MAXIMUM AND MINIMUM PROBLEMS IN FUNCTIONS OF QUADRATIC FORMS

by

ROY WESTWICK

A Thesis Submitted In Partial Fulfilment

Of The Requirements For The Degree Of

MASTER OF ARTS

In The Department of Mathematics

We accept this thesis as conforming to the standard required for candidates for the degree of MASTER OF ARTS.

THE UNIVERSITY OF BRITISH COLUMBIA

April 1957

ABSTRACT

Let A be an $n \times n$ hermitian matrix, let $E_2(a_1, \ldots, a_k)$ be the second elementary symmetric function of the letters $a_1, \ldots, a_k$ and let $C_2(A)$ be the second compound matrix of A. In this thesis the maximum and minimum of $\det \{(Ax_i, x_j)\}$ and $E_2 [(Ax_1, x_1), \ldots, (Ax_k, x_k)]$ and the minimum of $\sum_{1 \leq i_1 < i_2 \leq k} (C_2(A)x_{i_1} \wedge x_{i_2}, x_{i_1} \wedge x_{i_2})$ are calculated. The maxima and minima are taken over all sets of k orthonormal vectors in unitary n-space and $x_{i_1} \wedge x_{i_2}$ designates the Grassman exterior product. These results depend on the inequality $E_2(a_1, \ldots, a_k) \leq \binom{k}{2} \left(\frac{E_1(a_1, \ldots, a_k)}{k}\right)^2$ which is here established for arbitrary real numbers, and on the minimum of $E_2 (x_1, \ldots, x_k)$ where the minimum is taken over all values of $x_1, \ldots, x_k$ such that $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} \alpha_i$ and $\sum_{i=1}^{q} x_{s_i} \leq \sum_{i=1}^{q} \alpha_i$ for all sets of q distinct integers $s_1, \ldots, s_q$ taken from $1, \ldots, k$. Here $\alpha_1 \geq \ldots \geq \alpha_k$.

ACKNOWLEDGMENTS

I wish to express my sincere thanks to Dr. M.D. Marcus for the help he gave me when working on this thesis and to both Dr. Marcus and Dr. Moyls for their suggestions and their time taken up in reading through the thesis. I am also indebted to the National Research Council of Canada for the financial assistance that made this study possible.

TABLE OF CONTENTS

MAXIMUM AND MINIMUM PROBLEMS IN FUNCTIONS OF QUADRATIC FORMS

I INTRODUCTION

The main results contained in this thesis are the solutions to the problems of determining;

$$\min \text{ and } \max \det (Ax_i, x_j) , \ 1 \leq i, \ j \leq k;$$

$$\min \text{ and } \max E_2\left[(Ax_1, x_1) , \cdot \cdot \cdot , (Ax_k, x_k)\right];$$

$$\min \sum_{1\leq i_1<i_2\leq k} \left(C_2(A)x_{i_1} \wedge x_{i_2} , x_{i_1} \wedge x_{i_2} \right) ;$$

where A is an n by n non-singular Hermitian matrix and the vectors x_j are constrained to run over orthonormal (o.n.) sets.

To this end two lemmas and a corollary on elementary symmetric functions are established.

Definition 1.1 <u>Let</u> $1 \leq p \leq n$ <u>be positive integers. The elementary symmetric function of degree</u> p <u>on the</u> k <u>letters</u> $a_1, \cdot \cdot \cdot , a_k$ <u>is the coefficient of</u> t^{k-p} <u>in</u>

$$\prod_{i=1}^{k} (t + a_i)$$

and written as $E_p(a_1, \cdots, a_k)$.

It will also be convenient to define:

$$E_p(\hat{a}_i) = E_p(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_k).$$

Definition 1.2 Let $x_1, \cdots, x_p$ be any vectors in V_n where $p \leq n$ and,

$$x_1 = (x_{11}, \cdots, x_{1n})$$
$$x_p = (x_{p1}, \cdots, x_{pn}) .$$

Consider the matrix

$$(1.1) \quad \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ x_{p1} & \cdots & x_{pn} \end{pmatrix}.$$

Construct the $\binom{n}{p}$ possible p by p determinants from this matrix by selecting p columns at a time. Arrange these numbers in lexicographic order according to the manner of selection of the columns from (1.1). Define this $\binom{n}{p}$ vector to be $x_1 \wedge \cdots \wedge x_p \in V_{\binom{n}{p}}$.

Definition 1.3 For $p \leq n$ define

$$Q_{pn} = \left\{ \{i_1, \cdots, i_p\} \;\middle|\; 1 \leq i_1 < \cdots < i_p \leq n \right\}$$

that is Q_{pn} has as elements sets of p distinct integers taken from $1, 2, \cdots, n$ in increasing order.

By x_ω we shall mean

$$x_\omega = x_{i_1} \wedge \cdots \wedge x_{i_p} \quad ; (i_1, \cdots, i_p) \in Q_{pn} .$$

Definition 1.4 Let A be a linear transformation on V_n to V_n. Define the linear transformation $C_p(A)$ on $V_{\binom{n}{p}}$ to $V_{\binom{n}{p}}$ by defining its effect on the basis set $\left\{ e_\omega \;\middle|\; \omega \in Q_{pn} \right\}$ to be,

$$C_p(A)\, e_{i_1} \wedge \cdots \wedge e_{i_p} = Ae_{i_1} \wedge \cdots \wedge Ae_{i_p}$$

where e_s is the n vector with δ_{is} in the i^{th} entry.

For a detailed discussion of the theory and applications of Compound Matrices see [6] .

II THE GENERAL PROBLEM

In the rest of the thesis by A_i we shall mean (Ax_i, x_i) .

The general problem is to examine the algebraic structure of extremal sets of o.n. vectors for

$$(2.1) \qquad E_{r}(A_1, \cdots, A_k)$$

$$(2.2) \qquad \sum_{\omega \in Q_{kn}} (C_k(A)x_\omega, x_\omega)$$

and to compute the extreme values.

For A semi-definite both problems have been completely solved for both max and min. [3], [5]

The max and min for $E_1(A_1, \cdots, A_k)$ is known for A simply Hermitian [2].

For r = k in (2.1) we are dealing with the product of k quadratic forms. The invariance under A of $L(x_1, \cdots, x_k)$, the sub-space spanned by an extremal set x_j, has been obtained for A non-singular Hermitian and the extreme values have been calculated [4].

For k = n in (2.2) we are dealing with det (Ax_i, x_j) ; $1 \leq i, j \leq n$. Max min and min max results have been established for A definite [1].

III RESULTS

Theorem 3.1 Let A be non-singular Hermitian on V_n to V_n. Let $x_1, \cdots, x_k$ be an o.n. set of vectors in V_n. Let P be the orthogonal projection of V_n to $L(x_1, \cdots, x_k)$. Let $\lambda_1, \cdots, \lambda_k$ be the eigenvalues of PAP in L and $u_1, \cdots, u_k$ the corresponding eigenvectors. Then,

$$\det (Ax_i, x_j) = \prod_{j=1}^{k} \lambda_j = \prod_{j=1}^{k} (Au_j, u_j) \quad .$$

Proof:

Since x_i, x_j are in L

$$(Ax_i, x_j) = (APx_i, Px_j) = (PAPx_i, x_j)$$

Hence

$$\det (Ax_i, x_j) = \det (PAPx_i, x_j) \quad .$$

Now

$$x_\alpha = \sum_{j=1}^{k} (x_\alpha, u_j)u_j \quad .$$

Let

$$X = \begin{pmatrix} (x_1, u_1) & \cdots & (x_1, u_k) \\ \vdots & & \vdots \\ (x_k, u_1) & \cdots & (x_k, u_k) \end{pmatrix} = \left\{ (x_i, u_j) \right\}$$

$$\Gamma = \begin{pmatrix} \lambda_1 & & & \\ & \cdot & 0 & \\ & 0 & \cdot & \\ & & & \lambda_k \end{pmatrix}$$

$$(PAPx_\alpha, x_\beta) = (\sum_{j=1}^{k}(x_\alpha, u_j)\,\lambda_j u_j\,,\ \sum_{j=1}^{k}(x_\beta, u_j)u_j)$$

$$= \sum_{j=1}^{k}\lambda_j(x_\alpha, u_j)\overline{(x_\beta, u_j)}$$

$$= (X\Gamma X^*)_{(\alpha,\beta)}$$

Hence

$$\det(PAPx_\alpha, x_\beta) = \det(X\Gamma X^*) = \det(XX^*\Gamma)$$

$$= \det(\Gamma) = \prod_{j=1}^{k}\lambda_i$$

since

$$XX^* = \left\{ (\sum_{j=1}^{k}(x_\alpha, u_j)\overline{(x_\beta, u_j)})_{\alpha\beta} \right\}$$

$$= \left\{ (\sum_{j=1}^{k}(x_\alpha, u_j)u_j\,,\ \sum_{j=1}^{k}(x_\beta, u_j)u_j\,)_{\alpha\beta} \right\}$$

$$= \left\{ (x_\alpha, x_\beta)_{\alpha\beta} \right\}$$

$$= \left\{ (\delta_{\alpha\beta})_{\alpha\beta} \right\}$$

$$= I\,.$$

Theorem 3.2 Let $f(x_1, \cdot\cdot\cdot, x_k) = \det(Ax_i, x_j)$ where A is n by n non-singular Hermitian neither non-negative nor non-positive and where $x_1, \cdot\cdot\cdot, x_k$ is an o.n. set of vectors. If $x_1, \cdot\cdot\cdot, x_k$ is an extremal set then $L(x_1, \cdot\cdot\cdot, x_k)$ is invariant under A. $(k < n)$.

Proof: Define u_j and P as in theorem 1 for the subspace $L(x_1, \ldots x_k)$. Then

$$\det (Ax_i, x_j) = \det (PAPx_i, x_j) = \det(PAPu_i, u_j)$$

and

$$L(x_1, \ldots, x_k) = L(u_1, \ldots, u_k) \quad .$$

Assume $L(x_1, \ldots, x_k)$ is not invariant, and say $\rho = (Au_1, v) \neq 0$ where $v \in L^{\perp}(u_1, \ldots, u_k)$, $\|v\| = 1$.

Set

$$u'_1 = \frac{u_1 - t\rho v}{\sqrt{1 + t^2 |\rho|^2}}$$

$$u'_i = u_i \ , \ i = 2, \ldots, k \quad .$$

Consider $f(u'_1, \ldots u'_j)$

$$= \frac{1}{1 + t^2 |\rho|^2} \begin{vmatrix} (Au'_1, u'_1) & \cdots\cdots & (Au'_1, u'_k) \\ \vdots & & \\ (Au'_k, u'_1) & \cdots\cdots & (Au'_k, u'_k) \end{vmatrix}$$

$$= \frac{1}{1 + t^2 |\rho|^2} \begin{vmatrix} (Au'_1, u'_1) & (Au'_1, u_2) & \cdots & (Au'_1, u_k) \\ (Au_2, u'_1) & (Au_2, u_2) & & \\ \vdots & & \ddots & 0 \\ \vdots & 0 & & \ddots \\ (Au_k, u'_1) & & & (Au_k, u_k) \end{vmatrix}$$

Since the u_i are an extremal set we must have

$$\left(\frac{\partial f}{\partial t}\right)_{t = 0} = 0$$

Note that $(Au_j, u'_1)_{t=0} = (Au_j, u_1) = (PAPu_j, u_1) = 0$ for $j > 1$.

Note also that $\frac{\partial}{\partial t} (Au'_1, u'_1)_{t=0} = -2 |\rho|^2$

Hence

$$\left(\frac{\partial f}{\partial t}\right)_{t=0} = \begin{vmatrix} -2|\rho|^2 & \sim & \cdots & \sim \\ 0 & (Au_2, u_2) & & 0 \\ \vdots & & \ddots & \\ \vdots & 0 & & \ddots \\ 0 & & & (Au_k, u_k) \end{vmatrix}$$

All other terms are zero since differentiating the i^{th} row $(i > 1)$ and setting $t = 0$ gives rise to a determinant with all zeros in the i^{th} column. Hence

$$\left(\frac{\partial f}{\partial t}\right)_{t=0} = -2|\rho|^2 \left[\prod_{j=2}^{k} (Au_j, u_j)\right] = 0$$

and since $|\rho| \neq 0$

$$\prod_{j=2}^{k} (Au_j, u_j) = 0$$

Thus either max det $(Ax_i, x_j) = 0$ or min det $(Ax_i, x_j) = 0$ which implies either det $(Ax_i, x_j) \leq 0$ or det $(Ax_i, x_j) \geq 0$ for all choices of $x_1, \ldots, x_k$. Clearly det (Ax_i, x_j) takes on all possible products of k of the eigenvalues of A. This implies that the eigenvalues are all of the same sign since the product of any k of them must be of the same sign. But this implies that A is definite contrary to hypothesis. Therefore L is invariant under A. Therefore we have

$$\min\ (\max)\ \det\ (Ax_i, x_j) = \min\ (\max) \prod_{i=1}^{k} \lambda_{r_i}$$

where $(r_1, \ldots, r_k) \in Q_{kn}$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

Lemma 3.3 <u>For</u> $a_1, \ldots, a_k$ <u>arbitrary and real</u>

$$E_2(a_1, \ldots, a_k) \leq \binom{k}{2}\left(\frac{E_1(a_1, \ldots, a_k)}{k}\right)^2$$

Proof: Let

$$E_i = E_i(a_1, \ldots, a_k) \quad , i = 1, 2$$

$$E_i^{\wedge} = E_i(a_1, \ldots, a_{k-1}) \quad , i = 1, 2$$

$$f_k(a_k) = E_2 - \binom{k}{2}\left(\frac{E_1}{k}\right)^2 = E_2 - \frac{k-1}{2k} E_1{}^2 .$$

Assume the lemma is true for $k - 1$ numbers $a_1, \ldots, a_{k-1}$.

$$\frac{d}{da_k} f_k(a_k) = \frac{d}{da_k}\left[a_k E_1^{\wedge} + E_2^{\wedge} - \frac{k-1}{2k}\left[a_k + E_1^{\wedge}\right]^2 \right]$$

$$= E_1^{\wedge} - \frac{k-1}{k}(a_k + E_1^{\wedge})$$

and

$$\frac{d^2}{da^2} f_k(a_k) = -\frac{k-1}{k} < 0 \text{ everywhere.}$$

Hence $f_k(a_k)$ is a maximum at $\frac{d}{da_k} f_k(a_k) = 0$, that is when $a_k = \frac{1}{k-1} E_1^{\wedge}$.

At this point

$$f_k(a_k) = \frac{1}{k-1}(E_1^{\wedge})^2 + E_2^{\wedge} - \frac{k-1}{2k}\left(\frac{k}{k-1}\right)^2 (E_1^{\wedge})^2$$

$$= E_2^{\wedge} - \frac{k-2}{2(k-1)}(E_1^{\wedge})^2$$

$$= E_2^{\wedge} - \frac{k-2}{2}\left(\frac{E_1^{\wedge}}{k-1}\right)^2$$

$$\leq 0 .$$

For $k = 2$ we have $f_2 = a_1 a_2 - \left(\frac{a_1 + a_2}{2}\right)^2 = \frac{-a_1{}^2 - a_2{}^2}{4} \leq 0$.

Lemma 3.4 Let $\alpha_1 \geq \ldots \geq \alpha_k$. If

(1) $$\sum_{s=1}^{k} x_s = \sum_{s=1}^{k} \alpha_s = R$$

(2) $$\sum_{s=1}^{j} x_{i_s} \leq \sum_{s=1}^{j} \alpha_s \quad , (i_1, \ldots, i_j) \in Q_{jk}$$

$$1 \leq j \leq k-1$$

<u>then</u>

$$\text{Min } E_2(x_1, \ldots, x_k) = E_2(\alpha_1, \ldots, \alpha_k) \ .$$

Proof: The value is taken on by setting $x_i = \alpha_i$ in which case (2) holds because $\alpha_1 + \ldots + \alpha_j$ is the sum of the largest j α's while $x_{i_1} + \ldots + x_{i_j}$ will be the sum of an arbitrary set of j α's.

Either min $E_2(x_1, \ldots, x_k)$ is taken on when all of the inequalities (2) are strict, in which case

$$(3) \qquad \frac{\partial}{\partial x_i}\left[E\,(x_1, \ldots, x_k) - \mu\left(\sum_1^k x_i - R\right)\right] = 0$$

or it is attained when for some t, $1 \leq t \leq k - 1$,

$$(4) \qquad x_{i_1} + \ldots + x_{i_t} = \alpha_1 + \ldots + \alpha_t \ ,$$

that is, when one of the inequalities becomes an equality.

Case 1. If (3) is true for each i, then

$$E_1(\hat{x}_i) - \mu = 0$$

$$R = E_1(x_1, \ldots, x_k) = x_i + E_1(\hat{x}_i),$$

$$\mu = R - x_i \ ,$$

$$k\mu = kR - \sum_{i=1}^k x_i \ ,$$

$$k\mu = (k - 1)R \ ,$$

$$\mu = \frac{k-1}{k} R \ ,$$

$$x_i = \frac{R}{k} \ ,$$

$$E_2(x_1, \ldots, x_k) = \binom{k}{2}\left(\frac{R}{k}\right)^2$$

$$= \binom{k}{2}\left(\frac{E_1(\alpha_1 \ldots \alpha_k)}{k}\right)^2$$

$$\geq E_2(\alpha_1 \ldots \alpha_k) \quad \text{by lemma 3.3} \ .$$

That is, any relative min in the interior of the region defined by (1) and (2) is $\geq E_2(\alpha_1, \ldots \alpha_k)$.

(5) Assume that for each ω, $2 \leq \omega \leq k-1$, the lemma is true for all sets $a_1 \geq \ldots \geq a_\omega$.

Case 2. (4) applies.

Define E_2 of a single number to be zero. Then

$$E_2(x_1, \ldots, x_k) = E_2(x_{i_1}, \ldots x_{i_t}) + E_2(x_{i_{t+1}}, \ldots x_{i_k})$$
$$+ (x_{i_1} + \ldots + x_{i_t})(x_{i_{t+1}} + \ldots + x_{i_k}) \; .$$

(6)
$$= E_2(x_{i_1}, \ldots x_{i_t}) + E_2(x_{i_{t+1}}, \ldots, x_{i_k})$$
$$+ (\alpha_1 + \ldots + \alpha_t)(\alpha_{t+1} + \ldots + \alpha_k) \; , \; 1 \leq t \leq k-1 \; ,$$

since (4) and (1) imply that

(7)
$$x_{i_{t+1}} + \ldots + x_{i_k} = \alpha_{t+1} + \ldots + \alpha_k \; .$$

Now,

$$E_2(x_{i_1}, \ldots, x_{i_t}) \geq E_2(\alpha_1, \ldots, \alpha_t)$$

because by (4) and (2)

$$\sum_{s=1}^{t} x_{i_s} = \sum_{s=1}^{t} \alpha_s$$

$$\sum_{s=1}^{j} x_{\beta_s} \leq \sum_{s=1}^{j} \alpha_s \qquad (\beta_1, \ldots, \beta_j) \in (i_1, \ldots, i_t)$$
$$1 \leq j \leq t-1$$

we can apply the induction step (5) for $\omega = t$ and $a_i = \alpha_i$. For $t = 1$ we have by definition $E_2(x_1) = 0 = E_2(\alpha_1)$, $E_2(x_1) \geq E_2(\alpha_1)$.

Also

$$E_2(x_{i_{t+1}}, \ldots, x_{i_k}) \geq E_2(\alpha_{t+1}, \ldots, \alpha_k)$$

because by (2)

$$x_{i_1} + \dots + x_{i_t} + x_{t+1} + \dots + x_{t+j}$$

$$\leq \alpha_1 + \dots + \alpha_t + \dots + \alpha_{t+j}$$

and therefore by (4)

$$x_{\beta_{t+1}} + \dots + x_{\beta_{t+j}} \leq \alpha_{t+1} + \dots + \alpha_{t+j}$$

together with (7) we can apply the induction step (5) for $\omega = k - t$ and $a_1 = \alpha_{t+1}, \dots, a_{k-t} = \alpha_k$.

Hence by (6)

$$E_2(x_1, \dots x_k) \geq E_2(\alpha_1, \dots, \alpha_k) + E_2(\alpha_{t+1}, \dots, \alpha_k)$$
$$+ (\alpha_1 + \dots + \alpha_t)(\alpha_{t+1} + \dots + \alpha_k)$$
$$= E_2(\alpha_1, \dots, \alpha_k) \quad .$$

To complete the induction we need to show that for $\omega = 2$ the lemma is true. In this case

$$x_1 \leq \alpha_1 \quad ,$$
$$x_2 \leq \alpha_1 \quad ,$$
$$x_1 + x_2 = \alpha_1 + \alpha_2$$

Therefore $x_1 \geq \alpha_2$, $x_2 \geq \alpha_2$ and we must show that

$$\min E_2(x_1 \; x_2) = \alpha_1 \alpha_2 \quad .$$
$$E_2(x_1 \; x_2) = x_1 \; x_2 = x_1(\alpha_1 + \alpha_2 - x_1)$$

Since $\frac{d^2}{dx_1^2}\left[x_1(\alpha_1 + \alpha_2 - x_1)\right] = -1 < 0$ in the entire interval, $\alpha_2 \leq x_1 \leq \alpha_1$, the minima are attained at $x_1 = \alpha_2$ and $x_1 = \alpha_1$ and in both cases we get $\alpha_1 \alpha_2$.

Lemma 3.4 is an extension of a result contained in [5,] to the second elementary symmetric function. If the variables were restricted to positive values then the result is contained in the result referred to above.

Corollary 3.5 If $\alpha_2 \leq -(\alpha_2 + \ldots + \alpha_k)$; $\alpha_{i+1} \leq \alpha_i$; $i = 2, \ldots k$; $\sum_{i=1}^{k} x_i = 0$ and $\sum_{s=1}^{j} x_{i_s} \leq -\sum_{s=j+1}^{k} \alpha_s$, $(i_1, \ldots i_j) \in Q_{jk}$, $1 \leq j \leq k-1$ then

$$\text{Min } E_2(x_1, \ldots, x_k) = E_2\left[\alpha_2, \ldots, \alpha_k, -(\alpha_2 + \ldots + \alpha_k)\right]$$

Proof: Set $\alpha_1 = -(\alpha_2 + \ldots + \alpha_k)$. Then $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$.

$$\sum_{s=1}^{j} \alpha_s = -(\alpha_2 + \ldots + \alpha_k) + \alpha_2 + \ldots + \alpha_j$$

$$= -\sum_{s=j+1}^{k} \alpha_s$$

Therefore we have

$$\sum_{s=1}^{j} x_{i_s} \leq \sum_{s=1}^{j} \alpha_s$$

and since

$$\sum_{s=1}^{k} \alpha_s = -(\alpha_2 + \ldots + \alpha_k) + \alpha_2 + \ldots + \alpha_k = 0$$

we have

$$\sum_{s=1}^{k} x_s = \sum_{s=1}^{k} \alpha_s \quad .$$

Apply lemma 3.4 and we have the result.

Theorem 3.6 Let A be Hermitian with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Let $M = \max\left\{ \left|\sum_{s=1}^{k} \lambda_s\right| ; \left|\sum_{s=n-k+1}^{n} \lambda_s\right| \right\}$. Then

$$\max E_2(A_1, \ldots, A_k) = \binom{k}{2}\left(\frac{M}{k}\right)^2$$

<u>where</u> <u>max</u> <u>is</u> <u>taken</u> <u>over</u> <u>all</u> <u>sets</u> <u>of</u> k o.n. <u>vectors</u> $x_1, \ldots, x_k$.

Proof: By Lemma (3.1)

$$E_2(A_1, \ldots, A_k) \leq \binom{k}{2}\left(\frac{E_1(A_1, \ldots, A_k)}{k}\right)^2 .$$

By [2]

$$\sum_{s=n-k+1}^{n} \lambda_s \leq \sum_{s=1}^{k} A_s \leq \sum_{s=1}^{k} \lambda_s ,$$

which implies

$$E_2(A_1, \ldots, A_k) \leq \binom{k}{2}\left(\frac{M}{k}\right)^2 .$$

The value is attained since there exists k o.n. vectors $y_1, \ldots, y_k$ such that

$$(Ay_i, y_i) = \frac{\sum_{s=1}^{k} \lambda_s}{k} , \quad 1 \leq i \leq k ,$$

and k o.n. vectors $z_1, \ldots z_k$ such that

$$(Az_1, z_1) = \frac{\sum_{s=n-k+1}^{n} \lambda_s}{k} , \quad 1 \leq i \leq k .$$

Theorem 3.7 <u>Let</u> A <u>be</u> n x n <u>Hermitian</u> <u>with</u> <u>eigenvalues</u> $\lambda_1 > \ldots > \lambda_n$. <u>Let</u> $x_1, \ldots, x_{k+1}$ <u>be</u> <u>an</u> o.n. <u>set</u> <u>of</u> <u>vectors</u> <u>which</u> <u>minimizes</u> $E_2(A_1, \ldots A_{k+1})$. <u>Let</u> L <u>be</u> <u>the</u> <u>subspace</u> <u>spanned</u> <u>by</u> $x_1, \ldots x_{k+1}$. <u>Then</u> L <u>is</u> <u>invariant</u> <u>under</u> A <u>if</u>:

(a) <u>Either</u> <u>of</u>

$$(1) \quad \begin{array}{l} \lambda_1 + \ldots + \lambda_k < 0 \\ \lambda_n + \ldots + \lambda_{n-k+1} > 0 \end{array}$$

<u>is true or:</u>

(b) <u>both of</u>

$$(2) \qquad \begin{aligned} &\lambda_1 + \ldots + \lambda_k > 0 \\ &\lambda_n + \ldots + \lambda_{n-k+1} < 0 \end{aligned}$$

<u>and one of</u>

$$\begin{aligned}
&\text{(i)} \quad \lambda_1 > -(\lambda_n + \ldots + \lambda_{n-k+2}) \ , \\
&\text{(ii)} \quad -(\lambda_n + \ldots + \lambda_{n-k+2}) > \lambda_1 > -(\lambda_2 + \lambda_{n-k+3} \\
&\qquad + \ldots + \lambda_n) \ , \\
(3) \quad &\text{(iii)} \quad -(\lambda_2 + \ldots + \lambda_i + \lambda_{n-k+i+1} + \ldots + \lambda_n) > \lambda_1 > \\
&\qquad -(\lambda_2 + \ldots + \lambda_{i+1} + \lambda_{n-k+i+2} + \ldots + \lambda_n) \ , \\
&\qquad 2 \leq i \leq k-2 \ , \\
&\text{(iv)} \quad -(\lambda_2 + \ldots + \lambda_{k-1} + \lambda_n) > \lambda_1 > -(\lambda_2 + \ldots + \lambda_k),
\end{aligned}$$

<u>are satisfied.</u>

Note: The theorem states that for any Hermitian matrix with distinct eigenvalues and whose largest eigenvalue λ_1 is not equal to any of the k numbers

$$\begin{aligned}
&\text{(i)} \quad -(\lambda_2 + \ldots + \lambda_k) \\
&\text{(ii)} \quad -(\lambda_2 + \ldots + \lambda_i + \lambda_{n-k+i+1} + \ldots + \lambda_n) \\
&\qquad 2 \leq i \leq k-1 \\
&\text{(iii)} \quad -(\lambda_{n-k+2} + \ldots + \lambda_n)
\end{aligned}$$

L is invariant under A.

Proof: Assume L is not invariant under A and that $Ax_{k+1} \notin L$. Therefore there exists a unit vector $v \in L^{\perp}$ such that $(Ax_{k+1}, v) = \rho \neq 0$. Let

$$x'_i = x_i \quad \text{for } i = 1, \ldots, k \ ,$$

$$x'_{k+1} = \frac{x_{k+1} - t\rho v}{\sqrt{1 + t^2|\rho|^2}} \quad ,$$

$$A'_i = (Ax'_i, x'_i) \quad .$$

Clearly $x'_1, \ldots, x'_{k+1}$ is an o.n. set of vectors. Since $x_1, \ldots, x_{k+1}$ is a minimizing set for the function $E_2(A_1, \ldots, A_{k+1})$ we have

$$\left[\frac{\partial}{\partial t} E_2(A'_1, \ldots, A'_{k+1})\right]_{t=0} = 0$$

or

$$\left[\frac{\partial}{\partial t} (A'_{k+1})\right]_{t=0} \cdot E_1(\widehat{A'_{k+1}}) = 0 \quad .$$

Since

$$\left[\frac{\partial}{\partial t} (A'_{k+1})\right]_{t=0} = -2|\rho|^2 \neq 0$$

we have

$$E_1(\widehat{A'_{k+1}}) = 0 \quad ,$$

and we arrive at the result that

(4) $$\text{Min } E_2(A_1, \ldots A_{k+1}) = E_2(A_1, \ldots, A_k)$$

and

(5) $$E_1(A_1, \ldots, A_k) = 0$$

Case (a). If one of the expressions (1) is true then by [2] one of

$$\max \sum_1^k A_i < 0$$

$$\min \sum_1^k A_i > 0$$

is true which implies that $\sum_1^k A_i \neq 0$ for any o.n. set $x_1, \ldots, x_k$.

But this contradicts (5) which in turn implies that L is invariant under A.

Case (b). Both of (2) hold and one of (3) holds.

(6) $$\sum_{s=1}^{j} A\gamma_s \leq \sum_{s=1}^{j} \lambda_s \qquad \text{by } [2]$$

and

$$\sum_{s=1}^{j} A\gamma_s = -\sum_{s=j+1}^{k} A\gamma_s \, , \qquad \text{by } (5)$$

(7) $$\sum_{s=j+1}^{k} A\gamma_s = \sum_{s=1}^{k-j} A\beta_s \geqq \sum_{s=n-k+j+1}^{n} \lambda_s \, , \text{ by } [2]$$

$$\sum_{s=1}^{j} A\gamma_s \leq -\sum_{s=n-k+j+1}^{n} \lambda_s$$

The plan of the proof is to determine a lower bound for min $E_2(A_1, \ldots, A_k)$ under the restrictions (5), (6) and (7) and to show that this is greater than an attainable value of $E_2(A_1, \ldots, A_{k+1})$ which will be a contradiction of (4).

Case 1. (2) and (3i) holds.

Set $\alpha_{k-j} = \lambda_{n-j}$. Then $\displaystyle\sum_{s=n-k+j+1}^{n} \lambda_s = \sum_{s=j+1}^{k} \alpha_s$

which with (7) gives

(8) $$\sum_{s=1}^{j} A\gamma_s \leq -\sum_{s=j+1}^{k} \alpha_s \quad .$$

Also

(9) $$\alpha_{i+1} < \alpha_i \quad ,$$

and since $\alpha_2 = \lambda_{n-k+2} < \lambda_{n-k+1}$ and by (2), $\lambda_{n-k+1} < -(\lambda_{n-k+2} + \ldots + \lambda_n)$ we have

(10) $$\alpha_2 < -(\alpha_2 + \ldots + \alpha_k)$$

since $-(\alpha_2 + \ldots + \alpha_k) = -(\lambda_{n-k+2} + \ldots + \lambda_n)$.

Because (5), (8), (9) and (10) are valid we can apply corollary 3.5 to get

$$E_2(A_1, \ldots, A_k) \geq E_2\left[\alpha_2, \ldots, \alpha_k, -(\alpha_2 + \ldots + \alpha_k)\right]$$
$$= E_2\left[\lambda_{n-k+2}, \ldots, \lambda_n, -(\lambda_{n-k+2} + \ldots + \lambda_n)\right]$$
$$= E_2(\lambda_{n-k+2}, \ldots, \lambda_n) - (\lambda_{n-k+2} + \ldots + \lambda_n)^2 .$$

By choosing appropriate eigenvectors for the x_i an attainable value of $E_2(A_1, \ldots A_{k+1})$ is

$$E_2(\lambda_1, \lambda_{n-k+1}, \ldots, \lambda_n)$$
$$= \lambda_1 \lambda_{n-k+1} + (\lambda_1 + \lambda_{n-k+1})(\lambda_{n-k+2} + \ldots + \lambda_n)$$
$$+ E_2(\lambda_{n-k+2}, \ldots, \lambda_n)$$
$$= \lambda_1(\lambda_{n-k+2} + \ldots + \lambda_n) +$$
$$\lambda_{n-k+1}(\lambda_1 + \lambda_{n-k+2} + \ldots + \lambda_n)$$
$$+ E_2(\lambda_{n-k+2}, \ldots, \lambda_n)$$

and by (3i) and (2) this last expression is

$$< \lambda_1(\lambda_{n-k+2} + \ldots + \lambda_n) + E_2(\lambda_{n-k+2}, \ldots, \lambda_n)$$
$$- (\lambda_{n-k+2} + \ldots + \lambda_n)(\lambda_1 + \lambda_{n-k+2} + \ldots + \lambda_n)$$
$$= E_2(\lambda_{n-k+2}, \ldots, \lambda_n) - (\lambda_{n-k+2} + \ldots + \lambda_n)^2$$
$$\leq E_2(A_1, \ldots A_k)$$

which contradicts (4).

Cases 2 and 4 are special cases of 3 but in order not to get involved in any ingenious devices we have split it into three cases.

Case 2. (2) and (3ii) apply.

Set $\alpha_2 = -(\lambda_1 + \lambda_{n-k+3} + \ldots + \lambda_n)$ and $\alpha_{k-j} = \lambda_{n-j}$, $j = 0, 1, \ldots, k-3$. Then by (3ii) $\lambda_{n-k+2} < -(\lambda_1 + \lambda_{n-k+3} + \ldots + \lambda_n) = \alpha_2$ and since $\alpha_3 = \lambda_{n-k+3}$ we have $\alpha_3 < \alpha_2$. From this and because the λ_j's are ordered,

(11) $\qquad \alpha_{i+1} \leq \alpha_i \qquad i = 2, \ldots k-1.$

By (3ii) $\lambda_2 > -(\lambda_1 + \lambda_{n-k+3} + \ldots + \lambda_n) = \alpha_2$ and since $\lambda_2 < \lambda_1 = -(\alpha_2 + \ldots + \alpha_k)$ we have

(12) $\qquad \alpha_2 < -(\alpha_2 + \ldots + \alpha_k)$.

By definition of α_i we have $\sum\limits_{s=j+1}^{k} \alpha_s = \sum\limits_{s=n-k+j+1}^{n} \lambda_s$ for $2 \leq j \leq k-1$ while $-\sum\limits_{s=2}^{k} \alpha_s = \lambda_1$. Therefore by (6) and (7) we have

(13) $$\sum_{s=1}^{j} A\gamma_s \leq -\sum_{s=j+1}^{k} \alpha_s \quad .$$

By (5), (11), (12) and (13) we can apply corollary (3.5) to get

$$E_2(A_1, \ldots A_k) \geq E_2\left[\alpha_2, \ldots \alpha_k, -(\alpha_2 + \ldots + \alpha_k)\right]$$
$$= E_2\left[-(\lambda_1 + \lambda_{n-k+3} + \ldots + \lambda_n), \lambda_{n-k+3}, \ldots, \lambda_n, \lambda_1\right]$$
$$= E_2\left(\lambda_1, \lambda_{n-k+3}, \ldots, \lambda_n\right) - (\lambda_1 + \lambda_{n-k+3} + \ldots + \lambda_n)^2$$

An attainable value of $E_2(A_1, \ldots, A_{k+1})$ is

$$E_2(\lambda_1, \lambda_2, \lambda_{n-k+2}, \ldots, \lambda_n)$$
$$= \lambda_2\lambda_{n-k+2} + (\lambda_2 + \lambda_{n-k+2})(\lambda_1 + \lambda_{n-k+3} + \ldots + \lambda_n)$$
$$+ E_2(\lambda_1, \lambda_{n-k+3}, \ldots, \lambda_n)$$
$$= \lambda_{n-k+2}(\lambda_1 + \lambda_2 + \lambda_{n-k+3} + \ldots + \lambda_n)$$
$$+ \lambda_2(\lambda_1 + \lambda_{n-k+3} + \ldots + \lambda_n)$$
$$+ E_2(\lambda_1, \lambda_{n-k+3}, \ldots, \lambda_n)$$

and by (3ii) this expression is

$$< -(\lambda_1 + \lambda_{n-k+3} + \ldots + \lambda_n)(\lambda_1 + \lambda_2 + \lambda_{n-k+3} + \ldots + \lambda_n)$$
$$+ (\lambda_1 + \lambda_{n-k+3} + \ldots + \lambda_n)(\lambda_2)$$
$$+ E_2(\lambda_1, \lambda_{n-k+3}, \ldots, \lambda_n)$$
$$= E_2(\lambda_1, \lambda_{n-k+3}, \ldots, \lambda_n)$$
$$- (\lambda_1 + \lambda_{n-k+3}, \ldots, \lambda_n)^2$$

$$\leq E_2(A_1, \ldots, A_k)$$

which contradicts (4).

Case 3. (2) and (3iii) apply.

Set $\alpha_s = \lambda_s \qquad s = 2, \ldots, i$

$$\alpha_{i+1} = -(\lambda_1 + \ldots + \lambda_i + \lambda_{n-k+i+2} + \ldots + \lambda_n)$$

$$\alpha_{k-s} = \lambda_{n-s} \qquad s = 0, 1, \ldots, k-i-2 \;.$$

For $k-1 \geq j > i$, $\sum_{s=j+1}^{k} \alpha_s = \alpha_k + \ldots + \alpha_{j+1} =$

$\lambda_n + \ldots + \lambda_{n-k+j+1} = \sum_{s=n-k+j+1}^{n} \lambda_s$ and therefore by (7)

$$(14) \qquad \sum_{s=1}^{j} A_{\gamma_s} \leq -\sum_{s=j+1}^{k} \alpha_s \;.$$

For $i \geq j \geq 1$, $\sum_{s=j+1}^{k} \alpha_s =$

$(\alpha_k + \ldots + \alpha_{i+2}) + \alpha_{i+1} + \ldots + \alpha_{j+1} = \lambda_n + \ldots + \lambda_{n-k+i+2}$
$- (\lambda_1 + \ldots + \lambda_i + \lambda_{n-k+i+2} + \ldots + \lambda_n) + \lambda_{i+2} + \ldots + \lambda_{j+1}$
$= -(\lambda_1 + \ldots + \lambda_j) = -\sum_{s=1}^{j} \lambda_i$ and therefore by (6)

$$(15) \qquad \sum_{s=1}^{j} A_{\gamma_s} \leq -\sum_{s=j+1}^{k} \alpha_s \;.$$

We may combine (14) and (15) to get

$$(16) \qquad \sum_{s=1}^{j} A_{\gamma_s} \leq -\sum_{s=j+1}^{k} \alpha_s \qquad 1 \leq j \leq k-1 \;.$$

In order to show that $\alpha_{j+1} \leq \alpha_j$ we need only check that $\alpha_{i+2} \leq \alpha_{i+1} \leq \alpha_i$ since for all other indicies the α's are ordered in the same way as the λ's. By (3iii) and the definition of α_{i+1} we have

$$\alpha_{i+1} + \lambda_1 - \lambda_{n-k+i+1} > \lambda_1 > \alpha_{i+1} + \lambda_1 - \lambda_{i+1}$$

or

$$-\lambda_{n-k+i+1} > -\alpha_{i+1} > -\lambda_{i+1}$$

or

$$\lambda_{i+1} > \alpha_{i+1} > \lambda_{n-k+i+1}$$

and since $\alpha_i = \lambda_i > \lambda_{i+1}$, $\alpha_{i+2} = \lambda_{n-k+i+2} < \lambda_{n-k+i+1}$ we have

$$\alpha_{i+2} < \alpha_{i+1} < \alpha_i$$

so that

(17) $$\alpha_{j+1} \leq \alpha_j \qquad 2 \leq j \leq k-1 \ .$$

Also since $\lambda_1 = -(\alpha_2 + \ldots + \alpha_k)$ and $\alpha_2 = \lambda_2$ we have

(18) $$\alpha_2 < -(\alpha_2 + \ldots + \alpha_k) \ .$$

Because (5), (16), (17) and (18) are valid we can apply corollary 3.5 to get

$$E_2(A_1, \ldots, A_k) \geqq E_2\left[\alpha_2, \ldots, \alpha_k, -(\alpha_2 + \ldots + \alpha_k)\right]$$

$$= E_2\Big[\lambda_2, \ldots, \lambda_i, -(\lambda_1 + \ldots + \lambda_i + \lambda_{n-k+i+2} + \ldots + \lambda_n),$$

$$\lambda_{n-k+i+2}, \ldots, \lambda_n, \lambda_1\Big]$$

$$= E_2(\lambda_1, \ldots, \lambda_i, \lambda_{n-k+i+2}, \ldots, \lambda_n)$$

$$-(\lambda_1 + \ldots + \lambda_i + \lambda_{n-k+i+2} + \ldots + \lambda_n)^2 \ .$$

An attainable value of $E_2(A_1, \ldots, A_{k+1})$ is

$$E_2(\lambda_1, \ldots \lambda_{i+1}, \lambda_{n-k+i+1}, \ldots, \lambda_n)$$

$$= \lambda_{i+1}\lambda_{n-k+i+1} + E_2(\lambda_1, \ldots, \lambda_i, \lambda_{n-k+i+2}, \ldots \lambda_n)$$

$$+ (\lambda_{i+1} + \lambda_{n-k+i+1})(\lambda_1 + \ldots + \lambda_i + \lambda_{n-k+i+2} + \ldots + \lambda_n)$$

$$= \lambda_{n-k+i+1}(\lambda_1 + \ldots + \lambda_{i+1} + \lambda_{n-k+i+2} + \ldots + \lambda_n)$$

$$+ \lambda_{i+1}(\lambda_1 + \ldots + \lambda_i + \lambda_{n-k+i+2} + \ldots + \lambda_n)$$

$$+ E_2(\lambda_1, \ldots, \lambda_i, \lambda_{n-k+i+2}, \ldots, \lambda_n)$$

and by (3iii) this expression is

$$< -(\lambda_1 + \dots + \lambda_i + \lambda_{n-k+i+2} + \dots + \lambda_n)(\lambda_1 + \dots + \lambda_{i+1} + \lambda_{n-k+i+2} + \dots + \lambda_n) + (\lambda_1 + \dots + \lambda_i + \lambda_{n-k+i+2} + \dots + \lambda_n)\lambda_{i+1} + E_2(\lambda_1, \dots \lambda_i, \lambda_{n-k+i+2}, \dots, \lambda_n)$$

$$= -(\lambda_1 + \dots + \lambda_i + \lambda_{n-k+i+2} + \dots + \lambda_n)^2 + E_2(\lambda_1, \dots, \lambda_i, \lambda_{n-k+i+2}, \dots, \lambda_n)$$

$$\leq E_2(A_1, \dots, A_k)$$

which contradicts (4).

Case 4. (2) and (3iv) apply.

Set $\alpha_s = \lambda_s \qquad s = 2, \dots, k-1$

$\alpha_k = -(\lambda_1 + \dots + \lambda_{k-1})$.

By (6) and because $-\sum_{s=j+1}^{k} \alpha_s = -\sum_{s=j+1}^{k-1} \alpha_s - \alpha_k = \sum_{s=1}^{j} \lambda_s$

we have

$$\text{(19)} \qquad \sum_{s=1}^{j} A_{\gamma_s} \leq -\sum_{s=j+1}^{k} \alpha_s \quad , \quad 1 \leq j \leq k-1 \quad .$$

To show that $\alpha_{j+1} \leq \alpha_j$ we need only check it in case $j = k-1$. Since $\alpha_{k-1} = \lambda_{k-1} > \lambda_k$ and since by (3iv) we have $\lambda_k > -(\lambda_1 + \dots + \lambda_{k-1})$ we have $\alpha_k \leq \alpha_{k-1}$ and therefore

$$\text{(20)} \qquad \alpha_{j+1} \leq \alpha_j \qquad 2 \leq j \leq k-1 \quad .$$

Also because $\alpha_2 = \lambda_2$ and $\lambda_1 = -(\alpha_2 + \dots + \alpha_k)$ we have

$$\text{(21)} \qquad \alpha_2 < -(\alpha_2 + \dots + \alpha_k) \quad .$$

By (5), (19), (20) and (21) we can apply corollary 3.5 to get

$$E_2(A_1, \dots A_k) \geq E_2\left[\alpha_2, \dots, \alpha_k, -(\alpha_2 + \dots + \alpha_k)\right]$$

$$= E_2\left[\lambda_2, \dots, \lambda_{k-1}, -(\lambda_1 + \dots + \lambda_{k-1}), \lambda_1\right]$$

$$= -(\lambda_1 + \dots + \lambda_{k-1})^2 + E_2(\lambda_1, \dots \lambda_{k-1}) \quad .$$

An attainable value of $E_2(A_1, \dots A_{k+1})$ is $E_2(\lambda_1, \dots, \lambda_k, \lambda_n)$

$$= \lambda_k \lambda_n + (\lambda_k + \lambda_n)(\lambda_1 + \cdots + \lambda_{k-1})$$
$$+ E_2(\lambda_1, \ldots, \lambda_{k-1})$$
$$= \lambda_n (\lambda_1 + \cdots + \lambda_{k-1} + \lambda_k) + \lambda_k(\lambda_1 + \cdots + \lambda_{k-1})$$
$$+ E_2(\lambda_1, \ldots, \lambda_{k-1})$$
$$< -(\lambda_1 + \cdots + \lambda_{k-1})(\lambda_1 + \cdots + \lambda_k) + (\lambda_1 + \cdots + \lambda_{k-1})\lambda_k$$
$$+ E_2(\lambda_1, \ldots, \lambda_{k-1})$$
$$= -(\lambda_1 + \cdots + \lambda_{k-1})^2 + E_2(\lambda_1, \ldots, \lambda_{k-1})$$
$$\leq E_2(A_1, \ldots, A_k)$$

which contradicts (4) and the proof is complete.

Theorem 3.8 <u>Let</u> A <u>be</u> <u>Hermitian</u> <u>with</u> <u>eigenvalues</u> $\lambda_1 \geq \ldots \geq \lambda_n$. <u>Then</u>

$$\text{Min } E_2(A_1, \ldots A_n) = E_2(\lambda_1, \ldots \lambda_n)$$

<u>where</u> $x_1, \ldots, x_n$ <u>range</u> <u>over</u> <u>all</u> <u>sets</u> <u>of</u> n o.n. <u>vectors</u>.

Proof:

$$E_2(A_1, \ldots, A_n)$$
$$= \sum_{1 \leq i_1 < i_2 \leq n} (Ax_{i_1}, x_{i_1})(Ax_{i_2}, x_{i_2})$$
$$\geq \sum_{1 \leq i_1 < i_2 \leq n} \det (Ax_{i_s}, x_{i_t}) \qquad s = 1, 2 \quad t = 1, 2$$
$$= \sum_{1 \leq i_1 < i_2 \leq n} (C_2(A)x_{i_1} \wedge x_{i_2}, x_{i_1} \wedge x_{i_2})$$
$$= \text{trace } C_2(A)$$
$$= E_2(\lambda_1, \ldots, \lambda_n)$$

and the value is taken on.

Theorem 3.9 <u>Let</u> A <u>be</u> <u>Hermitian</u> <u>with</u> <u>eigenvalues</u> $\lambda_1 \geq \ldots \geq \lambda_n$. <u>Then</u> <u>for</u> $1 \leq k \leq n$,

$$\text{Min } E_2(A_1, \ldots A_k) = \text{Min } E_2(\lambda_{i_1}, \ldots, \lambda_{i_k}) \quad , \quad (i_1,\ldots,i_k) \in Q_{kn}$$

<u>where</u> $x_1, \ldots, x_k$ <u>run over all sets of</u> k o.n. <u>vectors</u>.

Proof: There exist sequences

$$\epsilon_j^i = \left\{ \epsilon_1^i, \ldots, \epsilon_m^i, \ldots \right\} , \quad i = 1, \ldots, n$$

such that for each j

(1) $$(\lambda_1 + \epsilon_j^1), \ldots, (\lambda_n + \epsilon_j^n)$$

satisfy the hypotheses of theorem 3.7 and also $\lim_{j\to\infty} \epsilon_j^i = 0$ for each i.

This is true because violation of the hypotheses implies a finite number of linear equalities to be valid which can be made into strict inequalities by arbitrary small changes in the λ_i's.

For each j let A_{ϵ_j} be the Hermitian matrix with eigenvalues (1), let $(A_{\epsilon_j})_i = (A_{\epsilon_j} x_i, x_i)$, and let $x'_1, \ldots, x'_k$ be a set of k o.n. vectors which minimizes $E_2\left[(A_{\epsilon_j})_1, \ldots, (A_{\epsilon_j})_k\right]$. By theorem 3.7 the subspace $L(x'_1, \ldots x'_k)$ is invariant under A_{ϵ_j} and if we minimize A_{ϵ_j} restricted to this subspace then by theorem 3.8 this minimum is equal to $E_2(\alpha_1, \ldots, \alpha_k)$ where α_i are the eigenvalues of A_{ϵ_j} restricted to $L(x'_1, \ldots, x'_k)$. Since $L(x'_1, \ldots, x'_k)$ is invariant under A_{ϵ_j} we must have that $\alpha_1, \ldots, \alpha_k$ is a choice of some k of the eigenvalues $\lambda_1+\epsilon_j^1, \ldots, \lambda_n+\epsilon_j^n$ say $\lambda_{i_1}+\epsilon_j^{i_1}, \ldots, \lambda_{i_k}+\epsilon_j^{i_k}$. By choosing the x_i to be eigenvectors we can attain all the values $E_2\left[(\lambda_{i_1} + \epsilon_j^{i_1}), \ldots, (\lambda_{i_k} + \epsilon_j^{i_k})\right]$ and therefore

$$\operatorname{Min} E_2\left[(A\epsilon_j)_1, \ldots, (A\epsilon_j)_k\right]$$
$$= \min_{(i_1,\ldots,i_k)\in Q_{kn}} E_2\left[(\lambda_{i_1} + \epsilon_j^{i_1}), \ldots, (\lambda_{i_k} + \epsilon_j^{i_k})\right]$$

Taking the limit as $j \to \infty$ gives the desired result.

Lemma 3.10 Let

$$f(x_1 \ldots x_k) = \sum_{\omega\in Q_{rk}} (C_r(A)\, x_\omega, x_\omega)$$

$$g(z_1 \ldots z_k) = E_r\left[(Az_1, z_1), \ldots, (Az_k, z_k)\right]$$

where A is n x n Hermitian and both $(x_1, \ldots, x_k)$ and $(z_1, \ldots, z_k)$ range over o.n. sets of vectors. Then

$$\operatorname{Rng}(f) \subset \operatorname{Rng}(g) .$$

Proof: Let P and u_i be defined as in Theorem 3.1 for $L(x_1, \ldots, x_k)$.

$$\begin{aligned} f(x_1 \ldots x_k) &= \sum_{\omega\in Q_{rk}} (C_r(A)\, x_\omega, x_\omega) \\ &= \sum_{\omega\in Q_{rk}} (C_r(PAP)x_\omega, x_\omega) \\ &= \operatorname{tr}\left[C_r(PAP)\right] \\ &= \sum_{\omega\in Q_{rk}} (C_r(PAP)\, u_\omega, u_\omega) \\ &= \sum_{1\le i_1<\ldots<i_r\le k} \det\left\{(PAP\, u_{i_s}, u_{i_t})\right\} \\ &= \sum_{1\le i_1<\ldots<i_r\le k} \prod_{s=1}^{r} (PAP\, u_{i_s}, u_{i_s}) \\ &= \sum_{1\le i_1<\ldots<i_r\le k} \prod_{s=1}^{r} (Au_{i_s}, u_{i_s}) \\ &= g(u_1, \ldots, u_k) . \end{aligned}$$

Theorem 3.11 Let A be Hermitian with eigenvalues $\lambda_1 \ge \ldots \ge \lambda_n$. Then

$$\operatorname{Min} \sum_{1\le i_1<i_2\le k} (C_2(A)x_{i_1} \wedge x_{i_2},\ x_{i_1} \wedge x_{i_2})$$

$$= \operatorname*{Min}_{1\le j_1<\ldots<j_k\le n} E(\lambda_{j_1}, \ldots, \lambda_{j_k})$$

where $x_1, \ldots, x_k$ range over all sets of k o.n. vectors.

Proof: By lemma 3.10

$$\sum_{1\le i_1<i_2\le k} (C_2(A)x_{i_1} \wedge x_{i_2},\ x_{i_1} \wedge x_{i_2})$$

$$\ge \operatorname{Min} E_2\left[(Az_1, z_1), \ldots, (Az_k, z_k)\right]$$

and by Theorem 3.9 this expression is

$$= \operatorname*{Min}_{(i_1,\ldots i_k)\in Q_{kn}} E_2(\lambda_{i_1}, \ldots, \lambda_{i_k}) \ .$$

If ξ_i is the eigenvector corresponding to λ_i then

$$\sum_{1\le i_1<i_2\le k} (C_2(A)\,\xi_{i_1} \wedge \xi_{i_2},\ \xi_{i_1} \wedge \xi_{i_2})$$

$$= \sum_{1\le i_1<i_2\le k} \begin{vmatrix} (A\,\xi_{i_1}, \xi_{i_1}) & (A\,\xi_{i_1}, \xi_{i_2}) \\ (A\,\xi_{i_2}, \xi_{i_1}) & (A\,\xi_{i_2}, \xi_{i_2}) \end{vmatrix}$$

$$= \sum_{1\le i_1<i_2\le k} \lambda_{i_1}\lambda_{i_2} = E_2(\lambda_{i_1}, \ldots, \lambda_{i_k}) \ .$$

Hence $\displaystyle\operatorname*{Min}_{(i_1,\ldots,i_k)\in Q_{kn}} E_2(\lambda_{i_1}, \ldots, \lambda_{i_k})$ is taken on.

BIBLIOGRAPHY

[1] A.R. Amir-Moez, Extreme Properties of Eigenvalues of a Hermitian Transformation and Singular Values of Sum and Product of Linear Transformations, Thesis for Doctor of Philosophy in Mathematics, University of California at Los Angeles, June 1955, pp. 24-25.

[2] Ky Fan, On a Theorem of Weyl Concerning the Eigenvalues of Linear Transformations I, Proc. Nat. Acad. Sci. U.S.A., vol 35 (1949), pp. 652-655.

[3] M. Marcus and J.L. McGregor, Extremal Properties of Hermitian Matrices, to appear in Can. J. Math.

[4] M. Marcus and B. Moyls, Maximum and Minimum Values for the Elementary Symmetric Functions of Hermitian Forms, submitted to J. London Math. Soc.

[5] A. Ostrowsky, Sur Quelques Applications des Fonctions Convexes et Concaves au sens de I. Schur, J. Math. Pures Appl., vol 30-31 (1951-52), page 265, Thm IV A.

[6] R.C. Thompson, Theory and Applications of Compound Matrices, Masters thesis in Mathematics, University of British Columbia, April 1956.