THE SOLUTION OF DIFFERENTIAL EQUATIONS THROUGH INTEGRAL EQUATIONS

by

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ABSTRACT

A method of writing the solution of a second order differential equation through a Volterra Integral Equation is developed. The method is applied to initial value problems, to special functions, and to bounded Quantum Mechanical problems. Some of the results obtained are original, and other results agree essentially with the work done previously by others.

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TABLE OF CONTENTS

INTRODUCTION .... (i)

CHAPTER ONE. THE GENERAL METHOD .... (1)

Section 1.1. Derivation Of The Volterra Integral Equation .... (1)

Section 1.2. Determination Of The Arbitrary Constants .... (3)

Section 1.3. The Volterra Integral Equation For The General Cauchy Problem .... (5)

Section 1.4. The Equation With The First Derivative Term Missing .... (10)

Section 1.5. Solution Of The Integral Equation ... (11)

Section 1.6. The General Solution Of A Related Integral Equation .... (16)

CHAPTER TWO. APPLICATION TO INITIAL VALUE PROBLEMS ... (20)

Section 2.1. Introduction .... (20)

Section 2.2. A Problem With A Perturbation In The First Derivative Term .... (20)

Section 2.3. Another Single Perturbation Problem . (22)

Section 2.4. Multiple Perturbation Problems ...... (23)

Section 2.5. Another Treatment Of The Problem In Section 2.2 .... (26)

CHAPTER THREE. APPLICATION TO SPECIAL FUNCTIONS ..... (28)

Section 3.1. Introduction .... (28)

Section 3.2. The Expansion Of The Solution Of The Confluent Hypergeometric Equation In Series of Bessel Functions .... (29)

TABLE OF CONTENTS (Continued)

Section 3.3. A Generalization Of The Problem .. (33)

Section 3.4. A Further Generalization .... (35)

Section 3.5. Solution Of Related Differential Equations Expressed In Terms Of Bessel Functions .... (37)

Section 3.6. Appendix To Chapter Three .... (39)

CHAPTER FOUR. PHYSICAL APPLICATIONS .... (42)

Section 4.1. Introduction .... (42)

Section 4.2. The Integral Equation For Bounded Quantum Mechanical Problems ...... (42)

Section 4.3. The Bounded Hydrogen Atom Problem .(45)

Section 4.4. The Bounded Rigid Rotator .... (49)

Section 4.5. The Ground Level Of The Bounded Rigid Rotator .... (51)

Section 4.6. Higher Levels Of The Bounded Rigid Rotator .... (54)

BIBLIOGRAPHY .... (57)

INTRODUCTION

The central theme of this thesis is the use of a Volterra Integral Equation to express the solutions of a second order differential equation in terms of known functions. By the procedure which is followed, it is then possible to derive certain properties of these solutions systematically. The idea originated with Cauchy, Liouville (7), and contemporaries in the early nineteenth century. In particular, Liouville transformed the equation

(a) $$u''(x) + \rho^2 u(x) = q(x)\, u(x)$$

into the integral equation of the second kind

(b) $$u(x) = u(0)\cos\rho x + \frac{1}{\rho} u'(0)\sin\rho x + \frac{1}{\rho}\int_0^x q(t)\sin\rho(x-t)\,u(t)\,dt.$$

The classical approach was to consider (a) as a non-homogeneous differential equation whose homogeneous part has known solutions $\cos\rho x$ and $\sin\rho x$, and to apply the method of variation of parameters or Laplace Transform theory to obtain the integral equation (b) in terms of these known functions. The equation (b) was used to study the asymptotic behaviour of the eigenvalues and the eigenfunctions of (a) for large ρ.

More recent investigators, notably Ikeda (4), Fubini (3), and Tricomi (15) have changed the viewpoint to that of comparing the unknown solutions of (a) with the known solutions of a distinct differential equation

(c) $$v''(x) + \rho^2 v(x) = 0,$$

or, in general, of comparing the solutions of

(d) $$u''(x) + P(x)\,u'(x) + Q(x)\,u(x) = 0$$

with the supposed known solutions $v_1(x)$ and $v_2(x)$ of

(e) $$v''(x) + R(x)\,v'(x) + S(x)\,v(x) = 0$$

through a Volterra Integral Equation. This approach will be used throughout the discussion.

In the first chapter, the integral equation associated with the equation (d) will be derived, and a procedure will be given for the determination of the arbitrary constants in order that the initial conditions be satisfied. The whole idea will be generalized to an n-th order linear differential equation, giving a result entirely analogous to the second order case. The appropriate existence theorems needed in later chapters will be proved.

The very nature of the method suggests that it be used to get expansions of solutions of certain differential equations in terms of better known solutions of other equations. In the second chapter, we use this idea to expand the solutions of the Confluent Hypergeometric Equation in Bessel Functions of the first and second kind. The computational value of such an expansion has been discussed by Karlin (9).

In the fourth chapter, we shall show that boundary value problems as well as initial value problems can be handled by adapting the method. In particular, the bounded Quantum Mechanical problems are discussed, and eigenvalues for the Hydrogen atom problem and for the rigid rotator problem are calculated.

CHAPTER ONE

THE GENERAL METHOD

1.1 Derivation of the Volterra Integral Equation.

The object is to express the solutions of the second order differential equation

(1.1.1) $$u''(x) + P(x)\,u'(x) + Q(x)\,u(x) = 0$$

in terms of the known solutions $v_1(x)$ and $v_2(x)$ of the equation

(1.1.2) $$v''(x) + R(x)\,v'(x) + S(x)\,v(x) = 0$$

by a Volterra Integral Equation. It is assumed that the u equation has the same singularities as the v equation. The result will be obtained by adapting the method of variation of parameters for solving non-homogeneous differential equations: let (1.1.1) be rewritten in the form

(1.1.3) $$u''(x) + R(x)\,u'(x) + S(x)\,u(x) = [R(x)-P(x)]\,u'(x) + [S(x)-Q(x)]\,u(x)$$

with supposed solution

(1.1.4) $$u(x) = c_1(x)\,v_1(x) + c_2(x)\,v_2(x)$$

where $v_1(x)$ and $v_2(x)$ satisfy (1.1.2). From equation (1.1.4),

(1.1.5) $$u'(x) = c_1(x)\,v_1'(x) + c_2(x)\,v_2'(x)$$

provided that

(1.1.6) $$c_1'(x)\,v_1(x) + c_2'(x)\,v_2(x) = 0,$$

and

(1.1.7) $$u''(x) = c_1(x)\,v_1''(x) + c_2(x)\,v_2''(x) + c_1'(x)\,v_1'(x) + c_2'(x)\,v_2'(x)$$

If (1.1.4), (1.1.5), and (1.1.7) are substituted into (1.1.3), the result is

(1.1.8) $$c_1'(x)\,v_1'(x) + c_2'(x)\,v_2'(x) = [R(x)-P(x)]\,u'(x) + [S(x)-Q(x)]\,u(x).$$

The solution of the linear algebraic equations (1.1.6) and (1.1.8) is

(1.1.9) $$c_1'(x) = \frac{\begin{vmatrix} 0 & v_2(x) \\ \Phi & v_2'(x) \end{vmatrix}}{\begin{vmatrix} v_1(x) & v_2(x) \\ v_1'(x) & v_2'(x) \end{vmatrix}} \qquad c_2'(x) = \frac{\begin{vmatrix} v_1(x) & 0 \\ v_1'(x) & \Phi \end{vmatrix}}{\begin{vmatrix} v_1(x) & v_2(x) \\ v_1'(x) & v_2'(x) \end{vmatrix}}$$

where

$$\Phi = \{R(x)-P(x)\}u'(x) + \{S(x)-Q(x)\}u(x) .$$

Hence,

(1.1.10) $$c_1(z) = \alpha_1 + \int_b^z -\frac{v_2(x)}{W(x)}\,\Phi\,dx$$

and

(1.1.11) $$c_2(z) = \beta_1 + \int_b^z \frac{v_1(x)}{W(x)}\,\Phi\,dx,$$

where α_1 and β_1 are constants of integration, b is a constant,[1] and the Wronskian of v_1 and v_2 is given by

(1.1.12) $$W(x) = \begin{vmatrix} v_1(x) & v_2(x) \\ v_1'(x) & v_2'(x) \end{vmatrix} .$$

If (1.1.10) and (1.1.11) are put into (1.1.4), it is seen that $u(z)$ must satisfy the integral equation

$$u(z) = \alpha v_1(z) + \beta v_2(z) + \int_b^z \frac{v_1(x)v_2(z) - v_1(z)v_2(x)}{W(x)}\{[R(x)-P(x)]u'(x) + [S(x)-Q(x)]u(x)\}dx$$

or

(1.1.13) $$u(z) = \alpha_1 v_1(z) + \beta_1 v_2(z) + \int_b^z N(z,x)\{R(x)-P(x)\}u'(x)dx + \int_b^z N(z,x)\{S-Q\}u\,dx$$

where

(1.1.14) $$N(z,x) = \begin{vmatrix} v_1(x) & v_2(x) \\ v_1(z) & v_2(z) \end{vmatrix} \Big/ W(x) = M(z,x)/W(x) .$$

---

1 If b is not an ordinary point of the differential equation (1.1.1), there results apply only for the solution $u(z)$ of (1.1.1) which is finite at b ; the definite integrals in (1.1.10) and (1.1.11) do not exist in general for the other solutions. However, by deleting the constant b from the lower limit of the integrals, the result (1.1.15) is expressed in terms of an indefinite integral, and the above restriction is removed.

In order to get (1.1.13) in the form of a Volterra Integral Equation, we remove the $u'(x)$ term by performing a partial integration: with

$$U = N(z,x)\,\{R(x)-P(x)\}$$

and

$$dV = u'(x)\,dx$$

we obtain

$$\begin{aligned} I &= \int_b^z N(z,x)\,\{R(x)-P(x)\}\,u'(x)\,dx \\ &= u(z)N(z,z)\{R(z)-P(z)\} - u(b)\frac{v_1(b)v_2(z)-v_1(z)v_2(b)}{W(b)}\{R(b)-P(b)\} - \int_b^z u\frac{\partial}{\partial x}\{N(R-P)\}dx \\ &= -\frac{u(b)v_1(b)}{W(b)}\{R(b)-P(b)\}\,v_2(z) + \frac{u(b)v_2(b)}{W(b)}\{R(b)-P(b)\}\,v_1(z) - \int_b^z u\frac{\partial}{\partial x}\{N(R-P)\}dx \end{aligned}$$

since $N(z,z)=0$ from (1.1.14). Thus,

$$I = \alpha_2 v_1(z) + \beta_2 v_2(z) - \int_b^z u(x)\frac{\partial}{\partial x}\{N(z,x)\,(R(x)-P(x))\}\,dx$$

where α_2 and β_2 are constants. Putting this into (1.1.13), we obtain finally

(1.1.15) $$u(z) = \alpha v_1(z) + \beta v_2(z) + \int_b^z K(z,x)\,u(x)\,dx\ ,$$

where

(1.1.16) $$K(z,x) = N(z,x)\,\{S(x)-Q(x)\} - \frac{\partial}{\partial x}\{N(z,x)(R(x)-P(x))\}$$

and $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$ are constants.

1.2 Determination of the Arbitrary Constants.

For initial value problems, the constants α and β can be calculated explicitly. In order to arrive at the result, we first need to develop a property of the function $K(z,x)$, stated in the

Lemma: $$K(z,z) = R(z) - P(z)\,.$$

Proof: From (1.1.14)

(1.2.1) $$M(z,z) = 0, \qquad N(z,z) = 0\,.$$

Also from (1.1.14),

$$\frac{\partial M(z,x)}{\partial x} = \frac{\partial}{\partial x}\begin{vmatrix} v_1(x) & v_2(x) \\ v_1(z) & v_2(z) \end{vmatrix} = \begin{vmatrix} v_1'(x) & v_2'(x) \\ v_1(z) & v_2(z) \end{vmatrix}.$$

Hence,

(1.2.2) $$\frac{\partial M(z,z)}{\partial x} = -\begin{vmatrix} v_1(z) & v_2(z) \\ v_1'(z) & v_2'(z) \end{vmatrix} = W(z),$$

$$\frac{\partial N(z,z)}{\partial x} = \left[\frac{W(x)\,\frac{\partial M}{\partial x}(z,x) - M(z,x)\,\frac{dW(x)}{dx}}{W(x)^2}\right]_{x=z}$$, or

(1.2.3) $$\frac{\partial N(z,z)}{\partial x} = -1,$$

where use is made of (1.2.1) and (1.2.2). From (1.1.16),

$$K(z,z) = \frac{\partial N(z,z)}{\partial x}\{R(z)-P(z)\} - N(z,z)\left[\frac{\partial}{\partial x}\{R(x)-P(x)\}\right]_{x=z}$$

The result now follows because of (1.2.1) and (1.2.3).

From (1.1.15),

(1.2.4) $$u(b) = \alpha v_1(b) + \beta v_2(b)$$

and

(1.2.5) $$u'(z) = \alpha v_1'(z) + \beta v_2'(z) + \int_b^z \frac{\partial K}{\partial z}(z,x)\,dx + K(z,z)\,u(z).$$

Upon use of the Lemma,

$$u'(z) = \alpha v_1'(z) + \beta v_2'(z) + \int_b^z \frac{\partial K(z,x)}{\partial z}\,dx + \{R(z)-P(z)\}\,u(z).$$

Hence, for $z = b$,

(1.2.6) $$u'(b) - \{R(b)-P(b)\}\,u(b) = \alpha v_1'(b) + \beta v_2'(b).$$

The solution of the linear algebraic equations (1.2.4) and (1.2.6) then gives α and β in terms of the initial values $u(b)$ and $u'(b)$:

(1.2.7) $$\alpha = \begin{vmatrix} u(b) & v_2(b) \\ \varphi(b) & v_2'(b) \end{vmatrix}\Big/ W(b)\ ; \qquad \beta = \begin{vmatrix} v_1(b) & u(b) \\ v_1'(b) & \varphi(b) \end{vmatrix}\Big/ W(b),$$

where

(1.2.8) $$\varphi(b) = u'(b) - \{R(b)-P(b)\}\,u(b).$$

This is the result that Ikeda (4) obtained by a different method.

1.3 The Volterra Integral Equation For The General Cauchy Problem.

The solution of the n-th order linear differential equation

(1.3.1) $$u^{(n)}(x) + \sum_{r=1}^{n} P_{n-r}(x)\, u^{(n-r)}(x) = 0$$

assuming that initial values $u^{(l)}(1)\ (l=1,2,\cdots (n-1))$ is to be expressed in terms of the linearly independent solutions $v_r(x)\ (r=1,2,\cdots n)$ of the equation

(1.3.2) $$v^{(n)}(x) + \sum_{r=1}^{n} R_{n-r}(x)\, v^{(n-r)}(x) = 0$$

through a Volterra Integral Equation. We suppose that the functions $P_{n-r}(x)$ and $R_{n-r}(x)$ are analytic for all required values of x . Let (1.3.1) be rewritten in the form

(1.3.3) $$u^{(n)}(x) + \sum_{r=1}^{n} R_{n-r}(x) u^{(n-r)}(x) = \sum_{r=1}^{n} \{R_{n-r}(x) - P_{n-r}(x)\} u^{(n-r)}(x),$$

with supposed solution

(1.3.4) $$u(z) = \sum_{r=1}^{n} c_r(z) v_r(z).$$

Following the method of Variation of Parameters, we have

$$u'(x) = \sum_{r=1}^{n} c_r(x) v_r'(x),$$

$$u''(x) = \sum_{r=1}^{n} c_r(x) v_r''(x),$$

$$\cdots\cdots$$

(1.3.5) $$u^{(n-1)}(x) = \sum_{r=1}^{n} c_r(x) v_r^{(n-1)}(x),$$

$$u^{(n)}(x) = \sum_{r=1}^{n} \{ c_r(x) v_r^{(n)}(x) + c_r'(x) v_r^{(n-1)}(x) \},$$

provided that the c_r's satisfy

$$\sum_{r=1}^{n} c_r'(x) v_r(x) = 0,$$

(1.3.6) $$\sum_{r=1}^{n} c_r'(x) v_r'(x) = 0,$$

$$\cdots\cdots$$

$$\sum_{r=1}^{n} c_r'(x) v_r^{(n-2)}(x) = 0 .$$

Putting (1.3.5) into (1.3.3), we get

(1.3.7) $$\sum_{r=1}^{n} c_r'(x) v_r^{(n-1)}(x) = \sum_{r=1}^{n} \{ R_{n-r}(x) - P_{n-r}(x)\} u^{(n-r)}(x) = \Phi .$$

Let $W(x)$ be the Wronskian of the n functions $v_r(x)\ (r=1,2,\cdots n)$

$$(1.3.8)\qquad W(x) = \begin{vmatrix} v_1(x) & v_2(x) & \cdots & v_n(x) \\ v_1'(x) & v_2'(x) & & v_n'(x) \\ \vdots & \vdots & & \vdots \\ v_1^{(n-1)}(x) & v_2^{(n-1)}(x) & & v_n^{(n-1)}(x) \end{vmatrix} .$$

Since $W(x) \neq 0$, the unique solution of the n algebraic equation (1.3.6) and (1.3.7) is

$$(1.3.9)\qquad c_r'(x) = \frac{1}{W(x)} \begin{vmatrix} v_1(x) & v_2(x) & \cdots & v_{r-1}(x) & 0 & v_{r+1}(x) & \cdots & v_n(x) \\ v_1'(x) & v_2'(x) & & v_{r-1}'(x) & 0 & v_{r+1}'(x) & & v_n'(x) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ v_1^{(n-1)}(x) & v_2^{(n-1)}(x) & & v_{r-1}^{(n-1)}(x) & \Phi & v_{r+1}^{(n-1)}(x) & & v_n^{(n-1)}(x) \end{vmatrix} .$$

Putting these values into (1.3.4), we obtain

$$(1.3.10)\qquad u(z) = \sum_{r=1}^{n} \alpha_r v_r(z) + \int_b^z \sum_{r=1}^{n} \begin{vmatrix} v_1(x) & v_2(x) & \cdots & v_{r-1}(x) & 0 & v_{r+1}(x) & \cdots & v_n(x) \\ v_1'(x) & v_2'(x) & & v_{r-1}'(x) & 0 & v_{r+1}'(x) & & v_n'(x) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ v_1^{(n-1)}(x) & v_2^{(n-1)}(x) & & v_{r-1}^{(n-1)}(x) & \Phi & v_{r+1}^{(n-1)}(x) & & v_n^{(n-1)}(x) \end{vmatrix} \frac{v_r(z)}{W(x)}\, dx ,$$

where α_r are the n constants of integration involved in computing the c_r's, and where b is a fixed constant. Summing the determinants under the integral sign in (1.3.10), we have

$$(1.3.11)\qquad u(z) = \sum_{r=1}^{n} \alpha_r v_r(z) + \int_b^z \begin{vmatrix} v_1(x) & v_2(x) & \cdots & v_n(x) \\ v_1'(x) & v_2'(x) & & v_n'(x) \\ \vdots & \vdots & & \vdots \\ v_1^{(n-1)}(x) & v_2^{(n-1)}(x) & & v_n^{(n-1)}(x) \\ v_1(z) & v_2(z) & & v_n(z) \end{vmatrix} \frac{\Phi}{W(x)}\, dx .$$

Writing

$$(1.3.12)\qquad N(z,x) = \frac{1}{W(x)} \begin{vmatrix} v_1(x) & v_2(x) & \cdots & v_n(x) \\ v_1'(x) & v_2'(x) & & v_n'(x) \\ \vdots & \vdots & & \vdots \\ v_1(z) & v_2(z) & & v_n(z) \end{vmatrix} = \frac{M(z,x)}{W(x)} ,$$

and replacing Φ by its value from (1.3.7), we get from (1.3.11),

$$(1.3.13)\qquad u(z) = \sum_{r=1}^{n} \alpha_r v_r(z) + \int_b^z N(z,x) \sum_{r=1}^{n} \left\{ R_{n-r}(x) - P_{n-r}(x) \right\} u^{(n-r)}(x)\, dx ,$$

or upon interchanging integration and summation,

$$(1.3.14)\quad u(z) = \sum_{r=1}^{n} \alpha_r v_r(z) + \sum_{r=1}^{n-1} \int_b^z N(z,x)\{R_{n-r}(x) - P_{n-r}(x)\} u^{(n-r)}(x)\, dx + \int_b^z N(z,x)\{R_0 - P_0\} u\, dx .$$

To transform this into a Volterra Integral Equation, we need the

Lemma. For $r = 1, 2, \cdots (n-1)$, and $k = 1, 2, \cdots (n-r-1)$, the following holds

$$(1.3.15)\qquad \left[\frac{\partial^{(k)}}{\partial x^{(k)}} \left\{ N(z,x) \left[R_{n-r}(x) - P_{n-r}(x) \right] \right\} \right]_{x=z} = 0 .$$

Proof. From the definition of (1.3.12),

$$M(z,z) = 0 \quad ;$$

$$\frac{\partial M(z,x)}{\partial x} = \begin{vmatrix} v_1(x) & \cdots & v_n(x) \\ v_1'(x) & & v_n'(x) \\ \vdots & & \vdots \\ v_1^{(n-1)}(x) & & v_n^{(n-1)}(x) \\ v_1(z) & & v_n(z) \end{vmatrix} ;$$

$$\frac{\partial^2 M(z,x)}{\partial x^2} = \begin{vmatrix} v_1(x) & \cdots & v_n(x) \\ v_1'(x) & & v_n'(x) \\ \vdots & & \vdots \\ v_1^{(n)}(x) & & v_n^{(n)}(x) \\ v_1(z) & & v_n(z) \end{vmatrix} + \begin{vmatrix} v_1(x) & \cdots & v_n(x) \\ v_1'(x) & & v_n'(x) \\ \vdots & & \vdots \\ v_1^{(n-1)}(x) & & v_n^{(n-1)}(x) \\ v_1(z) & & v_n(z) \end{vmatrix} ;$$

. . . .

$$\frac{\partial^{(n-2)} M(z,x)}{\partial x^{(n-2)}} = \begin{vmatrix} v_1(x) & \cdots & v_n(x) \\ v_1'(x) & & v_n'(x) \\ \vdots & & \vdots \\ v_1^{(2n-4)}(x) & & v_n^{(2n-4)}(x) \\ v_1(z) & & v_n(z) \end{vmatrix} + \begin{vmatrix} v_1(x) & \cdots & v_n(x) \\ v_1'(x) & & v_n'(x) \\ \vdots & & \vdots \\ v_1^{(2n-5)}(x) & & v_n^{(2n-5)}(x) \\ v_1(z) & & v_n(z) \end{vmatrix} + \cdots \begin{vmatrix} v_1(x) & \cdots & v_n(x) \\ v_1'(x) & & v_n'(x) \\ \vdots & & \vdots \\ v_1^{(n-1)}(x) & & v_n^{(n-1)}(x) \\ v_1(z) & & v_n(z) \end{vmatrix}$$

Each determinant involved in the partial derivatives

$$\partial M^{(j)}(z,x) / \partial x^{(j)} \qquad (j = 1, 2, \ldots (n-1))$$

contains a non-derived first row. Hence, for $x = z$, this first row becomes identical with the last row, and the determinant vanishes; therefore,

(1.3.16) $$\frac{\partial M(z,z)}{\partial x} = \frac{\partial^2 M(z,z)}{\partial x^2} = \cdots \frac{\partial^{(n-2)} M(z,z)}{\partial x^{(n-2)}} = 0 .$$

From (1.3.12) it follows that each of the partial derivatives

$$\frac{\partial N(z,z)}{\partial x}, \quad \frac{\partial^2 N(z,z)}{\partial x^2}, \quad \ldots \quad \frac{\partial^{(n-2)} N(z,x)}{\partial x^{(n-2)}},$$

contains only terms with

$$M(z,x), \quad \frac{\partial M(z,x)}{\partial x}, \quad \frac{\partial^2 M(z,x)}{\partial x^2}, \ldots \quad \frac{\partial^{(n-2)} M(z,x)}{\partial x^{(n-2)}}$$

as factors, so by (1.3.16)

(1.3.17) $$N(z,z) = \frac{\partial N(z,z)}{\partial x} = \frac{\partial^2 N(z,z)}{\partial x^2} = \cdots \frac{\partial^{(n-2)} N(z,z)}{\partial x^{(n-2)}} = 0 .$$

For $r = 1, 2, \ldots (n-1)$, and $K = 1, 2, \ldots n-r-1$, we have

$$\frac{\partial^{(K)}}{\partial x^{(K)}} \left\{ N(z,x) \left[R_{n-r}(x) - P_{n-r}(x) \right] \right\} = \sum_{p=0}^{K} \binom{K}{p} \frac{\partial^{(K-p)} N(z,x)}{\partial x^{(K-p)}} \frac{\partial^{(p)}}{\partial x^{(p)}} \left[R_{n-r}(x) - P_{n-r}(x) \right] .$$

Since each term in the summation has as a factor one of the partial derivatives $\partial^{(j)}N(z,x)/\partial x^{(j)}$ $(j=0,1,\cdots(n-2))$, equation (1.3.17) shows that the right side vanishes when $x=z$, and hence the Lemma is proved.

We now perform $(n-r)$ partial integrations upon the integrals on the right side of (1.3.14), and use the Lemma at each stage to simplify the integrated part:

$$\int_b^z N(z,x)\left\{R_{n-r}(x)-P_{n-r}(x)\right\}u^{(n-r)}(x)\,dx$$

$$=\left[N(z,x)\left\{R_{n-r}(x)-P_{n-r}(x)\right\}u^{(n-r-1)}(x)\right]_b^z-\int_b^z\frac{\partial}{\partial x}\left\{N(z,x)\left[R_{n-r}(x)-P_{n-r}(x)\right]\right\}u^{(n-r-1)}(x)\,dx$$

$$=\sum_{\ell=1}^{n}\alpha_{\ell,1}^{(r)}v_\ell(z)-\int_b^z\frac{\partial}{\partial x}\left\{N(z,x)\left[R_{n-r}(x)-P_{n-r}(x)\right]\right\}u^{(n-r-1)}(x)\,dx$$

$$=\sum_{\ell=1}^{n}\alpha_{\ell,1}^{(r)}v_\ell(z)-\left[\frac{\partial}{\partial x}\left\{N(z,x)\left[R_{n-r}(x)-P_{n-r}(x)\right]\right\}u^{(n-r-2)}(x)\right]_b^z+(-1)^2\int_b^z\frac{\partial^2}{\partial x^2}\left\{N(z,x)\left[R_{n-r}-P_{n-r}\right]\right\}u^{(n-r-2)}(x)\,dx$$

$$=\sum_{\ell=1}^{n}\alpha_{\ell,1}^{(r)}v_\ell(z)+\sum_{\ell=1}^{n}\alpha_{\ell,2}^{(r)}v_\ell(z)+(-1)^2\int_b^z\frac{\partial^2}{\partial x^2}\left\{N(z,x)\left[R_{n-r}(x)-P_{n-r}(x)\right]\right\}u^{(n-r-2)}(x)\,dx$$

$$=\quad\cdots\cdots\cdots\cdots$$

$$=\sum_{\ell=1}^{n}\alpha_{\ell,1}^{(r)}v_\ell(z)+\sum_{\ell=1}^{n}\alpha_{\ell,2}^{(r)}v_\ell(z)+\cdots\cdots+\sum_{\ell=1}^{n}\alpha_{\ell,n-r-1}^{(r)}v_\ell(z)$$
$$+(-1)^{n-r-1}\left[\frac{\partial^{(n-r-1)}}{\partial x^{(n-r-1)}}\left\{N(z,x)\left[R_{n-r}-P_{n-r}\right]\right\}u(x)\right]_b^z+(-1)^{n-r}\int_b^z\frac{\partial^{(n-r)}}{\partial x^{(n-r)}}\left\{N(z,x)\left[R_{n-r}-P_{n-r}\right]\right\}u(x)\,dx$$

$$=\sum_{\ell=1}^{n}\beta_\ell^{(r)}v_\ell(z)+\int_b^z(-1)^{n-r}\frac{\partial^{(n-r)}}{\partial x^{(n-r)}}\left\{N(z,x)\left[R_{n-r}(x)-P_{n-r}(x)\right]\right\}u(x)\,dx$$

where $\beta_\ell^{(r)}=\sum_{K=1}^{n-r}\alpha_{\ell,K}^{(r)}$, and all the $\alpha_{\ell,K}^{(r)}$ are constants depending upon b. Putting these values into the summation in (1.3.14), we get

$$u(z)=\sum_{r=1}^{n}\alpha_r v_r(z)+\sum_{r=1}^{n-1}\sum_{\ell=1}^{n}\beta_\ell^{(r)}v_\ell(z)+\sum_{r=1}^{n-1}\int_b^z(-1)^{n-r}\frac{\partial^{(n-r)}}{\partial x^{(n-r)}}\left\{N(z,x)\left[R_{n-r}-P_{n-r}\right]\right\}u(x)\,dx$$
$$+\int_b^z N(z,x)\left[R_0(x)-P_0(x)\right]u(x)\,dx,$$

(1.3.18) $$u(z)=\sum_{r=1}^{n}\gamma_r v_r(z)+\int_b^z K(z,x)\,u(x)\,dx,$$

where the γ_r are constants, and the kernel is given by

(1.3.19) $$K(z,x)=\sum_{r=1}^{n}(-1)^{n-r}\frac{\partial^{(n-r)}}{\partial x^{(n-r)}}\left\{N(z,x)\left[R_{n-r}(x)-P_{n-r}(x)\right]\right\}.$$

The arbitrary constants of integration γ_r are determined from the initial values $u^{(\ell)}(b)$. From (1.3.18),

$$u(b) = \sum_{r=1}^{n} \gamma_r v_r(b) ;$$

$$u'(b) = \sum_{r=1}^{n} \gamma_r v_r'(b) + K(b,b)u(b).$$

Now, the expansion of $\partial^{(n-1)}M(z,x)/\partial x^{(n-1)}$ as a sum of determinants contains exactly one term with the first row derived once, as well as the other rows down to the $(n-1)$st derived once, and contains all other terms with a non-derived first row. Hence, for $x = z$, it follows that

(1.3.20) $\quad \partial^{(n-1)} M(z, z) / \partial x^{(n-1)} = (-1)^{n-1} W(z)$.

From this and from the definition (1.3.12), it is seen that

$$\partial^{(n-1)} N(z,z) / \partial x^{(n-1)} = (-1)^{n-1}.$$

Substitution of this and the values (1.3.17) into (1.3.19) gives

(1.3.21) $\quad K(z, z) = R_{n-1}(z) - P_{n-1}(z)$.

Hence,

$$u'(b) = \sum_{r=1}^{n} \gamma_r v_r'(b) + \{R_{n-1}(b) - P_{n-1}(b)\} u(b), \quad \text{or}$$

$$f_1(b) = \sum_{r=1}^{n} \gamma_r v_r'(b).$$

Similarly, for the ℓ-th derivative, we get an expression of the form

(1.3.22) $\quad f_\ell(b) = \sum_{r=1}^{n} \gamma_r v_r^{(\ell)}(b)$,

where $f_\ell(b)$ depends upon the initial values $u^{(r-1)}(b)$, $R_{n-r}(b)$, and $P_{n-r}(b)$ $(r = 1,2, \cdots n;\ \ell = 0, 1, 2, \cdots (n-1))$. The solution of the linear equation (1.3.22) is

(1.3.23) $$\gamma_r = \frac{1}{W(b)} \begin{vmatrix} v_1(b) & v_2(b) & \cdots & v_{r-1}(b) & 0 & v_{r+1}(b) & \cdots & v_n(b) \\ v_1'(b) & v_2'(b) & & v_{r-1}'(b) & f_1(b) & v_{r+1}'(b) & & v_n'(b) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ v_1^{(n-1)}(b) & v_2^{(n-1)}(b) & & v_{r-1}^{(n-1)}(b) & f_{n-1}(b) & v_{r+1}^{(n-1)}(b) & & v_n^{(n-1)}(b) \end{vmatrix},$$

where $W(x)$ is the Wronskian of the n functions $v_r(x)$.

In conclusion, we have reduced the general Cauchy problem to the problem of solving the integral equation (1.3.20), where all the constants γ_r are given by (1.3.23) in terms of the Cauchy initial values.

1.4 The Equation With The First Derivative Term Missing.

This section applies to the case $n=2$ discussed in section 1.1. From equation (1.1.16), we see that the kernel will simplify to

(1.4.1) $$K(z,x) = N(z,x)\{S(x) - Q(x)\}$$

if $R(x)-P(x)\equiv 0$. We can arrange this in certain problems by choosing the v equation such that $P\equiv R$; however, in other cases of interest this choice is inconvenient, and instead we transform the variables so that the new functions $P(x)$ and $R(x)$ are zero in (1.1.1) and (1.1.2). We now show that the latter entails no loss of generality; that is, starting with the second order differential equation

(1.4.2) $$h''(t) + L(t)h'(t) + [M(t) + \lambda T(t)]h(t) = 0$$

with real, continuous, differentiable coefficient functions $L(t)$, $M(t)$, and $T(t)$, and non-negative $T(t)$, it is possible to change the variables so that (1.4.2) becomes

(1.4.3) $$u''(x) + [Q(x) + \lambda]u(x) = 0 .$$

First, introduction of the integrating factor

$$p(t) = \exp\left\{\int_0^t L(t)\,dt\right\}$$

enables (1.4.2) to be written in the self-adjoint form

(1.4.4) $$\frac{d}{dt}\left\{p(t)\frac{dh}{dt}\right\} + \{q(t) + \lambda\rho(t)\}h(t) = 0,$$

where $$q(t) = p(t)M(t); \quad \rho(t) = p(t)T(t).$$

If in (1.4.4) we make the trial substitution

(1.4.5) $$h(t) = f(x)u(x); \quad \frac{d}{dt} = \theta(x)\frac{d}{dx},$$

then, in order that the resulting u equation have the coefficients of u'' and λu the same, and in order that the u' term vanish, the functions $f(x)$ and $\theta(x)$ must satisfy the equations

$$2p\theta f' + [p\theta' + p'\theta] f = 0 \quad \text{, and} \qquad p\theta^2 = \rho ,$$

with solution $\theta = (\rho/p)^{1/2}$; $\quad f = (p\rho)^{-1/4}$.

Putting these into (1.4.5), we obtain the change of variables

(1.4.6) $$\mathcal{R}(t) = (p\rho)^{-1/4}\, u(x) \; ; \; \frac{d}{dt} = (\rho/p)^{1/2} \frac{d}{dx} \quad ,$$

which changes (1.4.2) into (1.4.3).

We now obtain special properties of the solutions of (1.4.3). Let two linearly independent solutions be $u_1(x)$ and $u_2(x)$:

$$u_1''(x) + [\, Q(x) + \lambda \,]\; u_1(x) = 0,$$

$$u_2''(x) + [\, Q(x) + \lambda \,]\; u_2(x) = 0 .$$

If we subtract u_1 times the second equation from u_2 times the first, we obtain

$$\frac{d}{dx} \{ u_1(x)\, u_2'(x) - u_2(x)\, u_1'(x) \} = 0 .$$

Integrating and then dividing both sides by $u_1(x)^2$, we obtain

$$\frac{u_1(x)\, u_2'(x) - u_2(x) u_1'(x)}{u_1(x)^2} = \frac{C}{u_1(x)^2} .$$

Since u_1 and u_2 are linearly independent, the constant C is different from zero, and it follows that

(1.4.7) $$u_2(x) = C\, u_1(x) \int^x \frac{dt}{[u_1(t)]^2} .$$

Of course, the Wronskian is given by

(1.4.8) $$W(x) = u_1(x) u_2'(x) - u_2(x)\, u_1'(x) = C .$$

1.5 Solution of The Integral Equation.

The integral equation (1.3.18) is to be solved by the method of successive approximations. By this method,

(1.5.1) $$u_p(z) = u_0(z) + \int_b^z K(z,x)\, u_{p-1}(x)\,dx$$

is the p-th approximation to (1.3.18), where

(1.5.2) $$u_0(z) = \alpha v_1(z) + \beta v_2(z).$$

The Liouville-Neumann Theorem (8) states that a sufficient condition for the sequence $\{u_p(z)\}$ to converge to the unique continuous solution of (1.3.18) is that $u_0(z)$ be continuous and $K(z,x)$ be bounded with only regularly distributed discontinuities. However, in the problems that we shall consider, the theorem cannot be applied because of the unbounded nature of the kernel, and special consideration is required.

In the type of problem to be discussed in Chapter 4, the differential equation to be solved has the form (1.4.3), with the first derivative term missing, and further $S-Q=\mu$ where μ is a constant. In this case, (1.1.16) gives

(1.5.3) $$K(z,x) = \mu\, N(z,x).$$

The integral equation (1.1.15) then has the form, (with $\beta = 0$),

(1.5.4) $$u(z) = \alpha v_1(z) + \mu \int_b^z N(z,x)\, u(x)\,dx,$$

where (1.1.14),(1.4.7), and (1.4.8) show that

(1.5.5) $$N(z,x) = v_1(x)\, v_1(z) \left[\int^z \frac{dt}{[v_1(t)]^2} - \int^x \frac{dt}{[v_1(t)]^2} \right].$$

The existence of the solution of the integral equation (1.5.4) under quite general conditions will now be proved.

Theorem 1 Suppose that $v_1(z)/\{(z-b)^p h(z)\}$ is analytic in the finite z-plane with zeros of order m_j at $z=b_j$ $(j=0,1,2,\cdots g)$, where $h(z)$ is an entire function without zeros, p is a non-negative number, and b_0 is defined as b . Then the sequence $\{u_n(z)\}$ of successive approximations associated with (1.5.4) converges to the unique continuous solution of (1.5.4) for all z.

Proof. Let (1.5.4) be rewritten in the form

(1.5.6) $$u(z) = \alpha v_1(z) + \mu v_1(z) \int_b^z L(z,x)\, u(x)\, dx ,$$

where

(1.5.7) $$L(z,x) = v_1(x) \int^z \frac{dt}{[v_1(t)]^2} - v_1(z) \int^x \frac{dt}{[v_1(t)]^2} .$$

By hypothesis, $v_1(z)$ may be expanded about any of the points in a series of the form

(1.5.8) $$v_1(x) = \alpha_0^{(j)} (x-b_j)^{m_j+p_j} + O[(x-b_j)^{m_j+p_j+1}] ,$$

with infinite radii of convergence, where we define

$$p_j = \begin{cases} p, & j=0, \\ 0, & j \neq 0. \end{cases}$$

Successive approximations to the solution of (1.5.6) are

(1.5.9) $$u_n(z) = u_{n-1}(z) + \alpha \mu^n v_1(z) f_n(z),$$

where

(1.5.10) $$f_n(z) = \int_b^z L(z,x)\, v_1(x)\, f_{n-1}(x)\, dx$$

and

$$u_0(z) = \alpha v_1(z) ; \qquad f_0(z) = 1 .$$

The proposed solution of (1.5.6) is then

(1.5.11) $$u(z) = \alpha\, v_1(z) \sum_{n=1}^{\infty} \mu^n f_n(z) .$$

The nature of the functions $f_n(z)$ will now be examined; from (1.5.7) and (1.5.8),

$$L(z,x)\, v_1(x) = v_1(x)^2 \frac{v_2(z)}{v_1(z)} - v_1(x) v_2(x),$$

(1.5.12) $$L(z,x)\, v_1(x) = \left[\alpha_0^{(j)} (x-b_j)^{2m_j+2p_j} + O(x-b_j)^{2m_j+2p_j+1}\right]\left[\beta_0^{(j)} (z-b_j)^{-2m_j-2p_j+1} + \cdots\right]$$
$$- \left[\alpha_0^{(j)} (x-b_j)^{2m_j+2p_j} + O(x-b_j)^{2m_j+2p_j+1}\right]\left[\beta_0^{(j)} (x-b_j)^{-2m_j-2p_j+1} + \cdots\right]$$

(1.5.13) $$L(z,x)\, v_1(x) = \left[\alpha_0^{(j)} (x-b_j)^{2m_j+2p_j} + \cdots\right]\left[\beta_0^{(j)} (z-b_j)^{-2m_j-2p_j+1} + \cdots\right] - \left[\hat{\alpha}_0^{(j)} (x-b_j) + \cdots\right].$$

Suppose now without loss of generality that the zeros are ordered according to increasing moduli

$$|b_j| \leq |b_{j+1}| \qquad (j = 0, 1, 2, \cdots (q-1))$$

Since the series on the right of (1.5.8) has an infinite radius of convergence, the first series involving x in

(1.5.13) is uniformly convergent for all finite x, and can be integrated termwise. Now, the function $v_1(x)\,v_2(x)$ can be represented in the circle $C\,:\; 0<|x-b_j|<R_j$, where

$$R_j = \min_{\text{over } \ell}\left\{ |b_j - b_{j+\ell}| - r \right\} \qquad (\ell = 1, 2, \cdots (g-j))$$

for an arbitrary small number r, by the product of the Laurent Series given in the second term on the right of equation (1.5.12). The product series represents either an analytic function in C, or a function with an isolated pole at b_j. Since the expansion shows that the latter is not the case, it follows that $v_1(x)\,v_2(x)$ is analytic in C. Application of this reasoning for increasing j until all the points b_j have been exhausted shows that $v_1(x)v_2(x)$ is analytic for all finite x, and representable by any of the uniformly convergent series given in the second term on the right of (1.5.12).

From (1.5.10) we then have

$$|f_1(z)| \leq \int_b^z |L(z,x)\, v_1(x)|\,|dx|$$

$$= \left|\left[\alpha_0^{(j)} \frac{(z-b_j)^{2m_j+2p_j+1}}{2m_j+2p_j+1} + \cdots\right]\left[\beta_0^{(j)}(z-b_j)^{-2m_j-2p_j+1} + \cdots\right] - \left[\hat{\alpha}_0^{(j)}\frac{(z-b_j)^2}{2} + \cdots\right]\right|,$$

$$(1.5.14)\quad |f_1(z)| \leq \left|\left[\hat{\alpha}_0^{(j)}\frac{(z-b_j)^2}{2m_j+2p_j+1} + \cdots \hat{\alpha}_\ell^{(j)}\frac{(z-b_j)^{2+\ell}}{2m_j+2p_j+1+\ell} + \cdots\right] - \left[\hat{\alpha}_0^{(j)}\frac{(z-b_j)^2}{2} + \cdots \hat{\alpha}_\ell^{(j)}\frac{(z-b_j)^{2+\ell}}{2+\ell} + \cdots\right]\right|.$$

By the same type of argument used above, the two series in square brackets on the right side represent functions $\hat{\alpha}(z)$ and $\bar{\alpha}(z)$ which are analytic in the finite plane. Further, the series formed from $\hat{\alpha}(z)$ by putting $j=0$, dividing the first term by $2m+2p+1$, dividing all the terms by $(z-b)^2$, and dropping superscripts is

$$S^* = \sum_{\ell=0}^{\infty} S_\ell^* \,(z-b)^\ell$$

with an infinite radius of convergence, where

$$(1.5.15)\qquad S_\ell^* = \begin{cases} \hat{\alpha}_0, & \ell = 0, \\ \hat{\alpha}_\ell/(2m+2p+1+\ell), & \ell \geq 1. \end{cases}$$

Since the analyticity of $\hat{\alpha}(z)$ and $\bar{\alpha}(z)$ has been established, we shall hereafter use the series for $j=0$, and drop superscripts. Equation (1.5.14) then gives

$$|f_1(z)| \le |\hat{\alpha}_0|\left(\tfrac{1}{2} - \tfrac{1}{2m+2p+1}\right)|z-b|^2 + \cdots |\hat{\alpha}_\ell|\left(\tfrac{1}{2+\ell} - \tfrac{1}{2m+2p+1+\ell}\right)|z-b|^{\ell+2} + \cdots$$

$$< \frac{2m+2p-1}{2}\, |z-b|^2 \left\{ |\hat{\alpha}_0| + \cdots |\hat{\alpha}_\ell| \frac{2}{2+\ell}\, \frac{|z-b|^\ell}{2m+2p+1+\ell} + \cdots \right\},$$

(1.5.16) $|f_1(z)| \;<\; \dfrac{2m+2p-1}{2}\, A_1\, |z-b|^2,$

where A_1 is a bound on the series

(1.5.17) $$S^{(1)} = \sum_{\ell=0}^{\infty} S_\ell^{(1)}\, |z-b|^\ell,$$

with

(1.5.18) $$S_\ell^{(1)} = \begin{cases} |\hat{\alpha}_0|, & \ell = 0, \\ 2|\hat{\alpha}_\ell| \,/\, (2+\ell)(2m+2p+1+\ell), & \ell \ge 1 \end{cases}.$$

Putting (1.5.16) into (1.5.10), we get

$$|f_2(z)| \le \frac{2m+2p-1}{2} A_1 \left\{ |\hat{\alpha}_0|\left(\tfrac{1}{4} - \tfrac{1}{2m+2p+3}\right)|z-b|^4 + \cdots |\hat{\alpha}_\ell|\left(\tfrac{1}{4+\ell} - \tfrac{1}{2m+2p+3+\ell}\right)|z-b|^{4+\ell} + \cdots \right\}$$

$$< \frac{(2m+2p-1)^2}{(2)\,(4)} A_1 |z-b|^4 \left\{ |\hat{\alpha}_0| + \cdots \quad |\hat{\alpha}_\ell| \frac{4}{4+\ell}\, \frac{|z-b|^\ell}{2m+2p+3+\ell} + \cdots \right\}$$

$$< \frac{(2m+2p-1)^2}{(2)\,(4)} A_1\, A_2\, |z-b|^4,$$

where A_2 is a bound on the series

$$S^{(2)} = \sum_{\ell=0}^{\infty} S_\ell^{(2)}\, |z-b|^\ell,$$

with

$$S_\ell^{(2)} = \begin{cases} |\hat{\alpha}_0|, & \ell = 0, \\ 4|\hat{\alpha}_\ell| \,/\, (4+\ell)(2m+2p+3+\ell), & \ell \ge 1 \end{cases}.$$

By induction,

$$|f_n(z)| < \frac{(2m+2p-1)^n}{(2)(4)\cdots(2n)}\, A_1 A_2 \cdots A_n\, |z-b|^{2n},$$

where A_n is a bound on the series

(1.5.19) $$S^{(n)} = \sum_{\ell=0}^{\infty} S_\ell^{(n)}\, |z-b|^\ell,$$

with

(1.5.20) $$S_\ell^{(n)} = \begin{cases} |\hat{\alpha}_0|, & \ell = 0, \\ 2n|\hat{\alpha}_\ell| \,/\, (2n+\ell)(2m+2p+2n-1+\ell), & \ell \ge 1. \end{cases}$$

Since a comparison of (1.5.20) with (1.5.15) shows that

$$S_\ell^{(n)} \le |S_\ell^{*}| \qquad \text{for all } \ell \text{ and } n,$$

it follows that the sequence of numbers $\{A_n\}$ is bounded above by some number A . Hence,

(1.5.21) $$|f_n(z)| < \frac{(2m+2p-1)^n}{2^n\, \underline{|n}}\, A^n\, |z-b|^{2n},$$

so that the series on the right of (1.5.11) is dominated by

(1.5.22) $\quad \exp\{(2m+2p-1)A\mu|z-b|^2\}$,

and the convergence is established. By the same reasoning as used in the Liouville-Neumann Theorem (8) $\lim_{n\to\infty} u_n(z)$ satisfies (1.5.6)

To show the uniqueness of the bounded solution, suppose that $\omega(z)$ is another bounded solution of (1.5.6),

(1.5.23) $\quad \omega(z) = \alpha v_1(z) + \mu v_1(z)\int_b^z L(z,x)\,\omega(x)\,dx.$

Since $\omega(z)$ is bounded, there exists a constant E such that

$$|u(z)-\omega(z)| < E\,|v_1(z)|$$

for all z. From (1.5.6) and (1.5.23),

(1.5.24) $\quad |u(z)-\omega(z)| < \mu|v_1(z)|\int_b^z |L(z,x)||u(x)-\omega(x)||dx|$

$$< \mu E|v_1(z)|\int_b^z |L(z,x)||v_1(x)||dx|$$

$$< \mu E|v_1(z)||f_1(z)|.$$

Putting this back into the integral on the right side of (1.5.24), we obtain

$$|u(z)-\omega(z)| < \mu^2 E\,|v_1(z)||f_2(z)|.$$

Continuing the process, we get at the n-th stage,

$$|u(z)-\omega(z)| < \mu^n E\,|v_1(z)||f_n(z)| \to 0 \text{ as } n\to\infty.$$

by (1.5.21), which proves the uniqueness.

We can also state the following

Corollary. The same result holds if the kernel has the form

$$K(z,x) = N(z,x)\,D(x),$$

where $\quad D(x) = D_j\,(x-b_j)^{s_j}$

for $(s_j = 1, 2, \cdots \quad)$, and $(j = 0, 1, 2, \cdots\ g\ )$.

1.6 The General Solution Of A Related Integral Equation.

In the footnote on page (2), it is observed that the integral equation (1.1.15) can be written

(1.6.1) $$u(z) = \alpha v_1(z) + \beta v_2(z) + \mu \int^z K(z,x)\, u(x)\, dx$$

in terms of an indefinite integral. It is this form (1.6.1) which will be used in Chapter 3, and hence it is of interest to examine the convergence of its solution by successive approximations of the type (1.5.1). Since both the second solution $v_2(z)$ and the kernel $K(z,x)$ may be unbounded at a finite point b, the general solution of (1.6.1) cannot be expected to exist for all z. However, in the next theorem we shall show that under certain conditions the solution does exists for all z excluded from a small circle Γ about b.

Theorem 2. Suppose that the function $v_1(z)/\{(z-b)^p\, h(z)\}$ is analytic in the finite z-plane[1] with zeros of order one at b_j $(j = 1,2,\cdots g)$, and with one zero of order $m>1$ at $b_0 = b$, where $h(z)$ is an entire function without zeros, and p is a non-negative number. Further, let $K(z,x) = N(z,x)\, D(x)$, where $D(x) = P_s\, (x-b)^s$ for positive integral s. Then the sequence $\{u_n(z)\}$ of successive approximations associated with (1.6.1) converges for all $z \notin \Gamma$ to the unique solution of (1.6.1).

Proof. Successive approximations to the solution of (1.6.1) are

(1.6.2) $$u_n(z) = u_{n-1}(z) + \mu^n \{\alpha v_1(z)\, g_n(z) + \beta h_n(z)\}$$

where

(1.6.3) $$g_n(z) = \int^z L(z,x)\, v_1(x)\, g_{n-1}(x)\, dx,$$

and

(1.6.4) $$h_n(z) = \int^z K(z,x)\, h_{n-1}(x)\, dx,$$

and where we define

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1 The finite z-plane refers to all values of z for which $|z| < |z_0|$, where z_0 is fixed.

(1.6.5) $$u_0(z) = \alpha v_1(z) + \beta v_2(z); \qquad g_0(z) = 1; \qquad h_0(z) = v_2(z).$$

The proposed solution of (1.6.1) is then

(1.6.6) $$u(z) = \alpha v_1(z) \sum_{n=0}^{\infty} \mu^n g_n(z) + \beta \sum_{n=0}^{\infty} \mu^n h_n(z).$$

Essentially the same procedure that was used in Theorem 1 gives

$$|u(z)| < \alpha |v_1(z)| \sum_{n=0}^{\infty} \mu^n \frac{A^n |z-b|^{2n} (2m+2p-1)^n}{2^n \lfloor n} + \beta \sum_{n=0}^{\infty} \mu^n \frac{B^n |z-b|^{2n}}{2^n \lfloor n}$$

for finite z excluded from Γ, from which the convergence follows for these values of z.

To show uniqueness of the general solution, suppose that $w(z)$ is a second solution of (1.6.1) which is finite for all finite z outside of Γ,

(1.6.7) $$w(z) = \alpha v_1(z) + \beta v_2(z) + \int^z K(z,x)\, w(x)\, dx.$$

From the finiteness of $v_1(z)$, $v_2(z)$, $u(z)$, and $w(z)$ outside Γ, it follows that there exist constants C_1 and C_2 so that

$$|u(z) - w(z)| < C_1 |v_1(z)| + C_2 |v_2(z)|.$$

From (1.6.1) and (1.6.7)

(1.6.8) $$|u(z) - w(z)| \leq \mu \int^z |K(z,x)|\, |u(x) - w(x)|\, |dx|$$
$$= \mu |v_1(z)| C_1 \int^z |L(z,x) v_1(x)|\, |dx| + \mu C_2 \int^z |K(z,x) v_2(x)|\, |dx|,$$

(1.6.9) $$|u(z) - w(z)| < \mu |v_1(z)| C_1 |g_1(z)| + \mu C_2 |h_1(z)|.$$

The substitution of (1.6.9) into the right side of (1.6.8) gives

$$|u(z) - w(z)| \leq \mu \int^z |K(z,x)|\, |\mu |v_1(x)| C_1 |g_1(x)| + \mu C_2 |h_1(x)||\, |dx|$$
$$= \mu^2 |v_1(z)| C_1 |g_2(z)| + \mu^2 C_2 |h_2(z)|.$$

Repetition of the process gives at the n-th stage

$$|u(z) - w(z)| < \mu^n |v_1(z)| C_1 |g_n(z)| + \mu^n C_2 |h_n(z)|$$
$$\to 0 \quad \text{as} \quad n \to \infty.$$

The following generalization can be proved by similar reasoning:

Theorem 3. Suppose that the function $v_1(z)/\{(z-b)^{\rho} h(z)\}$ is analytic in the finite z-plane with zeros of order m_j at $z=b_j$, where $h(z)$ is entire without zeros, and ρ is non-negative. Further, let $K(z,x)/N(z,x) = \sum_{\sigma=0}^{s} D_{\sigma} (x-b)^{\sigma}$ $(s=0,1,2,\cdots)$ Then the sequence of successive approximations of (1.6.1) converges for all values of z (in the finite plane) excluded from small circles Γ_j about b_j to the unique solution of (1.6.1).

CHAPTER TWO

APPLICATION TO INITIAL VALUE PROBLEMS

2.1 Introduction.

This chapter contains applications of the results of Chapter 1 to the type of initial value problem in which the differential equation to be solved differs from a known equation by terms containing small parameters. In particular, the integral equation (1.1.5) is used to obtain solutions as power series expansions in one or two of these parameters. Also, as a general result, the formulation of the general Cauchy problem as an integral equation, obtained in section 1.3, is used to give multiple power series expansions of higher order differential equations in several parameters. In the type of problem considered, the m-th successive approximation to the solution of the integral equation yields all the terms of the multiple power series having the sum of the powers of the various parameters less than or equal to m. Since the differential equations under consideration will be assumed to have no singular points, so that the kernel $K(z,x)$ and the second solution are bounded, the Liouville-Neumann Theorem guarantees the convergence of the general solution of the integral equation by successive approximations.

2.2 A Problem With A Perturbation In The First Derivative Term.

Suppose that the solution of the differential equation

$$u''(x) + 2s F(x)\, u'(x) + Q(x)\, u(x) = 0 \tag{2.2.1}$$

having the initial values $u(b)$ and $u'(b)$ is required, when

$v_1(x)$ and $v_2(x)$ are known to be solutions of

(2.2.2) $\qquad v''(x) + Q(x)\, v(x) = 0 \,.$

Equation (2.2.1) differs from (2.2.2) by a term containing a small parameter s . From (1.1.14), (1.1.16), and (1.4.8),

$$N(z,x) = \tfrac{1}{c}\left[v_1(x) v_2(z) - v_1(z) v_2(x) \right],$$
$$K(z,x) = 2s\left[\frac{\partial N(z,x)}{\partial x} F(x) + N(z,x)\, F'(x) \right]$$
$$= s\, L(z,x)$$

where $L(z,x)$ is independent of s . From (1.1.15) and (1.2.7),

(2.2.3) $\qquad u(z) = \alpha v_1(z) + \beta v_2(z) + s\int_b^z L(z,x)\, u(x)\, dx,$

where α and β are linear in s . With

$$u_0(z) = \alpha v_1(z) + \beta\, v_2(z) = h_0(z) + \tilde{h}_1(z)\, s,$$

the first approximation to (2.2.3) is

$$u_1(z) = h_0(z) + \tilde{h}_1(z)\, s + s\int_b^z L(z,x)\left[h_0(x) + \tilde{h}_1(x)\, s \right] dx$$
$$= h_0(z) + h_1(z)\, s + \tilde{h}_2(z)\, s^2.$$

The m-th approximation then has the form

$$u_m(z) = \sum_{\ell=0}^{m} h_\ell(z)\, s^\ell + \tilde{h}_{m+1}(z)\, s^{m+1},$$

giving the complete power series up to the term in s^m .

As a simple illustration, consider the problem, whose solution is easily obtained by other methods, in which $F(x)=1$, $Q(x) = n^2$ (for real n), are put into (2.2.1). In this case,

$$v_1(z) = \cos nz \,; \qquad v_2(z) = \sin nz\,; \qquad W(z) = C = 1;$$

$$K(z,x) = -2s\cos n(x-z).$$

If the initial conditions are $u(0) = 1,\ u'(0) = 0$ then (1.2.7) yields $\alpha = 1$, $\beta = \frac{2s}{n}$, and the integral equation is

$$u(z) = \cos nz + \frac{2s}{n}\sin nz - 2s\int_0^z \cos n(x-z)\, u(x)\, dx,$$

with first approximation

(2.2.4) $\qquad u_1(z) = \cos nz + \frac{s}{n}\left\{ \sin nz - nz\cos nz \right\} - \frac{2s^2}{n^2}\, nz \sin nz \,.$

This checks with the first two terms in the power series expansion of the known solution

(2.2.5) $$u(z) = \cos\omega z + \frac{s}{\omega}\sin\omega z,$$

where $$\omega = (n^2 - s^2)^{\frac{1}{2}}.$$

2.3 Another Single Term Perturbation Problem.

Consider the problem of finding the solution of the equation

(2.3.1) $$u''(x) + P(x)u'(x) + [T(x) + sG(x)]\,u(x) = 0$$

with the perturbation $sG(x)$ in the $u(x)$ term, having the initial values $u(b)$ and $u'(b)$, where $v_1(x)$ and $v_2(x)$ satisfy

(2.3.2) $$v''(x) + P(x)\,v'(x) + T(x)\,v(x) = 0.$$

Again, use of (1.1.14), (1.1.15), and (1.1.16) shows that the integral equation for the problem is

(2.3.3) $$u(z) = \alpha v_1(z) + \beta v_2(z) + s\int_b^z L(z,x)\,u(x)dx,$$

where, from (1.2.7), α and β as well as $L(z,x)$ are independent of the parameter s. The m-th approximation to the solution of (2.3.3) then has the form

(2.3.4) $$u_m(z) = \sum_{l=0}^{m} h_l(z)\, s^l.$$

For example, suppose that the solution of

(2.3.5) $$u''(x) + \{s(x^2+ax) + n^2\}\,u(x) = 0$$

with initial conditions

(2.3.6) $$u(0) = 1; \qquad u'(0) = 0$$

is required, where a and n are real constants. In this case, the kernel is

$$K(z,x) = \left[\frac{s}{n}\sin n(x-z)\right]\left[x^2+ax\right]$$

whence the integral equation is

(2.3.7) $$u(z) = \alpha\cos nz + \beta\sin nz + \frac{s}{n}\int_0^z \sin n(x-z)\{x^2+ax\}\,u(x)dx.$$

From (1.2.7), the initial conditions (2.3.6) give

$$\alpha = \frac{1}{n}\begin{vmatrix} 1 & 0 \\ 0 & n \end{vmatrix} = 1; \qquad \beta = \frac{1}{n}\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0,$$

and hence the first approximation to (2.3.7) is

$$u_1(z) = \cos nz + \frac{s}{n}\int_0^z \sin n(x-z)\, x^2 \cos nx \, dx + \frac{as}{2n}\int_0^z \sin n(x-z)\, x \cos nx \, dx$$

$$= h_0(z) + h_1(z)\, s,$$

where

$$h_0(z) = \cos nz$$

$$h_1(z) = \frac{-1}{4n^2} z^2\cos nz - \frac{a}{4n^2} z\cos nz - \frac{1}{6n} z^3 \sin nz - \frac{a}{4n} z^2 \sin nz - \frac{1}{4n^3} z\sin nz + \frac{a}{4n^3}\sin nz \, .$$

2.4 Multiple Perturbation Problems.

Consider the differential equation

(2.4.1) $$u''(x) + 2s\,F(x)\,u'(x) + [\,Q(x) + t\,G(x)\,]\,u(x) = 0,$$

containing two independent perturbing parameters s and t. Suppose the solution of

(2.4.2) $$v''(x) + Q(x)\,v(x) = 0$$

are known to be $v_1(x)$ and $v_2(x)$. Then, from (1.1.15) and (1.1.16), $u(x)$ satisfies the Volterra Integral Equation

(2.4.3) $$u(z) = \alpha\, v_1(z) + \beta\, v_2(z) + s\int_b^z L_1(z,x)\,u(x)\,dx + t\int_b^z L_2(z,x)\,u(x)\,dx,$$

where the functions

$$L_1(z,x) = \frac{\partial}{\partial x}\{N(z,x)\,F(x)\},$$

$$L_2(z,x) = -N(z,x)\,G(x)$$

are independent of s and t. If the initial values for $u(x)$ are $u(b)$ and $u'(b)$, it follows from (1.2.7) that the constants α and β are given by

$$\alpha = \frac{1}{C}\begin{vmatrix} u(b) & v_2(b) \\ u'(b) + 2s\,F(b)\,u(b) & v_2'(b) \end{vmatrix} = \alpha_1 + \alpha_2 s,$$

$$\beta = \frac{1}{C}\begin{vmatrix} v_1(b) & u(b) \\ v_1'(b) & u'(b) + 2s\,F(b)\,u(b) \end{vmatrix} = \beta_1 + \beta_2 s \, .$$

The first approximation to (2.4.3) then has the form

$$u_1(z) = h_{0,0}(z) + h_{1,0}(z)\,s + h_{0,1}(z)\,t + \tilde{h}_{1,1}(z)\,st + \tilde{h}_{2,0}(z)\,s^2,$$

which gives all the linear terms in the solution $u(z)$. Like-wise, at the m-th stage of approximation,

$$u_m(z) = \sum_{l=0}^{m} \sum_{i+j=l} h_{i,j}(z) s^i t^j + \sum_{i+j=m+1} \tilde{h}_{i,j}(z) s^i t^j,$$

giving all the contributions to terms in $u(z)$ having the sum of powers of s and t less than or equal to m.

As an example, consider the differential equation

(2.4.5) $$u''(x) + s(1+ax)u'(x) + [n^2 + tx]u(x) = 0,$$

of the type considered by Coveyou and Mulliken (1). In this case, the kernel is

$$K(z,x) = \frac{\sin n(x-z)}{n} tx - \frac{\partial}{\partial x}\left\{\frac{\sin n(x-z)}{n} s(1+ax)\right\}$$
$$= \frac{\sin n(x-z)}{n}\{tx - as\} - s\cos n(x-z)\{1+ax\}.$$

For the initial conditions $u(0)=1$, $u'(0)=0$, equations (2.4.4) give

$$\alpha = 1 \quad ; \qquad \beta = \frac{s}{n}.$$

Hence, the integral equation for the problem is

(2.4.6) $$u(z) = \cos nz + \frac{s}{n}\sin nz + \int_0^z \sin n(x-z)\frac{tx-as}{n} u(x)dx - s\int_0^z \cos n(x-z)\{1+ax\}u(x)dx.$$

With $u_0(z) = \cos nz + \frac{s}{n}\sin nz$, the first approximation to the solution of (2.4.6) is

(2.4.7) $$u_1(z) = h_{0,0}(z) + h_{1,0}(z)s + h_{0,1}(z)t + \tilde{h}_{1,1}(z)st + h_{2,0}(z)s^2,$$

where

$$h_{0,0}(z) = \cos nz,$$
$$h_{1,0}(z) = \frac{1}{2n}\sin nz + \frac{a}{4n} z\sin nz - \left(\frac{z}{2} + \frac{az^2}{4}\right)\cos nz,$$
$$h_{0,1}(z) = \frac{1}{4n^3}\sin nz - \frac{1}{4n^2} z\cos nz - \frac{1}{4n} z^2 \sin nz.$$

The first approximation (2.4.7) gives all the linear terms in the solution (2.4.6).

Consider now a general perturbation problem, in which the solution of the n-th order linear differential equation

(2.4.8) $$u^{(n)}(x) + \sum_{r=1}^{n} \left[R_{n-r}(x) + s_{n-r} F_{n-r}(x)\right] u^{(n-r)}(x) = 0$$

having the initial values $u^{(\ell)}(b)$; $(\ell = 0, 1, 2, \cdots (n-1))$ is required. Equation (2.4.8) is changed by terms containing small parameters S_{n-r} from the equation

(2.4.9) $$v^{(n)}(x) + \sum_{r=1}^{n} R_{n-r}(x)\, v^{(n-r)}(x) = 0$$

whose solutions $v_r(x)$; $(r = 1, 2, \cdots n)$ are supposed to be known. From (1.3.18), the integral equation for this problem is

(2.4.10) $$u(z) = \sum_{r=1}^{n} \gamma_r v_r(z) + \sum_{r=1}^{n} s_{n-r} \int_b^z L_{n-r}(z,x)\, u(x)\,dx,$$

where, from (1.3.19)

$$L_{n-r}(z,x) = (-1)^{n-r+1} \frac{\partial^{(n-r)}}{\partial x^{(n-r)}} \left\{ N(z,x)\, F_{n-r}(x) \right\} \quad (r = 1, 2, \cdots n).$$

Here, $N(z,x)$ is given in terms of $v_r(x)$; $(r = 1, 2, \cdots n)$ by (1.3.12). The constants γ_r, given by (1.3.23), will in general have the form

$$\gamma_r = \sum_{j_i = 0 \text{ or } 1} \eta^{(r)}_{j_0, j_1, \cdots j_{n-1}}\, s_0^{j_0} s_1^{j_1} \cdots s_{n-1}^{j_{n-1}} .$$

where the exponent on each of the parameters S_i is either one or zero, and the coefficients $\eta^{(r)}_{j_0, j_1, \cdots j_{n-1}}$ depend upon the initial values $u^{(\ell)}(b)$, $v_r^{(\ell)}(b)$, and $F^{(\ell)}_{n-r}(b)$, $(\ell = 0, 1, \cdots (n-1))$.

The m-th approximation of (2.4.10) is

(2.4.11) $$u_m(z) = \sum_{\ell=0}^{m} \sum_{\sum_{i=0}^{n-1} j_i = \ell} h_{j_0, j_1 \cdots j_{n-1}}(z) \prod_{i=0}^{n-1} s_i^{j_i} + \sum_{\sum_{i=0}^{n-1} j_i = m+1} \tilde{h}_{j_0, j_1 \cdots j_{n-1}}(z) \prod_{i=0}^{n-1} s_i^{j_i} .$$

This gives all the terms of $u(z)$ which have the sum of the exponents of the various parameters s_i less than or equal to m. The second summation on the right of (2.4.11) gives incomplete contributions to terms of $u(z)$ having the sum of exponents of s_i equal to $m+1$.

2.5 Another Treatment Of The Problem In Section 2.2

Consider the problem of finding the solution of the second order differential equation

(2.5.1) $$h''(t) + 2sF(t)h'(t) + n^2 h(t) = 0 \; ; \; (F(0) = c),$$

of the type (2.2.1) with $Q(t) = n^2$, for a real constant n, which satisfies the initial conditions

(2.5.2) $$h(0) = 1 \; ; \quad h'(0) = 0 \, .$$

Actually, the conditions (2.5.2) can be replaced by completely arbitrary ones without essentially changing the result. If the first derivative term is removed from (2.5.1) by the transformations

(2.5.3) $$h(t) = e^{-s\int_0^t F(t)dt} u(x) \; ; \qquad t = x,$$

obtained by following the procedure of section 1.4, the result is

(2.5.4) $$u''(x) + \{-sF'(x) + s^2 F(x)^2 + n^2\} u(x) = 0,$$

where $u(x)$ must satisfy the initial conditions

(2.5.5) $$u(0) = 1 \; ; \quad u'(0) = sc \, .$$

When (2.5.4) is compared with

(2.5.6) $$v''(x) + n^2 v(x) = 0,$$

with solutions

$$v_1(x) = \cos nx \, , \qquad v_2(x) = \sin nx \, ,$$

the resulting integral equation is

(2.5.7) $$u(z) = \cos nz + \frac{sc}{n}\sin nz - \int_0^z \sin n(x-z)\left[\frac{s^2}{n}F^2(x) + \frac{s}{n}F'(x)\right]u(x)dx \, .$$

With $u_0(z) = \cos nz + \frac{sc}{n}\sin nz$, the first approximation to the solution of (2.5.7) is

(2.5.8) $$\begin{aligned} u_1(z) = {} & \cos nz + s\left\{\int_0^z \cos n(2x-z)\, F(x)\, dx\right\} \\ & + s^2\left\{\frac{\sin nz}{2n}\int_0^z F(x)^2 dx - \frac{1}{2n}\int_0^z \sin n(2x-z)\, F^2(x)\, dx + \frac{c}{n}\int_0^z \sin n(2x-z)\, F(x)\, dx\right\} \\ & + s^3\left\{\frac{c}{2n^2}\int_0^z \cos n(2x-z)\, F^2(x)\, dx - \frac{c}{2n^2}\cos nz \int_0^z F(x)^2 dx\right\}. \end{aligned}$$

The sequence of successive approximations associated with (2.5.7) actually does converge to the unique continuous solution of (2.5.7) provided that the function $F(x)$ together with its first derivative are bounded for all values of the argument. The m-th approximation is of the form

$$u_m(z) = \sum_{\ell=1}^{m} k_\ell(z)\, s^\ell + \sum_{\ell=m+1}^{2m+1} \tilde{k}_\ell(z)\, s^\ell ,$$

where the powers of s higher than s^m, represented by the second summation, will in general receive contributions from later approximations.

As an example, suppose that $F(t) = 1 = C$ in (2.5.1); then (2.5.8) gives for the approximation ,

$$(2.5.9) \qquad u_1(z) = \cos nz + \frac{s}{n}\sin nz + \frac{s^2}{2n^2}\, nz \sin nz - \frac{s^3}{2n^3}\left\{ nz\cos nz - \sin nz \right\} .$$

In this particular example, the first approximation (2.5.9) gives all terms up to those containing s^3 , and likewise the m-th approximation gives all terms up to those containing s^{2m+1} , since there is no overlapping of terms at the successive stages of approximation. The result (2.5.9) checks with the expansion of the known solution (2.2.5) up to the s^3 term.

CHAPTER THREE

APPLICATION TO SPECIAL FUNCTIONS

3.1 Introduction.

The object in this chapter is to use the result of Section 1.1 to obtain expansions of special functions in series of better known functions. Ikeda (4) first used this method to expand $J_n(\alpha x)$ and $Y_n(\alpha x)$ in terms of $J_n(x)$ and $Y_n(x)$ respectively, where $J_n(x)$ and $Y_n(x)$ are the Bessel Functions of first and second kinds of order n. In addition to rederiving Ikeda's formal results, we have examined the convergence of the series; in particular, we have found that the $J_n(\alpha x)$ series converges for all x and all α, but that a restriction must be imposed upon α in order that the $Y_n(\alpha x)$ series converge, (for all x excluded from a neighbourhood of the origin.) For details, see M.A. Thesis of D.A. Trumpler (16).

More recently, F. Tricomi (15) has obtained expansions of the Confluent Hypergeometric Function in series of Bessel Functions. Using Laplace Transform methods, he arrived at an expansion for the well-behaved solution of the Confluent Hypergeometric Equation, and gave a four-term recurrence for the coefficients in the series. Also, by setting up an integral equation similar to that which we have derived in Section 1.1, he obtained asymptotic formulae, but no general expansions. In this chapter, we use the result of Section 1.1 to obtain the general solution of the Confluent Hypergeometric Equation as series in $J_n(x)$ and $Y_n(x)$, and as a special case, the well-behaved solution of this equation as a series in $J_n(x)$.

Further, we arrive at a similar series of Bessel Functions for the solution of a generalized Confluent Hypergeometric Equation. Theoretically, the procedure could be generalized to obtain expansions of various other functions in terms of known functions except for the computational difficulties in evaluating certain integrals involving the latter.

3.2 The Expansion Of The Solution Of The Confluent Hypergeometric Equation In Series Of Bessel Functions.

The object is to express the solution $W(a,c;t)$ of the Confluent Hypergeometric Equation

(3.2.1) $$t\,W''(t) + (c-t)\,W'(t) - a\,W(t) = 0$$

in terms of the solutions $J_n(t)$ and $Y_n(t)$ of Bessel's Equation

(3.2.2) $$t\,h''(t) + h'(t) + \left(t - \frac{n^2}{t}\right) h(t) = 0 .$$

We now proceed to set up an integral equation linking the solutions of (3.2.1) and (3.2.2). In order to obtain the simple expression (1.4.1) for the kernel, we use (1.4.6) to get the transformation

(3.2.3) $$h(t) = t^{-\frac{1}{2}}\, v(x); \qquad t = x,$$

which changes (3.2.2) into

(3.2.4) $$v''(x) + \left[\frac{\frac{1}{4} - n^2}{x^2} + 1\right] v(x) = 0 .$$

Likewise, we can remove the first derivative term from (3.2.1) by the change of variable

(3.2.5) $$W(t) = e^{\frac{t}{2}}\, t^{-\frac{c}{2}}\, V(t),$$

which changes (3.2.1) into

(3.2.6) $$V''(t) + \left\{-\frac{1}{4} + \frac{\mu}{t} - \frac{\ell(\ell+1)}{t^2}\right\} V(t) = 0,$$

where

(3.2.7) $\quad \ell = \frac{c}{2} - 1 \; ; \qquad \mu = \frac{c}{2} - a \, .$

The further transformation

(3.2.8) $\quad t = \frac{x^2}{4\mu} \; ; \qquad V(t) = \frac{x^{\frac{1}{2}}}{4\mu^{\frac{1}{4}}} \, u(x)$

changes (3.2.6) into

(3.2.9) $\quad u''(x) + \left\{ \frac{\frac{1}{4} - n^2}{x^2} - \frac{x^2}{\lambda^2} + 1 \right\} u(x) = 0 ,$

where

(3.2.10) $\quad \ell = \frac{n-1}{2}, \qquad n = 2\ell + 1 = c - 1, \qquad \lambda = 4\mu \, .$

We now use the result of Section 1.1 to write the solutions of (3.2.9) in terms of the known solutions $x^{\frac{1}{2}} J_n(x)$ and $x^{\frac{1}{2}} Y_n(x)$ of (3.2.4) by a Volterra Integral Equation. From (1.1.14, (1.1.16), and (3.2.3), we obtain

$$N(z,x) = \frac{x^{\frac{1}{2}} z^{\frac{1}{2}} \{ h_1(x) h_2(z) - h_1(z) h_2(x) \}}{x \{ h_1(x) h_2'(x) - h_2(x) h_1'(x) \}}$$

$$= x^{\frac{1}{2}} z^{\frac{1}{2}} \frac{\pi}{2} \{ J_n(x) Y_n(z) - J_n(z) Y_n(x) \},$$

upon taking $h_1(x) = J_n(x)$, $h_2(x) = Y_n(x)$, and using the identity (See Watson (17))

(3.2.11) $\quad J_n(x) \, Y_n'(x) - J_n'(x) Y_n(x) = 2/\pi x \quad ;$

(3.2.12) $\quad K(z,x) = \frac{\pi}{2} x^{\frac{1}{2}} z^{\frac{1}{2}} \{ J_n(x) Y_n(z) - J_n(z) Y_n(x) \} \frac{x^2}{\lambda^2} \, .$

The integral equation (1.1.15) is then[1]

(3.2.13) $\quad u(z) = \alpha z^{\frac{1}{2}} J_n(z) + \beta z^{\frac{1}{2}} Y_n(z) + \frac{1}{\lambda^2} z^{\frac{1}{2}} \frac{\pi}{2} \int^z [J_n(x) Y_n(z) - J_n(z) Y_n(x)] \, x^{\frac{1}{2}} x^2 u(x) \, dx \, .$

To obtain the solution of (3.2.13), we shall need the following special results:

(3.2.14) $\quad J_n(x) = \frac{2}{x} (n+1) J_{n+1}(x) - J_{n+2}(x) \; ;$

(3.2.15) $\quad Y_n(x) = \frac{2}{x} (n+1) Y_{n+1}(x) - Y_{n+2}(x) \; ;$

(3.2.16) $\quad I_\lambda = \frac{\pi}{2} \int^z [J_n(x) Y_n(z) - J_n(z) Y_n(x)] \left(\frac{x}{2}\right)^\lambda x \, J_{n+\lambda}(x) dx = \frac{1}{\lambda+1} \left(\frac{z}{2}\right)^{\lambda+1} J_{n+\lambda+1}(z) + C_1^{(\lambda)} J_n(z) + C_2^{(\lambda)} Y_n(z);$

(3.2.17) $\quad L_\lambda = \frac{\pi}{2} \int^z [J_n(x) Y_n(z) - J_n(z) Y_n(x)] \left(\frac{x}{2}\right)^\lambda x \, Y_{n+\lambda}(x) dx = \frac{1}{\lambda+1} \left(\frac{z}{2}\right)^{\lambda+1} Y_{n+\lambda+1}(z) + C_3^{(\lambda)} J_n(z) + C_4^{(\lambda)} Y_n(z),$

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1 See the footnote on page 2.

where λ is a positive integer, and $C_i^{(\lambda)}$ $(i=1,2,3,4)$ are constants of integration. The recurrence relations (3.2.14) and (3.2.15) are well-known (16), and the results (3.2.16) and (3.2.17) will be established in Section 3.6.

If we take $u_0(z) = z^{\frac{1}{2}}\{\alpha J_n(z) + \beta Y_n(z)\}$ and calculate the first few approximations of (3.2.13), it becomes apparent that the solution of (3.2.13) will have the form

$$(3.2.18) \qquad u(z) = z^{\frac{1}{2}} \sum_{r=0}^{\infty} \left[A_r \left(\tfrac{z}{2}\right)^r J_{n+r}(z) + B_r \left(\tfrac{z}{2}\right)^r Y_{n+r}(z) \right],$$

upon rearrangement of the terms in the series. We then substitute (3.2.18) into (3.2.13) and determine the necessary recurrence formulae for the coefficients A_r and B_r so that (3.2.13) is satisfied. The result of the substitution is

$$\sum_{r=0}^{\infty} \left[A_r \left(\tfrac{z}{2}\right)^r J_{n+r}(z) + B_r \left(\tfrac{z}{2}\right)^r Y_{n+r}(z) \right] = \alpha J_n(z) + \beta Y_n(z)$$

$$+ \left(\tfrac{2}{\lambda}\right)^2 \frac{\pi}{2} \int^z \left[J_n(x) Y_n(z) - J_n(z) Y_n(x) \right] x \left(\tfrac{x}{2}\right)^2 \sum_{r=0}^{\infty} \left[A_r \left(\tfrac{x}{2}\right)^r J_{n+r}(x) + B_r \left(\tfrac{x}{2}\right)^r Y_{n+r}(x) \right] dx .$$

Applying (3.2.14) and (3.2.15) to the Bessel Functions in the summation under the integral sign, and then using (3.2.16) and (3.2.17) , we obtain in turn

$$\sum_{r=0}^{\infty} \left[A_r \left(\tfrac{z}{2}\right)^r J_{n+r}(z) + B_r \left(\tfrac{z}{2}\right)^r Y_{n+r}(z) \right] = \alpha J_n(z) + \beta Y_n(z) + \left(\tfrac{2}{\lambda}\right)^2 \frac{\pi}{2} \int^z \left[J_n(x) Y_n(z) - J_n(z) Y_n(x) \right]$$

$$\cdot\, x \left[\sum_{r=0}^{\infty} A_r \left\{ (n+r+1) \left(\tfrac{x}{2}\right)^{r+1} J_{n+r+1}(x) - \left(\tfrac{x}{2}\right)^{r+2} J_{n+r+2}(x) \right\} + B_r \left\{ (n+r+1) \left(\tfrac{x}{2}\right)^{r+1} Y_{n+r+1}(x) - \left(\tfrac{x}{2}\right)^{r+2} Y_{n+r+2}(x) \right\} \right] dx$$

$$= \alpha_1 J_n(z) + \beta_1 Y_n(z) + \left(\tfrac{2}{\lambda}\right)^2 \sum_{r=0}^{\infty} A_r \left\{ \frac{n+r+1}{r+2} \left(\tfrac{z}{2}\right)^{r+2} J_{n+r+2}(z) - \frac{1}{r+3} \left(\tfrac{z}{2}\right)^{r+3} J_{n+r+3}(z) \right\} + B_r \left\{ \frac{n+r+1}{r+2} \left(\tfrac{z}{2}\right)^{r+2} Y_{n+r+2}(z) - \frac{1}{r+3} \left(\tfrac{z}{2}\right)^{r+3} Y_{n+r+3}(z) \right\},$$

where α_1 and β_1 are constants depending upon α, β, and $C_i^{(\lambda)}$ $(i=1,2,3,4\ ;\ \lambda=1,2,\cdots)$. Replacing r by $r+1$ in the first and third terms under the summation on the right side, we may rewrite this in the form

$$\sum_{r=0}^{\infty}\left[A_r\left(\tfrac{z}{2}\right)^r J_{n+r}(z)+B_r\left(\tfrac{z}{2}\right)^r Y_{n+r}(z)\right] = \alpha_1 J_n(z)+\beta_1 Y_n(z)+\left(\tfrac{2}{\lambda}\right)^2\left[\tfrac{n+1}{2}A_0\left(\tfrac{z}{2}\right)^2 J_{n+2}(z)+\tfrac{n+1}{2}B_0\left(\tfrac{z}{2}\right)^2 Y_{n+2}(z)\right]$$
$$+\left(\tfrac{2}{\lambda}\right)^2\sum_{r=0}^{\infty}\left\{\tfrac{n+r+2}{r+3}A_{r+1}-\tfrac{1}{r+3}A_r\right\}\left(\tfrac{z}{2}\right)^{r+3}J_{n+r+3}(z)+\left\{\tfrac{n+r+2}{r+3}B_{r+1}-\tfrac{1}{r+3}B_r\right\}\left(\tfrac{z}{2}\right)^{r+3}Y_{n+r+3}(z).$$

Equating separately the coefficients of $\left(\frac{z}{2}\right)^{r+3}J_{n+r+3}(z)$ and $\left(\frac{z}{2}\right)^{r+3}Y_{n+r+3}(z)$ on both sides, we obtain the recurrence relations

$$(3.2.19)\quad A_{r+3}=\left(\tfrac{2}{\lambda}\right)^2\left\{\tfrac{-1}{r+3}A_r+\tfrac{n+r+2}{r+3}A_{r+1}\right\};\quad B_{r+3}=\left(\tfrac{2}{\lambda}\right)^2\left\{\tfrac{-1}{r+3}B_r+\tfrac{n+r+2}{r+3}B_{r+1}\right\},$$

where

$$\begin{cases} A_0=\alpha_1\;; & B_0=\beta_1\;; \quad A_1=B_1=0\;; \\ A_2=\alpha_1\left(\tfrac{2}{\lambda}\right)^2\tfrac{n+1}{2}\;; & B_2=\beta_1\left(\tfrac{2}{\lambda}\right)^2\tfrac{n+1}{2}\;. \end{cases}$$

Putting $A_r=\alpha_1 a_r$, and $B_r=\beta_1 b_r$, we can rewrite (3.2.18) in the form

$$(3.2.20)\qquad u(z)=z^{\frac{1}{2}}\left[\alpha_1\sum_{r=0}^{\infty}a_r\left(\tfrac{z}{2}\right)^r J_{n+r}(z)+\beta_1\sum_{r=0}^{\infty}b_r\left(\tfrac{z}{2}\right)^r Y_{n+r}(z)\right],$$

where now

$$(3.2.21)\qquad \begin{cases} a_{r+3}=\left(\tfrac{2}{\lambda}\right)^2\left\{\tfrac{-1}{r+3}a_r+\tfrac{n+r+2}{r+3}a_{r+1}\right\}\;; \\ a_0=b_0=1\;; \quad a_1=b_1=0; \quad a_2=b_2=\left(\tfrac{2}{\lambda}\right)^2\tfrac{n+1}{2}\;; \\ a_r=b_r \qquad (r=3,4,\cdots\cdots) \end{cases}$$

Since the conditions of Theorem 2 of Section 1.6. are satisfied, the sequence of successive approximations to the solution of (3.2.13) converges for all finite z excluded from a small neighbourhood of the origin. However, the series (3.2.20) has been obtained by rearrangement of terms, so that the convergence of this series does not follow immediately. This question is treated in more detail in the M.A. Thesis of D.A. Trumpler (16). To obtain the solution which is finite at the origin, we set $\beta_1=0$ in (3.2.20). The constant α_1 is determined from the appropriate normalization. It can be shown (16) that the second solution of (3.2.9) is obtained by taking a ~~special~~

special values for α_1 and β_1 .

The solution $W(a,c;\, t)$ of (3.2.1) is obtained from (3.2.20) by changing back to the original variables, using (3.2.5) and (3.2.8) :

$$(3.2.22) \qquad W(a,c;\,t) = e^{\frac{t}{2}}\, t^{\frac{1}{4}-\frac{c}{2}}\, u\left(2\left[\tfrac{c}{2}-a\right]^{\frac{1}{2}} t^{\frac{1}{2}}\right),$$

where the constant λ in the recurrence (3.2.21) is given by

$$(3.2.23) \qquad \lambda = 2c - 4a .$$

3.3 A Generalization Of The Problem.

Consider a generalization of the Confluent Hypergeometric Equation

$$(3.3.1) \qquad V''(t) + \left\{-\tfrac{1}{4}\gamma_p\, t^{p-1} + \frac{\mu}{t} - \frac{\ell(\ell+1)}{t^2}\right\} V(t) = 0,$$

which is different from (3.2.6) in that the term $-\frac{1}{4}$ in (3.2.6) is replaced by $-\frac{1}{4}\gamma_p\, t^{p-1}$, where γ_p is a constant, ($p=1,2,\cdots$). As in (3.2.6), μ and ℓ are constants of the problem. The case discussed earlier ($p=1$) is related to the quantum mechanical problem for an harmonic oscillator in space. The more general form here (and the generalization considered in Section 3.4) could therefore be interpreted as an anharmonic oscillator in space. The change of variable (3.2.8) transforms (3.3.1) into

$$(3.3.2) \qquad u''(x) + \left\{\frac{\frac{1}{4}-n^2}{x^2} - \frac{x^{2p}}{\lambda^2} + 1\right\} u(x) = 0,$$

where now

$$(3.3.3) \qquad \lambda = \frac{(4\mu)^{\frac{p+1}{2}}}{\gamma_p^{\frac{1}{2}}} .$$

As in Section 3.2, we use the results of Section 1.1 to compare (3.3.2) with (3.2.4), and obtain the integral equation

$$(3.3.4) \qquad u(z) = \alpha z^{\frac{1}{2}} J_n(z) + \beta z^{\frac{1}{2}} Y_n(z) + \frac{1}{\lambda^2} z^{\frac{1}{2}} \frac{\pi}{2} \int^{z} \left[J_n(x) Y_n(z) - J_n(z) Y_n(x)\right] x^{\frac{1}{2}} x^{2p} u(x)\, dx .$$

In order to obtain the solution of (3.3.2) which is finite at the origin, we take $\beta = 0$; a simple modification would give the general solution, (as in Section 3.2). Following the method of Section 3.2, we look for the solution of (3.3.4) in the form

$$(3.3.5) \qquad u(z) = \alpha z^{\frac{1}{2}} \sum_{r=0}^{\infty} a_r \left(\frac{z}{2}\right)^r J_{n+r}(z)$$

and determine the necessary recurrence formulae for the coefficients a_r in order that (3.3.4) be satisfied. Putting (3.3.5) into (3.3.4), we get

$$(3.3.6) \qquad \sum_{r=0}^{\infty} a_r \left(\frac{z}{2}\right)^r J_{n+r}(z) = J_n(z) + \frac{1}{\lambda^2}\frac{\pi}{2}\int^{z}\left[J_n(x)Y_n(z) - J_n(z)Y_n(x)\right] x\, x^{2p} \sum_{r=0}^{\infty} a_r \left(\frac{x}{2}\right)^r J_{n+r}(x)\,dx .$$

In the evaluation of the integral, we need the result

$$(3.3.7) \qquad J_{n+r}(x) = \sum_{\ell=0}^{p} (-1)^{\ell} \left(\frac{2}{x}\right)^{p-\ell} \binom{p}{\ell} \left[(n+r+\ell+1)\cdots(n+r+p)\right] J_{n+r+p+\ell}(x),$$

where p is a positive integer. This will be proved in Section 3.6. Substituting (3.3.7) into (3.3.6) and using (3.2.15), we get, upon interchanging summation and integration,

$$\sum_{r=0}^{\infty} a_r \left(\frac{z}{2}\right)^r J_{n+r}(z) = J_n(z)$$
$$+ \left(\frac{2^p}{\lambda}\right)^2 \sum_{r=0}^{\infty}\sum_{\ell=0}^{p} a_r (-1)^{\ell} \left(\frac{z}{2}\right)^{r+p+\ell+1} \binom{p}{\ell} \frac{1}{r+p+\ell+1} \left[(n+r+\ell+1)\cdots(n+r+p)\right] J_{n+r+p+\ell+1}(z) .$$

The change of dummy $s = r - p + \ell$ leads to

$$(3.3.8) \begin{cases} \displaystyle\sum_{r=0}^{\infty} a_r \left(\frac{z}{2}\right)^r J_{n+r}(z) = J_n(z) + \left(\frac{2^p}{\lambda}\right)^2 \sum_{s=0}^{\infty}\sum_{\ell=0}^{p} a_{s+p-\ell} (-1)^{\ell} \left(\frac{z}{2}\right)^{s+2p+1} \binom{p}{\ell} \frac{1}{s+2p+1} \left[(n+s+p+1)\cdots(n+s+2p-\ell)\right] J_{n+s+2p+1}(z) \\ \displaystyle + \left(\frac{2^p}{\lambda}\right)^2 \sum_{s=0}^{p-1} a_s \left(\frac{z}{2}\right)^{s+p+1} \binom{p}{0} \frac{1}{s+p+1} \left[(n+s+1)\cdots(n+s+p)\right] J_{n+s+p+1}(z) \\ \displaystyle - \left(\frac{2^p}{\lambda}\right)^2 \sum_{s=0}^{p-2} a_s \left(\frac{z}{2}\right)^{s+p+2} \binom{p}{1} \frac{1}{s+p+2} \left[(n+s+2)\cdots(n+s+p)\right] J_{n+s+p+2}(z) \\ + \cdots \\ \displaystyle + \left(\frac{2^p}{\lambda}\right)^2 (-1)^{p-1} a_0 \left(\frac{z}{2}\right)^{s+2p} \binom{p}{p-1} \frac{1}{s+2p} \left[(n+s+p)\right] J_{n+s+2p}(z). \end{cases}$$

After the dummy of summation s in the right member has been replaced by r, equating the coefficients of $\left(\frac{z}{2}\right)^r J_{n+r}(z)$

gives the recurrence formula

(3.3.9) $$a_{r+2p+1} = \left(\frac{2p}{\lambda}\right)^2 \sum_{\ell=0}^{p} a_{r+p-\ell}(-1)^\ell \binom{p}{\ell} \frac{1}{r+2p+1}\left[(n+r+p+1)\cdots(n+r+2p-\ell)\right].$$

This gives a_{r+2p+1} in terms of $a_r, a_{r+1}, \cdots a_{r+p}$. In order to apply the recursion, however, we need the explicit forms of $a_0, a_1, \cdots a_{2p}$. From (3.3.8)

(3.3.10) $$a_0 = 1; \quad a_1 = a_2 = \cdots = a_p = 0.$$

Now, in (3.3.8), the only term on the right which contributes to $J_{n+p+1}(z)$ is the $s=0$ term in the first single summation. Hence,

$$a_{p+1} = a_0 \binom{p}{0} \frac{1}{p+1}\left[(n+1)\cdots(n+p)\right],$$

$$a_{p+1} = \frac{1}{p+1}\left[(n+1)\cdots(n+p)\right].$$

Similarly, we obtain

(3.3.11) $$a_{p+t} = (-1)^{t-1}\binom{p}{t-1}\frac{1}{p+t}\left[(n+t)(n+t+1)\cdots(n+p)\right]$$

for $t = 1, 2, \cdots p$. In summary, (3.3.10) and (3.3.11) give the coefficients $a_0, a_1, \cdots a_{2p}$, and (3.3.9) then gives all subsequent a_r $(r = 2p+1, 2p+2, \cdots)$.

2.4 A Further Generalization.

Consider a further generalization of the Confluent Hypergeometric Equation

(3.4.1) $$V''(t) + \left\{-\frac{1}{4}\sum_{g=1}^{p} \gamma_g t^{g-1} + \frac{\mu}{t} - \frac{\ell(\ell+1)}{t^2}\right\} V(t) = 0,$$

where the γ_g are constants ($g = 1, 2, \cdots p$; $p = 1, 2, \cdots$), and μ and ℓ are again constants. As in Section 3.3, the change of variable (3.2.8) transforms (3.4.1) into

(3.4.2) $$u''(x) + \left[\frac{\frac{1}{4} - n^2}{x^2} - \frac{1}{\lambda^2}\sum_{g=1}^{p} c_g x^{2g} + 1\right] u(x) = 0,$$

where

$$(3.4.3) \qquad \lambda = 4\mu \; ; \qquad c_g = (4\mu)^{g-1} / \gamma_g \, .$$

Using the results of Section 1.1 to compare (3.4.2) with (3.2.4), we get, for $\beta = 0$,

$$(3.4.4) \quad u(z) = \alpha z^{\frac{1}{2}} J_n(z) + \frac{1}{\lambda^2} z^{\frac{1}{2}} \frac{\pi}{2} \int^z \left[J_n(x) Y_n(z) - J_n(z) Y_n(x) \right] x^{\frac{1}{2}} \sum_{g=1}^{p} c_g x^{2g} u(x) dx \, .$$

Substitution of the proposed solution (3.3.5) into (3.4.4), and use of (3.3.7) and (3.2.15) gives

$$(3.4.5) \begin{cases} \sum_{r=0}^{\infty} a_r \left(\frac{z}{2}\right)^r J_{n+r}(z) = J_n(z) \\ + \sum_{r=0}^{\infty} \sum_{g=1}^{p} \sum_{l=0}^{g} c_g \left(\frac{2^g}{\lambda}\right)^2 a_r (-1)^l \left(\frac{z}{2}\right)^{r+g+l+1} \binom{g}{l} \frac{1}{r+g+l+1} \left[(n+r+l+1) \cdots (n+r+g) \right] \\ \qquad \times J_{n+r+g+l+1}(z) \, . \end{cases}$$

The successive changes of dummy $s = r - p + l$, and $t = s - 2p + 2g$ transform (3.4.5) into

$$(3.4.6) \begin{cases} \sum_{r=0}^{\infty} a_r \left(\frac{z}{2}\right)^r J_{n+r}(z) = J_n(z) + G_n(z) + H_n(z) \\ + \sum_{t=0}^{\infty} \sum_{g=1}^{p} \sum_{l=0}^{g} c_g \left(\frac{2^g}{\lambda}\right)^2 a_{t+2p-g-l} (-1)^l \left(\frac{z}{2}\right)^{t+2p+1} \binom{g}{l} \frac{1}{t+2p+1} \left[(n+t+2p-g+1) \cdots (n+t+2p-l) \right] J_{n+t+2p+1}(z) \, , \end{cases}$$

where

$$\begin{cases} H_n(z) = \sum_{g=1}^{p} c_g \left(\frac{2^g}{\lambda}\right)^2 \left\{ \sum_{s=0}^{g-1} a_s \left(\frac{z}{2}\right)^{s+g+1} \binom{g}{0} \frac{1}{s+g+1} \left[(n+s+1) \cdots (n+s+g) \right] J_{n+s+g+1} + \sum_{s=0}^{g-2} a_s \left(\frac{z}{2}\right)^{s+g+2} \binom{g}{1} \frac{1}{s+g+2} \right. \\ \left. \quad \times \left[(n+s+2) \cdots (n+s+g) \right] J_{n+s+g+2}(z) + \cdots a_0 (-1)^{g-1} \left(\frac{z}{2}\right)^{s+2g} \binom{g}{g-1} \frac{1}{s+2g} (n+s+g) J_{n+s+2g}(z) \right. , \end{cases}$$

and

$$\begin{cases} G_n(z) = c_1 \left(\frac{2}{\lambda}\right)^2 \sum_{t=0}^{2p-3} \sum_{l=0}^{1} a_{t+1-l} (-1)^l \left(\frac{z}{2}\right)^{t+3} \frac{1}{t+3} \left[(n+t+2) \cdots (n+t+2-l) \right] J_{n+t+3}(z) \\ \quad + c_2 \left(\frac{2^2}{\lambda}\right)^2 \sum_{t=0}^{2p-5} \sum_{l=0}^{2} a_{t+2-l} (-1)^l \left(\frac{z}{2}\right)^{t+5} \frac{1}{t+5} \left[(n+t+3) \cdots (n+t+4-l) \right] J_{n+t+5}(z) \\ \quad + \cdots \\ \quad + c_{p-1} \left(\frac{2^{p-1}}{\lambda}\right)^2 \sum_{t=0}^{1} \sum_{l=0}^{p-1} a_{t+p-1-l} (-1)^l \left(\frac{z}{2}\right)^{t+2p+1} \frac{1}{t+2p+1} \left[(n+t+p) \cdots (n+t+2p-2-l) \right] J_{n+t+2p-1}(z) \, . \end{cases}$$

Equating the coefficients of $J_{n+t+2p-1}(z)$ in (3.4.6) gives the recurrence relation

$$(3.4.7) \quad a_{t+2p+1} = \sum_{g=1}^{p} \sum_{l=0}^{g} c_g \left(\frac{2^g}{\lambda}\right)^2 (-1)^l \binom{g}{l} \frac{1}{t+2p+1} \left[(n+t+2p-g+1) \cdots (n+t+2p-l) \right] a_{t+2p-g-l} \, ,$$

where $a_0, a_1, \cdots a_{2p}$ are obtained from (3.4.6) in any given example. However, the complicated nature of the functions

$G_n(z)$ and $H_n(z)$ makes it inconvenient to get general expressions for these coefficients.

For finite numbers ρ, Theorem 3 of Section 1.6 shows that the sequence of successive approximations assosiated with the integral equation (3.4.4) actually converges to the unique continuous solution of (3.4.4). As in Section 3.2, however, we have rearranged terms in obtaining (3.3.5), so that further attention is required in order to establish the convergence.

3.5 Solutions of Related Differential Equations Expanded In Terms Of Bessel Functions.

In this section, it will be shown that the solutions of a number of important differential equations are related to the Confluent Hypergeometric Function through various changes of variable. Hence, the results of Section 3.2 can be used to express these solutions as series of Bessel Functions. Numerical values for these solutions could then be computed accurately by making use of the extensive tabulation of the Bessel Function (17), and in fact, for $\lambda << 2$, only a few terms of the rapidly convergent series (3.2.19) would be needed to guarantee accurate results (9).

(a) The Whittaker Function (18).

Putting $m = \ell + \frac{1}{2}$ into (3.2.6) gives

(3.5.1) $$V''(t) + \left\{ -\frac{1}{4} + \frac{\mu}{t} + \frac{\frac{1}{4} - m^2}{t^2} \right\} V(t) = 0,$$

which, by (3.2.5), has as its solution the Whittaker Function

(3.5.2) $$M_{\mu, m}(t) = e^{-\frac{t}{2}} t^{\frac{c}{2}} W(a, c; t)$$

Since (3.2.7) gives $m = \frac{c}{2} - \frac{1}{2}$, $\mu = \frac{c}{2} - a$,
equation (3.5.2) can be written

(3.5.3) $$M_{\mu,m}(t) = e^{-\frac{t}{2}} t^{m+\frac{1}{2}} W(m+\tfrac{1}{2}-\mu,\ 2m+1;\ t),$$

which is now in a form to which (3.2.21) can be applied.

(b) The Laguerre Function Density;

By the substitutions

(3.5.4) $$t = \frac{2r}{\mu}\ ; \qquad V(t) = L(r),$$

equation (3.2.6) becomes

(3.5.5) $$L''(r) + \left[\frac{2}{r} - \frac{\ell(\ell+1)}{r^2} - \frac{1}{\mu^2}\right] L(r) = 0,$$

with solution

(3.5.6) $$L_{\mu,\ell}(r) = M_{\mu,m}(t) = M_{\mu,\ell+\frac{1}{2}}\left(\frac{2r}{\mu}\right).$$

(c) The "Associated Hermite Equation".

The so-called Associated Hermite Equation

(3.5.7) $$T''(y) + \left[-\frac{(2\ell+\frac{1}{2})(2\ell+\frac{3}{2})}{y^2} - \frac{y^2}{16} + \mu\right] T(y) = 0,$$

obtained from (3.2.6) by the substitutions

(3.5.8) $$t = \frac{y^2}{4}\ ; \qquad V(t) = \left(\frac{y}{2}\right)^{\frac{1}{2}} T(y),$$

has the solution

(3.5.9) $$T(y) = \left(\frac{y}{2}\right)^{-\frac{1}{2}} M_{\mu,\ \ell+\frac{1}{2}}\left(\frac{y^2}{4}\right),$$

or

$$T(y) = \left(\frac{y}{2}\right)^{-\frac{1}{2}} e^{-\frac{y^2}{8}} \left(\frac{y^2}{4}\right)^{\ell+1} W\left(\ell+1-\mu,\ 2\ell+2;\ \frac{y^2}{4}\right).$$

(d) Hermite's Equation (18)

Hermite's Equation (or Weber's Equation) is

(3.5.10) $$D_j''(z) + \left[(j+\tfrac{1}{2}) - \frac{z^2}{4}\right] D_j(z) = 0,$$

which is related to (3.2.9) by the transformations

(3.5.11) $$\lambda = 2j+1\ ; \quad x = \left(\frac{\lambda}{2}\right)^{\frac{1}{2}} z\ ; \quad u(x) = D_j(z)\ ; \quad n = \tfrac{1}{2}.$$

Hence, the solution of (3.5.10) may be written

$$D_j(z) = u\left([j+\tfrac{1}{2}]^{\frac{1}{2}} z\right),$$

where in the expansion (3.2.19) for u , and in the recurrence relation (3.2.20),

$$n=\tfrac{1}{2}\,; \qquad \lambda = 2j+1 .$$

(e) The Equation For The Harmonic Oscillator In Space

This equation is

$$(3.5.12) \qquad H_n''(y) + \left[\frac{\frac{1}{4}-n^2}{y^2} - \frac{1}{16}y^2 + \mu\right] H_n(y) = 0,$$

which is related to (3.2.9) by the transformation

$$(3.5.13) \qquad x = \mu^{\frac{1}{2}} y, \qquad u(x) = H_n(y).$$

The solution of (3.5.12) is then

$$(3.5.14) \qquad H_n(y) = u(\mu^{\frac{1}{2}} y).$$

3.6 Appendix To Chapter Three.

(a) Proof Of (3.2.16) and (3.2.17).

Consider the integral

$$(3.6.1) \qquad \int^z J_n(x)\, J_{n+\lambda+1}(x)\cdot x^{\lambda+2}\,dx = \int^z [x^{-n+1} J_n(x)][x^{n+\lambda+1} J_{n+\lambda+1}(x)]\,dx$$

Since

$$(3.6.2) \qquad x^{-n+1} J_n(x) = -\frac{d}{dx}\left[x^{-n+1} J_{n-1}(x)\right]; \qquad \frac{d}{dx}\left[x^{n+\lambda+1} J_{n+\lambda+1}(x)\right] = x^{n+\lambda+1} J_{n+\lambda}(x)$$

a partial integration of (3.6.1) gives

$$\int^z J_n(x)\, J_{n+\lambda+1}(x)\cdot x^{\lambda+2}\,dx = \left[-J_{n-1} J_{n+\lambda+1} x^{\lambda+2}\right]^z + \int^z J_{n-1} J_{n+\lambda}\, x^{\lambda+2}\,dx$$

$$= \left[-J_{n-1} J_{n+\lambda+1} x^{\lambda+2}\right]^z + n\int^z J_n J_{n+\lambda} x^{\lambda+1}\,dx + \int^z J_n' J_{n+\lambda}\, x^{\lambda+2}\,dx,$$

using the relation (17)

$$(3.6.3) \qquad x J_{n-1}(x) = n J_n(x) + x J_n'(x) .$$

A partial integration of the last integral on the right gives

$$(3.6.4) \qquad \begin{aligned} \int^z J_n J_{n+\lambda+1}\, x^{\lambda+2}\,dx &= \left[-J_{n-1} J_{n+\lambda+1} x^{\lambda+2}\right]^z + n\int^z J_n J_{n+\lambda}\, x^{\lambda+1}\,dx \\ &+ \left[J_n J_{n+\lambda}\, x^{\lambda+2}\right]^z - \int^z J_n J_{n+\lambda}'\, x^{\lambda+2}\,dx - (\lambda+2)\int^z J_n J_{n+\lambda}\, x^{\lambda+1}\,dx . \end{aligned}$$

Upon use of the identity (17)

$$x J_{n+\lambda}'(x) = (n+\lambda)\, J_{n+\lambda}(x) - x J_{n+\lambda+1}(x),$$

(3.6.4) becomes

$$\int^z J_n J_{n+\lambda+1}\, x^{\lambda+2}\,dx = \left[J_n J_{n+\lambda} x^{\lambda+2} - J_{n-1} J_{n+\lambda+1} x^{\lambda+2}\right]^z$$

$$+ (n-\lambda-2)\int^z J_n J_{n+\lambda}\, x^{\lambda+1}\,dx + \int^z J_n J_{n+\lambda+1}\, x^{\lambda+2}\,dx - (n+\lambda)\int^z J_n J_{n+\lambda}\, x^{\lambda+1}\,dx .$$

Upon transposing terms in this equation, we get finally

(3.6.5) $$\int^z J_n(x) J_{n+\lambda}(x) x^{\lambda+1} dx = \frac{1}{2(\lambda+1)} \left[J_n(x) J_{n+\lambda}(x) x^{\lambda+2} - J_{n-1}(x) J_{n+\lambda+1}(x) x^{\lambda+2} \right]^z$$

Since the functions $Y_n(x)$ satisfy the same recurrence relations as $J_n(x)$, the following result is obtained in the same way as (3.6.5)

(3.6.6) $$\int^z Y_n(x) J_{n+\lambda}(x)\, dx = \frac{1}{2(\lambda+1)} \left[Y_n(x) J_{n+\lambda}(x) x^{\lambda+2} - Y_{n-1}(x) J_{n+\lambda+1}(x) x^{\lambda+2} \right]^z .$$

Upon use of (3.6.5) and (3.6.6), we get

$$I_\lambda = \frac{\pi}{2^{\lambda+2}(\lambda+1)} J_{n+\lambda+1}(z)\, z^{\lambda+2} \{ Y_{n-1}(z) J_n(z) - J_{n-1}(z) Y_n(z) \} + c_1^{(\lambda)} J_n(z) + c_2^{(\lambda)} Y_n(z).$$

Now, from (3.6.3) and (3.2.11),

(3.6.7) $$Y_{n-1}(z)\, J_n(z) - J_{n-1}(z)\, Y_n(z) = 2/\pi z,$$

and the result (3.2.16) follows.

Again, following the same procedure with $J_{n+\lambda+1}(x)$ replaced by $Y_{n+\lambda+1}(x)$, we get, instead of (3.6.5) and (3.6.6)

(3.6.8) $$\int^z J_n(x) Y_{n+\lambda}(x)\, x^{\lambda+1} dx = \frac{1}{2(\lambda+1)} \left[J_n(x) Y_{n+\lambda}(x) x^{\lambda+2} - J_{n-1}(x) Y_{n+\lambda+1}(x) x^{\lambda+2} \right]^z ;$$

(3.6.9) $$\int^z Y_n(x) Y_{n+\lambda}(x)\, x^{\lambda+1} dx = \frac{1}{2(\lambda+1)} \left[Y_n(x) Y_{n+\lambda}(x) x^{\lambda+2} - Y_{n-1}(x) Y_{n+\lambda+1}(x) x^{\lambda+2} \right]^z ,$$

from which the result (3.2.17) follows.

(b) Proof Of (3.3.7)

We need the following

Lemma. For $\ell = 1, 2, \cdots p$, the following relation holds

(3.6.10) $$\binom{p}{\ell-1}(\nu+\ell) + \binom{p}{\ell}(\nu+p+\ell+1) = \binom{p+1}{\ell}(\nu+p+1) .$$

We now prove the result (3.3.7) by finite induction upon p.

For $p=1$, (3.3.7) gives

(3.6.11) $$J_\nu(x) = \left(\frac{2}{x}\right)\binom{1}{0} \nu J_{\nu+1}(x) - \binom{1}{1} J_{\nu+2}(x),$$

where for convenience, we put $\nu = n+r \quad (r = 0,1,2,\cdots)$, which is correct by (3.2.14). Assuming the result (3.3.7) is true for $p = q$ we obtain, with the help of (3.2.14),

$$(3.6.12)\quad J_\nu(x) = T_0^{(1)} + \sum_{\ell=0}^{q}\left\{T_\ell^{(2)} + T_{\ell+1}^{(1)}\right\} + T_{q+1}^{(2)},$$

where

$$T_\ell^{(2)} = (-1)^{\ell-1}\left(\frac{2}{x}\right)^{q-\ell+1}\binom{q}{\ell-1}\left[(\nu+\ell)\cdots(\nu+q)\right]\left[-J_{\nu+q+\ell+1}(x)\right],$$

$$T_{\ell+1}^{(1)} = (-1)^{\ell}\left(\frac{2}{x}\right)^{q-\ell}\binom{q}{\ell}\left[(\nu+\ell+1)\cdots(\nu+q)\right]\left[\left(\frac{2}{x}\right)(\nu+q+\ell+1)J_{\nu+q+\ell+1}(x)\right]$$

$$(\ell = 0,1,2,\cdots q)$$

Hence,

$$T_\ell^{(2)} + T_{\ell+1}^{(1)} = (-1)^{\ell}\left(\frac{2}{x}\right)^{q-\ell+1}\left[(\nu+\ell+1)\cdots(\nu+q)\right]\left[\binom{q}{\ell-1}(\nu+\ell) + \binom{q}{\ell}(\nu+q+\ell+1)\right]J_{\nu+q+\ell+1}(x)$$

$$= (-1)^{\ell}\left(\frac{2}{x}\right)^{q-\ell+1}\left[(\nu+\ell+1)\cdots(\nu+q)\right]\left[\binom{q+1}{\ell}(\nu+q+1)\right]J_{\nu+q+\ell+1}(x)$$

upon application of the Lemma. Putting this into (3.6.12), we get

$$J_\nu(x) = \sum_{\ell=0}^{q+1}(-1)^{\ell}\left(\frac{2}{x}\right)^{q-\ell+1}\binom{q+1}{\ell}\left[(\nu+\ell+1)\cdots(\nu+q+1)\right]J_{\nu+q+\ell+1},$$

which completes the proof by induction.

CHAPTER FOUR

PHYSICAL APPLICATIONS

4.1 Introduction.

Although the method of Section 1.1 was originally designed for initial value problems, it can be adapted to solve boundary value problems. In this chapter, we shall discuss a type of boundary value problem which arises in Quantum Mechanics. Now, in the usual problems treated in Quantum Mechanics, it is required to find the solutions of the Schrodinger Wave Equation which satisfies a set of "natural boundary conditions", for which the position of the mass particle is unrestricted. The probability interpretation of the wave function then leads to the boundary conditions of finiteness at the singular points of the wave equation. If, however, the system under consideration is enclosed, then these conditions are replaced by the "artificial boundary conditions" that the wave function vanish at certain ordinary points of the differential equation. In fact, for these so-called bounded Quantum Mechanical problems, the boundary conditions require that the wave function vanish on some surface in finite three-space, such as a sphere or a cone. The corresponding physical condition is that there be an infinitely high and infinitely steep potential wall on this surface.

4.2 The Integral Equation For The Bounded Quantum Mechanical Problem.

Let generalized curvilinear coordinates x_1, x_2, and x_3

in three dimensional Euclidean space be chosen so that the surface on which the wave function ψ vanishes is $x_1 = c$, where c is a constant. We assume that the surface is of sufficiently simple nature that the Schrodinger Wave Equation is separable (12) in the chosen coordinates x_1, x_2, x_3. The space dependent wave equation is, for a particle of mass M,

(4.2.1) $$-\frac{\hbar^2}{2M}\nabla^2\psi = (E-V)\psi,$$

where $\hbar$ is Planck's constant divided by 2π, E is the energy constant, V is the potential energy, and ∇^2 is the Laplacian operator in the coordinate system x_1, x_2, x_3. The substitution

$$\psi(x_1, x_2, x_3) = X_1(x_1)\, X_2(x_2)\, X_3(x_3)$$

permits the separation of (4.2.1) into three ordinary differential equations for the functions $X_i(x_i)$. The equations for $X_2(x_2)$ and $X_3(x_3)$ have the same solutions as in the unbounded problem, and the latter are supposed known. The $X_1(x_1)$ equation has the form

(4.2.2) $$\frac{d}{dx_1}\left[p(x_1)\frac{dX_1}{dx_1}\right] + \left[g_\ell(x_1) + \lambda\rho(x_1)\right]X_1(x_1) = 0$$

where ℓ is the quantum number arising from the $X_2(x_2)$ equation. From Section 1.4, (4.2.2) can be transformed into

(4.2.3) $$u''(x) + \left[F_\ell(x) + \lambda\right]u(x) = 0,$$

which we suppose has the two singular points b and $\hat{b}$, with no other singular points between them. In the bounded problem, $u(x)$ must be continuous for all x satisfying

$$b \le |x| \le |x_0| < \hat{b}$$

and must vanish at x_0. Hence, the boundary conditions to be satisfied are

(4.2.4) $\quad u(b)$ Finite ; $\quad u(x_0) = 0$.

Following Section 1.1, we compare (4.2.3) with the equation

(4.2.5) $$v''(x) + [F_\ell(x) + \lambda_0]\, v(x) = 0,$$

where $v(x)$ satisfies the boundary conditions

(4.2.6) $\quad v(b)$ Finite ; $\quad v(\hat{b}) = 0$.

We suppose that a solution of (4.2.5) which is analytic in the finite plane is known to be

(4.2.7) $$v_1(x) = R_n^\ell(x),$$

and that the eigenvalue $\lambda_0(n)$ is known. From Section 1.4, the second solution of (4.2.5) and the Wronskian of the two solutions are given by

(4.2.8) $$v_2(x) = c\, R_n^\ell(x) \int^x \frac{dt}{[R_n^\ell(t)]^2},$$

and

(4.2.9) $$W(x) = c.$$

Hence, from (1.1.14), (1.1.15), and (1.1.16), the integral equation connecting the solutions of (4.2.3) and (4.2.5) is

(4.2.10) $$u(z) = \alpha R_n^\ell(z) + \beta v_2(z) + (\lambda_0 - \lambda) R_n^\ell(z) \int_b^z R_n^\ell(x) u(x) \int_x^z \frac{dt\, dx}{[R_n^\ell(t)]^2}.$$

The convergence of the solution of (4.2.10) by successive approximations is established by Theorem 1 of Section 1.5, since it has been assumed that $R_n^\ell(x)$ is analytic in the finite plane. The first of conditions (4.2.4) requires that $\beta = 0$. If we take $u_0(z) = \alpha R_n^\ell(z)$, the first approximations to the solution of (4.2.10) is

(4.2.11) $$u_1(z) = \alpha R_n^\ell(z)\left[1 - (\lambda - \lambda_0) \int_b^z [R_n^\ell(x)]^2 dx \int_x^z \frac{dt}{[R_n^\ell(t)]^2}\right].$$

By applying the second of (4.2.4) to (4.2.11), we get for the first approximations to the eigenvalue

$$\lambda^{(1)}_{\ell,n}(z_0) = \lambda_0 + \left[\int_b^{z_0} [R^{\ell}_n(x)]^2 dx \int_x^{z_0} \frac{d\tau}{[R^{\ell}_n(t)]^2} \right]^{-1} \tag{4.2.12}$$

The problem is then reduced to evaluating integrals of the type appearing on the right of (4.2.12). However, we are now prevented from continuing the general discussion because of our inability to obtain expressions for these integrals.

4.3 The Bounded Hydrogen Atom Problem.

The Dutch Physicists Michels, de Boer, and Bijl (10) were interested in the behaviour of gaseous matter under pressure, and in particular they wished to determine the effect of pressure upon the spectral lines of Hydrogen gas. In order that the mathematical problem be solved, it is assumed that the effect of pressure can be replaced by an infinitely high and infinitely steep potential wall on the surface of a sphere of finite radius z_0. Although physical objections to such an assumption have been pointed out, (de Groot and ten Seldam (2),) it is nevertheless useful to solve the quantum mechanical problem of finding the eigenfunctions and the eigenvalues for the Hydrogen atom wave equation, under the condition that the atom be enclosed in a sphere of radius z_0.

The $\Theta(\theta)$ and $\Phi(\varphi)$ parts of the wave equation clearly have solutions which are identical with the solutions corresponding to the natural boundary condition $\psi(\infty) = 0$. It remains to solve the radial part of the Hydrogen atom wave equation

$$u''(z) + \left[\frac{2}{z} - \frac{\ell(\ell+1)}{z^2} + \lambda \right] u(z) = 0, \tag{4.3.1}$$

under the artificial boundary conditions

(4.3.2) $u(0)$ Finite ; $u(z_0)=0$,

instead of the natural boundary conditions

(4.3.3) $u(0)$ Finite ; $u(\infty)=0$.

The conditions (4.3.3) give solutions of (4.3.1) easily by the Frobenius method; the eigenvalues are $\lambda_0(n)=\frac{-1}{n^2}$, for positive integers n, and the eigenfunctions are the Laguerre Function Densities.

However, (4.3.2) require that λ satisfy the equation $u(z_0;\ \lambda)=0$, where $u(z;\lambda)$ denotes the Laguerre Function Density (corresponding to the eigenvalue λ,) which is related to the Confluent Hypergeometric Function $W(a,c;t)$ (cf. equation (3.5.6)). Michels et al (10) have found approximations for the eigenvalues of the ground level, and de Groot and ten Seldam (2) have extended their method to the 2s and the 2p levels, giving graphs and tables for the shift in λ. Soon afterward, Sommerfeld and Welker (14) applied the formulae of Michels et al for values of z_0 equal to three and four times the Bohr radius. Also, Sommerfeld and Welker stressed the importance of a general investigation of the behaviour of the Confluent Hypergeometric Function near $z=\infty$.

Sommerfeld and Welker (14) have also discussed a graphical method for obtaining the eigenvalues, which gives accurately the curve $\lambda=\lambda(z_0)$ for small values of z_0. By this method, the known standard solutions $u(z;\ -\frac{1}{n^2})$ are plotted for various positive integral n, and the first zeros of these solutions are located. These functions are then solutions of the problem for the particular values $z_0^{(n)}$ of z_0. A graph of $z_0^{(n)}$ against n is drawn, and by interpolation, the value of $n(z_0)$ corresponding to a given value z_0 is taken from the graph. Then the ground level eigenvalue is $\lambda(z_0) = -1/n(z_0)^2$. The eigenvalues for higher levels are obtained by a similar procedure.

The preceeding gives an historic sketch of work done on the problem up to the present. We now proceed to give our own treatment, using the method of Section 4.2. The mathematical problem amounts to solving equation (4.3.1) under the boundary conditions (4.3.2), when we know that the solutions of the equation

(4.3.4) $$v''(z) + \left[\frac{2}{z} - \frac{\ell(\ell+1)}{z^2} + \lambda_0\right] v(z) = 0,$$

satisfying the boundary conditions

(4.3.5) $$v(0) \text{ Finite} \quad ; \qquad v(\infty) = 0$$

are

(4.3.6) $$v_1(z) = R^{\ell}_{n}(z) \quad , \text{ with } \quad \lambda_0(n) = -1/n^2.$$

Since the problem thus presented is of the same type considered in Section 4.2, the Volterra Integral Equation corresponding to equation (4.3.1) is (4.2.10), with first approximation (4.2.11). For the ground level,

$$\ell = 0, \qquad n = 1, \qquad \lambda_0 = -1,$$

$$R^{\ell}_{n}(z) = R^{0}_{1}(z) = z e^{-z},$$

and (4.2.11) gives

(4.3.7) $$u_1(z) = \alpha z e^{-z}\left[1 - (\lambda+1)\int_0^z x^2 e^{-2x}\,dx \int_x^z \frac{e^{2t}\,dt}{t^2}\right].$$

With $U = \int_x^z \frac{e^{2t}\,dt}{t^2}$, and $dV = x^2 e^{-2x}\,dx$,

a partial integration gives

$$I = \int_0^z x^2 e^{-2x}\,dx \int_x^z \frac{e^{2t}\,dt}{t^2} = \lim_{\epsilon\to 0}\left\{\left[-e^{-2x}\left(\frac{x^2}{2} + \frac{x}{2} + \frac{1}{4}\right)\int_x^z \frac{e^{2t}\,dt}{t^2}\right]_\epsilon^z - \int_\epsilon^z \left(\frac{1}{2} + \frac{1}{2x} + \frac{1}{4x^2}\right)dx\right\},$$

or

$$-I = \lim_{\epsilon\to 0}\left[-e^{-2\epsilon}\left(\frac{\epsilon^2}{2}+\frac{\epsilon}{2}+\frac{1}{4}\right)\int_\epsilon^z \frac{e^{2t}dt}{t^2} + \int_\epsilon^z\left[\frac{1}{2}+\frac{1}{2x}+\frac{1}{4x^2}\right]dx\right].$$

Now,

$$\int_\epsilon^z \frac{e^{2t}dt}{t^2} = \left[\frac{-1}{t} + 2\ln t + 2t + \sum_{n=3}^{\infty}\frac{2^n t^{n-1}}{(n-1)\,\underline{|n}}\right]_\epsilon^z$$

Hence,

$$\begin{aligned}-I = &\Big(\lim_{\epsilon\to 0}\Big[e^{-2\epsilon}\left(\frac{\epsilon^2}{2}+\frac{\epsilon}{2}+\frac{1}{4}\right)\Big(\frac{-1}{z}+2\ln z+2z+\sum_{n=3}^{\infty}\frac{2^n z^{n-1}}{(n-1)\,\underline{|n}}\\ &+\frac{1}{\epsilon}-2\ln\epsilon-2\epsilon-\sum_{n=3}^{\infty}\frac{2^n\epsilon^{n-1}}{(n-1)\,\underline{|n}}\Big)+\Big(\frac{z}{2}+\frac{\ln z}{2}-\frac{1}{4z}-\frac{\epsilon}{2}-\frac{\ln\epsilon}{2}+\frac{1}{4\epsilon}\Big)\Big]\\ = &-\frac{1}{2}\sum_{n=3}^{\infty}\frac{(2z)^{n-1}}{(n-1)\,\underline{|n}},\end{aligned}$$

and (4.3.7) gives

$$(4.3.8)\qquad u_1(z) = \alpha e^{-z}\left[z - \frac{\lambda+1}{4}\sum_{n=3}^{\infty}\frac{(2z)^n}{(n-1)\,\underline{|n}}\right].$$

Application of the first of the conditions (4.3.2) gives

$$(4.3.9)\qquad \lambda^{(1)}_{0,1}(z_0) = -1 + \frac{4z_0}{\sum_{n=3}^{\infty}\frac{(2z_0)^n}{(n-1)\,\underline{|n}}}.$$

The second approximation to the integral equation is obtained by putting (4.3.8) back into (4.2.11):

$$u_2(z) = u_1(z) + \lim_{\epsilon\to 0}\left\{\frac{(\lambda+1)^2}{4}\alpha z e^{-t}\int_\epsilon^z x e^{-2x}\sum_{n=3}^{\infty}\frac{(2x)^n}{(n-1)\,\underline{|n}}\int_x^z\frac{e^{2t}dt\,dx}{t^2}\right\}.$$

Again integrating by parts and letting $\epsilon\to 0$, we get

$$(4.3.10)\qquad u_2(z) = \alpha e^{-z}\left[z - \frac{\lambda+1}{4}\sum_{n=3}^{\infty}\frac{(2z)^n}{(n-1)\,\underline{|n}} + \frac{(\lambda+1)^2}{16}\sum_{n=4}^{\infty}\frac{n}{n-2}\sum_{r=n+1}^{\infty}\frac{(2z)^r}{(r-1)\,\underline{|r}}\right]$$

Upon making the approximation $(\lambda+1)^2 = (\lambda^{(1)}+1)^2$, where $\lambda^{(1)}$ is given by (4.3.9), and applying (4.3.2), we get for the second approximation to the eigenvalue,

$$(4.3.11)\qquad \lambda^{(2)}_{0,1}(z_0) = \lambda^{(1)}_{0,1}(z_0) + \frac{4z_0^2\sum_{n=4}^{\infty}\frac{n}{n-2}\sum_{r=n+1}^{\infty}\frac{(2z_0)^r}{(r-1)\,\underline{|r}}}{\left[\sum_{n=3}^{\infty}\frac{(2z_0)^n}{(n-1)\,\underline{|n}}\right]^3}.$$

The result (4.3.9) is in essential agreement with that obtained by de Groot and ten Seldam (2), and in fact the calculated eigenvalues fit the curve of Sommerfeld and Welker (cf. the bottom of page 46) better than the results of de Groot and ten Seldam. The values of $\lambda^{(2)}$ are within one percent of the correct value when z_0 is at least five times the Bohr radius.

Sums of the type appearing in (4.3.9) and (4.3.11) are most easily handled by using a method of de Groot and ten Seldam, which depends upon the properties of the exponential integral

$$Ei_m(x) = \int_1^x e^x x^{-m} dx,$$

which is tabulated (6).

4.4 The Bounded Rigid Rotator.

For the general rotator problem in three-space, a mass particle μ is restricted to rotate at a constant distance a from the origin, and for the bounded problem, μ is not allowed to enter a cone defined by an azimuthal angle. In other words, there is an infinitely high and infinitely steep potential wall on the surface of the cone, and in solving the quantum mechanical problem, it is required that the wave function vanish there. This rotator problem has been considered graphically by Sommerfeld and Hartmann (13), who used the "one-sided boundary conditions" that μ be restricted from entering only the lower nappe of the cone; that is, they applied the boundary condition that the wave function vanish only for $\theta = \theta_0$, where θ_0 is an angle near π . They obtained the eigenvalues graphically by constucting nodal curves analogous to the curves used by

Sommerfeld and Welker in the hydrogen atom problem, (cf. section 4.3) and also arrived at an analytic result for the ground level in the limiting case that θ_0 is near π. They gave references to the origin of the problem leading back to a paper by Pauling (11) in 1926.

In spherical polar coordinates r, θ, and φ, the Schrodinger wave equation for the rigid rotator is

$$-\frac{\hbar^2}{2\mu a^2}\frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right\}=E\psi,$$

where the fact that r has the constant value a has been used. The usual separation of the variables is

$$\psi(\theta,\varphi) = \Theta(\theta)\,e^{im\varphi} \qquad (m=0,\pm1,\pm2,\cdots)$$

where $\Theta(\theta)$ satisfies the equation

(4.4.1) $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[\lambda-\frac{m^2}{\sin^2\theta}\right]\Theta(\theta)=0$$

with $\lambda = \dfrac{2\mu a^2 E}{\hbar^2} = \dfrac{2J}{\hbar^2}E$.

By (1.4.6), the change of variable

(4.4.2) $$u(x) = \sin^{\frac{1}{2}}\theta\;\Theta(\theta) \quad ; \quad x=\theta$$

transforms (4.4.1) into

(4.4.3) $$u''(x)+\left[\frac{\frac{1}{4}-m^2}{\sin^2 x}+\lambda+\frac{1}{4}\right]u(x)=0 .$$

The boundary conditions to be satisfied by the solutions of (4.4.3) are

(4.4.4) $$u(0) \text{ finite} \quad ; \quad u(x_0)=0,$$

where $\frac{\pi}{2} < x_0 \leqslant \pi$.

Now, it is known that solutions of the equation

(4.4.5) $$v''(x)+\left[\frac{\frac{1}{4}-m^2}{\sin^2 x}+\lambda_0+\frac{1}{4}\right]v(x)=0$$

satisfying the conditions

(4.4.6) $\quad v(0)$ Finite ; $\quad v(\pi) = 0$

are

(4.4.7) $\quad v_1(x) = R_j^m(x) = \sin^{\frac{1}{2}} x \; P_j^m(\cos x),$

where $P_j^m(\xi)$ are the associated Legendre Functions, and that the corresponding eigenvalues are $\lambda_0(j) = j(j+1)$ for $j = 0, 1, 2, \cdots$. Hence, reference to (4.2.10) shows us that $u(z)$ satisfies the integral equation

(4.4.8) $$u(z) = \alpha \sin^{\frac{1}{2}} z \, P_j^m(\cos z) + (\lambda_0 - \lambda)\sin^{\frac{1}{2}} z \, P_j^m(\cos z)\int_0^z \sin^{\frac{1}{2}} x \, P_j^m(\cos x) u(x) \int_x^z \frac{dt \, dx}{\sin t\,[P_j^m(\cos t)]^2}$$

in which $\beta = 0$ by the first of conditions (4.4.4). The existence of the solution of this equation is guaranteed by Theorem 1 of Section 1.5.

4.5 The Ground Level Of The Bounded Rigid Rotator.

For the ground level,

$$m = j = \lambda_0(j) = 0; \qquad P_j^m(\cos x) = 1$$

so that (4.4.8) becomes

(4.5.1) $$u(z) = \alpha \sin^{\frac{1}{2}} z - \lambda \sin^{\frac{1}{2}} z \int_0^z \sin^{\frac{1}{2}} x \; u(x) \int_x^z \frac{dt}{\sin t} .$$

The first approximation to the solution of (4.5.1) is

(4.5.2) $$u_1(z) = \alpha \sin^{\frac{1}{2}} z \left[1 - \lambda \sin^{\frac{1}{2}} z \int_0^z \sin x \left\{ \ln \frac{1-\cos z}{\sin z} - \ln \frac{1-\cos x}{\sin x} \right\} dx \right]$$

or, upon making the subsitutions

(4.5.3) $$y = \cos x \quad , \qquad \omega = \cos z \quad ,$$

$$u_1 = \alpha (1-\omega^2)^{\frac{1}{4}} \left[1 + \frac{\lambda}{2} \int_1^\omega \left[\ln \frac{1-\omega}{1+\omega} - \ln \frac{1-y}{1+y} \right] dy \right.$$
$$= \alpha (1-\omega^2)^{\frac{1}{4}} \left[1 + \lambda \ln \frac{1+\omega}{2} \right]$$

or

(4.5.4) $$u_1(z) = \alpha \sin^{\frac{1}{2}} z \left[1 + \lambda \ln \frac{1+\cos z}{2} \right] .$$

Application of the second of conditions (4.5.4) gives for the first approximation to the ground level eigenvalue, (4.5.5) $$\lambda_{0,0}^{(1)}(z_0) = \frac{-1}{\ln\frac{\eta}{2}} ,$$

where

(4.5.6) $$\eta = 1+\cos z_0 .$$

The second approximation to the solution of (4.5.1) is obtained by putting (4.5.4) back into the right side of (4.5.1) :

$$u_2(z) = u_1(z) - \alpha\,\lambda^2 \sin^{\frac{1}{2}} z \int_0^z \sin x\,[\ln(1+\cos x) - \ln 2]\int_x^z \frac{dt}{\sin t}\,dx .$$

Again making the substitution (4.5.3), we obtain

$$u_2(z) = u_1(z) + \alpha(1-\omega^2)^{\frac{1}{4}}\frac{\lambda^2}{2}\int_1^\omega [\ln(1+y)-\ln 2][\ln\tfrac{1-\omega}{1+\omega} - \ln\tfrac{1-y}{1+y}]\,dy ,$$

or

(4.5.7) $$u_2 = u_1 + \alpha(1-\omega^2)^{\frac{1}{4}}\frac{\lambda^2}{2}\left\{\int_1^\omega \ln(1+y)[\ln\tfrac{1-\omega}{1+\omega} - \ln\tfrac{1-y}{1+y}]dy - 2\ln 2[\ln(1+\omega)-\ln 2]\right\}.$$

In order to evaluate the expression

(4.5.8) $$I = \ln\frac{1-\omega}{1+\omega}\int_1^\omega \ln(1+y)dy - \int_1^\omega \ln(1+y)\ln(1-y)dy + \int_1^\omega \ln^2(1+y)\,dy$$

on the right side of (4.5.7), we need the following results:

(4.5.9) $$\int_1^\omega \ln(1+y)dy = (1+\omega)\ln(1+\omega) - (1+\omega) - 2\ln 2 + 2 ;$$

(4.5.10) $$\int_1^\omega \ln^2(1+y)dy = (1+\omega)\ln^2(1+\omega) - 2(1+\omega)\ln(1+\omega) + 2(1+\omega) - 2\ln^2 2 + 4\ln 2 - 4 ;$$

(4.5.11) $$\int_1^\omega \ln(1+y)\ln(1-y)\,dy = (1+\omega)\ln(1+\omega)\ln(1-\omega) + (1-\omega-2\ln 2)\ln(1-\omega)$$
$$- (1+\omega)\ln(1+\omega) + (1+\omega)$$
$$+ 2\sum_{n=2}^{\infty}\left(\frac{1-\omega}{4}\right)^n + 2\ln 2 - 2 .$$

Putting these into (4.5.8) and simplifying, we obtain

(4.5.12) $$I = -2\ln(1+\omega) + \omega + 2\ln 2 - 1$$
$$- 2\sum_{n=2}^{\infty}\left(\frac{1-\omega}{4}\right)^n + 2\ln 2\,(\ln(1+\omega) - \ln 2) .$$

Since ω is close to -1, the summation on the right side of (4.5.12) is close to $-2\{\zeta(2)-1\}$, where $\zeta(s)$ is the Riemann Zeta-Function. Putting (4.5.12) into (4.5.7), and using the value $\zeta(2) = \pi^2/6$, we get

$$u_2 = u_1 + \alpha(1-\omega^2)^{\frac{1}{4}}\frac{\lambda^2}{2}\left[-2\ln(1+\omega) + \omega + 2\ln 2 - 1 - 2\left(\frac{\pi^2}{6} - 1\right)\right]$$

or

(4.5.13) $$u_2(z) = \alpha_1 \sin^{\frac{1}{2}} z \left\{ 1 + \lambda \ln \frac{1+\cos z}{2} - \lambda^2 \left[\ln \frac{1+\cos z}{2} + 1.145 - \frac{\cos z}{2} \right] \right\}.$$

By applying the second of conditions (4.4.4) to equations (4.5.13) and using the approximation $[\lambda_{0,0}]^2 = [\lambda_{0,0}^{(1)}]^2$, where $\lambda_{0,0}^{(1)}$ is given by (4.5.5), we obtain for the second approximation to the eigenvalue

(4.5.14) $$\lambda_{0,0}^{(2)}(z_0) = \lambda_{0,0}^{(1)}(z_0) + \frac{\ln \frac{\eta}{2} - \frac{\eta}{2} + 1.645}{(\ln \frac{\eta}{2})^3},$$

where η is given by (4.5.6).

Again, the third approximation to the eigenvalue is found to be

(4.5.15) $$\lambda_{0,0}^{(3)}(z_0) = \lambda_{0,0}^{(2)}(z_0) + \Delta \lambda_{0,0}^{(3)}(z_0),$$

where

$$0 > \Delta \lambda_{0,0}^{(3)}(z_0) > \frac{0.290 \ln \frac{\eta}{2} - 2.290}{2 (\ln \frac{\eta}{2})^4}.$$

In the limit $\eta \to 0$, equation (4.5.14) reduces to

(4.5.16) $$\lambda_{0,0}^{(S.H.)}(z_0) = \frac{-1}{\ln \frac{\eta}{2}} + \frac{1}{(\ln \frac{\eta}{2})^3},$$

which is the result obtained by Sommerfeld and Hartmann (13) by a different method. The following table gives, for small values of η, a comparison of the eigenvalues obtained from (4.5.16) and those from the first three approximations (4.5.5), (4.5.14) and (4.5.15).

TABLE I

$\pi - z_0$	η	$\lambda_{0,0}^{(1)}$	$\lambda_{0,0}^{(2)}$	$\lambda_{0,0}^{(3)}$	$\lambda_{0,0}^{(S.H.)}$
$0°48'$	10^{-4}	0.1010	0.1095	0.1091	0.1112
$2°34'$	10^{-3}	0.1315	0.1450	0.1443	0.1488
$8°6'$	10^{-2}	0.1888	0.2126	0.2102	0.2243
$11°27'$	2×10^{-2}	0.2171	0.2475	0.2436	0.2643
$18°11'$	5×10^{-2}	0.2711	0.3122	0.3032	0.3446
$25°51'$	10^{-1}	0.3338	0.3900	0.3660	0.4449

4.6 Higher Energy Levels Of The Bounded Rigid Rotator.

For the 1-1 level,

$$m = j = 1 \, ; \qquad \lambda_0 = 2 \, ; \qquad P_j^m\,(\cos x) = \sin x \, .$$

Putting these values into (4.4.8), and following the same procedure that was used in Section 4.5, we get for the first and second approximations to the eigenvalue

$$\lambda_{1,1}^{(1)}(z_0) = 2 - \frac{3}{\ln\frac{\eta}{2} - \frac{1}{\eta} + \frac{1}{2}} \qquad (\eta = 1 + \cos z_0), \tag{4.6.1}$$

$$\lambda_{1,1}^{(2)}(z_0) = \lambda_{1,1}^{(1)}(z_0) + \frac{\frac{11}{54}\frac{1}{\eta} - \frac{1}{9}\frac{\ln\frac{\eta}{2}}{2-\eta} + \frac{1}{9}\ln\frac{\eta}{2} - 0.188}{\left[\ln\frac{\eta}{2} - \frac{1}{\eta} + \frac{1}{2}\right]^3},$$

or for small values of η ,

$$\lambda_{1,1}^{(2)}(z_0) = \lambda_{1,1}^{(1)}(z_0) + \frac{\frac{11}{54}\frac{1}{\eta} + \frac{1}{18}\ln\frac{\eta}{2}}{\left[\ln\frac{\eta}{2} - \frac{1}{\eta} + \frac{1}{2}\right]^3} \, . \tag{4.6.2}$$

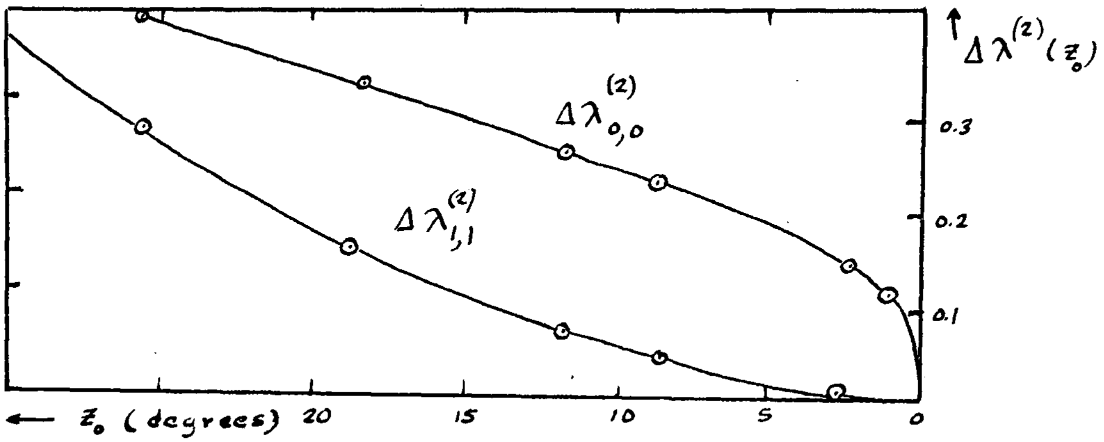
The rapid convergence of the successive approximations for small values of η is illustrated in the following table: $\left(\Delta\lambda_{1,1}^{(2)} = \lambda_{1,1}^{(2)} - 2\right)$

TABLE 2

$\pi - z_0$	η	$\lambda_{1,1}^{(1)}$	$\lambda_{1,1}^{(2)}$	$\Delta\lambda_{1,1}^{(2)}$
$2^\circ 34'$	10^{-3}	2.0029	2.0029	0.0029
$8^\circ 6'$	10^{-2}	2.0285	2.0291	0.0291
$11^\circ 27'$	2×10^{-2}	2.0549	2.0565	0.0565
$18^\circ 11'$	5×10^{-2}	2.1266	2.1343	0.1343
$25^\circ 51'$	10^{-1}	2.230	2.253	0.253
$36^\circ 50'$	2×10^{-1}	2.401	2.465	0.465

The following graph compares the results of TABLES 1 and 2:

GRAPH OF EIGENVALUES FOR THE (0,0) and (1,1) ENERGY LEVELS



The graphs illustrate that, as $\eta \to 0$, the increments in the eigenvalues of the bounded rotator approach those of the free rotator for the (0,0) and (1,1) energy levels. Further, there are vertical and horizontal tangents respectively to the (0,0) and (1,1) curves at the origin. For higher energy levels, we have calculated only the first approximation to the eigenvalues. The results are tabulated below.

TABLE 3

Level(m,j)	$P_j^m(\cos x)$	$\lambda_0(j)$	$\Delta\lambda^{(1)}_{m,j}$
(2,2)	$\sin^2 x$	6	$-3/\left(\ln\frac{\eta}{2} - \frac{1}{3}\frac{1}{\eta^2}\right)$
(3,3)	$\sin^3 x$	12	$-7/\left(\ln\frac{\eta}{2} - \frac{1}{15}\frac{2}{\eta^3}\right)$
(4,4)	$\sin^4 x$	20	$-9/\left(\ln\frac{\eta}{2} - \frac{1}{105}\frac{6}{\eta^4}\right)$
(r,r)	$\sin^r x$	$r(r+1)$	$-(2r+1)/\left(\ln\frac{\eta}{2} - \frac{1}{(3)(5)\cdots(2r+1)}\frac{K_r}{\eta^r}\right.$
(0,1)	$\cos x$	2	$-3/\left(\ln\frac{\eta}{2} - \frac{1}{\eta+1} + 1\right)$
(0,2)	$2-3\sin^2 x$	6	$-5/\left(\ln\frac{\eta}{2} - \frac{7}{9}\frac{1}{\eta^2}\right)$
(1,2)	$\sin x\cos x$	6	$-15/\left(\ln\frac{\eta}{2} - \frac{1}{\eta}\right)$
(2,3)	$\sin^2 x\cos x$	12	$-7/\left(\ln\frac{\eta}{2} - \frac{1}{105}\frac{1}{\eta^2}\right)$

In TABLE 3, the extreme right hand column involves only the highest power of η which appears in the exact formulae for $\lambda^{(1)}_{m,j}$. Also, for the (r,r) level, K_r is a constant which can be determined for each particular value of r; for example, $K_1 = 1$; $K_2 = 1$; $K_3 = 2$; $K_4 = 6$.

The eigenvalues obtained from the formulae in the extreme right hand column again approach those of the free rotator as $\eta \to 0$. The tangents to the curves are vertical for the (r,0) curves, ($r = 0, 1, 2, \cdots$), and horizontal for the other curves. The qualitative results are in agreement with those obtained graphically by Sommerfeld and Hartmann (13).

In conclusion, the last chapter gives a systematic procedure for reformulating a given quantum mechanical problem as a Volterra Integral Equation. Reference to Chapter 1 has shown that, under quite general conditions, the sequence of successive approximations associated with such an equation converges to the unique continuous solution of the equation. Although computational difficulty in evaluating certain definite integrals has prevented us from obtaining general results, we have nevertheless demonstrated the use of the method in evaluating eigenvalues for the Hydrogen atom problem and the bounded rigid rotator problem.

BIBLIOGRAPHY

1. R.R. Coveyou and T.W. Mulliken, U.S. Atom. En. Comm. Rep. AECD-2407, (1948)
2. S.R. de Groot and C.A. ten Seldam, Physica, 12, 669, (1946)
3. F. Fubini, Rend. Lincei, 6, 26, 253, (1937)
4. Y. Ikeda, Math. Zeit., 22 , 16, (1925)
5. E.L. Ince, Ordinary Differential Equations, Longmans and Green, (1927)
6. E. Jahnke and F. Emde, Tables of Functions with Formulae and Curves, Leipzig-Berlin, B.G. Teubner, (1938)
7. J. Liouville, J. de Math., 2, 24, (1837)
8. W.V. Lovitt, Linear Integral Equations, 1st edition, New York, McGraw-Hill (1924)
9. M. Karlin, J. Math. Phys., 28, 43, (1949)
10. A. Michels, J. de Boer, and A. Bijl, Physica, 4, 981 (1937)
11. E Pauling, Phys. Rev., 27, 568, (1926)
12. H.P. Robinson, Math. Ann., 98, 749, (1927)
13. A. Sommerfeld and H. Hartmann, Ann. Phys., 5, 37, 333, (1940)
14. A. Sommerfeld and H. Welker, Ann. Phys. 5, 32,56, (1938)
15. F. Tricomi, Ann. Mat. Pura. App., 4, 26, 141, (1947)
    F. Tricomi, Comment. Math. Helv., 22, 150, (1949)
    F. Tricomi, Ann. Mat. Pura. App. 4, 28, 263 (1949)
    F. Tricomi, Univ. e Politecnico Torino, Rend. Sem. Mat. 8, 7, (1949)

16. D.A. Trumpler, Masters Thesis, Univ. of B.C., (1953)

17. G.N. Watson, Bessel Functions, 2-nd edition, Camb. Univ. Press, (1952)

18. E.T. Whittaker and G.N. Watson, Modern Analysis, 4-th edition, Camb. Univ. Press, (1952)