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ON THE UNIQUENESS OF MULTIPLE TRIGONOMETRIC SERIES

ABSTRACT

The multiple trigonometric series \( \sum c_m \exp(i(m,x)) \), where \( m = (m_1, \ldots, m_n), x = (x_1, \ldots, x_n), (m,x) = m_1x_1 + \ldots + m_nx_n \), and \( m_i \geq 0 \) is said to be summable \((T, k)\) if the series \( \sum C_p(x) \) is summable \((C, k)\), where \( C_p(x) \) denotes the triangular sum \( \sum c_m \exp(i(m,x)) \), \( m_1 + \ldots + m_n = p \). The series is said to be bounded \((T, k)\) if the series \( \sum C_p(x) \) is bounded \((C, k)\).

Using the fact that the triangular summation of multiple trigonometric series considered is equivalent to the Cesàro summation of a single series of a particular form, this thesis obtains uniqueness theorems for multiple trigonometric series by first proving the required theorems for the single series.

It is shown that if the series \( \sum c_n \exp(int) \) is summable \((C, k)\) then the coefficients \( c_n \) are given in terms of the \( \rho^{k+2} \) integral defined by James [Trans. Amer. Math. Soc., vol 76, 1954, pp. 149-176, Section 8]. When the series is bounded \((C, k)\) a Fourier representation is obtained in terms of Burkill's \( C_{k+1}P \)-integral [Proc. London Math. Soc. (2), vol. 39, 1935, pp. 541-552].

It is shown that if \( f(t) \) is periodic and \( C_rP \)-integrable, then the definite \( C_rP \)-integral is a constant multiple of the definite \( \rho^{r+1} \)-integral of \( f(x) \). This gives a Fourier representation of the coefficients in terms of the \( \rho^{k+2} \)-integral when the series is bounded \((C, k)\).

These results are then extended to multiple trigonometric series. A representation for the coefficients in terms of the \( C_{k+1}P \)-integral is demonstrated if the series is bounded \((T, k)\). Finally, a uniqueness theorem is proved where the summability set is a countable set of \( n \)-tuples of the form \((x_{10}, \ldots, x_{n-10}, x_{ni})\) for fixed \( x_{10}, \ldots, x_{n-10} \) and \( i \geq 1 \).
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ON THE UNIQUENESS OF MULTIPLE TRIGONOMETRIC SERIES

by

GEORGE ELLIOT CROSS

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Abstract

The multiple trigonometric series \( \sum c_m \exp(i(m,x)) \), where \( m = (m_1, \ldots, m_n) \), \( x = (x_1, \ldots, x_n) \), \( (m,x) = m_1x_1 + \ldots + m_nx_n \), and \( m_j \geq 0 \), is said to be summable \((T,k)\) if the series \( \sum \infty \mathcal{C}(x) \) is summable \((C,k)\) where \( \mathcal{C}(x) \) denotes the triangular sum \( \sum c \exp(i(m,x)) \), \( m_1 + \ldots + m_n = p \). The series is said to be bounded \((T,k)\) if the series \( \sum \infty \mathcal{C}(x) \) is bounded \((C,k)\).

Using the fact that the triangular summation of the multiple trigonometric series considered is equivalent to the Cesàro summation of a single series of a particular form, this thesis obtains uniqueness theorems for multiple trigonometric series by first proving the required theorems for the single series.

It is shown that if the series \( \sum \infty c_n \exp(int) \) is summable \((C,k)\) then the coefficients \( c_n \) are given in terms of the \( \mathcal{C}^{k+2} \) - integral defined by James [Trans. Amer. Math. Soc. vol. 76 (1954) pp. 149-176, section 8]. When the series is bounded \((C,k)\) a Fourier representation is obtained in terms of Burkill's \( \mathcal{C}_{k+1}^P \) - integral [Proc. London Math. Soc. (2) vol. 39 (1935) pp. 541-552].

It is shown that if \( f(t) \) is periodic and \( \mathcal{C}_r^P \) - integrable, then the \( \mathcal{C}_r^P \) - (definite) integral is a constant multiple of the \( \mathcal{C}^{r+1} \) - (definite) integral of \( f(x) \). This gives a Fourier representation of the coefficients in terms of the \( \mathcal{C}^{k+2} \) - integral when the series is bounded \((C,k)\).
These results are then extended to multiple trigonometric series. A representation for the coefficients in terms of the $C_{k+1}^P$-integral is demonstrated if the series is bounded $(T,k)$. Finally a uniqueness theorem is proved where the summability set is a countable set of $n$-tuples of the form $(x_{10}, x_{20}, \ldots, x_{n-10}, x_{n1})$, for fixed $x_{10}, x_{20}, \ldots, x_{n-10}$, and $i = 1, 2, 3, \ldots$. 
In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representative. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Mathematics
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1. **Introduction**

Using a process of spherical summation introduced by Bochner [1], Cheng [4] has proved that if a multiple trigonometric series

\[ (1.1) \sum c_m \exp(i(m,x)) \]

is everywhere \((C,1)\) summable to zero by spherical means and satisfies a certain condition, then it vanishes identically. Shapiro [13, 14] has extended these results to allow exceptional sets of capacity zero. More generally Cheng has shown that if the series is everywhere \((C,1)\) summable by spherical means to a function \(f(x_1, x_2, \ldots, x_n)\) then under certain conditions the series is the Fourier series of \(f\).

This thesis is concerned with uniqueness theorems for multiple trigonometric series of a particular form summed by a "triangular" method. This method reduces the summation of the multiple series to that of a single series of the form

\[ (1.2) \sum_{n=0}^{\infty} c_n \exp(inx) \]

where \(c_n = a_n - i b_n\).

James [7] has shown that if the real part of series (1.2) is summable \((C,k)\) and an additional condition involving the imaginary part is satisfied, then the coefficients are given in modified Fourier form, where the integral involved is his \(P^{k+2}\) - integral. It is shown here that if series (1.2)
is summable (C,k) then the coefficients have a similar Fourier representation in terms of James' $\mathcal{C}^{k+2}$-integral.

Burkill [3] has shown that, if (1.2) is bounded (C,0) except on a countable set and if the series obtained by integrating series (1.2) once converges everywhere, then the coefficients can be written in Fourier form using the $C_1P$-integral. An analogous result is shown to be true in this thesis when (1.2) is bounded (C,k). The proof of this depends on a powerful result by Marcinkiewicz and Zygmund [8], and on generalizations of theorems by Verblunsky and Zygmund.

James has proved that if $f(x)$ is $C_rP$-integrable it is also $\mathcal{C}^{r+1}$-integrable and has given a representation of the $\mathcal{C}^{r+1}$-(indefinite) integral in terms of the $C_rP$-(indefinite) integral. The two preceding results suggest the relationship between the $\mathcal{C}^{r+1}$ and $C_rP$-(definite) integrals given in the theorem of Chapter 6.

Series (1.1) is said to be bounded (T,k) if the corresponding series (1.2) is bounded (C,k). It is shown that if (1.1) is bounded (T,k) for all values $(x_1 + t, \ldots, x_n + t)$ then the coefficients are given by repeated integrals of dimension $n+1$, the inner integral being a $C_{k+1}P$-integral.

It is then shown that if there is a countable set of $n$-tuples such that series (1.1) is summable (T,k) to 0 for all values $(x_{10} + t, \ldots, x_{n-1} 0^+ t, x_{n1} t)$, $i = 1, 2, 3, \ldots$, then the series vanishes identically.
2. Notation and Definitions

Consider the multiple trigonometric series

\[ \sum c_m \exp(i(m,x)), \]

where \( c_m \) may be complex, \( m = (m_1, \ldots, m_n) \), \( (m,x) = m_1 x_1 + \ldots + m_n x_n \), and the summation is over all non-negative integers \( m_j \).

**DEFINITION 2.1.** The series is said to be summable \((T,k)\) to sum \( f(x) \) if the series

\[ \sum_{p=0}^{\infty} C_p(x) \]

is summable \((C,k)\) to \( f(x) \), where \( C_p(x) \) denotes the triangular sum

\[ \sum c_m \exp(i(m,x)) \]

where the summation is over all non-negative integers \( m_j \) such that \( m_1 + \ldots + m_n = p \) and \( (x) = (x_1, \ldots, x_n) \).

**DEFINITION 2.2** The series \((2.1)\) is said to be bounded \((T,k)\) at \( (x) \) if the series \((2.2)\) is bounded \((C,k)\) at \( (x) \).

**DEFINITION 2.3.** Let \( g(t) \) be a function defined in the interval \([a,b]\). If, for a given \( t_0 \) in \([a,b]\),

\[ g(t_0 + h) = c_0 + c_1 h + c_2 h^2/2! + \ldots + c_k h^k/k! + o(h^k), \]

where the numbers \( c_j = c_j(t_0) \) are independent of \( h \), then \( c_k \) is called the \( k \)-th de la Vallee Poussin derivative of \( g \) at the point \( t_0 \) and is denoted by \( g(k)(t_0) \).

**DEFINITION 2.4.** If \( g(k)(t_0) \) exists for \( 0 \leq k \leq n - 1 \), define \( \gamma_n(t_0, h) \) by
\[ h^{n/n!} \gamma_n(t_0, h) = g(t_0 + h) - g(t_0) - \sum_{k=1}^{n-1} \left[ \frac{h^k}{k!} \right] g^{(k)}(t_0), \]

and let

\[ \Delta g_n(t_0) = \lim \sup_{h \to 0} \gamma_n(t_0, h) \]

\[ \iota g_n(t_0) = \lim \inf_{h \to 0} \gamma_n(t_0, h) \]

DEFINITION 2.5. Let \( \varrho(t) = f(x_{10} + t, \ldots, x_{n0} + t) = f(x_0 + t) = u(x_0 + t) + i v(x_0 + t). \) If for a given \( t_0, \)

\[ \varrho(t_0 + t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2/2! + \ldots + (\alpha_r + \epsilon_r t)t^r/r! \]

for \( 1 \leq r \leq k, \) where \( \epsilon_r t \to 0 \) with \( t, \) and the \( \alpha_i \) are complex constants independent of \( t, \) then \( \alpha_k \) will be called the \( k \)-th generalized derivative of \( \varrho \) at the point \( (x_0 + t_0) \) and will be denoted by \( \varrho^{(k)}(x_0 + t_0). \) It will be convenient to write \( \varrho^{(k)}(t_0) \) for \( \varrho^{(k)}(x_0 + t_0) \) wherever it will cause no misunderstanding.

Thus if \( U(t) = u(x_0 + t) \) and \( V(t) = v(x_0 + t) \) then

\[ \varrho^{(k)}(t_0) = U^{(k)}(t_0) + i V^{(k)}(t_0) \]

where \( U^{(k)}(t_0) \) and \( V^{(k)}(t_0) \) denote the \( k \)-th de la Vallee Poussin derivatives of \( U(t) \) and \( V(t) \) at \( (x_0 + t_0). \)

For reference purposes some definitions and theorems required for the development of the \( C_0^m \)-integral [6] and the \( C_0^P \)-integral [2] are given below.

Let \( u(t) \) be a real valued function of the single variable \( t \) defined in \([a, b]\) and let \( a_i, 1 = 1, 2, \ldots, n \) be fixed points such that \( a = a_1 < a_2 < \ldots < a_n = b. \)
DEFINITION 2.6. The functions $Q(t)$ and $q(t)$ are called major and minor functions, respectively, of $u(t)$ over $(a_1) = (a_1, a_2, \ldots, a_n)$ if

(2.4.1) $Q(t)$ and $q(t)$ are continuous in $[a, b]$, and, for $1 \leq k \leq n - 1$, each $Q(k)(t)$, $q(k)(t)$ exists and is finite in $(a, b)$;

(2.4.2) $Q(a_1) = q(a_1) = 0$, $i = 1, \ldots, n$, where $a = a_1 < \ldots < a_n = b$;

(2.4.3) $\int Q(n)(t) \geq u(t) \geq \Delta q(n)(t)$, in $(a, b)$;

(2.4.4) $\int Q(n)(t) \neq -\infty$, $\Delta q(n)(t) \neq +\infty$ in $(a, b)$.

DEFINITION 2.7. For each major and minor function of $u(t)$ over $(a_1)$, the functions defined by

$$Q^*(t) = (-1)^{r+n}Q(t), \quad q^*(t) = (-1)^{r+n}q(t), \quad a_r \leq t < a_{r+1},$$

are called associated major and minor functions, respectively, of $u(t)$ over $(a_1)$.

DEFINITION 2.8. Let $c$ be a point of $(a_1, a_n)$ such that $c \neq a_1, i = 1, 2, \ldots, n$. Then $u(t)$ is said to be $\gamma^n$-integrable over $(a_1; c)$ if for every $\epsilon > 0$ there is a pair $(Q(t), q(t))$ satisfying conditions (2.4.1) - (2.4.4) such that

$$|Q(c) - q(c)| < \epsilon.$$ 

THEOREM 2.1. If $u(t)$ is $\gamma^n$-integrable over $(a_1; c)$ there is a function $U^*(t)$ which is the inf of all associated major functions of $u(t)$ over $(a_1)$ and the sup of all associated minor functions.
DEFINITION 2.9. If \( u(t) \) is \( C^n \) - integrable over \( (a_i; c) \) and if \( U^*(t) \) is the function of Theorem 2.1, define \( U(t) \) by
\[
(-1)^{r_n} U(t) = U^*(t), \quad \text{when } a_r \leq t < a_{r+1}.
\]
If \( a_s < c < a_{s+1} \), the \( C^n \) - integral of \( u(t) \) over \( (a_i; c) \) is defined to be \((-1)^{s+n} U(c)\). Since \((-1)^{s+n} U(a_i) = U(a_i) = 0\), the integral is defined to be zero if \( c = a_i \), \( i = 1, 2, \ldots, n \).

The notation is
\[
(-1)^{s+n} U(c) = \int_{(a_i)}^c u(t) \, dt.
\]

THEOREM 2.2. If \( u(t) \) is \( C^n \) - integrable over \( (a_i; c) \), it is also \( C^n \) - integrable over \( (a_i; t) \) for every \( t \) in \([a, b]\).

If \( U(t) \) is the function of Definition 2.9, then, for \( a_s \leq t < a_{s+1} \),
\[
(-1)^{s+n} U(t) = \int_{(a_i)}^t u(t) \, dt.
\]

The \( C^n \) - integral is defined by induction. Suppose that for \( n \geq 1 \) the \( C^n - 1P \) - integral has been defined taking as the \( C^0P \) - integral the Perron integral \([9, p. 201]\).

Assuming that \( u(t) \) is \( C^n - 1P \) - integrable, let
\[
C_n(u, t, t + h) = (n/h^n)C_n - 1P \int_t^{t+h} (t + h - f)^{n-1} u(f) \, df.
\]

DEFINITION 2.10. The function \( u(t) \) is said to be \( C^n \) - continuous at \( t_0 \) if \( C_n(u, t_0, t_0 + h) \to u(t_0) \) as \( h \to 0 \).

DEFINITION 2.11. The upper and lower \( C^n \) - derivatives of \( u(t) \) denoted by \( C^n D^+ u(t) \) and \( C^n D^- u(t) \), respectively, are
defined to be the lim sup and the lim inf, respectively, as $h \to 0$ of the expression

$$(n + 1/h)(C_n(u, t, t + h) - u(t)).$$

**DEFINITION 2.12.** If $C_n^D u(t) = C_n^D N u(t)$, their common value is defined to be the $C_n$-derivative of $u(t)$ and is denoted by $C_n^D u(t)$.

**DEFINITION 2.13.** The function $M(t)$ is said to be a $C_n$-major function of $u(t)$ over $[a, b]$ if

1. $M(t)$ is $C_n$-continuous;
2. $M(a) = 0$;
3. $C_n^D M(t) \geq u(t)$, p.p in $[a, b]$;
4. $C_n^D M(t) > -\infty$ in $[a, b]$.

A $C_n$-minor function $m(t)$ is defined in a similar way.

**DEFINITION 2.14.** If, for every $\epsilon > 0$, there is a pair $M(t), m(t)$ satisfying the conditions of Definition 2.13 and such that $|M(b) - m(b)| < \epsilon$, then $u(t)$ is said to be $C_nP$-integrable over $[a, b]$.

**DEFINITION 2.15.** Let $I(b) = \text{lower bound of all } M(b)$ and $J(b) = \text{upper bound of all } m(b)$. For a $C_nP$-integrable function $u(t)$ the bounds have a common limit $[2]$ which is called the definite $C_nP$-integral of $u(t)$ over $[a, b]$.

Suppose that $\phi(t) = u(t) + 1 v(t)$ where $u(t)$ and $v(t)$ are $C_nP$-integrable and in the notation of Theorem 2.2
\((-1)^s + n U(c) = \int_{(a_i)}^c u(t) \, dt\)

\((-1)^s + n V(c) = \int_{(a_i)}^c v(t) \, dt.\)

If \(a_s \leq c < a_s + 1\), the \(\mathcal{Y}^n\) - integral of \(\mathcal{O}(t)\) over \((a_i; c)\) is defined to be \((-1)^s + n U(c) + i(-1)^s + n V(c)\), i.e.

\[\int_{(a_i)^c} \mathcal{O}(t) \, dt = \int_{(a_i)}^c u(t) \, dt + i \int_{(a_i)}^c v(t) \, dt = \]

\[(-1)^s + n [U(c) + i V(c)] = (-1)^s + n \Phi(c).\]

The \(C_{np}\) - integral of \(\mathcal{O}(t)\) is defined in an analogous way.

When a series of the form

\[\sum_{n=1}^{\infty} \frac{c_n \exp(int)}{(in)^j}\]

is summable \((C, r - j - 1)\) to a function, say, \(F^r - j(t) - i G^r - j(t)\), where \(r\) and \(j\) are positive integers, it will be convenient to write simply,

\[(2.6) \sum_{n=1}^{\infty} \frac{c_n \exp(int)}{(in)^j} = F^r - j(t) - i G^r - j(t), \quad (C, r - j - 1),\]

without stating explicitly that \(F^r - j(t) - i G^r - j(t)\) is defined by (2.6).
3. The Integrated Series

A solution of the representation problem for the series

\[ \sum_{n=1}^{\infty} c_n \exp(int) \]

in terms of the \( k+2 \)-integral, when the series is summable \((C,k)\), involves the following theorem stated by Zygmund [16, p. 226, problem 12] to be proved as an exercise:

**THEOREM 3.1.** Let the series (3.1) be summable \((C,\alpha)\), for a fixed \( \alpha = 0, 1, 2, \ldots \) and \( t = t_0 \), to sum \( s, |s| < \infty \). Let \( r \) be an integer \( > \alpha + 1 \). Then the series integrated term by term \( r \) times converges uniformly for all \( t \) to a continuous function \( \Psi(t) \), and \( \Psi(r)(t_0) \) exists and is equal to \( s \).

When the series (3.1) is bounded \((C,k)\) a Fourier representation for the coefficients in terms of the \( C_{k+1} \)-integral can be obtained using a generalization of Theorem 3.1.

**THEOREM 3.2.** Let the series (3.1) be bounded \((C,\alpha)\), for a fixed \( \alpha = 0, 1, 2, \ldots \) and \( t \in E, |E| > 0 \). If \( r = \alpha + 2 \), then for each \( t \in E \),

\[ \sum_{n=1}^{\infty} \frac{c_n \exp(int)}{(in)^J} = F^r - J(t) - \frac{1}{C^r - J(t)}, \quad (C, r - j - 1), \]
where \( j = 1, 2, \ldots, r - 1 \), and

\[
(3.2.r) \quad \sum_{n=1}^{\infty} c_n \exp(int) = F(t) - i G(t) = \mathcal{O}(t),
\]

where the last series is absolutely and uniformly convergent for all \( t \). In addition, \( \mathcal{O}(s)(t) \) exists and is finite for \( 0 \leq s \leq r - 1, t \in E \), and

\[
(3.3) \quad \mathcal{O}(t + h) = \mathcal{O}(t) + h \mathcal{O}(1)(t) + \ldots + \left[ h^r - \frac{1}{(r-1)!} \right] \mathcal{O}(r-1)(t) + \left[ \omega(t,h)/r! \right] h^r
\]

where \( \omega(t,h) = O(1) \) as \( h \to 0 \). Furthermore,

\[
(3.4) \quad \mathcal{O}(\alpha + 2 - j)(t) = \sum_{n=1}^{\infty} c_n \exp(int) \quad (C, \alpha + 1 - j)
\]

for \( 0 < j \leq r \), and \( \mathcal{O}(r)(t) \) exists p.p. in \( E \).

Proof. It is clear that \( c_n \exp(int) = O(n^\infty) \) and this is sufficient to guarantee the convergence property of series (3.2.r). The summability \( (C, \alpha) \) of series (3.2.1) and the summability of series (3.2.2), (3.2.3), \ldots, (3.2.r - 1) follows from two theorems by Hardy \( [5, \text{Theorem 71}, \text{p.} 128 \text{and Theorem 76, p.} 131, \text{respectively}] \).

To obtain (3.3) and (3.4) it may be assumed without loss of generality that \( t = 0 \). Let \( \Upsilon(u) = \exp(iu)/(iu)^r \), \( \Theta(h) = \sum_{\nu=0}^{r-1} (ih)^\nu/\nu! \), \( \lambda(h) = \exp(\nu h) - \Theta(h) \), and for \( (ih)^r \) any sequence \( \{u_m\} \) let \( \Delta u_n = \Delta^1u_n = u_n - u_n + 1 \).
\[ \Delta^j u_n = \Delta (\Delta^j - 1 u_n). \] Then Zygmund's proof [16, pp. 260 - 261], with condition \( s_n = o(n^\infty) \) replaced by \( s_n = O(n^\infty) \) yields

\[ (3.5) \quad \mathcal{O}(h) = \sum_{\nu=0}^{r-1} (A_{\nu} / \nu! h^\nu) + h^r R(h), \]

where \( A_{\nu} = \sum s_n^\alpha \Delta^{\alpha+1} (n^\nu) - r \) and \( R(h) = \sum s_n^\alpha \Delta^{\alpha+1} \lambda(nh) \) both converge absolutely, and \( R(h) = O(1) \) as \( h \to 0 \). Thus

\[ \mathcal{O}(t + h) = \mathcal{O}(t) + h \mathcal{O}_1(t) + \ldots + \]

\[ (h^r - 1/(r-1)! \mathcal{O}_{(r-1)}(t) + \left[ \mathcal{O}(t,h)/r! \right] h^r, \]

where \( \mathcal{O}(t,h) = O(1) \) as \( h \to 0 \). It follows from a theorem due to Marcinkiewicz and Zygmund [8, Lemma 7, p. 15] that \( \mathcal{O}_r(t) \) exists p.p. in \( E \).

Equation (3.5) gives \( \mathcal{O}_{(r-j)}(0) = A_{r-j} = \sum s_n^\alpha \Delta^{\alpha+1} (n^\nu)^{-j} \), and since the \((C, \alpha)\) sum of the series \( \sum c_n^\alpha / (n)^j \) equals the \((C, 0)\) sum of the series \( \sum s_n^\alpha \Delta^{\alpha+1} (n^{-1}) \), [5, p. 128], (3.4) is established.

**THEOREM 3.3.** If under the hypotheses of Theorem 3.2, the set \( E \) is an open interval, then

\[ (3.6) \quad C_a^D F(a) = S_{(a+1)}(t), \]

\[ C_a^D G(a) = S_{(a+1)}(t), \]

\[ 0 \leq a \leq \alpha, \quad t \in E, \quad C_a^D F(a+1) \]

\[ C_a^D G(a+1) \]

\[ \text{are finite for } t \in E, \text{ and} \]
For the proof a lemma is required.

**LEMMA.** If \( \sum_{n=1}^{\infty} a_n \) is summable \((C, r + 1)\), where \( r > -1 \), then a necessary and sufficient condition that it should be bounded \((C, r)\) is that \( B^r_n = O(n^r + 1) \) where \( b_n = n a_n \) and \( B^0_n, B^1_n, B^2_n, \ldots \) are formed from the \( b_n \) as \( A^0_n, A^1_n, A^2_n, \ldots \) are from the \( a_n \) (cf. 5, p. 96).

**Proof.** It is easy to verify that
\[
(n + r + 1)A^r_n = B^r_n + (r + 1)A^r_{n+1},
\]
and hence that
\[
A^r_n = (r + 1/n + r + 1)(B^r_n/(r + 1) + A^r_{n+1}),
\]
and the lemma follows.

The relations (3.6) will be proved by induction and in virtue of the symmetry of the enunciation only the first of the relations will be considered.

The result is well known for \( a = 0 \) [15, Lemma 5, p. 206]. By the lemma, series (3.2.1) is bounded \((C, \alpha - 1)\), series (3.2.2) is bounded \((C, \alpha - 2)\), \ldots , series (3.2.\( r - 1 \)) is bounded \((C, 0)\). Assume that the relation holds for all \( s < \alpha \) and hence that \( F^s(t) \) is the \( C_sP \) - integral of \( F(s + 1)(t) \) for all \( s < \alpha \). Then \((\alpha - 1)\) integrations by parts gives
\[ C_\xi DF^\alpha (t) = \lim_{h \to 0} \left[ \frac{1}{h^{\alpha + 1}} \left( \frac{1}{h} \int_{t}^{t + h} (t + h - f)^{\alpha} df - F^\alpha (t) \right) \right] = \lim_{h \to 0} \left[ (\alpha + 1)!/h^{\alpha + 1} \right] \left[ F(t + h) - F(t) - \sum_{k = 1}^{\alpha} \frac{h^k}{k!} F_k(t) \right]. \]

Hence

\[ C_\xi DF^\alpha (t) = \lim_{h \to 0} \left[ (\alpha + 1)!/h^{\alpha + 1} \right] \left[ F(t + h) - F(t) - \sum_{k = 1}^{\alpha} \frac{h^k}{k!} F_k(t) \right] = F(\alpha + 1)(t), \text{ by Theorem 3.2.} \]

It follows from (3.3) that

\[ (\alpha + 2)!/h^{\alpha + 2} \left[ F(t + h) - F(t) - \sum_{k = 1}^{\alpha + 1} \frac{h^k}{k!} F_k(t) \right] = 0(1), \]

as \( h \to 0 \) and so \( C_\xi \alpha + 1 DF^\alpha (\alpha + 1)(t) \) and \( C_\xi \alpha + 1 DF^\alpha (\alpha + 1)(t) \) are finite.

Finally,

\[ C_\xi \alpha + 1 DF^\alpha (\alpha + 1)(t) = \lim_{h \to 0} \left[ (\alpha + 2)!/h^{\alpha + 2} \right] \left[ F(t + h) - F(t) - \sum_{k = 1}^{\alpha + 1} \frac{h^k}{k!} F_k(t) \right] \]

if the limit on the right hand side exists. By Theorem 3.2 \( C_\xi \alpha + 1 DF^\alpha (\alpha + 1)(t) \) exists p.p. in \( E \) and is equal to \( F(\alpha + 2)(t) \).

**THEOREM 3.4.** Suppose that the real part of series (3.1) is bounded \((C, \alpha), (x \in E, |E| > 0, \alpha > -1)\). Then the
series (3.1) is summable (C, \( \alpha + \delta \)), \( \delta > 0 \), almost everywhere in \( E \) to sum \( \varphi _{(r)}(t) \), where \( \varphi (t) = F(t) - 1 \) \( G(t) \) is the function of Theorem 3.2 and \( r = \alpha + 2 \).

Proof. Marcinkiewicz and Zygmund have proved [8, Theorems 2 and 3] that if the real part of series (3.1) is bounded (C,\( \alpha \)), (\( x \in E \), \( |E| > 0 \), \( \alpha > -1 \)), then the series (3.1) is bounded (C,\( \alpha \)) and summable (C, \( \alpha + \delta \)), \( \delta > 0 \), p.p. in \( E \).

Now, to fix ideas take \( \alpha = 1 \). In view of the proof of Theorem 3.2, the series
\[
\sum_{n=1}^{\infty} \frac{-a_n \sin(nt) + b_n \cos(nt)}{n^3} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{a_n \cos(nt) + b_n \sin(nt)}{n^3}
\]
are uniformly convergent everywhere and hence are the Fourier series of \( F(t) \) and \( G(t) \) respectively. It follows also from Theorem 3.2 that \( F(3)(t) \) and \( G(3)(t) \) exist almost everywhere in \( E \). By a well known theorem (Cf. [16], p. 257) the Fourier series of \( F(t) \) and \( G(t) \) differentiated term by term three times are, almost everywhere in \( E \), summable (C,k), \( k > 3 \) to the values \( F(3)(t) \) and \( G(3)(t) \) respectively.

Hence series (3.1) is summable (C, \( 1 + \delta \)), \( \delta > 0 \), p.p. in \( E \) and also summable (C,4) to \( \varphi _{(3)}(t) \) p.p. in \( E \). It follows that the series (3.1) is almost everywhere summable (C, \( 1 + \delta \)) to sum \( \varphi _{(3)}(t) \).
4. The Expression of Coefficients in Terms of the $\phi^{k+2}$ - Integral

The main result of this chapter may be formulated as follows:

THEOREM 4.1. If $k = 2m$, let $\int_k = (2m + 2)! / [(m + 1)!]^2$ and let $(\alpha_1)$ be the set

$$(4.1) \quad -(2m + 2)\pi, -2m\pi, \ldots, -2\pi, 2\pi, \ldots, (2m + 2)\pi;$$

if $k = 2m + 1$, let $\int_k = (2m + 3)! / [(m + 1)!(m + 2)!]$ and let $(\alpha_1)$ be the set

$$(4.2) \quad -(2m + 2)\pi, -2m\pi, \ldots, -2\pi, 2\pi, \ldots, 2m\pi,$$

$$(2m + 2)\pi, (2m + 4)\pi.$$ 

Then if series $(3.1)$ is summable $(C,k)$ for all $t$ to a function $\mathcal{O}(t) = u(t) + iv(t)$, $|\mathcal{O}(t)| < \infty$, the coefficients of the series are given by

$$(4.3) \quad c_p = \frac{\int_k}{(2\pi)^{k+2}} \int_0^{(\alpha_1)} \mathcal{O}(t) \exp(-ipt) dt.$$ 

For the proof several preliminary theorems must be proved, the first one of which is the analogue of a theorem by James [6, Theorem 5.4, p. 159].

THEOREM 4.2. If $F(k+2)(t)$ exists and is finite in $(a,b)$ then it is $\phi^{k+2}$ - integrable over $(a_1; t)$ where $a = a_1 < a_2 < \ldots < a_n = b$. If $a_s \leq t < a_{s+1}$, then
(4.4) \[ (-1)^s + k + 2 \int_{a_1}^{t} F(k + 2)(t) d_k + 2t = \]

\[ F(t) - \sum_{i=1}^{k+2} \lambda(t; a_i) F(a_i) \]

where \[ \lambda(t; a_i) = \prod_{r \neq i} \frac{(t - a_r)}{a_i - a_r}. \]

**Proof.** The function

\[ Q(t) = F(t) - \sum_{i=1}^{k+2} \lambda(t; a_i) F(a_i) \]

is continuous and each \( Q(n) \) exists and is finite in \([a, b] \), \( 1 \leq n \leq k + 2 \). Furthermore, \( Q(a_i) = 0, i = 1, 2, 3, \ldots, k + 2 \), and

\[ \int Q(k + 2)(t) = F(k + 2)(t) = \Delta Q(k + 2)(t), \]

\[ \int Q(k + 2)(t) = -\infty, \Delta Q(k + 2)(t) = +\infty, \]

in \((a, b)\).

Thus \( Q(t) \) is both a major and a minor function of \( F(k + 2)(t) \) over \((a_i)\). It follows that \( F(k + 2)(t) \) is \( 0^{o}k + 2 \) integrable over \((a_i; t)\), and (4.4) follows from the definition of the \( 0^{o}k + 2 \) integral.

**THEOREM 4.3.** If the series (3.1) is summable \((C,k)\) for all \( t \) in \([a,b]\) to a function \( \mathcal{U}(t) = u(t) + v(t) \), where \([a,b]\) is a finite, closed interval and \( |\mathcal{U}(t)| < \infty \), then \( \mathcal{U}(t) \) is \( 0^{o}k + 2 \) integrable over \((a_i; t)\) for every \( t \) in \([a,b]\) where \( a = a_1 < \ldots < a_n = b \).

**Proof.** Series (3.1) integrated term by term \( k + 2 \) times
converges uniformly to a continuous function $\psi(t) = H(t) + iK(t)$. By Theorem 3.1 $H_r(t)$ and $K_r(t)$ exist and are finite for all $t$ in $[a,b]$, $1 \leq r \leq k + 2$, and $H_{(k+2)}(t) = u(t), K_{(k+2)}(t) = v(t)$. By the previous theorem $H_{(k+2)}(t) = u(t)$ and $K_{(k+2)}(t) = v(t)$ are $\mathcal{C}^{k+2}$-integrable. Hence $u(t) + iv(t) = \mathcal{O}(t)$ is $\mathcal{C}^{k+2}$-integrable over $(a_1; t)$.

**LEMMA.** Under the hypotheses of Theorem 4.3, $\mathcal{O}(t)\exp(-ipt), p \geq 1$, is $\mathcal{C}^{k+2}$-integrable over $(a_1; t)$.

**Proof.** By hypothesis

$$\sum_{n=0}^{\infty} c_n \exp(int) = \mathcal{O}(t), \quad (C,k),$$

for all $t$ in $[a,b]$. It follows that

$$\sum_{n=0}^{\infty} c_n \exp(i(n-p)t) = \mathcal{O}(t)\exp(-ipt), \quad (C,k),$$

for all $t$ in $[a,b]$, and therefore that

$$(4.5) \quad \sum_{n=p}^{\infty} c_n \exp(i(n-p)t) = -c_0 \exp(-ipt) - c_1 \exp(-i(1-p)t) - \ldots - c_p \exp(-it) + \mathcal{O}(t)\exp(-ipt) = \Theta(t), \quad (C,k).$$

By Theorem 4.3, $\Theta(t)$ is $\mathcal{C}^{k+2}$-integrable over $(a_1; t)$ for every $t$ in $[a,b]$. It follows that $\mathcal{O}(t)\exp(-ipt)$ is $\mathcal{C}^{k+2}$-integrable since each of the other functions on the right hand side of (4.5) is $\mathcal{C}^{k+2}$-integrable.
Theorem 4.1 can now be proved.

Proof. Consider first the expression for \( c_0 \) when \( k \) is even. It follows from Theorem 4.2 and the proof of Theorem 4.3 that,

\[
(4.6) \quad (-1)^{\frac{k+2}{2}} + k + 2 \int_{(0)}^t \Omega(t) d_k + 2t = (-1)^{\frac{k+2}{2}} \int_{(0)}^t \Omega(t) d_k + 2t = H(t) - \sum_{i=1}^{\frac{k+2}{2}} \lambda(t; \alpha_i) H(\alpha_i) +
\]

\[
i \left[ K(t) - \sum_{i=1}^{\frac{k+2}{2}} \lambda(t; \alpha_i) K(\alpha_i) \right] = Q(t) + i R(t),
\]

where \( \alpha_i \) is the set \((4.1)\). The function \( H(t) + i K(t) \) may be written in the form \( \Psi(t) + H_1(t) + i H_2(t) \), where \( \Psi(t) = c_0 t^{k+2}/(k+2)! \) and \( H_1(t) \) and \( H_2(t) \) are periodic with period \( 2\pi \). If \( t = 0 \), the right hand side of \((4.6)\) becomes

\[
(4.7) \quad \Psi(0) - \sum_{i=1}^{\frac{k+2}{2}} \lambda(0; \alpha_i) \Psi(\alpha_i) + [H_1(0) + i H_2(0)]
- \sum_{i=1}^{\frac{k+2}{2}} \lambda(0; \alpha_i) \left[ H_1(0) + i H_2(0) \right].
\]

This reduces to

\[
\frac{c_0}{(k+2)!} \prod_{i=1}^{\frac{k+2}{2}} (-\alpha_i) = (-1)^{\frac{k+2}{2}} \frac{c_0 (2\pi)^{k+2}}{\Gamma(k)}
\]

(cf. James, [7], p. 106).
Hence

\[ c_0 = \frac{\sqrt{k}}{(2\pi)^k + 2} \int_0^\infty \theta(t) d_k + 2t. \]

The expression for \( c_0 \) when \( k \) is odd can be obtained in a similar manner since

\[(4.8) \quad (-1) \left[ \frac{(k+1)/2}{r} \right] + k + 2 \int_0^t \theta(t) d_k + 2t = \]

\[-1 \left[ \frac{(k+1)/2}{r} \right] + 1 \int_0^t \theta(t) d_k + 2t = \]

\[\Psi(t) + H_1(t) + 1 H_2(t), \]

\[\frac{\alpha (k+1)/2 \leq t \leq \frac{\alpha [k+1)/2] + 1}{\alpha_1} \]

where \((\alpha_1)\) is the set \((4.2)\).

To obtain the formulae for \( c_p, p \geq 1 \), rewrite equation \((4.5)\) in the form

\[(4.9) \quad \sum_{r=0}^\infty a_r \exp(irt) = \Theta(t), \quad (C,k) \]

where \( r = n - p, n = p, p + 1, \ldots \), and \( a_r = a_n - p = c_n \).

The constant term in \((4.9)\) is \( a_0 = c_p \), and for \( k \) even,

\[
\frac{c_p}{(k+2)!} \prod_{1=1}^{k+2} (-\alpha_1) = \]

\[-1 \left[ \frac{(k+2)/2}{r} \right] + k + 2 \int_0^\infty \theta(t) d_k + 2t = \]
\[ (-1)^{(k+2)/2} \int_{x_1}^{0} \varphi(t) \exp(-ipt) \, dk + 2^t, \]

since

\[ \int_{x_1}^{0} \exp(-imt) \, dk + 2^t = 0. \]

It follows that

\[ c_p = \frac{\int_{k}^{\infty}}{(2\pi)^{k+2}} \int_{x_1}^{0} \varphi(t) \exp(-ipt) \, dk + 2^t. \]

For \( k \) odd the argument is similar.

**Remark.** The proof of Theorem 4.1 is simpler than the proof of the analogous result by James [7, pp. 105 - 106] since the results on the formal multiplication of trigonometric series are not needed.
5. The Expression of Coefficients in Terms of the $C^{\alpha+1}_n$ - Integral

**THEOREM.** Suppose that the series (1.2) is bounded $(C, \alpha)$, for a fixed $\alpha = 0, 1, 2, \ldots$ in $(-\pi, \pi)$ and let $r = \alpha + 2$. If $F(t)$ and $G(t)$ are the functions of Theorem 3.2, the coefficients of series (1.2) are given by

\begin{align}
(5.1) \quad a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} F(r)(t)\cos(nt)dt = \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} G(r)(t)\sin(nt)dt,
\end{align}

\begin{align}
(5.2) \quad b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} F(r)(t)\sin(nt)dt = \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} G(r)(t)\cos(nt)dt,
\end{align}

where the integrals are $C^{\alpha+1}_n$ - integrals.

**Proof.** To fix ideas take $\alpha = 2$. It follows from Theorem 3.2, Theorem 3.3, and the definition of the $C_n$ - integral that

\[ F^S(t) - F^S(-\pi) = F_S(t) - F_S(-\pi) = C_S \int_{-\pi}^{t} F(s+1)(x)dx, \]

$0 \leq s \leq 3$. Hence, using the property of integration by parts for the $C_n$ - integral [2],

\[ C_{2n} \int_{-\pi}^{\pi} F(4)(t)\cos(nt)dt = C_{2n} \int_{-\pi}^{\pi} F(3)(t)\sin(nt)dt = \]

\[ C_{1n} \int_{-\pi}^{\pi} F(2)(t)n^2\cos(nt)dt = -C_{0n} \int_{-\pi}^{\pi} F(1)(t)n^3\sin(nt)dt = \]
\[
C_0^P \int_{-\pi}^{\pi} F(t) n^4 \cos(nt) \, dt = a_n \pi,
\]
since \[
\sum_{n=1}^{\infty} \left[ a_n \cos(nt) + b_n \sin(nt) \right] / n^4
\]
is the uniformly convergent Fourier series of \( F(t) \). The other results follow similarly.
6. The $C_k + l^P$-integral and the $\gamma^k + 2^P$-integral

It has been shown by James [6] that $C_k + l^P$-integrability implies $\gamma^k + 2^P$-integrability and that the indefinite $\gamma^k + 2^P$-integral is equal to an $(r + 1)$-fold integral in which the innermost integral is an indefinite $C_r^P$-integral, the next one an indefinite $C_r + l^P$-integral, and so on, the outermost integral being an indefinite $C_0^P$-integral. A relationship between the definite $\gamma^k + 2^P$-integral and the definite $C_k + l^P$-integral suggests itself since, when series (1.2) is summable $(C, k)$, the theorem in Chapter 5 and Theorem 4.1 give analogous expressions for the coefficients $c_n$, one in terms of the $\gamma^k + 2^P$-integral, the other in terms of the $C_k + l^P$-integral.

**THEOREM.** Suppose that $f(t)$ is periodic with period $2\pi$ and $C_r^P$-integrable over $(-\pi, \pi)$. Let $(\alpha_i)$ be the set

(6.1) \[-r\pi, -(r - 2)\pi, \ldots , -2\pi, 2\pi, \ldots , (r + 2)\pi,\]

when $r$ is even, and the set

(6.2) \[-(r + 1)\pi, -(r - 1)\pi, \ldots , -2\pi, 2\pi, \ldots , (r - 1)\pi, (r + 1)\pi,\]

when $r$ is odd. If

$$F_r(x) = C_r^P \int_{-r\pi}^{x} f(t) dt,$$

$$F_k(x) = C_k^P \int_{-r\pi}^{x} F_{k+1}(t) dt, \quad 0 \leq k \leq r - 1,$$
\[ F(x) = F_0(x), \]

then

\[ \left[ \frac{r - 1}{(2\pi)^r} \right] \int_{r}^{0} f(t)dt + 1 = C_r P \int_{-\pi}^{\pi} f(t)dt. \]

**Proof.** Consider the case when \( r \) is odd. (The proof for \( r \) even is similar.) It is known [6, p. 168] that

\[ (-1)^{[r+1]/2} + r + 1 \int_{0}^{\pi} f(t)dt + 1 = \]

\[ F(0) - \sum_{i=1}^{r+1} \lambda (0; \alpha_i) F(\alpha_i). \]

It is easy to verify that

\[ F_r(x) = C_r P \int_{-r}^{x} f(t)dt = \frac{x}{2\pi} \int_{-\pi}^{\pi} f(t)dt + G_2(x) = x \mu + G_2(x), \]

where \( G_2(x) \) is periodic with period \( 2\pi \). It follows that

\[ F(x) = \left[ \mu x^r + 1/(r+1)! \right] + G_0(x) + \text{(a polynomial in } x \text{ of degree } r), \]

where \( G_0(x) \) is periodic with period \( 2\pi \).

The right hand side of (6.3) thus reduces to

\[ (-1)^{(r+1)/2} \mu (2\pi)^r + 1/\left[ \int_{r-1}^{\pi} \right] \text{(cf. p. 18). This yields} \]

\[ \left[ \frac{r - 1}{(2\pi)^r + 1} \right] \int_{0}^{\pi} f(t)dt + 1 \int_{-\pi}^{\pi} \frac{1}{C_r P} \int_{-\pi}^{\pi} f(t)dt, \]

the required result.
7. **Uniqueness Theorems for Multiple Trigonometric Series**

Since the triangular method of summation defined in Chapter 2 reduces the summation of a multiple trigonometric series to the Cesàro summation of a single series, it is possible to use the results of the preceding chapters to obtain uniqueness theorems for multiple trigonometric series.

**THEOREM 7.1.** If the series \( \sum \) is summable \((T,k)\) to a function \( \mathcal{O}(x_1, x_2, \ldots, x_n, t) \equiv \mathcal{O}(x,t), \quad |\mathcal{O}(x,t)| < \infty, \) for all values \( (x_1 + t, x_2 + t, \ldots, x_n + t) \) where \( 0 \leq x_1, x_2, \ldots, x_n, t \leq 2\pi \) then the coefficients of the series are given by

\[
C_{m_1, m_2, \ldots, m_n} = \frac{1}{(2\pi)^n} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathcal{O}(x,t) e^{-ipt} d_1 \cdots dx_n,
\]

where \( m_1 + \ldots + m_n = p, \quad \int d_1 \) and \( \alpha_1 \) are defined as in Theorem 4.1, and the \( n \) outer integrals are Riemann integrals.

**Proof.** For simplicity in writing, the proof will be given for \( n = 2 \).

By hypothesis,

\[
C_\infty + \sum_{p=1}^\infty C_p(x_1, x_2)e^{ipt} = \mathcal{O}(x_1, x_2, t), \quad (C,k),
\]

for all \( t \) and \( 0 \leq x_1, x_2 \leq 2\pi \). It follows from Theorem 4.1
\[ C_p(x_1, x_2) = \sum_{m_1 + m_2 = p} c_{m_1 m_2} e^{i(m_1 x_1 + m_2 x_2)} = \frac{\int_k}{(2\pi)^k + 2} \int_0^0 \mathcal{U}(x_1, x_2, t) e^{-ipt} d_k + 2t, \]

with \(0 \leq x_1, x_2 \leq 2\pi\). Now let \(m\) and \(n\) be any two positive integers such that \(m + n = p\). Multiplying both sides of (7.2) by \(e^{-imx_1}\) and integrating with respect to \(x_1\) over \((0, 2\pi)\) yields

\[ (2\pi)(c_{m, p-m}) e^{i(p-m)x_2} = \]

\[ 2\pi c_{mn} e^{inx_2} = \]

\[ \int_0^{2\pi} e^{-imx_1} \int_k \int_0^0 \mathcal{U}(x_1, x_2, t) e^{-ipt} d_k + 2t dx_1. \]

Then, multiplying both sides of (7.3) by \(e^{-inx_2}\) and integrating with respect to \(x_2\) over \((0, 2\pi)\) gives

\[ (2\pi)^2 c_{mn} = \]

\[ \int_0^{2\pi} \int_0^{2\pi} e^{-i(mx_1 + nx_2)} \int_k \int_0^0 \mathcal{U}(x_1, x_2, t) e^{-ipt} d_k + 2t dx_1 dx_2, \]

the required result.
A more general result is obtainable in a similar way in virtue of the theorem of Chapter 5.

**THEOREM 7.2.** If series

\[ \sum c_m e^{i(m,x)}, \quad m_j > 0, \]

is bounded \((T,k)\) for all values \((x_1 + t, \ldots, x_n + t), 0 \leq x_1, x_2, \ldots, x_n, t \leq 2\pi,\) and if \(r = k + 2,\) then the series

\[ \sum \frac{C(x_1, x_2, \ldots, x_n) e^{ipt}}{(ip)^r} \]

converges uniformly for all \(t\) and each \((x)\) to a continuous function \(\Psi(t),\) and the coefficients of the series are given by

\[ (7.5) \quad c_{m_1, \ldots, m_n} = \]

\[ \frac{1}{(2\pi)^n} \int_0^{2\pi} \ldots \int_0^{2\pi} \left( \frac{e^{-i(m,x)}}{2\pi} \int_0^{2\pi} \Psi(r)(t)e^{-ipt} dt dx_1 \ldots dx_n, \right) \]

where the inner integral is a \(C^k + 1^P\) integral.

**Proof.** (for \(n = 2\)) The convergence property of the integrated series is obvious.

The theorem of Chapter 5 gives \(C_p(x_1, x_2) = \)

\[ (7.6) \quad \sum_{m_1 + m_2 = p} c_{m_1, m_2} e^{i(m_1 x_1 + m_2 x_2)} = \]

\[ \frac{1}{2\pi} \int_0^{2\pi} \Psi(r)(t)e^{-ipt} dt, \]
and (7.5) follows from (7.6) in the same way that (7.1) follows from (7.2).

A uniqueness theorem which is stronger than Theorem 7.1 may be obtained in a very interesting manner. It is stronger in the sense that the set of $n$-tuples $(x_1, x_2, \ldots, x_n)$ in the hypothesis is reduced to a countable set.

**THEOREM 7.3.** If for a countable set $A$ of $n$-tuples $(x_{10}, x_{20}, \ldots, x_n, x_{n1})$, for fixed $x_{10}, x_{20}, \ldots, x_n - 10$ and $i = 1, 2, 3, \ldots$, series (1.1) is summable $(T, k)$ to 0 for all values $(x_{10} + t, x_{20} + t, \ldots, x_{n1} + t)$, $i = 1, 2, 3, \ldots$, $0 \leq t \leq 2\pi$, then the series vanishes identically.

**Proof.** Only the case $n = 3$ will be considered.

In view of the proof of Theorem 7.1, it is clear that

\[(7.7) \sum_{m_1 + m_2 + m_3 = p} c_{m_1} m_1 e^{i(m_1 x_{10} + m_2 x_{20} + m_3 x_{31})} = 0,\]

for fixed $x_{10}, x_{20}$ and $i = 1, 2, 3, \ldots$. This gives

\[\sum_{m_1 + m_2 + m_3 = p} \left[ c_{m_1} m_1 e^{i(m_1 x_{10} + m_2 x_{20})} \right] e^{im_3 x_{31}} = \]
\[
\sum_{m_1 + m_2 + m_3 = p} \left[ c_{m_1, m_2, m_3} \right] e^{im_3 x_{31}} = 0
\]

\[
\sum_{m_1 + m_2 = 0} \left[ c_{m_1, m_2, p - m_1 - m_2} \right] e^{ipx_{31}} e^{-i(m_1 + m_2)x_{31}} = 0,
\]

\[
e^{ipx_{31}} \sum_{m_1 + m_2 = 0} \left[ c_{m_1, m_2, p - m_1 - m_2} \right] e^{-i(m_1 + m_2)x_{31}} = 0,
\]

\[i = 1, 2, 3, \ldots\] It follows that the equation

\[
(7.8) \sum_{m_1 + m_2 = 0}^{p} c_{m_1, m_2, p - m_1 - m_2} \left[ e^{-1u} \right]^{m_1 + m_2} = 0
\]

has a countable number of roots, \(x_{31}, x_{32}, x_{33}, \ldots\). But the left-hand side of (7.8) is a polynomial of degree \(p\) and cannot have a countable number of zeros. The only possibility is that \(c_{m_1, m_2, p - m_1 - m_2} \equiv c_{m_1, m_2, m_3} = 0\), and hence that \(c_{m_1, m_2, m_3} = 0\) for all \(m_1, m_2, m_3\) such that \(m_1 + m_2 + m_3 = p\).
BIBLIOGRAPHY


