A METHOD FOR FINDING THE ASYMPTOTIC

BEHAVIOUR OF A FUNCTION FROM ITS

LAPLACE TRANSFORM

by

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Abstract

In many practical problems, particularly in circuit analysis, the Laplace Transform method is used to solve linear differential equations. When only the asymptotic behaviour at infinity of the solution is of interest, it is not necessary to find the exact solution. We have developed a method for finding the asymptotic behaviour of a function directly from its Laplace transform. The method is a generalization of one given by Doetsch [5,6].

The behaviour of a function $F(t)$ for large $t$ depends upon the singularities of its transform $f(s)$ on the line to the right of which $f(s)$ is regular. The asymptotic behaviour of $F(t)$ is expressed in terms of comparison functions $G_k(t)$ whose transforms have the same singularities as $f(s)$. We have considered singularities such as $1/s^{v+1}$, $(\ln s)^n/s^{v+1}$, $1/s^{v+1} \ln s$, $e^{-Ks}/s^{v+1}$, $(\ln s)^n e^{-Ks}/s^{v+1}$, or $e^{-Ks}/s^{v+1} \ln s$. The first two have been studied extensively by Doetsch.
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Introduction

The Laplace transform of a function \( F(t) \) is defined to be

\[
f(s) = \int_0^\infty e^{-st} F(t) dt
\]

provided the integral exists in some right half-plane \( R(s) > s \).\textsuperscript{1}

The transform is used extensively in circuit analysis and in the study of automatic control systems to solve linear differential equations or Volterra type integral equations. In such cases one first finds the transform of the desired solution, then the solution itself can usually be found by means of the inversion formula

\[
F(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{st} f(s) ds, \quad x > s
\]

Unless \( f(s) \) has only poles, the evaluation of the inversion integral can be quite laborious. In problems where only the asymptotic behaviour of the solution is of interest, it is not necessary to find the exact solution for all \( t \). It is the purpose of this thesis to give a method whereby the asymptotic behaviour of \( F(t) \) can be determined directly from its transform \( f(s) \) without calculating the inversion integral.

Before considering any theory, we shall outline the history of the Laplace Transform. Then we shall prove
several theorems which give the asymptotic behaviour of $F(t)$ for particular transforms $f(s)$. Later we shall extend these theorems to more general cases. A few applications will be given in the last chapter.
Chapter 1
History of the Laplace Transform

The results of applying the Laplace Transform to linear differential equations with constant coefficients are in many respects similar to those of applying an operational calculus. Consequently, the development of the Laplace Transform is closely connected with that of operational calculus.

The idea of replacing an operator by a symbol goes as far back as the time of Leibnitz (1710) who noticed the resemblance between the formulas for the nth derivative of a product and the nth power of a sum. Later Lagrange (1772) expressed the Taylor series in the purely symbolic form,

\[ f(x+h) = e^{hD}f(x) \quad \text{where } D = \frac{d}{dx} \]

Calculations were carried out exactly as though D were an algebraic quantity, not an operator, and questions of validity were just avoided.

Between 1782 and 1812, Laplace developed the new idea that to a function \( y(k) \) defined for equidistant arguments (i.e., \( k = 0,1,2\ldots \)) as object function, there is a resultant function

\[ u(t) = \sum_{k=0}^{\infty} y(k)t^k \]

If the summation extends to negative powers then to \( y(k+1) \) corresponds the function \( u(t)/t \). The process of
forming finite differences of the object function results in a purely algebraic process on the resultant function, namely multiplication by \((1/t - 1)\). Laplace saw in this the real basis for the Lagrange formula when \(t = e^{-D}\). Given a difference equation for \(y(k)\), he formed the corresponding algebraic equation and solved it. Then he tried to express \(y(k)\) in terms of \(u(t)\). This introduced the problem of how the coefficients of a power series are determined by the function. Cauchy's formula for the coefficients was not known until 1831, but Euler (1793) had derived an expression for the coefficients of what was later called a Fourier Series. Through the substitution \(t = e^{i\omega}\), Laplace formed the series

\[
(1.2) \quad u(e^{i\omega}) = U(\omega) = \sum_{k=0}^{\infty} y(k) e^{ik\omega}
\]

to which he applied Euler's formula and obtained

\[
(1.3) \quad y(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(\omega)e^{-ki\omega}d\omega
\]

By this method he was able to express the unknown function in terms of a definite integral. The method was purely formal since the series expansion of \(u(t)\) might not converge for \(t\) on the unit circle and the true sense of complex integration was not known until the time of Cauchy (1814).
Dissatisfied that his formula should contain imaginary quantities, he set

\[ y(k) = \int t^{k-1} \phi(t) dt \]  

(1.4)

where the path of integration was to be determined. Except for the factor \(1/2\pi i\), this is the Cauchy formula for the coefficients and is in the form of a Mellin Transform, which can be derived from the Laplace transform by the substitution \( z = e^{-t} \). Then he went back and set the unknown function

\[ y(s) = \int e^{-st} U(t) dt \]  

(1.5)

which was suggested by the form of (1.3) but still continued to use (1.4).

The first research on the transform

\[ A(t, \phi) = \int e^{tx} \phi(x) dx \]  

(1.6)

is due to Euler (1737) who applied it to the integration of differential equations. But because Laplace's results were better known, the transform \( \int_0^\infty e^{-sx} \phi(x) dx \) was later called after him.

Guided by the work of Laplace, Abel (1824) set up the functional relationship

\[ \phi(x, y, x, \ldots) = \int e^{xu+yv+zp+\ldots} f(u, v, p, \ldots) du dv dp \ldots = g\{f\}, \text{ say,} \]  

(1.7)

which is exactly the reverse of Laplace's definition. He
derived several fundamental properties, such as

$$(1.8) \quad \mathcal{L}\{u^nf(u)\} = \frac{d^n}{dx^n} \mathcal{L}\{f(u)\}.$$ 

The Laplace transform

$$(1.9) \quad \int_0^\infty e^{-st}F(t)dt = \int_0^\infty e^{-iyt} e^{-xt} F(t)dt$$

can be considered as the Fourier transform of $e^{-xt}F(t)$ where $F(t) = 0$ for $t < 0$. With Cauchy's work on complex integration and the development of the Fourier Integral Theorem (1820), the results could be applied to the Laplace transform.

Apparelly not much more was done until Heaviside (1850-1925) developed an operational calculus independently of what had been done previously. He was an electrical engineer without a University education but his curious methods led him to important results. Most of his work was in electro-magnetic theory where it was necessary to solve linear differential equations with constant coefficients. By formally substituting $p^n$ for $d^n/dt^n$, $p^{-n}$ for $\int^n (dt)^n$ the linear differential equation was reduced to an algebraic equation. Then he applied various rules which he had developed for obtaining solutions, the most important ones being the Shift Rule, the Expansion Theorem, and the interpretation of the Irrational Operator.

In some cases, Heaviside's rules are arbitrary. For example, $p/p-a l$ can be expanded in two ways, in
either an ascending or a descending power series. The first corresponds to taking derivatives of a constant and was therefore interpreted as being equal to zero; the second, to successive integrations of a constant and was interpreted by Heaviside as being equal to $e^{at}$. In [8] Heaviside states "It is in its generality a rather difficult and obscure matter. I have not succeeded in determining the amount of latitude that is permissible in the purely algebraic treatment of operators." Consequently, he used his method subject to independent tests for guidance but never gave any reasons for what he did. To those trying to follow him, his attack seemed full of inconsistencies.

Most writers refer to his method as the 'Heaviside operational calculus', except for Gardner and Barnes [7], along with Murnaghan [11], who call it the 'Cauchy-Heaviside operational calculus'. According to them, Cauchy about 1812 developed an operational calculus based on the Laplace and Fourier transforms, formally identical with parts of Heaviside's method. Consequently, they state "It is now clear that Cauchy had not only supplied the original operational calculus of the type considered but had derived it using the Laplace transformation. He had thereby supplied a basis for its rigorous treatment." Heaviside is given credit for the extensive applications only.
Heaviside was followed by many who tried to make his rules consistent and rigorous. One of the first of these was Carson (1919) who reduced the problem of inversion to the purely mathematical problem of solving the integral equation

\[ f(p) = \int_0^\infty e^{-pt} h(t) dt \]

for \( h(t) \). But then, according to Bush (1924) the method was no longer suitable for an engineer, so he attempted to define a system of operating rules which would avoid the Heaviside inconsistencies but which would keep his simplicity of method.

Bromwich (1916), took an entirely different point of view. He assumed that the solution of a differential equation could be expressed as a contour integral in the complex plane of the form

\[ \phi(t) = \frac{1}{2\pi i} \int_{C-\infty}^{C+\infty} p(\lambda)e^{\lambda t} d\lambda \]

where the path of integration was to the right of all the singularities of \( p(\lambda) \) and \( p(\lambda) \) was to be determined from the differential equation and initial conditions. This approach had been suggested earlier by Cauchy who had solved differential equations by means of contour integrals.

It was not until the time of Levy (1926), that the two points of view were reconciled. He
showed that, under certain conditions, the solution of

\[(1.12) \quad h(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} f(p) dp\]

was

\[(1.13) \quad f(p) = \int_0^\infty e^{-pt} h(t) dt \quad \text{and vice versa}\]

Of the many others who also tried to place Heaviside's method on a firm basis, some of the more prominent ones are Cohen, Berg, Carslow, Jaeger, Marsh, and Josephs.

Those who have considered the Laplace transform from a more theoretical point of view are Jeffries, van der Pol, Doetsch, McLachlan, and Churchill.

The Laplace transform is usually considered as a Riemann or a Lebesque integral but it can also be considered as a Stieltje's integral, \[\int_0^\infty e^{-st} \Phi(t) dt.\]

A complete development of the theory for this transform is given by Widder \[15\].

An extensive bibliography of the historical development can be found in Gardner and Barnes \[7\].
Chapter 2
Development of the Theory

The asymptotic behaviour of a function can be expressed in several ways.

(i) If $F(t) - G(t) = o(1)$, as $t \to \infty$, then $F(t)$ is defined to be difference asymptotic to $G(t)$.

(ii) If $F(t)/G(t) = 1 + o(1)$, as $t \to \infty$, then $F(t)$ is defined to be quotient asymptotic to $G(t)$.

In some instances both definitions are satisfied but this is not true generally. For example, consider the function $[t]$, here $[t] \sim t$ in the quotient asymptotic sense only. Finally,

(iii) $F(t) \sim a_0 + a_1/t + a_2/t^2 + \ldots$

if $t^n[F(t) - a_0 - a_1/t - a_2/t^2 \ldots - a_n/t^n] \to 0$ as $t \to \infty$.

This is the Poincare [12] definition of the asymptotic expansion of $F(t)$ about infinity. The series itself may diverge everywhere but if $t$ is large, the terms decrease very rapidly at first and the error in approximating to $F(t)$ by the sum of the first $n$ terms is of lower order than the last term considered as $t \to \infty$.

An operational method for determining the asymptotic behaviour of a function was first developed by Heaviside but his method did not always lead to correct results. The following example illustrates his method.
To find the current \( I \) entering a cable of distributed resistance \( R \) and capacity \( C \), if an emf \( e^{-j\omega t} \) is impressed at time \( t = 0 \), he set up the linear differential equation, replaced \( \frac{d}{dt} \) by \( p \), and obtained

\[
(2.1) \quad I = h(p) = \sqrt{\frac{C}{R}} \cdot \frac{\omega p^{3/2}}{\omega^2 + p^2} \cdot l
\]

Expanding in a power series of \( p \)

\[
(2.2) \quad h(p) = \sqrt{\frac{C}{R}} \cdot \frac{p^{3/2}}{\omega} \left(1 - \frac{p^2}{\omega^2} + \frac{p^4}{\omega^4} + \ldots \right) \cdot l
\]

and using the operational rules

\[
\sqrt{p} \cdot l \rightarrow l/\sqrt{\pi t} \\
p^{n+1/2} \cdot l \rightarrow p^n \cdot l/\sqrt{\pi t} + d^n/dt^n \cdot l/\sqrt{\pi t}
\]

he obtained

\[
(2.3) \quad I \rightarrow \sqrt{\frac{C}{R} \pi t} \left(1/2\omega t - 1.3.5/(2\omega t)^3 + \ldots \right)
\]

Heaviside knew that the steady state solution was

\[
\sqrt{\omega C/2R} (\cos \omega t + \sin \omega t)
\]

so he came to the conclusion that the method had given the transient distortion.

Obviously the Heaviside approach is inadequate and a theory must be developed which will give both the steady state solution and the transient distortion. For this the Laplace transform method is more satisfactory, since the conditions are precisely known under which
various operations are valid. Also, it includes Heaviside's method as a special case.

We recall that the function $f(s)$ is said to be the Laplace transform of the object function $F(t)$ if

$$f(s) = \int_0^\infty e^{-st} F(t) dt, \quad \text{Re}(s) > s_0$$

$$= \mathcal{L}\{F(t)\}$$

In what follows we will consider only Riemann integrals, improper if necessary. The function $f(s)$ is determined uniquely by $F(t)$ but to a given $f(s)$ there correspond an infinite number of object functions which, however, differ from each other only by null functions. The particular object function which is continuous throughout or right continuous (i.e., $\lim_{h \to 0} f(x+h)$ existo) is uniquely determined. We shall consider only such functions, and then a one-to-one correspondence exists between the class of all object functions $F(t)$ whose Laplace transforms exist and the class of all resultant functions $f(s)$.

The relationship between the asymptotic behaviour of the two classes is of particular interest here. Two basic types of theorem describing properties of the transform can be given; these are the Abelian and Tauberian theorems.

The Abelian theorems predict the asymptotic behaviour of $f(s)$ at particular points from the behaviour
of $F(t)$. An example would be the theorem which states that if $\int_{t}^{\infty} F(u)du \to L$ as $t \to \infty$, then $f(s) \to L$, as $s \to 0$ inside the sector $S$: $\arg s \leq \pi/2$.

The Tauberian theorems predict the asymptotic behaviour of $F(t)$ at either zero or infinity from the behaviour of $f(s)$, but they require that certain restrictions be placed on $F(t)$. The reason for this can be seen from the following example. If $F(t)$ has the limit $A$ as $t \to \infty$, then, according to an Abelian theorem, $f(s) \sim A/s$, as $s \to 0$ inside the sector $S$. If $F(t)$ itself has no limit but the Cesaro mean of order $k$

$$U_k(t) = k! \left( \int_{t}^{\infty} F(t)(dt)^k/t^k \right)$$

has the limit $A$ as $t \to \infty$, then by means of the convolution integral it can be shown that $f(s) \sim A/s$, as $s \to 0$ inside the sector $S$. The function $f(s)$ is therefore insensitive to whether or not the limit of $F(t)$ exists. If the only information is that $f(s) \sim A/s$, as $s \to 0$ inside a sector $S$, then it is impossible to predict that $F(t) \sim A$, as $t \to \infty$. For reasons like this, the Tauberian theorems are more complicated than the corresponding Abelian theorems.

From now on we shall assume that $f(s)$ is known and the asymptotic behaviour for large $t$ of $F(t)$ is to be determined. It would seem that a Tauberian theorem could be applied but because $F(t)$ is unknown,
we do not know whether or not it will satisfy the necessary restrictions. To avoid this difficulty we consider an inversion formula such that \( F(t) = L^{-1}\{f(s)\} \).

Then the roles of the two classes of functions - the class of object functions and the class of resultant functions - are interchanged. If conditions are imposed on \( f(s) \) to ensure that there exists a function \( F(t) \) such that \( L\{F(t)\} = f(s) \), it is possible to apply a theorem, like an Abelian theorem, to \( f(s) \) which Doetsch [5, p. 224] calls an 'Indirect Abelian theorem' and obtain the asymptotic behaviour of \( F(t) \) without any restrictions on \( F(t) \) itself.

For the Laplace transform the inversion formula that we shall consider is

\[
(2.5) \quad F(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{st} f(s) ds.
\]

Conditions are known which ensure that \( L\{F(t)\} = f(s) \) for the \( F(t) \) defined by (2.5) [5, p. 126]. For our purpose it is sufficient to assume an \( F(t) \) exists such that \( L\{F(t)\} = f(s) \). The following theorem is stated without proof [5, p. 107].

**Theorem 2-1**

Let \( L\{F(t)\} = f(s) \) and let \( f(s) \) be regular for \( R(s) > s_0 \). If \( \int_{-\infty}^{\infty} e^{iyt} f(x+iy) dy \) converges uniformly for \( t > T \) and a fixed \( x > s_0 \), where \( s = x + iy \), then
(2.6) \( F(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{ts} f(s) ds \) for \( t \geq T \).

Under the rather mild condition of the theorem it can be shown that

\[ \int_{T_1}^{T_2} dt \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{ts} f(s) ds = \int_{T_1}^{T_2} F(t) dt \]

for \( T_2 > T_1 > T \)

from which the result follows.

Without loss of generality we may assume that \( f(s) \) is regular in the half-plane \( \Re(s) > 0 \) and has at least one singularity on the line \( \Re(s) = 0 \). This can always be done by considering \( \mathcal{L} \{ e^{-so} F(t) \} \) instead of \( \mathcal{L} \{ F(t) \} \).

We state the following theorem without proof.

**Theorem 2.2.**

Let \( f(s) \) satisfy the following conditions:

(i) \( F(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{ts} f(s) ds \) exists for \( x > 0 \) and is such that \( \mathcal{L} \{ F(t) \} = f(s) \).

(ii) \( f(s) \) is analytic and regular for \( \Re(s) > -a \), some \( a > 0 \), except for a finite number of singularities on \( \Re(s) = 0 \).

(iii) \( f(s) \to 0 \) as \( |s| \to \infty \) for \( \Re(s) \geq -a \).

(iv) \( \int_{-\infty}^{\infty} e^{iyt} f(-a + iy) dy \) converges uniformly for \( t \geq T \) then

(2.7) \( F(t) = \frac{1}{2\pi i} \int_{c} e^{ts} f(s) ds \)
(where C is any contour from \((-a-i\infty)\) to \((-a+i\infty)\) such that all singularities of \(f(s)\) are to the left of it) and the contribution from portions along \(R(s) = -a\) are \(o(e^{-at})\) as \(t \to \infty\).

From now on we shall say that \(f(s)\) belongs to the class A if it satisfies the conditions of this theorem.

We shall assume first that \(f(s)\) has only one singularity on the line \(R(s) = 0\), namely at \(s = 0\). Then \(C\) is a contour of the type in Fig. 1.

![Figure 1](image)

The singularity that \(f(s)\) may have at the origin can be either a pole, a branch point or an essential singularity. We shall show that each type of singularity of the transform at the origin leads to a certain behaviour of the object function at infinity.

Consider \(G(t,v) = t^v/(v+1)\). This function has the transform \(1/s^{v+1}\) for \(R(v) > -1\), and \(R(s) > 0\). For all other \(v\) the integral defining the transform does
not converge at the origin.

Define

\[
G(t) = \begin{cases} 
0, & 0 \leq t < 1 \\
\frac{t}{\Gamma(v+1)}, & t \geq 1, \, v \neq -1, -2, \ldots 
\end{cases}
\]

Then \( L\{G(t)\} = \int_0^\infty e^{-st} \frac{t^v}{\Gamma(v+1)} \, dt \).

For \( R(v) > -1 \), 
\[
L\{G(t)\} = \int_0^\infty e^{-st} \frac{t^v}{\Gamma(v+1)} \, dt - \int_0^\infty e^{-st} \frac{t^v}{\Gamma(v+1)} \, dt
\]

\[= \frac{1}{s} v + \text{integral function}.\]

For \( R(v) < -1 \), we can choose an integer \( n \) such that 
\(-1 < R(v+n) < 0 \). Integrating by parts \( n \) times we obtain

\[
L\{G(t)\} = e^{-s} \left\{ \frac{1}{\Gamma(v+2)} \cdots \frac{1}{\Gamma(v+n+1)} \right\} + s^n \int_0^\infty e^{-st} \frac{t^v}{\Gamma(v+1)} \, dt
\]

The first term is an integral function of \( s \); the integral can be treated the same as before and so

\[
L\{G(t)\} = \frac{1}{s} v + \text{integral function}, \, R(s) > 0, \, v \neq -1, -2, \ldots
\]

Furthermore, \( L\{G(t)\} \) satisfies conditions (i)-(iii) of Theorem 2.2. Because the integral defining the transform converges absolutely and \( G(t) \) is of bounded variation, the inversion formula holds \([6, \, p. \, 360]\). By applying the same procedure as Doetsch \([6, \, p. \, 366]\) we can show that
(2.9) \[ L \{ G(t) \} = e^{i(v+1)\psi} \int_{-\infty}^{\infty} e^{-se^{i\psi}r} r^v/(v+1) \, dr \]
\[ + i \int_{0}^{\pi} e^{-x\cos\theta + y\sin\theta - i(x\sin\theta + y\cos\theta)} \theta^{i(v+1)} \, d\theta, \quad s = x + iy \]
\[ = 1/s^{v+1} + \text{integral function} \]

which defines the analytic continuation into the half-plane $R(se^{i\psi}) > 0$. The first integral tends to zero as $I(se^{i\psi}) \to \infty$ for each fixed $R(se^{i\psi}) > 0$; the second integral tends to zero as $y \to \infty$ for all $x$ provided $-\pi/2 < \psi < \pi/2$ and tends to zero as $y \to -\infty$ for all $x$ provided $0 \leq \psi < \pi/2$. Therefore $L\{G(t)\} \to 0$ as $|s| \to \infty$ for $R(s) \geq -a$. We have not been able to show that it satisfies condition (iv) of Theorem 2.2 but shall assume it does so. Then $L\{G(t)\}$ belongs to the class $A$.

The essential property of $G(t,v)$ used to get the above results is that it satisfies the equation

(2.10) \[ \frac{d}{dt} G(t,v+1) = G(t,v) \]

Many special functions are known to satisfy this equation \[14\] and the same method may be applied to them. Another example will serve to illustrate what new results can be obtained.

It is known that \( (\xi/k)^{\nu} J_{\nu} \left( 2\sqrt{kt} \right) \) satisfies equation (2.10) and has the transform $k^{-\nu/2}$ for $R(\nu) > -1$, $R(s) > 0$. Define

(2.11) \[ G(t) = \begin{cases} 0 & 0 \leq t < 1 \\
(\xi/k)^{\nu} J_{\nu} \left( 2\sqrt{kt} \right) & t > 1 \end{cases} \]

and proceed as before.
Then
\begin{equation}
(2.12) \quad L\{G(t)\} = e^{-\frac{K}{s}} + \text{integral function, } R(s) > 0
\end{equation}
for all \( v \) and furthermore it belongs to the class \( A \) if we assume condition (iv) of Theorem 2.2.

In these two examples the above arguments also apply when \( G(t,v) \) is replaced by \( d^n/dv^n G(t,v) \) or \( \int G(t,v)dv \).

Thus we have generated a number of functions whose transforms have the types of singularity at the origin that were mentioned earlier. The functions \( G(t) \) and their transforms (except for the integral functions) are given in the table on the following page.
<table>
<thead>
<tr>
<th>( G(t) )</th>
<th>( g(s) )</th>
<th>Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\frac{t}{K_1})^{\nu} ), ( \nu \geq 1 )</td>
<td>( \frac{1}{s^{\nu+1}} )</td>
<td>( \nu \neq -1, -2, \ldots )</td>
</tr>
<tr>
<td>( \begin{cases} \frac{d}{dy} \ln t^\nu \left( v+1 \right), \nu &gt; 1 \end{cases} )</td>
<td>( \frac{u \nu}{s^{\nu+1}} )</td>
<td>( \nu \neq -1, -2, \ldots )</td>
</tr>
<tr>
<td>( m = 1, 2, \ldots )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \begin{cases} \frac{d}{dy} \ln \left( \frac{t}{K_2} \right)^{\nu} \right), \nu \geq 1 )</td>
<td>( \frac{e^{-\nu}}{s^{\nu+1}} )</td>
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<td>( \frac{e^{-\nu}}{s^{\nu+1}} )</td>
<td>( m = 1, 2, \ldots )</td>
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<td>( \frac{e^{-\nu}}{s^{\nu+1}} )</td>
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<td>( \frac{e^{-\nu}}{s^{\nu+1}} )</td>
<td>none</td>
</tr>
<tr>
<td>( \begin{cases} \frac{d}{dy} \ln \left( \frac{t}{K_6} \right)^{\nu} \right), \nu \geq 1 )</td>
<td>( \frac{e^{-\nu}}{s^{\nu+1}} )</td>
<td>none</td>
</tr>
</tbody>
</table>
The asymptotic behaviours of the first four functions in this table are well known, but a few remarks should be made about the last two. Function (6) belongs to the class of functions \( \int_0^\infty G(v) \, t^v / \Gamma(v+1) \, dv \) which have been studied extensively by Colombo [4].

If \( L\{G(t)\} = g(s) \) exists for \( R(s) > 0 \), then the Laplace transforms of these functions exists and

\[
L\left\{ \int_0^\infty G(v) \, t^v / \Gamma(v+1) \, dv \right\} = g(\ln s) / s, \quad R(s) > 1
\]

In particular

\[
L\left\{ \int_0^\infty \frac{t^{\nu+n}}{\Gamma(\nu+n+1)} \, v^\nu \, n! \, dv \right\} = \frac{1}{s^{\nu+n}} (\ln s)^{n+1},
\]

for \( R(\nu) > -1, R(n) > -1, R(1) > 0 \)

For \( m = 0 \), it can be shown by means of a contour integral representation that

\[
(2.13) \int_0^\infty \frac{t^{\nu+u}}{\Gamma(\nu+u+1)} \, du = e^t \sim -t^\nu / \ln \Gamma(\nu+1), \quad t \to \infty, \quad \nu \neq -1, -2, -3.
\]

\[
\sim (-1)^{-\nu} (\nu+1)! \, t^\nu / (\ln t)^2,
\]

\[
t \to \infty, \quad \nu = -1, -2, -3.
\]

Function (7) is related to this class of functions in that

\[
\int_\nu^\infty t^{-\nu} J_{\nu}(2\sqrt{t}) \, dt = \int_\nu^\infty J_{\nu}(2\sqrt{t-u}) \, du \int_0^\infty u^\nu / \Gamma(\nu+1) \, du
\]

Again by means of a contour integral it can be shown that for \( k \neq 0 \)
The functions given in this table all belong to the class A and may be used to express the asymptotic behaviour of our unknown function $F(t)$. If from $F(t)$ we subtract one of the functions $G(t)$ of Table I whose transform has the same singularity at $s = o$ as that of $f(s)$, then according to Theorem 2.2.

\[
(2.14) \quad \int_{0}^{\infty} \left( \frac{t}{x} \right)^{\nu} J_{\nu} \left( 2 \pi k t \right) d\nu - e^{t-K}
\]

\[
- \frac{1}{2\pi i} \int_{C} \frac{e^{s t}}{s} ds, \quad \text{as } t \to \infty
\]

Because the integrand is now regular for $R(s) > -a$, the integration can be taken along the line $R(s) = -a$ and the difference

\[
F(t) - G(t) = o(e^{-at}), \quad \text{as } t \to \infty.
\]

If none of the functions $G(t)$ of Table I has a transform with the same singularity at $s = o$ as that of $f(s)$, then $L\{F(t) - G(t)\}$ will not be regular at the origin. Consequently we need the following theorem.

**Theorem 2.3.**

Let $f(s)$ belong to the class A and have one singularity on $R(s) = o$ at $s = o$; let $L \{G(t)\} = g(s) + \text{integral function}$, where $G(t)$ is a function from Table I. If $f(s) - g(s)$ is not regular at $s = o$ and
\[ f(s) - g(s) = o\left\{ |\psi(s)| \right\} \text{ as } |s| \to 0, \text{ then} \]

\[(2.16) \quad |F(t) - G(t)| < A \int_{C_1} e^{t\psi(s)} |ds|, \text{ as } t \to \infty \]

where \( C_1 \) is the portion of the contour \( C \) of FIG. 1 lying inside a sufficiently small circle about the origin.

**Proof.**

Given an \( \varepsilon > 0 \), there exists a sufficiently small \( R \) such that if \( |s| \leq R \), \( |f(s) - g(s)| < \varepsilon |\psi(s)| \).

Choose \( a < R \). Then the contour integral can be divided into two parts; \( C_1 \) the portion lying inside the circle of radius \( R \) and \( C_2 \) the portion lying outside this circle. Because \( L\{F(t) - G(t)\} \) belongs to the class \( A \), the integral along \( C_2 \) is \( o(e^{-at}) \) as \( t \to \infty \). Therefore, by (2.15)

\[(2.17) \quad |F(t) - G(t)| = \left| \frac{1}{2\pi i} \int_{C_1} e^{t\psi(s)}(f(s) - g(s)) \text{ integral function} ds \right| + o(e^{-at}) \]

But \( \int_{C_1} e^{t\psi(s)} \text{ integral function} ds = o(e^{-at}) \), as \( t \to \infty \), and so,

\[ |F(t) - G(t)| = \left| \frac{1}{2\pi i} \int_{C_1} e^{t\psi(s)}(f(s) - g(s)) ds + o(e^{-at}) \right| < \varepsilon /2\pi \int_{C_1} |e^{t\psi(s)}| ds, \text{ as } t \to \infty \]

which is the result of our theorem.

From this theorem it follows that if

\[ f(s) = o\left\{ |s/\lambda|^2 \right\}, \lambda \text{ real, then} \]
The theory developed in this chapter gives difference asymptotic relationships. A function which for all $T$ has discontinuities for $t \geq T$ cannot be difference asymptotic to a continuous function as was seen earlier for $[t]$. Since our comparison functions are all continuous for $t > 1$, $F(t)$ must be continuous for large $t$. 

(2.18) $F(t) = o(t^{-\lambda-1})$, as $t \to \infty$
Chapter 3

Generalizations of the Theory

We shall now extend Theorem 2.3 to include the case when \( f(s) \) has a finite number of singularities on the line \( R(s) = 0 \), and the case when \( f(s) \) has an asymptotic expansion about the singularity at \( s = 0 \). Lastly, we shall show how the asymptotic behaviour of \( F(t) \) can be obtained for various arguments of \( t \).

Suppose \( f(s) \) has a finite number of singularities on the line \( R(s) = 0 \) at \( s = s_k, k = 1, 2, \ldots, n \), and suppose that \( f(s) \sim g_k(s) \), as \( s \to s_k \). If \( f(s) \) belongs to the class \( \mathcal{A} \), then

\[
F(t) = \frac{1}{2\pi i} \int_{C''} e^{st} f(s) ds,
\]

where \( C'' \) is a contour of the type in FIG. 2.

![FIG. 2](attachment:figure2.png)

The comparison functions in Table 1 all have transforms whose singularities are at the origin. If
s is replaced by \( s = s_k \), then \( g(s-s_k) \) has a singularity at \( s = s_k \), and the corresponding comparison function will be \( e^{s_k t}G(t) \). We shall say \( g(s) \) is a function from Table 1 even though its singularity occurs at \( s_k \) instead of the origin. With the aid of Theorem 2.3 the following theorem is readily proved.

**Theorem 3.1**

Let \( f(s) \) belong to the class A and let \( f(s) \) have singularities at \( s_k \), \( k = 1, 2, \ldots n \) with \( R(s_k) = 0 \). If \( f(s) \sim g_k(s) \) as \( s \to s_k \), where \( g_k \) is a function from Table 1, then

\[
F(t) = \sum_{k=1}^{n} C_k(t) = \frac{1}{2\pi i} \oint_{C^n} e^{ts} \left\{ f(s) - \sum_{k=1}^{n} g_k(s) \right\} ds
\]

\[
= A \left\{ \sum_{k=1}^{n} \oint_{C_k} e^{ts} g_k(s) |ds| \right\}, \quad \text{as} \quad t \to \infty
\]

where \( C_k \) is the portion of \( C^n \) lying inside a sufficiently small circle about \( s_k \).

Because we can neglect at most a finite number of terms that are \( o(e^{-at}) \), \( f(s) \) can have only a finite number of singularities on the line \( R(s) = 0 \).

It may happen that \( f(s) \) has an asymptotic expansion about the singularity at \( s = 0 \). In this case we can prove Theorem 3.2.
Theorem 3.2

Let $f(s)$ belong to the class $A$. If

$$f(s) = \sum_{j=1}^{n} g_j(s) + o \left\{ g_n(s) \right\}, \quad as \quad s \to 0$$

where $g_j(s)$ is a function from Table 1 then

$$F(t) - \sum_{j=1}^{n} G_j(t) = A \left\{ 1/2\pi \int_{C} |e^{ts} g_n(s)| ds \right\}, \quad as \quad t \to 0$$

A combination of the above two cases is also possible.

Theorem 3.3

Let $f(s)$ belong to the class $A$ and let $f(s)$ have singularities at $s_k, k = 1, 2, ..., n$, with $R(s_k) = 0$. If

$$f(s) = \sum_{j=1}^{m_k} g_{jk}(s) + o \left\{ g_{m_k}(s) \right\}, \quad as \quad s \to s_k$$

where $g_{jk}(s)$ is a function from Table 1, then

$$F(t) - \sum_{k=1}^{n} \sum_{j=1}^{m_k} G_{jk}(t) < A \left\{ \sum_{k=1}^{n} \int_{C_k} |g_{m_k}(s) e^{ts}| ds \right\}, \quad as \quad t \to 0$$

where $C_k$ is the portion of $C^n$ lying inside a sufficiently small circle about the singularity $s = s_k$.

This last theorem has been proved earlier by Sutton, [13], for particular functions $f(s)$. He assumed that about each singularity, $f(s)/s$ could be
expanded in a convergent power series such that the contour integral along the circles about the singularities tended to zero as the radius of these circles approached zero.

So far we have considered $F(t)$ only for real values of $t$, but the theory can be generalized to include complex values as has been done by Doetsch.

If the path of integration of $\int_0^{\infty} e^{-st}F(t)dt$ is changed to a line passing through the origin and making an angle $\varphi$ with the real axis, and the integral converges for some $s$, we will obtain a function which depends on $\varphi$, namely

$$L^\varphi\{F\} = e^{i\varphi} \int_0^{\infty} e^{-e^{i\varphi}sr}F(re^{i\varphi})dr.$$  

It can be shown that the integral has a half-plane of convergence and that, if $L^\varphi\{F\}$ and $L^\varphi_0\{F\}$ have a common region of convergence, then $L^\varphi\{F\} = L^\varphi_0\{F\}$ in that region. Therefore, by letting $\varphi$ vary we may generate the analytic continuation of $f(s)$.

Except in the trivial case, $F(t) = 0$, $f(s)$ will have a set of singularities which can be enclosed in a convex region. The convex hull is defined to be the intersection of all these regions. A line tangent to the convex hull is called a "supporting line" and forms the boundary line of the region in which $L^\varphi\{F\}$ is regular, where $\varphi$ is the angle that the normal to this "supporting
line" makes with the real axis. The asymptotic behaviour of $F(t)$ for $t = re^{i\varphi}$, $r \to \infty$ can then be found from the singularities on this line.

Starting from an $F(t)$ defined for $t > 0$, we therefore obtain $f(s) = \mathcal{L}\{F(t)\}$. In particular, if $f(s)$ has a finite number of singularities, we may obtain the analytic continuation of $F(t)$ for all arguments of $t$ by letting the "supporting line" rotate about the convex hull.
Chapter 4

Applications

We shall conclude with a discussion of three examples which illustrate the method we have developed. The examples are chosen to illustrate certain obvious generalizations of the method.

Consider first the function

\[ (4.1) \quad F(t) = \int_{t}^{\infty} \frac{J_0(y)}{y} \, dy \]

which has the transform

\[ (4.2) \quad L\{F(t)\} = f(s) = \ln \left( \sqrt{s^2 + 1} - s \right) / s \]

This function belongs to the class A and has branch points at \( s = \pm i \). Since \( s^2 + 1 - s \) is never zero, \( \ln(s^2 + 1 - s) \) may be expanded in a power series. Near \( s = -i \),

\[ (4.3) \quad f(s) \sim \sqrt{2} \, \pi \frac{e^{\pi}}{2} \left\{ (s+i)^{\frac{1}{2}} - (i)^{\frac{1}{2}} (s+i)^{\frac{3}{2}} \left( 1 + \frac{1^2}{2.3!} \right) + (i)^3 (s+i)^{\frac{3}{2}} \left( 1 + \frac{1^2}{2.3!} + \frac{1^2 3^2}{2.3.5!} \right) \right\} \]

Near \( s = i \), \( f(s) \) behaves like the complex conjugate of the above expansion. From 2.18 and Theorem 3.1 it follows that
In the above example the remainder term was of the form $o\left\{ s^{\lambda} \right\}$. To show that our method applies even in more complicated cases, consider the following transform belonging to the class $A$.

$$f(s) = \frac{e^{\frac{k}{s^2}}}{s(1+s^2)} = \frac{e^{\frac{k}{s^2}}}{s} \sum_{j=0}^{\infty} (\frac{1}{s^2})^j s^j + o\left\{ s^{-\lambda} e^{\frac{k}{s^2}} \right\}, \text{as } s \to 0$$

By Theorem 3.2 and case (3) of Table 1

$$F(t) - \sum_{j=0}^{\infty} (\frac{1}{s^2})^j \int_{c_1} e^{\frac{ts}{s^2}} s^{j-1} ds = o\left\{ \int_C e^{\frac{ts}{s^2}} s^{n-1} ds \right\}, \text{as } t \to \infty$$

Put $s = \frac{u}{\sqrt{k}}$, $ds = \frac{1}{\sqrt{k}} du$, and obtain

$$\int_C e^{\frac{ts}{s^2}} s^{n-1} ds = \frac{1}{\sqrt{k}} \int_C e^{\frac{\sqrt{k}}{u}} u^{n-1} |du|$$

Assume $Re(\sqrt{k}) > 0$. For $C^*$ take the contour of FIG. 3, and then the greatest contribution to the integral will come from a small interval near $u = 1$. Divide $C^*$ into two parts, I and II, as shown.
On $I$, $u = e^{i\theta}$

$$
\int_{I} |e^{\frac{\sqrt{t}}{k} \left(u + \frac{1}{u}\right)} u^{\frac{n-1}{2}}/du| = \left\{ \int_{-\delta}^{\delta} 1 e^{\frac{2\sqrt{t}}{k} \cos \theta} \right\} d\theta
$$

$$
= \left\{ \int_{-\delta}^{\delta} e^{\frac{2\sqrt{t}}{k} \left(1 - \frac{\theta^2}{2}\right)} \right\} d\theta \left\{ 1 + O(2\frac{\sqrt{t}}{k} \delta^2) \right\}
$$

$$
= \left\{ \frac{e^{\frac{2\sqrt{t}}{k} \delta}}{(kt)^{\frac{1}{2}}} \int_{-\delta}^{\delta} e^{-\frac{\theta^2}{2}} d\theta \right\} \left\{ 1 + O(2\frac{\sqrt{t}}{k} \delta^2) \right\}
$$

$$
= \left\{ \frac{\sqrt{\pi}}{(kt)^{\frac{1}{2}}} \int_{-\delta}^{\delta} e^{-\frac{\theta^2}{2}} d\theta \right\} \left\{ 1 + O(2\frac{\sqrt{t}}{k} \delta^2) \right\}
$$

Choose $\delta = t^{-\frac{1}{6}}$, then $\sqrt{t} \delta^4$ is small but $\sqrt{t} \delta^2$ is large and our integral is equal to

(4.8) $$\int_{I} |e^{\frac{\sqrt{t}}{k} \cos \theta}| d\theta \leq \frac{\pi}{2} \left| e^{\frac{2\sqrt{t}}{k} \cos \theta} \right|$$
which is of lower order than (4.8). Similarly it can be shown that the integral along the remainder of $C^*$ is of lower order. Combining these results with (4.8), (4.7), and (4.6) we have

$$f(t) \sim \sum_{\kappa} \frac{\kappa}{\kappa!} (2\pi i)^{1/2} f(\frac{t}{\pi}) + o(1)$$

This result is still true if $R(\sqrt{t}) = o$.

An example of a function which has poles on the line $R(s) = o$ is

$$f(s) = \ln s/1 + s^2.$$ 

The function is regular for $R(s) > o$, but has poles at $s = \pm i$ and a logarithmic singularity at the origin. The convex hull in this case is simply a straight line from $i$ to $-i$. For $0 < \arg t < \pi$ only the singularity at $s = -i$ contributes to the asymptotic behaviour and $F(t) \sim \frac{\pi}{2} e^{-it}$; similarly for $-\pi < \arg t < 0$, only the singularity at $s = i$ contributes and $F(t) \sim \frac{\pi}{2} e^{it}$; but for real values of $t$ all three singularities contribute.

Near $s = o$, $f(s) = \ln s \sum_{\kappa = o} (\frac{t}{\kappa})^\kappa s^{2\kappa} + o(s^{-1} s^{2k+1})$
Then by Theorem 3.1

$$\text{(4.11) } F(t) = \frac{\pi}{t} e^{it} - \frac{\pi}{t} e^{-it} + \sum_{k=0}^{\infty} \frac{(-1)^k A_k}{t^{2k+1}}$$

$$= \frac{1}{2\pi i} \int_{C} e^{ts} \left\{ \frac{\ln s}{1+s^2} - \frac{\pi s}{1+s^2} - \ln s \sum_{k=0}^{\infty} \frac{(e^i s)^k}{k} \right\} ds$$

The integrand is now regular at $s = \pm i$, so we obtain

$$\left| \frac{1}{2\pi i} \int_{C} e^{ts} \left\{ \frac{\ln s}{1+s^2} - \sum_{k=0}^{\infty} \frac{\ln s (e^i)^k}{k} \right\} ds \right|$$

$$\leq A \left\{ \int_{C} |e^{ts} s^{-2m+1}| |ds| \right\} , \text{ as } t \to \infty$$

$$= 0 \left\{ \frac{1}{t^{2m+2}} \right\}$$

and so

$$\text{(4.12) } F(t) \sim \frac{\pi}{2} \cos t - \frac{\pi}{t} \left(1 - \frac{2}{t^2} + \frac{4}{t^4} \cdots \right)$$

In a practical problem $(\pi/2) \cos t$ would be interpreted as the steady state solution and the series as the transient distortion.

We are now in a position to see why Heaviside's method breaks down in the case considered earlier. In our notation $f(s)$ is equivalent to $\mathfrak{h}(p)/p$; therefore

$$f(s) = \sqrt{\frac{c}{R}} \frac{\omega T_s}{S^2 + \omega^2}$$
This function has a branch point at $s = 0$ and poles at $s = i\omega$. By expanding in a power series Heaviside completely neglected the poles. If these are taken into account the correct result is obtained.

Under special conditions Heaviside's operational method gives the correct result. The method given here is more general in that it can deal with functions like $\Delta n s$ or $e^{ks}$ which do not have an expansion about the origin and it removes the apparent contradictions and arbitrariness of Heaviside's method.
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