PAIRS OF MATRICES WITH PROPERTY L

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ABSTRACT

Let A and B be n-square complex matrices with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $\mu_1, \mu_2, \cdots, \mu_n$ respectively. The matrices A and B are said to have property L if any linear combination $aA + bB$, with a, b complex, has as eigenvalues the numbers $a\lambda_i + b\mu_i$, $i = 1,2,\cdots,n$.

A theorem of Dr. M. D. Marcus, which gives a necessary and sufficient condition such that two matrices A and B have property L in terms of the traces of various power-products of A and B, is proved.

This theorem is used to investigate the conditions on B for the special cases $n = 2$, 3, and 4, when A is in Jordan canonical form.

The final result is a theorem which gives a necessary condition on B for A and B to have property L when A is in Jordan canonical form.

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1. Introduction.

Definition. Let A and B be n-square complex matrices with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $\mu_1, \mu_2, \cdots, \mu_n$ respectively. The matrices A and B are said to have property L if any linear combination $aA + bB$, with a, b complex, has as eigenvalues the numbers $a\lambda_i + b\mu_i$, $i = 1, 2, \cdots, n$.

In 1952 T. S. Motzkin and Olga Taussky [1] proved the following theorem:

Let the n-square matrices A and B have property L. Let t be the number of different eigenvalues of A and assume that all the eigenvalues λ_i of A are arranged in sets of equal ones. Let m_i be the multiplicity of the eigenvalues λ_i of A and assume that there are m_i independent eigenvectors corresponding to each λ_i. Let μ_i be the corresponding eigenvalues of B. Let $B^* = P^{-1}BP$ where $A^* = P^{-1}AP$ is in Jordan canonical form.

Then

$$B^* = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1t} \\ B_{21} & B_{22} & \cdots & B_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ B_{t1} & B_{t2} & \cdots & B_{tt} \end{pmatrix}$$

where B_{ii} is an m_i-square matrix ($i = 1, 2, \cdots, t$) and

$$|xI - B^*| = \prod_{i=1}^{t} |xI - B_{ii}|.$$

Furthermore we have $\sum b^*_{ik} b^*_{ki}$ where the summation is over all $i<k$ with (i,k) outside of every B_{ii} ($i = 1, 2, \cdots, t$).

In the above theorem A is restricted in that it is similar to a diagonal matrix. In the present paper we consider the structure of B^* for more general types of A. We first present a result of Professor M. Marcus which gives a necessary and sufficient condition that A and B have property L in terms of the trace of various power-products of A and B. We then determine the resulting special conditions on B^* for the cases $n = 2, 3$ and 4. We conclude with a general necessary condition that $\tilde{A}$ and B have property L.

2. A Necessary and Sufficient Condition for Property L.

Theorem 1. A and B have property L if and only if

$$(1)\quad \operatorname{trace}\Big(\sum_{\substack{x_1+\cdots+x_r = p \\ y_1+\cdots+y_r = k-p \\ x_1 \geq 0,\ x_2,\cdots,x_r > 0 \\ y_r \geq 0,\ y_{r-1},\cdots,y_1 > 0}} A^{x_1}B^{y_1}\cdots A^{x_r}B^{y_r} \Big) = \binom{k}{p}\Big(\sum_{i=1}^{n} \lambda_i^p \mu_i^{k-p}\Big),\quad 1 \leq p \leq k,\ 1 \leq k \leq n,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $\mu_1, \mu_2, \cdots, \mu_n$ are eigenvalues of A and B respectively.

Proof. If A and B have property L, then

$$\operatorname{trace}(aA + bB) = \sum_{i=1}^{n} (a\lambda_i + b\mu_i)$$

and

$$\operatorname{trace} (aA + bB)^k = \sum_{i=1}^{n} (a\lambda_i + b\mu_i)^k \quad (k = 1, 2, \cdots, n).$$

Expanding both sides of the second identity, we have

$$a^k \operatorname{trace} A^k + \sum_{p=1}^{k-1} a^p b^{k-p} \operatorname{trace}\Big(\sum_{\substack{x_1+\cdots+x_r = p \\ y_1+\cdots+y_r = k-p \\ x_1 \geq 0,\ x_2,\cdots,x_r > 0 \\ y_r \geq 0,\ y_{r-1},\cdots,y_1 > 0}} A^{x_1}B^{y_1}\cdots A^{x_r}B^{y_r} \Big) + b^k \operatorname{trace} B^k$$

$$= a^k \sum_{i=1}^{n} \lambda_i^k + \sum_{p=1}^{k-1} a^p b^{k-p} \binom{k}{p} \sum_{i=1}^{n} \lambda_i^p \mu_i^{k-p} + b^k \sum_{i=1}^{n} \mu_i^k.$$

The coefficients of $a^p b^{k-p}$ $(p = 0, 1, \cdots, k)$ in both sides of the identity must be equal, so we have the following conditions:

$$\operatorname{trace} A^k = \sum_{i=1}^{n} \lambda_i^k, \qquad \operatorname{trace} B^k = \sum_{i=1}^{n} \mu_i^k \text{ and}$$

$$\operatorname{trace}\Big(\sum_{\substack{x_1+\cdots+x_r=p\\ y_1+\cdots+y_r=k-p\\ x_1\geq 0,\ x_2,\cdots,x_r>0\\ y_r\geq 0,\ y_{r-1},\cdots,y_1>0}} A^{x_1}B^{y_1}\cdots A^{x_r}B^{y_r}\Big) = \binom{k}{p}\Big(\sum_{i=1}^{n}\lambda_i^p\,\mu_i^{k-p}\Big),\quad 1\leq p\leq k,\ 1\leq k\leq n.$$

Conversely, if

$$\operatorname{trace}\Big(\sum A^{x_1}B^{y_1}\cdots A^{x_r}B^{y_r}\Big) = \binom{k}{p}\Big(\sum_{i=1}^{n}\lambda_i^p\,\mu_i^{k-p}\Big),\quad 1\leq p\leq k,\ 1\leq k\leq n,$$

then $\operatorname{trace}(\,aA + bB\,)^k$

$$= \operatorname{trace}\Big(\,a^kA^k + \sum_{p=1}^{k-1}a^pb^{k-p}\sum A^{x_1}B^{y_1}\cdots A^{x_r}B^{y_r} + b^kB^k\,\Big)$$

$$= a^k\operatorname{trace}A^k + \sum_{p=1}^{k-1}a^pb^{k-p}\operatorname{trace}\Big(\sum A^{x_1}B^{y_1}\cdots A^{x_r}B^{y_r}\Big) + b^k\operatorname{trace}B^k$$

$$= \sum_{i=1}^{n}\Big(\,a^k\lambda_i^k + \sum_{p=1}^{k-1}a^pb^{k-p}\binom{k}{p}\lambda_i^p\,\mu_i^{k-p} + b^k\mu_i^k\,\Big)$$

$$= \sum_{i=1}^{n}(\,a\lambda_i + b\mu_i)^k.$$

$$\therefore\quad \operatorname{trace}(\,aA + bB\,)^k = \sum_{i=1}^{n}(\,a\lambda_i + b\mu_i)^k,\quad 1\leq k\leq n.$$

For brevity of notation we now let

$$aA + bB = C \quad\text{and}\quad a\lambda_i + b\mu_i = \gamma_i\quad (\,i = 1,2,\cdots,n\,).$$

Suppose C has eigenvalues δ_i, $(\,i = 1,2,\cdots,n\,)$, we want to prove that the δ_i's are exactly the γ_i's.

Since there exists a unitary matrix U such that

$$U^{-1}C\,U = \begin{bmatrix}\delta_1 & & * \\ & \delta_2 \ddots & \\ 0 & & \delta_n\end{bmatrix} \quad\text{and}\quad U^{-1}C^kU = \begin{bmatrix}\delta_1^k & & * \\ & \delta_2^k \ddots & \\ 0 & & \delta_n^k\end{bmatrix},\quad k = 1,2,\cdots,n,$$

are upper triangular, then

$$\sum_{i=1}^{n}\gamma_i^k = \operatorname{trace}C^k = \operatorname{trace}U^{-1}C^k\,U = \sum_{i=1}^{n}\delta_i^k = S_k,\quad 1\leq k\leq n.$$

By Newton's formula for symmetric functions, we have

$$p_1 = S_1,$$

$$p_2 = \tfrac{1}{2}(\,S_1^2 - S_2\,),$$

$$\cdots\cdots,$$

$$p_k = \text{rational function of } S_1, S_2, \cdots, S_k,$$

where p_k, $1 \leq k \leq n$, are the elementary function of the n variables $\delta_1, \delta_2, \cdots, \delta_n$, and $f(x) = x^n - p_1 x^{n-1} + \cdots + (-1)^{n-1} p_n = 0$ is the characteristic equation of C. But $S_k = \sum_{i=1}^{n} \gamma_i^k$, $1 \leq k \leq n$, so p_k, $1 \leq k \leq n$, are also the elementary functions of $\gamma_1, \gamma_2, \cdots, \gamma_n$. Hence these are also the n roots of the characteristic equation $f(x) = 0$ of C. Since $f(x) = 0$ is of degree n and has only n roots, tthe δ_i's are exactly the γ_i's. Therefore $\gamma_i = a\lambda_i + b\mu_i$ ($i = 1, 2, \cdots, n$) are the eigenvalues of $C = aA + bB$. This completes the proof of the theorem.

3. Application.

In this section we shall determine explicitly necessary and sufficient conditions on the coefficients of an n-square complex matrix B, when $n = 2, 3$ and 4, such that A and B have property L by using the Theorem 1. We assume that A is in Jordan cannonical form.

Case 1. When $n=2$, suppose $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ has eigenvalues μ_1 and μ_2

There are three cases to be considered.

(i) Let $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\lambda_1 \neq \lambda_2$, then A and B have property L if and only if $b_{12} b_{21} = 0$.

Proof. By Theorem 1, A andB have property L if and only if the following relations hold:

$$(2) \quad \begin{aligned} \text{trace } (B) &= b_{11} + b_{22} = \mu_1 + \mu_2, \\ \text{trace } (AB) &= b_{11}\lambda_1 + b_{22}\lambda_2 = \mu_1\lambda_1 + \mu_2\lambda_2. \end{aligned}$$

$$\text{or} \quad \begin{aligned} (\mu_1 - b_{11}) + (\mu_2 - b_{22}) &= 0, \\ (\mu_1 - b_{11})\lambda_1 + (\mu_2 - b_{22})\lambda_2 + &= 0. \end{aligned}$$

Since $\begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix} = \lambda_2 - \lambda_1 \neq o$,

$\mu_1 = b_{11}$ and $\mu_2 = b_{22}$.

Now $\mu_1\mu_2 = b_{11}\, b_{22} - b_{12}\, b_{21} = \mu_1\mu_2 - b_{12}\, b_{21}$.

$\therefore\ b_{12}\, b_{21} = o$.

Conversely if $b_{12}b_{21} = o$, then $\mu_1 = b_{11}$, $\mu_2 = b_{22}$ and (2) holds.

(ii) Let $A = \begin{bmatrix} \lambda_1 & 1 \\ o & \lambda_1 \end{bmatrix}$; then A and B have property L if and only if $b_{21} = o$.

Proof. $\begin{cases} \text{trace } (B) = b_{11} + b_{22} = \mu_1 + \mu_2 , \\ \text{trace } (AB) = \lambda_1(b_{11} + b_{22}) + b_{21} = \lambda_1(\mu_1 + \mu_2). \end{cases}$

$\therefore\ b_{21} = o$.

This is clearly necessary and sufficient.

(iii) If $A = \lambda I$, then, for any B, A and B have property L.

Proof. The characteristic equation of $\alpha A + \beta B$ is

(3) $|\alpha A + \beta B - \sigma I| = o$,

or $|\alpha \lambda I + \beta B - \sigma I| = o$.

If $\beta = o$, then the eigenvalues of αA are both $\alpha\lambda$.

If $\beta \neq o$, (3) becomes

$$\left| B - \left(\frac{\sigma}{\beta} - \frac{\alpha\lambda}{\beta} \right) I \right| = o \quad .$$

If B has eigenvalues μ_1 and μ_2, then each $\mu_i = \frac{\sigma_i}{\beta} - \frac{\alpha\lambda}{\beta}$ and $\sigma_i = \alpha\lambda + \beta\mu_i$, $i = 1, 2$.

Case II . n = 3.

We let $B = (b_{ij})$, $(ij = 1,2,3)$ have eigenvalues μ_1, μ_2 and μ_3.

There are six cases to be considered, but we shall give the proof only for the first case and state the necessary and sufficient conditions for the rest without giving their proofs.

(i) Let $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, where λ_1, λ_2 and λ_3 are different, then

A and B have property L if and only if

$$(4)\quad \begin{cases} \mu_1 = b_{11}, \quad \mu_2 = b_{22}, \mu_3 = b_{33}; \\ b_{12}b_{21} : b_{31}b_{13} : b_{23}b_{32} = (\lambda_1 - \lambda_2) : (\lambda_3 - \lambda_1) : (\lambda_2 - \lambda_3). \end{cases}$$

Proof If A and B have property L, then from (1),

$$(5)\quad \begin{cases} \text{trace } B = b_{11} + b_{22} + b_{33} = \mu_1 + \mu_2 + \mu_3 . \\ \text{trace } AB = b_{11}\lambda_1 + b_{22}\lambda_2 + b_{33}\lambda_3 = \mu_1\lambda_1 + \mu_2\lambda_2 + \mu_3\lambda_3 , \\ \text{trace } A^2B = b_{11}\lambda_1^2 + b_{22}\lambda_2^2 + b_{33}\lambda_3^2 = \mu_1\lambda_1^2 + \mu_2\lambda_2^2 + \mu_3\lambda_3^2 . \\ \text{trace } B^2 = \sum_{i=1}^{3}\sum_{k=1}^{3} b_{ik}b_{ki} = \sum_{i=1}^{3}\mu_i^2 , \\ \text{trace } AB^2 = (\sum_{k=1}^{3} b_{1k}b_{k1})\lambda_1 + (\sum_{k=1}^{3} b_{2k}b_{k2})\lambda_2 + \\ \qquad + (\sum_{k=1}^{3} b_{3k}b_{k3})\lambda_3 = \mu_1^2\lambda_1 + \mu_2\lambda_2 + \mu_3\lambda_3 . \end{cases}$$

For example, the third equation of (5) is derived from (1) when k=3, p=2; in this case (1) becomes.

$$\binom{3}{2}\sum_{i=1}^{3}\lambda_i^2\mu_i = \text{trace } (A^2 B + ABA + BA^2)$$
$$= \text{trace } A^2B + \text{trace } ABA + \text{trace } BA^2$$
$$= 3\,\text{trace } A^2B,$$

since, for any matrices M,N, trace MN = trace N M .

From the first three equations of (5) we have

$$\begin{cases} (\mu_1 - b_{11}) + (\mu_2 - b_{22}) + (\mu_3 - b_{33}) = o, \\ \lambda_1(\mu_1 - b_{11}) + \lambda_2(\mu_2 - b_{22}) + \lambda_3(\mu_3 - b_{33}) = o, \\ \lambda_1^2(\mu_1 - b_{11}) + \lambda_2^2(\mu_2 - b_{22}) + \lambda_3^2(\mu_3 - b_{33}) = o. \end{cases}$$

Since $\begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \neq o,$

$$(6) \quad \mu_1 = b_{11}, \; \mu_2 = b_{22}, \; \mu_3 = b_{33}.$$

From the last two equations of (5) and (6), we have

$$\begin{cases} (b_{12}b_{21} + b_{13}b_{31})\lambda_1 + (b_{21}b_{12} + b_{23}b_{32})\lambda_2 + (b_{31}b_{13} + b_{32}b_{23})\lambda_3 \\ \quad = o, \\ b_{12}b_{21} + b_{31}b_{13} + b_{23}b_{32} = o; \end{cases}$$

or $\begin{cases} b_{23}b_{32}\lambda_1 + b_{13}b_{31}\lambda_2 + b_{12}b_{21}\lambda_3 = o, \\ b_{23}b_{32} + b_{13}b_{31} + b_{12}b_{21} = o. \end{cases}$

$$\therefore b_{23}b_{32} : b_{13}b_{31} : b_{12}b_{21} = (\lambda_2 - \lambda_3) : (\lambda_3 - \lambda_1) : (\lambda_1 - \lambda_2)$$

Hence a necessary condition is

$$(4) \quad \begin{cases} \mu_1 = b_{11}, \mu_2 = b_{22}, \mu_3 = b_{33}, \\ b_{23}b_{32} : b_{13}b_{31} : b_{12}b_{21} = (\lambda_2 - \lambda_3) : (\lambda_3 - \lambda_1) : (\lambda_1 - \lambda_2). \end{cases}$$

Conversely, if (4) hold, then

$$\text{trace } AB = \lambda_1 b_{11} + \lambda_2 b_{22} + \lambda_3 b_{33} = \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3,$$

$$\text{trace } A^2B = \lambda_1^2 b_{11} + \lambda_2^2 b_{22} + \lambda_3^2 b_{33} = \lambda_1^2\mu_1 + \lambda_2^2\mu_2 + \lambda_3^2\mu_3,$$

(4) Cont'd

$$\operatorname{trace} AB^2 = \lambda_1 (b_{11}^2 + b_{12} b_{21} + b_{13} b_{31}) + \lambda_2 (b_{21} b_{12} + b_{22}^2 + b_{23} b_{32}) + \lambda_3 (b_{31} b_{13} + b_{32} b_{23} + b_{33}^2)$$

$$= \lambda_1 b_{11}^2 + \lambda_2 b_{22}^2 + \lambda_3 b_{33}^2 + b_{12} b_{21} (\lambda_1 + \lambda_2) + b_{23} b_{32} (\lambda_2 + \lambda_3) + b_{31} b_{13} (\lambda_3 + \lambda_1)$$

$$= \lambda_1 \mu_1^2 + \lambda_2 \mu_2^2 + \lambda_3 \mu_3^2 + k (\lambda_1^2 - \lambda_2^2) + k (\lambda_2 - \lambda_3)(\lambda_2 + \lambda_3) + k (\lambda_3 - \lambda_1)(\lambda_3 + \lambda_1)$$

$$= \lambda_1 \mu_1^2 + \lambda_2 \mu_2^2 + \lambda_3 \mu_3^2 .$$

Hence by theorem 1, A and B have property L.

(ii) Let $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ where $\lambda_1 \neq \lambda_3$, then A and B have property L if and only if

$$\begin{cases} \mu_1 + \mu_2 = b_{11} + b_{22}, \quad \mu_3 = b_{33}, \\ b_{13} b_{31} + b_{23} b_{32} = 0 . \end{cases}$$

(iii) Let $A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, $\lambda_1 \neq \lambda_3$, then A and B have property L if and only if

$$\begin{cases} \mu_1 + \mu_2 = b_{11} + b_{22}, \quad \mu_3 = b_{33}, \\ b_{21} = 0 , \\ (b_{13} b_{31} + b_{23} b_{32})(\lambda_1 - \lambda_3) = b_{23} b_{31} . \end{cases}$$

(iv) Let $A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$ then A and B have property L

if and only if

$$\begin{cases} b_{21} = 0 , \\ b_{23} b_{31} = 0 . \end{cases}$$

(v) Let $A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$ then A and B have property L

if and only if

$$\begin{cases} b_{31} = 0 , \\ b_{32} = - b_{21} , \\ b_{21} (b_{11} - b_{33}) = 0 . \end{cases}$$

(vi) Let $A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda I_3$, then, for any B , A and B have property L .

Case III, $n = 4$:

In this case we let $B = (b_{ij})$, $(i,j = 1,2,3,4)$, have eigenvalues μ_1, μ_2, μ_3 and μ_4 . There are fourteen cases to be considered, but we shall only state the necessary and sufficient condition for these cases without giving their proofs.

(i) Let $A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$ where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are different, then A and B have property L if and only if

$$\begin{cases} \mu_i = b_{ii}, \quad i = 1,2,3,4, \\ \sum\limits_{\substack{i,j=1 \\ i<j}}^{4} b_{ij}b_{ji}(\lambda_i+\lambda_j) = 0, \\ \sum\limits_{\substack{i,j=1 \\ i<j}}^{4} b_{ij}b_{ji}(\lambda_i^2+\lambda_i\lambda_j+\lambda_j^2) = 0, \\ \sum\limits_{i=1}^{4} \Big(\sum\limits_{j,k=1}^{4} b_{ij}b_{jk}b_{ki} - b_{ii}^3 \Big)\lambda_i = 0. \end{cases}$$

(ii) Let $A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$ where λ_1, λ_3 and λ_4 are different, then A and B have property L if and only if

$$\begin{cases} \mu_i = b_{ii}, \quad i = 3,4, \\ b_{13}b_{31}+b_{14}b_{41}+b_{23}b_{32}+b_{24}b_{42}+b_{34}b_{43} = 0, \\ (b_{31}b_{13}+b_{32}b_{23}+b_{34}b_{43})(\lambda_3-\lambda_1)+(b_{41}b_{14}+b_{42}b_{24}+b_{43}b_{34})(\lambda_4-\lambda_1) \\ = 0. \end{cases}$$

(iii) Let $A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$ where $\lambda_1 \neq \lambda_3$, then A and

B have property L if and only if

$$\begin{cases} \mu_1+\mu_2 = b_{11}+b_{22} , \\ \mu_1^2+\mu_2^2 = \sum_{k=1}^{4} b_{1k}b_{k1}+ \sum_{k=1}^{4} b_{2k}b_{k2} , \\ \mu_1^3+\mu_2^3 = \sum_{k=1}^{4} \sum_{i=1}^{4} b_{1k}b_{ki}b_{i1}+ \sum_{k=1}^{4} \sum_{i=1}^{4} b_{2k}b_{ki}b_{i2} , \\ b_{13}b_{31}+b_{14}b_{41}+b_{23}b_{32}+b_{24}b_{42} = 0. \end{cases}$$

(iv) Let $A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$, where $\lambda_1 \neq \lambda_4$, then A and B have property L if and only if

$$\begin{cases} \mu_4 = b_{44} , \\ \sum_{k=1}^{3} b_{4k}b_{k4} = 0, \\ \sum_{k=1}^{3} \sum_{i=1}^{3} b_{4k}b_{ki}b_{i4} = 0. \end{cases}$$

(v) Let $A = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$, then A and B have property L if and only if

$$\begin{cases} b_{41} = 0 , \\ b_{31}+b_{42} = 0 , \\ b_{21}+b_{32}+b_{43} = 0 , \\ \sum_{k=1}^{4} (b_{2k}b_{k1}+b_{3k}b_{k2}+b_{4k}b_{k3}) = 0, \\ \sum_{k=1}^{4} \sum_{i=1}^{4} (b_{2k}b_{ki}b_{i1}+b_{3k}b_{ki}b_{i2}+b_{4k}b_{ki}b_{i3}) = 0, \\ (b_{11}-b_{44})b_{31}-b_{43}b_{21} = 0 . \end{cases}$$

(vi) Let $A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$, then A and B have property L

if and only if

$$\begin{cases} b_{31} = 0 \,, \\ b_{21}+b_{32} = 0 \,, \\ \sum_{k=1}^{4} (b_{2k}b_{k1}+b_{3k}b_{k2}) = 0 \,, \\ \sum_{k=1}^{4} \sum_{i=1}^{4} (b_{2k}b_{ki}b_{i1}+b_{3k}b_{ki}b_{i2}) = 0 \,, \\ b_{34}b_{41} = 0. \end{cases}$$

(vii) Let $A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$, then A and B have property L

if and only if

$$\begin{cases} b_{21}+b_{43} = 0 \,, \\ \sum_{k=1}^{4} (b_{2k}b_{k1}+b_{4k}b_{k3}) = 0 \,, \\ \sum_{k=1}^{4} \sum_{i=1}^{4} (b_{2k}b_{ki}b_{i1}+b_{4k}b_{ki}b_{i3}) = 0 \,, \\ b_{23}b_{41}-b_{43}b_{21} = 0 \end{cases}$$

(viii) Let $A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$, then A and B have property L

if and only if

$$\begin{cases} b_{21} = 0 \,, \\ \sum_{k=1}^{4} b_{2k}b_{k1} = 0 \,, \\ \sum_{k=1}^{4} \sum_{i=1}^{4} b_{2k}b_{ki}b_{i1} = 0 \,. \end{cases}$$

(ix) Let $A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$, then A and B have property L

if and only if

$$\begin{cases} \mu_3 = b_{33}, \mu_4 = b_{44} \\ b_{21} = 0 , \\ 3\left(\sum\limits_{k=1}^{4} b_{3k}b_{k3} - b_{33}^2\right)(\lambda_3^2-\lambda_1^2) + 3\left(\sum\limits_{k=1}^{4} b_{4k}b_{k4} - b_{44}^2\right)(\lambda_4^2-\lambda_1^2) - \\ \quad -2(b_{13}b_{31}+b_{23}b_{32})(\lambda_1-\lambda_3)^2 - 2(b_{14}b_{41}+b_{24}b_{42})(\lambda_1-\lambda_4)^2 + \\ \quad +8\lambda_1 \sum\limits_{k=1}^{4} b_{2k}b_{k1} + 4(\lambda_3 b_{23}b_{31} + \lambda_4 b_{24}b_{41}) = 0, \\ \left(\sum\limits_{k=1}^{4}\sum\limits_{i=1}^{4} b_{3k}b_{ki}b_{i3} - b_{33}^3\right)(\lambda_1-\lambda_3) + \left(\sum\limits_{k=1}^{4}\sum\limits_{i=1}^{4} b_{4k}b_{ki}b_{i4} - b_{44}^3\right)(\lambda_1-\lambda_4) - \\ \quad - \sum\limits_{k=1}^{4}\sum\limits_{i=1}^{4} b_{2k}b_{ki}b_{i1} = 0. \end{cases}$$

(x) Let $A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}$, where $\lambda_1 \neq \lambda_3$, then A and B

have property L if and only if

$$\begin{cases} b_{21} = 0 , \\ \mu_3 + \mu_4 = b_{33} + b_{44} , \\ \sum\limits_{k=1}^{4} b_{2k}b_{k1} = (b_{13}b_{31} + b_{14}b_{41} + b_{23}b_{32} + b_{24}b_{42})(\lambda_1-\lambda_3) , \\ \mu_3^2 + \mu_4^2 = b_{33}^2 + 2b_{34}b_{43} + b_{44}^2 , \\ \sum\limits_{i,k=1}^{4} b_{2k}b_{ki}b_{i1} = \Big(\sum\limits_{i,k=1}^{4} b_{3k}b_{ki}b_{i3} + \sum\limits_{i,k=1}^{4} b_{4k}b_{ki}b_{i4} - b_{33}^3 - 4b_{33}b_{34}b_{44} + \\ \quad + b_{33}b_{44}^2 + b_{33}^2 b_{44} - b_{44}^3 - 4b_{34}b_{44}b_{43}\Big)(\lambda_1-\lambda_3) . \end{cases}$$

(xi) Let $A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}$, where $\lambda_1 \neq \lambda_3$, then A and B

have property L if and only if

$$
\begin{cases}
b_{21} = 0 \, , \\
b_{43} = 0 \, , \\
\mu_3 + \mu_4 = b_{33} + b_{44} \, , \\
\left(\sum\limits_{k=1}^{4} b_{3k} b_{k3} + \sum\limits_{k=1}^{4} b_{4k} b_{k4}\right)(\lambda_3 - \lambda_1) + \sum\limits_{k=1}^{4} b_{2k} b_{k1} + \sum\limits_{k=1}^{4} b_{4k} b_{k3} = \\
= (\mu_3^2 + \mu_4^2)(\lambda_3 - \lambda_1) \, , \\
\left(\sum\limits_{k=1}^{4} b_{2k} b_{k1} - \sum\limits_{k=1}^{4} b_{4k} b_{k3}\right)(\lambda_1 - \lambda_3) + 2 b_{23} b_{41} - \\
-(b_{13}b_{31} + b_{23}b_{32} + b_{14}b_{41} + b_{24}b_{42})(\lambda_1 - \lambda_3)^2 = 0 \, , \\
\left(\sum\limits_{i,k=1}^{4} b_{3k} b_{ki} b_{i3} + \sum\limits_{i,k=1}^{4} b_{4k} b_{ki} b_{i4}\right)(\lambda_3 - \lambda_1) + \sum\limits_{i,k=1}^{4} b_{2k} b_{ki} b_{i1} + \\
+ \sum\limits_{i,k=1}^{4} b_{4k} b_{ki} b_{i3} = (\mu_3^3 + \mu_4^3)(\lambda_3 - \lambda_1) \, .
\end{cases}
$$

(xii) Let $A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$, $\lambda_1 \neq \lambda_4$, then A and B have property L if and only if

$$
\begin{aligned}
& b_{21} = 0 \, , \; \mu_4 = b_{44} \, , \\
& \left(\sum_{k=1}^{4} b_{4k} b_{k4} - b_{44}^2\right)(\lambda_4 - \lambda_1) + \sum_{k=1}^{4} b_{2k} b_{k1} = 0, \\
& \left(\sum_{k=1}^{4} b_{4k} b_{k4} - b_{44}^2\right)(\lambda_4 + \lambda_1) + b_{24} b_{41} = 0 \, , \\
& \left(\sum_{i,k=1}^{4} b_{4k} b_{ki} b_{i4} - b_{44}^3\right)(\lambda_4 - \lambda_1) + \sum_{i,k=1}^{4} b_{2k} b_{ki} b_{i1} = 0.
\end{aligned}
$$

(xiii) Let $A = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} = \lambda I$, then, for any B, A and B

have property L.

(xiv) Let $A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$, $\lambda_1 \neq \lambda_4$, then A and B have property L if and only if

$$\begin{cases} b_{31} = 0 , \\ b_{21} + b_{32} = 0 , \\ \mu_4 = b_{44} , \\ \left(\sum_{k=1}^{4} b_{4k} b_{k4} - b_{44}^2\right)(\lambda_4 - \lambda_1) + \sum_{k=1}^{4} b_{2k} b_{k1} + \sum_{k=1}^{4} b_{3k} b_{k2} = 0 , \\ \left(\sum_{k=1}^{4} b_{4k} b_{k4} - b_{44}\right)(\lambda_4 - \lambda_1)^2 + (b_{24} b_{41} + b_{34} b_{42})(\lambda_4 - \lambda_1) + \\ \quad + b_{34} b_{41} = 0, \\ \left(\sum_{i,k=1}^{4} b_{4k} b_{ki} b_{i4} - b_{44}^3\right)(\lambda_4 - \lambda_1) + \sum_{i,k=1}^{4} b_{2k} b_{ki} b_{i1} + \\ \quad + \sum_{i,k=1}^{4} b_{3k} b_{ki} b_{i2} = 0 . \end{cases}$$

4. <u>The Necessary Condition on B such that A and B have Property L.</u>

<u>Theorem 2.</u> Let $A = \mathrm{diag}\,(A_1, A_2, \cdots, A_r)$ where

$$A_i = \begin{pmatrix} A_{i1} & 0 & \cdots & 0 \\ 0 & A_{i2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{is_i} \end{pmatrix} \quad \text{and} \quad A_{ij} = \begin{pmatrix} u_i & 1 & 0 \cdots & 0 \\ 0 & u_i & 1 \cdots & 0 \\ & & \ddots & 1 \\ 0 & 0 & 0 \cdots & u_i \end{pmatrix}.$$

Here A is of order n, A_i is of order n_i, and A_{ij} is of order m_{ij}, where $m_{i1} + m_{i2} + \cdots + m_{is_i} = n_i$, $n_1 + n_2 + \cdots + n_r = n$, and $u_1, u_2, \cdots, u_r$ are different eigenvalues of A.

And let

$$B = \begin{bmatrix} B_{11} & B_{12} \cdots & B_{1r} \\ B_{21} & B_{22} \cdots & B_{2r} \\ \cdots\cdots & \cdots\cdots & \cdots \\ B_{r1} & B_{r2} \cdots & B_{rr} \end{bmatrix}$$

be an n-square matrix with eigenvalues $v_1, v_2, \cdots, v_n$, where $B_{ii} = (b^{(1)}_{ij})$ is of order n_i.

If A and B have property L, then

$$\sum_{j=1}^{n_1} v_j = \sum_{j=1}^{n_1} b^{(1)}_{jj}, \quad \sum_{j=1}^{n_2} v_{n_1+j} = \sum_{j=1}^{n_2} b^{(2)}_{jj}, \ldots, \sum_{j=1}^{n_r} v_{n-n_r+j} = \sum_{j=1}^{n_r} b^{(r)}_{jj},$$

$$\sum_{j=k+1}^{m_{11}} b^{(1)}_{j\ j-k} + \sum_{j=k+1}^{m_{12}} b^{(1)}_{m_{11}+j,\ m_{11}+j-k} + \cdots + \sum_{j=k+1}^{m_{1s_1}} b^{(1)}_{n_1-m_{1s_1}+j,\ n_1 m_{1s_1}+j-k}$$

$$= 0, \ (k = 1, 2, \cdots, n_1 - 1),$$

$$\sum_{j=k+1}^{m_{21}} b^{(2)}_{j\ j-k} + \sum_{j=k+1}^{m_{22}} b^{(2)}_{m_{21}+j,\ m_{21}+j-k} + \cdots + \sum_{j=k+1}^{m_{2s_2}} b^{(2)}_{n_2-m_{2s_2}+j,\ n_2-m_{2s_2}+j-k}$$

$$= 0, \ (k = 1, 2, \cdots, n_2 - 1),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\sum_{j=k+1}^{m_{r1}} b^{(r)}_{j\ j-k} + \sum_{j=k+1}^{m_{r2}} b^{(r)}_{m_{r1}+j,\ m_{r1}+j-k} + \cdots + \sum_{j=k+1}^{m_{rs_r}} b^{(r)}_{n_r-m_{rs_r}+j,\ n_r-m_{rs_r}+j-k}$$

$$= 0, \ (k = 1, 2, \cdots, n_r - 1).$$

Proof: There is no loss of generality in assuming that the orders of A_{ij} ($j = 1, 2, \cdots, s_i$) are so arranged such that

$$m_{i1} \leq m_{i2} \leq \cdots \leq m_{is_i} \qquad (i = 1, 2, \cdots, r).$$

One can show easily by induction that

$$A^k = \begin{pmatrix} A_1^k & 0 & \cdots & 0 \\ 0 & A_2^k & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & A_r^k \end{pmatrix} \quad \text{where} \quad A_i^k = \begin{pmatrix} A_{i1}^k & 0 & \cdots & 0 \\ 0 & A_{i2}^k & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & A_{is_i}^k \end{pmatrix}$$

$$\text{and} \quad A_{ij}^k = \begin{pmatrix} u_i^k & \binom{k}{1}u_i^{k-1} & \binom{k}{2}u_i^{k-2} & \cdots & 1 & 0 \cdots 0 \\ 0 & u_i^k & \binom{k}{1}u^{k-1} \cdots & \binom{k}{k-1}u_i & 1 \cdots & 0 \\ 0 & \cdots & u_i^k & \cdots & & 1 \\ 0 & 0 & \cdots & \cdots & & u_i^k \end{pmatrix} \quad 1 \leq k < m_{ij}-1$$

$$\text{and} \quad A_{ij}^k = \begin{pmatrix} u_i^k & \binom{k}{1}u_i^k & \cdots & \binom{k}{m_{ij}-1}u_i^{k-m_{ij}+1} \\ 0 & u_i^k & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & u_i^k \end{pmatrix} \quad k \geq m_{ij}-1.$$

Now trace (A^kB)

$$= \text{trace} \begin{pmatrix} A_1^k & 0 & \cdots & 0 \\ 0 & A_2^k & \cdots & 0 \\ & \cdots\cdots & & \\ 0 & 0 & \cdots & A_r^k \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1r} \\ B_{21} & B_{22} & \cdots & B_{2r} \\ & \cdots\cdots & & \\ B_{r1} & B_{r2} & \cdots & B_{rr} \end{pmatrix}$$

$$= \text{trace}\,(A_1^kB_{11}) + \text{trace}\,(A_2^kB_{22}) + \text{trace}\,(A_3^kB_{33}) + + \cdots + \text{trace}\,(A_r^kB_{rr}) \quad (k = 1,2,\ldots n-1)$$

where $B_{ii} = (b_{ij}^{(i)})$ $(i = 1,2,\cdots r)$ is of order n_i.

Furthermore,

$$\operatorname{trace}(A_i^k B_{ii}) =$$

$$= \operatorname{trace}\begin{pmatrix} A_{i1}^k & 0 & \cdots & 0 \\ 0 & A_{i2}^k & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & A_{is_i}^k \end{pmatrix}\begin{pmatrix} b_{11}^{(i)} & b_{12}^{(i)} & \cdots & b_{1n_i}^{(i)} \\ b_{21}^{(i)} & b_{22}^{(i)} & \cdots & b_{2n_i}^{(i)} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n_i1}^{(i)} & b_{n_i2}^{(i)} & \cdots & b_{n_in_i}^{(i)} \end{pmatrix}$$

$$= \operatorname{trace}\left(A_{i1}^k \begin{pmatrix} b^{(i)} & \cdots & b_{1\,m_{i1}}^{(i)} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ b_{m_{i1}1}^{(i)} & \cdots & b_{m_{i1}m_{i1}}^{(i)} \end{pmatrix} \right) +$$

$$+ \operatorname{trace}\left(A_{i2}^k \begin{pmatrix} b_{m_{i1}+1,\ m_{i1}+1}^{(i)} & \cdots & b_{m_{i1}+1\ m_{i1}+m_{i2}}^{(i)} \\ \cdot & \cdots & \cdot \\ b_{m_{i1}+m_{i2}\ m_{i1}+1}^{(i)} & \cdots & b_{m_{i1}+m_{i2}\ m_{i1}+m_{i2}}^{(i)} \end{pmatrix} \right) +$$

$$+ \cdots +$$

$$+ \operatorname{trace}\left(A_{is_i} \begin{pmatrix} b_{n_i-m_{is_i}+1,\, n_i-m_{is_i}+1}^{(i)} & \cdots & b_{n_i-m_{is_i}+1\ n_i}^{(i)} \\ \cdot & \cdots & \cdot \\ b_{ni\ ni-m_{is_i}+1}^{(i)} & \cdots & b_{n_i\ n_i}^{(i)} \end{pmatrix} \right)$$

$$= \left(\sum_{j=1}^{m_{i1}} b_{jj}^{(i)}\right)u_i^k + \left(\sum_{j=2}^{m_{i1}} b_{jj-1}^{(i)}\right)\binom{k}{1}u_i^{k-1} + \cdots + \left(\sum_{j=k}^{m_{i1}} b_{jj-k+1}^{(i)}\right)u_i +$$

$$+ \sum_{j=k+1}^{m_{i1}} b_{jj-k}^{(i)} + \left(\sum_{j=1}^{m_{i2}} b_{m_{i1}+j,\ m_{i1}+j}^{(i)}\right)u_i^k +$$

$$+ \left(\sum_{j=2}^{m_{i2}} b^{(i)}_{m_{i1}+j\ m_{i1}+j-1} \right) \binom{k}{1} u_i^{k-1} + \cdots + \sum_{j=k+1}^{m_{i2}} b^{(i)}_{m_{i1}+j\ m_{i1}+j-k}$$

$$+ \quad \cdot \quad \cdot \quad \cdot \quad +$$

$$+ \left(\sum_{j=1}^{m_{is_i}} b^{(i)}_{n_i-m_{is_i}+j,\, n_i-m_{is_i}+j} \right) u_i^k +$$

$$+ \left(\sum_{j=2}^{m_{is_i}} b^{(i)}_{n_i-m_{is_i}+j,\ n_i-m_{is_i}+j-1} \right) \binom{k}{1} u_i^{k-1} +$$

$$+ \cdots + \sum_{j=k+1}^{m_{is_i}} b^{(i)}_{n_i-m_{is_i}+j,\ n_i-m_{is_i}+j-k}$$

$$= \left(\sum_{j=1}^{n_i} b^{(i)}_{jj} \right) u_i^k + \left(\sum_{j=2}^{m_{i1}} b^{(i)}_{jj-1} + \sum_{j=2}^{m_{i2}} b^{(i)}_{m_{i1}+j,\ m_{i1}+j-1} + \right.$$

$$\left. + \cdots + \sum_{j=2}^{m_{is_i}} b^{(i)}_{n_i-m_{is_i}+j,\ n_i-m_{is_i}+j-1} \right) \binom{k}{1} u_i^{k-1} +$$

$$+ \cdots + \left(\sum_{j=k+1}^{m_{i1}} b^{(i)}_{jj-k} + \sum_{j=k+1}^{m_{i2}} b^{(i)}_{m_{i1}+j,\ m_{i1}+j-k} + \cdots + \right.$$

$$\left. + \sum_{j=k+1}^{m_{is_i}} b^{(i)}_{n_i-m_{is_i}+j,\ n_i-m_{is_i}+j-k} \right),$$

where $\sum_{j=k+1}^{m_{is_i}} b^{(i)}_{t+j,\ t+j-k} = 0$, when $k = m_{ij}$, and $k = 1, 2, \cdots, n-1$, $i = 1, 2, \cdots r$.

Since A and B have property L , by Theorem 1 we have the following condition :

$$\text{Trace } (B) = \sum_{j=1}^{n_1} b^{(1)}_{jj} + \sum_{j=1}^{n_2} b^{(2)}_{jj} + \cdots + \sum_{j=1}^{n_r} b^{(r)}_{jj}$$

$$= \sum_{j=1}^{n_1} v_j + \sum_{j=1}^{n_2} v_{n_1+j} + \cdots + \sum_{j=1}^{n_r} v_{n-n_r+j} ,$$

$$\text{Trace } (A^k B) = \sum_{i=1}^{r} \text{trace } (A_i^k B_{ii})$$

$$= \sum_{i=1}^{r} \Big[\left(\sum_{j=1}^{n_i} b^{(i)}_{jj} \right) u_i^k + \left(\sum_{j=2}^{m_{i1}} b^{(i)}_{jj-1} + \right.$$

$$+ \sum_{j=2}^{m_{i2}} b^{(i)}_{m_{i1}+j,\, m_{i1}+j-1} + \cdots +$$

$$+ \sum_{j=2}^{m_{is_i}} b^{(i)}_{n_i-m_{is_i}+j,\ n_i-m_{is_i}+j-1})\binom{k}{1}u_i^{k-1} + \cdots +$$

$$+ \Big(\sum_{j=k+1}^{m_{i1}} b^{(i)}_{j\ j-k} + \cdots + \sum_{j=k+1}^{m_{i2}} b^{(i)}_{m_{i1}+j,\ m_{i1}+j-k} +$$

$$+ \cdots + \sum_{j=k+1}^{m_{is_i}} b^{(i)}_{n_i-m_{is_i}+,\ n_i-m_{is_i}+j-k} \Big)\Big]$$

$$= u^k\Big(\sum_{j=1}^{n_1} v_j\Big) + u^k\Big(\sum_{j=1}^{n_2} v_{n_1+j}\Big) + \cdots + u^k\Big(\sum_{j=1}^{n_r} v_{n-n_r+j}\Big),$$

$$(k = 1,2,\cdots n-1).$$

The determinant of the coeffcients of the above n simultaneous linear equation is as follows :

$$D^{\dagger} = \begin{vmatrix} 1 & 0 & \cdots & 0 \cdots & 1 & 0 \cdots & 0 & \cdots & 1 & 0 & \cdots 0 \\ u_1 & \binom{1}{1} & \cdots & 0 \cdots & u_2 & \binom{1}{1} \cdots & 0 & \cdots & u_r & \binom{1}{1} & \cdots\ 0 \\ u_1^2 & \binom{2}{1}u & \binom{2}{2} \cdots & 0 & & \cdots & 0 & \cdots & & & \cdots\ 0 \\ \cdot\ \cdot\ \cdot & & & & & \cdots & & & \cdots & & \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & & \binom{n_1-1}{n_1-1} & & \cdots & \binom{n_2-1}{n_2-1} & & \cdots & & \binom{n_r-1}{n_r-1} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & & & & \cdots & & & \cdots & & \\ u_1^{n-1} & & \cdots & & u_2^{n-1} & \cdots & & & u_r^{n-1} & \cdots & \end{vmatrix}$$

$$= (u_1-u_2)^{n_1n_2}(u_1-u_3)^{n_1n_3}\cdots(u_{r-1}-u_r)^{n_{r-1}n_r}$$

$$\neq 0 .$$

Hence the solutions is

† The value of D was found by L. Schendal in 1891. See the book " The Theory of Determinants in the Historical Order of Development ", Vol. 4 ,p.178-180.

$$\sum_{j=1}^{n_1} v_j = \sum_{j=1}^{n_1} b_{jj}^{(1)}, \quad \sum_{j=1}^{n_2} v_{n_1+j} = \sum_{j=1}^{n_2} b_{jj}^{(2)}, \cdots, \quad \sum_{j=1}^{n_r} v_{n-n_r+j} = \sum_{j=1}^{n_r} b_{jj}^{(r)},$$

$$\sum_{j=k+1}^{m_{11}} b_{j\,j-k}^{(1)} + \sum_{j=k+1}^{m_{12}} b_{m_{11}+j\ m_{11}+j-k}^{(1)} + \cdots + \sum_{j=k+1}^{m_{1s_1}} b_{n_1-m_{1s_1}+j\ n_1-m_{1s_1}+j-k}^{(1)}$$

$$= 0 \ , \ (k = 1,2,\cdots,n_1-1) \ ,$$

$$\sum_{j=k+1}^{m_{21}} b_{j\,j-k}^{(2)} + \sum_{j=k+1}^{m_{22}} b_{m_{21}+j\ m_{21}+j-k}^{(2)} + \cdots + \sum_{j=k+1}^{m_{2s_2}} b_{n_2-m_{2s_2}+j\ n_2-m_{2s_2}+j-k}^{(2)}$$

$$= 0 \ , \ (k = 1,2,\cdots,n_2-1) \ ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{j=k+1}^{m_{r1}} b_{j\,j-k}^{(r)} + \sum_{j=k+1}^{m_{r2}} b_{m_{r1}+j\ m_{r1}+j-k}^{(r)} + \cdots + \sum_{j=k+1}^{m_{rs_r}} b_{n_r-m_{rs_r}+j\ n-m_{rs_r}+j-k}^{(r)}$$

$$= 0 \ , \ (k = 1,2,\cdots,n_r-1) \ .$$

The theorem is proved.

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