VALUATIONS OF POLYNOMIAL RINGS

by

Ronald Alvin Macauley

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF ARTS

in the Department

of

MATHEMATICS

We accept this thesis as conforming to the standard required from candidates for the degree of MASTER OF ARTS.

Members of the Department ~~of~~ Mathematics

THE UNIVERSITY OF BRITISH COLUMBIA
April, 1951

Abstract

If R is a field on which all (non-archimedean) valuations are known, then all valuations on $R[x]$, where x is transcendental over R, are also known. Ostrowski described such valuations of $R[x]$ by means of pseudo-convergent sequences in the algebraic completion of A of R. MacLane later showed that if all valuations of R are discrete, then any valuation V of $R[x]$ can be represented by certain "key" polynomials in $R[x]$. The present paper exhibits the connection between these two treatments. This is achieved by first determining keys for the valuation which a pseudo-convergent sequence defines on $A[x]$, and then relating these keys to those for V.

Acknowledgment

The writer wishes to express his thanks to Dr. B.N. Moyls of the Department of Mathematics at the University of British Columbia for his advice and guidance. His numerous criticisms and suggestions proved invaluable in the preparation of this thesis.

-

1. Introduction. A non-archimedean valuation V, hereafter simply called a valuation, of an integral domain R is a single-valued mapping of the elements of R into the real numbers and $+\infty$ such that:

1) Va is a unique finite real number for $a \neq 0$,
2) $V0 = +\infty$,
3) $V(ab) = Va + Vb$ for all $a, b \in R$,
4) $V(a + b) \geq \min\{Va, Vb\}$ for all $a, b \in R$.

An extremely important property of these valuations is that, if $Va \neq Vb$, then $V(a + b) = \min\{Va, Vb\}$; hence, if $V(a + b) > Va$, $Va = Vb$.

Ostrowski, and later MacLane, attacked the problem of finding all extensinns of valuations on an integral domain R to the ring of polynomials $R[x]$, where x is transcendental over R. MacLane's results are based on the assumption that all valuations of R are discrete; that is, the real numbers used as values form an isolated point set. It is the purpose of this paper to provide a connection between the valuations of Ostrowski and MacLane on $R[x]$, where R is a field with only discrete valuations.

Definition 1.1: Let R be a field with a valuation V. The sequence $\{\alpha_i\}$, where $\alpha_i \in R$, is a pseudo-convergent sequence with respect to V if $V(\alpha_i - \alpha_{i-1}) < V(\alpha_{i+1} - \alpha_i)$ for all $i > N$, some fixed positive integer.

If $\{\alpha_i\}$ is a pseudo-convergent sequence with respect to

V , then the sequence $\{V\alpha_i\}$ is eventually strictly monotone increasing or eventually attains a constant value; as long as 0 is not a limit of $\{\alpha_i\}$, $\{V\alpha_i\}$ converges to a finite limit. This is important, for it is essentially this property that Ostrowski uses to extend V on R to W on $R[x]$, where W is a valuation of $R[x]$. He shows ([1], III, page 371) that if $f(x) \varepsilon R[x]$, then $\{f(\alpha_i)\}$ is also a pseudo-convergent sequence. This implies that $\{Vf(\alpha_i)\}$ is convergent to a finite limit except when $\{\alpha_i\}$ converges to a root of $f(x)$. Hence, if $\{\alpha_i\}$ is a pseudo-convergent sequence possessing no limit in the algebraic completion A of R , the function W on $R[x]$ defined by $Wf(x) = \lim_{i\to\infty} Vf(\alpha_i)$ is a valuation ([1], section 65, page 374) of $R[x]$. Further, ([1], IX, page 378) every valuation of $R[x]$ may be obtained by means of some pseudo-convergent sequence in A . The pseudo-convergent sequences in A are valued by an extension of V on R to A ; this extension always exists ([1], II, page 300) . This last reference implies that any valuation of $R(x)$ may be extended to $A(x)$. Hence, if all valuations of $A[x]$ are found, all valuations of $R[x]$ are automatically found. This result is of prime importance to the development of the theory in this paper.

<u>Definition 1.2</u>: Let K be an integral domain with a valuation V . Two elements $a, b \varepsilon K$ are equivalent with respect to V , written $a \sim b\ (V)$, if $V(a - b) > Va$.

<u>Definition 1.3</u>: For $a, b \varepsilon K$, a equivalence divides b in V if there exists $c \varepsilon K$ such that $b \sim ca\ (V)$; notation:

$a|b\ (V)$.

If V is any valuation of $R[x]$ which reduces to a discrete valuation V_0 of R, MacLane ([2]) represents V by the following inductive method: a value $V_1x = Vx = \mu_1$ is assigned to x. Then for any polynomial $f(x) \varepsilon R[x]$, $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_0$, a function V_1 on $R[x]$ is defined by $V_1f(x) = \min_i\{V_0a_i + i\mu_1\}$. This function V_1 may be shown to be a valuation of $R[x]$ such that $V_1 \leq V$; that is, $V_1g(x) \leq Vg(x)$ for all $g(x) \varepsilon R[x]$. The value V_1 is called a first stage value and is symbolized by $V_1 = [V_0, V_1x = \mu_1]$. Either $V_1 = V$, that is, $V_1g(x) = Vg(x)$ for all $g(x) \varepsilon R[x]$, or there exists an $f(x) \varepsilon R[x]$ such that $V_1f(x) < Vf(x)$. If that latter is the case, $\varphi_2(x) \varepsilon R[x]$ is chosen such that φ_2 is a monic polynomial of the smallest degree satisfying $V_1\varphi_2 < V\varphi_2$. This polynomial satisfies, over V_1 , MacLane's conditions for a key polynomial.

<u>Definition 1.4</u>: Let W be any valuation of $R[x]$. A polynomial $\varphi \varepsilon R[x]$ is a key polynomial over the value W if:

(i) φ is equivalence irreducible - $\varphi|a(x)b(x)\ (W)$ implies either $\varphi|a(x)\ (W)$ or $\varphi|b(x)\ (W)$,

(ii) φ is minimal - $\varphi|a(x)\ (W)$ implies $\deg a(x) \geq \deg \varphi$,

(iii) φ is monic.

It is shown ([2], theorem 4.2) that if a key polynomial φ over W is assigned a value $\mu = W'\varphi > W\varphi$, then the function W' on $R[x]$ defined by

$$W'f(x) = \min_i \{Wf_i(x) + i\mu\},$$

where $f(x) = f_n(x)\,\varphi^n + f_{n-1}(x)\,\varphi^{n-1} + \ldots + f_o(x)$, $\deg f_i(x) < \deg \varphi$, is a valuation of $R[x]$. Further, $W \leq W'$ and $Wf(x) < W'f(x)$ if and only if $\varphi | f(x)\ (W)$; in particular, $Wf(x) = W'f(x)$ if $\deg f(x) < \deg \varphi$ ([2], theorem 5.1). In the original valuation V, the polynomial φ_2 chosen above will define a valuation V_2 on $R[x]$ if assigned the value $\mu_2 = V\varphi_2 > V_1\varphi_2$. The value V_2 satisfies $V_2 \leq V$ and and $V_2f(x) = V_1f(x) = Vf(x)$ for all $f(x) \varepsilon R[x]$ such that $\deg f(x) < \deg \varphi_2$. The second-stage value V_2 is symbolized by $V_2 = [V_o,\ V_1x = \mu_1,\ V_2\varphi_2 = \mu_2]$. As before, either $V_2 = V$ or there exists a monic polynomial φ_3 of minimum degree satisfying $V_2\varphi_3 < V\varphi_3$. Again, if φ_3 exists it is a key polynomial over V_2 and may be used to define a valuation V_3 such that $V_3 \leq V$ and $V_3f(x) = V_2f(x) = Vf(x)$ for all $f(x) \varepsilon R[x]$ with $\deg f(x) < \deg \varphi_3$. The third-stage value V_3 is symbolized by $V_3 = [V_o,\ V_1x = \mu_1,\ V_2\varphi_2 = \mu_2,\ V_3\varphi_3 = \mu_3]$. MacLane shows ([2], theorem 8.1) that if this procedure is continued, equality will occur after either a finite or countable number of steps. In the first case V will have a representation

$$V = V_k = [V_O,\ V_1x = \mu_1, V_2\varphi_2 = \mu_2,\ \ldots\ ,\ V_k\varphi_k = \mu_k],$$

and is called an inductive value. In the latter case

$$V = V_\infty = [V_o,\ V_1x = \mu_1,\ V_2\varphi_2 = \mu_2,\ \ldots\ ,\ V_k\varphi_k = \mu_k,\ \ldots],$$

where $V_\infty f(x) = \lim_{k\to\infty} V_kf(x)$, and V is called a limit value. Hence each valuation of $R[x]$ may be represented by one of these two cases if every valuation of R is discrete.

The key polynomials defining the above inductive and limit values satisfy:

(iv) $\varphi_i \sim \varphi_{i-1}\ (V_{i-1})$ is false for all $i \geq 2$,

(v) $\deg \varphi_i \geq \deg \varphi_{i-1}$ for all $i \geq 2$.

Now since every valuation of $R[x]$ is either an inductive value or the limit of a sequence of inductive values it is necessary only to consider key polynomials which also satisfy (iv) and (v). For this reason it will be assumed in this paper that a key polynomial is a polynomial satisfying (i), (ii), (iii), (iv) and (v).

The representation of a valuation on $R[x]$ is not necessarily unique, but if one additional restriction is placed on the key polynomials the representation becomes unique when V_0 on R is discrete. Let $V_k = [V_0, V_1 x = \mu_1, \ldots, V_k\varphi_k = \mu_k]$ be an inductive valuation of $R[x]$. The V_k value of $f(x) \in R[x]$ is found from the expansion

$$f(x) = f_n(x)\,\varphi_k^n + f_{n-1}(x)\,\varphi_k^{n-1} + \ldots + f_0(x),$$

where $\deg f_i(x) < \deg \varphi_k$. By expanding each $f_i(x)$ in powers of φ_{k-1} and the coefficients of this expansion again in powers of φ_{k-2} and continuing these expansions, finally the expression

$$f(x) = \sum_j a_j x^{m_{1j}} \varphi_2^{m_{2j}} \ldots \varphi_k^{m_{kj}},$$

where $a_j \in R$ and each

$$m_{ij} < \frac{\deg \varphi_{i+1}}{\deg \varphi_i},$$

is obtained. Furthermore, $V_k f(x) = \min_j \left\{ V_k\left[a_j x^{m_{1j}} \varphi_2^{m_{2j}} \ldots \varphi_k^{m_{kj}}\right]\right\}$.

Now, the elements of R may be partitioned into classes of equivalent elements with respect to V_o and a representative may chosen from each class. In particular, the element 1 is to be chosen as a representative. These representatives are called the V_o-representatives. If in the above expansion of $f(x)$ each a_j is a V_o-representative and all terms have the minimum value $V_k f(x)$, then $f(x)$ is called homogeneous in V_k. Every polynomial $f(x) \varepsilon R[x]$ is equivalent in V_k to one and only one homogeneous polynomial $h(x) \varepsilon R[x]$ ([2], lemma 16.2); $h(x)$ is called the homogeneous part of $f(x)$. An inductive or limit value is called a homogeneous value if each φ_i, $i \geq 2$, is homogeneous in V_{i-1}. MacLane has shown ([2], theorem 16,3 and 16.4) that any inductive or limit value constructed from a discrete V_o may be represented by one and only one homogeneous inductive or limit value.

The inductive and limit values of $R[x]$ will always be considered to be homogeneous values.

2. The relation between the valuations of Ostrowski and MacLane will first be established on $A[x]$, where A is an algebraically complete field.

It will be found convenient, in this section and future sections, to remove the condition that a MacLane value has first key x. It is necessary only that the first key be linear and monic. The properties of MacLane values will be preserved.

Every valuation V of $A[x]$ may be defined by some pseudo-convergent sequence $\{\alpha_i\}$, with respect to V_o on A, which does not possess a limit in A ; $Vf(x)$ is defined as

$$Vf(x) = \lim_{i\to\infty} V_o f(\alpha_i) .$$

These pseudo-convergent sequences may be divided into two types. To obtain the desired classification, a pseudo-limit is defined.

<u>Definition 2.1</u>: An element $\alpha \varepsilon A$ is pseudo-limit of the pseudo-convergent sequence $\{\alpha_i\}$, where $\alpha_i \varepsilon A$, with respect to the valuation V_o if $V_o(\alpha-\alpha_i) = \delta_i$, where $\delta_i < \delta_{i+1}$ for $i >$ some integer N .

<u>Note</u>: "Pseudo-limit" as defined here is not the same as that defined by Ostrowski.

Now $V_o(\alpha_i - \alpha_{i+1}) = \gamma_i$, where $\gamma_i < \gamma_{i+1}$ for $i >$ some integer N'. Since $\gamma_i = V_o(\alpha_i - \alpha_{i+1}) = V_o[(\alpha_i - \alpha) + (\alpha - \alpha_{i+1})]$ it follows that, for $i > N$, $\gamma_i = \delta_i$.

The pseudo-convergent sequences are now divided into two classes:

(1) $\{\alpha_i\}$ possesses a pseudo-limit in A ,

(2) $\{\alpha_i\}$ does not possess a pseudo-limit in A ,

<u>Theorem 2.2</u>: If the pseudo-convergent sequence $\{\alpha_i\}$, with respect to V_o, has a pseudo-limit $\alpha \varepsilon A$, then the Ostrowski valuation V of $A[x]$ defined by $\{\alpha_i\}$ is the same as the first-stage valuation V_1 defined by $V_1 = [V_0, V_1(x - \alpha) = \gamma]$,

where $$\gamma = \lim_{i\to\infty} \gamma_i = \lim_{i\to\infty} V_0(\alpha_i - \alpha_{i+1}) .$$

<u>Proof</u>: It is sufficient to consider a monic linear polynomial $x - \beta$ in $A[x]$. Since $\alpha_i - \beta = (\alpha_i - \alpha) + (\alpha - \beta)$ and $V_o(\alpha_i - \alpha) = \gamma_i$, either $V_o(\alpha_i - \beta) = \gamma_i$ or $V_o(\alpha_i - \beta) = V_0(\alpha - \beta)$, for i sufficiently large.

Hence

$$V(x-\beta) = \lim_{i\to\infty} V_0(\alpha_i - \beta) = \min\left\{\gamma, V_0(\alpha-\beta)\right\} = V_1(x-\beta) .$$

Theorem 2.3: Given a finite inductive value $V = [V_0, V(x-\alpha) = \gamma]$ on $A[x]$, a pseudo-convergent sequence $\{\alpha_i\}$ with pseudo-limit $\alpha \varepsilon A$ can be found such that

$$\gamma = \lim_{i\to\infty} \gamma_i$$

Proof: Let $a \neq 0$ in A be chosen such that $Va = d > \theta$. Then there exists a real number σ such that $\sigma d = \gamma$. A sequence of integers $\{n_i\}$ can be found such that

$$\frac{n_1}{10} < \frac{n_2}{10^2} < \ldots < \frac{n_i}{10^i} < \ldots$$

and

$$\lim_{i\to\infty} \frac{n_i}{10^i} = \sigma .$$

Let β_i be any one of the roots of $x^{10^i} - a$. Then $d = Va = V\beta_i^{10^i} = 10^i V\beta_i$ or $V\beta_i = 1/10^i\, d$. Hence, the sequence $\{V\beta_i^{n_i}\}$ is a strictly increasing sequence with

$$\lim_{i\to\infty} V\beta_i^{n_i} = \lim_{i\to\infty} \frac{n_i}{10^i} d = \sigma d = \gamma .$$

Let α_i be defined by $\alpha_i = \beta_i^{n_i} + \alpha$. Since

$$V(\alpha_i - \alpha_{i+1}) = V(\beta_i^{n_i} - \beta_{i+1}^{n_{i+1}}) = V\beta_i^{n_i}, \quad \{\alpha_i\}$$

is a pseudo-convergent sequence and since $V(\alpha_i - \alpha) = V\beta_i^{n_i}$, α is a pseudo-limit of this sequence. By Theorem 2.2, the sequence $\{\alpha_i\}$ can have no limit in A, since the Ostrowski value defined by $\{\alpha_i\}$ is also defined by the finite value V.

On combining Theorem 2.2 and Theorem 2.3 an equivalence is obtained between valuations defined by pseudo-convergent sequences with pseudo-limits and the inductive values of MacLane.

<u>Theorem 2.4</u>: If $\{\alpha_i\}$ is a pseudo-convergent sequence with respect to V_0, with no pseudo-limit in A, then the Ostrowski valuation V defined by $\{\alpha_i\}$ is the same as the MacLane limit value $V' = [V_0, V_1(x-\alpha_1) = \gamma_1, \ldots, V_i(x-\alpha_i) = \gamma_i, \ldots]$, where $\gamma_i = V_0(\alpha_i-\alpha_{i+1})$. Also, in a MacLane valuation V' the sequence $\{\alpha_i\}$ is pseudo-convergent with no limit in A and the Ostrowski value V defined by $\{\alpha_i\}$ is equal to V'.

<u>Proof</u>: If necessary remove a finite number of terms from the beginning of the pseudo-convergent sequence $\{\alpha_i\}$ and renumber the α's, so that $\gamma_i = V_0(\alpha_i-\alpha_{i+1})$ is strictly increasing. Since $V_0(\alpha_{i+n}-\alpha_i) = \gamma_i$ for all $n \geq 1$, $V(x-\alpha_i) = \lim_{n\to\infty} V_0(\alpha_{i+n}-\alpha_i) = \gamma_i$. Let V_1 be defined by $V_1 = [V_0, V_1(x-\alpha_1) = \gamma_1]$, then V_1 defines a first-stage value of $A[x]$ such that $V_1 \leq V$. Now, from MacLane's inductive argument used in the introduction, it follows immediately that $V' = [V_0, V_1(x-\alpha_1) = \gamma_1, \ldots, V_i(x-\alpha_i) = \gamma_i, \ldots]$ is a MacLane valuation and also satisfies $V' \leq V$. If there exists $x - \beta \in A[x]$ such that $V'(x-\beta) < V(x-\beta)$ $= \lim_{i\to\infty} V_0(\alpha_i-\beta)$, then there exists a positive integer N such that $V'(x-\beta) < V_0(\alpha_i-\beta)$ for all $i > N$. Therefore, from $V'(x-\beta) = \lim_{i\to\infty}[\min\{\gamma_i, V_0(\alpha_i-\beta)\}]$ it follows that $\lim_{i\to\infty}\gamma_i$ $< V_0(\alpha_i-\beta)$ for $i > N$. Hence, $V_0(\alpha_{i+1}-\beta) = V_0[(\alpha_{i+1}-\alpha_i) + (\alpha_i-\beta)] = \gamma_i \leq \lim_{i\to\infty}\gamma_i$, for $i > N$, which is a contradiction. Thefefore $V' = V$. Suppose now that V' is a MacLane valuation. From $x - \alpha_{i+1} = (x-\alpha_i) + (\alpha_i-\alpha_{i+1})$ it follows that $V_0(\alpha_i-\alpha_{i+1}) \geq V_i(x-\alpha_i) = \gamma_i$, for otherwise

$$V_i[(x-\alpha_{i+1}) - (\alpha_i-\alpha_{i+1})] = V_i(x-\alpha_i) = \gamma_i > V_0(\alpha_i-\alpha_{i+1})$$

and so $x - \alpha_{i+1} \sim \alpha_i - \alpha_{i+1} (V_i)$, which contradictsthe minimal condition (ii) of definition 1.4 for a key polynomial over V_i. Therefore $V_i(x - \alpha_{i+1}) = \gamma_i$ and since $V_i(x - \alpha_{i+1}) < V_{i+1}(x-\alpha_{i+1})$, $\gamma_i < \gamma_{i+1}$ for all $i \geq 1$. Now

$$V_0(\alpha_i - \alpha_{i+1}) = V'\left[(x - \alpha_{i+1}) - (x - \alpha_i)\right] = \gamma_i ;$$

hence $\{\alpha_i\}$ is a pseudo-convergent sequence. If β were a limit of $\{\alpha_i\}$, then $\lim_{i\to\infty} V_0(\alpha_i - \beta) = \infty$. Let $k > i$ be chosen such that $V_0(\alpha_k - \beta) > \gamma_i$, then

$$V_0(\alpha_i - \beta) = V_0\left[(\alpha_i - \alpha_k) + (\alpha_k - \beta)\right] = \gamma_i.$$

Therefore

$$V'(x - \beta) = \lim_{i\to\infty}\left[\min\left\{\gamma_i,\ V_0(\alpha_i - \beta)\right\}\right] = \lim_{i\to\infty} V_0(\alpha_i - \beta) = \infty;$$

but V' is a finite value. Hence $\{\alpha_i\}$ has no limit in A and will, therefore, define an Ostrowski valuation which, by first part of theorem 2.4., must be the same as V'.

NOTE; If $\{\alpha_i\}$ has a pseudo-limit $\alpha \in A$, then V' may also be represented by

$$V'' = \left[V_0,\ V''(x - \alpha) = \lim_{i\to\infty} \gamma_i\right].$$

The results of this section now provide a connection between the two methods of valuation $A[x]$.

In sections 4 and 5 it will be shown how a MacLane valuation of $A[x]$ reduces to a MacLane valuation of $R[x]$, that is, the key polynomials and their assigned values will be found for the reduced valuation on $R[x]$, and conversely how to extend a value on $R[x]$ to $A[x]$. The connection between an Ostrowski valuation of $R[x]$ and a MacLane valuation of $R[x]$ will then be clear.

3. The key polynomials defining the restriction of a valuation of $A[x]$ to $R[x]$ are intimately related to the key polynomials used by MacLane ([3]) to extend a valuation V_o on R to a valuation W of $R(\alpha)$, a separable extension of R. For this reason a description of the methods used by MacLane and the essential results will now be given.

As a particular example, consider the inductive value

$$V_k = \left[V_o,\ V_1x = \mu_1, V_2\varphi_2 = \mu_2,\ \dots\ ,\ V_k\varphi_k = \mu_k\right]$$

of $R[x]$ and reassign to φ_k the value $+\infty$. This defines a new, generalized valuation

$$V = \left[V_o,\ V_1x = \mu_1,\ V_2\varphi_2 = \mu_2,\ \dots\ ,\ V_{k-1}\varphi_{k-1} = \mu_{k-1},\ V\varphi_k = \infty\right]$$

of $R[x]$. The generalized valuation V satisfies all the conditions of a valuation except that elements other than 0 are assigned the value $+\infty$. If α is a root of φ_k, the valuation V will define a valuation W on $R(\alpha)$. This is immediately seen upon noticing that

$$R(\alpha) \cong \frac{R[x]}{(\varphi_k)}$$

and defining W by $Wf(\alpha) = Vf(x)$. If the φ_i, for $2 \leq i \leq k$, above are homogeneous in the preceding inductive value V_{i-1}, MacLane has shown ([3], theorem 5.3) that this extension W of V_o is the only extension of V_o to $R(\alpha)$.

To facilitate the discussion of the remainder of this section and in view of sections 4 and 5, it is convenient at this point to define the terms projection and effective degree.

<u>Definition 3.1</u>: Let $V_k = \left[V_o,\ V_1x = \mu_1,\ \dots\ ,\ V_k\varphi_k = \mu_k\right]$ be an

inductive value of $R[x]$. If

$$G(x) = g_n(x)\,\varphi_k^n + g_{n-1}(x)\,\varphi_k^{n-1} + \dots + g_o(x)\ ,$$

where $\deg g_i(x) < \deg \varphi_k$, is a polynomial in $R[x]$, then the projection of V_k with respect to $G(x)$ is $\alpha - \beta$, written $\operatorname{proj}(V_k) = \alpha - \beta$, where α and β are the maximum and minimum values respectively of i such that

$$V_k G(x) = V_k\left[g_i(x)\,\varphi_k^i\right].$$

Definition 3.2: The effective degree of $G(x)$ in φ_k is α : written $D_\varphi G(x) = \alpha$.

Let W be a valuation of $R(\alpha)$, where W is an extension of V_o on R and α has minimal polynomial $G(x) \in R[x]$. By the isomorphism $R(\alpha) \cong R[x]/(G(x))$ it is clear that a generalized valuation V on $R[x]$ may be defined by $Vf(x) = Wf(\alpha)$. The valuation V assigns the value $+\infty$ only to the members of the ideal $(G(x))$. It would seem natural to construct V as MacLane does for finite valuations; that is, for valuations which assign the value $+\infty$ only to 0. As before, a first-stage value $V_1 = \left[V_o,\ V_1 x = \mu_1\right]$, where $\mu_1 = Vx \neq \infty$, is defined; again $V_1 \leq V$. It is worth noting that $\operatorname{proj}(V_1) > 0$. For if $\operatorname{proj}(V_1) = 0$ then would be only one term in

$$G(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_o$$

with minimum value, and $VG(x) = V_1 G(x) \neq \infty$. To define a secon-stage value a monic polynomial $f(x)$ of minimal degree satisfying $Vf(x) > V_1 f(x)$ is chosen. If $f(x)$ is not homogeneous in V_1, then its homogeneous part is to be chosen. Denote this homogeneous part by φ_2. As was mentioned in the introduction, φ_2 is a key polynomial over V_1. The second stage value V_2 is then defined by $V_2 = \left[V_0, V_1 x = \mu_1, V_2\varphi_2 = \mu_2\right]$, where $\mu_2 \neq V\varphi_2$. Now, if $G(x)$ is a homogeneous key over V_1, then φ_2 is chosen as $G(x)$ and $\mu_2 = \infty$. That $G(x)$ is a monic polynomial of minimal degree satisfying $VG(x) > V_1G(x)$ will follow from lemmas 3.3 and 3.4.

<u>Lemma 3.3</u>: Let V_k be a k-th stage inductive value of $R[x]$ satisfying:

(1) $V_k f(x) \leq Vf(x)$ for all $f(x) \in R[x]$,

(2) $\deg f(x) < \deg \varphi_k$ implies $V_k f(x) = Vf(x)$,

(3) $V_k\varphi_i = V\varphi_i = \mu_i$ for $1 \leq i \leq k$.

If ψ is a monic polynomial of minimal degree satisfying $V_k\psi < V\psi$, then $V_k f(x) < Vf(x)$ implies $\psi | f(x) \quad (V_k)$.

<u>Proof:</u> Let $f(x)$ have the quotient remainder expression $f(x) = q(x)\psi + r(x)$, where $\deg r(x) < \deg \psi$. Then

$$V_k[f - q\psi] = V[f - q\psi] \geq \min\{Vf, V[q\psi]\} > \min\{V_k f, V_k[q\psi]\},$$

because of (2), the choice of ψ and the assumption (1) for $q(x)$. Hence $\psi | f(x) \quad (V_k)$.

<u>Lemma 3.4:</u> Let V_k be an inductive value of $R[x]$. Any polynomial $G(x) \in R[x]$ has an equivalence decomposition

$$G(x) \sim e(x)\, \varphi_k^{t_0}\, \psi_1^{t_1}\, \psi_2^{t_2} \cdots \psi_r^{t_r} \quad (V_k),$$

where each ψ_i is a homogeneous key over V_k, $t_0 \geq 0$ and $t_i > 0$ for $1 \leq i \leq r$, and $e(x)$ is an equivalence unit, that is, $D_\varphi e(x) = 0$ in V_k. This decomposition is unique except for equivalence units.

<u>Proof</u>: Cf. [3], theorem 4.2.

Now suppose φ_2 is a homogeneous monic polynomial of minimal degree satisfying $V_1\varphi_2 < V\varphi_2$ when $G(x)$ is a homogeneous key over V_1. Since $V_1G(x) < VG$, $G(x) \sim h(x)\varphi_2$ (V_1) by lemma 3.3. But $G(x) \sim G(x)$ (V_1), and therefore by lemma 3.4 $G(x) = \varphi_2$. Hence in this case the value V is given by

$$V = \left[V_0,\ V_1x = \mu_1,\ VG(x) = \infty\right].$$

If $G(x)$ is not a homogeneous key over V_1, then the second-stage is given by

$$V_2 = \left[V_0,\ V_1x = \mu_1,\ V_2\varphi_2 = \mu_2\right],$$

where $V_2 \leq V$. It is noticed again that $\mathrm{proj}\,(V_2) > 0$ for otherwise $VG(x) \neq \infty$. Also $\varphi_2 | G(x)$ (V_1) by lemma 3.3. MacLane's inductive process is repeated until $G(x)$ does become a homogeneous key over some V_k or, if this does not occur, it is repeated a countable number of steps. In the former case

$$V = \left[V_0,\ V_1x = \mu_1,\ \ldots,\ V_k\varphi_k = \mu_k,\ VG(x) = \infty\right]$$

by the preceding argument. Also $\varphi_i | G(x)$ (V_{i-1}) for all i such that $2 \leq i \leq k$, and $\mathrm{proj}\,(V_i) > 0$ for $1 \leq i \leq k$. If a countable number of steps are required, then

$$V = V' = \left[V_0,\ V_1x = \mu_1,\ \ldots,\ V_k\varphi_k = \mu_k,\ \ldots\right].$$

Certainly, $V_k \leq V$ and $\varphi_k | G(x)$ (V_{k-1}) for all $k \geq 2$. Since each φ_k is minimal over V_{k-1}, $\deg \varphi_k \leq \deg G(x)$. So from some point on all the keys will have the same degree. In this case it can be shown ([2], lemma 6.3) that the value group of V' is discrete if the value group of R is discrete; that is, the real numbers used as values for V' form an isolated point set. If $V'f(x) < Vf(x)$ for some $f(x) \in R[x]$, then $V_kf(x) < Vf(x)$ for all $k > 0$ by the monotone increasing character of the inductive values and so $\varphi_{k+1} | f(x) \ (V_k)$. Therefore $V_k f(x) < V_{k+1} f(x)$ for all $k > 0$ (Cf. the introduction). But since the value group of V' is discrete,

$$Vf(x) \geq \lim_{k \to \infty} V_k f(x) = \infty .$$

Therefore only polynomials in $(G(x))$ could satisfy $V'f(x) < Vf(x)$ but since $V'f(x) = \infty = Vf(x)$ for $f(x) \varepsilon (G(x))$, $V = V'$.

It is seen that every discrete V_0 of R may be extended to a finite separable extension $R(\alpha)$ of R by MacLane's inductive process, where the homogeneous keys can be further restricted to satisfy the conditions proj $(V_i) > 0$ for $i > 0$ and $\varphi_i | G(x) \ (V_{i-1})$ for $i \geq 2$, where $G(x)$ is the minimal poly- of α . In fact, it is not difficult to see that these restrictions are necessary.

From the preceding arguments it follows almost immediately that every such sequence of values constructed by these restricted keys will give a valuation of $R(\alpha)$.

The construction of such a sequence of values may be accomplished in a systematic manner. Let $V_1 = [V_0, V_1 x = \mu_1]$ be a first-stage value such that proj $(V_1) > 0$ for $G(x)$. If V_{i-1} has been defined, the next key φ_i is chosen as any one of the ψ_j occurring in the unique equivalence decomposition of lemma 3.4. The corresponding value μ_i is chosen so that proj $(V_i) > 0$ and $\mu_i > V_{i-1}\varphi_i$. In the sequence of valuations so defined, each V_i is called an i-th approximant to $G(x)$. MacLane not only shows that every such sequence of values defines a valuation

W on $R(\alpha)$ which is an extension of V_0, but that :

(1) if $G(x)$ eventually becomes a homogeneous key over the inductive value V_k, then the i-th approximant is unique for $2 \leq i \leq k$ and also, the value μ_1 is unique ([3], theorem 5.3) . This implies that V_9 may be extended to $R(\alpha)$ in only one way ([3], theorem 10.1) ,

(2) if a countable sequence of keys are required, then there is at most a finite number of different sequences that can be constructed. Hence, V_0 on R may be extended to $R(\alpha)$ in at most a finite number of ways ([3], theorem 10.1) .

4. The reduction, or restriction, of an inductive value W of $A[x]$ to $R[x]$ will first be found; following theorem 4.7 the reduction of a limit value will be found. These results will be established by mathematical induction.

<u>Theorem 4.1</u>: If $W = [W_0,\ W(x - \alpha) = \gamma]$ is any inductive value of $A[x]$ with $W_0\alpha \geq \gamma$, then $W = W' = [W_0,\ W'x = \gamma]$.

<u>Proof</u>: Let $x - \beta \in A[x]$. If $W_0(\alpha - \beta) < \gamma$, then $W_0(\alpha - \beta) = W_0\beta$; and

$$W(x - \beta) = \min\{\gamma,\ W_0(\alpha - \beta)\} = \min\{\gamma,\ W_0\beta\} = W'(x - \beta) .$$

If $W_0(\alpha - \beta) \geq \gamma$, then $W_0\beta \geq \min\{W_0(\alpha - \beta),\ W_0\alpha\} \geq \gamma$, and $W(x - \beta) = \gamma = W'(x - \beta)$.

An immediate consequence of theorem 4.1 is:

<u>Theorem 4.2:</u> If $W = [W_0,\ W(x - \alpha) = \gamma]$ is any inductive value of $A[x]$ with $W_0\alpha \geq \gamma$, then $V = [V_0,\ V_1x = \gamma]$ is the reduction of W to $R[x]$ providing $V_0 = W_0$ on R .

<u>Theorem 4.3</u>: Let $W = [W_o, W(x - \alpha) = \gamma]$ be an inductive value of $A[x]$ with $\gamma > W_o\alpha$, then $V_1 = [V_o, V_1 x = W_o\alpha]$, where $V_o = W_o$ on R, is the first-stage of the reduction of W to $R[x]$. There exist polynomials $f(x) \varepsilon R[x]$ such that $V_1 f(x) < Wf(x)$.

<u>Proof</u>: The value of x is $Wx = \min\{\gamma, W_o\alpha\} = W_o\alpha$. Therefore $W_1 = [W_o, W_1 x = W_o\alpha]$ is a first-stage value to W; $W_1 \leq W$. Hence $V_1 = [V_o, V_1 x = W_o\alpha]$ is the first-stage of the reduction to $R[x]$. Let $G(x) = (x - \alpha)(x - \beta_1) \dots (x - \beta_t)$ be the minimal polynomial of α in $R[x]$. Since $W_1(x - \alpha) < W(x - \alpha)$ and $W_1(x - \beta_i) \leq W(x - \beta_i)$, $V_1 G(x) = W_1 G(x) < WG(x)$.

Theorem 4.3 shows that for $\gamma > W_o\alpha$ at least one more key is necessary to obtain the correct reduction of W to $R[x]$.

<u>Lemma 4.4</u>: Let $W = [W_o, W(x - \alpha) = \gamma]$ be any inductive value of $A[x]$. A polynomial $f(x) \varepsilon A[x]$ is equivalence divisible by $x - \alpha$ in W if and only if $W_o f(\alpha) > Wf(x)$.

<u>Proof</u>: Let $f(x) = f_n(x - \alpha)^n + f_{n-1}(x - \alpha)^{n-1} + \dots + f_o$ be the expansion of $f(x)$ in powers of $x - \alpha$; $f_i \varepsilon A$. Since $Wf(x) = \min\{W_o f_i + i\gamma\}$, $W_o f_o \geq Wf(x)$; and because $f_o = f(\alpha)$, the relation $W_o f(\alpha) \geq Wf(x)$ always holds. Suppose $W_o f(\alpha) > Wf(x)$. Then $W[f(x) - \{f_n(x - \alpha)^n + \dots + f_1(x - \alpha)\}] = W_o f(\alpha) > Wf(x)$ and, therefore, $f(x) \sim f_n(x-\alpha)^n + \dots + f_1(x-\alpha)$ in W; that is, $x - \alpha \mid f(x)$ in W. Suppose, now, $f(x) \sim q(x)(x - \alpha)$ in W. Then $f(x) = q(x)(x - \alpha) + h(x)$, where $Wh(x) > Wf(x)$. But, since h_o, the last term in the expansion of $h(x)$, is f_o it follows that

$W_o f_o = W_o h_o \geq Wh(x) > Wf(x)$; that is, $W_o f(\alpha) > Wf(x)$.

In the results to follow the polynomials φ_i and the real numbers μ_i will be the homogeneous key polynomials and their values which are used by MacLane to extend a value V_o on R to a value W_o on $R(\alpha)$ (§3) . Since the value W_o on A , and therefore on $R(\alpha)$, is given, V_o will be the restriction of W_o to R . However, there exist φ_i and μ_i defining the extension of this V_o to the given W_o on $R(\alpha)$.

Theorem 4.5: Let the polynomials φ_i and the numbers μ_i be the keys and values which define the extension of V_o on R to W_o on $R(\alpha)$. The k-th stage of the reduction of $W = [W_o,\ W(x - \alpha) = \gamma]$ to $R[x]$ is given by

$$V_k = [V_o,\ V_1 x = \mu_1,\ V_2\varphi_2 = \mu_2,\ \ldots\ V_k\varphi_k = \mu_k],$$

provided that $x - \alpha | \varphi_i$ in W is false for all i in the interval $1 \leq i \leq k$.

Proof: By Theorem 4.3 and lemma 4.4 this theorem is true for $k = 1$. Suppose the result is true up to $k - 1$, then $V_{k-1} = [V_o,\ V_1 x = \mu_1,\ \ldots\ ,\ V_{k-1}\varphi_{k-1} = \mu_{k-1}]$ is the (k-1)st stage of the reduced value and $V_{k-1} f(x) \leq Wf(x)$ for all $f(x) \in R[x]$. Let $f(x) \in R[x]$ be any polynomial such that $\deg f(x) < \deg \varphi_k$. Then $V_{k-1}f(x) = W_o f(\alpha)$, because in extending V_o on R to W_o on $R(\alpha)$ the value $W_o f(\alpha)$ is defined to be $V_{k-1}f(x)$ when $\deg f(x) < \deg \varphi_k$. But, since $Wf(x) \geq V_{k-1}f(x) = W_o f(\alpha) \geq Wf(x)$ by lemma 4.4, it is concluded that $Wf(x) = V_{k-1}f(x)$ for all $f(x) \in R[x]$ such that $\deg f(x) < \deg \varphi_k$. However, $W\varphi_k = W_o\varphi_k(\alpha) = \mu_k > V_{k-1}\varphi_k$, since $x - \alpha | \varphi_k$ in W is false, and φ_k may then be chosen the next key over V_{k-1} with value $V_k\varphi_k = \mu_k$ (Cf. introduction). Also, the value

$$V_k = \left[V_o, V_1x = \mu_1, \dots, V_{k-1}\varphi_{k-1} = \mu_{k-1}, V_k\varphi_k = \mu_k\right]$$

satisfies the relation $V_kf(x) \leq Wf(x)$ for all $f(x) \in R[x]$ and is, therefore, a k-th stage of the restriction of W to $R[x]$.

Lemma 4.6: If φ_i are the keys used to extend V_o on R to W_o on $R(\alpha)$, then there exists an i such that $x - \alpha | \varphi_i$ in $W = \left[W_o, W(x - \alpha) = \gamma\right]$.

Proof: There are two cases to consider:

(a) W_o is found by an inductive value - then the last key is $G(x)$, the minimal polynomial for α in $R[x]$, which is divisible by $x - \alpha$ and, therefore equivalence divisible by $x - \alpha$ in W,

(b) W_o is found by a limit value - if there exists no i such that $x - \alpha | \varphi_i$ in W, then by theorem 4.5 every φ_i, with value μ_i, occurs in the reduction of W to $R[x]$. But this implies the value of $G(x)$ is $+\infty$; while $WG(x) < +\infty$.

Lemma 4.6 implies the existence of a first key φ_{k+1} which is equivalence divisible by $x - \alpha$ in W. By theorem 4.5,

$$V_k = \left[V_o, V_1x = \mu_1, \dots, V_k\varphi_k = \mu_k\right]$$

is the k-th stage of the reduction of W to $R[x]$. There are two possibilities for the k-th stage value of φ_{k+1}, either

$V_k\varphi_{k+1} < W\varphi_{k+1}$ or $V_k\varphi_{k+1} = W\varphi_{k+1}$.

Theorem 4.7: Let $W = \left[W_o; W(x-\alpha) = \gamma\right]$, with $\gamma > W_o\alpha$, be given on $A[x]$. Let $\{V_i\}$ be the sequence of approximants to $G(x)$, the minimal polynomial of α, defining the extension of V_0 on R to W_0 on $R(\alpha)$. If φ_{k+1} is the first key in these approximants such that $x - \alpha \mid \varphi_{k+1}(W)$, then the reduction of W to $R[x]$ is given by:

(1) $V = \left[V_0, V_1 x = \mu_1, \ldots, V_k\varphi_k = \mu_k, V\varphi_{k+1} = W\varphi_{k+1}\right]$ when $V_k\varphi_{k+1} < W\varphi_{k+1}$,

(2) $V_k = \left[V_0, V_1 x = \mu_1, \ldots, V_k\varphi_k = \mu_k\right]$ when $V_k\varphi_{k+1} = W\varphi_{k+1}$.

Proof: As in theorem 4.5, $V_k f(x) = W_0 f(\alpha) = Wf(x)$ for all $f(x) \in R[x]$ such that $\deg f(x) < \deg \varphi_{k+1}$. Since in (1) $V_k\varphi_{k+1} < W\varphi_{k+1}$, φ_{k+1} may be chosen as the next key with value $W\varphi_{k+1}$; this gives $V \leq W$ on $R[x]$. If V_k in (1) and V_k in (2) are both denoted by V', the two results may be given by one proof. Suppose the existence of

$$f(x) = f_n(x)\,\varphi_{k+1}^{n} + f_{n-1}(x)\,\varphi_{k+1}^{n-1} + \ldots + f_0(x)$$
$$= f_0(x) - g(x)\,\varphi_{k+1},$$

where $\deg f_i(x) < \deg \varphi_{k+1}$ and $f(x) \in R[x]$, a monic polynomial of minimum degree such that $V'f(x) < Wf(x)$. Then $V'f_0(x) \geq V'f(x)$. If $V' = V_k$, this is immediate from the definition of $Vf(x)$. If $V' = V_k$, then φ_{k+1} is a key polynomial over V_k since it defines an approximant V_{k+1}. Suppose $V_k f_0(x) < V_k f(x)$, then $V_k\left[f_0(x) - g(x)\,\varphi_{k+1}\right] = V_k f(x) > V_k f_0(x)$ and $f_0(x) \sim g(x)\,\varphi_{k+1}(V_k)$, which contradicts the minimal condition of the key φ_{k+1}. Now, $V'g(x) = Wg(x)$ because $\deg g(x) < \deg f(x)$; therefore $V'\left[f(x) - f_0(x)\right] = V'(-g(x)) + V'\varphi_{k+1} = W(-g(x)) + W\varphi_{k+1} = W\left[f(x)-f_0(x)\right]$. That $V'f_0(x) = V'f(x)$ may be seen by assuming $V'f_0(x) > V'f(x)$; for then

$$V'f(x) = V'[f(x) - f_o(x)] = W[f(x) - f_o(x)] \geq \min\{Wf(x), Wf_o(x)\} > V'f(x) .$$

Now $W[f_o(x) - g(x)\ \varphi_{k+1}] = Wf(x) > V'f(x) = V'f_o(x) = Wf_o(x)$; hence $f_o(x) \sim g(x)\varphi_{k+1}(W)$. Since $x - \alpha | \varphi_{k+1}(W)$, $x - \alpha | f_o(x) \ (W)$. But, since $\deg f_o(x) < \deg \varphi_{k+1}$ this contradicts $W_o f(\alpha) = Wf_o(x)$; therefore $V' = W$ on $R[x]$ in either (1) or (2) .

On combining theorems 4.2 and 4.7 and lemma 4.4 a picturesque description of the reduction can be given in terms of the size of γ . For this purpose consider $W = [W_o, W(x - \alpha) = \gamma]$ on $A[x]$ with $\gamma < W_o\alpha$ and examine the reduction to $R[x]$ as γ continuously increases. For $\gamma < W_o\alpha$ the reduction is $V_1' = [V_o, V_1'x = \gamma]$. As γ increases to $W_o\alpha = \mu_1$ the reduction increases to $V_1 = [V_o, V_1x = \mu_1]$. When γ is just larger than $W_o\alpha$, the key φ_2 of the second approximant to $G(x)$, the minimal polynomial of α , is needed. The reduction is $V_2' = [V_o, V_1x = \mu_1, V_2'\varphi_2 = \mu]$. As γ increases again the value $V_2'\varphi_2 = \mu$ increases to $W\varphi_2 = W_o\varphi_2(\alpha) = \mu_2$ and the reduction increases to $V_2 = [V_o, V_1x = \mu_1, V_2\varphi_2 = \mu_2]$. If this process is continued it is seen that as γ increases the reduction sweeps through the approximants to $G(x)$ which describe the extension of V_o to W_o on $R(\alpha)$. The only difference between the reduction and the corresponding approximant defined by the same keys is that the value assigned to the last key in the reduction may be less than its value in the approximant. But as γ is increased this value will increase to the corresponding approximant value if the last key is not $G(x)$. For then γ would have to increase to ∞

It will now be shown what this reduction is when W is a limit value. In the remainder of this section W is defined by

$$W = \left[W_0,\ W_1(x - \alpha_1) = \gamma_1,\ \ldots\ ,\ W_i(x - \alpha_i) = \gamma_i,\ \ldots\right],$$

where the pseudo-convergent sequence $\{\alpha_i\}$ has no pseudo-limit in A. It should be noticed that

$$W_i = \left[W_0,\ W_1(x - \alpha_1) = \gamma_1,\ \ldots\ ,\ W_i(x - \alpha_i) = \gamma_i\right]$$

can also be represented by $W_i = \left[W_0,\ W_i(x - \alpha_i) = \gamma_i\right]$. For, to find $W_i f(x)$, $f(x)$ is expanded in powers of $x - \alpha_i$ and the coefficients are valued with W_{i-1}. But the coefficients are in A and are therefore actually valued by W_0.

<u>Lemma 4.8</u>: The value $W_i = W'_{i+1} = \left[W_0,\ W'_{i+1}(x - \alpha_{i+1}) = \gamma_i\right]$ on $A[x]$.

<u>Proof</u>: Let $x - \beta \in A[x]$. If $W_0(\alpha_i - \beta) < \gamma_i$ then
$W_0(\alpha_{i+1} - \beta) = W_0\left[(\alpha_{i+1} - \alpha_i) + (\alpha_i - \beta)\right] = W_0(\alpha_i - \beta)$;
$W'_{i+1}(x - \beta) = \min\{\gamma_i, W_0(\alpha_{i+1} - \beta)\} = \min\{\gamma_i, W_0(\alpha_i - \beta)\}$
$= W_i(x - \beta)$. If $W_0(\alpha_i - \beta) \geq \gamma_i$, then
$W_0(\alpha_{i+1} - \beta) \geq \min\{\gamma_i, W_0(\alpha_i - \beta)\} = \gamma_i$, and
$W'_{i+1}(x - \beta) = \gamma_i = W_i(x - \beta)$.

<u>Lemma 4.9</u>: For each $x - \beta \in A[x]$ there exists a positive integer N such that $W(x - \beta) = W_i(x - \beta)$ for all $i \geq N$.

<u>Proof</u>: If no such N exists, then $W(x - \beta) > W_i(x - \beta)$ for all $i > 0$ since $W_i \leq W_{i+1}$. Now $W_0(\alpha_{i+1} - \beta) > W_i(x - \beta)$ for otherwise from $x - \beta = (x - \alpha_{i+1}) + (\alpha_{i+1} - \beta)$ it follows that $\gamma_i = W_i(x - \alpha_{i+1}) \geq W_0(\alpha_{i+1} - \beta)$; and therefore $\gamma_{i+1} > W_0(\alpha_{i+1} - \beta)$. Hence, $W(x - \beta) = W_0(\alpha_{i+1} - \beta)$ $= W_{i+1}(x - \beta)$, which contradicts $W(x - \beta) > W_i(x - \beta)$ for all $i > 0$. Since $W_0(\alpha_{i+1} - \beta) > W_i(x - \beta) = W_i(x - \alpha_{i+1}) = \gamma_i$, therefore

$$W_0(\alpha_{i+1} - \beta) = W_0\left[(\alpha_{i+1} - \alpha_{i+2}) + (\alpha_{i+2} - \beta)\right] = \gamma_{i+1}$$

for all $i > 0$. But this implies β is a pseudo-limit of $\{\alpha_i\}$.

Theorem 4.10: The $k(i)$ keys occurring in the reduction of $W_i = \left[W_0, W_i(x - \alpha_i) = \gamma_i\right]$ on $A[x]$ to $R[x]$ are the first $k(i)$ keys in the reduction of

$$W = \left[W_0, W_1(x - \alpha_1) = \gamma_1, \ldots, W_i(x - \alpha_i) = \gamma_i, \cdots\right]$$

on $A[x]$ to $R[x]$. Also, the values μ_ν, for $1 \leq \nu \leq k(i)$, in the reduction of W_i are the first $k(i) - 1$ values for the keys in the reduction of W.

Proof: By lemma 4.9, for any given $f(x) \varepsilon R[x]$ there exists an N such that $Wf(x) = W_i f(x)$ for all $i \geq N$. If $V_{k(i)}$ is the reduction of W_i, then $V_{k(i)} f(x) = Wf(x)$ for all $i \geq N$. Hence, the sequence of values $\{V_{k(i)}\}$ on $R[x]$ gives every polynomial in $R[x]$ the correct W value. It is only necessary then to show that the $k(i)$ keys in the reduction of W_i are the first $k(i)$ keys in the reduction of W_{i+1} and that the values, with the possible exception of $\mu_{k(i)}$, are the same. By lemma 4.8, W_i and $W'_{i+1} = \left[W_0, W'_{i+1}(x - \alpha_{i+1}) = \gamma_i\right]$ define the same valuation of $A[x]$. Hence, they will have the same reduction on $R[x]$ and, because the keys in the reductions are homogeneous, each reduction will be identical with respect to keys and values ([2], theorem 16.4). As the value of $x - \alpha_{i+1}$ is increased from γ_i to γ_{i+1} the value $\mu_{k(i)}$ might increase and the keys, if any, appearing in $V_{k(i+1)}$ but not in $V_{k(i)}$ are used to augment $V_{k(i)}$ to $V_{k(i+1)}$. These are the only changes that can happen; and at least one of these changes must happen. The truth of this follows from the discussion immediately after theorem 4.7

and the fact that the minimal polynomial of α_{i+1} definitely increases as the value of $x - \alpha_{i+1}$ increases.

Theorem 4.11: The reduction of a limit value W on $A[x]$ to $R[x]$, as described in theorem 4.10, is a limit value.

Proof: Suppose the reduction is an inductive value $V_k = \left[V_0, V_1x = \mu_1, \ldots, V_k\varphi_k = \mu_k\right]$. By lemma 4.9 there exists a smallest i such that $W_i\varphi_k = W\varphi_k = \mu_k$. The reduction of W_i to $R[x]$ must be V_k since this is the first stage in which φ_k assumes the value μ_k and $W_i \leq W$. But for $G(x)$, the minimal polynomial of α_{i+1} in $R[x]$, $V_kG(x) = W_iG(x) < W_{i+1}G(x) \leq WG(x)$. Since by assumption $V_k = W$, this contradiction establishes the theorem.

5. In section 4 the connection between a value W of $A[x]$ and its reduction to $R[x]$ was established. The converse problem will now be solved; that is, given a value V of $R[x]$, to find an extension of V to $A[x]$. First, however, it will be shown that a value W on $A[x]$ may be written in a standard form.

In the following theorem the notation

$$V = \left[V_0, V_1x = \mu_1, \ldots, V_k\varphi_k = \mu_k, \qquad\right]$$

is to mean that a least the keys up to φ_k occur in the representation of V; however, V may be an inductive value with keys past φ_k, or V may even be a limit value. This notation will also be used for

$$W = \left[W_0, W_1x = \mu_1, \ldots, W_k(x - \alpha_k) = \gamma_k, \qquad\right]$$

Lemma 5.1: If a value W' of $A[x]$ reduces to

$$V = \left[V_0, V_1x = \mu_1, \ldots, V_k\varphi_k = \mu_k, \qquad\right]$$

on $R[x]$, then V may be extended to

$W = \left[W_o,\ W_1x = \mu_1,\ W_2(x - \alpha_2) = \gamma_2,\ \dots,\ W_k(x - \alpha_k) = \gamma_k,\qquad\right]$
on $A[x]$ where:

(1) $\varphi_i\ (\alpha_i) = 0$ for $i = 1, 2, \dots, k$,

(2) $\gamma_i = W'(x - \alpha_i)$ for $i = 1, 2, \dots, k$,

(3) $W_o\beta = W'\beta$ for all $\beta \varepsilon A$,

(4) $W'f(x) \geq Wf(x)$ for all $f(x) \varepsilon A[x]$.

Further, the reduction of $W_i = \left[W_o,\ W_i(x - \alpha_i) = \gamma_i\right]$ is $V_i = \left[V_o,\ V_1x = \mu_1,\ \dots,\ V_i\varphi_i = \mu_i\right]$ for $i = 1, 2, \dots, k$.

<u>Proof</u>: Let W_o on A be defined by $W_o\beta = W'\beta$ for $\beta \varepsilon A$ and W_1 defined by $W_1 = \left[W_o,\ W_1x = \mu_1\right]$. The reduction of W_1 is certainly $V_1 = \left[V_o,\ V_1x = \mu_1\right]$. Let

$$f(x) = \sum_i a_i x^i,\quad f(x) \varepsilon A[x],$$

then $W'f(x) \geq \min\left\{W_oa_i + iW'x\right\} = W_1f(x)$. The value W_1 satisfies (2), (3), (4) and also (1) since $\varphi_1 = x$ and $\alpha_1 = 0$. Assume theorem true up to W_{k-1}. Then W_{k-1} reduces to V_{k-1} and $W'\varphi_k > V_{k-1}\varphi_k = W_{k-1}\varphi_k$. For a key to augment W_{k-1} let a factor $x - \alpha_k$ of φ_k be chosen so that $W'(x-\alpha_k) > W_{k-1}(x-\alpha_k)$ (Cf. introduction) and $W'(x - \alpha_k) \geq W'(x - \beta)$ for any factor $x - \beta$ of φ_k such that $W'(x - \beta) > W_{k-1}(x - \beta)$. Now define W_k by $W_k = \left[W_{k-1},\ W_k(x - \alpha_k) = \gamma_k = W'(x - \alpha_k)\right]$. For $x - \beta$, any factor of φ_k,

$$W_k(x - \beta) = \min\left\{\gamma_k,\ W_o(\alpha_k - \beta)\right\}, \text{ and}$$
$$W'(x - \beta) \geq \min\left\{\gamma_k,\ W_o(\alpha_k - \beta)\right\}.$$

The inequality cannot hold; for, then,

$$W'(x - \beta) > \gamma_k = W_k(x - \alpha_k) \geq W_k(x - \beta) \geq W_{k-1}(x - \beta),$$

which contradicts the choice of $x - \alpha_k$. Therefore $W'(x - \beta) = W_k(x - \beta)$ for all factors of φ_k; so $W'\varphi_k = W_k\varphi_k$. Since $\varphi_k(\alpha_k) = 0$, certainly $x - \alpha_k | \varphi_k$ in W_k.

This means the reduction will use only keys φ_i for $i \leq k$. But $W_k\varphi_k = W'\varphi_k = \mu_k > W_{k-1}\varphi_k$, hence the reduction of W_k must be V_k.

Theorem 5.2: If a value $W' = \left[W_o, W'(x - \beta) = \delta\right]$ on $A[x]$ reduces to $V_k = \left[V_o, V_1x = \mu_1, \ldots, V_k\varphi_k = \mu_k\right]$ on $R[x]$, then there exists an α, such that $\varphi_k(\alpha) = 0$, and a γ so that $W' = W = \left[W_o, W(x - \alpha) = \gamma\right]$. Also $\gamma = \delta$.

Proof: Let V_k be extended to W on $A[x]$ as in theorem 5.1, where $\alpha = \alpha_k$, then $W \leq W'$. If there exists an $x - \theta \in A[x]$ such that $W(x - \theta) < W'(x - \theta)$, then for $G(x) \in R[x]$, where $G(\theta) = 0$, it follows that

$$V_kG(x) = WG(x) < W'G(x) = V_kG(x); \quad \text{hence,} \quad W = W'.$$

Also, since $\gamma = W(x - \alpha) = W'(x - \alpha) \leq \delta$ and

$$\delta = W'(x - \beta) = W(x - \beta) \leq \gamma,$$

then $\gamma = \delta$.

Theorem 5.3: Let $W' = \left[W_o, W_1(x - \beta_1) = \delta_1, \ldots, W_i(x - \beta_i) = \delta_i, \ldots\right]$ be a value on $A[x]$ which reduces to

$$V = \left[V_o, V_1x = \mu_1, \ldots, V_k\varphi_k = \mu_k, \ldots\right] \text{ on } R[x],$$

then W' may be represented by

$$W = \left[W_o, W_1x = \mu_1, W_2(x - \alpha_2) = \gamma_2, \ldots, W_k(x - \alpha_k) = \gamma_k \ldots\right]$$

where:

(1) $\varphi_k(\alpha_k) = 0$ for all $k \geq 1$,

(2) reduction of W_k to $R[x]$ is V_k,

Proof: The proof is similar to that for theorem 5.2.

From theroems 5.2 and 5.3 it is seen that every valuation of $A[x]$ may be put into a form such that each α_k is a root of the corresponding key φ_k appearing in the reduction of the valuation to $R[x]$. This information indicates how a valuation V of $R[x]$ may be extended to some valuation of $A[x]$. It will now be shown how this extension can be accomplished.

Lemma 5.4: Let α be a root of some polynomial $\varphi \varepsilon A[x]$, then a valuation $W = \left[W_o, W(x - \alpha) = \gamma\right]$ can be defined on $A[x]$ such that φ has a prescribed value μ. The value γ is uniquely determined.

Proof: Let $\varphi = \beta_n(x - \alpha)^n + \beta_{n-1}(x - \alpha)^{n-1} + \ldots + \beta_1(x - \alpha)$, where $\beta_i \varepsilon A$, then $W\varphi = \min_i \left\{ W_o \beta_i + i\gamma \right\}$. Let the numbers γ_i be defined by $W_o \beta_i + i\gamma_i = \mu$ for $i = 1, 2, \ldots, n$ and γ defined by $\gamma = \max_i \gamma_i$. For this value of γ, $W_o \beta_i + i\gamma \geq \mu$ for $i = 1, 2, \ldots, n$ and the equality holds for at least one value of i. Suppose there were two values γ and γ', with $\gamma > \gamma'$, with the desired property. Since $\varphi(\alpha) = 0$, there exists an $i \neq 0$ such that

$$\mu = W_o \beta_i + i\gamma > W_o\beta_i + i\gamma'.$$

So W defined by γ' would give φ a value $W\varphi < \mu$. Therefore the value $W(x - \alpha)$ is unique.

Lemma 5.5: In an inductive value

$$V_k = \left[V_o, V_1x = \mu_1, \ldots, V_{k-1}\varphi_{k-1} = \mu_{k-1}, V_k\varphi_k = \mu_k\right] \text{ of } R[x]$$

the V_i for $1 \leq i < k$ are the complete, and only, set of approximants to φ_k.

Proof: This follows immediately from [3], theorem 5.3 (Cf. end of §3) if V is defined by

$$V = \left[V_o, V_1x = \mu_1, \ldots, V_{k-1}\varphi_{k-1} = \mu_{k-1}, V\varphi_k = \infty\right].$$

Theorem 5.6: Let W_o on A be an extension of V_o on R, then $V_k = \left[V_o, V_1x = \mu_1, \ldots, V_k\varphi_k = \mu_k\right]$ on $R[x]$ may be extended to $W = \left[W_o, W(x - \alpha) = \gamma\right]$ on $A[x]$, where $\varphi_k(\alpha) = 0$.

Proof: By lemma 5.4 the value $W(x - \alpha) = \gamma$ is uniquely determined from $W\varphi_k = \mu_k$. Now, W defined by this γ reduces to

$$V_\ell' = \left[V_o, V_1x = \mu_1, \ldots, V_{\ell-1}\varphi_{\ell-1} = \mu_{\ell-1}, V_\ell\varphi = \nu\right],$$

where $\ell \leq k$, by theorem 4.7 since $\varphi_k(\alpha) = 0$. But $W\varphi_k = \mu_k$ and, so, $V_\ell'\ \varphi_k = \mu_k = V_k\varphi_k$. This implies $\ell = k$ and $\nu = \mu_k$. That is, the reduction of W is V_k or W is the extension of V_k.

Since theorem 5.2 claims every valuation W may be defined by some α where $\varphi_k(\alpha) = 0$, it is seen that for a given W_o the maximum number of extensions of V_k on $R[x]$ to W on $A[x]$ is the degree of φ_k; and, every extension may be found by the method of theorem 5.6.

Theorem 5.7: Let W_o on A be an extension of V_o on R and let $V = \left[V_o, V_1x = \mu_1, \ldots, V_k\varphi_k = \mu_k, \ldots\right]$ be a limit value of $R[x]$. The value V may be extended to the MacLane value

$$W = \left[W_o, W_1x = \mu_1, W_2(x - \alpha_2) = \gamma_2, \ldots, W_k(x - \alpha_k) = \gamma_k, \ldots\right]$$

on $A[x]$ where:

(1) $\varphi_i(\alpha_i) = 0$ for $i = 1, 2, \ldots$,

(2) γ_i is uniquely determined by the factor $x - \alpha_i$ and the value μ_i of φ_i,

(3) $D_\varphi(x - \alpha_i) = 1$ in $x - \alpha_{i-1}$ for all $i > 1$ (Cf. definition 3.2),

(4) W_i reduces to V_i on $R[x]$ for $i = 1, 2, \ldots$.

Proof: It will be possible to formally construct the sequence of values $\{W_i\}$ if it can be shown that there always exists a factor of φ_i satisfying (3). The construction of W_i with this factor may be accomplished by the method of theorem 5.6. However, it will be necessary to show that W defined by this sequence of values is actually a MacLane value. In order to prove that property (3) may be satisfied it will be shown that, given a value W_k which satisfies (1), (2), and (4) a value W_{k+1} can be defined over W_k also satisfying (1), (2) and (4) and such that $D_\varphi(x - \alpha_{k+1}) = 1$ in $x - \alpha_k$. Then, since W_1 satisfies (1), (2) and (4) it follows by induction that property (3) can be satisfied for all $i > 1$. Let W_k satisfy (1), (2) and (4) and let W_k' be defined by $W_k' = \left[W_o, W_k'(x - \alpha_k) = \gamma\right]$ with $\gamma < \gamma_k$ but such that $V_{k-1}\varphi_k < W_k'\varphi_k = \mu < \mu_k$. Then as $\gamma \to \gamma_k$, $\mu \to \mu_k$. Therefore, if φ_{k+1} is expanded in powers of φ_k, as $\gamma \to \gamma_k$ the value of φ_{k+1} must continuously increase, since proj $V_k > 0$ with respect to φ_{k+1}. So, if φ_{k+1} is expanded in powers of $x - \alpha_k$, the value of φ_{k+1} must also continuously increase as $\gamma \to \gamma_k$. This could only happen if $D_\varphi\varphi_{k+1} \geq 1$ in $x - \alpha_k$. Hence, there exists a factor $x - \alpha_{k+1}$ of φ_{k+1} such that $D_\varphi(x - \alpha_{k+1}) = 1$ in $x - \alpha_k$. Now W_{k+1} may be defined by $W_{k+1} = \left[W_k, W_{k+1}(x - \alpha_{k+1}) = \gamma_{k+1}\right]$ where W_{k+1} satisfies (1), (2), (3) and (4). It only remains to show that W defined by this sequence of values is a MacLane value. Since $D_\varphi(x - \alpha_{k+1}) = 1$ in $x - \alpha_k$, $\gamma_k = W_k(x - \alpha_k) = W_k(x - \alpha_{k+1})$

$$\leq W_o(\alpha_k - \alpha_{k+1}) ;$$

that is, $\gamma_k \leq W_o(\alpha_k - \alpha_{k+1})$. Let W_{k+1}' be defined by

$$W_{k+1}' = \left[W_o, W_{k+1}'(x - \alpha_{k+1}) = \gamma_k\right],$$

then $W_k = W_{k+1}'$; for, let $x - \beta \in A[x]$, then

$$W_k(x - \beta) = \min\{\gamma_k, W_o(\alpha_k - \beta)\} \quad \text{and}$$
$$W'_{k+1}(x - \beta) = \min\{\gamma_k, W_o(\alpha_{k+1} - \beta)\} \quad \text{but}$$
$$W_o(\alpha_{k+1} - \beta) \geq \min\{W_o(\alpha_{k+1} - \alpha_k), W_o(\alpha_k - \beta)\} \geq \min\{\gamma_k, W_o(\alpha_k - \beta)\}$$

Therefore W_k and W'_{k+1} have the same reduction V_k on $R[x]$. Hence, $\gamma_{k+1} > \gamma_k$ in order that W_{k+1} reduce to V_{k+1}. Let

Let φ_k have the expansion

$$\varphi_k = f_n(x - \alpha_k)^n + f_{n-1}(x - \alpha_k)^{n-1} + \dots + f_1(x - \alpha_k) \text{ ,then}$$
$$W_k\varphi_k = \min_i{}_o\{W_o f_i + i\gamma_k\} \quad \text{and}$$
$$W_{k+1}\varphi_k \geq \min_i\{W_o f_i + iW_{k+1}(x - \alpha_k)\} .$$

Let i, which cannot be zero, be chosen such that a minimum term is actually obtained in the second inequality; then

$$W_o f_i + i\gamma_k \geq W_k\varphi_k = W_{k+1}\varphi_k \geq W_o f_i + iW_{k+1}(x - \alpha_k) .$$

Therefore, since $i \neq 0$, $\gamma_k \geq W_{k+1}(x - \alpha_k)$. Now,

$$W_o(\alpha_k - \alpha_{k+1}) \geq \min\{W_{k+1}(x - \alpha_{k+1}), W_{k+1}(x - \alpha_k)\} , \quad \text{but}$$
$$W_{k+1}(x - \alpha_{k+1}) = \gamma_{k+1} > \gamma_k \geq W_{k+1}(x - \alpha_k), \text{ and, so,}$$
$$W_o(\alpha_k - \alpha_{k+1}) = W_{k+1}(x - \alpha_k) \leq \gamma_k .$$

But it is known from above that $W_o(\alpha_k - \alpha_{k+1}) \geq \gamma_k$. Therefore $W_o(\alpha_k - \alpha_{k+1}) = \gamma_k$ and since $\gamma_{k+1} > \gamma_k$ for all $k \geq 1$ the sequence $\{\alpha_i\}$ is pseudo-convergent. The sequence has no pseudo-limit since W reduces to a limit value; otherwise V would be an inductive value ([2], theorem 16.4). There can be no limit for the sequence $\{\alpha_i\}$ in A since V is a finite value; alternatively, every limit of pseudo-convergent sequence is a pseudo-limit. Therefore, W is a MacLane valuation of $A[x]$ satisfying properties (1), (2), (3) and (4).

Because of theorem 5.3 and since for every limit value W of $A[x]$, $D_\varphi(x - \alpha_{k+1}) = 1$ in $x - \alpha_k$, every extension of a limit value V of $R[x]$ to $A[x]$ may be found as in Theorem 5.7.

Bibliography

1. A. Ostrowski, Untersuchen zur arithmetischen Theorie der Körper, Mathematische Zeitschrift, vol. 39 (1934), pp. 269 - 404.

2. S. MacLane, A construction for Absolute Values in Polynomial Rings, Transactions of the American Mathematical Society, vol. 40 (1936), pp. 363 - 395.

3. S. MacLane, A construction for Prime Ideals as Absolute Values of an Algebraic Field, Duke Mathematical Journal, vol. 2 (1936), pp. 492-510.