A GENERALIZATION OF THE FIRST PLÜCKER FORMULA

BY

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Abstract of

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The first Plücker formula from algebraic geometry gives the class of an algebraic curve in terms of the order and the singularities of the curve. Here a study is made of real, differentiable curves with a view to finding the corresponding result for such curves. The class of a point P with respect to a real, differentiable curve C is defined to be the number of tangents of C which pass through P. First it is shown how the class of P depends on its position relative to C, then it is shown how the class of P depends on the nature, numbers, and relative positions of the singularities of C.

In the last Chapter the results are applied to classify real, differentiable curves of class three. It is found that a curve of class three must contain one of the following three combinations of singularities:

(1) One cusp and one inflection point.

(2) One cusp and one double tangent.

(3) Three cusps.

A GENERALIZATION OF THE FIRST PLÜCKER FORMULA

CHAPTER I

1.1 Introduction. The first Plücker formula from algebraic geometry expresses the class of an algebraic curve in terms of its order and its singularities. In this thesis an attempt is made to find the corresponding result for a real, differentiable curve. In the first three chapters a method is developed which may be used in studying the effects of singularities on the class of such a curve. In the last chapter this method is used to classify differentiable, closed curves of class 3.

1.2 The projective plane. The space considered in this thesis is the projective plane. The chief properties of the projective plane are:

(i) Every pair of lines in the projective plane has a point in common.

(ii) Lines in the projective plane are closed.

(iii) The projective plane is locally affine. This means that any finite region of the projective plane has the properties of the affine plane.

1.3 Curves. A curve is defined to be a single-valued continuous mapping of the projective line. Such a mapping may be interpreted in two ways. In the first interpretation the points of the projective line are mapped into the points of the curve. In the other interpretation the points of the projective line are mapped into the tangents of the curve. Each of these curves is called the dual of the other.

1.4 Order. The order of a curve is the greatest number of points common to the curve and any straight line.

1.5 Differentiability. Tangent. Let a secant a intersect a curve C(s) in points C(s1)# and C(s2), where s is the curve parameter. Let s2 approach s1. If as s2 approaches s1, a approaches a limit which is independent of the manner in which s2 approaches s1, then C(s) is differentiable at C(s1). The limit approached by a is the tangent to C(s) at C(s1).

Throughout this thesis it is assumed that curves are of finite order and differentiable at every point.

1.6 The principle of duality. It can be proved that any proposition concerning lines and points which has been proved for differentiable curves, holds in the dual. That is, the proposition holds if the roles of lines and points are interchanged.

1.7 Dual differentiability. Scherk[1] has proved that if a curve is differentiable its dual is differentiable.

The dual of the definition given in 1.5 is:

Let A be the point common to two tangents t(s1) and

Due to typing difficulties, subscripts are placed on the same lines as the letters to which they are affixed.

1 P. Scherk, Czechoslovakian Journal of Mathematics and Physics, Prague, 1936.

t(s2) of the curve whose tangent is t(s). Let s2 approach s1. If as s2 approaches s1, A approaches a limit which is independent of the manner in which s2 approaches s1, then the curve is differentiable in the dual sense. The limit approached by A is the point of contact of t(s1).

If a curve C is differentiable (by 1.5) then, by the principle of duality, the dual curve of C is differentiable in the dual sense. But, by Scherk's result, the dual curve is differentiable (by 1.5), so its dual, the curve C, is differentiable in the dual sense. In other words, if a curve is differentiable it is differentiable in both senses.

1.8 Elementary Arc. An elemntary arc is an open part of a curve which has at most two points in common with a straight line.

It has been shown by Hjelmslev[2] that a curve of finite order is made up of a finite number of elementary arcs.

1.9 Lines of support and lines of intersection. A line h which has a point P in common with a curve C is a line of intersection of C at P if in any neighborhood of P, however small, there exist points of C on both sides of h. Otherwise h is a line of support.

Scherk[1] showed that if one non-tangent of a curve C at a point P is a line of intersection at P, then all non-

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2. J. Hjelmslev, Om Grundlaget for laeren om simple Kurver, Nyt Tidsk.f. Math., 1907.

tangents at P are lines of intersection and if one non-tangent at P is a line of support at P, then all non-tangents at P are lines of support.

1.10 Characteristic. The characteristic of a point P is given by a pair of numbers determined as follows:

(a) The first number is one or two according as non-tangents at P are lines of intersection or lines of support.

(b) The second number is one or two chosen so that:

(i) If the tangent at P is a line of intersection then the sum of the numbers is odd.

(ii) If the tangent at P is a line of support then the sum of the numbers is even.

1.11 Classification of curve points. The classification of curve points is given in the first two columns of the following table. It can be shown that the dual characteristics are those which are given in the third column.

POINT	CHARACTERISTIC	DUAL CHARACTERISTIC	DUAL POINT
ordinary point	(1, 1)	(1, 1)	ordinary point
cusp	(2, 1)	(1, 2)	inflection point
inflection point	(1, 2)	(2, 1)	cusp
bill cusp	(2, 2)	(2, 2)	bill cusp

It can also be shown that every interior point of an elementary arc is an ordinary point of the curve of which the arc is a part.

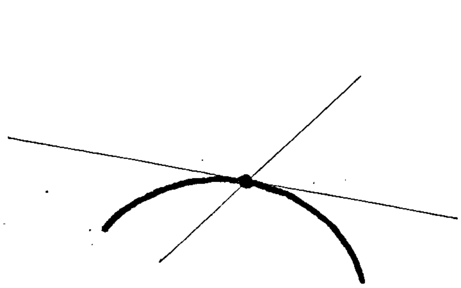
1.12 <u>Nodes and double tangents</u>. If in the mapping defined in 1.3, two different points of the projective line are mapped onto the same point of the curve then the point is called a node. The dual of a node is a double tangent. A double tangent occurs when two different points of the projective line are mapped into the same tangent of the curve.

1.13 <u>Singularities</u>. Cusps, inflection points, bill cusps, nodes, and double tangents will be referred to as singularities.

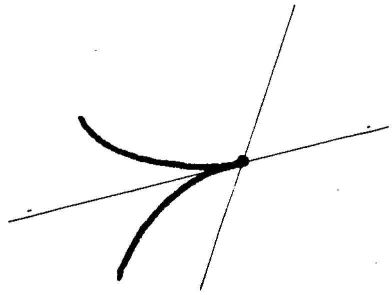
1.14 <u>Critical tangents</u>. Tangents at inflection points and bill cusps will be called critical tangents.

1.15 <u>The class of a point</u>. The class of a point P with respect to a curve C is the number of tangents to C which pass through P.

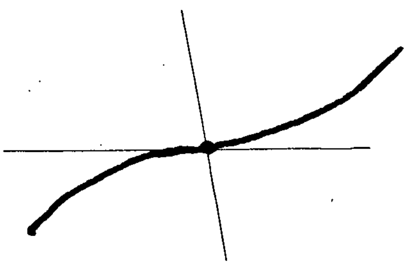
The tangent to a curve C at a point P on C will be counted as one tangent through P.



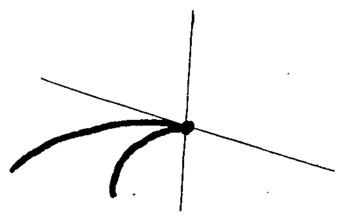
Ordinary point (1,1)



Cusp (2,1)



Inflection point (1,2)



Bill cusp (2,2)

Node

Double tangent

CHAPTER II

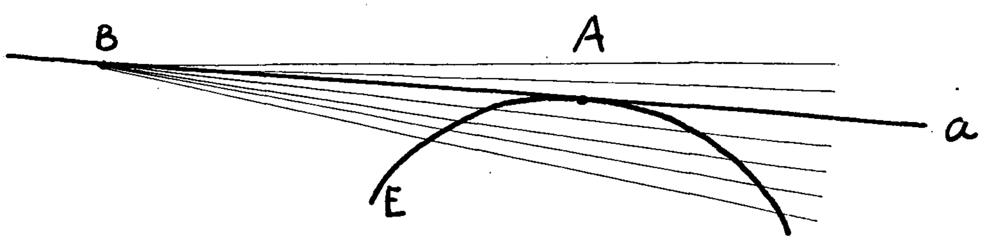
Given an algebraic curve Ca, the class with respect to Ca of points in the plane is constant, depending only on the singularities of Ca. In other words the same number of tangents to Ca, real or imaginary, pass through every point in the plane. For a real, differentiable curve C, however, the class with respect to C of a point in the plane depends not only on the singularities of C but also on the location of the point. In this chapter it will be shown that for a given curve C, the curve C and its critical tangents divide the plane into regions such that all points in a given region have the same class with respect to C.

2.1 <u>Theorem</u>. On any point on any interior tangent of an elementary arc there exist lines which contain two points of the arc.

Proof. Let A be an interior point on an elementary arc E. Let a be the tangent at A. Let B be any point on a. The problem is to show that there exist lines through B which contain two points of E.

If B=A the line through B and any other point of E is a line through B which contains two points of E. Therefore the theorem is true if B=A.

Suppose B≠A. Since A is an ordinary point, a is a line of support and there exists a neighborhood N of A such



that all points of E in N lie on one side of a. Let a moving point P trace E. E is continuous so as P approaches A, P intersects all lines of the pencil on B which lie sufficiently near a and on the same side of a as E. As P recedes from A, P remains on the same side of a and so intersects the same lines of the pencil on B. Thus there exist lines through B which contain two points of E. This proves the theorem completely.

2.2 <u>Theorem.</u> An end-tangent to an elementary arc E cannot contain an interior point of E.

Proof. Suppose the tangent a at an end-point A of E contains an interior point P of E. Since E is continuous and differentiable there exists a tangent a' near a which contains a point P' on E and in a neighborhood of P. Let N be a neighborhood of the point of contact of a' which does not contain P'. By Theorem 2.1 there exists a line h on P' which contains two points of the part of E which lies in N. Then h contains three points of E. By the definition of an elementary arc this is impossible. Therefore an end-tangent of E cannot contain an interior point of E.

According to the definition (1.8) of an elementary arc, an end-tangent a of an elementary arc E may contain both end-points of E. In the remainder of this thesis it will be convenient to assume that an end-tangent to an elementary arc E does not contain both end-points of E.

2.3 Theorem. If P is any point on an elementary arc E then the class of P with respect to E is one.

Proof. E is differentiable at P; therefore there exists at least one tangent of E which passes through P, namely: the tangent at P itself.

Suppose a second tangent a of E passes through P. Let A be the point of contact of a. By Theorem 2.2, A is an interior point of E. Let N be a neighborhood of A which does not contain P. By Theorem 2.1, there exists a line h through P which contains two points of that part of E which lies in N. Then h contains three points of E. By the definition of an elementary arc this is impossible. Therefore exactly one tangent of E passes through P. Therefore the class of P with respect to E is one.

2.4 Theorem. On any line through any interior point of an elementary arc there exist points which contain two tangents to the arc.

Proof. This is the dual of Theorem 2.1 and is therefore true by the principle of duality.

2.5A Lemma. Suppose a secant s contains two points S1 and S2 of a curve C. Let S3 be an ordinary point of C. If S1 and S2 both approach S3 along C then s approaches the tangent at S3.

Proof. Let E be a finite elementary arc of which S3 is an interior point. Let N be a neighborhood of S3 which does not contain the end-points of E. As S1 and S2 approach S3 along that part of E within N they will be the end-points of an elementary arc E' which is contained in E. Since s contains only the end points of E', E' will be entirely on one side of s. The remaining part of E in N will lie on the side of s opposite to E'. As S1 and S2 approach S3, E' vanishes and the part of E in N lies entirely on one side of s. Hence s approaches the tangent at S3.

The dual of this lemma is:

Suppose S is the point common to two tangents s1 and s2 of a curve C. Let s3 be the tangent at an ordinary point of C. If s1 and s2 both approach s3 along C then S approaches the point of contact of s3.

2.5 Theorem. Let C be a curve and let h be a line which intersects C in an ordinary point A. Let a rolling tangent t roll along C. As t rolls through A the intersection of t with h reverses.

Proof. Let E be any elementary arc of C of which A is an interior point. Let P be a point on h which has class 2 with respect to E. Let t(s1) and t(s2), where s is the curve parameter, be the two tangents to E which meet in P. E may be chosen arbitrarily small so it may be assumed that some points on h have class zero with respect to E. Then as t rolls along E, t does not intersect all points of h yet t intersects P twice. Therefore there exists a value s3 (s1<s3<s2) for which the intersection of t(s) with h reverses.

Let P approach the reversal point. Then s1 approaches s3 and s2 approaches s3 and both t(s1) and t(s2) approach t(s3). But from the dual of Lemma 2.5A, as t(s1) and t(s2) approach a common tangent their intersection approaches a curve point. Therefore the point of reversal of the intersection of t with h is a curve point, namely: A. This proves the theorem.

2.6 Theorem. Let A be an ordinary or an inflection point of a curve C. Let h be any line on A other than the tangent at A, and let B be any point on h other than A. Then, for a sufficiently small neighborhood N of A, any line h' of the pencil on B which is sufficiently close to h intersects the part of C contained in N exactly once.

Proof. Let a point P trace C. Since h is a line of intersection at A, P will cross lines on both sides of h. If N is sufficiently small, h has no point other than A in common with the part of C contained in N; hence, P will cross, once, lines of the pencil on B which are sufficiently close to h.

2.7 Theorem. Let A be a cusp or a bill cusp on a curve C. Let Na be the part of C in a sufficiently small neighborhood of A. Let h be any line on A other than the tangent at A and let B be any point on h other than A. Then a line h' of the pencil on B, which is sufficiently close to h, intersects Na twice or not at all depending on the side of h on which h' lies.

Proof. Let a point P trace Na. Since h is a line of support at A, Na lies entirely on one side of h so as P traces Na, P will not cross h. As P approaches A, since C is continuous, P will cross, once, all lines of the pencil on B which are sufficiently close to h and on one side of h. As P recedes from A, P will cross, once more, the same lines of the pencil on B. This proves the theorem.

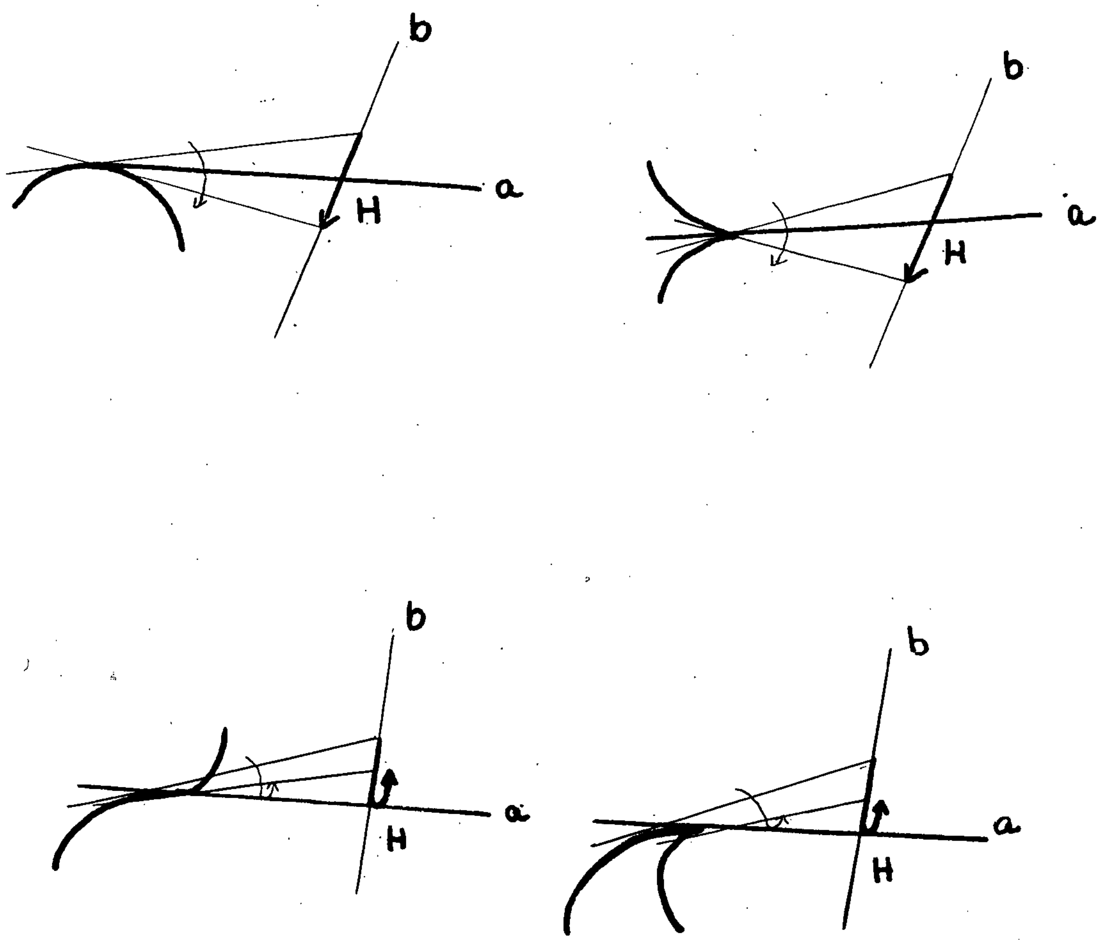
2.8 Theorem. Let a be tangent to a curve C at an ordinary point or a cusp. Let Na be the part of C in a sufficiently small neighborhood of the point of contact of a. Let H be any point on a other than its point of contact and let b be any line on H other than a. Then any point H' on b which is sufficiently close to H has class one with respect to Na.

Proof. This is the dual statement of Theorem 2.6 and is therefore true by the principle of duality.

2.9 Theorem. Let a be a critical tangent to a curve C. Let Na be the part of C in a sufficiently small neighborhood of the point of contact of a. Let H be any point on a other than its point of contact and let b be any line on H other than a. Then a point H' on b, which is sufficiently close to H, has class two or zero with respect to Na depending on the side of H on which H' lies.

Proof. This is the dual statement of 2.7 and is therefore true by the principle of duality.

DIAGRAMS ILLUSTRATING THEOREMS 2.8 AND 2.9



In the following corollaries a is a tangent to a curve C, H is any point on a other than its point of contact, and Na is the part of C in a suitably chosen neighborhood of the point of contact of a.

2.10 Corollary. If a is a non-critical tangent there exists a neighborhood of H in which every point has class one with respect to Na.

This follows from 2.8.

2.11 Corollary. If a is a critical tangent, then in any neighborhood of H, however small, there exist points with class zero with respect to Na and points with class two with respect to Na.

This follows from 2.9.

2.12 Corollary. If a moving point P crosses a at H then if a is a non-critical tangent the class of P with respect to Na does not change but if a is a critical tangent the class of P with respect to Na changes by two.

This follows from 2.8 and 2.9.

2.13 Theorem. If a fixed line h intersects a curve C in ordinary points only, then as a tangent t rolls along C, its intersection with h reverses if and only if:

(i) t rolls through an inflection point.

or (ii) t rolls through a bill cusp.

or (iii) t rolls through a point common to C and h.

Proof. This theorem follows immediately from 2.5 and 2.12.

Suppose A is an interior point of an elementary arc E. Let h be any line through A other than the tangent at A. Let a tangent t roll over E and let I be the intersection of t with h. As t rolls over E, I moves along h to A, reverses at A, and moves back along h. I does not reverse again, by theorem 2.13, and since A is a point on E the class of A with respect to E is one, so I does not return to A. Thus no point on h is covered by I more than twice. It is also seen that there exists a neighborhood N of A such that points of h which lie in N and on one side of E are covered twice whilepoints of h on the other side of E are not covered by I. This discussion leads to the two following theorems:

2.14 Theorem. If E is an elementary arc and if P is any point in the plane then the class of P with respect to E is not greater than two.

Proof. If P lies on E then, by Theorem 2.3, the class of P with respect to E is one.

Suppose P does not lie on E. Let h be the line common to P and an interior point of E. From the foregoing discussion, no point on h has class greater than two with respect to E. Therefore the class of P with respect to E is not greater than two.

2.15 Theorem. If a moving point P crosses an elementary arc E at an interior point then the class of P with respect to E changes by two.

Proof. Assume that as P crosses E it moves along a line h.

From the discussion preceeding 2.14, points of h near E and on one side of E have class two with respect to E while points of h near E and on the other side of E have class zero with respect to E. Therefore as P moves along h across E its class with respect to E changes by two.

2.16 <u>Theorem</u>. If a moving point P crosses an end-tangent b of an elementary arc E at any point other than the point of contact of b or the point common to b and the other end-tangent of E, than the class of P with respect to E changes by one.

Proof. Let H be any point on b which is not on E or the other end-tangent of E. Let h be any line through H other than b. Let a tangent t start at b and roll over E and let I be the point of intersection of t with h. As t rolls over E, I moves off from H covering points of h near H and on one side of b. I may or may hot return to H.

If I does not return to H, then points of h near b and on one side of b have class one with respect to E, while points of h near b and on the other side of b have class zero with respect to E.

Suppose I does return once to H. I will not stop at H since H is not a point on the other end-tangent of E and, by Theorem 2.13, I will not reverse at H. Therefore I will move through H covering once all points of h near b and on both sides of b. So if I returns once to H, points of h near b and on one side of b have class two with respect to E, while points of h near b and on the other side of b have class one.

I cannot return to H a second time since the class of H with respect to E cannot exceed two.

It is seen that points of h near b and on one side of b have class one greater than points of h on the other side of b. Therefore if a point P moves across b its class with respect to E changes by one.

2.17 Theorem. Let E be an elementary arc. Let P be a point which is not contained in E or in either of the end-tangents of E. If the class of P with respect to E is c (c=0, 1, or 2) then in any neighborhood of P, which contains no points of E or of the end-tangents of E, all points have class c with respect to E.

Proof. Let N be a neighborhood of P which contains no points of E or of the end-tangents of E. Let h be any line containing P. Let a tangent t roll over E and let I be the intersection of t and h. N contains no points of the end-tangents of E, so the end-points of I are not contained in N. I contains no point of E, so the reversal points of I, if any exist, lie outside of N. Therefore every time I covers a point of h in N it must cover all points of h in N. Therefore since P has class c with respect to E then all points of h in N have class c with respect to E. It follows that all points in N have class c with respect to E.

2.18 Corollary. A moving point P changes its class with respect to an elementary arc E if and only if it crosses E or an end-tangent of E.

2.19 Theorem. A curve C and the critical tangents to C divide the plane into regions of uniform class with respect to C.

Proof. Suppose a point P moves about the plane. From 2.18, the only way in which P can change its class with respect to an elementary arc is by crossing either the arc itself or one of its end-tangents.

Since C is composed of a finite number of elementary arcs the class of P with respect to C is the sum of its classes with respect to the component arcs of C. Hence the only ways in which the class of P with respect to C can change are:

(a) P may cross C itself, thereby changing its class with respect to the elementary arc in the neighborhood of the crossing point.

(b) P may cross a tangent to C, thus changing its class with respect to the two elementary arcs in the neighborhood of the point of contact of the tangent. It follows from 2.12 that the class of P with respect to C changes only if the tangent crossed is a critical tangent.

Therefore the curve C and its critical tangents divide the plane into regions of equal class with respect to C.

CHAPTER III

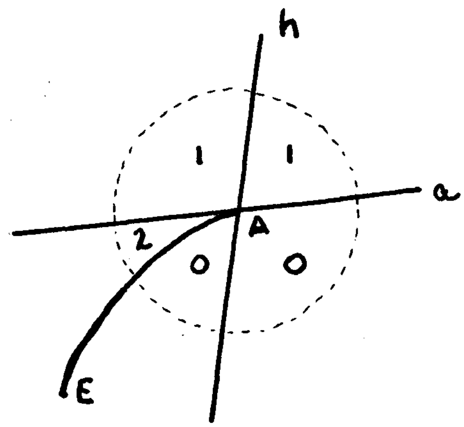
At the outset of Chapter II it was stated that for a differentiable curve C, the class of P varies with the location of P relative to C. In Chapter II we established regions such that in a given region the class of P is constant and such that as P crosses from one region into an adjoining region its class increases or decreases by two. In this part we wish to obtain a method for finding the class of points in each region.

3.1 Theorem. Let E be an elementary arc with an end-point A. Let a be the tangent at A. Let h be any line through A other than a. Let N be a neighborhood of A which contains no point of the other end-tangent of E. If h contains a point B of E other than A choose N small enough to exclude B. Then a and h divide N into four parts as follows:

(a) Two parts, those on the opposite side of a from E, in which all points have class one with respect to E.

(b) One part, on the same side of a as E but on the opposite side of h, in which all points have class zero with respect to E.

(c) One part, on the same side of a as E and on the same side of h as E, in which points not on E have class zero or two with respect to E depending on the side of E on which they lie.



Proof. If a point P moves about within N the only way in which it can change its class with respect to E is by crossing either E or a.

Suppose P moves within the half of N which lies on the same side of a as E. As P moves across E its class with respect to E changes by two. Therefore, except when P is on E itself, the class of P with respect to E can never be one. Therefore E divides this half of N into two parts, one in which all points have class two with respect to E and one in which all points have class zero with respect to E. To prove (b) and (c) it will be sufficient to show that h sub-divides the part in which all points have class zero with respect to E.

Let a tangent t start at a and roll over the part of E which is contained in N. The point of intersection I of t with h starts at A and moves along h. I will not reverse, and it will not return to A since A has class one with respect to E. Therefore no point on h is covered twice by I. Therefore h contains no points of class two with respect to E and so the points of h which lie in N and on the same side of a as E have class zero with respect to E. This proves (b) and (c).

Suppose P moves across a into the half of N which lies on the side of a opposite to E. By 2.16, the class of P will change by one. But before crossing a the class of P was either zero or two. Then, since the class of P cannot exceed two, the class of P after crossing a must be one. This proves (a).

Given a curve C, we wish to obtain a method for finding the class of points in each of the regions defined in Chapter II. Suppose we know the class of some point S. Then we can find the class of any other point A if we let a moving point P move from S to A and note the changes in the class of P as it crosses C and the critical tangents of C. The problem is to find the class of a point S.

The method will be to choose an ordinary point S on C, let a moving point P start at S, trace C, and return to S. At any point in the tracing we will let k denote the class of P with respect to the part of C traced up to that point. If we can keep count of k throughout the tracing then as P arrives back at S, k will give us the class of S with respect to the complete curve C.

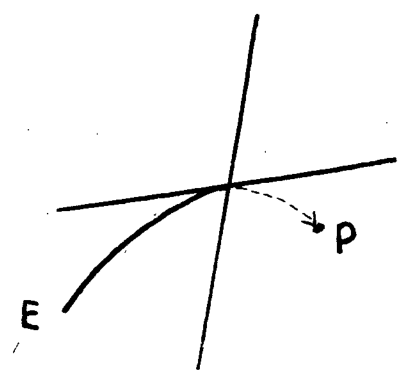
It will be convenient to think of P as continually moving out of one elementary arc of C into the adjoining elementary arc. Changes in k will be considered in three parts,

(a) the point being traced, (b) the elementary arc E just traced, and (c) that portion of C which preceeds E.

(a) <u>The point being traced</u>. We will think of P as continually "picking up" a tangent, namely: the tangent at the point being traced.

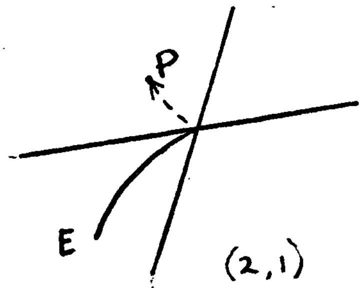
(b) <u>The elementary arc E just traced</u>. Let A denote the end-point of E.

(i) Suppose P traces an ordinary point. Since an ordinary point has characteristic (1,1) then as P traces an ordinary point, P does not cross the tangent at A but P does cross any other line h through A so, by 3.1 (b), P "loses" a tangent to E. But P "picks up" the tangent at the point being traced. The net result is that k does not change.

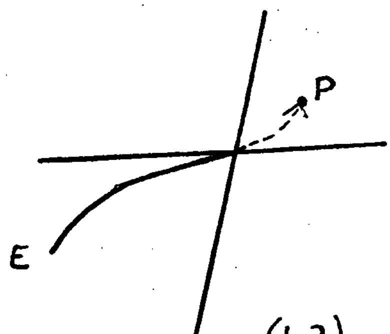


(1, 1)

(ii) Suppose P traces a cusp (2,1) or an inflection point (1,2). In this case P crosses the tangent at A thus, by 3.1 (a), P "retains" one tangent to E. But P "picks up" the tangent at the point being traced. The net result is that k increases by one.

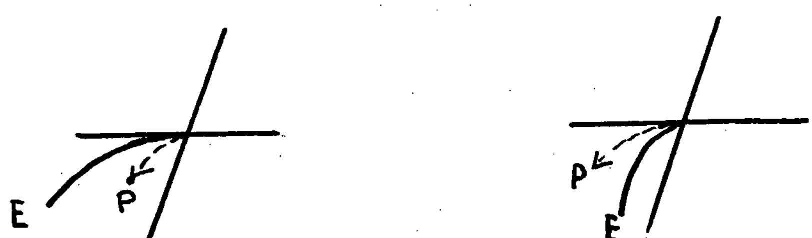


(2,1)



(1,2)

(iii) Suppose P traces a bill cusp (2,2). In this case P does not cross either the tangent at A or any other line h through A. Therefore, by 3.1 (c), P either "picks up" one tangent to E or "loses" one tangent to E. But P "picks up" the tangent at the point being traced. The net result is that either k increases by two or k does not change.



(2,2)

(c) <u>That portion of C which preceed E</u>. Here Theorems 2.16 and 2.19 apply. That is, if P crosses a part of the curve which has been traced, or a critical tangent to that part of the curve, then by 2.19, k increases or decreases by two. If P crosses the tangent at the starting point S, then the class of P with respect to the elementary arc of which S is an end point increases or decreases by one, depending on the position of crossing and the direction of crossing. This follows from 2.16.

3.2 <u>Summary</u>. If a point P starts at an ordinary point S and traces a curve C and if, at any point in the tracing, k is the class of P with respect to the part of C traced up to that point, then k changes according to the following:

(1) If P traces a cusp or an inflection point, k increases by one.

(2) If P traces a bill cusp, k either does not change or increases by two.

(3) If P crosses the part of C which has been traced, k increases by two or decreases by two.

(4) If P crosses a critical tangent of that part of C which has been traced, k increases or decreases by two.

(5) If P crosses the tangent at the starting point, k increases or decreases by one.

CHAPTER IV

4.1 <u>The class of a curve.</u> The class of a curve C is given by the greatest number of tangents to C which contain a common point.

In this chapter closed, differentiable curves of class three are considered. The methods of Chapters II and III are used to classify such curves.

Throughout this chapter C=C(s) ($-\infty < s \leq \infty$) denotes a closed differentiable curve of class three. t=t(s) denotes the tangent to C(s). Ns denotes the part of C in any suitably chosen neighborhood of the point C(s).

4.2 <u>Theorem.</u> A critical tangent to C contains no ordinary points of C.

Proof. Suppose t(s1) is a critical tangent to C and suppose t(s1) contains an ordinary point C(s2) of C. By Theorem 2.4, there exists a point P on t(s1) such that the class of P with respect to Ns2 is two. By Theorem 2.17, there exists a neighbrohood M of P wherein all points have class two with respect to Ns2. By 2.11, there exists a point P' in M such that the class of P' with respect to Ns1 is two. Then the class of P' with respect to C is at least four. This is impossible since C is a curve of class three. Therefore a critical tangent to C contains no ordinary point of C.

4.3 Theorem. The class of an ordinary point on C cannot exceed two.

Proof. Suppose an ordinary point C(s1) has class at least three with respect to C. Let the three tangents containing C(s1) be t(s1), t(s2), and t(s3). By 4.2, these are non-critical tangents. By 2.10 there exists a neighborhood M of C(s1) in which all points have class one with respect to each of Ns2 and Ns3. But, by Theorem 2.15, M contains a point P whose class with respect to Ns1 is two. Then P has class at least four with respect to C. This is impossible since C is a curve of class 3. Therefore the class of an ordinary point on C cannot exceed two.

4.4 Theorem. Suppose a moving point P traces the curve C. Let C(0) be the starting point and let k(s1) denote the class of the point C(s1) with respect to that part of C(s) for which $0 \leq s \leq s1$. As P traces C, k is always at least one since there is a tangent at the point P itself and, from 4.3, k cannot exceed two. Therefore k is always either one or two and k cannot change by more than one.

4.5 Corollary. It follows from 4.4 and 3.2 that the only ways in which k can change are:

(i) If P traces an inflection point k increases by one.

(ii) If P traces a cusp k increases by one.

(iii) If P crosses t(0) either k increases by one or k decreases by one.

4.6 Corollary. If C(s1) is a cusp or an inflection point then

$$\text{for } s \leq s1 \text{ and } |s-s1| < \epsilon,\ k(s)=1$$

$$\text{and for } s > s1 \text{ and } |s-s1| < \epsilon,\ k(s)=2$$

since as P traces C(s1), k increases by one.

4.7 Theorem. If C(s1) is an inflection point or a cusp and C(s3) is an inflection point or a cusp, then for some value s2 where $s1 < s2 < s3$, C(s2) lies on t(0).

Proof. By 4.6, k(s)=2 for $s > s1$ and $|s-s1| < \epsilon$. Also by 4.6, k(s)=1 for $s < s3$ and $|s-s3| < \epsilon$. Hence there exists some point C(s2) ($s1 < s2 < s3$) where k decreases by one. By 4.5 the point C(s2) must lie on t(0).

4.8 A K2. A curve of order two on a projective one-space, which is denoted by the symbol K2, is defined to be a single-valued continuous mapping of the projective line onto the projective line where no point is covered more than twice.

A well-known property of a K2 is that it has at most two reversals.

4.9 Theorem. Let C(s1) be an ordinary point on C with tangent t(s1) where t(s1) contains no singularities of C. Let the tangent t roll once over C. Then the intersection of t with t(s1) has at most two reversals.

Proof. Consider the following mapping:

let I(s) be the intersection of t(s) with t(s1) for $s \neq s1$,

let I(s1)=C(s1).

I(s) defines a single-valued mapping of the points of C onto the line t(s1). I(s) is continuous for $s \neq s1$ because of the differentiability of C and I(s) is also continuous for s=s1 since, by the dual differentiability of C, $\lim_{s \to s_1} I(s)=C(s1)$. Further, no point on t(s1) is covered more than twice since three coincident values of I, say I(s2), I(s3) and I(s4), would imply four concurrent tangents namely: t(s1), t(s2), t(s3), and t(s4). Thus I(s) generates a curve on t(s1) which satisfies the condition for a K2. Therefore I(s) has at most two reversals. This proves the theorem.

4.10 <u>Theorem</u>. The only singularities that can occur in C are cusps, inflection points and double tangents.

Proof. It follows from 4.4 that C can contain no nodes or bill cusps since in the tracing of either of these points k would change by two. The only singularities that remain are cusps, inflection points, and double tangents.

4.11 <u>Theorem</u>. C has at least one cusp and, in any case C has an odd number of cusps.

Proof. Consider the dual problem.

Let $\overline{t}(s)$ be the tangent to a closed curve $\overline{C}(s)$ of order three. Let $\overline{C}(0)$ be an ordinary point. Let I(s) be the intersection of $\overline{t}$ with a fixed line h which does not pass through any singularities of $\overline{C}$. Since $\overline{C}$ is closed and continuous, and $\overline{t}$ is continuous, then I generates on h a continuous closed curve which has an even number of reversals. If a point I(s1) is a reversal point on h then, by 2.13, either $I(s1)=\overline{C}(s1)$ or $\overline{C}(s1)$ is an inflection point. (By the dual of 4.10 $\overline{C}$ contains no bill cusps).

Let the number of reversals of I be U(even). Since $\bar{C}$ is closed and of odd order, the number of points common to $\bar{C}$ and h is odd and at least one. Let the number of points common to $\bar{C}$ and h be V(odd). Let W be the number of inflection points on $\bar{C}$. Then from the foregoing

$$W + V(\text{odd and at least one}) = U(\text{even}),$$

whence W is odd and at least one. Therefore $\bar{C}$ has at least one inflection point and in any case an odd number of inflection points. This, dualized, gives the statement which was to be proved.

4.12 Theorem. If A is a cusp or an inflection point of a curve C, then in any neighborhood N of A there exist points which have class three with respect to that part of C which lies in N.

Proof. Let E1 and E2 be simple arcs in N which are joined at A. We know from 3.2 that there exists a curve point P on E1 which contains a tangent t of E2. By 2.15, a point P' on t which lies sufficiently near P will have class two with respect to E1. Then P' has class three with respect to that part of C which lies in N.

The dual of this theorem is:

If a is the tangent at an inflection point or a cusp of a curve C, then there exist lines in a neighborhood, N of a which intersect, three times, that part of C which lies in N.

4.13 Theorem. C cannot have one cusp alone without any other singularity.

Proof. Consider the dual problem.

Suppose $\overline{C}$ is a closed curve of order three with one inflection point and no other singularity. Let the inflection point be $\overline{C}(0)$ and, with this point as starting point, let a point P trace $\overline{C}$. P cannot cross $\overline{t}(0)$ because if P crossed $\overline{t}(0)$ in a point $\overline{C}(s1)$ then, by the dual of 4.12, a line in the neighborhood of $\overline{t}(0)$ would cut $\overline{C}$ four times, three times in the neighborhood of $\overline{C}(0)$ and once in the neighborhood of $\overline{C}(s1)$.

Consider k (the class of P with respect to the part traced) as P approaches $\overline{C}(0)$. Since P has traced no singularities and has not crossed $\overline{t}(0)$, then k=1 as P approaches $\overline{C}(0)$. But from 3.2, k must be at least two. Therefore $\overline{C}$ cannot have one inflection point alone without any other singularities.

The dual of the result is:

A closed curve C of class three cannot have one cusp alone without any other singularity.

4.14 <u>Theorem</u>. C cannot have:

(a) An inflection point and more than one cusp.

or (b) More than one inflection point.

or (c) More than three cusps.

Proof. Let h be the tangent at an ordinary point of C which contains no singularities of C. Let a tangent t roll over C. Suppose C contains a total of n cusps and inflection points. Take the point of contact of h as the starting point and let P trace C. Then, by 4.7, C will cross h n-1 times. By 2.13 (iii), each of the n-1 crossing points causes a reversal of the intersection of t with h. By 2.13 (i), each inflection point causes an additional reversal so, if m is the number of inflection points on C, the number of reversals may be expressed $(n-1)+m$. By 4.9, this must not exceed two. That is we must have $(n-1)+m \leq 2$. By checking each of (a), (b), and (c) with this formula it is seen that the theorem is true.

4.15 Theorem. C has at most one double tangent because the point of intersection of two double tangents would have class four with respect to C.

4.16 Theorem. C cannot have a triple tangent since a point of contact of a triple tangent would be a curve point with class three with respect to C. This is impossible by 4.3.

4.17 Theorem. If C has a double tangent then C has at most one cusp.

Proof. Suppose C has a double tangent and more than one cusp. Taking one of the points of contact of the double tangent as C(0), let P trace C. Then by 4.7, P must cross t(0) at some point say C(s1). Then C(s1) has class three with respect to C. This is impossible by 4.3. Therefore the theorem is true.

4.18 Theorem. C cannot have a double tangent and an inflection point.

Proof. Suppose C has a double tangent and an inflection point. By Theorem 4.2, the inflection tangent cannot coincide with the double tangent. Let A be the point common to the double tangent and the inflection tangent. Let a point P move along the double tangent. The class of P will be at least two. Therefore as P passes through A, by corollary 2.12, the class of P changes either from two to four or from four to two. Both changes are impossible since C is a curve of class three. Therefore the theorem is true.

From the foregoing we conclude that any curve of class three must contain one of the following three combinations of singularities:

(a) One cusp and one inflection point.

(b) One cusp and one double tangent.

(c) Three cusps.

It remains to show the existence of curves of class three for each of the types (a), (b) and (c). This will be done by examples.

The method will be to consider curves which, from algebraic geometry, are know to be of order three. Then by using the duality principle we will confirm the above classification of curves of class three.

(a) <u>One cusp and one infledtion point</u>. Consider the third order curve

$$x_2^2 x_3 = x_1^3. \qquad (1)$$

In the neighborhood of $(0,0,1)$, (1) becomes

$$y^2 = x^3. \qquad (2)$$

The curve (2) passes through $(0,0)$ where it has slope zero.

Then the tangent to (2) at (0,0) is

$$y=0.$$

Inspection of (2) shows that (2) lies entirely on the right of $x=0$ and since $x=0$ is not tangent to (2) at (0,0), then non-tangents to (2) at (0,0) are lines of support.

It is also seen that (2) is symmetrical with respect to the tangent $y=0$ so the tangent at (0,0) is a line of intersection. Then the characteristic of (0,0) is (2,1) and (0,0) is a cusp. By 4.11, (1) also has an inflection point, so (1) is a curve of order three with one inflection point and one cusp. Its dual is a curve of class three with one cusp and one inflection point.

(b) <u>One cusp and one double tangent.</u> Consider the third order curve

$$x_2^2 x_3 = x_1^3 + x_1^2 x_3. \tag{3}$$

In the neighborhood of (0,0,1) this curve becomes

$$y^2 = x^2(x+1). \tag{4}$$

Differentiating,

$$y' = \frac{\pm(3x+2)}{2\sqrt{x+1}}. \tag{5}$$

Inspection of (4) and (5) shows that (4) passes through the origin and, at the origin, the curve has two distinct tangents. The point (0,0) then, is a node.

The curve (3) is a curve of order three with one inflection point and one node. Its dual is a curve of class three with one cusp and one double tangent.

(c) Three cusps. Consider the third order curve

$$x_2^2 x_3 = x_1(x_1^2 + x_3^2). \tag{6}$$

In the neighborhood of (0,0,1) the curve becomes

$$y^2 = x(x^2 + 1). \tag{7}$$

Differentiating,

$$y' = \frac{\pm(3x^2 + 1)}{2\sqrt{x(x^2+1)}}. \tag{8}$$

By inspection we see that (7) is symmetrical with respect to the x-axis, that (7) has no points on the left of the y-axis, and that (7) cuts the x-axis at the point (0,0) only. From this it follows that, at the point (0,0), the x-axis is a line of intersection and the y-axis is a line of support. Equation (8) shows that the y-axis is the tangent to (7) at (0,0), therefore the characteristic of the point (0,0) is (1,1) so (0,0) is an ordinary point.

We have seen that a curve of class three must contain one of the three combinations of singularities (a), (b), and (c) given on Page 32. By duality, a curve of order three must contain one inflection point and one cusp, or one inflection point and one node, or three inflection points.

The curve (6) must have at least two singularities. Therefore (7) must have at least one singularity since it contains all the points of (6) except the point at infinity. The point (0,0) is not a singularity. Therefore by symmetry any singularities in (7) must occur in pairs. But the only singularity that can occur twice in a curve of order three is an inflection point. Therefore (7) has three inflection points and its dual curve will be a curve of class three with three cusps.