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THE ELEMENTARY FUNCTION THEORY OF AN HYPERCOMPLEX VARIABLE AND THE THEORY OF CONFORMAL MAPPING IN THE HYPERBOLIC PLANE

by

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ABSTRACT OF THESIS

The present thesis is based on a paper by Bencivenga . In this paper the author develops a theory of function for the dual and bireal variables . He constructs the "retto" and "hyperbolic" planes for the geometric representation of the dual and bireal variables , respectively , and establishes a type of conformal mapping of these planes into themselves by means of differentiable functions of the variable . Further , in each of these planes he proves the analogue for the Cauchy integral theorem of the complex plane . Finally he shows that functions of the dual and bireal variable which possess all derivatives at a given point of the plane may be expanded in a Taylor series about that point . In the first chapter we give a summary of this paper .

Bencivenga's dual and bireal number systems , and also the complex number system , are two-dimensional cases of the n - dimensional associative , commutative linear algebra with unit element . In chapter II we generalize Bencivenga's function theory to functions over the above mentioned linear . An important class of results from the theory of functions of a complex variable are not generalizable , since they depend on the field properties peculiar to the complex algebra .

In chapter III we undertake a detailed study of the hyper- -bolic plane with particular reference to the conformal properties of differentiable functions of the bireal variable , as a special case of conformal transformation of the hyperbolic plane , we study the bilinear transformation . We find that

the rectagular hyperbola is the geom~~a~~etrical form which is invariant under this transformation of the hyperbolic plane . Singularities play a larger role in this theory than in the case of the analgous transformation theory of the complex plane .

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CHAPTER I

Introduction

In a paper entitled "Sulla Rappresentazione Geometrica Delle Algebre Doppie Dotate Di Modulo" * , U. Bencivenga has given a geometric representation and function theory for dual and bireal numbers . It is the purpose of the present thesis to investigate the function theory for a more general form of hypercomplex variable , and to develop the theory of conformal mapping in the plane of the bireal variable . We will give first a summary of the results of Bencivenga's work which will be required in later chapters .

I.I The Dual Number System .

The dual numbers are defined by $z = x + y w$, where x and y are real and 1 , w are basis elements which have the multiplication table :

	1	w
1	1	w
w	w	0

* Atti. Accad. Sci. Napoli Ser(3) V.2, No.7 (I946)

RETTO PLANE

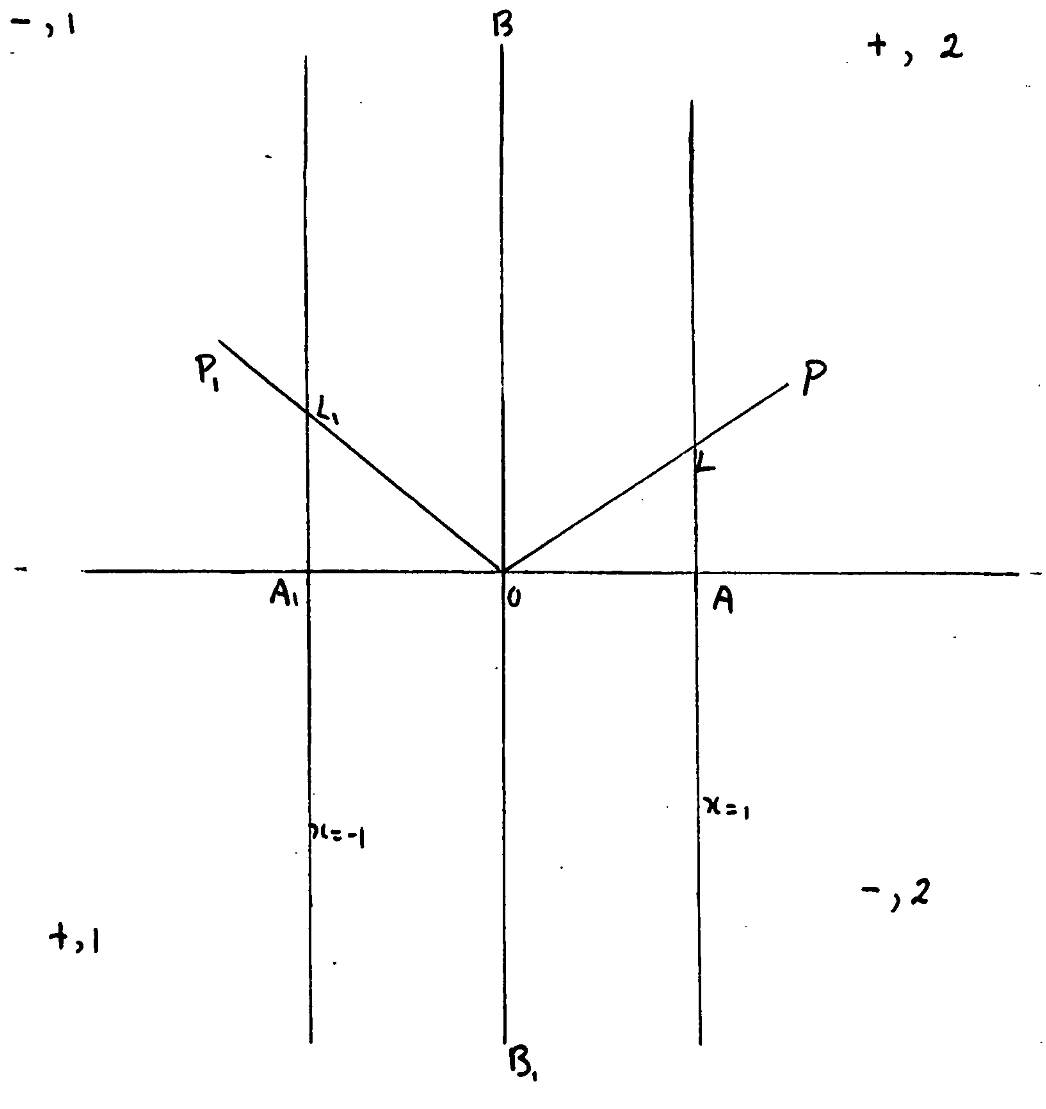

Figure I

Chapter I

These numbers form a commutative linear algebra over the real numbers . Moreover , the algebra of dual numbers is isomorphic with that of the 2×2 matrices :

$$z = x + yw \longleftrightarrow \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} .$$

The modulus of z is defined by

$$|z| = \sqrt{\begin{vmatrix} x & y \\ 0 & x \end{vmatrix}} = |x| .$$

I.2 Representation of Dual Numbers in the "Retto" Plane .

The retto plane consists of all points of the cartesian plane with the distance $\rho(P_1, P_2)$ between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ defined by $\rho(P_1, P_2) = |x_1 - x_2|$

This metric is symmetric and satisfies the triangle inequality. However , $\rho(P_1, P_2) = 0$ does not imply that $P_1 = P_2$.

In figure I , the vector OP defines a rettilinear angle whose magnitude is given by twice the area of triangle LOA , whose algebraic sign is positive , and which bears the sub--script 2 . Magnitudes of angles in other quadrants are deter--mined in the same manner , with algebraic signs and subscripts according to figure I . Addition of rettilinear angles is de--fined by $\phi_s + \psi_r = (\phi + \psi)_{s+r}$,

where subscripts are taken modulo 2 . Rettilinear sine and cosine are defined by

$$\operatorname{sinr} \phi_s = (-1)^s \phi \quad , \qquad \operatorname{cosr} \phi_s = (-1)^s ,$$

and satisfy the addition formulae :

$$\sin(\phi_s + \psi_n) = \sin\phi_s \cos\psi_n + \cos\phi_s \sin\psi_n$$

$$\cos(\phi_s + \psi_n) = \cos\phi_s \cos\psi_n .$$

The dual number $z = x + yw$ is represented in the retto plane by the point (x, y) , where x and y are signed Euclidean distances from the y- and x-axes , respectively .

If $|z| = r$, $am(z) = \phi_s$; then

$$x = r\cos\phi_s , \quad y = r\sin\phi_s , \text{ and}$$

$$z = x + yw = r(\cos\phi_s + w\sin\phi_s).$$

I.3 Elementary Operation Formulae .

Bencivenga establishes the following set of multiplication , division , and power formulae :

I. $r_1(\cos\phi_s + w\sin\phi_s) \cdot r_2(\cos\psi_n + w\sin\psi_n)$

$$= r_1 r_2 \left[\cos(\phi_s+\psi_n) + w\sin(\phi_s+\psi_n)\right].$$

2. For an integer $n \geq 0$:

$$\left[r(\cos\phi_s + w\sin\phi_s)\right]^n = r^n\left[\cos(n\phi)_{ns} + w\sin(n\phi)_{ns}\right].$$

3. For rational $\frac{p}{q} \geqslant 0$:

$$\left[r(\cos\phi_s + w\sin\phi_s)\right]^{\frac{p}{q}} = r^{\frac{p}{q}}\left[\cos\left(\tfrac{p}{q}\phi\right)_x + w\sin\left(\tfrac{p}{q}\phi\right)_x\right],$$

where $qx \equiv ps$, mod 2 .

From this equation for x , it follows that for numbers lying in quadrants designated by the subscript I , even roots do not exist within the dual system .

4. For irrational $\mu > 0$:

$$\left[r(\cos\phi_s + W\sin\phi_s)\right]^{\mu} = r^{\mu}\left[\cos(\mu\phi_s) + W\sin(\mu\phi_s)\right]$$

has a merely formal significance .

5. For $r \neq 0$

$$\frac{1}{r(\cos\phi_s + W\sin\phi_s)} = \frac{1}{r}(\cos\phi_s - W\sin\phi_s)$$

$$= \frac{1}{r}\left[\cos(-\phi_s) + W\sin(-\phi_s)\right].$$

6. Formulae 2 , 3 , 4 , 5 may be incorporated in the general formula :

$$\left[r(\cos\phi_s + W\sin\phi_s)\right]^{x} = r^{x}\left[\cos(x\phi_s) + W\sin(x\phi_s)\right]$$

valid for all real x .

I.4 Functions of the Dual Variable .

Functions over the dual numbers are defined in the usual manner : a function $F(z)$ is defined over a set of dual numbers when a method is given for uniquely determining a second dual number to correspond , as image, to any given member of the set . The mapping is expressible :

$$F(z) = F_1(x,y) + F_2(x,y)\cdot W ,$$

where F_1 , F_2 are real functions of x and y .

By definition , $F(z)$ is differentiable at z if there exists

$$\phi(z) = \phi_1(x,y) + \phi_2(x,y)\cdot W ,$$

such that

$$\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + w\left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy\right) = (\phi_1 + w\phi_2)(dx + w\,dy),$$

assuming the existence and continuity of the first partials of F_1 and F_2 with respect to x and y . From this defini- -tion Bencivenga derives the "Cauchy - Riemann" equations :

$$\frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial y}, \qquad (\text{I.4I})$$

$$\frac{\partial F_1}{\partial y} = 0,$$

which , with the existence and continuity of the partials , are necessary and sufficient conditions for the existence of the derivative $F'(z)$ of $F(z)$ at a point z . The real components of a differentiable function $F(z)$ are of the form

$$F_1 = F_1(x) \qquad F_2 = yF_1'(x) + \Psi(y)$$

and the derivative is given :

$$F'(z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial x}\cdot w \qquad (\text{I.42})$$

A power series $\sum_{k=0}^{\infty} A_k z^k$ will define a function of z if the corresponding real (component) series both converge . Writing $A_k = a_k + b_k w$, we require the convergence of

$$\sum a_k x^k \quad \text{and} \quad \sum b_k x^k .$$

That is , $\sum A_k z^k$ converges within some open region of the retto plane : $|x| < \Omega$, which is bounded by the "modulus curves" $x = \Omega$, $x = -\Omega$.

Bencivenga establishes the Taylor expansion about a point $z = z_0$:

$$F(z_0+z) = \sum_{k=0}^{\infty} \frac{F^{(k)}(z_0)}{k!} z^k \qquad (\text{ I.43 })$$

for functions possessing all derivatives at z_0 . The expans--ion is valid for all points z within the region of converg--ence of the right member of (I.43) .

The line - integral $\int_{c\,z_0}^{z} F(z)\,dz$ over a path C in the retto plane is defined in the usual manner : Take a decompo--sition of the curve segment : $z_0, z_1, \ldots, z_{i-1}, z_i, \ldots, z_m = z$, and a point ζ_i on each segment z_{i-1}, z_i .
Form the sum

$$S_\sigma = \sum_{i=1}^{m} F(\zeta_i)(z_i - z_{i-1}) .$$

Refine the decomposition σ by increasing m and allowing z_i to approach z_{i-1} , in the Euclidean sense . Denoting the refinement : $|\sigma| \to 0$, the integral is defined :

$$\int_{c\,z_0}^{z} F(z)\,dz = \lim_{|\sigma|\to 0} S_\sigma . \qquad (\text{ I.44 })$$

Bencivenga shows that if $F(z)$ is continuous on C the integral exists , and is given by the formula :

$$\int_{c\,z_0}^{z} F(z)dz = \int_{c\,z_0}^{z} [F_1(x)+wF_2(x,y)](dx+w\,dy) = \int_{c\,z_0}^{z} F_1dx + w\int_{c\,z_0}^{z}(F_1dy+F_2dx) . \qquad (\text{ I.45 })$$

Bencivenga next proves the Cauchy integral theorem for this case : If $F(z)$, is differentiable on the closed contour and at all points of the interior bounded by C , then

$$\oint_C F(z)\,dz = 0. \qquad (\text{I.46})$$

Consequently , within any such region $\int_{z_0\,\Gamma}^{z} F(z)\,dz$ defines a differentiable function $I(z)$ which is independent of the path Γ , and has the derivative : $I'(z) = F(z)$.

I.5 Conformal Representation .

The author proves that a differentiable function of a dual variable maps the retto plane into itself so that rettilinear angles are preserved . The mapping will , in general , fail to be conformal at those points for which the differentiability of thefunction fails .

I.6 The Bireal Number System .

The bireal numbers are defined $z = x + yu$, where x , y are real and basis elements 1 , u have multiplication table

	1	u
1	1	u
u	u	1

The bireal system is a commutative linear algebra over the reals , isomorphic with the 2×2 matrix algebra :

$$z = x + yu \longleftrightarrow \begin{bmatrix} x & y \\ y & x \end{bmatrix} .$$

The modulus of z is defined by

$$|z| = \sqrt{\left|\begin{vmatrix} x & y \\ y & x \end{vmatrix}\right|} = \sqrt{|x^2 - y^2|} .$$

The author constructs a theory of the bireal variable and its functions similar to that for the dual variable ; the differ--ences in the parallel theories are confined to details of proof and formulation .

I.7 Representation of Bireal Numbers in the Hyperbolic Plane .

The hyperbolic plane consists of all points of the cartesian plane with the distance $\rho(P_1, P_2)$ between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ defined by

$$\rho(P_1, P_2) = \sqrt{|(x_1 - x_2)^2 - (y_1 - y_2)^2|} .$$

For this metric $\rho(P_1, P_2) = 0$ does not imply $P_1 = P_2$, and , furthermore , the triangle inequality fails . All points on the rectangular hyperbolas $x^2 - y^2 = \pm 1$ are unit distance from the origin .

Hyperbolic angles are defined in terms of the unit modulus curve $x^2 - y^2 = \pm 1$. In order that an hyperbolic angle shall define a unique vector originating at the origin , it is

HYPERBOLIC PLANE

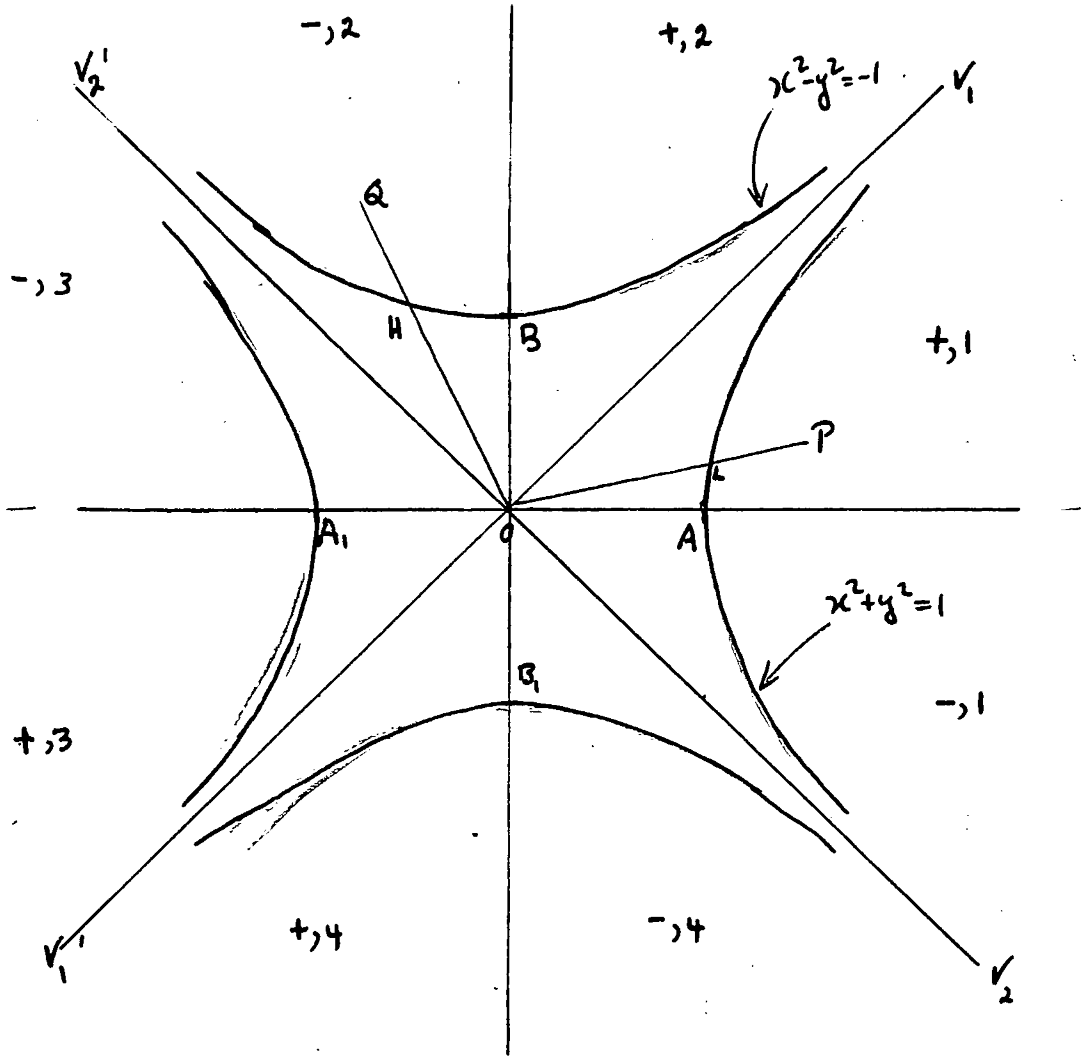

Figure 2

Chapter I

necessary to specify angles by magnitude , algebraic sign ,and subscript : Quadrants of the hyperbolic plane (figure 2) are signed according to the same scheme as for the retto plane. Quadrants $V_2 O V_1$, $V_1 O V_2'$, $V_2' O V_1'$ and $V_1' O V_2$ are distinguished by subscripts I , 2 , 3 , 4 respectively . In figure 2 , OP defines hyperbolic angle ϕ_1 , where ϕ is positive and equal in magnitude to twice afea bounded by x- axis , OP , and $x^2 - y^2 = 1$; OQ defines ψ_2 where $-\psi$ is equal to twice area bounded by y- axis , OQ , and $x^2 - y^2 = -1$. Similarly every other vector origin--ating at the origin defines a unique hyperbolic angle .

Addition of hyperbolic angles is defined by the equation $\phi_s + \psi_n = (\phi + \psi)_t$ and the matrix :

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

From s , n determine t as follows :
Find s in column I , and n in row 4 ; find t at the inter--section of the row and column so determined . For example , if $s = 2$, $n = 3$ then $t = 4$.

The sine and cosine functions for this plane are defined as follows : $\cosh \phi_1 = \cosh \phi$, $\sinh \phi_1 = \sinh \phi$, where $\cosh \phi$, $\sinh \phi$ are the ordinary hyperbolic sine and cosine . Functions of angles in other quadrants are defined according to scheme :

$$\cosh \phi_2 = \sinh \phi_1 \qquad \sinh \phi_2 = \cosh \phi_1$$

$$\cosh \phi_3 = -\cosh \phi_1 \qquad \sinh \phi_3 = -\sinh \phi_1 \qquad (\text{I.7I})$$

$$\cosh \phi_4 = -\cosh \phi_2 \qquad \sinh \phi_4 = -\sinh \phi_2$$

With definitions (I.7I) the addition formulae :

$$\sinh(\phi_s + \psi_r) = \sinh \phi_s \cosh \psi_r + \cosh \phi_s \sinh \psi_r ,$$
$$(\text{I.72})$$
$$\cosh(\phi_s + \psi_r) = \cosh \phi_s \cosh \psi_r + \sinh \phi_s \sinh \psi_r ,$$

are satisfied by every pair ϕ_s , ψ_r . Furthermore , if $|Z| = r$, $am(z) = \phi_s$, then

$$x = r \cosh \phi_s \qquad y = r \sinh \phi_s$$
$$(\text{I.73})$$
$$z = x + yu = r(\cosh \phi_s + u \sinh \phi_s),$$

where $Z = x + yu$ is represented in the hyperbolic plane by point (x, y) , x and y being the signed Euclidean distances of the point from the y- and x-axes , respectively.

I.8 Elementary Operations on the Bireal Numbers .

The formulae of section (I.3) have exact analogues in the bireal system :

I. $r_1(\cosh \phi_s + u \sinh \phi_s) \cdot r_2(\cosh \psi_r + u \sinh \psi_r)$

$$= r_1 r_2 \left[\cosh(\phi_s + \psi_r) + u \sinh(\phi_s + \psi_r)\right].$$

2. For an integer $n \geqq 0$:

$$\left[r(\cosh \phi_s + u \sinh \phi_s)\right]^n = r^n \left[\cosh(n\phi)_{ns} + u \sinh(n\phi)_{ns}\right].$$

3. For rational $\frac{p}{q} \geqq 0$

$$\left[r\left(\cosh\phi_s + u\sinh\phi_s\right)\right]^{\frac{p}{q}} = r^{\frac{p}{q}}\left[\cosh\left(\frac{p}{q}\phi_s\right) + u\sinh\left(\frac{p}{q}\phi_s\right)\right],$$

where $\frac{p}{q}\phi_s = \left(\frac{p}{q}\phi\right)_x$, $\sigma(q,x) = \sigma(p,s)$,

$\sigma(p,s) = s + s + \cdots\cdots + s$ to p terms , the add-ition being carried out according to the above matrix rule for subscripts .

4. For irrational $\mu > 0$:

$$\left[r\left(\cosh\phi_s + u\sinh\phi_s\right)\right]^{\mu} = r^{\mu}\left[\cosh(\mu\phi_s) + u\sinh(\mu\phi_s)\right]$$

has a merely formal significance .

5. For $r \neq 0$

$$\frac{1}{r\left(\cosh\phi_s + u\sinh\phi_s\right)} = \frac{1}{r}\left[\cosh(-\phi_s) + u\sinh(-\phi_s)\right].$$

6. Formulae 2 , 3 , 4 , 5 may be incorporated in the general formula :

$$\left[r\left(\cosh\phi_s + u\sinh\phi_s\right)\right]^{x} = r^{x}\left[\cosh(x\phi_s) + u\sinh(x\phi_s)\right]$$

for all real x .

For some purposes it is convenient to replace the basis 1 , u by the equivalent basis V_1 , V_2 :

$$\begin{aligned} 1 &= V_1 + V_2 \\ u &= V_1 - V_2 \end{aligned} \qquad \text{(I.74)}$$

The V_1 , V_2 multiplication table is

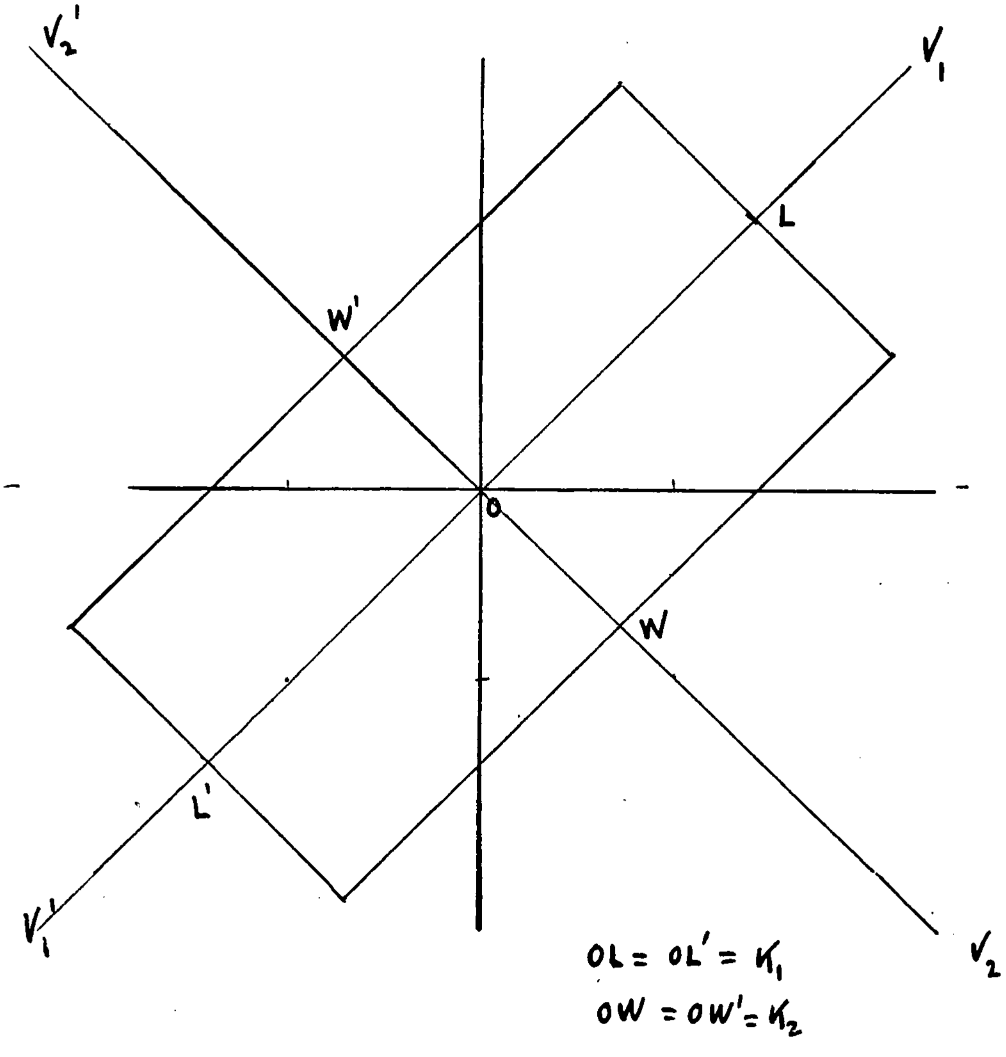

Figure 3

Chapter I

	V_1	V_2
V_1	V_1	0
V_2	0	V_2

z
is expressed in the V_1, V_2 system :

$$z = x + yu = (x+y)V_1 + (x-y)V_2 \tag{I.75}$$

I.9 Functions of the Bireal Variable .

A power series in the bireal variable z :

$$\sum_{K=0}^{\infty} A_K z^K = \sum_{K=0}^{\infty} (a_K V_1 + b_K V_2)(sV_1 + tV_2)^K$$
$$= \sum_{K=0}^{\infty} a_K s^K \cdot V_1 + \sum_{K=0}^{\infty} b_K t^K \cdot V_2$$

defines a function of z if the real series $\sum_{K=0}^{\infty} a_K s^K$, $\sum_{K=0}^{\infty} b_K t^K$ both converge . If the radii of convergence of $\sum_{K=0}^{\infty} a_K s^K$, $\sum_{K=0}^{\infty} b_K t^K$ are K_1, K_2 respectively, then the region of convergence for the bireal series $\sum_{K=0}^{\infty} A_K z^K$ is the interior of a rectangle in the hyperbolic plane(figure3).

Functions of a bireal variable and differentiability of such functions are defined in exactly the same manner as for the dual variable (section I.4). The necessary and sufficient conditions for differentiability at a point in this case are : (I) the existence and continuity of the first partial deriv--atives with respect to x and y of the function at the point in question .

(2) that the Cauchy - Riemann equations

$$\frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial y} \quad , \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \qquad (\text{I.91})$$

be satisfied at the point , by the function

$$F(z) = F_1(x,y) + F_2(x,y)\, u \, .$$

The derivative , if it exists , is given by the formula

$$F'(z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial x} \cdot u \, . \qquad (\text{I.92})$$

If we transform to the V_1 , V_2 algebra , a differentiable function takes the form

$$f(z) = f_1(s) \cdot V_1 + f_2(t) \cdot V_2 \qquad (\text{I.93})$$

where $z = s V_1 + t V_2$.

The author establishes the Taylor expansion :

$$F(z_0 + z) = \sum_{K=0}^{\infty} \frac{F^{(K)}(z_0)}{K!} z^K \, , \qquad (\text{I.94})$$

valid for some rectangular region about a point z_0 at which all derivatives of $F(z)$ exist .

The line - integral $\int_{C\,z_0}^{z} F(z)\, dz$ of a function of a bireal variable is defined in the same manner as for the dual variable (Section I . 4) . The author shows that if is continuous on C , the integral may be decomposed :

$$\int_{z_0 \, C}^{z} F(z)\, dz = \int_{z_0 \, C}^{z} (F_1 + F_2 \cdot u)(dx + dy \cdot u)$$

$$= \int_{z_0 \, C}^{z} F_1 dx + F_2 dy \quad + \quad u \int_{z_0 \, C}^{z} F_1 dy + F_2 dx . \qquad (\text{I.95})$$

The Cauchy integral theorem :

" $$\oint_C F(z)\, dz = 0$$

if $F(z)$ is differentiable on C and at every point within region bounded by C . " is proved by the author for bireal functions .

Finally , Bencivenga proves the conformal property of differentiable bireal functions : " A differentiable function of the bireal variable maps the hyperbolic plane into itself with the preservation of hyperbolic angles ." The mapping will, in general , fail to be conformal at those points at which the differentiability of the function fails .

CHAPTER II

Function Theory of a Hypercomplex Variable .

In this chapter we will develop the function theory , analogous to that of Bencivenga , for any linear algebra over the real numbers which is associative , commutative , and possesses a unit element . We shall see that the generalizations of differentiability , Taylor development of functions , the Cauchy integral theorem , and conformal representation are consequences of the fact that the algebra forms a commutative ring with unit element; on the other hand , we shall find that there is another class of results in the theory of functions of a complex variable which cannot be generalized . These are consequences of the field properties of the algebra of complex numbers , and therefore pertain only to the theory of functions of a complex variable .

2.I Classification of Linear Algebras over the Real Numbers which are Associative , Commutative and Possess a Unit Element .

Theorem 2.II The only independent binary associative commutative linear algebras with unit element over the real numbers are the complex , dual and bireal number systems, any other binary form is expressible in terms of one of these independent forms .

Proof : The general binary form is given by $x = x_1 + x_2 \varepsilon$

where x_1 , x_2 are real , and the basis $1, \varepsilon$ has the multiplication table

	1	ε
1	1	ε
ε	ε	$\alpha+\beta\varepsilon$

α , β being real numbers . From the multiplication table we see that ε is a root of a quadratic equation with real coe--fficients , namely

$$\varepsilon^2 - \beta\varepsilon - \alpha = 0, \qquad (2.\mathrm{II})$$

which may be written

$$\left(\varepsilon - \frac{\beta}{2}\right)^2 = \frac{\beta^2+4\alpha}{4} = \gamma, \text{ say .}$$

Let $(\varepsilon_1, \varepsilon_2, \cdots \cdots, \varepsilon_n)$ denote the algebra with basis $\varepsilon_1, \varepsilon_2, \ldots\ldots, \varepsilon_n$, where $\varepsilon_1 = 1$.

(i) If $\gamma > 0$, let $u = \dfrac{\varepsilon - \frac{\beta}{2}}{\sqrt{\gamma}}$.

Then $(1, \varepsilon)$ is equivalent to $(1, u)$, which is the bireal algebra since $u^2 = 1$.

(ii) If $\gamma = 0$, let $W = \varepsilon - \frac{\beta}{2}$.

Since $W^2 = 0$, $(1, \varepsilon)$ is equivalent to the dual algebra $(1, W)$.

(iii) If $\gamma < 0$, let $i = \dfrac{\varepsilon - \frac{\beta}{2}}{\sqrt{-\gamma}}$

Then $i^2 = -1$, and therefore $(1, \varepsilon)$ is equivalent to the complex algebra $(1, i)$.

<u>Theorem 2.12</u> Of the three binary algebras, the complex algebra alone forms a field. The other two form merely rings with unit element.

Proof : The binary algebra $(1, \varepsilon)$ forms a field if and only if $\varepsilon^2 - \beta\varepsilon - \alpha = 0$ is irreducible in the real field Δ. This condition is satisfied if and only if $\beta^2 + 4\alpha < 0$,

in which case the algebra $(1, \varepsilon)$ is equivalent to the complex algebra $(1, i)$.

Then* $\Delta(\varepsilon) = \Delta(i)$.

In the cases (i) and (ii) of the theorem 2.II, where $(1, \varepsilon)$ is equivalent to $(1, u)$ and $(1, W)$ respectively, $\varepsilon^2 - \beta\varepsilon - \alpha = 0$ is reducible.

Hence $\Delta(\varepsilon)$, and therefore the equivalent $\Delta(u)$ and $\Delta(W)$ are not fields.

* By $\Delta(\varepsilon_1, \ldots\ldots, \varepsilon_n)$ we will mean the algebra obtained by adjoining the elements $\varepsilon_1, \ldots\ldots, \varepsilon_n$ to the real field. Thus, for example, $\Delta(i)$ will denote the complex number field since it is obtained by adjoining the element i to the real number field.

In fact , $1+u$ is a divisor of zero in the bireal algebra , and W is a divisor of zero in the dual algebra ; since

$$(1+u)(1-u) = 0 ,$$

and

$$W^2 = 0 .$$

Theorem 2.13 The complex algebra is the only associative , commutative linear algebra with unit element over the real numbers , which forms a field .

Proof : Suppose the algebra $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, where $\varepsilon_1 = 1$, over the real field Δ , forms the adjunction field

$$\Delta(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \Delta(\varepsilon_2, \varepsilon_3 \ldots, \varepsilon_n) .$$

Any polynomial $p(\varepsilon_2)$ in ε_2 with real coefficients in Δ is factorable in the complex field $\Delta(i)$. Therefore

$$\Delta(\varepsilon_2) < \Delta(i) .$$

Any polynomial $q(\varepsilon_3)$ with coefficients in $\Delta(i)$ is factorable in $\Delta(i)$. Therefore $\Delta(i, \varepsilon_3) < \Delta(i)$,

So that $\Delta(\varepsilon_2, \varepsilon_3) < \Delta(i)$.

Continuing the argument , we have finally

$$\Delta(\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n) < \Delta(i).$$

Hence if $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ forms a field , it is a subfield of the complex field or the complex field itself .

But if γ, δ are real and not zero then $(1, i)$ and $(\gamma, \delta i)$ are equivalent bases over the real numbers .

Hence $\Delta(i) < \Delta(\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n)$, So that $\Delta(\varepsilon_2, \varepsilon_3, \dots \varepsilon_n) = \Delta(i)$,

Proving that any algebra $(\varepsilon_1, \varepsilon_2, \dots\dots, \varepsilon_n)$ which forms a field over the real numbers must form the complex field .

2.2 Hypercomplex Algebra

Let $\varepsilon_1, \varepsilon_2, \dots\dots, \varepsilon_n$, with $\varepsilon_1 = 1$, be the basis of an associative , commutative linear algebra over the reals . Then an element of the algebra may be written in the form

$$x = \sum_{\nu=1}^{n} x_\nu \varepsilon_\nu ,$$

where the x_ν are real numbers . The hypercomplex algebra so defined possesses the unit element $\varepsilon_1 = 1$. Moreover ,let the multiplication table for the basis elements be given by

$$\varepsilon_L \varepsilon_K = \varepsilon_K \varepsilon_L = \sum_{i=1}^{n} C^i_{LK} \varepsilon_i .$$

This table is therefore defined by the $\frac{n^2(n+1)}{2}$ real constants

$$C^i_{LK} = C^i_{KL} .$$

Since multiplication is associative , we have $(\varepsilon_K \varepsilon_L)\varepsilon_p = \varepsilon_K(\varepsilon_L \varepsilon_p)$

and so
$$\left(\sum_{i=1}^{n} C^i_{KL} \varepsilon_i\right)\varepsilon_p = \varepsilon_K\left(\sum_{i=1}^{n} C^i_{Lp} \varepsilon_i\right)$$

$$\sum_{i,j=1}^{n} C^i_{KL} C^j_{ip} \varepsilon_j = \sum_{i,j=1}^{n} C^i_{Lp} C^j_{Ki} \varepsilon_j .$$

The necessary and sufficient conditions for associative multiplication are therefore

$$\sum_{i=1}^{n} c_{kl}^{i} c_{ip}^{j} = \sum_{i=1}^{n} c_{lp}^{i} c_{ki}^{j} . \qquad (2.2I)$$

The equations (2.2I) impose $\frac{n^3(n+1)}{2}$ conditions on the $\frac{n^2(n+1)}{2}$ constants . As examples of algebras of more than two dimensions satisfying these conditions , we note the following :

(i) The algebra

	ε_1	ε_2	ε_3
ε_1	ε_1	ε_2	ε_3
ε_2	ε_2	ε_1	ε_3
ε_3	ε_3	ε_3	0

(ii) The class of linear algebras for which the basis

$(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ forms an abelian group .

(iii) Take an irreducible polynomial $p(x)$ of degree n , with coefficients in the real field Δ . BY ring adjunc--tion , adjoin the n roots $\theta_1, \theta_2, \ldots, \theta_n$ to form the linear algebra $\Delta[\theta_1, \theta_2, \ldots, \theta_n]$ with basis

$1, \theta_1, \theta_2, \ldots, \theta_n$

2.3 Matrix Representation of a Hypercomplex Number ..

The hypercomplex number $x = \sum_{\nu=1}^{n} x_\nu \varepsilon_\nu$ will define a unique $n \times n$ matrix $M(x)$ whose r^{th} row vector is formed from the real coefficients of

$$\varepsilon_r x = \sum_{\nu, s=1}^{n} C_{r\nu}^{s} x_\nu \varepsilon_s .$$

Thus

$$M(x) = \left[\sum_{\nu=1}^{n} C_{r\nu}^{s} x_\nu \right] . \qquad (2.3I)$$

It is well known that there exists an isomorphism under addi--tion and multiplication defined by

$$x \longleftrightarrow M(x).$$

The modulus function of x is the determinant

$|M(x)|$ of $M(x)$, the actual modulus being given by

$$|x| = \sqrt{\left|\, |M(x)| \,\right|} .$$

This representation will enable us to study the hypercomplex variable through the properties of the corresponding system of matrices----- a fact which will be exploited in the theory of conformal mapping .

2.4 Functions, Continuity, Differentiability and Convergence.

In the following development of the theory of a function of a hypercomplex variable , two properties of the hypercomplex numbers are of fundamental importance :

(i) The hypercomplex numbers form a commutative ring with unit element .

(ii) The base field of the hypercomplex system is the field of real numbers .

A <u>function</u> $f(x)$ of the hypercomplex variable

$x = \sum_{\nu=1}^{n} x_\nu \varepsilon_\nu$ is defined to be a single valued mapping of the space $(x_1 , \cdots , x_n)$ into itself . It can be expressed in the form

$$f(x) = \sum_{\nu=1}^{n} f_\nu(x_1, \cdots, x_n)\, \varepsilon_\nu ,$$

where the $f_\nu(x_1 , \cdots , x_n)$ are real functions of the variables $x_1 , \cdots , x_n$.

$f(x)$ is said to be <u>continuous</u> at $x^{(1)} = \sum_{\nu=1}^{n} x_\nu^{(1)} \varepsilon_\nu$

if each of the real functions $f_\nu(x_1 , \cdots , x_n)$ is continuous in the real variables $x_1 , \cdots , x_n$ at $x^{(1)}$.

If each component f_ν of $f(x)$ possesses all first partial derivatives at $x^{(1)}$ then $f(x)$ possesses a <u>differential</u> at $x^{(1)}$ and we can write

$$d\,f(x) = \sum_{\nu=1}^{n} d f_\nu \cdot \varepsilon_\nu \qquad \text{for} \quad x = x^{(1)}.$$

If $f(x)$ possesses a differential at $x^{(1)}$, and there exists a differential coefficient $\phi(x)$

such that
$$d\,f(x) = \phi(x)\,dx \qquad (2.4I)$$

at $x = x^{(1)}$, then $f(x)$ is said to be <u>differentiable</u> at $x^{(1)}$.

A power series

$$\sum_{\kappa=0}^{\infty} A_\kappa x^\kappa = \sum_{\nu=1}^{n} \Big(\sum_{\lambda_1 \cdots \lambda_n} a^{\nu}_{\lambda_1 \cdots \lambda_n} x_1^{\lambda_1} \cdots x_n^{\lambda_n} \Big) \varepsilon_\nu$$

defines a hypercomplex number $f(x)$ at x if each of the component real series converges at x.

In this case we say that $\sum_{\kappa=0}^{\infty} A_\kappa x^\kappa$ <u>converges</u> at x.

$\sum_{\kappa=0}^{\infty} A_\kappa x^\kappa$ converges over a region if it converges at every point of the region ; and <u>converges uniformly</u> over the region if each component real series converges uniformly over the same region . Within its region of uniform convergence, the series $\sum_{\kappa=0}^{\infty} A_\kappa x^\kappa$ defines a continuous function of x.

If limit $\lim\limits_{\substack{x_1 \\ \vdots \\ x_n}\Big\}\to 0} f_\nu(x_1, \cdots, x_n) = L_\nu$

and $$L = \sum_{\nu=1}^{n} L_\nu \mathcal{E}_\nu$$ then we shall say

that $$\lim_{x \to 0} f(x) = L.$$

2.5 Generalized Cauchy - Riemann Equations .

We shall now generalize the Cauchy-Riemann equations of complex variable function theory . We prove the following theorem :

<u>Theorem 2.5I</u> The necessary and sufficient conditions that $f(x)$ be differentiable at $x = x^{(1)}$ are :

(i) that the first partial derivatives of the f_ν with respect to the variables $x_1, \cdots, x_n$ exist and be continuous at $x = x^{(1)}$,

(ii) that the "Cauchy-Riemann" equations

$$\frac{\partial f_\kappa}{\partial x_L} = \sum_{\nu=1}^{n} C_{\nu L}^{\kappa} \frac{\partial f_\nu}{\partial x_1}, \qquad \begin{matrix} \kappa = 1, \cdots, n \\ L = 1, \cdots, n \end{matrix} \qquad (2.5I)$$

be satisfied at $x = x^{(1)}$.

Proof : $f(x)$ is differentiable at $x = x^{(1)}$ if there exists a function $\phi(x)$ such that

$$df = \phi(x)\, dx \qquad at \quad x = x^{(1)}.$$

Writing this equation in expanded form :

$$\sum_{\nu=1}^{n} df_\nu \varepsilon_\nu = \Big(\sum_{\nu=1}^{n} \phi_\nu \varepsilon_\nu\Big)\Big(\sum_{\nu=1}^{n} dx_\nu \varepsilon_\nu\Big) = \sum_{\nu,\mu,i=1}^{n} C_{\nu\mu}^{\,i}\, \phi_\nu\, dx_\mu\, \varepsilon_i .$$

Equating the K^{th} components :

$$df_K = \sum_{L=1}^{n} \frac{\partial f_K}{\partial x_L} dx_L = \sum_{\nu,\mu=1}^{n} C_{\nu\mu}^{\,K}\, \phi_\nu\, dx_\mu .$$

Equating coefficients of the independent dx_L ;

$$\frac{\partial f_K}{\partial x_L} = \sum_{\nu=1}^{n} C_{\nu L}^{\,K}\, \phi_\nu . \qquad (2.52)$$

Since $\varepsilon_1 = 1$, $\varepsilon_L = \varepsilon_1 \varepsilon_L = \sum_{K=1}^{n} C_{1L}^{\,K} \varepsilon_K$,

So that $C_{1L}^{\,K} = C_{L1}^{\,K} = \delta_L^K$, the Kronecker delta.

Setting $L = 1$ in (2.52) and applying this result :

$$\frac{\partial f_K}{\partial x_1} = \sum_{\nu=1}^{n} C_{\nu 1}^{\,K}\, \phi_\nu = \sum_{\nu=1}^{n} \delta_\nu^K\, \phi_\nu = \phi_K . \qquad (2.53)$$

Combining (2.52) and (2.53) we have the Cauchy-Riemann equations (2.5I)

Corollary (i) If $f(x)$ is differentiable at x , then (by equation (2.53)) the derivative of $f(x)$ is given by the formula :

$$f'(x) = \sum_{\nu=1}^{n} \frac{\partial f_\nu}{\partial x_1} \varepsilon_\nu . \qquad (2.54)$$

Corollary (ii) If $f(x)$ is differentiable at x , and all second partials of the f_ν with respect to $x_1 , \cdots , x_n$ exist at x , then the second derivative of $f(x)$ with respect to x exists at x , and is given by the formula :

$$f''(x) = \sum_{\nu=1}^{n} \frac{\partial^2 f_\nu}{\partial x_1^2} \varepsilon_\nu .$$

To prove this it is only necessary to show that , under the hypothesis ,

$$f'(x) = \sum_{\nu=1}^{n} \frac{\partial f_\nu}{\partial x_1} \varepsilon_\nu$$

satisfies the Cauchy-Riemann equations (2.5I) , and then to apply equation (2.54) to $f'(x)$. We must show that

$$\frac{\partial}{\partial x_L}\left(\frac{\partial f_\kappa}{\partial x_1}\right) = \sum_{\nu=1}^{n} c_{L\nu}^{\kappa} \frac{\partial}{\partial x_1}\left(\frac{\partial f_\nu}{\partial x_1}\right),$$

which is equivalent to

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f_\kappa}{\partial x_L}\right) = \frac{\partial}{\partial x_1} \sum_{\nu=1}^{n} c_{L\nu}^{\kappa} \frac{\partial f_\nu}{\partial x_1},$$

which is merely the result of partial differentiation with respect to x_1 of the Cauchy-Riemann equations for $f(x)$.

This result is immediately generalizable to

Corollary (iii) If $f(x)$ is differentiable at x, and all partials of the f_ν up to the m^{th} order exist at x, then the m^{th} derivative of $f(x)$ with respect to x exists at x and is given :

$$f^{(m)}(x) = \sum_{\nu=1}^{n} \frac{\partial^m f_\nu}{\partial x_1^m} \varepsilon_\nu \, . \qquad (2.55)$$

Corollary (iv) Assuming the existence of the higher partial derivatives involved, all higher derivatives of the f_ν with respect to $x_1, \ldots\ldots, x_n$,

of a differentiable function $f(x)$, are expressible in terms of partial derivatives of the same order with respect to x_1.

For, differentiating

$$\frac{\partial f_\kappa}{\partial x_L} = \sum_{\nu=1}^{n} c^\kappa_{L\nu} \frac{\partial f_\nu}{\partial x_1}$$

with respect to x_p, we obtain

$$\frac{\partial^2 f_\kappa}{\partial x_p \partial x_L} = \sum_{\nu=1}^{n} c^\kappa_{L\nu} \frac{\partial}{\partial x_p}\left(\frac{\partial f_\nu}{\partial x_1}\right) = \sum_{\nu,\lambda=1}^{n} c^\kappa_{L\nu} \, c^\nu_{p\lambda} \frac{\partial^2 f_\lambda}{\partial x_1^2} \, .$$

Differentiating this with respect to x_s :

$$\frac{\partial^3 f_\kappa}{\partial x_s \partial x_p \partial x_\iota} = \sum_{\nu,\lambda=1}^{m} c^{\kappa}_{\iota\nu}\, c^{\nu}_{p\lambda} \frac{\partial}{\partial x_s}\left(\frac{\partial^2 f_\lambda}{\partial x_1^2}\right)$$

$$= \sum_{\nu,\lambda,\mu=1}^{m} c^{\kappa}_{\iota\nu}\, c^{\nu}_{p\lambda}\, c^{\lambda}_{s\mu} \frac{\partial^3 f_\mu}{\partial x_1^3} ,$$

and continuing in this way , any m^{th} order partial deriva--tive of f_κ is expressible in terms of the m^{th} order derivatives of the f_ν with respect to x_1 .

2.6 Analytic Functions

We will say that $f(x)$ is analytic at $x = x^{(1)}$ if all derivatives : $f'(x)$, $f''(x)$, $f'''(x)$, ...
exist at $x = x^{(1)}$.

Theorem 2.6I The necessary and sufficient conditions for $f(x)$ to be analytic at x are :

(i) that $f(x)$ be differentiable at x ,

(ii) that each component $f_\nu(x_1, \ldots x_m)$ possesses all partial derivatives with respect to $x_1, \ldots, x_m$ at x .

Proof : By theorem 2.5I , corollary (iii) the conditions of the theorem guarantee the existence of all m^{th} order derivatives with respect to x :

$$f^{(m)}(x) = \sum_{\nu=1}^{n} \frac{\partial^m f_\nu}{\partial x_1^m} \varepsilon_\nu .$$

The conditions are therefore sufficient .

If $f^{(m)}(x)$ exists , then , by the last mentioned equation,

each $\dfrac{\partial^m f_\nu}{\partial x_1^m}$ exists for $\nu = 1, \cdots\cdots, n$.

Then by theorem 2.51 , corollary (iv) all m^{th} order partials with respect to the variables $x_1, \ldots\ldots, x_n$ must exist . Hence the conditions of the theorem are necessary .

<u>Theorem 2.62</u> If $f(x)$ is differentiable at x , then so is $x f(x)$.

Proof : Let $F(x) = x f(x)$.

Then $$\Delta F = (x + \Delta x) f(x + \Delta x) - x f(x)$$

$$= (x + \Delta x)\left[f(x) + f'(x)\Delta x + \eta(x, \Delta x)\Delta x \right] - x f(x) ,$$

where $$\lim_{\Delta x \to 0} \eta(x, \Delta x) = 0 .$$

Since , by hypothesis , $f(x)$ is differentiable at x :

$$\Delta F = \left[x f'(x) + f(x) \right] \Delta x + \eta_1(x, \Delta x)\Delta x ,$$

where $\lim_{\Delta x \to 0} \eta_1(x, \Delta x) = 0$.

Therefore $F'(x) = x f'(x) + f(x)$.

Corollary (i) Every polynomial in x is analytic .

For , x itself is analytic , so by the theorem x^2, x^3, ..., x^K, ... are analytic . Any hyper-complex constant a is analytic , so by the theorem $a x^K$ is analytic. . Since a finite sum satisfies the Cauchy-Riemann equations if each component function does , any polynomial in x is differentiable . Since the derivative of a polynom--ial is again a polynomial , all derivatives of a polynomial exist , so that every polynomial is analytic .

Corollary (ii) Within its region of uniform convergence, the series

$$\sum_{K=0}^{\infty} A_K x^K$$

is an analytic function .

For , each term of the series satisfies the Cauchy-Riemann equations , hence the series itself satisfies these equations . Since the real series converge uniformly , all their partials with respect to x_1 , ..., x_m exist .

2.7 On the Relation of Differentiable to Analytic Functions .

Every differentiable function of a complex variable is analytic at the point in question . This result of the theory of functions of a complex variable is a consequence of the Cauchy integral formula , which in turn rests on the field properties of complex algebra . If the hypercomplex variable is other than the complex variable , then a function may be differentiable at a point and yet fail to beanalytic at the same point . We give the following example of this situation where the variable is bireal :

Let $$f(x) = \frac{1}{4}\left[(x_1+x_2)^2 + (x_1+x_2)^2 u\right] \quad , x_1+x_2 \leqq 1$$

and $$g(x) = \frac{1}{2}\left[(x_1+x_2-\tfrac{1}{2}) + (x_1+x_2-\tfrac{1}{2})u\right] \quad , x_1+x_2 \geqq 1 .$$

Now define $$F(x) = \begin{cases} f(x) & , x_1+x_2 \leqq 1 \\ g(x) & , x_1+x_2 \geqq 1 \end{cases}$$

$$F(x) = F_1(x_1,x_2) + F_2(x_1,x_2)\, u$$ satisfies the Cauchy - Riemann equations

$$\frac{\partial F_1}{\partial x_1} = \frac{\partial F_2}{\partial x_2} ,$$

$$\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1} ,$$

at every point of the hyperbolic plane . Since all the first partials of F_1 and F_2 with respect to x_1 and x_2 exist and are continuous at every point of the plane , $F(x)$ is differentiable at every point of the plane , by theorem (2.5I) The derivative of $F(x)$ is by theorem (2.5I) ,corollary(i),

$$F'(x) = \begin{cases} \frac{1}{2}\left[(x_1+x_2) + (x_1+x_2)u\right] , & x_1+x_2 \leqq 1 , \\ \frac{1}{2}\left[1 + u \right] , & x_1+x_2 \geqq 1 . \end{cases}$$

But the second derivative $F''(x)$ fails to exist on the line

$x_1 + x_2 = 1$.

Hence on this line , $F(x)$ is differentiable but not analytic.

The identity of differentiable and analytic functions does not necessarily hold for an algebra other than the complex algebra . In the theory of functions of a complex variable , this identity belongs to the class of results which are deriv--ed from the field properties of the algebra .

2.8 Taylor Series .

<u>Theorem 2.8I</u> If $f(x)$ is analytic at the point $x = X$, then the expansion

$$f(X+x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(X)}{m!} x^m , \tag{2.8I}$$

is valid for some region about X .

To prove this , write :

$$L(x) = f(X+x) = \sum_{\nu=1}^{n} f_\nu(X_1+x_1, \dots, X_n+x_n)\, \varepsilon_\nu , \tag{2.82}$$

$$R(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(X)}{m!} x^m = \sum_{\nu=1}^{n} R_\nu(x_1, \dots, x_n)\, \varepsilon_\nu . \tag{2.83}$$

$R(x)$ will be an analytic function over a certain region about $x = 0$, within which the series (2.83) is uniform--ly convergent . The proof will consist in the identification of $L(x)$ with $R(x)$ over this region .

Since $R(x)$ is analytic at $x = 0$, we have by (2.83) :

$$R^{(m)}(0) = f^{(m)}(X) . \tag{2.84}$$

Since $L(x)$ is analytic at $x = 0$, we apply theorem(2.5I), corollary (iii) to (2.82) :

$$L^{(m)}(0) = \sum_{\nu=1}^{n} \frac{\partial^m}{\partial X_1^m} f_\nu (X_1, \dots X_n)\, \varepsilon_\nu \, , \qquad (2.85)$$

Since $\quad f(X+x) = \sum_{\nu=1}^{n} f_\nu (X_1 + x_1, \dots, X_n + x_n)\, \varepsilon_\nu$

is analytic at $x = 0$:

$$f^{(m)}(X) = \sum_{\nu=1}^{n} \frac{\partial^m}{\partial X_1^m} f_\nu (X_1, \dots, X_n)\, \varepsilon_\nu \, , \qquad (2.86)$$

From the equations (2.84) , (2 .85) , (2.86), therefore :

$$L^{(m)}(0) = R^{(m)}(0) \, . \qquad (2.87)$$

Differentiating (2.82) and (2.83) by rule of theorem (2.5I) , corollary (iii) ,we have :

$$L^{(m)}(x) = \sum_{\nu=1}^{n} \frac{\partial^m f_\nu}{\partial x_1^m}\, \varepsilon_\nu \, ,$$

$$R^{(m)}(x) = \sum_{\nu=1}^{n} \frac{\partial^m R_\nu}{\partial x_1^m}\, \varepsilon_\nu \, ,$$

and so by (2.87) we have

$$\left.\frac{\partial^m f_\nu}{\partial x_1^m}\right]_{x=0} = \left.\frac{\partial^m R_\nu}{\partial x_1^m}\right]_{x=0}, \quad \nu = 1, \cdots, n , \qquad (2.88)$$

By theorem (2.5I) , corollary (iv) , all partials, of all orders , of the f_ν are expressible in terms of partials of the f_ν with respect to x_1 ; hence by (2.88) all par--tial derivatives , of all orders , of the f_ν and R_ν are equal, at $x = 0$.

Also $L(0) = R(0)$ implies that $f_\nu = R_\nu$ at $x = 0$. Hence by the theory of real functions :

$$f_\nu(x_1, \cdots, x_n) = R_\nu(x_1, \cdots, x_n)$$

within the region of uniform convergence of f_ν , R_ν .

Therefore

$$L(x) \equiv R(x) \qquad (2.89)$$

over the intersection of all the regions of convergence of the f_ν and R_ν . That is , (2.8I) holds over some region about X .

2.9 Line Integrals .

The line integral of a function $f(x)$ over a curve C in space $(x_1, \ldots\ldots, x_n)$ is defined in usual manner : Let curve C be defined by the parametric equations :

$$x_\nu = x_\nu(t) \quad , \nu = 1, \cdots, n .$$

Let x^0, X be initial and terminal points of C . Make a decomposition of C by subdivisions at points $x^0, x^{(1)}, \ldots\ldots x^{(m)} = X$ and take inter- -mediate poimts $\xi^{(\kappa)} = \sum_{\nu=1}^{n} \xi_\nu^{(\kappa)} \varepsilon_\nu$

such that $x_\nu^{(\kappa-1)} \leqq \xi_\nu^{(\kappa)} \leqq x_\nu^{(\kappa)}$ or $x_\nu^{(\kappa-1)} \geqq \xi_\nu^{(\kappa)} \geqq x_\nu^{(\kappa)}$.

The line-integral is defined :

$$\int_{x^0, C}^{X} f(x)\,dx = \lim_{\substack{m \to \infty \\ \text{all } x^{(\kappa)} - x^{(\kappa-1)} \to 0}} \sum_{\kappa=1}^{m} f(\xi^{(\kappa)})\,(x^{(\kappa)} - x^{(\kappa-1)}) \cdot \quad (2.9I) .$$

Since

$$f(x)\,dx = \left(\sum_{\nu=1}^{n} f_\nu \varepsilon_\nu\right)\left(\sum_{\nu=1}^{n} dx_\nu \varepsilon_\nu\right)$$

$$= \sum_{\kappa,\lambda,i=1}^{n} c^{\,i}_{\kappa\lambda} f_\kappa\, dx_\lambda\, \varepsilon_i$$

assuming that $f(x)$ is continuous on C , we may write :

$$\int_{x^0\,C}^{X} f(x)\,dx = \sum_{\nu=1}^{n}\left(\int_{x^0\,C}^{X} \sum_{\kappa,\lambda=1}^{n} c^{\,\nu}_{\kappa\lambda} f_\kappa\, dx_\lambda\right)\varepsilon_\nu \,. \qquad (2.92)$$

<u>Theorem 2.9I</u> Generalized Cauchy Integral Theorem :

Let $f(x)$ be analytic within the region

$$x^{(1)}_\nu \leqq x_\nu \leqq x^{(2)}_\nu \quad , \quad \nu = 1, \cdots, n \quad .$$

Let C be a simple closed curve within this region . Then

$$\oint_C f(x)\,dx = 0 \,. \qquad (2.93)$$

To prove this , we decompose the integral into its real components :

By (2.92) we must prove

$$\oint_C \sum_{\kappa,\lambda=1}^{n} c^{\,i}_{\kappa\lambda} f_\kappa\, dx_\lambda = 0 \,, \; i = 1, \cdots, n \,. \qquad (2.93)$$

Since , by hypothesis , all partial derivatives with respect to x_1 , ... , x_n of the f_K exist and are continuous over the region in which C is embedded , the necessary and sufficient conditions for (2.93) are :

$$\frac{\partial}{\partial x_K}\sum_{\alpha=1}^{n} c_{\alpha L}^{i} f_\alpha = \frac{\partial}{\partial x_L}\sum_{\alpha=1}^{n} c_{K\alpha}^{i} f_\alpha ,$$

$$\text{i.e.}\quad \sum_{\alpha=1}^{n} c_{\alpha L}^{i}\frac{\partial f_\alpha}{\partial x_K} = \sum_{\alpha=1}^{n} c_{K\alpha}^{i}\frac{\partial f_\alpha}{\partial x_L} . \qquad (2.94)$$

Applying the Cauchy -Riemann conditions

$$\frac{\partial f_\alpha}{\partial x_K} = \sum_{\beta=1}^{n} c_{K\beta}^{\alpha}\frac{\partial f_\beta}{\partial x_1}$$

to (2.94) , we obtain :

$$\sum_{\alpha,\beta=1}^{n} c_{\alpha L}^{i} c_{K\beta}^{\alpha}\frac{\partial f_\beta}{\partial x_1} = \sum_{\alpha,\beta=1}^{n} c_{K\alpha}^{i} c_{L\beta}^{\alpha}\frac{\partial f_\beta}{\partial x_1} . \qquad (2.94)$$

But equations (2.94) hold if

$$\sum_{\alpha=1}^{n} c_{\alpha L}^{i} c_{K\beta}^{\alpha} = \sum_{\alpha=1}^{n} c_{K\alpha}^{i} c_{K\beta}^{\alpha} \qquad (2.95)$$

But equations (2.95) are merely the associativity conditions (2.2I) . This proves equations (2.93) as a consequence of the Cauchy - Riemann conditions and the associativity of the algebra , and hence equation (2.93) of the theorem .

2.IO Conformal Mapping .

In this section we seek a generalization of the notion of conformal mapping which has been established for different--iable functions of complex , dual and bireal variables .

The angle between line-elements dx, δx in the complex plane is defined by its cosine function as follows :

$$\Omega(dx, \delta x) = \frac{\begin{vmatrix} dx_1 & dx_2 \\ -\delta x_2 & \delta x_1 \end{vmatrix}}{\left| \begin{vmatrix} dx_1 & dx_2 \\ -dx_2 & dx_1 \end{vmatrix} \cdot \begin{vmatrix} \delta x_1 & \delta x_2 \\ -\delta x_2 & \delta x_1 \end{vmatrix} \right|^{\frac{1}{2}}} .$$

Let $y = y(x)$ be a differentiable function of the complex variable x , and

$$dy = y'(x)\, dx ,$$

$$\delta y = y'(x)\, \delta x .$$

Then the law of conformal mapping for a function of a complex variable states that

$$\Omega(dy, \delta y) = \Omega(dx, \delta x)$$

at every point x_0 for which $y(x)$ is differentiable and

$$|y'(x)| \neq 0 .$$

For the bireal variable , the hyperbolic angle between the line - elements $dx, \delta x$ in the hyperbolic plane is de--fined by the hyperbolic cosine function :

$$\Omega(dx, \delta x) = \frac{\begin{vmatrix} dx_1 & dx_2 \\ \delta x_2 & \delta x_1 \end{vmatrix}}{\left| \begin{vmatrix} dx_1 & dx_2 \\ dx_2 & dx_1 \end{vmatrix} \begin{vmatrix} \delta x_1 & \delta x_2 \\ \delta x_2 & \delta x_1 \end{vmatrix} \right|^{\frac{1}{2}}} ,$$

and the law of conformality , proved by Bencivenga , may be expressed as follows :

At every point x for which $y = y(x)$ is different--iable and $|y'(x)| \neq 0$,

$$\Omega(dy, \delta y) = \Omega(dx, \delta x).$$

Finally , the right-cosine fûnction for elements in the retto plane is given

$$\Omega(dx, \delta x) = \frac{\begin{vmatrix} dx_1 & dx_2 \\ 0 & \delta x_1 \end{vmatrix}}{\left| \begin{vmatrix} dx_1 & dx_2 \\ 0 & dx_1 \end{vmatrix} \begin{vmatrix} \delta x_1 & \delta x_2 \\ 0 & \delta x_1 \end{vmatrix} \right|^{\frac{1}{2}}} ,$$

and the law of conformal mapping states that at every point

x for which $y = y(x)$ is differentiable and $|y'(x)| \neq 0$

$$\Omega(dy, \delta y) = \Omega(dx, \delta x).$$

Let $M(x)$ denote the matrix (2.3I) corresponding to the hypercomplex number x , and let $|M(x)|$ be its determinant . Let $M\binom{\delta x}{dx}$ be the matrix obtained from $M(dx)$ by replacing the first row vector of $M(dx)$ by the first row vector of $M(\delta x)$. With this notation , the angle function in the above three cases is expressible by the single formula :

$$\Omega(dx, \delta x) = \frac{\left| M\binom{dx}{\delta x} \right|}{\left| |M(dx)| \cdot |M(\delta x)| \right|^{\frac{1}{2}}},$$

and the law of conformal mapping reads :

At every point x for which $y = y(x)$ is different--iable and $|y'(x)| \neq 0$, we have

$$\Omega(dy, \delta y) = \Omega(dx, \delta x).$$

In the above cases the function $\Omega(dx, \delta x)$ is symmetric , i.e.

$$\Omega(dx, \delta x) = \Omega(\delta x, dx).$$

We now seek a formula for $\Omega(dx, \delta x)$ where the variable $x = \sum_{\nu=1}^{n} x_\nu \varepsilon_\nu$ is the general hypercomplex variable . The required expression must reduce to the above forms for the cases that x is the complex , bireal or dual variable . It is also desirable that it remain symmetric .

As above , let $M(x)$ be the matrix (2.3I) corresponding to $x = \sum_{\nu=1}^{n} x_\nu \varepsilon_\nu$ and let $M\binom{\delta x}{dx}$ be the matrix obtained from $M(dx)$ by replacing its first row by that of $M(\delta x)$. A function fulfilling the required conditions is :

$$\Omega(dx, \delta x) = \frac{\frac{1}{2}\left\{ \left| M\binom{dx}{\delta x} \right| + \left| M\binom{\delta x}{dx} \right| \right\}}{\Big| \left| M(dx) \right| \cdot \left| M(\delta x) \right| \Big|^{\frac{1}{2}}} \qquad (2.\text{IOI})$$

Theorem 2.IOI General Law of Conformal Mapping :

If the function $y = y(x)$ of the variable

$x = \sum_{\nu=1}^{n} x_\nu \varepsilon_\nu$ is differentiable at $x = x^{(1)}$

and $\left| y'(x^{(1)}) \right| \neq 0$,

then $\Omega(dy, \delta y) = \Omega(dx, \delta x)$ at $x = x^{(1)}$.

(2.IO2)

We prove this by reducing the expression for $\Omega(dy, \delta y)$ to that for $\Omega(dx, \delta x)$, using the hypothesis that $y(x)$ has a derivative whose modulus does not vanish at the point $x = x^{(1)}$.

Since $y(x)$ is differentiable at $x = x^{(1)}$, we may write $dy = y'(x)\,dx$ at this point . It follows from the isomorphism $x \longleftrightarrow M(x)$ that

$$M(dy) = M\left(y'(x)\,dx\right) = M\left(y'(x)\right) M(dx). \qquad (\ 2.\mathrm{I}03\ )$$

Multiplication is commutative for the algebra and therefore , by the isomorphism , for the matrices . Therefore

$$M(dy) = M(dx)\, M\left(y'(x)\right). \qquad (\ 2.\mathrm{I}04)$$

Also it follows that

$$M\begin{pmatrix}\delta y\\ dy\end{pmatrix} = M\begin{pmatrix}\delta x\\ dx\end{pmatrix} M\left(y'(x)\right), \qquad (\ 2.\mathrm{I}05)$$

since both members of (2.I05) are obtained fromthe correspond -ing members of (2.I04) by an equivalent replacement of the first row vector .

We have

$$\Omega(dy, \delta y) = \frac{\frac{1}{2}\left\{ \left| M\binom{dy}{\delta y} \right| + \left| M\binom{\delta y}{dy} \right| \right\}}{\left| \, |M(dy)| \cdot |M(\delta y)| \, \right|^{1/2}}$$

$$\frac{\frac{1}{2}\left\{ |M(y')| \cdot \left| M\binom{dx}{\delta x} \right| + |M(y')| \left| M\binom{\delta x}{dx} \right| \right\}}{\left| \, |M(y')| \, |M(dx)| \cdot |M(y')| \, |M(\delta x)| \, \right|^{1/2}}$$

Since $\quad |y'(x^{(1)})| \neq 0 \quad$, by hypothesis ,

then $\quad M\left(y'(x^{(1)})\right) \neq 0$

Hence the above equation reduces at $\quad x = x^{(1)}$

to

$$\Omega(dy, \delta y) = \frac{\frac{1}{2}\left\{ \left| M\binom{dx}{\delta x} \right| + \left| M\binom{\delta x}{dx} \right| \right\}}{\left| \, |M(dx)| \cdot |M(\delta x)| \, \right|^{1/2}}$$

$$= \Omega(dx, \delta x)$$

which is the equation (2.102) , required by the theorem .

The theorems which have been generalized in this chapter require as hypothesis merely the commutative ring properties of the algebra and the existence of a unit element . In sec--tion (2.7) we have encountered one property which is not generalizable , namely the identity of differentiable and ana--lytic functions . All results of complex variable function theory requiring field properties as hypothesis will not be generalizable to ring algebras . To this class belong the "residue theorems" and the whole theory of point singularities in the theory of functions of a complex variable .

CHAPTER III

Conformal Representation in the Hyperbolic Plane .

Bencivenga has shown that a function of a bireal variable maps the hyperbolic plane into itself in such a manner , that at those points for which the derivative of the function exists and its modulus does not vanish , hyperbolic angles are preserved in the mapping . In this section we study the conformal mapping of the hyperbolic plane in more detail , and , in particular , we attempt a systematic treatment of the bilinear transformation of the hyperbolic plane .

3.1 Geometry of the Hyperbolic Plane .

The point (x,y) of the hyperbolic plane represents the bireal number $z = x + yu$. Many of the Euclidean theorems of the complex plane have analogues in the hyperbolic plane . In this correspondence , Euclidean distance $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ will be replaced by hyperbolic distance

$$\sqrt{|(x_1 - x_2)^2 - (y_1 - y_2)^2|}$$

circular angles by hyperbolic angles , and the circle

$(x - x_0)^2 + (y - y_0)^2 = a^2$ by the rectangular hyperbola

$$(x - x_0)^2 - (y - y_0)^2 = \pm a^2$$

3.2 Length of a Hyperbolic Line Segment .

The length of a hyperbolic line segment of a curve C may be defined as follows :

Let the parametric equations of a curve C be

$$x = x(t) \qquad , \qquad y = y(t)$$

and let the parameter t increase monotonely from t_0 to t . Then $z = z(t) = x(t) + y(t) \cdot u$

will also be an equation of C . Now make a decomposition σ of the curve by letting t take the set of values

$$t_0 < t_1 < t_2 < \cdots\cdots < t_{n-1} < t_n = t ,$$

and let $|\sigma|$ denote the $\max (t_\nu - t_{\nu-1})$.

Form the σ – sum

$$S_\sigma = \sum_{\nu=1}^{n} | z(t_\nu) - z(t_{\nu-1}) | \qquad (3.21)$$

Then the hyperbolic length of the line segment is defined to be

$$S = \lim_{|\sigma| \to 0} S_\sigma , \qquad (3.22)$$

if this exists independently of the decomposition σ .

<u>Theorem 3.21</u> The hyperbolic length of the segment of the curve

$$x = x(t) \quad , \quad y = y(t) \qquad \text{for} \quad t_0 \leq t \leq t_1$$

is given by

$$\int_{t_0}^{t_1} \sqrt{\left| x'^2(t) - y'^2(t) \right|} \; . \, dt \,,$$

provided (i) $x(t)$ and $y(t)$ possess continuous first der--ivatives and (ii) $x'^2(t) - y'^2(t) \neq 0$

for $t_0 \leqq t \leqq t_1$.

Proof : A point for which $x'^2(t) - y'^2(t) = 0$ is called a singular point . Suppose that conditions (i) and (ii) are satisfied . Then

$$|z(t_\nu) - z(t_{\nu-1})|^2 = \left| \left(x(t_\nu) - x(t_{\nu-1})\right)^2 - \left(y(t_\nu) - y(t_{\nu-1})\right)^2 \right|$$

and , applying the law of the mean , this is equal to

$$\left| x'^2(\bar{t}_\nu) - y'^2(\bar{\bar{t}}_\nu) \right| (t_\nu - t_{\nu-1})^2$$

where

$$t_{\nu-1} \leqq \bar{t}_\nu \leqq t_\nu \,, \qquad t_{\nu-1} \leqq \bar{\bar{t}}_\nu \leqq t_\nu \,.$$

Now write

$$A(t) = x'^2(t) - y'^2(t) \,.$$

Then $|z(t_\nu) - z(t_{\nu-1})|^2 = \left| A(\bar{t}_\nu) + \varepsilon \right| (t_\nu - t_{\nu-1})^2$, where ε tends to zero uniformly over the closed interval $[t_0 , t_1]$ as $t_\nu - t_{\nu-1} \to 0$, and hence

$$S_\sigma = \sum_{\nu=1}^{n} \sqrt{|A(\bar{t}_\nu) + \varepsilon|} \; (t_\nu - t_{\nu-1})$$

$$= \sum_{\nu=1}^{n} \sqrt{|A(\bar{t}_\nu)|} \; \left| 1 + \frac{1}{2A(\bar{t}_\nu)} \varepsilon + \cdots \right| (t_\nu - t_{\nu-1}) \qquad (3.23)$$

since $A(t)$ does not vanish in the imterval , and since it is continuous , it is bounded away from zero . Now ε tends uniformly to zero , and so we have , by (3.21) , (3.22) and (3.23) ,

$$S = \lim_{|\sigma| \to 0} \sum_{\nu=1}^{n} \sqrt{|A(\bar{t}_\nu)|} \; (t_\nu - t_{\nu-1})$$

$$= \int_{t_0}^{t} \sqrt{|A(t)|} \; dt$$

$$= \int_{t_0}^{t} \sqrt{|x'^2(t) - y'^2(t)|} \cdot dt ,$$

as stated by the theorem .

<u>Corollary</u> As neighbouring points approach coincidence in a non-singular region of the curve , the ratio of arc-length to chord-length (both in hyperbolic metric) tends to unity .

Proof : The hyperbolic length of the chord is

$$C = \sqrt{\left|\left(x(t) - x(t_0)\right)^2 - \left(y(t) - y(t_0)\right)^2\right|}$$

$$= \sqrt{\left|x'^2(\bar{t}) - y'^2(\bar{\bar{t}})\right|}$$

where $t_0 \leqq \bar{t} \leqq t \quad , \quad t_0 \leqq \bar{\bar{t}} \leqq t$

Thus $$C = \sqrt{\left|A(\bar{t}) + \varepsilon\right|} \cdot (t - t_0)$$

where ε tends uniformly to zero as $t - t_0$ tends to zero .

The hyperbolic arc length is

$$S = \int_{t_0}^{t} \sqrt{|A(t)|}\, dt$$

$$= \sqrt{\left|A(\bar{\bar{\bar{t}}})\right|} \cdot (t - t_0)$$

where $t_0 \leqq \bar{\bar{\bar{t}}} \leqq t$, by the law of the mean for integrals .

Therefore $$\lim_{t - t_0 \to 0} \frac{C}{S} = \lim_{t - t_0 \to 0} \frac{\sqrt{|A(\bar{t}) + \varepsilon|}}{\sqrt{|A(\bar{\bar{\bar{t}}})|}} = 1$$

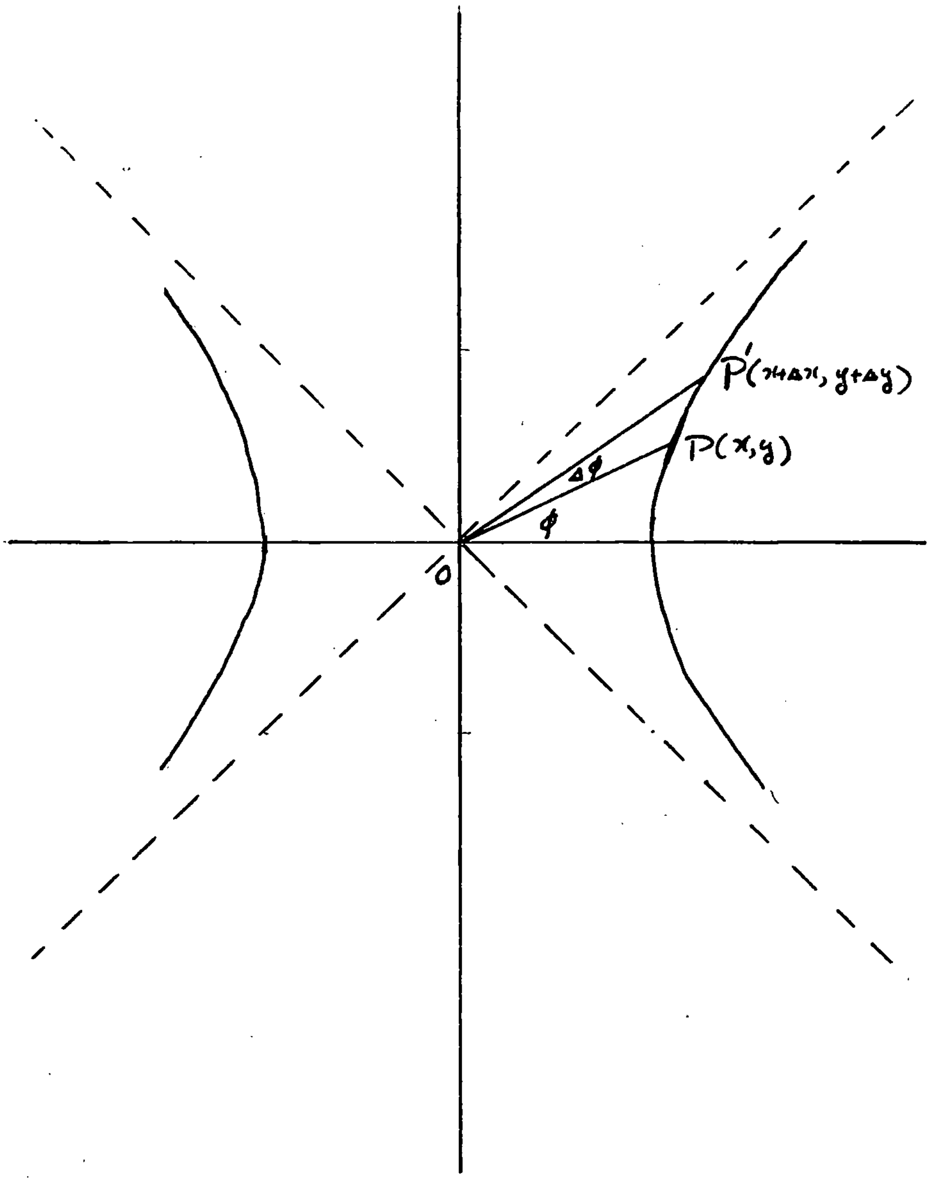

Figure I

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3.3 Rectangular Hyperbola .

The curve $(x-x_0)^2-(y-y_0)^2 = a^2$, together with its conjugate

$(x-x_0)^2-(y-y_0)^2 = -a^2$ play in the hyperbolic plane the role which the circle plays in the complex plane . We refer to (x_0, y_0) as the centre and to a as the radius of the hyperbola . The radius is the constant hyper-bolic distance of any point on the hyperbola from the centre . Since all hyperbola's entering into this subject are rectang-ular , with axes parallel to the coordinate axes , we refer to a rectangular hyperbola of this type simply as an"hyperbola" .

<u>Theorem 3.31</u> An hyperbolic arc of hyperbolic length s subtends an hyperbolic angle of magnitude

$$\phi = \frac{s}{r}$$

at the centre of the hyperbola of radius r .

Proof : Let the point $P(x,y)$ on the hyperbola (fig. 1) determine the radius vector OP making an angle ϕ with the x - axis , and let the neighbouring point

$P'(x+\Delta x, y+\Delta y)$ on the hyperbola determine the radius vector OP' making an angle $\phi + \Delta\phi$ with the x-axis. Since

$$\cosh(\phi+\Delta\phi) = \frac{x+\Delta x}{r} \quad , \quad \sinh(\phi+\Delta\phi) = \frac{y+\Delta y}{r},$$

and , since we are working in quadrant I , then

$$\sinh \Delta\phi = \sinh[(\phi+\Delta\phi)-\phi]$$

$$= \sinh(\phi+\Delta\phi)\cosh\phi - \cosh(\phi+\Delta\phi)\sinh\phi$$

$$= \frac{y+\Delta y}{r}\cdot\frac{x}{r} - \frac{x+\Delta x}{r}\cdot\frac{y}{r}$$

$$= \frac{x\Delta y - y\Delta x}{r^2}. \tag{3.31}$$

By the corollary to theorem (3.21) , the hyperbolic length Δs of the hyperbolic element of arc PP' is asymptotically equivalent to the chord length of PP' (in the hyperbolic metric) as $\Delta\phi$ tends to zero :

$$\Delta s \cong \sqrt{|(\Delta x)^2-(\Delta y)^2|}\,.$$

Therefore (3.31) becomes

$$\frac{\sinh\Delta\phi}{\Delta s} \cong \frac{1}{r^2}\,\frac{x\Delta y - y\Delta x}{\sqrt{|(\Delta x)^2-(\Delta y)^2|}}\,. \tag{3.32}$$

Differentiating the equation of the hyperbola

$$x^2 - y^2 = r^2$$

with r constant we get

$$x\Delta x - y\Delta y = 0$$

and hence

$$\frac{x\Delta y - y\Delta x}{\sqrt{|(\Delta x)^2-(\Delta y)^2|}} = \frac{x\left(\frac{x\Delta x}{y}\right) - y\Delta x}{\sqrt{|(\Delta x)^2 - \left(\frac{x\Delta x}{y}\right)^2|}} = \sqrt{|x^2-y^2|} = r$$

(3.32) then becomes $\dfrac{\sinh \Delta\phi}{\Delta s} \cong \dfrac{1}{r}$,

But
$$\sinh \Delta\phi = \Delta\phi + \frac{(\Delta\phi)^3}{3!} + \frac{(\Delta\phi)^5}{5!} + \cdots$$

and so $\sinh \Delta\phi \cong \Delta\phi \cong d\phi$.

Moreover , since $\Delta s \cong ds$,

we have $d\phi \cong \dfrac{ds}{r}$,

and therefore
$$\phi = \int_{s_1}^{s_2} \frac{ds}{r} = \frac{s_2 - s_1}{r} = \frac{s}{r},$$

where s is the arc length subtending ϕ at the centre .

<u>Theorem 3.32</u> Sine law for triangles :

Let sides a_1 , a_2 , a_3 of a triangle have hyperbolic lengths ρ_1 , ρ_2 , ρ_3 respectively , and let the interior angle defined by a_i , a_j be denoted by $\psi_{s_{ij}}^{i,j}$.

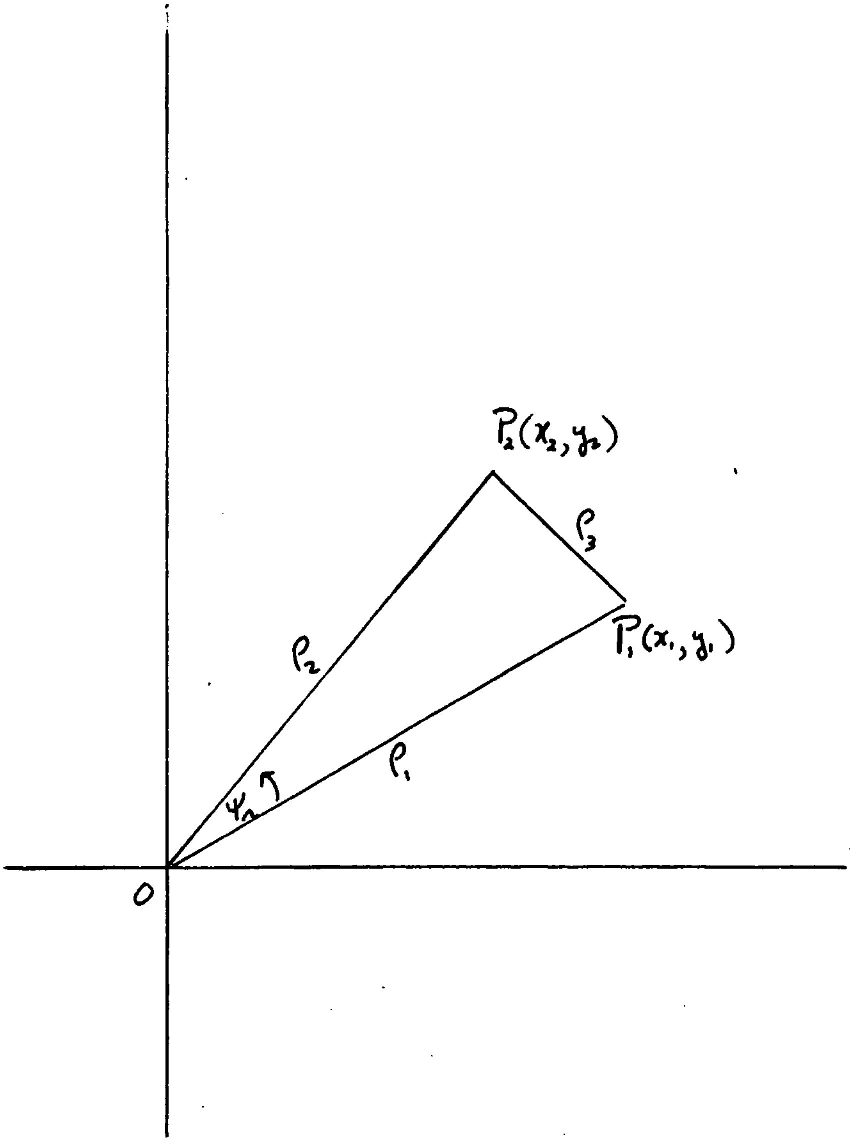

Figure 2

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Then :

$$\frac{|\sinh \psi^{23}_{s_{23}}|}{\rho_1} = \frac{|\sinh \psi^{31}_{s_{31}}|}{\rho_2} = \frac{|\sinh \psi^{12}_{s_{12}}|}{\rho_3} .$$

This law is proved by first developing the formula for the area of the triangle .

Let OP_1 (fig. 2) be of length ρ_1 and define angle ϕ_s , and let OP_2 be of length ρ_2 and define angle θ_t .

The angle measured from OP_1 to OP_2 is $\psi_n = \theta_t - \phi_s$.

Then

$$\tfrac{1}{2}\rho_1\rho_2 |\sinh \psi_n| = \tfrac{1}{2}\rho_1\rho_2 \left|\sinh \theta_t \cosh \phi_s - \cosh \theta_t \sinh \phi_s\right|$$

$$= \tfrac{1}{2}\left|\rho_2 \sinh \theta_t \, \rho_1 \cosh \phi_s - \rho_2 \cosh \theta_t \, \rho_1 \sinh \phi_s\right|$$

$$= \tfrac{1}{2}\left| y_2 x_1 - x_2 y_1\right| = A,$$

where A is area of the triangle OP_1P_2 .

From the area formula , $A = \tfrac{1}{2}\rho_1\rho_2 |\sinh \psi_n|$, the sine law follows immediately , on equating the three express--ions for area :

$$\tfrac{1}{2}\rho_1\rho_2 |\sinh \psi^{12}_{s_{12}}| = \tfrac{1}{2}\rho_2\rho_3 |\sinh \psi^{23}_{s_{23}}| = \tfrac{1}{2}\rho_3\rho_1 |\sinh \psi^{31}_{s_{31}}|$$

Therefore

$$\frac{|\sinh \psi_{s_{12}}^{12}|}{\rho_3} = \frac{|\sinh \psi_{s_{23}}^{23}|}{\rho_1} = \frac{|\sinh \psi_{s_{31}}^{31}|}{\rho_2}.$$

3.4 Hyperbolic Orthogonality .

We must give three definitions .

Vectors : To each number $x + yu = r(\cosh \phi_s + u \sinh \phi_s)$ corresponds a vector originating at the origin of the hyper--bolic plane , and defined completely by the modulus r and angle ϕ_s . Further , each vector originating at the origin determines a unique angle ϕ_s .

Diagonal Lines : The asymptotes of any rectangular hyper--bola : $(x - x_0)^2 - (y - y_0)^2 = a^2$ will be said to constitute a pair of diagonal lines in the hyperbolic plane . Thus to every distinct point of the plane corresponds one pair of diagonal lines .

Hyperbolic Orthogonality : Two vectors in the hyperbolic plane are mutually orthogonal if the hyperbolic tangent of the angle between them is infinite .

Theorem 3.41 Two vectors , corresponding to angles ϕ_s , θ_t respectively , are mutually orthogonal if and only if $\tanh \phi_s \cdot \tanh \theta_t = 1$.

Proof : The condition $\tanh(\phi_s - \theta_t) = \infty$

may be written
$$\frac{\tanh\phi_s - \tanh\theta_t}{1 - \tanh\phi_s \tanh\theta_t} = \infty . \qquad (3.41)$$

If the numerator is finite and not zero , the last equation is equivalent to : $\tanh\phi_s \cdot \tanh\theta_t = 1 .$

If the numerator is infinite then at least one of the compon--ents of this sum is infinite . Suppose $\tanh\phi_s = \infty$. Then, if $\tanh\theta_t \neq 0$, the denominator is also infi--nite so that the quotient is not infinite , as required . Hence it is necessary that $\tanh\theta_t = 0$, and we may assign the value 1 to the indeterminate form ;

$$\tanh\phi_s \cdot \tanh\theta_t = \infty \cdot 0 = 1 .$$

If the numerator is 0, then $\tanh\phi_s = \tanh\theta_t$;

let
$$x_1 + y_1 u = r_1(\cosh\phi_s + u \sinh\phi_s)$$
$$x_2 + y_2 \cdot u = r_2(\cosh\theta_t + u \sinh\theta_t)$$

$$x_1 = r_1 \cosh\phi_s \qquad y_1 = r_1 \sinh\phi_s$$
$$x_2 = r_2 \cosh\theta_t \qquad y_2 = r_2 \sinh\theta_t$$
$$\tanh\phi_s = \frac{y_1}{x_1} \qquad \tanh\theta_t = \frac{y_2}{x_2}$$

Equation (3.41) could be satisfied only if $\tanh\phi_s \cdot \tanh\theta_t = 1$,

and then , only if we assign the value ∞ to the indeter--minant form $\frac{0}{0}$.

The equations $\tanh\phi_s = \tanh\theta_t$

and $\tanh\phi_s \tanh\theta_t = 1$

of this special case , give

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}$$

$$\frac{y_1}{x_1} \cdot \frac{y_2}{x_2} = 1$$

Either

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = 1$$

or

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = -1$$

Hence both vectors lie on the same diagonal line through the origin , and have the same sense .

By a unit vector associated with a given vector we mean the vector of unit hyperbolic length defined by the same hyper--bolic angle .

<u>Theorem (3.42)</u> : Two vectors are mutually orthogonal if and only if their unit vectors are mutually reflections of one another in one or other of the diagonal lines through the

origin .

For if we write $$x_1 + y_1 u = r_1(\cosh\phi_s + u\sinh\phi_s)$$

$$x_2 + y_2 u = r_2(\cosh\theta_t + u\sinh\theta_t)$$

the condition $$\tanh\phi_s \tanh\theta_t = 1$$

gives $$\frac{y_1}{x_1}\cdot\frac{y_2}{x_2} = 1$$

or $$\frac{y_1}{x_1} = \frac{x_2}{y_2}$$

which expresses the symmetry with respect to one of the dia--gonal lines through the origin , as stated in the theorem .

<u>Theorem 3.43</u> : Cosine law for triangles :

Define a "length function" of line segment joining (x_1, y_1) and (x_2, y_2) to be $\overline{\rho^2} = (x_1 - x_2)^2 - (y_1 - y_2)^2$ then if sides a_1 , a_2 , a_3 of a triangle have lengths ρ_1 , ρ_2 , ρ_3 respectively , and length functions $\overline{\rho_1^2}$, $\overline{\rho_2^2}$, $\overline{\rho_3^2}$ respectively , and ψ_n is angle included by a_1 , a_2 :

$$\overline{\rho_3^2} = \overline{\rho_1^2} + \overline{\rho_2^2} - 2\rho_1\rho_2\cosh\psi_n .$$

Proof : For the triangle of fig. 2 :

$$2\rho_1\rho_2 \cosh\psi_r = 2\rho_1\rho_2 \left(\cosh\theta_t \cosh\phi_s - \sinh\theta_t \sinh\phi_s\right)$$

$$= 2\rho_1\rho_2 \left(\frac{x_2}{\rho_2}\frac{x_1}{\rho_1} - \frac{y_2}{\rho_2}\frac{y_1}{\rho_1}\right)$$

$$= 2\left(x_1 x_2 - y_1 y_2\right).$$

Then since $\overline{\rho_1^2} = x_1^2 - y_1^2$, $\overline{\rho_2^2} = x_2^2 - y_2^2$,

$$\overline{\rho_3^2} = (x_2 - x_1)^2 - (y_2 - y_1)^2,$$

and since $(x_2 - x_1)^2 - (y_2 - y_1)^2 = (x_1^2 - y_1^2) + (x_2^2 - y_2^2) - 2(x_1x_2 - y_1y_2)$

we have $\overline{\rho_3^2} = \overline{\rho_1^2} + \overline{\rho_2^2} - 2\rho_1\rho_2\cosh\psi_r.$

<u>Corollary</u> : If sides a_1, a_2 are mutually orthogonal, then

$$\overline{\rho_3^2} = \overline{\rho_1^2} + \overline{\rho_2^2}.$$

For ,
$$\frac{\sinh\psi_r}{\cosh\psi_r} = \tanh\psi_r = \infty.$$

Since the triangle is defined by three points in the finite plane , $\sinh\psi_r$ cannot be infinite because of the sine law (theorem 3.32) . Hence $\cosh\psi_r = 0$

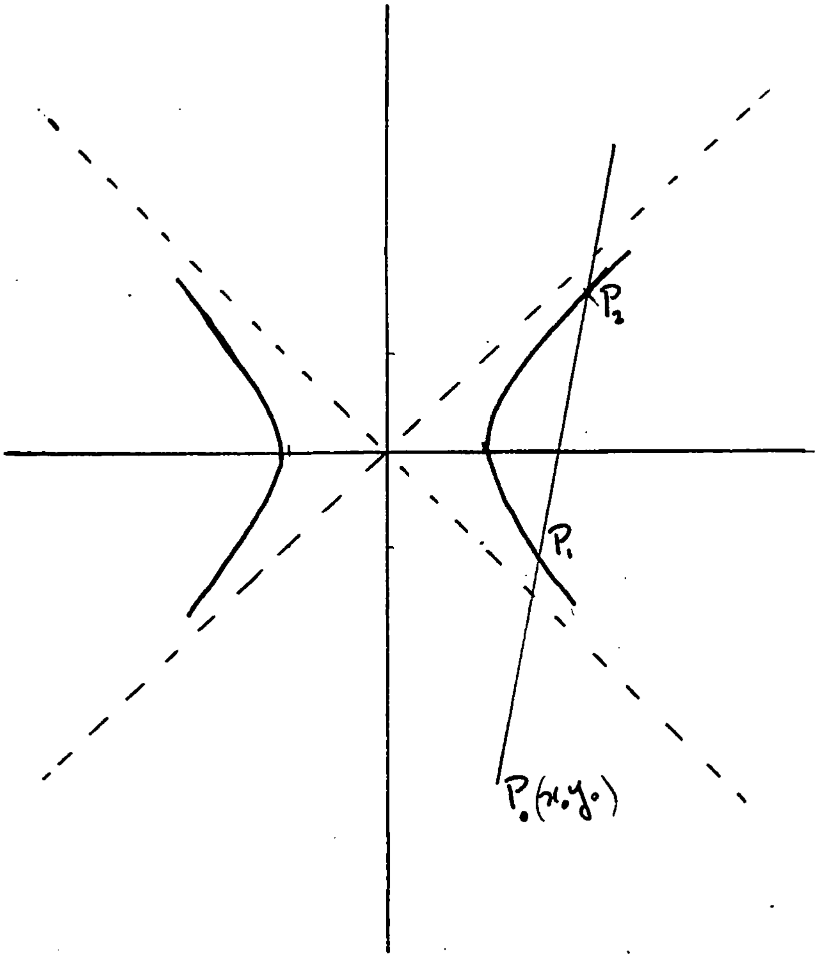

Figure 3

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By the angle between two curves intersecting at a point we mean the hyperbolic angle between the respective tangents to the curves at the point of intersection .

<u>Theorem 3.44</u> : The radius vector of an hyperbola intersects the hyperbola orthogonally .

Note that if ϕ_s is angle determined by the vector from origin to point (x, y) , then :

$$\tanh \phi_s = \frac{y}{x}$$

Hence the hyperbolic tangent of the angle ϕ_s of a vector is merely its "slope" as understood in Euclidean plane geometry.

The slope of tangent at (x, y) on the hyperbola

$$x^2 - y^2 = r^2$$

is given by

$$\frac{dy}{dx} = \frac{x}{y} = \tanh \theta_t .$$

where θ_t is angle made by the tangent and the positive x -axis . But the slope of the radius vector to (x, y) on the hyperbola is : $\tanh \phi_s = \frac{y}{x}$

Hence $\tanh \phi_s \cdot \tanh \theta_t = 1$, which proves the orthogonality stated in the theorem .

In the following theorem we distinguish a positive sense $P_0 L$ from a negative sense $L P_0$ along a line in the plane (Fig. 3) . We assign a positive sign to sense $P_0 L$

when we assign an angle ϕ_s to P_0L, and thus regard it as a vector .

Theorem 3.45 : Let a pencil of lines through any point P_0 of plane , whose vectors lie in one quadrant (bounded by diagonal lines) , cut an hyperbola in points P_1 , P_2 . Then the product $P_0P_1 \cdot P_0P_2$ is constant over the members of the pencil . The transition from one quadrant to an adjacent quadrant results in a mere sign chance in the product .

Proof : Let $P(x,y)$ be a point on P_0L . Let PL define angle ϕ_s and let ρ be directed hyperbolic distance of P from P_0 .

Then

$$\begin{aligned} x &= x_0 + \rho \cosh \phi_s \\ y &= y_0 + \rho \sinh \phi_s \end{aligned} \qquad (3.42)$$

Substituting the expressions (3.42) in $x^2 - y^2 = r^2$

we obtain : $(\cosh^2 \phi_s - \sinh^2 \phi_s)\rho^2 + 2(x_0 \cosh \phi_s - y_0 \sinh \phi_s)\rho + x_0^2 - y_0^2 - r^2 = 0$

But $\cosh^2 \phi_s - \sinh^2 \phi_s = \pm 1$ depending on the quadrant of ϕ_s

So that if the roots are ρ_1 , ρ_2 then

$$\rho_1 \rho_2 = \pm (x_0^2 - y_0^2 - r^2)$$

which is constant over a pencil of lines PL lying in one quadrant . Since $\left| \cosh^2\phi_s - \sinh^2\phi_s \right| \equiv 1$

and product $\rho_1 \rho_2$ depends on sign of $\cosh^2\phi_s - \sinh^2\phi_s$,

the value of $\rho_1 . \rho_2$ changes sign on transition from one quadrant to an adjacent quadrant .

3.5 Analytic Relations of Bireal Variables .

Theorem 3.51 Euler Theorem : A bireal variable is ex--pressible exponentially in terms of its modulus and amplitude:

$$x + yu = r(\cosh\phi_s + u\sinh\phi_s) = re^{u\phi_s}$$

where the factor $e^{u\phi_s}$ obeys the rules of an exponential function .

Setting $x = u\phi$ in $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{u\phi} = 1 + \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \cdots + u\left(\phi + \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \cdots\right)$$

$$= \cosh\phi + u\sinh\phi \qquad (3.51)$$

Writing $e^{u\phi_s} = \cosh\phi_s + u\sinh\phi_s$ (by definition)

$$\phi_1 = \phi$$

and applying relations :

$$\cosh \phi_2 = \sinh \phi_1$$
$$\sinh \phi_2 = \cosh \phi_1$$
$$\cosh \phi_3 = -\cosh \phi_1$$
$$\sinh \phi_3 = -\sinh \phi_1$$
$$\cosh \phi_4 = -\cosh \phi_2$$
$$\sinh \phi_4 = -\sinh \phi_2$$

We obtain :
$$e^{u\phi_s} = \lambda(s)\, e^{u\phi} \qquad (3.52)$$

where $\lambda(1) = 1$, $\lambda(2) = u$, $\lambda(3) = -1$, $\lambda(4) = -u$

The relations

$$(\cosh \phi_s + u \sinh \phi_s)(\cosh \psi_r + u \sinh \psi_r) = \cosh(\phi_s + \psi_r) + u \sinh(\phi_s + \psi_r)$$

$$\frac{1}{\cosh \phi_s + u \sinh \phi_s} = \cosh \phi_s - u \sinh \phi_s = \cosh(-\phi_s) + u \sinh(-\phi_s)$$

imply the exponential rules :

$$e^{u\phi_s} \cdot e^{u\psi_r} = e^{u(\phi_s + \psi_r)}$$
$$\frac{e^{u\phi_s}}{e^{u\psi_r}} = e^{u(\phi_s - \psi_r)} \qquad (3.53)$$

From (3.51)and (3.52) :

$$\frac{d}{d\phi} e^{u\phi_s} = \frac{d}{d\phi} \lambda(s) e^{u\phi} = \lambda(s) u e^{u\phi} = u e^{u\phi_s} \qquad (3.54)$$

CONVERGENCE OF POWER SERIES .

By convergence of an hyperbolic series we mean convergence of both real series .

A series of hyperbolic terms :

$$\sum_{\nu=0}^{\infty} z_\nu = \sum_{\nu=0}^{\infty} \rho_\nu e^{u\phi_s^\nu} = \sum_{\nu=0}^{\infty} \rho_\nu \cosh \phi_{s_\nu}^\nu + u \sum_{\nu=0}^{\infty} \rho_\nu \sinh \phi_{s_\nu}^\nu$$

is not dominated by the absolute series $\sum_{\nu=0}^{\infty} \rho_\nu$ as in the analogous case of a complex series , because

$$|\cosh \phi_{s_\nu}^\nu| \gtreqless 1 \quad , \quad |\sinh \phi_{s_\nu}^\nu| \gtreqless 1$$

For the same reason , for the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

of a bireal variable $z = x + yu$, there exists no radius of convergence . As Bencivenga shows , the region of convergence is bounded not by an hyperbola , but by a rectan--gle .

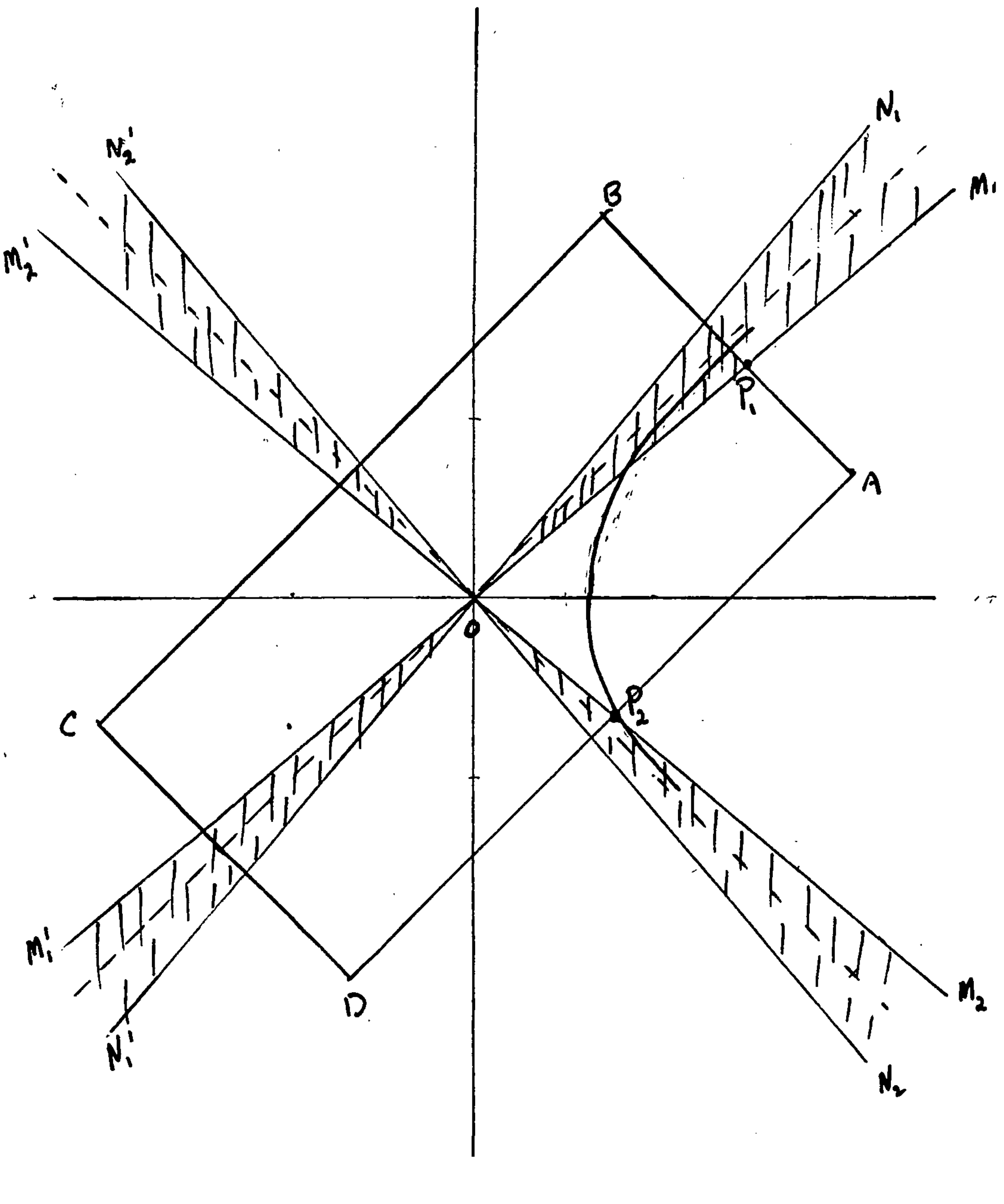

Figure 4

Chapter III

Theorem 3.52 : For a region of the plane defined by

$$|am(z)| \leqq \Omega$$

the Taylor expansion $\quad f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} z^m$

has a radius of convergence $a(\Omega)$.

We prove this geometrically : The series converges within some rectangular region $ABCD$ with sides parallel to diagonal lines through the origin O which is at the inter-section of diagonals AC and BD (Fig. 4) , as Bencivenga shows . In figure 4 let the unshaded parts of the plane represent a region $|am(z)| \leqq \Omega$.

Of the two points P_1 , P_2 at the intersection of boundary lines of region $|am(z)| \leqq \Omega$ with rectangle of con-vergence , let one of them , say P_2 be closest to O , in the hyperbolic metric . Then P_2 determines a unique hyperbola $x^2 - y^2 = a_1^2$ which passes through P_2 and such that P_1 is either on the hyperbola or lies to the side remote from the origin . Similarly determine the hyperbolas of radii a_2 , a_3 , a_4 respectively in the other three quad-rants . Any point z of the region $|am(z)| \leqq \Omega$ and such that $|z| < \min(a_1, a_2, a_3, a_4)$, lies inside the rectangle of convergence . The required radius of convergence corresponding to Ω is :

$$a(\Omega) = \min(a_1, a_2, a_3, a_4)$$

We now prove that an analytic function of a bireal variable maps the hyperbolic plane into itself conformally, by apply--ing the Taylor expansion and the Euler theorem. We employ the following notation :

Let $$z = x + yu = r e^{u\phi_s}$$

$$\lim |z| = 0 \quad \text{means} \quad r \to 0$$

$$\lim am(z) = 0 \quad \text{means} \quad \phi \to 0$$

$$\lim z = 0 \quad \text{means} \quad x \to 0 \text{ and } y \to 0$$

<u>Theorem 3.53</u> : The function $\omega = f(z)$ of the birealvar--iable z maps the z-plane conformally into the ω-plane at every point z at which $f(z)$ is analytic and $|f'(z)| \neq 0$. At all such points the mapping is biunique and the magnification and mapping angle are

$$|f'(z)| \quad , \quad am(f'(z)) \quad \text{respectively}.$$

Proof : Expanding $f(z)$ in Taylor series about z_0 we have $\omega - \omega_0 = A(z-z_0) + B(z-z_0)^2 + C(z-z_0)^3 + \cdots$

Writing $z - z_0 = r e^{u\phi_s}$, $A = a e^{u\alpha_p}$, $\omega - \omega_0 = \rho e^{\theta_t}$.

This becomes $\rho e^{\theta_t} = ar e^{u(\alpha_p+\phi_s)} + Br^2 e^{2u\phi_s} + Cr^3 e^{3u\phi_s} + \cdots$

Since, by hypothesis, $a = |A| = |f'(z_0)| \neq 0$,

$$\rho e^{\theta_t} = ar e^{u(\alpha_p+\phi_s)} \left[1 + \frac{B}{a} r e^{u(\phi_s-\alpha_p)} + \frac{C}{a} r^2 e^{u(2\phi_s-\alpha_p)} + \cdots \right]$$

We now impose the condition that

$$|am(z-z_0)| = |\phi| \leqq \Omega \qquad (3.55)$$

With the restriction (3.55) the series

$$\psi(r,\phi) = \frac{B}{a} r e^{u(\phi_s-\alpha_p)} + \frac{C}{a} r^2 e^{u(2\phi_s-\alpha_p)} + \cdots$$

has a radius of convergence $a(\Omega)$. Then, since the terms of the series have a common factor r ,

$$\rho e^{u\theta_t} = ar e^{u(\alpha_p+\phi_s)} \left[1 + \psi(r,\phi) \right] \qquad (3.56)$$

where $\lim_{r \to 0} \psi(r,\phi) = 0$

That is , ψ approaches zero uniformly with respect to ϕ .

(3.56) implies :

$$\begin{aligned} \rho &= a r \left(1 + \mu(r,\phi) \right) \\ \theta_t &= \alpha_p + \phi_s + \nu(r,\phi) \end{aligned} \qquad (3.57)$$

where the real functions $\mu(r,\phi)$, $\nu(r,\phi)$ each tend to zero , uniformly with respect to ϕ ,as r approaches zero .

At $r = 0$ we have

$$\begin{aligned} \rho &= a r \\ \theta_t &= \alpha_p + \phi_s \end{aligned} \qquad (3.58)$$

So that the magnification is $a = |f'(z_0)|$ and the angle of the mapping is $\alpha_p = am\,(f'(z_0))$.

Write $\omega = f(z)$ in the form :

$$\omega_1 + \omega_2 \cdot u = f_1(x,y) + f_2(x,y) \cdot u$$

Equations $\omega_1 = f_1(x,y)$, $\omega_2 = f_2(x,y)$

are uniquely soluble for x , y if

$$\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} \neq 0$$

Applying the Cauchy - Riemann equations, this condition reads

$$\left|\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_1}{\partial x} \end{vmatrix}\right| = |f'(z)|^2 \neq 0$$

Hence the mapping is biunique at points z for which $f(z)$ is differentiable and $|f'(z)| \neq 0$

It remains to remove the restriction (3.55) :
Since Ω may be chosen as large as we please , we can chose it to exceed any given finite $|\,am\,(z-z_0)\,|$

Therefore the theorem is proved for any finite value of

$|\,am\,(z-z_0)\,|$, and it remains to treat the case :

$|\,am\,(z-z_0)\,| = \infty$. Since $|f'(z_0)| \neq 0$ the mapping is biunique and hence to the mapping

$z_0 \to \omega_0$, $z \to \omega$ there corresponds the unique inverse mappings $\omega_0 \to z_0$, $\omega \to z$.

In this case $|am(z-z_0)| = \infty$ implies $|am(\omega-\omega_0)| = \infty$,

for if $|am(\omega-\omega_0)|$ were finite then $|am(z-z_0)|$

would also be finite . Therefore in this case also , the mapping is conformal , so that restriction (3.55) has been removed , to complete the proof .

Point at infinity

Let $z = x + yu$

By $z \to 0$ or $\lim z = 0$ we mean $x \to 0$ and $y \to 0$

By $z \to \infty$ or $\lim z = \infty$ we mean $x \to \infty$ or (and) $y \to \infty$

We regard ∞ as a single point added to the finite hyperbolic plane ; any variable $z = x + yu$ approaches this point at infinity as either x or y (or both together) tend to infinity on the real line .

Theorem 3.54 : Assuming that as a variable $z = x + yu$ tends to zero or infinity , it does so along a curve , the slope of whose tangent tends to a limit (finite or infinite), then

$$\lim_{z \to 0} \left(\frac{1}{z}\right) = \infty \quad , \quad \lim_{\substack{z \to \infty \\ \frac{x}{y} \to L \neq \pm 1}} \left(\frac{1}{z}\right) = 0$$

Proof :

$$\frac{1}{z} = \frac{x}{x^2 - y^2} - \frac{y}{x^2 - y^2} \cdot u \qquad (3.59)$$

By hypothesis, $\frac{x}{y}$ tends to a limit (finite or infinite):

$$\lim \frac{x}{y} = L$$

or $\quad x = y(L + \varepsilon) \quad$, where $\quad \varepsilon \to 0$

Substituting in (3.59) :

$$\frac{1}{z} = \frac{1}{y}\left[\frac{L+\varepsilon}{(L+\varepsilon)^2 - 1} - \frac{1}{(L+\varepsilon)^2 - 1} \cdot u\right]$$

If $L^2 \neq 1$,

$$\frac{1}{z} \sim \frac{1}{y}\left[\frac{L+\varepsilon}{L^2 - 1} - \frac{1}{L^2 - 1} \cdot u\right]^* \qquad (3.510)$$

If x , but not y , tends to infinity then L is infinite ; if y tends to infinity then both components of right member .....

* By $x_1 + y_1 u \sim x_2 + y_2 u$ we mean $\frac{x_1}{x_2} \to 1$, $\frac{y_1}{y_2} \to 1$

of (3.510) tend to zero . Hence $\frac{1}{z}$ tends to zero as z tends to infinity . As y tends to zero at least the second compon--ent of right member of (3.510) tends to infinity . This proves the theorem for $L^2 \neq 1$.

If $L = \pm 1$, $$\frac{1}{z} = \frac{1}{y}\left[\frac{\pm 1 + \varepsilon}{\varepsilon^2 \pm 2\varepsilon} - \frac{1}{\varepsilon^2 \pm 2\varepsilon} u \right]$$

so that $$\frac{1}{z} \sim \frac{1}{y}\left[\frac{\pm 1}{\pm 2\varepsilon} - \frac{1}{\pm 2\varepsilon} u \right] \qquad (3.511)$$

as $y \to 0$ both components of right member of (3.511) tend to infinity . This completes the proof , since statement of theoremrules out case $z \to \infty$ when $L^2 = 1$

3.6 Bilinear Transformation .

As a special case of the conformal transformation of the hyperbolic plahe we shall discuss in detail the bilinear tran--sformation :

$$\omega = \frac{\alpha z + \beta}{\gamma z + \delta}$$ where $\alpha, \beta, \gamma, \delta$

are bireal constants subject to the condition

$$\left\| \begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix} \right\| \neq 0 \qquad (3.61)$$

<u>Theorem 3.61</u> : If the single point at infinity is added to the finite hyperbolic plane to give the complete hyperbolic plane , then the bilinear transformation

$$\omega = \frac{\alpha z + \beta}{\gamma z + \delta}$$

maps the complete hyperbolic z -plane , with points for which $|\gamma z + \delta| = 0$, $\gamma z + \delta \neq 0$ excluded , biuniquely on the complete ω -plane ,with points for which

$$|\gamma\omega - \alpha| = 0 \quad , \quad \gamma\omega - \alpha \neq 0 \qquad \text{excluded .}$$

Proof : If $\dfrac{d\omega}{dz} = \dfrac{\alpha\delta - \beta\gamma}{(\gamma z + \delta)^2}$ exists

then $\left|\dfrac{d\omega}{dz}\right| \neq 0$ by condition (3.61).

$\dfrac{d\omega}{dz}$ fails to exist if $|\gamma z + \delta| = 0$; that

is if $|\gamma z + \delta|^2 = |(\gamma_1 + \gamma_2 u)(x + yu) + (\delta_1 + \delta_2 u)|^2$

$$= (\gamma_1^2 - \gamma_2^2)(x^2 - y^2) + 2(\gamma_1\delta_1 - \gamma_2\delta_2)x + 2(\gamma_2\delta_1 - \gamma_1\delta_2)y + (\delta_1^2 - \delta_2^2) = 0. \qquad (3.62)$$

Now a conic $Ax^2 + By^2 + 2Ex + 2Fy + C = 0$ is degenerate if

$$\begin{vmatrix} A & 0 & E \\ 0 & B & F \\ E & F & C \end{vmatrix} = 0$$

Therefore, since

$$\begin{vmatrix} \gamma_1^2 - \gamma_2^2 & 0 & \gamma_1\delta_1 - \gamma_2\delta_2 \\ 0 & -(\gamma_1^2 - \gamma_2^2) & \gamma_2\delta_1 - \gamma_1\delta_2 \\ \gamma_1\delta_1 - \gamma_2\delta_2 & \gamma_2\delta_1 - \gamma_1\delta_2 & \delta_1^2 - \delta_2^2 \end{vmatrix} = 0$$

(3.62) is the equation of a pair of diagonal lines inter--secting at the point $\left(-\dfrac{\gamma_1\delta_1 - \gamma_2\delta_2}{\gamma_1^2 - \gamma_2^2} \, , \, \dfrac{\gamma_2\delta_1 - \gamma_1\delta_2}{\gamma_1^2 - \gamma_2^2} \right)$

provided that $|\gamma|^2 = |\gamma_1^2 - \gamma_2^2| \neq 0$

$$\gamma z + \delta = 0 \quad at \quad z = -\frac{\delta}{\gamma} = -\frac{\delta\bar{\gamma}}{\gamma\bar{\gamma}}$$

$$= -\frac{(\delta_1 + \delta_2 u)(\gamma_1 - \gamma_2 u)}{\gamma_1^2 - \gamma_2^2} = -\frac{\gamma_1\delta_1 - \gamma_2\delta_2 + (-\delta_1\gamma_2 + \gamma_1\delta_2)u}{\gamma_1^2 - \gamma_2^2}$$

provided that $|\gamma|^2 \neq 0$

That is , $\gamma z + \delta = 0$ at the intersection of lines (3.62) , in the case that $|\gamma| \neq 0$.

The inverse of $\omega = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, if it exists , is given by

$$z = \frac{-\delta\omega + \beta}{\gamma\omega - \alpha} \qquad (3.63)$$

Case $|\gamma| \neq 0$:

$$\lim_{z \to -\frac{\delta}{\gamma}} \frac{\alpha z + \beta}{\gamma z + \delta} = \infty$$ i.e. ω tends to ∞

$$\lim_{\omega \to \infty} \frac{-\delta\omega + \beta}{\gamma\omega - \alpha} = -\frac{\delta}{\gamma}$$

Hence the one to one correspondence :

$$z = -\frac{\delta}{\gamma} \longleftrightarrow \omega = \infty$$

and similarly :

$$\omega = \frac{\alpha}{\gamma} \longleftrightarrow z = \infty$$

Case $|\gamma| = 0$: By condition (3.61) :

$$|\alpha| \neq 0 \quad \text{and} \quad |\delta| \neq 0$$

The pair of excluded lines (3.62) then reduce to a single straight line as follows :

Set $\gamma_2 = \gamma_1 \neq 0$ in (3.62) : $x + y + \frac{\delta_1 + \delta_2}{2\gamma_1} = 0$.
Then $\gamma z + \delta = \frac{\delta_1 - \delta_2}{2}(1 - u) \neq 0$.
Set $\gamma_2 = -\gamma_1$ in (3.62) : $x - y + \frac{\delta_1 - \delta_2}{2} = 0$
Then $\gamma z + \delta = \frac{\delta_1 - \delta_2}{2}(1 - u) \neq 0$,

where we have assumed $\gamma \neq 0$.

For $\gamma = 0$, since $|\alpha| \neq 0$ and $|\delta| \neq 0$ by (3.61), the mapping is defined over the finite z -plane , and over the finite ω -plane , for every point .

Hence for $|\gamma| = 0$ we may assign the correspondence

$$z = \infty \longleftrightarrow \omega = \infty .$$

We call the pair of diagonal lines for which $|\gamma z + \delta| = 0$ the singular lines of the z -plane , and similarly

$|\gamma\omega - \alpha| = 0$ defines the singular lines of the ω -plane . All points of the singular lines , except their intersection (in the finite plane , or at infinity , if the singular lines reduce to a single line) , are excluded from the mapping . In the case that $\gamma = 0$, there is no singular line in either of the finite z or finite ω -planes .

We may then think of the singular lines as reducing to the single point at infinity .

3.7 Bilinear Transformation of the Rectangular Hyperbola .

Let A, B, C be bireal constants and

$$x = x_1 + x_2 u \quad , \quad \bar{x} = x_1 - x_2 u \ ,$$

a bireal variable and its conjugate . Then

$$(A + \bar{A}) x \bar{x} + B x + \bar{B} \bar{x} + C + \bar{C} = 0 \qquad (3.71)$$

is the equation of a true or degenerate rectangular hyperbola with axes parallel to x- and y- axes , or of a single straight line (which will be classed as an hyperbola) . For , writing $A = A_1 + A_2 u$ etc., (3.71) may be written :

$$\left(x_1 + \frac{B_1}{2A_1}\right)^2 - \left(x_1 - \frac{B_2}{2A_1}\right)^2 = \frac{B_1^2 - B_2^2 - 4A_1 C_1}{4A_1^2} ,$$

if $A_1 \neq 0$, or $B_1 x_1 + B_2 x_2 + C_1 = 0$,

if $A_1 = 0$.

<u>Theorem 3.71</u> : The bilinear transformation of the hyper- -bolic plane :

$$x = \frac{\alpha y + \beta}{\gamma y + \delta}$$

transforms an hyperbola in the x -plane into an hyperbola in the y -plane and conversely , where "hyperbola" denotes a rectangular hyperbola with axes parallel to coordinate axes or a straight line .

To prove this we apply the bilinear transformation to (3.71): Since for any two bireal numbers a and b :

$$\overline{a \pm b} = \bar{a} \pm \bar{b} \quad , \quad \overline{ab} = \bar{a} \cdot \bar{b}$$

$$\overline{\left(\frac{a}{b}\right)} = \frac{\bar{a}}{\bar{b}} \quad , \quad \bar{\bar{a}} = a .$$

Then

$$x = \frac{\alpha y + \beta}{\gamma y + \delta} \quad \textit{implies} \quad \bar{x} = \frac{\bar{\alpha}\bar{y} + \bar{\beta}}{\bar{\gamma}\bar{y} + \bar{\delta}} .$$

Substituting these expressions for x and $\bar{x}$ in (3.71) the result is :

$$\begin{aligned} &\left[\alpha\bar{\alpha} (A+\bar{A}) + \alpha\bar{\gamma} B + \bar{\alpha}\gamma\bar{B} + \gamma\bar{\gamma} (C+\bar{C}) \right] y\bar{y} \\ &+ \left[\alpha\bar{\beta} (A+\bar{A}) + \alpha\bar{\delta} B + \bar{\beta}\gamma\bar{B} + \gamma\bar{\delta} (C+\bar{C}) \right] y \\ &+ \left[\bar{\alpha}\beta (A+\bar{A}) + \bar{\alpha}\delta\bar{B} + \beta\bar{\gamma} B + \delta\bar{\gamma} (C+\bar{C}) \right] \bar{y} \\ &+ \left[\beta\bar{\beta} (A+\bar{A}) + \beta\bar{\delta} B + \bar{\beta}\delta\bar{B} + \delta\bar{\delta} (C+\bar{C}) \right] = 0, \end{aligned}$$

which is of same form as (3.71) . (3.72)

Theorem 3.72 : A bilinear transformation , which maps a finite point of an hyperbola into the point at infinity , transforms the hyperbola into a straight line .

Proof : Let y_1, y_2, y_3, y_4 be images of x_1, x_2, x_3, x_4 respectively . Then the cross - ratio :

$$\frac{(y_1-y_4)(y_3-y_2)}{(y_1-y_2)(y_3-y_4)} = \frac{(x_1-x_4)(x_3-x_2)}{(x_1-x_2)(x_3-x_4)}$$

is invariant under the transformation . Thus the bilinear trans-formation is determined by making y_1, y_2, y_3 correspond to x_1, x_2, x_3 respectively : then the image y , of any fourth point x is given by

$$\frac{(y_1-y)(y_3-y_2)}{(y_1-y_2)(y_3-y)} = \frac{(x_1-x)(x_3-x_2)}{(x_1-x_2)(x_3-x)}.$$

Now let $y_3 \to \infty$ (in the sense of theorem (3.54))

i,e. for $y_3 = \infty$:

$$\frac{y_1-y}{y_1-y_2} = \frac{(x_1-x)(x_3-x_2)}{(x_1-x_2)(x_3-x)}$$

or $$x = \frac{\alpha y + \beta}{\gamma y + \delta}$$

where
$$\alpha = x_3 (x_1 - x_2)$$
$$\beta = y_1 x_2 (x_3 - x_1) - x_1 y_2 (x_3 - x_2)$$
$$\gamma = x_1 - x_2$$
$$\delta = y_1 (x_3 - x_1) - y_2 (x_3 - x_2)$$

From these we see that

$$\alpha \bar{\alpha} = x_3 \bar{x}_3 \gamma \bar{\gamma} , \alpha \bar{\gamma} = x_3 \gamma \bar{\gamma} , \bar{\alpha} \gamma = \bar{x}_3 \gamma \bar{\gamma} . \qquad (3.73)$$

The transform of $(A + \bar{A}) x \bar{x} + B x + \bar{B} \bar{x} + c + \bar{c} = 0$

is a straight line if and only if (by (2.72)) :

$$\alpha \bar{\alpha} (A + \bar{A}) + \alpha \bar{\gamma} B + \bar{\alpha} \gamma \bar{B} + \gamma \bar{\gamma} (c + \bar{c}) = 0,$$

which , on application of (3.73) , reduces to

$$x_3 \bar{x}_3 (A + \bar{A}) + x_3 B + \bar{x}_3 \bar{B} + c + \bar{c} = 0,$$

which is merely the statement that x_3 lies on the original hyperbola .

Corollary (1) Given a pencil of hyperbolas each member of which passes through distinct points P and Q ; then a bilinear transformation mapping Q into the point at infin-ity maps the pencil into a pencil of straight lines all pass-ing through P', the image of P. (the points P and Q will be refered to as the poles of the pencil.)

Corollary (2) Given a pencil of hyperbolas each member of which touches at P, then a bilinear transformation, mapp-ing P into the point at infinity, maps the pencil into a pencil of parallel straight lines.

Corollary (3) Every pencil H of hyperbolas through two points P, Q (distinct or coincident, one or both of which may be at infinity) determines a unique pencil K orthogonal to it. Pencils H, K are orthogonal conjugates in the sense that either pencil determines the other uniquely. Any two members of either pencil determines the system H, K uniquely.

Proof: A bilinear transformation T mapping Q into the point at infinity transforms H into a pencil of lines inter-secting in Q', the image of Q. There exists a unique system K of hyperbolas concentric at Q'. The inverse transformation T^{-1} maps K' into the pencil K which is orthogonal to H (By theorem (3.44) and conform-ality). Finally, any two members of a straight line pencil through Q' determine the line-pencil, and hence the equivalent hyperbolic pencil.

3.8 Bilinear Equivalence .

Any two systems , each of which is the image of the other under some bilinear transformation and its inverse , will be said to be bilinearly equivalent .

This relation is reflexive, symmetric and transitive and thus a proper equivalence relation .

Theorem 3.81 : Any two hyperbolic pencils (one or both of which may be a line-pencil through a point in the finite plane), each having two distinct poles , are bilinearly equivalent .

Proof : Let pencil H have distinct poles P and Q . There exists a transformation T_H mapping Q into infinity and P into P' , the intersection point of line pencil H' . (theorem (3.72) , corollary (1)) Let second pencil K have distinct poles R , S . There exists T_K mapping R into infinity and S into P' . The product $T_H T_K^{-1}$ maps H into K .

Theorem 3.82 : Every hyperbolic pencil with two distinct poles is bilinearly equivalent to a concentric system of hyperbolas .

Proof : The given pencil H with two distinct poles P and Q determines a unique orthogonal hyperbolic pencil K (theorem (3.72) corollary (3)). Since the poles of H are distinct , so are the poles of K , for otherwise the pencil and conjugate could be mapped into a system of parallel straight lines with a second system of lines (hyperbolically) orthogonal to it , which could then be mapped into two tangen--tial pencils , each having common point of tangency at some point T (theorem (3.72) , corollary (2)). Then P and Q would both map into T by the product transformation , contrary to the properties of the bilinear transformation . Therefore K may be mapped into a line pencil passing through a point R , the centre of the concentric system H' .

<u>Corollary</u> : If the hyperbolic pencil H has two poles so also has its orthogonal conjugate .

<u>Theorem 3.83 ;</u> Let the bilinear transformation T map an hyperbolic pencil with distinct poles P , Q in the x -plane into a concentric system in the x'-plane . Then P , Q lie on the singular lines of the x -plane with respect to T .

For if P , Q had images P' , Q' in the x'-plane, P' , Q' would have to be common to all members of the con--centric pencil . But members of a concentric pencil have no common points , hence P and Q must be points which have no images under T .

Example : $x = \dfrac{x' + u}{x'}$ transforms the concentric system

$$x_1^2 - x_2^2 = a \quad , \quad -\infty < a < \infty$$

into pencil

$$(x_1'^2 - x_2'^2)(1-a) - 2x_2' - 1 = 0$$

whose poles : $\frac{1}{2} - \frac{1}{2}u$, $-\frac{1}{2} - \frac{1}{2}u$

lie on the singular lines $x_1'^2 - x_2'^2 = 0$.

<u>Theorem 3.84</u> : Every hyperbolic pencil , the members of which are all tangent at a point T , is bilinearly equivalent to a pencil of parallel straight lines . Hence every two tangential pencils (i.e. pencils for which the poles coincide) are bilinearly equivalent .

The first statement is merely corollary (2) of theorem(3.72). The second statement follows as in theorem (3.81) ,since a given tangential pencil may be mapped into a given pencil of parallel lines by determining the transformation such that the point of tangency maps into infinity and any two other points on same hyperbola map into two points on the same line of the given parallel line-pencil .

<u>Theorem 3.85</u> : Every bilinear transformation maps diagonal lines into diagonal lines.

It is obvious from the conformal property that this must be true under the mapping of any differentiable function .

3.9 Interlocked Systems .

Two hyperbolas whose axes are mutually orthogonal are con--centric or they intersect . However , two hyperbolas whose axes are parallel may be so situated that they are neither concentric , nor do they intersect . Consider the two hyper--bolas with parallel axes

$$(x-\alpha)^2 - (y-\beta)^2 = \ell^2$$

$$x^2 - y^2 = a^2 .$$

Solving for y : $y = \dfrac{2\alpha x + \Omega}{2\beta}$, where $\Omega = \ell^2 - a^2 + \beta^2 - \alpha^2$.

Eliminating y : $4(\beta^2-\alpha^2)x^2 - 4\alpha\Omega x - \Omega^2 - 4a^2\beta^2 = 0$.

The discriminant vanishes for $\Omega^2 + 4(\beta^2-\alpha^2)a^2 = 0$.

Write $\beta^2 - \alpha^2 = \theta$, then $\Omega = \ell^2 - a^2 + \theta$, and

last equation reduces to $\theta^2 + 2(a^2+\ell^2)\theta + (a^2-\ell^2)^2 = 0$,

of which roots are : $\theta_1 = -(a-\ell)^2$, $\theta_2 = -(a+\ell)^2$.

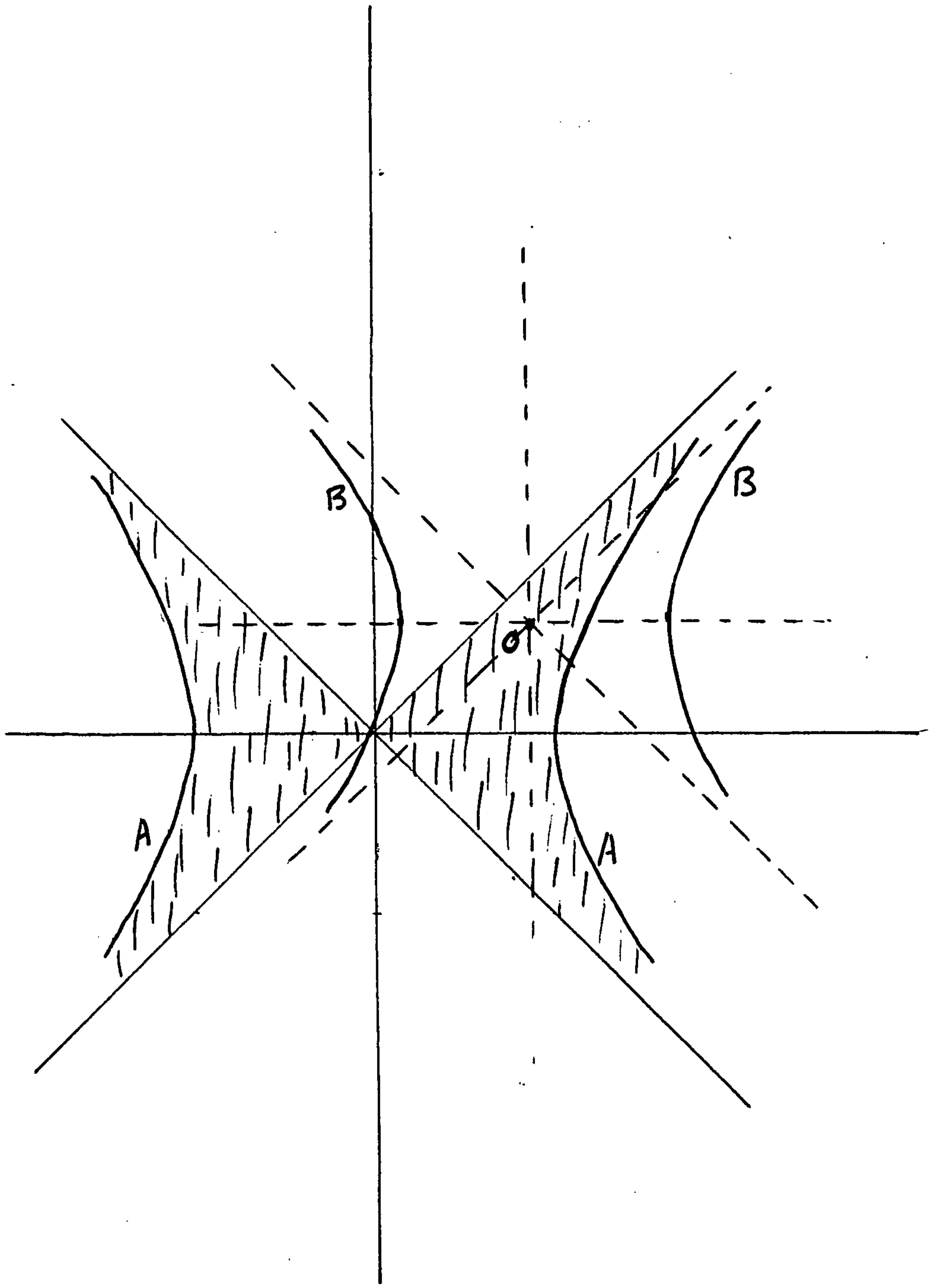

Figure 5

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Hence the discriminant vanishes for :

$$\alpha^2 - \beta^2 = (a - \ell)^2 \quad \text{or} \quad \alpha^2 - \beta^2 = (a + \ell)^2 .$$

In each case the left member is the square of the distance between centres (assuming $\alpha \geqslant \beta$) and the right member is the square of the difference or square of the sum of the radii . From the geometry , the hyperbolas intersect (i.e. discriminant > 0) for large values of $\alpha^2 - \beta^2$. It follows that the discriminant is < 0 , that is the hyperbolas have no point in common for

$$(a - \ell)^2 < \alpha^2 - \beta^2 < (a + \ell)^2$$

One of the hyperbolas then has position with respect to the other as illustrated in Fig. 5 . The centre O of B must lie within the shaded area . Two such hyperbolas will be said to be "interlocked" .

<u>Theorem 3.91</u> : A bilinear transformation maps an interlocked system into an interlocked system .

Proof : Let a bilinear transformation be applied to a system of two interlocked hyperbolas A and B . Let the asymptotes be a , ℓ respectively, and denote the corresponding image figures by the corresponding primed letters . If A', B' intersect at P' , then also A' intersects a' and B' intersects ℓ' at P' (since intersection can occur only on singular lines) This means that one diagonal of a' coincides with one diagonal of ℓ' (diagonal lines a, ℓ map into diagonal lines a', b' by theorem (3.85)). This means that one diagonal from a and a parallel diagonal from b are both excluded from the one to one mapping, contrary to the properties of the bilinear transformation. Hence A', B' do not intersect.

Theorem 3.92 : The orthogonal trajectories of an interlocked system form an interlocked system.

For suppose a pair of trajectories intersected at distinct points P and Q. Apply a transformation mapping P into infinity, then the original system will be transformed to a system concentric at Q'. The concentric system will in turn map into a pencil with two distinct poles (theorem (3.82)) which is contrary to theorem (3.91).

Suppose a pair of trajectories tangent at T. These trajectories will map into parallel lines, hence the original system will map into a pencil of parallel lines. This latter system will map, in turn, into a tangential pencil, contrary to theorem (3.91).

SUMMARY OF HYPERBOLIC PENCILS : There are three systems of hyperbolic pencils.

(i) Concentric system, with bilinearly equivalent forms : pencil with distinct poles, line pencil through a finite point.

(ii) Tangential system, with pencil of parallel lines, bilinearly equivalent to it.

(iii) Interlocked system.

Each is a closed system under the bilinear transformation, Orthogonal trajectories of each system belong to the same system .

3.10 Inverse Points .

Let a transversal from any point Q_0 of the plane cut an hyperbola in points Q_1 , Q_2 . Denoting the hyperbolic distance between Q_0 and Q_1 by $|Q_0 Q_1|$ (always a positive real number) we have shown that

$$|Q_0 Q_1| \cdot |Q_0 Q_2|$$

is the same for all transversals from Q_0 (theorem 3.45). In the special case that Q_1 , Q_2 coincide at T , $Q_0 T$ is tangent to the hyperbola at T , and product is $|Q_0 T|^2$. For all transversals from Q_0 :

$$|Q_0 Q_1| \cdot |Q_0 Q_2| = |Q_0 T|^2 \qquad (3.101)$$

Theorem 3.101 : For every point P of the hyperbolic plane there exists an inverse point with respect to a given hyperbola . P and its inverse P' lie on the same radius vector such that the product of their hyperbolic distances from the centre is the square of the radius of the hyperbola . If P' is the inverse of P , then P is the inverse of P'. A bilinear transformation maps a pair of inverse points into a pair of inverse points .

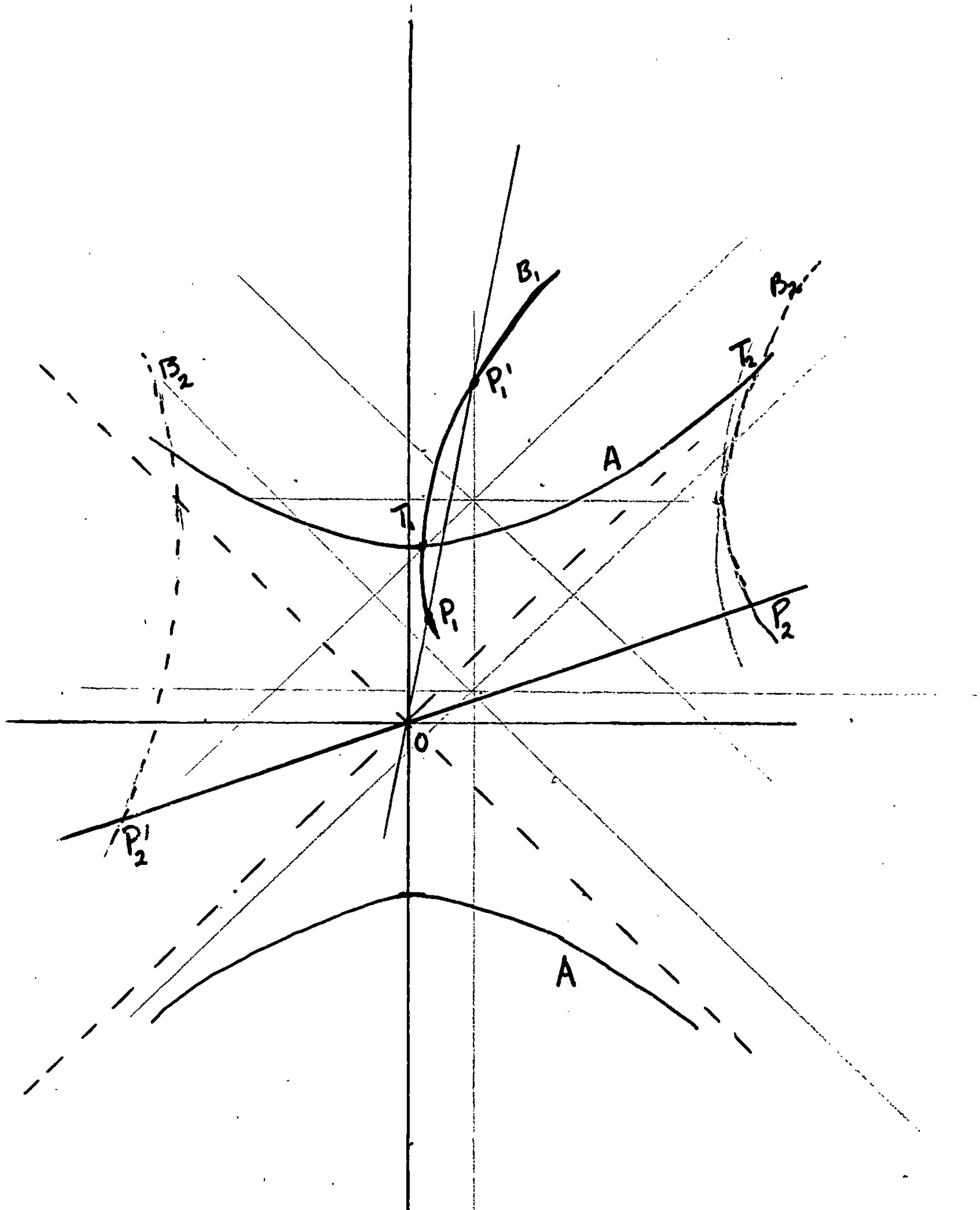

Figure 6

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Proof : Consider an hyperbola A of radius a , (fig. 6), and place the origin O at the centre of A . Let P_1 be any point of quadrant occupied by a branch of A . Through P_1 draw an hyperbola B_1 orthogonal to A cutting OP_1 (produced if necessary) at P_1' , and A at T_1 . Since OT_1 is tangent to B_1 at T_1 :

$$|OP_1| \cdot |OP_1'| = |OT_1|^2 = a^2 .$$

Thus P_1 determines P_1' , and conversely , independently of any particular orthogonal trajectory B_1 : a second hyper--bola B_1' , through P_1 and orthogonal to A , will intersect B_1 at the same point P_1' .

Under any bilinear transformation , the transform of an hy--perbola and two orthogonal transversals is again an hyperbola and two orthogonal transversals . That is , the image points of P_1 , P_1' will again be related to one another as inverse points relative to the transform of A .

Now consider P_2 , lying in a quadrant not occupied by a branch of A . Through P_2 draw hyperbola B_2 , ortho--gonal to A cutting OP_2 at P_2' and A at T_2 . From the geometry of the situation , the two branches of B_2 lie on ppposite sides of the axis of A , so that OP_2 , OP_2' are oppositively directed . Again from theorem (3.45),

$$|OP_2| \cdot |OP_2'| = |OT_2|^2 = a^2 ,$$

So that P_2, P_2' are inverse points with respect to A, since one determines the other independently of the orthogonal hyperbola B_2. The argument of previous paragraph shows that the images of P_2, P_2' under any bilinear transformation are again related as inverse points with respect to transform of A.

Corollary : Since the centre and the point at infinity form a pair of inverse points with respect to an hyperbola, a bilinear transformation maps the centre of any hyperbola into the centre of its transform if and only if it maps the point at infinity into itself.

3.11 Example of an Interlocked Pencil.

We apply the theory of inverse points to the problem of determining an interlocked hyperbolic pencil and its orthogonal conjugate from two given members of the pencil :

The pair :

$$x^2 - y^2 + 9 = 0 \qquad (3.111)$$

$$x^2 - y^2 - 2y + 5 = 0 \qquad (3.112)$$

have no point in common, that is they are interlocked. Let $P(\xi, 0)$ be any point, not the origin, on the x-axis. Then $P_1\left(-\frac{9}{\xi}, 0\right)$ is the inverse of P with respect to hyperbola (3.111) . Therefore every hyperbola through P and P_1 is orthogonal to (3.111) .

$$P_2 \left(\frac{6\xi}{1-\xi^2} \, , \, \frac{5+\xi^2}{1-\xi^2} \right)$$

is the inverse of P with respect to (3.112) , assuming that $\xi^2 \neq 0$ or 1 .

The three points P, P_1, P_2 define an hyperbola :

$$x^2 - y^2 + ax + by + c = 0$$

which is orthogonal to both (3.111) and (3.112) .

Evaluating a, b, c in terms of the coordinates of P, P_1, P_2 the orthogonal trajectory is :

$$x^2 - y^2 + \frac{9-\xi^2}{\xi} x - 4y - 9 = 0 .$$

Hence the pencil orthogonal to (3.111) and (3.112) is

$$x^2 - y^2 + Kx - 4y - 9 = 0 \qquad (3.113)$$

in parameter K .

The differential equation of family (3.113) is

$$x^2 + y^2 + 4y + 9 - 2x(y+2)y' = 0 . \qquad (3.114)$$

The differential equation of the family of orthogonal traject--ories is obtained by replacing $y' = \frac{dy}{dx}$ by $x' = \frac{dx}{dy}$ in (3.114) :

$$x^2 + y^2 + 4y + 9 - 2x(y+2)x' = 0 ,$$

or $\quad x^2 + Y^2 + 5 - 2xYx' = 0$, where $Y = y+2$,

which , on integration , gives :

$$Y^2 - x^2 - 5 + kY = 0$$

i.e. $\quad y^2 - x^2 + (4+k)y + 2k - 1 = 0 .$

Finally , set $\quad k = m - 4$,

$$x^2 - y^2 - my + 9 - 2m = 0 \qquad (3.115)$$

(3.115) is the interlocked pencil defined by (3.111) and (3.112) which correspond to $m = 0$, $m = 2$ respectively .