$ and, therefore, be positive, which means that $\lambda(B) < \lambda(A)$.
To complete the proof we also note that
\[ |\lambda(c) - \lambda(B)| = | \int_B^C \mu \cdot \frac{1}{t} [\alpha_i, \nu_i(t)] \cdot \nu_i(t) \cdot d\alpha | \leq \]
\[ \leq \mu M |\alpha(b) - \alpha(c)| \leq \mu M \delta = \mu^2 \frac{M^2}{L} [\exp LT - 1] = O(\mu^2) \]

Now we have
\[ |\{ \lambda(c) - \lambda(A) \} - \mu W| \leq |\lambda(c) - \lambda(B)| + |\{ \lambda(B) - \lambda(A) \} - \mu W| = \]
\[ = |\lambda(c) - \lambda(B)| + |Q - \mu W| = O(\mu^2) \]

It follows that for sufficiently small \( \mu \) the quantity \( \lambda(C) - \lambda(A) \) will have the same sign as \( \mu W \). This yields \( \lambda(C) < \lambda(A) \) and the theorem is proven.

From the proof of the theorem the following corollary is immediate.

**Corollary.**

Let \( A = (x_0, v_0) \in \mathbb{R} \), \( W(\lambda_0) \) and \( C(x_1, v_1) \) be defined as in the theorem and let \( W(\lambda_0) < 0 \). Then for sufficiently small \( \mu \) the inequality
\[ \sqrt{\alpha_i^2 + \nu_i^2} > \sqrt{\alpha_0^2 + \nu_0^2} \]
will hold.

Using the results just developed we are now in a position to make some statements about the existence of closed trajectories of (2.1.1) for sufficiently small values of \( \mu \). It is quite clear that if such trajectories exist then they should be sought in the vicinity of such trajectories.
\( \Gamma_\lambda \) of (2.1.2) for which \( W(\lambda) = 0 \). Let us plot the function \( W(\lambda) \) vs. \( \lambda \) (see Fig. 3). Suppose \( W(\lambda) \) has a zero for some particular value of \( \lambda \), say \( \lambda = \lambda^* \neq 0 \).

Suppose this zero is isolated and such that in a neighbourhood of \( \lambda^* \)
\[
W(\lambda) > 0 \quad \text{for} \quad \lambda > \lambda^*
\]
\[
W(\lambda) < 0 \quad \text{for} \quad \lambda < \lambda^*
\]

Consider a thin ring around the origin bounded by the trajectories \( \Gamma_{\lambda^*-\epsilon} \) and \( \Gamma_{\lambda^*+\epsilon} \) of (2.1.2) corresponding to the values of \( \lambda \) equal to
\[
(\lambda^*-\epsilon) \quad \text{and} \quad (\lambda^*+\epsilon)
\]

If \( \epsilon \) is sufficiently small we shall have
\[
W(\lambda^*-\epsilon) < 0 \quad \text{and} \quad W(\lambda^*+\epsilon) > 0
\]

Let \( A_1 \) and \( A_2 \) be the points of intersection with the positive v-semiaxis of the trajectories \( \Gamma_{\lambda^*+\epsilon} \) and \( \Gamma_{\lambda^*-\epsilon} \) of (2.1.2).
By Theorem II.2 it follows that, for sufficiently small values of \( \mu \), the trajectories \( T_1 \) and \( T_2 \) of (2.1.1) passing through \( A_1 \) and \( A_2 \), respectively, will cross again the positive \( v \)-semiaxis at points \( C_1 \) and \( C_2 \), respectively, such that \( C_1 \) lies below \( A_1 \) and \( C_2 \) lies above \( A_2 \) (see Fig. 5). This establishes a region \( K \) (shaded on Fig. 5) bounded by arcs \( A_1C_1 \) and \( A_2C_2 \) of \( T_1 \) and \( T_2 \), respectively, and by segments \( A_1C_1 \) and \( A_2C_2 \). It is clear that a trajectory of (2.1.2) passing at \( t = t_0 \) through a point inside \( K \) will remain inside \( K \) as \( t \to \infty \) and so it will satisfy the conditions of Theorem I.1. It follows that for sufficiently small values of \( \mu \) the system (2.1.1) will possess a non-trivial periodic solution. The closed trajectory corresponding to this periodic solution will approach, as \( \mu \to 0 \), the closed trajectory \( T_{\lambda^*} \) of (2.1.2) corresponding to \( \lambda = \lambda^* \).

Suppose now we are dealing with a zero \( \lambda = \lambda^* \neq 0 \) of \( W(\lambda) \) such that in a neighbourhood of \( \lambda^* \)

\[
W(\lambda) < 0 \quad \text{for} \quad \lambda > \lambda^*
\]

and

\[
W(\lambda) > 0 \quad \text{for} \quad \lambda < \lambda^*
\]

In this case we can construct, in the same fashion as before, a region \( K \) which is trajectory of (2.1.1) can enter. If we now make the change of independent variable

\[ t = -t' \]
in the system (2.1.1) we can easily see that the new system, viz.

\[
\begin{align*}
\frac{dw}{dt'} &= -\nu \\
\frac{dw}{dt'} &= \mu \cdot f(x,\nu) \cdot \nu + q(x) \cdot \nu
\end{align*}
\]

(2.1.12)

will have the same trajectories as (2.1.1) but their direction will change. Since the region K will satisfy the conditions of Theorem I.1 for the system (2.1.12) it follows that the system (2.1.12) and, therefore, system (2.1.1) as well, will possess a non-trivial periodic solution if \( \mu \) is sufficiently small.

We can now summarize our results in the following Theorem:

Theorem II.3

For sufficiently small values of \( \mu \) the system (2.1.1) will possess non-trivial periodic solutions reducing, as \( \mu \to 0 \), to such closed trajectories \( \Gamma_{\lambda^*} (\lambda^* \neq 0) \) for which \( W(\lambda^*) = 0 \) and for which the quantity \( W(\lambda) \cdot (\lambda - \lambda^*) \) does not change sign in some open neighbourhood of \( \lambda^* \).

The investigation conducted thus far has not considered the question of stability nor the related question of the number of periodic solutions arising in the vicinity of the curve \( \Gamma_{\lambda^*} \). These questions can be, at least partially, answered in the case when the function \( W(\lambda) \) is continuously differentiable.
In order to consider this problem we introduce the following definition of stability, sometimes also called the strong, or asymptotic stability.

**Definition.**

A periodic solution \( \varphi(t) \) of (1.1.2) is called stable if \( \varepsilon > 0 \) can be found such that every solution \( \psi(t) \) of (1.1.2) satisfying

\[
\|\psi(t_0) - \varphi(t_0)\| \leq \varepsilon
\]

will possess the following property: For every point \( \varphi(t^*) \) of the periodic solution \( \varphi(t) \) a sequence \( \{t_n\} (n = 0,1,2, \ldots) \) where \( t_n \to \infty \) as \( n \to \infty \) can be found such that

\[
(2.1.13) \quad \lim_{n \to \infty} \|\psi(t_n) - \varphi(t^*)\| = 0
\]

It was already proved by Poincaré that a non-trivial periodic solution of (2.1.1) will be stable if the following condition is satisfied.

\[
J = \int_0^T \left[ v \frac{\partial f(x,v)}{\partial v} + f(x,v) \right] \, dt > 0
\]

where \( x = x(t) \) and \( v = v(t) \) represent the solution under consideration and \( T \) is its period.

We shall now show that the function \( \frac{dw}{dx} \) is nothing else than the integral \( J \), as defined above, taken along a curve \( \Gamma \lambda \). Indeed, if \( [x_1(\lambda), 0] \) and \( [x_2(\lambda), 0] \) are
the points of intersection of $\Gamma_\lambda$ with the negative and positive $x$-semiaxes, respectively, and if

$$v^+(\infty, \lambda) = + \sqrt{2\lambda - 2} \text{G}(\infty)$$

$$v^-(\infty, \lambda) = - \sqrt{2\lambda - 2} \text{G}(\infty)$$

then (2.1.10) can be written as:

$$W(\lambda) = \int_{x_1(\lambda)}^{x_2(\lambda)} f [x, v^+(x, \lambda)] \cdot v^+(x, \lambda) \cdot dx - \int_{x_1(\lambda)}^{x_2(\lambda)} f [x, v^-(x, \lambda)] \cdot v^-(x, \lambda) \cdot dx$$

Differentiating with respect to $\lambda$ and remembering that $v^+(x, \lambda)$ and $v^-(x, \lambda)$ vanish at $x_1(\lambda)$ and $x_2(\lambda)$ we obtain

$$\frac{dW}{d\lambda} = \int_{x_1(\lambda)}^{x_2(\lambda)} \left\{ \frac{\partial f [x, v^+]}{\partial v^+} \cdot \frac{dv^+}{d\lambda} \cdot v^+ + f [x, v^+] \cdot \frac{dv^+}{d\lambda} \right\} dx - \int_{x_1(\lambda)}^{x_2(\lambda)} \left\{ \frac{\partial f [x, v^-]}{\partial v^-} \cdot \frac{dv^-}{d\lambda} \cdot v^- + f [x, v^-] \cdot \frac{dv^-}{d\lambda} \right\} dx$$

But we have $\frac{dv^+}{d\lambda} = \frac{1}{v^+}$ and $\frac{dv^-}{d\lambda} = \frac{1}{v^-}$. Substituting this in the above formula and changing back to the curvilinear integral we get

$$\frac{dW}{d\lambda} = \oint \left\{ \frac{\partial f [x, v]}{\partial v} + \frac{f [x, v]}{v} \right\} \cdot dx$$

Finally, if we take $t$ as the independent variable and put $dx = v \cdot dt$ we obtain
where $T$ is the period of our solution.

Consider now again a closed trajectory $\Gamma_{\lambda^*}$ of (2.1.2) such that

$$W(\lambda^*) = 0 \quad \frac{dW}{d\lambda}(\lambda^*) > 0$$

We have already seen that for sufficiently small values of $\mu$ there will exist a closed trajectory $\Gamma$ of (2.1.1) within an arbitrarily thin ring around $\Gamma_{\lambda^*}$. Because of the continuity of $\frac{dW}{d\lambda}$ it can be shown (we shall omit the detailed proof) that the value of $\int \mathcal{L} \cdot d\mathbf{t}$ computed along $\Gamma$ will have the same sign as $\frac{dW}{d\lambda}(\lambda^*)$, provided $\mu$ is sufficiently small. From this we may conclude that the resulting periodic solution of (2.1.1) will be stable. Since it is a well known theorem (see, for instance [7]) that two adjacent closed trajectories cannot be both stable it also follows that the curve $\Gamma_{\lambda^*}$ will generate a unique stable periodic solution of (2.1.1).

Remark. If $\varphi(t)$ is a solution of an autonomous system then $\varphi(t + t^*)$, where $t^*$ is arbitrary, is also a solution. In order to avoid reference to this fact every time we consider the uniqueness question, we shall henceforward assume that by saying that a solution $\varphi(t)$ is unique we shall mean that there are no other solutions except those of the form $\varphi(t + t^*)$. 
It is not hard to see that those zeros of $W(\lambda)$ for which $\frac{dW}{d\lambda} < 0$ will be similarly associated with unstable periodic solutions of (2.1.1) where the definition of an unstable periodic solution is the same as that of the stable one except that the sequence \( \{t_n^\prime\} \) is replaced by another one, \( \{t_n\} \) such that \( t_n^\prime \to -\infty \) as \( n \to \infty \).

It is to be remarked that the above theory is insufficient to treat the case when both $W(\lambda)$ and $\frac{dW}{d\lambda}$ vanish simultaneously for some value of $\lambda$.

2.2. Perturbations of systems with stable periodic solutions.

In this section we shall discuss the case of small perturbations of non-conservative systems of type (1.1.1). Suppose that the unperturbed (generating) system possesses a periodic solution corresponding to a closed trajectory $\mathcal{L}$ in the phase-plane. If we perturb this system by introducing an additional small term in the differential equation (1.1.1) then the question arises whether the perturbed system will also possess one or many periodic solutions, their respective trajectories reducing to $\mathcal{L}$ as the relative magnitude of the perturbation decreases to zero. This question may be partially answered by the following theorem:

Theorem II.4.

Consider the differential equation

\[
(\text{2.2.1}) \quad \frac{d^2x}{dt^2} + \varphi(x, \nu) \cdot \nu + q(x) \cdot x = 0 \quad \nu = \frac{dx}{dt}
\]
where \( f(x, v) \) and \( g(x) \) are continuous, bounded and Lipschitzian functions in an open region \( R \) around the origin. Let there be no other singularities in \( R \) except the origin.

If the equation (2.2.1) possesses a stable non-trivial periodic solution whose trajectory \( \Omega \) lies in \( R \) then the differential equation

\[
(2.2.2) \quad \frac{d^2 x}{dt^2} + [f(x, v) + \mu \cdot h(x, v)] \cdot v + q(x) \cdot x = 0 \quad \forall = \frac{dx}{dt}, \mu > 0
\]

will also possess a non-trivial periodic solution provided \( h(x, v) \) is continuous, bounded and Lipschitzian in \( R \) and \( \mu \) is sufficiently small.

Proof. Let \( X(t) = [x_0(t), v_0(t)] \) (where \( v_0(t) = \frac{dx_0}{dt} \)) be the stable periodic solution of (2.2.1) under consideration and \( \Omega \) - the closed trajectory associated with it. Choose a point \( X(t_0) = [x_0(t_0), v_0(t_0)] \) on \( \Omega \) and consider the point \( A = (a, b) \) lying in the interior of \( \Omega \) on the radiusvector through \( X(t_0) \) such that

\[
\| X(t_0) - (a, b) \| = \varepsilon, < \varepsilon
\]

where \( \varepsilon \) is the positive constant as defined for \( X(t_0) \) by the condition of stability (2.1.13).
Let \( \varphi(t) = [x_1(t), v_1(t)] \)
and \( \psi(t) = [x_2(t), v_2(t)] \)
(where \( v_i(t) = \frac{dx_i}{dt} \quad i = 1, 2 \)
be the solutions of (2.2.1) and (2.2.2), respectively, such that
\[
\begin{align*}
x_1(t_0) &= x_2(t_0) = a \\
v_1(t_0) &= v_2(t_0) = b
\end{align*}
\]
Let us follow the trajectory \( \Gamma_1 \) described in the phase-plane by \( \varphi(t) \). Let \( A' \) be the point where \( \Gamma_1 \) crosses the radiusvector through \( A \) after one complete revolution (see Fig. 6). It is obvious that \( A' \) lies between \( X(t_0) \) and \( A \); otherwise the stability condition (2.1.13) could not hold.

As usual, we set
\[
\lambda(x, v) = \frac{1}{2} v^2 + \int_0^x q(y) \cdot \gamma \cdot dy
\]
and note that
\[
\lambda(A') - \lambda(A) = p > 0
\]
Consider now the trajectory \( \Gamma_2 \) defined by \( \psi(t) \).
We shall show that after a complete revolution around the origin this trajectory shall cross the radiusvector through \( A \) at a point \( A'' \) such that \( \lambda(A'') > \lambda(A) \). Let the constant \( T \) be defined by means of
\[
A' = \varphi(t_0 + T)
\]
Now, in exactly the same way as in Lemma II.1 we can deduce that for all \( t \) such that \( t_0 \leq t \leq t_0 + T \) we shall have

\[
\| \psi(t) - \psi(t_0) \| \leq \frac{M}{L} \left[ \exp(LT) - 1 \right] = \delta
\]

where \( M = \sup h(x, v) \) and \( L \) is a constant.

The inequality (2.2.3) means that the point \( A'' = \psi(t_0 + T) \) will lie within a circle of radius \( \delta \) around \( A' \).

In order to estimate \( \lambda(A'') - \lambda(A) \) we use Lemma I.3.

Putting \( dx = v dt \) in (1.5.3) we transform the curvilinear integral into an ordinary one and obtain for our case:

\[
\lambda(A'') - \lambda(A) = - \int_{t_0}^{t_0 + T} \left[ \ell \left( x_1, v_1 \right) + \mu \cdot \eta (x_2, v_2) \right] v_z^2 \cdot dt
\]

Similarly, we can write

\[
\lambda(A') - \lambda(A) = - \int_{t_0}^{t_0 + T} \ell \left( x_0, v_0 \right) \cdot v_z^2 \cdot dt
\]

We shall also need the inequality

\[
| \lambda(A'') - \lambda(A') | = \int_{(x_0, v_0) \in R} \left[ \ell \left( x_1, v_1 \right) + \mu \cdot \eta (x_2, v_2) \right] v_z^2 d\alpha \leq (N + \mu M) k \cdot \delta
\]

where

\[
N = \sup_{(x, v) \in R} | \ell (x, v) |
\]

\[
k = \sup_{(x, v) \in R} | v |
\]

Now we can write

\[
\left| \left[ \lambda(A') - \lambda(A) \right] - \left[ \lambda(A'') - \lambda(A') \right] \right| =
\]
where \( K \) is the Lipschitz constant for \( f(x, v) \) in \( \mathbb{R} \).

Using (2.2.3) once again we finally obtain

\[
\left| \lambda(A^n) - \lambda(A) \right| - \left| \lambda(A') - \lambda(A) \right| \leq
\]

\[
\leq (N + \mu M) k \delta + \mu M k^2 T + K \delta k^2 T + N.2k \delta T = \mu \cdot D
\]

where \( D \) is a quantity which remains bounded as \( \mu \to 0 \).

It follows that for \( \mu \) sufficiently small the quantity \( \lambda(A^n) - \lambda(A) \) will have the same sign as \( \lambda(A') - \lambda(A) \) and, therefore, be positive. It follows that \( A^n \) is farther from the origin than \( A \) and, therefore, must lie between \( X(t_0) \) and \( A \), as stated.
Let us now choose a point \( C = (c, d) \) belonging to the exterior of \( \Omega \) and on the radiusvector through \( X(t_0) \) (see Fig. 7), such that

\[ \| X(t_0) - (c,d) \| = \varepsilon < \varepsilon \]

An entirely analogous calculation shows that the trajectory \( \Gamma_2 \) of (2.2.2) passing through \( C \) will, after a full revolution around the origin, cut the radiusvector through \( X(t_0) \) at a point \( C'' \) closer to the origin than \( C \).

It may be noted that in this case one has to ensure that the relevant trajectories remain within \( R \). It is easy to see, however, that by choosing first \( \varepsilon_2 \) and then \( \mu \) sufficiently small this can always be achieved.

It is now clear that we have established a ring-shaped domain \( R' \) (shaded area on Fig. 7) which the trajectories of (2.2.2) can enter (via the segments \( CC'' \) and \( AA'' \)) but cannot leave. The existence of a non-trivial periodic solution follows immediately from Theorem I.1.
The question which arises next is whether a periodic solution which exists under the conditions of Theorem II.4 will be unique and whether it will be stable. This question can be answered using the additional assumption that the stability integral

\[ J = \int_0^T \left[ \nu \cdot \frac{\partial f(x,\nu)}{\partial \nu} + f(x,\nu) \right] \, dt. \tag{2.2.4} \]

has a positive value for the original (unperturbed) stable periodic solution of (2.2.1) (here \( T \) is the period of this solution).

It was already mentioned that \( J > 0 \) represents a sufficient condition for stability. It is not a necessary condition. One can actually construct differential equations having stable periodic solutions with \( J = 0 \). However, such cases are rather artificial and in the equations arising from physical problems the condition \( J > 0 \) will hold for stable periodic solutions in practically all cases.

It may also be pointed out that the condition \( J > 0 \) may be often established without actually solving the equation and evaluating the integral (see, for instance, \[7\], the section on uniqueness theorems).

With these remarks we shall now proceed to prove the following theorem.

Theorem II.5

If \( \frac{\partial f(x,\nu)}{\partial \nu} \) and \( \frac{\partial \lambda(x,\nu)}{\partial \nu} \) exist and are bounded in \( R \) and if the periodic solution \( X(t) \) of (2.2.1) is such
that the stability integral $J$ associated with it is positive then the periodic solution of (2.2.2) described in Theorem II.4 is unique and stable.

Proof. Let $\Omega$ be the closed trajectory corresponding to $X(t) = [x_0(t), v_0(t)]$ and let $T$ be its period. Similarly, let $\Omega_1$ and $T_1 = T_1(\mu)$ be the trajectory and the period for the periodic solution of (2.2.2) under consideration.

From the proof of Theorem II.4 it is clear that given $\epsilon' > 0$ and sufficiently small, we can find a value of $\mu$ such that $\Omega_1$ will possess at least one point $Q$ lying within the circle of radius $\epsilon'$ centered at the point $P = [x_0(t_0), v_0(t_0)]$, with $v_0(t_0) \neq 0$, on $\Omega$.

Let $X_1(t) = [x_1(t), v_1(t)]$ be the periodic solution of (2.2.2) satisfying

$$[x_1(t_0), v_1(t_0)] = Q$$

and having, therefore, $\Omega_1$ for its trajectory and $T_1$ for its period.

At the time $t = t_0 + T$ the vector function $X(t)$ will return to the point $P$ while $X_1(t)$ will be at some distance $\rho_1$ from $Q$ where $\rho_1 \to 0$ as $\mu \to 0$. This readily follows from the considerations used in the existence proof. Since $v_0(t_0) \neq 0$ we can choose $\mu$ sufficiently small so that $v_1(t_0) \neq 0$ and then decrease $\mu$ again, if necessary, to fulfill $\rho_1 < 1/2 |v_1(t_0)|$. Let $T_1 = T + t(\mu)$; then from
\[ \frac{dx}{dt} = v \]

follows

\[ x(t_0 + \tau) \]

\[ T(\mu) = \int_{x_i(t_0 + \tau)}^{x_i(t_0 + \tau)} \frac{dx}{\nu} \]

where the integral is a curvilinear integral taken along \( \Omega_i \) in the direction of increasing \( t \).

Throughout the integration we have

\[ |v| \geq |v_i| - \rho_i \geq \frac{|v_1|}{2} > 0 \]

so that

\[ |T(\mu)| \leq \frac{|x_i(t_0 + \tau) - x_i(t_0 + \tau)|}{|v_1|-\rho_1} \leq \frac{\rho_1}{|v_1|-\rho_1} \]

It follows that \( T(\mu) \to 0 \) as \( \mu \to 0 \). In other words, \( T(\mu) \) is continuous in \( \mu \) in the neighbourhood of \( \mu = 0 \).

Having established this, we are now in a position to show that the solution \( X_1(t) \) is stable. Let \( J \) and \( J_1 \) be the stability integrals for \( X(t) \) and \( X_1(t) \), respectively, viz.

\[ J = \int_{0}^{T} \left[ v_0 \cdot \frac{\partial f}{\partial v} (x_0, v_0) + f(x_0, v_0) \right] dt \]

\[ J_1 = \int_{0}^{T} \left[ v_1 \cdot \frac{\partial f}{\partial v} (x_1, v_1) + f(x_1, v_1) \right] dt \]

where \( f_1(x, v) = f(x, v) + \mu \cdot h(x, v) \).

We shall show that \( |J - J_1| \to 0 \) as \( \mu \to 0 \). Indeed,
Using the fact that \( \frac{\partial f}{\partial x} \), \( \frac{\partial h}{\partial x} \), \( \frac{\partial f}{\partial v} \), and \( h(x, v) \) are bounded and that \( f(x, v) \) is Lipschitzian, we have

\[
|v_i \frac{\partial f}{\partial v}(x_i, v_i) - v_0 \frac{\partial f}{\partial v}(x_0, v_0)| \leq \|X_i(t) - X(t)\| \leq K_1' \cdot \mu \\
|f_i(x_i, v_i) - f_i(\infty, v_i)| \leq |f_i(x_i, v_i) - f_i(\infty, v_0)| + \mu |h_i(x_i, v_i)| \leq \|X_i(t) - X(t)\| + K_2' \cdot \mu \leq K_2' \cdot \mu
\]

where \( K_1 \) and \( K_2 \) are some constants independent of \( \mu \).

Because of the continuity of \( T_1 = T_1(\mu) \) we can also write

\[
\int_0^T |v_i \frac{\partial f}{\partial v}(x_i, v_i) + f_i(x_i, v_i)| \, dt \leq \int_0^T K_3 \cdot \mu \, dt \leq K_3 \cdot \mu
\]

where \( K_3 \) is independent of \( \mu \).

It is now clear that \( |J_i - J| \leq K_4 \mu \) for some constant \( K_4 \) and, therefore,

\[
|J_i - J| \to 0 \quad \text{as} \quad \mu \to 0
\]
Since \( J > 0 \) we conclude that for sufficiently small \( \mu \) we have \( J > 0 \).

We have therefore shown that all periodic solution of (2.2.2) generated by \( X(t) \) must lie within a thin ring (whose width approaches zero as \( \mu \to 0 \)) around the trajectory \( \mathcal{L} \) of \( X(t) \) and that all these solutions are stable.

But it is a well-known theorem (see, for instance, [7]) that two closed stable trajectories having a common singularity and no other ones in their interiors cannot be adjacent, that is, the ring-shaped region bounded by these trajectories must contain at least one closed trajectory which is not stable. Since this is not possible in our case we conclude that the trajectory \( \mathcal{L} \) is unique, and so is \( X(t) \).

In conclusion, we shall remark that the theorems II.4 and II.5 will remain valid if we replace "stable" by "unstable" and replace \( J > 0 \) by \( J < 0 \). The proofs are completely analogous to those above except that all trajectories are followed in the direction of decreasing \( t \). As an alternative, one may also introduce the new independent variable \( t' = -t \) and apply Theorems II.4 and II.5 directly to the new differential system.
CHAPTER THREE: Some special cases of self-excited oscillations.

3.1. Preliminary remarks.

In this chapter we shall state and prove several theorems establishing the existence of periodic solutions of differential equation (1.1.1) for some special cases, which cannot be treated as small perturbations of already known equations.

The proofs of all of these theorems are based on the same general principle. We proceed to show that all trajectories which start at points sufficiently far from the origin have the general shape of spirals convolving inwards. At the same time the conditions $f(0, 0) < 0$ and $g(0) > 0$ permit us to use Lemma I.4 to show that trajectories starting sufficiently close to the origin must recede from it. This permits us to establish a closed region free of singularities which satisfied the conditions of Poincaré-Bendixson theorem and ensures the existence of a periodic solution.

At the same time we also establish that all solutions are bounded. This is also an important property in applications.

It should be noted that in all cases the conditions imposed upon functions $g(x)$ and $f(x, v)$ are sufficient, but by no means necessary for the existence of periodic solutions. It is because of the rather restricted nature of the problem, as well as because of great variety of possible
behaviour patterns of non-linear systems that these conditions may appear to be fairly numerous or complicated.

3.2. Cases when $f(x, v)$ is bounded

Theorem III.1

Consider the differential equation

$$\frac{d^2x}{dt^2} + f(x, v) \cdot v + q(x) \cdot x = 0 \quad \frac{dx}{dt} = v$$

Let the functions $f(x, v)$ and $g(x)$ be locally Lipschitzian and satisfy the following conditions

1. $q(x) > 0$ for all $x$
2. $q(x) \to \infty$ as $|x| \to \infty$
3. $f(0, 0) < 0$
4. $f(x, v) \geq -M$ ($M > 0$) for $v > -b (b > 0)$, all $x$;
   $f(x, v) \geq N > M$ for $v \leq -b$, all $x$.

Then the equation (3.2.1) possesses a non-trivial periodic solution.

Proof. Define

$$\varphi(v) = \begin{cases} 
-M & \text{for } v > -b \\
N - (M+N) \frac{v+2b}{b} & \text{for } -b \leq v \leq -2b \\
N & \text{for } v < -2b 
\end{cases}$$

It is clear that $\varphi(v)$ is also locally Lipschitzian and satisfies

$$\varphi(v) \leq f(x, v)$$
for all \((x, v)\). We shall prove the existence of a non-trivial periodic solution for the equation

\[
(3.2.2) \quad \frac{d^2 x}{dt^2} + q(x) v + q(x) \cdot x = 0 \quad v = \frac{dx}{dt}
\]

Then the existence of a non-trivial periodic solution of (3.2.1) will follow from Theorem 1.2.

Let \(L(a) = \inf g(x)\). Let us choose \(a > 0\) large enough to satisfy

\[
(6) \quad L(a) \cdot a - 2Mb > 0
\]

\[
(7) \quad a (N-M) > \max \left(2b, \frac{8Mb^2}{L(a) \cdot a - 2Mb}\right)
\]

and

\[
(8) \quad G(-a) = \int_{-a}^{x} q(x) \cdot x \, dx > 2b^2
\]

This is always possible in view of (1) and (2). Let us consider the point \(A(-a, -2b)\) and the trajectory \(T\) of (3.2.2) through it. If we follow \(T\) in the direction of increasing \(t\) we can see that it must cross the \(x\)-axis at some point \(C\) such that \(x(c) < -a\). To show this draw the straight line \(AB\) (see Fig. 8) with the slope

\[
m = M - L(a) \cdot \frac{a}{2b} < 0
\]

The line \(AB\) intersects the \(x\)-axis at \(B\) with the abscissa

\[
x(8) = -a - \frac{4b^2}{L(a) \cdot a - 2Mb}
\]
On the other hand the slope \( \frac{dw}{dx} \) of \( T \) for \( x \leq -a \), and \( v \leq 0 \) satisfies

\[
\frac{dw}{dx} = -q'(v) - q(x) \cdot \frac{x}{v} \leq M - L(a) \cdot \frac{a}{2b} = m
\]

as follows from (1), (2) and (5).

From this result it is clear that \( T \) runs above \( AB \) and, therefore, cuts the \( x \)-axis at a point \( C(-a^*, 0) \) with

\[
x(3) \leq -a^* < -a
\]

Let us now consider the behaviour of \( T \) in the second quadrant. The slope of \( T \) is given now by

\[
(3.2.3) \quad \frac{dw}{dx} = M - q(x) \cdot \frac{x}{v}
\]
This slope is greater than the slope
\[ \frac{dw_i}{d\alpha} = -q(\alpha) \cdot \frac{x_i}{x} \]
of the curve \( \Gamma \) defined by
\[ (3.2.4) \quad v_1(\alpha) = \sqrt{\frac{2G(\alpha) - 2G(-a^*)}{\alpha}} \]
where \( G(\alpha) = \int_0^1 q(y) \cdot y \cdot dy \).

The curve \( \Gamma \) passes through point \( C \), runs through the second quadrant with \( v_1(x) \) increasing monotonically and crosses the \( v \)-axis at \([0, v_1(0)]\). From \(-a^* < -a\) follows that \( G(-a^*) > G(-a) \) and then, by (8) and (3.1.4), that \( v_1(0) > 2b \).

Since the slope of \( T \) is greater than the slope of \( \Gamma \) but bounded from above for \(-a^* \leq x \leq 0\) (except in the neighbourhood of the \( x \)-axis), it follows that \( T \) lies above \( \Gamma \) and must intersect the \( v \)-axis at a point \( E \) with \( v(E) > 2b \).

In the first quadrant the slope of \( T \) is bounded from above by \( M \), therefore \( T \) must remain below the straight line
\[ v = Mx + b_0 \]
for some \( b_0 > 0 \). This leaves \( \frac{x}{v} \) bounded away from zero and from (2) it follows that the slope of \( T \) will become and remain negative for sufficiently large \( x \). Hence \( T \) will descend and cross the \( x \)-axis at a point denoted by \( G(c^*, 0) \) (see Fig. 8).
Let $F(c, 2b)$ be the point in the first quadrant where $T$ intersects the horizontal line $v = 2b$ for the first time. Without loss of generality we may assume that $c$ satisfies the inequality

$$c (N-M) > 2b$$

Indeed, if this were not true we could find $c_1 > 0$ satisfying (3.2.5) and consider a trajectory $T_1$ through $F_1(c_1, 2b)$ and trace it back through the first, second and third quadrant, thus defining points $D_1, C_1$ and $A_1$ corresponding to the points $D, C$ and $A$ on trajectory $T$. Clearly $A_1$ would satisfy the conditions (6), (7) and (8). In this case we could apply all our considerations to $T_1$ instead of $T$.

Let us now reflect the arc $ACDEFG$ of $T$ in the $x$-axis to obtain the "image" $T'$ of $T$. Let $A', D', E'$ and $F'$ be the images of points $A, D, E$ and $F$, respectively.

Let us consider the point $H(c^*, -2b)$ and the trajectory $S$ through it. Let us follow $S$ in the direction of increasing $t$, that is to the left of $H$. We shall proceed to show that $S$ must intersect the $v$-axis at a point $J$ which lies above $E'$.

Let $v, v', \frac{dv}{dx}$ and $\frac{dv'}{dx}$ be the ordinates and the slopes of $S$ and $T'$, respectively, within $[0, c^*]$. 
We have

\[ \frac{d\varphi}{dx} = -\varphi(\varphi) - \varphi(\infty) \cdot \frac{\infty}{\varphi} \tag{3.2.6} \]

\[ \frac{d\varphi'}{dx} = \varphi(-\varphi') - \varphi(\infty) \cdot \frac{\infty}{\varphi'} \tag{3.2.7} \]

where the last equation follows easily from the fact that \( T' \) is an image of \( T \). Therefore,

\[ \frac{d\varphi'}{dx} - \frac{d\varphi}{dx} = \varphi(-\varphi') + \varphi(\varphi) + \varphi(\infty) \cdot \frac{\infty}{\varphi'} \left( \frac{1}{\varphi} - \frac{1}{\varphi'} \right) \]

Since both \( \varphi \) and \( \varphi' \) are negative, it follows that whenever \( \varphi \leq \varphi' \) we have

\[ \frac{d\varphi'}{dx} - \frac{d\varphi}{dx} \geq \varphi(-\varphi') + \varphi(\varphi) \tag{3.2.8} \]

If, in addition, \( \varphi \leq -2b \) this becomes

\[ \frac{d\varphi'}{dx} - \frac{d\varphi}{dx} \geq -M + N > 0 \tag{3.2.9} \]

In particular, it follows that if \( S \) intersects \( T' \) within \([0, c]\), it must do so in the direction indicated by arrows on Fig. 8. This fact will be used later.

Let us now return to the point \( H \) and follow the trajectory \( S \) to the left from it. If \( S \) intersects the line \( x = c \) above the point \( F' \) then, in view of the preceding remark, it will remain above \( T' \) throughout the interval \([0, c]\) and, consequently, will intersect the \( \varphi \)-axis at a point \( J \) lying above \( E' \). We shall therefore assume that \( \varphi < \varphi' \) at \( x = c \). If \( S \) intersects \( T \) within \([0, c]\) then our statement is proven, so we shall also make the assumption that \( \varphi < \varphi' \leq -2b \) throughout the interval \([0, c]\).
In this case let \( x^* \) be the least abscissa in 
\([c, c^*]\) for which \( v(x^*) = -2b \). Then, since \( v'(x^*) < 0 \)

\[ (3.2.10) \quad v'(x^*) - v(x^*) \leq -v(x^*) = 2b \]

Now, between \( x = 0 \) and \( x = x^* \) we have \( v(x) \leq -2b \), as well as \( v < v' \) so that the inequality \((3.2.9)\) is applicable.

Integrating \((3.2.9)\) between \( x \rightarrow 0 \) and \( x = x^* \) we obtain

\[ v'(x^*) - v(x^*) - v'(0) + v(0) \geq \int_0^{x^*} (N - M) \cdot dx \]

hence

\[ v(0) - v'(0) \geq (N - M) \cdot x^* + \left[ v(x^*) - v'(x^*) \right] \geq (N - M) \cdot c - 2b > 0 \]

as follows from \((3.2.5)\). This result contradicts the hypothesis that \( v < v' \) throughout the interval \([0, c]\). Consequently \( S \) must cross \( T' \) and the point \( J \) must lie above \( E' \), as shown on Fig. 8.

Let us now return to the point \( A \) and trace the trajectory \( T \) in the direction of decreasing \( t \), that is, to the right. By arguments similar to those employed in tracing \( T \) through the second quadrant we can easily see that \( T \) must lie below the line \( v = -2b \) and since its slope is bounded from below, it must intersect the \( v \)-axis at some point \( K \).

We shall show that \( K \) must lie below \( E' \). This result follows immediately if the point \( D' \) lies above \( A \) since it is easy to check that the inequality \((3.2.9)\) will also hold in
[-a, 0] and this will ensure that $T$ will remain below $T'$ within this interval. We need, therefore, to consider only the case when $D'$ lies below $A$.

First, we shall derive an inequality for the distance $DA'$ which is equal to the distance $D'A$. For this purpose consider the arcs $CD$ and $CA'$ of $T$ and $T'$ respectively. Let $v, v', \frac{dv}{dx}$ and $\frac{dv'}{dx}$ be the ordinates and slopes of $T$ and $T'$, respectively, within the interval $[-a^*, -a]$. We can easily derive that

\[(3.2.11) \quad \frac{dv}{dx} - \frac{dv'}{dx} = -\varphi(v) - \varphi(-v') + \varphi(x) \cdot \frac{1}{v} - \frac{1}{v'} \]

Let $x^{**}$ be the greatest abscissa in the interval $[-a^*, -a]$ for which $v = v'$. Then we have $v \geq v'$ inside the interval $[x^{**}, -a]$ and, since both $v$ and $v'$ are positive and $x < 0$, (3.2.11) becomes

\[(3.2.12) \quad \frac{dv}{dx} - \frac{dv'}{dx} \leq -\varphi(v) - \varphi(-v') \leq M + M = 2M \]

Therefore

\[
DA' = \int_{x^{**}}^{-a} \left( \frac{dv}{dx} - \frac{dv'}{dx} \right) dx \leq 2M (-a - x^{**})
\]

But

\[
x^{**} \geq -a^* \geq \infty(\beta) = -a - \frac{4b^2}{L(a) \cdot a - 2Mb}
\]

Consequently

\[
DA' \leq 2M \cdot \frac{4b^2}{L(a) \cdot a - 2Mb}
\]

Since $D'A = DA'$ we can write
We can now return to our considerations about the arc $AK$ of $T$ in the third quadrant. Let us denote again by $v, v', \frac{dv}{dx}$ and $\frac{dv'}{dx}$ the ordinates and slopes of $T$ and $T'$, respectively, and assume that $T$ remains above $T'$ throughout the interval $[-a, 0]$. Then the inequality (3.2.) will be again applicable. Integrating (3.2.9) between $x = -a$ and $x = 0$ we obtain

$$v'(c) - v(c) - v'(-a) + v(-a) \geq (N - M) a$$

But, by (3.2.13)

$$v(-a) - v'(-a) = \overline{DA} \leq \frac{8Mb^2}{L(a) \cdot a - 2Mb}$$

so that

$$v'(c) - v(c) \geq (N - M) a - \frac{8Mb^2}{L(a) \cdot a - 2Mb} > 0$$

as follows from (7). This contradicts our hypothesis and shows that $K$ must lie below $E'$.

Let us now consider the region $R'$ bounded by the arc $KCEG$ of $T$, by the vertical segment $GH$, by the arc $HJ$ of $S$ and by the vertical segment $JK$. As outlined in the proof of Lemma I.4 we can construct, using the fact that $\varphi(0) < 0$, a neighbourhood of the origin which no trajectory can enter. Excluding this neighbourhood from $R'$ we obtain a ring-shaped region $R$ free of singularities which satisfies
the condition of Theorem 1.1 and the proof is complete.

An additional remark can be made with regard to condition (2). This condition was used essentially only once, in order to show that the trajectory $T$ running through the first quadrant must finally attain a negative slope and cross the $x$-axis. If we replace (2) by a weaker condition

\[(2') \quad g(x) \geq k \quad \text{for } |x| \text{ sufficiently large}\]

then it can be shown that the argument will still hold provided that

\[M < 2 \sqrt{k}\]

where $M$ is defined as in (4).

Theorem III.2

Consider the differential equation

\[\frac{d^2x}{dt^2} + \left[h(\infty, v) + \mu \cdot X(x)\right] \cdot v + q(x) \cdot x = 0 \quad v = \frac{dx}{dt}\]

Let the functions $h$, $X$, and $g$ be locally Lipschitzian and satisfy the following conditions:

\[
(1^*) \quad q_x(\infty) > 0 \quad \text{for all } \infty \\
(2^*) \quad q_x(-\infty) = q_x(\infty) \\
(3^*) \quad q_x(\infty) \to \infty \quad \text{as } \infty \to \infty \\
(4^*) \quad h_x(0, 0) < 0 \\
(5^*) \quad h_x(\infty, v) \geq N > 0 \quad \text{for } |v| \geq b, \text{ all } \infty; \\
\quad h_x(\infty, v) \leq -M (M > 0) \quad \text{for } |v| < b, \text{ all } \infty.
\]
Then the equation \((3.2.14)\) possesses at least one non-trivial periodic solution for all values of the parameter \(\mu\).

Proof. Let us define

\[ f(v) = \begin{cases} 
- M & \text{for } |v| < b \\
N & \text{for } |v| > 2b \\
- M + (M+N) \frac{v-b}{b} & \text{for } b < v < 2b \\
N - (M+N) \frac{v+2b}{b} & \text{for } -2b < v < -b
\end{cases} \]

The function \(f(v)\) is locally Lipschitzian and satisfies the inequality

\[ (8^*) \quad f(v) \leq \lambda(x,v) \quad \text{for all } (x,v) \]

It follows from Theorem 1.2 that it will then suffice to prove the existence of a non-trivial periodic solution of the equation

\[ (3.2.15) \quad \frac{d^2x}{dt^2} + \left[ f(v) + \mu \cdot X(x) \right] \cdot v + q(x) \cdot x = 0 \]

Let us choose \(a > 0\) sufficiently large to satisfy

\[ (9^*) \quad L \cdot a - 2b (M+N) > 0 \quad \text{where } L = \inf_{x \geq a} q(x) > 0 \]

\[ (10^*) \quad a > \frac{4N b^2}{N} \cdot \frac{1}{L \cdot a - 2b (M+N)} \]
Let us consider the point \( P = (-a, -2b - \mu K a) \) in the third quadrant of the phase-plane and the trajectory \( T^* \) of (3.2.15) through it (see Fig. 9). If we follow \( T^* \) in the direction of decreasing \( t \), that is to the right, we can see, using (1*), (5*), (6*) and (7*) that its slope,

\[
\frac{dw}{dx} = -\xi(x) - \mu \cdot X(x) - q(x) \cdot \frac{x}{v}
\]

is bounded from above by \( \mu K \) and from below by \( -N - \mu K - L' \cdot \frac{\bar{a}}{2b} \)

where \( L' = \sup_{x<a} g(x) \). It follows that \( T^* \) must lie between the straight lines \( PP_1 \) and \( PP_2 \) with the slopes \( \mu K \) and \( (-N-\mu K-L' \cdot \frac{\bar{a}}{2b}) \) respectively and, consequently, must intersect the \( v \)-axis at a point denoted by \( A \). It is also clear that \( v \leq -2b \) for all points on the arc \( PA \) of \( T^* \).

If we now follow \( T^* \) from \( P \) in the direction of increasing \( t \) we see that \( \frac{dw}{dx} \) is bounded from above by \( (M + \mu K) \) and, therefore, \( T^* \) must lie above the straight line through \( P \) with that slope. This leaves the ratio \( \frac{v}{v} \) (which is non-negative in the third quadrant) bounded away from zero and then from (3*) it follows that, for \( |x| \) sufficiently large,
\[ \frac{dw}{dx} \] will become and remain negative. Hence \( T^* \) must cross the x-axis at a point denoted by \( C = (-a^*, 0) \). We also notice that if \( B \) is the point intersection of \( T^* \) with the line \( v = -2b \) (the one with the largest abscissa if there are more than one such points) then we have

\[ a^* > |\infty(B)| > a \]

Let us now consider the point \( Q = (a, 2b + \mu Ka) \) in the first quadrant and the trajectory \( T^{**} \) through it (see Fig. 10). By employing arguments analogous to those above we arrive at the conclusion that \( T^{**} \) must cross the v-axis at a point \( D \) and that for all points on the arc \( DQ \) of \( T^{**} \) we shall have \( v \geq 2b \).

Also, \( T^{**} \) must cross the x-axis at some point \( F = (a^{**}, 0) \) and the horizontal line \( v = 2b \) at some point \( E \) such that \( a^{**} > \infty(\bar{E}) > a \).
This done, we define \( c = \max(a^*, a^{**}) \) and consider the point \( C_1 = (-c, 0) \) and the trajectory \( T_1 \) through it (see Fig. 11).

Let us follow \( T_1 \) in the direction of decreasing \( t \) through the third quadrant. It is quite obvious, in view of the preceding analysis, that \( T_1 \) must cross the line \( v = -2b \) at some point \( B_1 = (-a_1, -2b) \) with \( a_1 > |\alpha(\theta)| > a \) and also must also must cross the \( v \)-axis at some point \( A_1 \). Moreover, all points on the arc \( B_1A_1 \) of \( T_1 \) must satisfy \( v \leq -2b \). Indeed, if \( c = a^* \) this is so because \( T_1 \) coincides with \( T^* \). If, however, \( c > a^* \) then \( B_1 \) must lie to the left of \( B \) and the slopes of all trajectories crossing the segment \( B_1B \) satisfy (on \( B_1B \)) the inequality

\[
\frac{dw}{dx} \leq -N + \mu K + \left( L \cdot \frac{a}{2b} \right) \mu K - \frac{L a}{2b} < \left( M + \mu K \right) - \frac{L a}{2b} < 0
\]

(where \((9^*)\) was used again). From this immediately follows that \( T_1 \) must remain below the line \( v = -2b \) between \( x = -a \), and \( x = x(B) \).

Before we go further we shall find an upper bound for the quantity \((c - a_1)\) which will be needed later. Let us draw through \( B_1 \) a straight line \( B_1D_1 \) with the slope

\[
m = M + \mu K - L \cdot \frac{a}{2b} < 0
\]

This line crosses the \( x \)-axis at a point \( D_1 \) with the abscissa
Since \( m \) is an upper bound for the slope \( \frac{dv}{dx} \) of arc \( C_1B_1 \) of \( T_1 \), this arc has no points below the segment \( D_1B_1 \), so \( C_1 \) can not lie to the left of \( D_1 \). This yields

\[
(3.2.16) \quad c - a, \leq |x(D_1) + a| = \frac{4b^2}{L \cdot a - 2b(M + \mu K)}
\]

Let us now reflect the arc \( C_1B_1A_1 \) of \( T_1 \) in the \( v \)-axis to obtain the arc \( C_2B' A_1 \) of the "image-trajectory" \( T'_1 \), points \( C_2 = (c, 0) \) and \( B'_1 = (a_1, -2b) \) being the images of \( C_1 \) and \( B_1 \), respectively.

Let us consider the trajectory \( T_2 \) passing through \( C_2 \) and follow it in the direction of increasing \( t \), viz. through the fourth quadrant to the left. After \( T_2 \) leaves the immediate neighbourhood of \( C_2 \) its slope becomes bounded from below and it follows that \( T_2 \) must intersect the \( v \)-axis at some point \( F_2 \). We shall show that \( F_2 \) lies above \( A_2 \).

To prove our assertion let \( v, v' \), \( \frac{dv}{dx} \) and \( \frac{dv'}{dx} \) be the ordinates and the slopes of \( T_2 \) and \( T'_1 \), respectively. Then

\[
\frac{dv}{dx} = -\frac{f(v) - \mu \cdot x(\infty) - q(\infty)}{v}
\]

and

\[
\frac{dv'}{dx} = \frac{f(v') + \mu \cdot x(-\infty) + q(-\infty)}{v'} = \frac{f(v') - \mu \cdot x(\infty) - q(\infty)}{v'}
\]

so that

\[
(3.2.17) \quad \frac{dv'}{dx} - \frac{dv}{dx} = f(v') + f(v) + q(\infty) \cdot \infty \cdot (\frac{1}{v} - \frac{1}{v'})
\]
Let $E_2$ be the point where $T_2$ intersects the vertical line $x = a_1$. If $E_2$ is below $B_1$ then we shall show that the distance $B_1E_2$ is bounded and estimate its bound.

Assuming that $E_2$ is below $B_1$ let $x_2$ be the least value of $x$ in the interval $[a_1, c]$ for which $v(x) = v'(x)$. Between $x = a_1$ and $x = x_2$ we have $v \leq v'$ so that the third term on the r.h.s. of (3.2.17) is positive and we have

$$\frac{dw'}{dx} - \frac{dw}{dx} = -M - M = -2M$$

Integrating between $x = a_1$ and $x = x_2$ we get

$$[v'(x_1) - v(x_2)] - [v'(a_1) - v(a_1)] \leq -2M (x_2 - a_1)$$

Remembering that $v'(x_2) = v(x_2)$ and $x_2 \leq C$ and using (3.2.16) we obtain

$$v'(a_1) - v(a_1) \leq 2M (c - a_1) \leq \frac{8M \mu^2}{L \cdot a - 2b(M + \mu \kappa)}$$

Let us now consider the interval $[0, a_1]$ and make the hypothesis that $T_2$ remains below $T_1$ throughout the interval. Then we have $v < v' \leq 2b$ and (3.2.17) becomes

$$\frac{dw'}{dx} - \frac{dw}{dx} \geq f(v') + f(v) \geq 2N$$

Integrating between $x = 0$ and $x = a_1$, we obtain

$$[v'(a_1) - v(a_1)] - [v'(0) - v(0)] \geq 2Na_1$$
or
\[ v'(0) - v(0) \leq v'(a_i) - v(a_i) - 2Na_i \leq \frac{\theta N b^2}{L \cdot a - 2b(M + \mu K)} - 2Na \]

But the r.h.s. of this inequality is negative because of \((9^* )\), so that \(v'(0) < v(0)\). This, however, contradicts our hypothesis. It follows that \(T_2\) must cross \(T_1^*\) in the interval \([0, a_1]\). But then \(T_2\) must remain (as \(t\) increases) above \(T_1^*\) because all trajectories crossing the arc \(A_1B_1^*\) of \(T_1^*\) must do so in the direction shown by the arrows on Fig. 11; that is, they can only enter the region bounded by the arc \(A_1B_1^*\) of \(T_1^*\) and the lines \(x = 0\) and \(v = -2b\). This follows from \((3.2.17)\) by putting \(v = v'\) and remembering that \(f(v') = f(v) = N\) on the arc \(A_1B_1^*\) so that

\[ \frac{d\omega'}{dx} - \frac{d\omega}{dx} = 2N > 0 \]

This proves that \(F_2\) must lie above \(A_1\) provided \(E_2\) is below \(B_1^*\). If, however, \(E_2\) is above \(B_1^*\) or coincides with it, then \(T_2\) must clearly remain above \(T_1^*\) throughout the interval \([0, a_1]\) (except, perhaps, at \(x = a_1\)) because of the arguments just stated above.

Now we can perform an exactly analogous construction for the upper half of the phase-plane by tracing \(T_2\) from \(C_2\) in the direction of decreasing \(t\) till it intersects the
v-axis at $A_2$, reflecting the arc $C_2A_2$ of $T_2$ in the $v$-axis to obtain the image-arc $C_1A_2$ of $T_2$ (see Fig. 12), then continuing $T_1$ beyond $C_1$ and proving that its intersection $F_1$ with the $v$-axis must lie below $A_2$.

We can now see that the region $R$ bounded by the arcs $A_1C_1G_1$ and $A_2C_2F_2$ of $T_1$ and $T_2$, respectively, and by the segments $A_2F_1$ and $A_1F_2$ has the property that no trajectory of (3.2.15) can leave it as $t \to \infty$. We can now conclude our proof by applying Lemma I.4 and the Theorem I.1
in the same fashion as in the preceding Theorem.

It can also be shown by considerations similar to those employed in the proof of Theorem III.1 that if we replace the condition \( (3^*) \) by the weaker condition

\[ q(x) \geq k \quad \text{for } |x| \text{ suff. large} \]

then the theorem will still hold for values of \( \mu \) satisfying

\[ \mu < \frac{2\sqrt{k}}{k} \]

**Theorem III.3**

Consider the differential equation

\[ \frac{d^2 x}{dt^2} + \left[ f(x, v) + \mu \cdot q(x, v) \right] \cdot v + q(x) \cdot x = 0 \quad v = \frac{dx}{dt} \]

Let the functions \( f, q \) and \( g \) be locally Lipschitzian and satisfy the following conditions

1. \( (1^{**}) \quad q(x) > 0 \quad \text{for all } x \)
2. \( (2^{**}) \quad q(x) \to \infty \quad \text{as } |x| \to \infty \)
3. \( (3^{**}) \quad f(0, 0) < 0 \)
4. \( (4^{**}) \quad f(x, v) \geq N > 0 \quad \text{for } |x| \geq a, \text{ all } v; \quad f(x, v) \leq -M (M > 0) \quad \text{for } |x| < a, \text{ all } v. \)
5. \( (5^{**}) \quad g(x, -v) = -g(x, v) \)
6. \( (6^{**}) \quad |g(x, v)| \leq L \quad \text{for all } (x, v) \)
7. \( (7^{**}) \quad \lim_{|v| \to \infty} g(x, v) \text{ exists for each } x \)
Then the differential equation (3.2.18) will possess at least one non-trivial periodic solution for all values of $\mu$.

Proof. Let us choose $v_0 > 0$ such that

$$\left(8^{* *}\right) \quad d = \sup \left| \varphi(x, v_1) - \varphi(x, v_2) \right| < \frac{N}{2}$$

for all $v_1, v_2$ such that

$$|v_1| \geq v_0 \quad |v_2| \geq v_0 \quad \text{and} \quad -a \leq x \leq a$$

This is always possible in view of $\left(7^{* *}\right)$. After this let us choose $x_0 > 0$ such that

$$\left(9^{* *}\right) \quad N(x_0 - a) > \frac{(2M + \frac{N}{2}) v_0^2}{\kappa} \left[ \exp \frac{2\alpha k}{v_0^2} - 1 \right]$$

where

$$\kappa = \sup \left| q(x) \cdot x \right| \quad \text{for} \quad -a \leq x \leq a$$

Finally, increase $v_0$, if necessary, to satisfy the inequalities

$$\left(10^{* *}\right) \quad \frac{K_1}{\kappa} \left(2M + \frac{N}{2}\right) \left[ \exp \frac{2\alpha k}{v_0^2} - 1 \right] < \frac{N}{2}$$

where

$$K_1 = -\inf \left[ q(x) \cdot x \right] \quad \text{for} \quad -x_0 \leq x \leq a$$

and

$$\left(11^{* *}\right) \quad d_1 = \sup \left| \varphi(x, v_1) - \varphi(x, v_2) \right| < \frac{N}{2}$$

for all $(x, v_1)$ and $(x_2, v_2)$ such that

$$|v_1| \geq v_0 \quad |v_2| \geq v_0 \quad \text{and} \quad -x_0 \leq x \leq a$$
This increase in $v_0$ will not violate the inequalities $(8^{**})$ and $(9^{**})$. The first assertion is evident while the second follows from the fact that the r.h.s. of $(9^{**})$ is equal to

$$\left(2M + \frac{N}{2}\right)\left(2a + \frac{a^3K}{v_0^3} + \frac{2a^5K^2}{3v_0^6} + \ldots\right)$$

and, consequently, decreases as $v_0$ increases.

Having done this, let us consider the point $S^* = (-x_0, v_0)$ in the second quadrant of the phase plane. Let us draw through $S^*$ a straight line $S^*B$ (see Fig. 13) with the slope

$$m = \mu L + \frac{K_2}{v_0}$$

where $K_2 = \sup \{g(x)x\}$ for $-x_0 < x < a$.

The slope of a trajectory of $(3.2.18)$ is given by

$$\frac{dy}{dx} = -q'(\infty, v) - \mu q(x, v) - q(\infty) \frac{\infty}{v}$$

A comparison between $(3.2.19)$ and $(3.2.20)$ shows that the slope of a trajectory passing through a point on $S^*B$ with $x \in [-x_0, a]$ is less than $m$, so that all trajectories cross $S^*B$ in this interval as it is indicated by arrows on Fig. 13. It is now clear that if we
consider a trajectory $T$ through $B = (a, v_0 + mx_0 + ma)$ and trace it to the left from $B$ then we shall have $v \geq v_0$ throughout the whole interval $[-x_0, a]$. Let us assume that $T$, when followed to the left, crosses the $x$-axis at a point $P$. The case where no such point exists will be discussed later.

Let us now follow the trajectory $T$ to the right of $B$. We shall show that it will cross the $x$-axis. Indeed, its slope is given by (3.2.20) and we see that for $x \geq a$ and $v \geq 0$ the first and the third terms on the r.h.s. are negative while the second term is bounded by $\mu L$. Using the condition $(2\ast\ast)$ and repeating the arguments employed in the proofs of the preceding two theorems we conclude that $T$ must cross the $x$-axis at a point which we shall denote by $C$.

Let us now reflect the arc $PSABC$ of $T$ in the $x$-axis to obtain its image $PS'A'B'C$ (see Fig. 14), denoted by $T'$. We can now make the following observation, all trajectories of (3.2.18) crossing $T'$ between $P$ and $A'$ or between $B'$ and $C$ do so in the direction indicated by the arrows, viz. They enter the region $R_0$ enclosed by $T$ and $T'$. To see this we compare the slope of a trajectory given by (3.2.20) with the slope of $T'$ given by
Using the condition \((5^{**})\) we see that the difference is

\[
\frac{dw'}{dx} - \frac{dw}{dx} = \phi(x,-\nu) + \phi(x,\nu)
\]

The r.h.s. of this equality is greater than \(2N > 0\) for all points of \(T'\) between \(P\) and \(A'\) or between \(B'\) and \(C\) and our assertion follows.

Let us now consider the trajectory \(T''\) of \((3.2.18)\) through the point \(B'\) and follow it to the left. At \(B'\) the trajectory \(T''\) must enter the region \(R_0\) described before, but between \(B'\) and \(A'\) the trajectory \(T''\) may cross and recross \(T'\) any number of times. We shall assume that \(T''\) finally leaves \(R_0\) at a point \(F\) (see Fig. 15). The other possibility is of no interest since it would mean that \(T''\) never leaves \(R_0\) as \(t \to \infty\) and the proof of the theorem will follow immediately.

Let \(v', v'', \frac{dw'}{dx}\) and \(\frac{dw''}{dx}\) denote the ordinates and the slopes of \(T'\) and \(T''\), respectively. Using \((4^{**})\)
and $(8 \ast \ast)$ we get

\[
\frac{d\psi}{dx} - \frac{d\psi}{dx}'' = f(x, -\psi') + f(x, \psi'') + \mu \left[ \psi(x, \psi''') - \psi(x, \psi') \right] + \\
+ q_{0}(x) \cdot \left( \frac{1}{\psi''} - \frac{1}{\psi'} \right) \geq -2m - \frac{N}{2} - \frac{K}{\nu_{0}^{2}} (\psi' - \psi'')
\]

where $K = \sup |g(x) \cdot x|$ for $-a \leq x \leq a$. Denoting $\psi' - \psi'' = \psi'$,

\[
2m + \frac{N}{2} = \sigma, \quad \frac{K}{\nu_{0}^{2}} = \rho
\]

the above inequality becomes

\[
\frac{d\psi}{dx} \geq -\psi - \psi
\]

Adding $\rho \psi$ to both sides and multiplying by $\exp \rho x \cdot dx$

we obtain

\[
\exp \rho x \cdot d\psi + \psi \cdot \rho \cdot \exp \rho x \cdot dx \geq -\psi \cdot \exp \rho x \cdot dx
\]

Integrating between $-a$ and $x(F)$ and remembering that at $x = x(F) \ \psi = 0$ we get

\[
-\psi(-a) \cdot \exp (-\rho a) \geq \frac{\psi}{\rho} \left[ \exp (-\rho a) - \exp \rho x(F) \right]
\]

or

\[
\psi(-a) \leq \frac{\psi}{\rho} \left[ \exp \rho \{x(F) + a\} - 1 \right]
\]

Substituting the expressions for $\psi$ and $\rho$ and using $x(F) + a < 2a$ we finally obtain

\[
\psi(-a) = \psi'(-a) - \psi(-a) \leq \frac{(2m + \frac{N}{2}) \nu_{0}^{2}}{K} \left[ \exp \frac{2ak}{\nu_{0}^{2}} - 1 \right] = \varepsilon
\]
This inequality states, in fact, that the vertical distance \( \overline{AT} \) (see Fig. 15) does not exceed \( \varepsilon \).

Let us now follow \( T^n \) beyond \( A^n \) to the left. We shall show that \( T^n \) must intersect the arc \( S'A' \) of \( T' \).

To prove this we make the hypothesis that this is not so, that is \( T^n \) remains below \( T' \) throughout the interval \([-\infty, -a] \). As before, we compute the difference \( \frac{dw'}{dx} - \frac{dw''}{dx} \) and apply the conditions \((4^{**})\) and \((11^{**})\) to derive the following inequality, valid in \([-\infty, -a]\):

\[
(3.2.22) \quad \frac{dw'}{dx} - \frac{dw''}{dx} \geq 2N - \frac{N}{2} - \frac{K_1}{\nu^2} (v' - v'')
\]

At \( x = -a \) we have \( (v' - v'') \leq \varepsilon \) and, therefore

\[
\left[ \frac{dw'}{dx} - \frac{dw''}{dx} \right] = \frac{3N}{2} - \frac{K_1}{K} (2N + \frac{N}{2}) \left[ \exp \left( \frac{2ak}{\nu^2} \right) - 1 \right] > \frac{3N}{2} - \frac{N}{2} = N > 0
\]

where the condition \((10^{**})\) was applied. Since

\[
\frac{d}{dx} (v' - v'') > 0 \text{ at } x = -a,
\]

the difference \( (v' - v'') \) will start decreasing as we follow \( T^n \) to the left. This in turn will increase \( \frac{d}{dx} (v' - v'') \) as follows from \((3.2.22)\) and it is obvious that the inequality

\[
(3.2.23) \quad \frac{d}{dx} (v' - v'') > N
\]

will hold throughout the interval \([-\infty, -a]\). Integrating \((3.2.23)\) from \( x = -x_0 \) to \( x = -a \) and putting
\[ v'(-a) - v''(-a) \leq \varepsilon \quad \text{we obtain} \]
\[ v'(-\infty) - v''(-\infty) < \varepsilon - N(x_0 - a) = \]
\[ = \frac{v_o^2}{K} (2M + \frac{N}{2}) \left[ \exp \frac{2ak}{v_o} - 1 \right] - N(x_0 - a) < 0 \]
as follows from (9**). This result contradicts our hypothesis.

It follows that there exists a point \( Q \) on the arc \( A'S' \) of \( T' \) where \( T'' \) crosses \( T' \) and re-enters the region \( R_0 \).

Let \( R \) be the region bounded by the arc \( PSABC \) of \( T \), by arc \( CB' \) of \( T' \), by arc \( B'Q \) of \( T'' \) and, finally, by arc \( QP \) of \( T' \) (see Fig. 16). From the considerations stated before it is clear that no trajectory of (3.2.18) can leave \( R \). We may now apply Lemma 1.4 and the Theorem 1.1 in the usual fashion and the proof is complete.

It remains to consider the case when the point \( P \) does not exist. As before, we construct the image \( T' \) of our trajectory \( T \) (see Fig. 17) and consider the trajectory \( T'' \) through \( B' \). Using the condition (2***) and repeating the arguments already used before we can easily show that \( T'' \)
must intersect the x-axis at some point $P''$. Let us follow $T''$ beyond $P''$. Except for a neighbourhood of $P''$ the slope of $T''$ is bounded and therefore $T''$ must cross the $v$-axis and enter the first quadrant. From there on we use the same argument as for the trajectory $T$ to show that $T''$ will cross the x-axis at some point $F''$. It is easy to see that $F''$ must lie to the left of $C$ since otherwise $T''$ would have had to cross $T$ which is impossible. The region $R'$ bounded by the arc $B'A''P''F''$ of $T''$, by the segment $F''C$ and by the arc $CB'$ of $T'$ is equivalent to $R$ as far as our purpose is concerned and the theorem follows in this case as well.

As before, we also have the following corollary:

**Corollary.**

If the condition (2***) is replaced by

$$q(x) \geq k \quad \text{for } |x| \text{ suff. large}$$

then the theorem will hold for the values of $\mu$ satisfying

$$\mu < \frac{2\sqrt{k}}{L}$$
3.3 A case when \( f(x, v) \) is not bounded

Theorem III.4

Consider the differential equation

\[
\frac{d^2 x}{dt^2} + f(x, v) \cdot v + g(x) \cdot x = 0 \quad \text{with} \quad v = \frac{dx}{dt}
\]

Let the functions \( f \) and \( g \) be locally Lipschitzian and satisfy the following conditions:

(a) \( q(\infty) > 0 \)

(b) \( q(-\infty) = q(\infty) \)

(c) \( f(x, 0) < 0 \)

(d) \( q(x, v) \equiv N > 0 \) except, perhaps, in some closed neighbourhood \( R \) of the origin and in the region for which

\[
0 < \frac{v}{x} < -m
\]

where \( m \) is a positive constant

(e) there exists \( x_\circ > 0 \) and \( b \neq 0 \) such that

\[
q(x) \geq \left( \frac{2m^2}{N} \right)^2 + b^2 \quad \text{for} \quad x = \infty,
\]

Then the equation (3.3.1) possesses a non-trivial periodic solution.

Proof. With \( x_1 \) and \( b \) defined as in (e), let us denote

\[
\frac{2m^2}{N} = a^2
\]

and choose \( x_\circ > x_1 \), large enough to satisfy the inequality

\[
(3.3.2) \quad b^2 x_\circ^2 - x_1^2 (a^4 + b^4) > 0
\]
Let us consider the curves defined by

\[(3.3.3) \quad \frac{1}{2} \nu^2 + G(\nu) = \lambda\]

where \[G(\nu) = \int_0^\nu q(y) \cdot y \cdot dy\]

and \(\lambda\) is a non-negative parameter. These curves are trajectories of the differential equation

\[\frac{d^2 \nu}{dt^2} + q(\nu) \cdot \nu = 0\]

and, because of the conditions (a) and (b) they are closed curves around the origin, symmetric with respect to both coordinate axes. They will be called "\(\lambda\)-curves". It is easy to see that we can always choose \(x_0^n > 0\) large enough so that the region \(R\) will lie entirely within the interior of the \(\lambda\)-curve given by

\[\frac{1}{2} \nu^2 + G(\nu) = G(\nu_{0^n})\]

Let us now denote \(x_0 = \max(x_0^1, x_0^n)\) and consider the point \(A = (-x_0, 0)\) in the phase-plane (see Fig. 18). Let us draw through \(A\) a \(\lambda\)-curve \(L\) and a trajectory \(T\) of \(3.3.1\). Let \(B\) be the point where \(L\) intersects the ray \(v = -mx\) and let \(D' = (-x_0, mx_0)\) and let \(H = (x_0, 0)\). We shall follow
the trajectory $T$ in the direction of increasing $t$ and show that it must cross the $v$-axis and then descend and cross the $x$-axis at some point $P$ lying between the origin and the point $H$.

As we start from $A$, the trajectory $T$ must remain to the right of the vertical line $DA$ and so must cross the ray $v = -MW$ at some point $C$. If $C$ is closer to the origin than $B$ (or coincides with $B$) then our assertion follows immediately since on the arc $BH$ of $L$ we have $f(x, v) \geq N > 0$ and it is easy to see that $T$ cannot leave the interior of $L$ via the arc $BH$.

We shall, therefore, assume that $C$ lies between $D$ and $B$. Then, clearly $\lambda(C) \leq \lambda(D)$. We also have

$$\lambda(D) - \lambda(A) = \frac{1}{2} [v(D)]^2 + G(\infty) - G(\infty) =$$

$$= \frac{1}{2} [v(D)]^2 = \frac{1}{2} MW^2 \xi_0^2$$

and, therefore,

$$(3.3.4) \quad \lambda(C) - \lambda(A) \leq \frac{1}{2} MW^2 \xi_0^2$$

Let us follow the trajectory $T$ beyond $C$. Because of the conditions (a) and (d) it easily follows that $T$ must cross the $v$-axis, then descend and cross the $x$-axis at some point $P$. From an argument employed before it is clear that if $T$ crosses the arc $BH$ of $L$ then $P$ must lie
to the left of $H$ and our assertion will be proven. To demonstrate that this will indeed be the case we shall make the contrary hypothesis and assume that $T$ lies above the arc $BH$ of $L$ and crosses the line $x = x_0$ at some point $F$ lying above $H$. This hypothesis is illustrated on Fig. 18.

Let $E$ be the intersection of $T$ with the $v$-axis. From Lemma I.3 and the condition (d) it readily follows that

\[(3.3.5) \quad \lambda(E) < \lambda(c)\]

Let $v_1 = v_1(x)$ and $v_2 = v_2(x)$ represent the equations of curves $T$ and $L_1$ respectively, in the interval $[0, x_0]$. Using our hypothesis we have

\[(3.3.6) \quad \lambda(E) - \lambda(F) = \int_0^{x_0} f(x, v_1) \cdot v_1 \cdot dx \geq N \int_0^{x_0} v_1 \cdot dx - N \cdot \Omega\]

where $\Omega$ is the area under the curve $L$ in the first quadrant. We are now going to show that

\[(3.3.7) \quad \Omega = \frac{\pi a^2}{4} \cdot x_0^2 = \frac{1}{2} \cdot \frac{m^2}{N} \cdot x_0^2\]

To see this let us consider the ellipse $L'$ defined by

\[(3.3.8) \quad v^2 = v_3^2(x) = a^2 \cdot \sqrt{x_0^2 - x^2}\]
The area under $L'$ in the first quadrant is equal to \( \frac{\pi a^2}{4} \cdot x_0^2 \); hence, if we show that $v_2 \geq v_3$ for $0 \leq x \leq x_3$, the inequality (3.3.7) will follow:

Indeed, we have

$$v_2^2 = 2 \cdot \lambda(h) - 2 \cdot G(x) = 2 \cdot G(x_0) - 2 \cdot G(x)$$

Together with (3.3.8) this yields

$$\delta = v_3^2 - v_2^2 = 2 \cdot G(x_0) - 2 \cdot G(x) - a^4 \cdot x_0^2 + a^4 \cdot x^2 =$$

$$= 2 \int_0^x q(y) \cdot y \, dy - 2 \int_0^{x_0} q(y) \cdot y \, dy - 2 \int_x^{x_0} q(y) \cdot y \, dy + 2 \int_0^{x_0} a^4 \cdot y \, dy =$$

$$= 2 \int_0^x [q(y) - a^4] \cdot y \, dy$$

We have to show that $\delta > 0$ for all $x \in [0, x_0]$. If $x \in [x_1, x_0]$ this follows immediately from the condition (e) which can be rewritten as

$$q(x) - a^4 \geq b^2 > 0 \quad \text{for} \quad x \geq x_1$$

If, however, $x \in [0, x_1]$ then we can write

$$\delta = 2 \int_0^{x_1} [q(y) - a^4] \cdot y \, dy + 2 \int_{x_1}^{x_0} [q(y) - a^4] \cdot y \, dy \geq$$

$$\geq 2 \int_0^{x_1} [-a^4] \cdot y \, dy + 2 \int_{x_1}^{x_0} b^2 \cdot y \, dy =$$

$$= - a^4 \cdot x_1^2 + b^2 \cdot (x_0^2 - x_1^2) =$$
where the conditions (e) and (3.3.2), as well as the fact that \( x_0 > x_0' \), were used. The validity of (3.3.7) is, therefore, established.

Now using (3.3.4), (3.3.5), (3.3.6) and (3.3.7) we obtain

\[
\lambda(F) - \lambda(A) = \left[ \lambda(F) - \lambda(E) \right] + \left[ \lambda(E) - \lambda(c) \right] + \left[ \lambda(c) - \lambda(A) \right] < \\
< - N \cdot \Omega + \frac{1}{2} m^2 x_0^2 < - \frac{1}{2} m^2 x_0^2 + \frac{1}{2} M^2 x_0^2 = 0
\]

Since \( \lambda(A) = \lambda(H) \) it follows that \( \lambda(F) < \lambda(H) \). But, according to our hypothesis, \( F \) lies above \( H \) on the same vertical, so that \( \lambda(F) > \lambda(H) \). The contradiction proves that the hypothesis is false and, consequently, that \( T \) must intersect the arc \( BH \) of \( L \). It follows that the point \( P \) lies to the left of \( H \).

If we now consider the trajectory \( T' \) of (3.3.1) passing through \( H \) then we can show, in the same fashion as above, that \( T' \) must cross the \( v \)-axis and then ascend and cross the \( x \)-axis at a point \( Q \) lying to the right of \( A \) (see Fig. 19).

The region bounded by the arcs \( AP \) and \( HQ \) of \( T \) and \( T' \), respectively, and the segments \( AQ \) and \( PH \) possesses
the property that no trajectory can leave it as \( t \to \infty \). The existence of a non-trivial periodic solution follows readily now by application of Lemma I.4 and the Theorem I.1.
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