ON CERTAIN RINGS OF E-VALUED CONTINUOUS FUNCTIONS

by

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Let \( C(X,E) \) denote the set of all continuous functions from a topological space \( X \) into a topological space \( E \). R. Engelking and S. Mrówka [2] proved that for any \( E \)-completely regular space \( X \) [Definition 1.1], there exists a unique \( E \)-compactification \( v_E X \) [Definitions 2.1 and 3.1] with the property that every function \( f \) in \( C(X,E) \) has an extension \( \overline{f} \) in \( C(v_E X,E) \).

It is proved that if \( E \) is a \((*)\)-topological division ring [Definition 5.5] and \( X \) is an \( E \)-completely regular space, then \( v_E X \) is the same as the space of all \( E \)-homomorphisms [Definition 5.3] from \( C(X,E) \) into \( E \). Also, we establish that if \( E \) is an \( H \)-topological ring [Definition 6.1] and \( X, Y \) are \( E \)-compact spaces [Definition 2.1], then \( X \) and \( Y \) are homeomorphic if, and only if, the rings \( C(X,E) \) and \( C(Y,E) \) are \( E \)-isomorphic [Definition 5.3]. Moreover, if \( \tau \) is an \( E \)-isomorphism from \( C(X,E) \) onto \( C(Y,E) \) then \( \overline{\tau}(\tau) \) is the unique homeomorphisms from \( Y \) onto \( X \) with the property that \( \tau(f) = f \circ \overline{\tau}(\tau) \) for all \( f \) in \( C(X,E) \), where \( \tau \) is the identity mapping on \( X \) and \( \overline{\tau} \) is a certain mapping induced by \( \tau \). In particular, the development of the theory of \( C(X,E) \) gives a unified treatment for the cases when \( E \) is the space of all real numbers or the space of all integers.

Finally, for a topological ring \( E \), the bounded subring \( C^*(X,E) \) of \( C(X,E) \) is studied. A function \( f \) in \( C(X,E) \) belongs to \( C^*(X,E) \) if for any \( 0 \)-neighborhood \( U \) in \( E \), there exists
a 0-neighborhood $V$ in $E$ such that $f[X] \cdot V \subset U$ and $V \cdot f[X] \subset U$.

The analogous results for $C^*(X, E)$ follow closely the theory of $C(X, E)$; namely, for any $E^*$-completely regular space $X$ [Definition 9.5], there exists an $E^*$-compactification $\nu_E^*X$ of $X$ such that every function $f$ in $C^*(X, E)$ has an extension $\overline{f}$ in $C^*(\nu_E^*X, E)$; when $E$ is the space of all rationals, real numbers, complex numbers, or the real quaternions, $\nu_E^*X$ is just the space of all $E$-homomorphisms from $C^*(X, E)$ into $E$. This is also valid for a topological ring $E$ which satisfies certain conditions. Also, two $E^*$-compact spaces [Definition 10.1] $X$ and $Y$ are homeomorphic if, and only if, the rings $C^*(X, E)$ and $C^*(Y, E)$ are $E$-isomorphic, where $E$ is any $H^*$-topological ring [Definition 12.8].
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INTRODUCTION

Let \( C(X,E) \) denote some kind of algebraic system of all continuous functions from a topological space \( X \) into a topological space \( E \). During the past twenty years extensive work has been done on \( C(X,R) \), where \( R \) is the space of all real numbers. In 1960, Gillman and Jerison [3] gave a systematic study of the ring \( C(X,R) \) on an arbitrary topological space \( X \). They are concerned with the relations between the algebraic properties of \( C(X,R) \) and the topological properties of the space \( X \). The ring \( C(X,Z) \) (where \( Z \) is the space of all integers) of all integer-valued continuous functions on a topological space \( X \) has been studied by R. S. Pierce [13] and S. Mrówka [10]. Their works paralleled known results in the theory of real-valued continuous functions. Mrówka, for example, showed that [10, theorem 2 and theorem 3]

(a) Every non-zero (ring -) homomorphism \( \varphi \) from \( C(X,Z) \) into \( Z \) can be written in the form (*) \( \varphi(f) = f(p) \) for every \( f \) in \( C(X,Z) \), where \( p \) is a fixed point of \( X \), if and only if \( X \) is \( Z \)-compact.

(b) If \( X \) and \( Y \) are \( Z \)-compact spaces and the rings \( C(X,Z) \) and \( C(Y,Z) \) are isomorphic, then the spaces \( X \) and \( Y \) are homeomorphic.

The object of this thesis is to study the relations between the algebraic properties of the function ring \( C(X,E) \), where \( E \) is a topological ring, and the topological structures of the spaces \( X \) and \( E \). In particular, we give a uniform treatment for the cases when \( E = R, Z, \) the space of all rational numbers,
the space of all complex numbers and the space of all real quaternions.

In Chapter 0, we quote some well-known results which will often be used in the thesis. In Chapter I, we give a survey of the properties of the class of all E-completely regular spaces and the class of all E-compact spaces [2, 4 and 9].

In Chapter II, we deal with the ring \( C(X, E) \), where \( E \) is a topological ring. R. Engelking and S. Mrówka [2] proved that for any E-completely regular space \( X \), there exists a unique E-compactification \( \nu_E X \) of \( X \) such that every function \( f \) in \( C(X, E) \) has an extension \( \overline{f} \) in \( C(\nu_E X, E) \). It is proved that if \( E \) is a (*)-topological division ring [Definition 5.5] and \( X \) is an E-completely regular space, then \( \nu_E X \) is the same as the space of all E-homomorphisms [Definition 5.3] from \( C(X, E) \) into \( E \). Also, we establish that [Theorem 6.7], if \( E \) is an H-topological ring [Definition 6.1], and \( X, Y \) are E-compact spaces [Definition 2.1], then the rings \( C(X, E) \) and \( C(Y, E) \) are E-isomorphic [Definition 5.3], if, and only if \( X, Y \) are homeomorphic. This implies the well known result [3, Theorem 8.3]: Two real compact spaces \( X, Y \) are homeomorphic if and only if \( C(X, R) \) and \( C(Y, R) \) are isomorphic; and [10, Theorem 3]: Two \( Z \)-compact spaces \( X, Y \) are homeomorphic if and only if the rings \( C(X, Z) \) and \( C(Y, Z) \) are isomorphic. We also establish that [Theorem 7.2] if \( E \) is an H-topological ring, \( X, Y \) are E-compact spaces, and \( t \) is an E-isomorphism from \( C(X, E) \) onto \( C(Y, E) \), then \( \overline{t}(\tau) \) is the unique homeomorphism from \( Y \) onto \( X \) such that \( t(f) = f \circ \overline{t}(\tau) \).
for all $f$ in $C(X,E)$, where $\tau$ is the identity mapping on $X$, and $\tau$ is a certain mapping induced by $t$. This generalizes the result of L. E. Pursell [14, Theorem 2.1].

In Chapter III, we consider the bounded subring $C^*(X,E)$ of $C(X,E)$, where $E$ is a topological ring. We say that a function $f$ in $C(X,E)$ belongs to $C^*(X,E)$ if for any zero-neighborhood $U$ in $E$, there exists a zero-neighborhood $V$ in $E$ such that $f[X] \cdot V \subseteq U$ and $V \cdot f[X] \subseteq U$. We obtain analogous results for $C^*(X,E)$ which closely follow the theory of $C(X,E)$; namely, for any $E^*$-completely regular space $X$ [Definition 9.5], there exists an $E^*$-compactification $v^*_EX$ of $X$ such that every function $f$ in $C^*(X,E)$ has an extension $\tau$ in $C^*(v^*_EX,E)$; when $E$ is the space of all rationals, real numbers, complex numbers or the real quaternions, $v^*_EX$ is just the space of all $E$-homomorphism from $C^*(X,E)$ into $E$. This is also valid for a topological ring which satisfies certain conditions. Also, two $E^*$-compact spaces [Definition 10.1] $X$ and $Y$ are homeomorphic if, and only if the rings $C^*(X,E)$ and $C^*(Y,E)$ are $E$-isomorphic, where $E$ is any $H^*$-topological ring [Definition 12.8].
CHAPTER 0

PRELIMINARIES

Unless explicitly stated, all topological spaces in consideration are assumed to be Hausdorff. Thus, the abbreviation "space" always means "Hausdorff topological space". Therefore, when we "construct" a space, we must check that it is a Hausdorff space. We assume also a basic knowledge of a general topology and abstract algebra.

In this chapter, we set forth some conventions in notation and terminology, and record some preliminary results. One should refer to [3] and [6] for those undefined terminologies.

$C(X,Y)$ will denote the set of all continuous functions from the space $X$ into the space $Y$. For each $y$ in $Y$, we shall denote by $\chi$ the constant function $\chi(x) = y$ for every $x$ in $X$; and $Y = \{y : y \in Y\}$.

0.1 Definition. We say that a subset $\mathcal{F}$ of $C(X,Y)$ determines the topology of $X$ if $\{\phi^{-1}[G] : G$ is an open subset in $Y$ and $\phi \in \mathcal{F}\}$ is a subbase for the topology of $X$.

0.2 Lemma. [3, p. 42]

(a) Let $\mathcal{F}$ be a family of mappings from a space $X$ into a space $Y$ that determines the topology of $X$. A mapping $\sigma$ from a space $S$ into $X$ is continuous if and only if the composite function $\phi \circ \sigma$ is continuous for every $\phi$ in $\mathcal{F}$.
(b) A mapping \( \sigma \) from a space into a product \( X = \prod_\alpha X_\alpha \) is continuous if and only if \( \pi_\alpha \circ \sigma \) is continuous for each projection \( \pi_\alpha \).

0.3 Definitions. Suppose that \( F \) is a family of functions such that each member \( f \) of \( F \) is on a topological space \( X \) into a space \( Y_f \). Then there is a natural mapping \( \sigma \) from \( X \) into the product \( \prod \{ Y_f : f \in F \} \) which is defined by mapping a point \( x \) of \( X \) into a member of the product whose \( f \)-coordinate is \( f(x) \), i.e. \( \sigma(x)_f = f(x) \) for each \( f \) in \( F \). We shall call \( \sigma \) the evaluation map.

We say that \( F \) distinguishes (or separates) points of \( X \) if for each pair of distinct points \( x \) and \( y \) of \( X \), there is \( f \) in \( F \) such that \( f(x) \neq f(y) \). The family \( F \) distinguishes (or separates) points and closed sets of \( X \) if for each closed set \( A \) of \( X \) and each point \( x \) of \( X \sim A \) there is \( f \) in \( F \) such that \( f(x) \notin \text{cl}f[A] \).

0.4 Remark. Since the space \( X \) is assumed to be Hausdorff, \( F \) separates points and closed sets of \( X \) implies \( F \) separates points of \( X \).

0.5 Lemma. [6, p. 116] Let \( F \) be a family of continuous functions, each member \( f \) being on a topological space \( X \) to a topological space \( Y_f \). Then:

(a) The evaluation map \( \sigma \) is a continuous function on \( X \) to the product space \( \prod \{ Y_f : f \in F \} \).
(b) The function \( \sigma \) is an open map of \( X \) onto \( \sigma[X] \) if \( \mathcal{F} \) distinguishes points and closed sets of \( X \).

(c) The function \( \sigma \) is one to one if and only if \( \mathcal{F} \) distinguishes points of \( X \).

By virtue of Lemma 0.5 and Remark 0.4, we have:

0.6 Corollary. Let \( \mathcal{F} \) be the family of functions given in Lemma 0.5. Then the evaluation map \( \sigma \) is a homeomorphism from \( X \) onto \( \sigma[X] \) if \( \mathcal{F} \) distinguishes points and closed sets of \( X \).

0.7 Lemma. [3, p. 92] If \( \varphi \) is a continuous function from a space \( S \) into a space \( Y \) whose restriction \( \varphi|_X \) to a dense subset \( X \) is a homeomorphism, then \( \varphi[S \sim X] \subset Y \sim \varphi[X] \).

0.8 Lemma. [3, p. 5] Let \( X \) be a dense subset of the Hausdorff spaces \( S \) and \( T \). If the identity mapping on \( X \) has continuous extensions \( \sigma \) from \( S \) into \( T \) and \( \tau \) from \( T \) into \( S \), then \( \sigma \) is a homeomorphism from \( S \) onto \( T \) and \( \sigma \leftarrow = \tau \).
CHAPTER I

E-COMPLETELY REGULAR SPACES AND E-COMPACT SPACES

§1. E-Completely Regular Spaces.

1.1 Definition. Let \( E \) be a space. A space \( X \) is \( E \)-completely regular if \( \bigcup_{n=1}^{\infty} C(X,E^n) \) separates points and closed sets of \( X \).

1.2 Proposition. Suppose \( X \) is \( E \)-completely regular. Then \( C(X,E) \) separates the points of \( X \).

Proof. Let \( x, y \) be two distinct points of \( X \). Since \( X \) is Hausdorff and \( E \)-completely regular, there is \( f \) in \( C(X,E^n) \) for some \( n \) such that \( f(x) \notin \text{cl}(f(y)) \). Hence \( f(x) \neq f(y) \). Thus for some \( 1 \leq i \leq n \), \((\pi_i \cdot f)(x) \neq (\pi_i \cdot f)(y) \). But \( \pi_i \cdot f \in C(X,E) \). This implies that \( C(X,E) \) separates the points of \( X \).

1.3 Definition. Let \( E \) be a space and \( X \) be a subset of a space \( Y \). We say that \( X \) is \( C(Y,E) \)-embedded if every function in \( C(X,E) \) can be extended to a function in \( C(Y,E) \).

1.4 Theorem. The following statements are equivalent.

(a) \( X \) is \( E \)-completely regular.

(b) \( X \) is homeomorphic with a \( C(E^{C(X,E)},E) \)-embedded subspace of \( E^{C(X,E)} \) under the evaluation map.

(c) \( X \) is homeomorphic with a subset of \( E^\alpha \) for some cardinal number \( \alpha \).
(d) $C(X,E)$ determines the topology of $X$.

Proof. (a) $\rightarrow$ (b). Let $\sigma$ be the evaluation map from $X$ into $E^{C(X,E)}$. We shall show that $\sigma$ is a homeomorphism and $\sigma[X]$ is $C(E^C(X,E),E)$-embedded.

By hypothesis, $X$ is $E$-completely regular, so $C(X,E)$ separates points of $X$ by Proposition 1.2. By Lemma 0.5, $\sigma$ is a one to one continuous map.

$\sigma$ is an open mapping from $X$ onto $\sigma[X]$. For let $G$ be a non-empty open subset of $X$. For each point $p$ in $G$ there is $h$ in $C(X,E^n)$ for some integer $n$ such that $h(p) \notin \text{cl} h[X \sim G]$. Let $h_i$ ($i = 1, 2, \ldots, n$) be the $i$-th coordinate function of $h$; i.e. $h_i = \pi_i \circ h$. Then $h_i \in C(X,E)$ ($i = 1, 2, \ldots, n$). Let $\pi(h_1, h_2, \ldots, h_n)$ be the projection from $E^{C(X,E)}$ into $E \{h_1, h_2, \ldots, h_n\} = E^n$. Then $N = \sigma[X] \cap \pi(h_1, \ldots, h_n)^{-1}[E^n \sim \text{cl} h[X \sim G]]$ is an open set in $\sigma[X]$ containing the point $\sigma(p)$, since $\pi(h_1, \ldots, h_n)(\sigma(p)) = (h_1(p), h_2(p), \ldots, h_n(p)) = h(p) \notin \text{cl} h[X \sim G]$. Furthermore, if $q \in N$ then $q = \sigma(x)$ for some $x$ in $X$ and $\pi(h_1, \ldots, h_n)(q) = \pi(h_1, \ldots, h_n)(\sigma(x)) = (h_1(x), \ldots, h_n(x)) = h(x) \notin \text{cl} h[X \sim G]$.

Hence $x \in G$ and $\sigma(x) = q \in \sigma[G]$. Therefore $\sigma(p) \in N \subset \sigma[G]$, and $\sigma[G]$ is open in $\sigma[X]$. This completes the proof that
\( \sigma \) is a homeomorphism.

Next, for each \( g \) in \( C(\sigma[X],E) \), \( g \cdot \sigma \) is in \( C(X,E) \). Thus \( \pi_{g \cdot \sigma} \in C(E^C(X,E),E) \) and \( \pi_{g \cdot \sigma}|_{\sigma[X]} = g \). Therefore, \( \sigma[X] \) is a \( C(E^C(X,E),E) \)-embedded subset of \( E^C(X,E) \).

(b) \( \rightarrow \) (c). The proof is trivial. (Let \( \alpha \) be the cardinal of \( C(X,E) \)).

(c) \( \rightarrow \) (d). Since \( X \) is homeomorphic with a subset of \( E^\alpha \) for some \( \alpha \), we may regard \( X \) as a subset of \( E^\alpha \). Then the topology of \( X \) is determined by the set \( \mathcal{P} \) of all projections from \( X \) into \( E \). But \( \mathcal{P} \subset C(X,E) \), hence the topology determined by \( \mathcal{P} \) is contained in the topology induced by \( C(X,E) \), and the latter is the smallest topology for \( X \) in which every member of \( C(X,E) \) is continuous. Thus the topology of \( X \) is determined by \( C(X,E) \).

(d) \( \rightarrow \) (a). Let \( F \) be a closed subset of \( X \) and \( p \in X \setminus F \). Since \( C(X,E) \) determines the topology of \( X \), there is a subbasic open set \( U = \bigcap_{i=1}^{n} f_i^{-1}[G_i] \), where \( f_i \in C(X,E) \) and \( G_i \) are open subsets of \( E \), such that \( p \in U \) and \( U \cap F = \emptyset \).

Let \( f \) be the evaluation map from \( X \) into \( E^{\{f_1, \ldots, f_n\}} = E^n \).

Then \( f \in C(X,E^n) \) by Lemma 0.5. \( V = \bigcap_{i=1}^{n} \pi_{f_i}^{-1}[G_i] \) is an open set in \( E^n \) and \( f(p) \in V \). Moreover, \( V \cap f[F] = \emptyset \), for if there exists \( x \in F \) such that \( f(x) \in V \), then \( x \in U \), and...
this contradicts $U \cap F = \emptyset$. Thus $f(p) \notin \text{cl}f[F]$ and $f \in C(X, E^n)$. Therefore, $X$ is $E$-completely regular.

The following two corollaries are immediate consequences of Theorem 1.4.

1.5 Corollary. Any subspace of an $E$-completely regular space is $E$-completely regular.

1.6 Corollary. An arbitrary product of $E$-completely regular spaces is $E$-completely regular.

1.7 Definition. A space $X$ is $O$-dimensional if it has a base consisting of open and closed sets.

1.8 Corollary. A space $X$ is $O$-dimensional if and only if $X$ is $E$-completely regular for any $O$-dimensional space $E$ with $\text{card } E \geq 2$.

Proof. Suppose that $X$ is $O$-dimensional and let $E$ be a space consisting of more than one point. Suppose $F$ is closed in $X$ and $p \in X \sim F$. Since $X$ is $O$-dimensional, there exists an open and closed subset $U$ of $X$ such that $p \in U$ and $U \cap F = \emptyset$. Define $f: X \to E$ by $f[U] = \{a\}$ and $f[X \sim U] = \{b\}$ where $a$, $b$ are fixed distinct points of $E$. Then $f \in C(X, E)$ and $f(p) \notin \text{cl}f[F]$. Thus $X$ is $E$-completely regular.

Conversely, let $E$ be a $O$-dimensional space and $X$ be an $E$-completely regular space. By Theorem 1.4, $C(X, E)$ determines the topology of $X$. Let $\mathcal{A}$ be a base of $E$
consisting of open and closed sets. Then

\[ \mathfrak{B} = \{ f^{-1}[A] : A \in \mathfrak{A}, f \in C(X,E) \} \]

is a base for \( X \) and each element in \( \mathfrak{B} \) is open and closed. Thus \( X \) is 0-dimensional.

1.9 Remarks. Urysohn [16] showed that there exists a countable Hausdorff space \( E \) such that the only real-valued continuous functions are constant functions. Such a space \( E \) is evidently not completely regular. But it is clearly an \( E \)-completely regular space, for the identity mapping on \( E \) is in \( C(E,E) \) and it separates points and closed sets of \( E \).

Suppose \( E_1 \) is \( E_2 \)-completely regular. Then every \( E_1 \)-completely regular space \( X \) is \( E_2 \)-completely regular. For let \( F \) be a closed subset of \( X \) and \( p \in X \setminus F \). Since \( X \) is \( E_1 \)-completely regular, there exists \( f \) in \( C(X,E_1^n) \) for some finite integer \( n \) such that \( f(p) \notin \text{cl}f[F] \). By Corollary 1.6, \( E_1^n \) is \( E_2 \)-completely regular, so there exists \( h \) in \( C(E_1^n,E_2^m) \) for some finite integer \( m \) such that \( h(f(p)) \notin \text{cl}h [\text{cl}f[F]] \supset \text{cl}(h \cdot f)[F] \). But \( h \cdot f \in C(X,E_2^m) \).

Hence \( X \) is \( E_2 \)-completely regular. Thus, we have:

(a) If \( E_1 \) is \( E_2 \)-completely regular then every \( E_1 \)-completely regular space is \( E_2 \)-completely regular.

There are spaces \( E_1 \) and \( E_2 \) such that \( E_1 \) is \( E_2 \)-completely regular but not conversely. For instance, take
12.

Let $E_1 = \{a, b\}$ with discrete topology and $E_2 = [0, 1]$ with usual topology. Then $E_1$ is $E_2$-completely regular but $E_2$ is not $E_1$-completely regular since $C(E_2, E_1)$ consists only of constant functions.

Suppose $E$ is completely regular. Similar to the proof of (a), every $E$-completely regular space is completely regular. But, the class of all $E$-completely regular spaces may be properly contained in the class of all completely regular spaces. For instance, take $E = \{a, b\}$ with discrete topology. Then $\{a, b\}$ is completely regular. The interval $[0,1]$ is completely regular, but it is not $E$-completely regular. For the case $E = \mathbb{R}$ (the space of all real numbers), a space $X$ is $\mathbb{R}$-completely regular if and only if it is completely regular, since $\mathbb{R}$ is completely regular and evidently, every completely regular space is $\mathbb{R}$-completely regular.

J. de Groot showed that there exists a subset $E$ of the Euclidean plane which contains more than one point and has the property that each continuous function of $E$ into itself is either the identity or a constant mapping [2, p. 435].

Consider this set $E$ and let $x_1, x_2 \in E$ and $x_1 \neq x_2$. Let $p = (x_1, x_2)$ in $E^2$ and $F = \{(x_1, x_1), (x_2, x_2)\} \subseteq E^2$. Then $F$ is closed in $E^2$ and $p \notin F$. Suppose there exists a continuous mapping $f$ of $E^2$ into $E$ such that $f(p) \notin \text{cl}_E[F]$. Then the mapping $f$ restricted to the set $A = \{(x, x_2)\} \subseteq E$ is non-constant, otherwise $f(p) = f(x_2, x_2)$,
and this contradicts \( f(p) \notin \text{cl} f[F] \). The function \( f|_A \) can be regarded as a function in \( C(E,E) \), and since it is non-constant, it is the identity, i.e. \( f(x,x_2) = x \) for every \( x \) in \( E \). In particular, \( f(p) = f(x_1,x_2) = x_1 \). Similarly, \( f|_B \) where \( B = \{(x_1,x): x \in E\} \) is non-constant and \( f|_B \in C(E,E) \), thus \( f(x_1,x) = x \) for every \( x \) in \( E \), and in particular, \( f(p) = f(x_1,x_2) = x_2 \). This leads to a contradiction. Therefore, \( C(E^2,E) \) does not separate points and closed sets of \( E^2 \) although \( E^2 \) is \( E \)-completely regular.

Thus, in general, the fact that \( \bigcup_{n=1}^{\infty} C(X,E^n) \) separates points and closed sets of a space \( X \) does not imply that \( C(X,E) \) separates points and closed sets of \( X \).

1.10 Definition. We say that an operation \( \Theta \) on \( C(X,E) \) is defined pointwisely provided that there exists an operation \( \Theta \) on \( E \) such that \((f \Theta g)(p) = f(p) \Theta g(p)\) for every \( f, g \) in \( C(X,E) \) and every point \( p \) in \( E \).

We can extend the above definition for pointwisely defined operations to operations which have more than two arguments as follows.

1.11 Definitions. An algebraic structure \( E \) is a couple \( [E; \{0_0,0_1,\ldots,0_\xi,\ldots\}^\xi<\alpha] \) where the \( 0_\xi \) (\( \xi<\alpha \)) are operations on the set \( E \). The type of \( E \) is the set \( \{n_\xi: \xi<\alpha\} \) where \( n_\xi \) denotes the number of arguments of the operation \( 0_\xi \).
An algebraic structure $E$ will be called a topological algebraic structure provided that a Hausdorff topology for $E$ is given such that all operations on $E$ are continuous.

Given a space $X$ and a topological algebraic structure $E$, $C(X,E)$ becomes an algebraic structure of the same type as $E$ if the operations on $C(X,E)$ are defined pointwisely. Suppose $A_1$ and $A_2$ are two algebraic structures of the same type. Then a homomorphism from $A_1$ into $A_2$ is a mapping from $A_1$ into $A_2$ which preserves all the operations on $A_1$. A one-to-one homomorphism from $A_1$ into $A_2$ is called an isomorphism.

1.12 Theorem. Let $E$ be a space. For any topological space $X$ there exists an $E$-completely regular space $Y$ and a continuous map $\tau$ from $X$ onto $Y$ such that the induced mapping $\tau': g \mapsto g \circ \tau$ is a one-to-one map from $C(Y,E)$ onto $C(X,E)$. Moreover, if $E$ is a topological algebraic structure, then $\tau'$ preserves all pointwisely defined operations, i.e., $\tau'$ is an isomorphism from $C(Y,E)$ onto $C(X,E)$.

Proof. Define $x \equiv x'$ in $X$ to mean that $f(x) = f(x')$ for every $f \in C(X,E)$. Then $\equiv$ is an equivalence relation on $X$. Let $Y$ be the set of all equivalence classes of $X$ under $\equiv$. Define a mapping $\tau$ from $X$ onto $Y$ as follows: $\tau(x)$ is the equivalence class that contains $x$ for each $x$ in $X$. 

With each $f$ in $C(X,E)$, associate a function $g$ in $E^Y$ as follows: $g(y)$ is the common value of $f(x)$ at every point $x$ of $y$. Thus $f = g \cdot \tau$. Let $C'$ denote the family of all such functions $g$; i.e. $g \in C'$ if and only if $g \cdot \tau \in C(X,E)$. Now, endow $Y$ with the weak topology induced by $C'$. Then $C' \subset C(Y,E)$. The continuity of $\tau$ follows from Lemma 0.2. Therefore, for any $g$ in $C(Y,E)$, $g \cdot \tau$ is in $C(X,E)$ and hence $g \in C'$. Thus $C(Y,E) = C'$ and so the topology of $Y$ is determined by $C(Y,E)$. It follows from Theorem 1.4 that $Y$ is $E$-completely regular and it remains to check that $Y$ is Hausdorff.

Given $y, y'$ in $Y$ and $y \neq y'$, there exists $x, x' \in X$ and $x \in y, x' \in y'$. Therefore $x \neq x'$, this means there is an $f$ in $C(X,E)$ such that $f(x) \neq f(x')$. Let $g$ be the function in $C'$ associated with $f$. Then $g(y) = f(x) \neq f(x') = g(y')$. Since $E$ is Hausdorff, let $U, V$ be disjoint open sets in $E$ containing $g(y)$ and $g(y')$ respectively. Then $g^{-1}[U]$ and $g^{-1}[V]$ are disjoint open sets in $Y$ containing $y$ and $y'$ respectively.

The induced mapping $\tau'$ is an onto map: for each $f$ in $C(X,E)$, let $g$ be the function in $C'$ associated with $f$, then $\tau'(g) = g \cdot \tau = f$.

$\tau'$ is one-to-one: let $g_1, g_2 \in C(Y,E)$ and $\tau'(g_1) = \tau'(g_2)$ i.e. $g_1 \cdot \tau = g_2 \cdot \tau$ on $X$. Since $\tau[X] = Y$, $g_1 = g_2$. 
Suppose $E$ is a topological algebraic structure. Then $C(X,E)$ and $C(Y,E)$ are algebraic structures of the same type if the operations on them are defined pointwisely. It can easily be checked that $\tau'$ is an isomorphism from $C(Y,E)$ onto $C(X,E)$.

1.13 Remark. Let $E$ be a topological algebraic structure. As a consequence of the foregoing theorem, algebraic properties that hold for all $C(X,E)$ with $X$ $E$-completely regular, hold just as well for all $C(X,E)$, with $X$ arbitrary. Perhaps, this is a reason for studying $E$-completely regular spaces, for we are interested in the connections between the algebraic structure of $C(X,E)$ and the topological properties of $X$.

§2. E-Compact Spaces.

2.1 Definition. Let $E$ be a given space. A space $X$ is $E$-compact if $X$ is $E$-completely regular and there does not exist any space $Y$ which contains $X$ as a proper dense $C(Y,E)$-embedded subset of $Y$.

2.2 Remark. It is easy to see that $E$-compactness is a topological invariant.

2.3 Lemma. An arbitrary closed subset $F$ of $E^\alpha$ where $\alpha$ is any cardinal number, is $E$-compact.

Proof. Suppose that $F$ is a closed subset of $E^\alpha$. Then $F$ is $E$-completely regular by Corollary 1.5. If $F$ is not $E$-compact then there exists a space $T$ which contains $F$ as
17.

a proper dense $C(T,E)$-embedded subset. Each projection $\pi_i$ from $F$ into $E$ has an extension $\pi_i^*$ in $C(T,E)$. Define a mapping $h : p \to (\pi_i^*(p))_i$ from $T$ into $E^\alpha$. Then $h \in C(T,E^\alpha)$ and $h|_F$ is the identity on $F$. Therefore, $h[T] = h[cl_T F] \subset clh[F] = cl F = F$. Thus $h$ is in $C(T,F)$. By Lemma 0.7, $h[T \sim F] \subset F \sim h[F] = F \sim F = \emptyset$. Hence $T = F$. This contradicts that $F$ is a proper subset of $T$.

2.4 Corollary. Given a space $E$, then $E^\alpha$ is $E$-compact for any cardinal number $\alpha$.

Proof. Since $E^\alpha$ is closed in itself, by Lemma 2.3, $E^\alpha$ is $E$-compact.

2.5 Theorem. Given any space $X$, there exists an $E$-compact space $W$ and a continuous mapping $\theta$ from $X$ into a dense $C(W,E)$-embedded subset of $W$ such that the induced mapping $\theta' : h \to h \circ \theta$ is one-one from $C(W,E)$ onto $C(X,E)$. Furthermore, if $E$ is a topological algebraic structure then $\theta'$ is an isomorphism from $C(W,E)$ onto $C(X,E)$ with respect to all pointwisely defined operations.

Proof. In virtue of Theorem 1.12, there exists an $E$-completely regular space $Y$ and a continuous mapping $\tau$ from $X$ onto $Y$ such that the induced mapping $\tau' = g \to g \circ \tau$ is one-one from $C(Y,E)$ onto $C(X,E)$. Since $Y$ is $E$-completely regular, by Theorem 1.4 (b), $\sigma[Y]$ is a $C(E^{C(Y,E),E})$-embedded
subset of $E^C(Y,E)$ where $\sigma$ is the evaluation map from $Y$ into $E^C(Y,E)$.

For each $h$ in $C(\sigma[Y],E)$, $g = h \circ \sigma \in C(Y,E)$ and $\pi_g \circ \sigma = g = h \circ \sigma$. Thus $h = \pi_g|_{\sigma[Y]}$. Therefore, $C(\sigma[Y],E)$ consists precisely of the restrictions to $\sigma[Y]$ of all the projections $\pi_g$ ($g \in C(Y,E)$). Now a function in $C(\sigma[Y],E)$ may have many continuous extensions to all of $E^C(Y,E)$, but all of these extensions must agree on $\sigma[Y]$ and hence also on its closure $\text{cl}\sigma[Y]$ since $E$ is Hausdorff. Thus the process of extension provided a one-one mapping, namely, $h \rightarrow \pi_h \circ |_{\text{cl}\sigma[Y]}$ from $C(\sigma[Y],E)$ onto $C(\text{cl}\sigma[Y],E)$. Since $\text{cl}\sigma[Y]$ is a closed subset of the product space $E^C(Y,E)$, it is $E$-compact by Lemma 2.3.

Let $W = \text{cl}\sigma[Y]$ and $\Theta = \sigma \circ \tau$. Then $\Theta[X] = (\sigma \circ \tau)[X] = \sigma[Y]$ which is dense and $C(W,E)$-embedded in $W$. Also the mapping $\Theta^\prime: h \rightarrow h \circ \Theta$ is one-one from $C(W,E)$ onto $C(X,E)$.

If $E$ is a topological algebraic structure then it is easy to check that $\Theta^\prime$ is an isomorphism from $C(W,E)$ onto $C(X,E)$.

2.6 Theorem. Given a space $E$. The following statements are equivalent.
(a) $X$ is $E$-compact.
(b) $X$ is homeomorphic with a $C(\mu^C(X,E),E)$-embedded subset of $E^C(X,E)$ under the evaluation map $\sigma$, and $\sigma[X]$
is closed in \( E^C(X, E) \).

(c) \( X \) is homeomorphic to a closed subset of \( E^\alpha \) for some cardinal number \( \alpha \).

Proof. (a) \( \rightarrow \) (b). Suppose \( X \) is \( E \)-compact. Then \( X \) is \( E \)-completely regular, by Theorem 1.4 (b), the evaluation map \( \sigma: X \rightarrow E^C(X, E) \) is a homeomorphism, and \( \sigma[X] \) is \( C(E^C(X, E), E) \)-embedded in \( E^C(X, E) \). If \( \text{cl}\sigma[X] \neq \sigma[X] \) then \( \text{cl}\sigma[X] \) contains \( \sigma[X] \) as a proper dense \( C(\text{cl}\sigma[X], E) \)-embedded subset. This means that \( \sigma[X] \) is not \( E \)-compact. But \( \sigma[X] \) as a homeomorphic image of an \( E \)-compact space \( X' \) is \( E \)-compact. Thus \( \text{cl}\sigma[X] = \sigma[X] \).

(b) \( \rightarrow \) (c). The proof is trivial.

(c) \( \rightarrow \) (a). We know that \( E \)-compactness is a topological invariant and by Lemma 2.3, closed subsets in \( E^\alpha \) are \( E \)-compact, therefore \( X \) is \( E \)-compact if it is homeomorphic with some closed subset in \( E^\alpha \) for some \( \alpha \).

The following two corollaries are immediate consequences of the above theorem.

2.7 Corollary. Closed subsets of an \( E \)-compact space are \( E \)-compact.

2.8 Corollary. Arbitrary products of \( E \)-compact spaces are again \( E \)-compact.

2.9 Remark. Let \( I = [0, 1] \) and \( R \) be the space of all real numbers. Then
(a) The following are equivalent.

(i) X is completely regular.
(ii) X is I-completely regular.
(iii) X is R-completely regular.

(b) [3, p. 160]. A space is (Hewitt) real-compact if and only if it is homeomorphic with a closed subspace of a product of real lines.

(c) [6, p. 118]. A space is compact if and only if it is homeomorphic with a closed subspace of \( I^\alpha \) for some \( \alpha \).

Proof. (a) (i) \( \rightarrow \) (ii). Suppose X is completely regular. Then \( C(X,I) \) separates points and closed sets of X, hence X is I-completely regular.

(ii) \( \rightarrow \) (iii). Since \( C(X,I^n) \subseteq C(X,R^n) \) \( (n = 1,2,\ldots) \), I-complete regularity of X implies R-complete regularity of X.

(iii) \( \rightarrow \) (i). The proof is given in Remark 1.9.

In view of the above remark and Theorem 2.6, we have:

2.10 Proposition. (a) A space X is compact if and only if it is I-compact.

(b) A space X is (Hewitt) real-compact if and only if it is R-compact.
§3. The Existence of the Maximal E-Compactification \( v_E X \) of an E-Completely Regular Space \( X \).

3.1 Definition. By an E-compactification of \( X \) we mean an E-compact space \( Y \) which contains \( X \) as a dense subset.

3.2 Lemma. Let \( X \) be a dense subset of a space \( T \). Then (a) is equivalent to (b).

(a) \( X \) is \( C(T,E) \)-embedded.

(b) \( X \) is \( C(T,Y) \)-embedded for any E-compact space \( Y \).

Proof. (a) \( \rightarrow \) (b). Suppose \( Y \) is E-compact. In view of Theorem 2.6 we may regard \( Y \) as a closed subset of some \( E^\alpha \). For each \( g \) in \( C(X,Y) \), let \( g_i \) be the i-th coordinate function of \( g \), i.e. \( g_i = \tau_i \cdot g \) (\( i \leq \alpha \)) where \( \tau_i \) are projections. Then \( g_i \in C(X,E) \). By hypothesis, it has an extension \( g_i^* \) in \( C(T,E) \). Let \( g^* \) be a function from \( T \) into \( E^\alpha \) whose i-th coordinate function is \( g_i^* \). Then \( g^* \in C(T,E^\alpha) \) and \( g^* \) is an extension of \( g \). In fact \( g^* \in C(T,Y) \), for \( g^*[T] = g^*[clX] \subseteq clg^*[X] = clg[X] \subseteq clY = Y \). Therefore \( g^* \in C(T,Y) \) is the extension of \( g \) for \( g \) in \( C(X,Y) \).

(b) \( \rightarrow \) (a). By Lemma 2.4, \( E \) itself is E-compact. Thus (a) is a special case of (b).

3.3 Theorem. Every E-completely regular space \( X \) has an E-compactification \( v_E X \) such that \( (*) \): \( X \) is \( C(v_E X,Y) \)-embedded for any E-compact space \( Y \). Furthermore, the space
$v^X$ is uniquely determined by $X$ in the sense that if an $E$-compactification $T$ of $X$ has property (\*) then there exists a homeomorphism of $v^X$ onto $T$ that leaves $X$ pointwise fixed.

Proof. Suppose $X$ is $E$-completely regular. By Theorem 1.4 (b), the evaluation map $\sigma$ from $X$ into $E^C(X,E)$ is a homeomorphism and $\sigma[X]$ is $C(E^C(X,E),E)$-embedded. Identify $X$ and $\sigma[X]$, then $X$ is a $C(E^C(X,E),Y)$-embedded subset of $E^C(X,E)$ for any $E$-compact space $Y$ (by Lemma 3.2). Therefore $X$ is $C(\text{cl}X,Y)$-embedded for any $E$-compact space $Y$. $\text{cl}X$ being a closed subset of the $E$-compact space $E^C(X,E)$ is $E$-compact (by Corollary 2.7). Take $v^X = \text{cl}X$. Then $v^X$ is an $E$-compactification of $X$ with property (\*).

Suppose $T$ is any $E$-compactification of $X$ with property (\*). The identity mapping on $X$ has continuous extensions $\theta$ from $v^X$ into $T$ and $\tau$ from $T$ into $v^X$. By Lemma 0.8, $\theta$ is a homeomorphism from $v^X$ onto $T$ and clearly $\theta(p) = p$ for every $p$ in $X$.

3.4 Remark. For any $E$-completely regular space $X$, let $\mathcal{F}(X,E)$ be the collection of all $E$-compactifications of $X$. Define an order $\leq$ on $\mathcal{F}(X,E)$ as follows: let $Y, T \in \mathcal{F}(X,E)$. Then $Y \leq T$ means that there exists a continuous function $f$ from $T$ into $Y$ such that $f|_X$ is the identity on $X$. 
It is easy to check that \((\mathcal{F}(X,E), \leq)\) is a partially ordered set. Furthermore, \(T \leq \nu_X\) for all \(T\) in \(\mathcal{F}(X,E)\), since the identity map on \(X\) has a continuous extension from \(\nu_X\) into \(T\). We shall therefore call \(\nu_X\) the maximal \(E\)-compactification of \(X\).

3.5 Remark. The space \(\nu_X\) is characterized as an \(E\)-compactification of \(X\) in which \(X\) is \(C(\nu_X,E)\)-embedded. Evidently, the mapping \(f \rightarrow f^*\) (where \(f^*\) is the extension of \(f\)) of \(C(X,E)\) onto \(C(\nu_X,E)\) preserves all pointwisely defined operations.

3.6 Corollary. Let \(X\) be an \(E\)-completely regular space. Then \(\text{cl}_E[X]\) is a "model" of \(\nu_X\) where \(\sigma\) is the evaluation map from \(X\) into \(E^C(X,E)\).

Proof. This can be seen in the proof of Theorem 3.3.

3.7 Corollary. Let \(S \subseteq X\) where \(X\) is \(E\)-completely regular. If \(S\) is \(C(X,E)\)-embedded then \(\text{cl}_{\nu_X}S = \nu_ES\).

Proof. Since \(\text{cl}_{\nu_X}S\) is an \(E\)-compactification of \(S\) in which \(S\) is \(C(\text{cl}_{\nu_X}S,E)\)-embedded, \(\text{cl}_{\nu_X}S = \nu_ES\).

3.8 Theorem. An arbitrary intersection of \(E\)-compact subsets of an \(E\)-completely regular space is \(E\)-compact.
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Proof. Let $Y_\alpha$ be a family of E-compact subsets of an E-completely regular space $Y$ and let $X = \cap_\alpha Y_\alpha$. For each $\alpha$, the identity mapping $\tau$ from $X$ into $Y_\alpha$ has a continuous extension from $\nu E X$ into the E-compact space $Y_\alpha$. As $\tau$ can have only one continuous extension from $\nu E X$ into $Y$, these extensions all coincide; hence this common extension $\tau^*$ carries $\nu E X$ into $\cap_\alpha Y_\alpha$ i.e. into $X$. By Lemma 0.7, $\tau^*[\nu E X \sim X] \subset X \sim \tau^*[X] = X \sim X = \emptyset$. Hence $X = \nu E X$. This proves that $X$ is E-compact.

3.9 Theorem. Let $\tau$ be a continuous mapping from an E-compact space $X$ into an E-completely regular space $Y$. Then the total preimage of each E-compact subset of $Y$ is again E-compact.

Proof. Let $F$ be an E-compact subset of $Y$, and let $S = \tau^{-1}[F]$. Because $X$ is E-compact, the identity map $\sigma$ on $S$ has a continuous extension to a mapping $\sigma^*: \nu E S \rightarrow X$. Also, $\tau|_S$ has a continuous extension $(\tau|_S)^*: \nu E S \rightarrow F$.

Since $S$ is dense in $\nu E S$, both these extensions are determined by their values on $S$. Now, $\tau|_S = \tau \cdot \sigma$, and therefore, $(\tau|_S)^* = (\tau \cdot \sigma)^* = \tau \cdot \sigma^*$. But by Lemma 0.7, $\sigma^*[\nu E S \sim S] \subset X \sim S$ so that $(\tau \cdot \sigma^*)[\nu E S \sim S] \subset Y \sim \tau[S] = Y \sim F$, whereas $(\tau|_S)^*[\nu E S \sim S] \subset F$. Therefore, $\nu E S \sim S = \emptyset$ or $S = \nu E S$, so $S$ is E-compact.
3.10 Problem: Let $X$ be an $E$-completely regular but not an $E$-compact space. Find some conditions on $X$ or $E$ so that the smallest element in $(\mathcal{F}(X,E), \leq)$ exists.

3.11 Theorem. Suppose $E_1, E_2$ are two topological spaces such that $E_1$ is $E_2$-compact and $E_2$ is $E_1$-completely regular. Then $\nu_{E_1}X \supset \nu_{E_2}X$ for every $E_1$-completely regular space $X$.

Proof. Since $E_1$ is $E_2$-completely regular, $E_1$-complete regularity of a space $X$ implies $E_2$-complete regularity of $X$. Thus, given any $E_1$-completely regular space $X$, $\nu_{E_2}X$ exists, and since $E_1$ is $E_2$-compact, every function $f$ in $C(X, E_1)$ has a unique extension $f^*$ in $C(\nu_{E_2}X, E_1)$.

Since $E_2$ is $E_1$-completely regular, the $E_2$-compact space $\nu_{E_2}X$ is $E_1$-completely regular, so $\nu_{E_1}(\nu_{E_2}X)$ exists, and it is an $E_1$-compactification of $X$ such that every function $g$ in $C(\nu_{E_2}X, E_1)$ has an extension in $C(\nu_{E_1}(\nu_{E_2}X), E_1)$. But, for any function $f \in C(X, E_1)$, $f = g|_X$ for some $g$ in $C(\nu_{E_2}X, E_1)$, so $f$ has an extension in $C(\nu_{E_1}(\nu_{E_2}X), E_1)$.

Hence $\nu_{E_1}(\nu_{E_2}X) = \nu_{E_1}X$. Therefore, $\nu_{E_2}X \subset \nu_{E_1}X$.

If $R$ is the real numbers and $C$ is the complex numbers then $R$ being a closed subset of $C$ is $C$-compact, and $C$ is $R$-compact since $C = R^2$. The class of all
completely regular spaces are precisely those spaces that are $R$-completely regular or $C$-completely regular. By Theorem 3.11, for any completely regular space $X$, $\nu_R X = \nu_C X$.

§4. Induced Mapping

Let $E$ be a topological space and $X, Y$ be $E$-completely regular spaces. Suppose $\tau$ is a continuous function from $X$ into $Y$. The mapping $\tau'$ from $C(Y,E)$ into $C(X,E)$ defined by $\tau'(g) = g \cdot \tau$ ($g \in C(Y,E)$) is called the induced mapping of $\tau$. The following proposition can be proved easily [3, p. 141].

4.1 Proposition. (a) $\tau'(e) = e$ for every $e \in E$.

(b) $\tau'$ determines the mapping $\tau$ uniquely.

(c) $\tau'$ is onto if and only if $\tau$ is a homeomorphism whose image is $C(Y,E)$-embedded in $Y$.

(d) Suppose $E$ is $O$-dimensional. Then $\tau'$ is one-one (into) if and only if $\tau[X]$ is dense in $Y$.

As a consequence of (c), (d) of Proposition 4.1, we have:

4.2 Corollary. Let $E$ be a $O$-dimensional space, $X$ an $E$-compact space and $Y$ an $E$-completely regular space. Let $\tau: X \to Y$ be continuous and $\tau': C(Y,E) \to C(X,E)$ be its induced mapping. Then $\tau'$ is a one-one, onto map if and only if $\tau$ is a homeomorphism of $X$ onto $Y$. 
Let $E$ be a topological ring and $X$ be an $E$-completely regular space. Then the maximal $E$-compactification $\nu^*_EX$ exists. The main problem in this section is to find some models of $\nu^*_EX$. We are able to show that under certain conditions given on the topological ring $E$, $\nu^*_EX$ is just the set of all "$E$-homomorphisms" from $C(X,E)$ into $E$.

5.1 Definition. A Hausdorff topological space $E$ is said to be a topological ring if $E$ itself is a ring and both addition and multiplication are continuous functions from $E^2$ into $E$.

5.2 Remark. A topological ring $E$ is a topological algebraic structure. The type of $E$ is $\{2,2\}$.

5.3 Definition. Suppose that $E$ is a subring of each of the rings $E_1$ and $E_2$. A ring homomorphism from $E_1$ into $E_2$ is said to be an $E$-homomorphism if its restriction to $E$ is the identity mapping on $E$. A one-one $E$-homomorphism is called an $E$-isomorphism.

Suppose $E$ is a topological ring and $X$ an $E$-completely regular space. Then $X$ is homeomorphic with
σ[X] where σ is the evaluation map from X into $P = E^{C(X,E)}$ (Theorem 1.4 (b)) and $ν_{EX} = cl_pσ[X]$ (Corollary 3.6). For each point x in X, σx is the mapping $(σx)(f) = f(x)$ from $C(X,E)$ into E. Moreover, σx is an E-homomorphism, since $(σx)(e) = e(x) = e, \forall e \in E$ and $(σx)(f+g) = (f+g)(x) = f(x) + g(x) = (σx)(f) + (σx)(g), (σx)(fg) = (fg)(x) = f(x)g(x) = (σx)(f) \cdot (σx)(g), (f,g \in C(X,E))$. Denote the set of all E-homomorphisms from $C(X,E)$ into E by $H(X,E)$. Then $σ[X] \subset H(X,E)$ and since the elements in $H(X,E)$ are mappings from $C(X,E)$ into E, $H(X,E) \subset E^{C(X,E)} = P$. If $cl_pσ[X] = H(X,E)$ then $H(X,E)$ is a model of $ν_{EX}$. This raises the question: under what conditions does $H(X,E) = cl_pσ[X]$? To answer this, first of all, we show the following lemma.

5.4 Lemma. For any topological space X and any topological ring E, $H(X,E)$ is a closed subset of $E^{C(X,E)}$.

Proof. For any $f, g$ in $C(X,E)$, let

$$A(f,g) = \{p \in E^{C(X,E)} : p(f+g) = p(f) + p(g)\},$$

where $p(f)$ is the image of $f$ under the map $p$. $A(f,g) \neq \emptyset$ because $A(f,g) \supset σ[X]$. We shall show that $A(f,g)$ is closed in $E^{C(X,E)}$. For any $q \in E^{C(X,E)} \sim A(f,g), q(f+g) \neq q(f) + q(g)$. Since E is Hausdorff, there exist disjoint open sets $U, V$ in E such that $q(f+g) \in U$ and $q(f)+q(g) \in V$. Since the operation + is a continuous function from $E \times E$
into $E$, there exist neighborhoods $V_1, V_2$ of $q(f), q(g)$, respectively, such that $V_1 + V_2 \subseteq V$. The set

$W = \pi_{f+g}[U] \cap \pi_f[V_1] \cap \pi_g[V_2]$ is a neighborhood of $q$ in $E^C(X, E)$, and for each $p \in W$, we have $p(f+g) \neq p(f) + p(g)$ since $p(f+g) \in U$, $p(f) + p(g) \in V_1 + V_2 \subseteq V$ and $U \cap V = \emptyset$. Therefore $W \cap A(f, g) = \emptyset$. Hence $A(f, g)$ is a closed subset of $E^C(X, E)$.

Similarly, for any $f, g$ in $C(X, E)$, the set

$M(f, g) = \{p \in E^C(X, E): p(fg) = p(f)p(g)\}$ is closed in $E^C(X, E)$.

Also, for each $e$ in $E$, $\{p \in E^C(X, E): p(e) = e\}$ is a closed subset of $E^C(X, E)$. We observe that

$H(X, E) = (\bigcap_{f, g \in C(X, E)} A(f, g) \cap M(f, g)) \cap (\bigcap_{e \in E} \{p \in E^C(X, E): p(e) = e\})$.

Therefore $H(X, E)$ is a closed subset of $E^C(X, E)$.

5.5 Definition. A topological division ring $E$ with unity 1 is said to be a (*)-topological division ring if:

(a) there is a continuous function $x \to x^*$ from $E$ into itself such that $xx^* + yy^* = 0$ implies $x = y = 0$, where 0 is the zero element in $E$.

(b) the function $x \to x^{-1}$ is continuous for $x \neq 0$ in $E$.

$(x^{-1}$ denote the multiplicative inverse of $x$ in $E)$.

5.6 Examples of (*)-Topological Division Rings.
(1) The ring $\mathbb{R}$ of all real numbers with the usual topology.
(2) The ring $\mathbb{Q}$ of all rational numbers with the relative topology induced by $\mathbb{R}$.
(3) The complex numbers with the usual topology.
(4) The real quaternion ring $\mathbb{H}$ with topology so that it is homeomorphic with the product $\mathbb{R}^4$. To be precise, we shall usually identify $\mathbb{H}$ with $\mathbb{R}^4$, the elements 1, $i$, $j$, $k$ of the basis of $\mathbb{H}$ being identified respectively with $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$ of the basis of $\mathbb{R}^4$.

In (1) and (2), $x^* = x$. In (3) and (4), $x^*$ is the conjugate of $x$.

Suppose $E$ is a (*)-topological division ring.

For each $f$ in $C(X,E)$, define a function $f^*$ by $f^*(x) = [f(x)]^*$ for every $x$ in $X$. Then $f^* \in C(X,E)$ since $x \rightarrow x^*$ is continuous on $E$.

If $f \in C(X,E)$ and $0 \not\in f[X]$ then the function $f^{-1}$ defined by $f^{-1}(x) = [f(x)]^{-1}$ for every $x$ in $X$ is well-defined and since $x \rightarrow x^{-1}$ is continuous for $x \neq 0$ in $E$, $f^{-1} \in C(X,E)$.

For any $f$ in $C(X,E)$, denote $Z(f) = \{x \in X : f(x) = 0\}$.

5.7 Lemma. Suppose $X$ is any topological space, and $E$ is a (*)-topological division ring with unity 1. Then (1) For any $f_1, f_2, \ldots, f_n$ in $C(X,E)$ ($n = 1, 2, \ldots$), there
exist functions \( g_1, g_2, \ldots, g_n \) in \( C(X, E) \) such that
\[
Z(f_1g_1 + \ldots + f_ng_n) = \bigcap_{i=1}^{n} Z(f_i).
\]

(2) Suppose \( p \) is a non-zero homomorphism from \( C(X, E) \) into \( E \). Then for \( h \) in \( C(X, E) \), \( p(h) = 0 \) implies \( Z(h) \neq \emptyset \).

Proof. (1) We prove (1) by induction.

For \( n = 1 \), take \( g_1 \) to be the constant function 1.

Let \( n = k \). By the induction hypothesis, we may assume that there are functions \( h_1, \ldots, h_{k-1} \) in \( C(X, E) \) such that
\[
Z(f_1h_1 + \ldots + f_{k-1}h_{k-1}) = \bigcap_{i=1}^{k-1} Z(f_i).
\]
Let \( h = f_1h_1 + \ldots + f_{k-1}h_{k-1} \). Since \( E \) is a \((\ast)\)-topological division ring, \( h(x)h^*(x) + f_n(x)f_n^*(x) = 0 \) if and only if \( h(x) = f_n(x) = 0 \). Thus \( Z(hh^* + f_n f_n^*) = Z(h) \cap Z(f_n) \). Consequently, \( Z(f_1g_1 + \ldots + f_ng_n) = \bigcap_{i=1}^{n} Z(f_i) \) where

\[
\begin{align*}
g_1 &= h_1 h^*, \\
g_2 &= h_2 h^*, \\
&\vdots \\
g_{n-1} &= h_{n-1} h^* \\
g_n &= f_n
\end{align*}
\]

(2) Suppose \( h \in C(X, E) \) and \( Z(h) = \emptyset \). Then \( h^{-1} \in C(X, E) \) and \( hh^{-1} = 1 \). Hence \( h \) cannot belong to any proper ideal of \( C(X, E) \). If \( p \) is a non-zero homomorphism from \( C(X, E) \) into \( E \) then \( \text{Ker} \ p = \{ f \in C(X, E) : p(f) = 0 \} \) is a proper ideal of \( C(X, E) \). Thus \( h \notin \text{Ker} \ p \) or \( p(h) \neq 0 \).

We shall now answer the question: When does \( H(X, E) = \text{cl}_p \sigma[X] \)?
Theorem. Suppose $E$ is a (*)-topological division ring, and $X$ is any topological space. Then $H(X, E) = \text{cl}_P \sigma[X]$ where $P = E^C(X, E)$, and $\sigma$ is the evaluation map from $E$ into $E^C(X, E)$.

Proof. We have seen that $\sigma[X] \subset H(X, E)$ and $H(X, E)$ is a closed subset of $P$ (Lemma 5.4) for any space $X$ and any topological ring $E$.

An arbitrary basic neighborhood of a point $p$ of $H(X, E)$ is a set

$$\bigcap_{k=1}^{n} \{q \in H(X, E) : q(f_k) \in N_k\}$$

where $N_k$ is a neighborhood of $p(f_k)$ in $E$, $(k = 1, 2, \ldots, n)$.

By Lemma 5.7 (1), there exist $g_1, \ldots, g_n$ in $C(X, E)$ such that

$$Z(\sum_{k=1}^{n} (f_k - p(f_k))g_k) = \bigcap_{k=1}^{n} Z(f_k - p(f_k)).$$

Let $h = \sum_{k=1}^{n} (f_k - p(f_k))g_k$. Since $p$ is an $E$-homomorphism, $p(h) = 0$. By (2) of Lemma 5.7, $Z(h) \neq \emptyset$.

Let $x \in Z(h) = \bigcap_{k=1}^{n} Z(f_k - p(f_k))$. Then $f_k(x) = p(f_k)$ $(k = 1, 2, \ldots, n)$. But $(\sigma x)(f_k) = f_k(x)$, hence

$$\sigma x(f_k) = p(f_k) \quad (k = 1, 2, \ldots, n).$$

Therefore, $\sigma x$ is in the
neighborhood of \( p \). Thus \( \mathcal{H}(X, E) = \text{cl}_P \sigma[X] \).

5.9 Theorem. Suppose \( E \) is a \((*)\)-topological division ring with unity, and \( X \) is an \( E \)-completely regular space. Then the set \( \mathcal{H}(X, E) \) of all \( E \)-homomorphisms from \( C(X, E) \) into \( E \) is a model of \( \nu_E X \).

Proof. This is a consequence of Corollary 3.6 and Theorem 5.8.

5.10 Remark. Theorem 5.8 holds for any topological ring \( E \) such that (1), (2) of Lemma 5.7 are valid.

For \( E = \mathbb{Z} \), the ring of all integers, (1) of Lemma 5.7 holds, for given \( f_i \in C(X, \mathbb{Z}) \) \( (i = 1, \ldots, n) \), take \( g_i = f_i \) \( (i = 1, 2, \ldots, n) \). Then \( \mathcal{Z}(\sum_{i=1}^{n} f_i^2) = \bigcap_{i=1}^{n} \mathcal{Z}(f_i) \). By [10, §5,(v)], (2) of Lemma 5.7 holds. Therefore, in view of Remark 5.10, we have:

5.11 Corollary. For any \( \mathbb{Z} \)-completely regular space \( X \), the set \( \mathcal{H}(X, \mathbb{Z}) \) of all \( \mathbb{Z} \)-homomorphisms from \( C(X, \mathbb{Z}) \) into \( \mathbb{Z} \) is a model of \( \nu_\mathbb{Z} X \).

§6. Representation Theorem of \( E \)-Homomorphisms and its Applications.

6.1 Definition. A topological ring \( E \) is said to be an \( \mathcal{H} \)-topological ring if \( \mathcal{H}(X, E) = \nu_E X \) for any \( E \)-completely regular space \( X \), i.e. \( \mathcal{H}(X, E) = \text{cl}_P \sigma[X] \) where \( P = E^{C(X, E)} \) and \( \sigma \) is the evaluation map from \( X \) into \( P \).
We have seen in Section 5 that the ring \( \mathbb{Z} \) and any \((*)\)-topological division rings are H-topological rings.

In this section, we shall assume that \( E \) is always an H-topological ring.

6.2 Theorem. Suppose \( X \) is an \( E \)-completely regular space. Then \( X \) is \( E \)-compact if and only if every \( E \)-homomorphism \( \theta \) from \( C(X,E) \) into \( E \) is fixed in \( X \), i.e., there exists a unique point \( x \) in \( X \) such that \( \theta(f) = f(x) \) for every \( f \) in \( C(X,E) \).

Proof. By Theorem 2.6, the \( E \)-completely regular space \( X \) is \( E \)-compact if and only if \( X \) is homeomorphic with 
\[ \sigma[X] \subset \mathbb{P} = E^{C(X,E)} \] under the evaluation map \( \sigma \), and \( \sigma[X] \) is closed in \( \mathbb{P} \). And since \( E \) is an H-topological ring, \( X \) is \( E \)-compact if, and only if, \( \sigma[X] = H(X,E) \). But \( \sigma[X] = H(X,E) \) means that, given any \( \theta \) in \( H(X,E) \), there exists a point \( x \) in \( X \) such that \( \sigma(x) = \theta \). The point \( x \) in \( X \) is uniquely determined by \( \theta \), since \( \sigma \) is a homeomorphism. Thus, \( \theta(f) = \sigma(x)(f) = f(x) \) for every \( f \) in \( C(X,E) \).

When \( E = \mathbb{Z}, \mathbb{Q} \) or \( \mathbb{R} \), it can be easily checked that every non-zero homomorphism from \( C(X,E) \) into \( E \) is an \( E \)-homomorphism. Therefore, as special cases of Theorem 6.2, we have the following corollary.

6.3 Corollary. (1) [10, Theorem 2]. Every non-zero homomorphism \( \varphi: C(X,\mathbb{Z}) \) into \( \mathbb{Z} \) can be written in the form \( \varphi(f) = f(p_{\varphi}) \) for every \( f \) in \( C(X,\mathbb{Z}) \) where \( p_{\varphi} \) is a fixed point in \( X \),
if and only if $X$ is $N$-compact. (Observe that the space $N = \{1, 2, \ldots, n, \ldots\}$ is homeomorphic with $Z$, hence $N$-compact is equivalent to $Z$-compact.)

(2) [1, theorem 1]. Every non-zero homomorphism $\varphi: C(X, Q) \to Q$ can be written in the form $\varphi(f) = f(p_Q)$ for all $f$ in $C(X, Q)$ where $p_Q$ is a fixed point in $X$, if and only if $X$ is $Q$-compact. (Observe that the space $N$ is closed in $Q$, therefore it is $Q$-compact. This implies every $N$-compact space is $Q$-compact).

(3) [3, p. 142]. A space $X$ is real-compact if and only if, to each non-zero homomorphism $\varphi$ from $C(X, R)$ into $R$, there corresponds a point $x$ in $X$ such that $\varphi(f) = f(x)$ for all $f$ in $C(X, R)$.

6.4 Corollary. Suppose that $X$ is an $E$-completely regular space. Then for each $\theta$ in $H(X, E)$ there exists a unique point $x$ in $\nu_EX$ such that $\theta(f) = \overline{f}(x)$ for all $f$ in $C(X, E)$, where $\overline{f}$ is the extension of $f$ in $C(\nu_EX, E)$.

Proof. Given $\theta$ in $H(X, E)$, we can define a mapping $\overline{\theta}$ from $C(\nu_EX, E)$ into $E$ by $\overline{\theta}(\overline{f}) = \theta(f)$ for every $\overline{f}$ in $C(\nu_EX, E)$. Clearly $\overline{\theta}$ belongs to $H(\nu_EX, E)$. Since $\nu_EX$ is $E$-compact, by Theorem 6.2 there exists a unique point $x$ in $\nu_EX$ such that $\overline{\theta}(\overline{f}) = \overline{f}(x)$ for all $\overline{f}$ in $C(\nu_EX, E)$. Thus, $\theta(f) = \overline{f}(x)$ for all $f$ in $C(X, E)$. 
As an application, we shall examine the problem of determining when a given E-homomorphism from $C(Y,E)$ into $C(X,E)$ is induced by some continuous mapping from $X$ into $Y$.

6.5 Theorem. Let $t$ be an E-homomorphism from $C(Y,E)$ into $C(X,E)$. If $Y$ is E-completely regular, then there exists a unique continuous mapping $\tau$ from $X$ into $\nu_{v^E}Y$ such that $t(g) = \overline{g} \circ \tau$ for every $g$ in $C(Y,E)$, where $\overline{g}$ is the extension of $g$ in $C(\nu_{v^E}Y,E)$.

Proof. For each $x$ in $X$, the mapping $g \mapsto (tg)(x)$ is an E-homomorphism from $C(Y,E)$ into $E$. By Corollary 6.4, there exists a unique point $\tau x$ in $\nu_{v^E}Y$ such that $(tg)(x) = \overline{g}(\tau x)$ for all $g$ in $C(Y,E)$. The mapping $\tau$ from $X$ into $\nu_{v^E}Y$ thus defined, evidently satisfies $tg = \overline{g} \circ \tau$ for all $g$ in $C(Y,E)$. Since $tg$ is continuous and $C(\nu_{v^E}Y,E)$ determines the topology of $\nu_{v^E}Y$, by Lemma 0.2 $\tau$ is continuous. The uniqueness of $\tau$ follows from Proposition 4.1 (b).

6.6 Remark. In Theorem 6.5, if $Y$ is E-compact then $Y = \nu_{v^E}Y$. Therefore $\tau$ is a continuous mapping, from $X$ into $Y$ such that $tg = g \circ \tau$ for all $g$ in $C(Y,E)$.

6.7 Theorem. Suppose that $X$ and $Y$ are E-compact spaces. Then the ring $C(X,E)$ is E-isomorphic with the ring $C(Y,E)$ (i.e., $C(X,E)$ and $C(Y,E)$ are isomorphic under an E-isomorphism) if and only if $X$ and $Y$ are homeomorphic.
Proof. Suppose \( t \) is an E-isomorphism from \( C(Y,E) \) onto \( C(X,E) \). Then there exists continuous functions \( \tau \) from \( X \) into \( Y \) and \( \sigma \) from \( Y \) into \( X \) such that 
\[
(tg)(x) = g(\tau x)
\]
for every \( x \) in \( X \) and \( g \) in \( C(Y,E) \) and 
\[
(t^{-1}f)(y) = f(\sigma y)
\]
for \( y \) in \( Y \) and \( f \) in \( C(X,E) \). Then for each \( y \) in \( Y \), we have: 
\[
g(y) = (t^{-1}(tg))(y) = (tg)(\sigma y) = g(\tau(\sigma y))
\]
for every \( g \) in \( C(Y,E) \). Since \( C(Y,E) \) separates points of \( Y \), \( (\tau \circ \sigma)(y) = y \) for all \( y \) in \( Y \). Similarly, \( (\sigma \circ \tau)(x) = x \) for all \( x \) in \( X \).

\( \tau \) is one-one: for if \( \tau x_1 = \tau x_2 \) for \( x_1, x_2 \) in \( X \) then 
\[
x_1 = \sigma(\tau x_1) = \sigma(\tau x_2) = x_2.
\]

\( \tau \) is onto: given \( y \) in \( Y \) then \( \sigma(y) \) is in \( X \) and \( \tau(\sigma y) = y \).

\( \tau \) has a continuous inverse, namely, the mapping \( \sigma \). Hence \( \tau \) is a homeomorphism from \( X \) onto \( Y \).

Conversely, if \( \tau \) is a homeomorphism from \( X \) onto \( Y \), then the induced mapping \( \tau' : g \to g \circ \tau \) is evidently an E-isomorphism from \( C(Y,E) \) onto \( C(X,E) \).

6.8 Remark. If \( X \) is an E-completely regular space but not an E-compact space, then \( X \) and \( v_E X \) are not homeomorphic. But \( C(X,E) \) is E-isomorphic with \( C(v_E X,E) \) under the mapping 
\[
f \to \overline{f}
\]
where \( f \in C(X,E) \) and \( \overline{f} \) is the extension of \( f \) in \( C(v_E X,E) \). Therefore, the class of all E-compact spaces is a maximal class of spaces for which Theorem 6.7 holds.
6.9 Remark. In view of Theorem 6.5 and Theorem 6.7, for any E-compact space $X$, every E-isomorphism from the ring $C(X,E)$ onto itself is the induced mapping of a unique homeomorphism from $X$ onto itself. The correspondence establishes an anti-isomorphism from the group of all E-automorphisms on $C(X,E)$ onto the group of all homeomorphisms on $X$. To see this, suppose $t_i$ $(i = 1, 2)$ are E-automorphisms on $C(X,E)$ and $\tau_i$ $(i = 1, 2)$ are homeomorphisms on $X$, such that $t_i(g) = g \cdot \tau_i$ $(g \in C(X,E), i = 1, 2)$. Then $(t_1 \circ t_2)(g) = t_1(t_2g) = t_1(g \cdot \tau_2) = (g \cdot \tau_2) \cdot \tau_1 = g \cdot (\tau_2 \cdot \tau_1)$, $(g \in C(X,E))$. Hence the E-automorphism $t_1 \circ t_2$ corresponds with the homeomorphism $\tau_2 \cdot \tau_1$.

§7. Construction of the Homeomorphism From $Y$ onto $X$ Determined by an E-Isomorphism From $C(X,E)$ Onto $C(Y,E)$.

Let $E$ be an H-topological ring and $X$, $Y$ be E-compact spaces. Then according to Theorem 6.5 and Theorem 6.7, for any E-isomorphism $t$ from $C(X,E)$ onto $C(Y,E)$ there is a unique homeomorphism $\tau$ from $Y$ onto $X$ such that $t(f) = f \circ \tau$ for $f$ in $C(X,E)$. In this section, we shall show that $\tau$ is the image of the identity mapping on $X$ under a certain isomorphism induced by $t$.

Let $\alpha$ be any cardinal number. For each $g$ in $C(X,E^\alpha)$, define $\overline{t}(g)$ in $C(Y,E^\alpha)$ such that the $i$-th coordinate function of $\overline{t}(g)$ is $t(\tau_i \cdot g)$ where $\tau_i$ is
the projection from $E^\alpha$ into the $i$-th coordinate space $E$.

Hence $\pi_i \cdot \bar{t}(g) = t(\pi_i \cdot g) = (\pi_i \cdot g) \cdot \tau = \pi_i \cdot (g \cdot \tau)$

$(i \leq \alpha)$. Therefore

7.1 $\bar{t}(g) = g \cdot \tau$

and it is easy to see that $\bar{t}$ is an $E^\alpha$-isomorphism from $C(X,E^\alpha)$ onto $C(Y,E^\alpha)$, and $(\bar{t}g)[Y] = g[X]$.

Since $X$ and $Y$ are $E$-completely regular spaces, we can regard them as subspaces of some product $E^\alpha$. Let $\pi$ be the identity mapping on $X$, hence $\pi_i \cdot \pi = \pi_i|_X$ $(i \leq \alpha)$.

By (7.1),

$\bar{t}(\pi) = \pi \cdot \tau = \tau$

Hence:

7.2 Theorem. Suppose $E$ is an $H$-topological ring and $X, Y$ are $E$-compact spaces. If $t$ is an $E$-isomorphism from $C(X,E)$ onto $C(Y,E)$, then $\bar{t}(\pi)$ is the unique homeomorphism from $Y$ onto $X$ such that $t(f) = f \cdot \bar{t}(\pi)$ for all $f$ in $C(X,E)$.

To end this section, we shall examine when every homomorphism is an $E$-homomorphism and give some consequences of Theorem 6.7 and Theorem 7.2.

7.3 Lemma. Suppose $E$ is a topological ring with a unity $1$ and that the zero homomorphism and the identity mapping are the only homomorphisms from $E$ to $E$. Then every homomorphism $t$ from $C(X,E)$ into $C(Y,E)$ is an $E$-homomorphism.

Proof. For each $y$ in $Y$, the correspondence $e \rightarrow (te)(y)$
is a homomorphism from E into E. By hypothesis, it is either the zero homomorphism or the identity mapping on E. But since 1 is the multiplicative unity in C(X,E), and t is onto, \( t(1) \) is the multiplicative unity in C(Y,E), i.e., \( t(1) = 1 \). Therefore, \( 1 - (t(1))(y) = 1 \). Hence \( e - (te)(y) \) is not the zero homomorphism, so it is the identity mapping on E. We have: \( (te)(y) = e \) for all \( e \) in E. For \( e \) fixed, we have \( (te)(y) = e \) for all \( y \) in Y. Thus \( te = e \) for all \( e \) in E, and \( t \) is an \( E \)-homomorphism.

7.4 Lemma. (a) [3, Theorem 0.22]. The only non-zero homomorphism of R into itself is the identity.

(b) The only non-zero homomorphism of Z (the integers) into itself is the identity.

(c) The only non-zero homomorphism of Q (the rationals) into itself is the identity.

Proof. Similar to the proof of Theorem 0.22 in [3].

The following two theorems are consequences of Theorem 6.7, Lemma 7.3 and Lemma 7.4.

7.5 Theorem. [3, Theorem 8.3]. Two real compact spaces \( X \) and \( Y \) are homeomorphic if and only if \( C(X,R) \) and \( C(Y,R) \) are isomorphic.

7.6 Theorem. [10, Theorem 2]. Two \( Z \)-compact spaces \( X \) and \( Y \) are homeomorphic if and only if \( C(X,Z) \) and \( C(Y,Z) \) are isomorphic.
7.7 Theorem. Two Q-compact spaces X and Y are homeomorphic if and only if \( C(X, Q) \) and \( C(Y, Q) \) are isomorphism.

7.8 Lemma [6, Problem J, p. 103]. Any two open convex subsets of the n-Euclidean space \( \mathbb{R}^n \) are homeomorphic.

Since \( \mathbb{R}^n \) itself is an open convex set, and \( \mathbb{R}^n \) is an R-compact space (Corollary 2.4), any open convex subset of \( \mathbb{R}^n \) is an R-compact space. By Theorem 7.2 and Lemmas 7.3, 7.4, we have:

7.9 Theorem [14, Theorem 2.1]. If X and Y are open convex subsets of \( \mathbb{R}^n \) (n finite) and \( t \) is an isomorphism from \( C(X, \mathbb{R}) \) onto \( C(Y, \mathbb{R}) \) then \( \overline{t}(\pi) \) is the unique homeomorphism from Y onto X with \( t(f) = f(\overline{t}(\pi)) \) for all \( f \) in \( C(X, \mathbb{R}) \), where \( \pi \) is the identity mapping on X and \( \overline{t} \) is the isomorphism from \( C(X, \mathbb{R}^n) \) onto \( C(Y, \mathbb{R}^n) \) defined by \( \pi_i \cdot \overline{t}(g) = t(\pi_i \cdot g) \) (1 \( \leq i \leq n \)) for all \( g \) in \( C(X, \mathbb{R}^n) \).

8.1 Definition. A subset $S$ of a topological ring $E$ is right bounded if for any neighborhood $U$ of $0$, there exists a neighborhood $V$ of $0$ such that $V \cdot S \subseteq U$ where $V \cdot S$ is the set $\{v \cdot s : v \in V, s \in S\}$. Left-boundedness is similarly defined, and a subset of $E$ is bounded if it is both left and right bounded.

8.2 Proposition. Any discrete topological ring $E$ is bounded in itself.

Proof. Since $\{0\}$ is a $0$-neighborhood and $\{0\} \cdot E = E \cdot \{0\} = \{0\}$, which is contained in any $0$-neighborhood, $E$ is bounded in itself.

8.3 Proposition. (a) Any subset of a bounded set in a topological ring $E$ is bounded.

(b) The union of a finite number of bounded subsets of a topological ring $E$ is bounded.

Proof. The proof is trivial.

8.4 Proposition. If $S$ and $T$ are bounded subsets of a topological ring, so are $S+T$ and $S \cdot T$.

Proof. Suppose $U$ is any $0$-neighborhood. Since $(x, y) = x + y$
is continuous on $E^2$, in particular at the point $(0,0)$, there exists a O-neighborhood $W$ such that $W + W \subset U$. Since $S$ and $T$ are bounded, there exists a O-neighborhood $V$ such that $V \cdot S$, $V \cdot T$, $S \cdot V$ and $T \cdot V$ are contained in $W$. Therefore, $V \cdot (S + T) \subset V \cdot S + V \cdot T \subset W + W \subset U$ and $(S + T) \cdot V \subset S \cdot V + T \cdot V \subset W + W \subset U$. Hence $S + T$ is bounded in $E$.

To see that $S \cdot T$ is right-bounded, let $U$ be any O-neighborhood, and $V$ be a O-neighborhood such that $V \cdot T \subset U$. Since $S$ is bounded, there exists a O-neighborhood $W$ such that $W \cdot S \subset V$. Then, $W \cdot (S \cdot T) = (W \cdot S) \cdot T \subset V \cdot T \subset U$. Hence $S \cdot T$ is right-bounded. Left-boundedness of $S \cdot T$ can be proved similarly.

8.5 Proposition. The closure $clS$ of a bounded subset $S$ in a topological ring $E$ is bounded in $E$.

Proof. Given any O-neighborhood $U$, let $V$ be a O-neighborhood such that $V - V \subset U$. Since $S$ is bounded and $(x,y) \rightarrow xy$ is continuous at $(0,0)$, there exists a O-neighborhood $W$ such that $W \cdot S \subset V$ and $W \cdot W \subset V$. For any $y$ in $clS$, $y + W$ is a neighborhood of $y$, hence $(y + W) \cap S \neq \emptyset$ i.e., there exists $w \in W$ and $s \in S$ such that $y + w = s$. For any $p \in W$, $py = ps - pw \in W \cdot S - W \cdot W \subset V - V \subset U$. Hence $W \cdot clS \subset U$. This proves that $clS$ is right-bounded. Left-boundedness of $clS$ can be proved similarly.
8.6 Proposition. Any compact subset \( K \) of a topological ring \( E \) is bounded in \( E \).

Proof. Given any 0-neighborhood \( U \) in \( E \) and any point \( x \) in the compact set \( K \), since \( (x,y) \to xy \) is continuous from \( E \times E \) into \( E \), we can find neighborhoods \( V(x) \) and \( W(x) \) of \( x \) and 0 respectively such that \( V(x) \cdot W(x) \subset U \). A finite number of the \( V \)'s cover \( K \). Let \( N \) be the intersection of the corresponding \( W \)'s. We have \( K \cdot N \subset U \). Hence \( K \) is left-bounded. Similarly, we can show that \( K \) is right-bounded.

Suppose \( E \) is a topological ring. Then \( E^\alpha \) becomes a topological ring provided the operations on \( E^\alpha \) are defined pointwisely. Therefore, a bounded subset of \( E^\alpha \) can be defined as in Definition 8.1.

8.7 Proposition. Suppose \( E \) is a topological ring. A subset \( W \) of \( E^\alpha \) is bounded in \( E^\alpha \) if and only if \( \pi_i[W] \) is a bounded subset of \( E \) for every projection \( \pi_i \) (\( i \leq \alpha \)).

Proof. Suppose that \( W \) is a bounded subset of \( E^\alpha \). Given any 0-neighborhood \( U \) in \( E \), for any fixed projection \( \pi_k \) (\( k \leq \alpha \)), \( \pi_k[U] \) is a 0-neighborhood in \( E^\alpha \). We know that there exists a 0-neighborhood \( V \subset E^\alpha \) such that \( W \cdot V \subset \pi_k[U] \). Hence

\[
\pi_k[W] \cdot \pi_k[V] = \pi_k[W \cdot V] \subset \pi_k[\pi_k[U]] = U.
\]
But $\pi_k[V]$ is a 0-neighborhood in $E$, so $\pi_k[W]$ is left-bounded. Right-boundedness of $\pi_k[W]$ can be proved similarly.

Suppose $W$ is a subset of $E^\alpha$ such that $\pi_i[W]$ is bounded in $E$ for each $i \leq \alpha$. Given any basic 0-neighborhood $\cap_{k=1}^n \pi_{j_k}^k [N_{j_k}]$ in $E^\alpha$ where $N_{j_k}$ are 0-neighborhoods in $E$. Let $N = \cap_{k=1}^n N_{j_k}$, then $N$ is a 0-neighborhood in $E$. Since $\pi_{j_k}^k [W]$ (for $k \leq n$) are bounded in $E$, there exists a 0-neighborhood $V$ in $E$ such that $\pi_{j_k}^k[W] \cdot V \subset N$ for $k = 1, 2, \ldots, n$. Then $W \cdot \cap_{k=1}^n \pi_{j_k}^k[V] \subset \cap_{k=1}^n \pi_{j_k}^k[N] \subset \cap_{k=1}^n \pi_{j_k}^k[N_{j_k}]$.

Therefore, $W$ is left-bounded in $E^\alpha$. Right-boundedness of $W$ can be shown similarly.

§9. $E^*$-Completely Regular Spaces.

9.1 Definition. Let $X$ be any topological space and $E$ a topological ring. A function $f$ in $C(X, E)$ is said to be bounded if $f[X]$ is bounded in $E$.

In view of Proposition 8.4, the set $C^*(X, E)$ of all bounded functions in $C(X, E)$ is a subring of $C(X, E)$.

9.2 Definition. Let $E$ be a topological ring. A space $X$ is said to be $E$-pseudocompact if $C(X, E) = C^*(X, E)$. 
Since (Proposition 8.6) any compact subset of a topological ring $E$ is bounded in $E$, and a continuous image of a compact set is compact, we have:

9.3 Proposition. A compact space is $E$-pseudocompact for any topological ring $E$.

9.4 Remark. It is clear that if a topological ring $E$ is bounded in itself then any space $X$ is $E$-pseudocompact. In particular, by Proposition 8.2 the space $Z$ of all integers is a bounded ring. Thus every space $X$ is $Z$-pseudocompact. Thus, our definition of $Z$-pseudocompactness is different from the "$Z$-pseudocompactness" defined in [13]. This is because boundedness with respect to the norm in a normed ring implies the boundedness as we have defined it, but not conversely. For instance, the space $Z$ is not bounded with respect to the usual norm, but by Proposition 8.2, it is bounded in itself. However, for the space $R$ of real numbers the two notions coincide. Therefore $R$-pseudocompactness coincides with the pseudocompactness as defined in [3]. In fact, to be a bounded continuous function from a space $X$ into $R$ is equivalent to being a bounded continuous function from $X$ into $R$ in the usual sense.

9.5 Definition. Let $E$ be a topological ring. A space $X$ is said to be $E^*$-completely regular if $\bigcup_{n=1}^{\infty} C^*(X, E^n)$ separates points and closed sets in $X$. 
9.6 Lemma. Let $E$ be a topological ring. Then

(a) $f \in C^*(X,E^a)$ if and only if $\pi_i \cdot f \in C^*(X,E)$ for every $i \leq a$.

(b) Suppose $X$ is $E^*$-completely regular, then $C^*(X,E)$ separates the points of $X$.

Proof. (a) This is a consequence of Proposition 8.7.

(b) Given $x \neq y$ in $X$. Since $X$ is Hausdorff, $\{y\}$ is closed. Thus, there is $f$ in $C^*(X,E^n)$ such that $f(x) \neq f(y)$, and hence there exists some $i$ ($1 \leq i \leq n$) with $\left(\pi_i \cdot f\right)(x) \neq \left(\pi_i \cdot f\right)(y)$. By (a) $\pi_i \cdot f \in C^*(X,E)$. Therefore $C^*(X,E)$ separates the points of $X$.

9.7 Definition. Let $E$ be a topological ring and $X$ a subset of a space $Y$. We say that $X$ is $C^*(Y,E)$-embedded if every function $f$ in $C^*(X,E)$ has an extension to a function in $C^*(Y,E)$.

9.8 Lemma. Suppose $E$ is a topological ring and $X$ an $E^*$-completely regular space. Let $P^* = E^0(X,E)$ and $\sigma^*: X \to P^*$ be the evaluation map which is defined by $(\sigma^*(x))_f = f(x)$. Then

(a) $\sigma^*$ is a homeomorphism from $X$ onto $\sigma^*[X]$.

(b) $\sigma^*[X]$ is bounded in $P^*$.

(c) $\text{cl}_P^*\sigma^*[X]$ is bounded in $P^*$, and $\sigma^*[X]$ is $C^*(\text{cl}_P^*\sigma^*[X],E)$-embedded in $\text{cl}_P^*\sigma^*[X]$. 
Proof. (a) By Lemma 0.5 (c), \( \sigma^* \) is one-one since (Lemma 9.6 (b)) \( C^*(X,E) \) separates the points of \( X \).

By Lemma 0.5 (a), \( \sigma^* \) is continuous.

To see that \( \sigma^* \) is an open mapping, let \( G \) be a non-empty open subset of \( X \). For each point \( p \) in \( G \), by \( E^* \)-complete regularity of \( X \), there exists some \( h \) in \( C^*(X,E^n) \) such that \( h(p) \in clh[X \sim G] \). By Lemma 9.6 (a) \( \tau_i \cdot h = f_i \in C^*(X,E) \) for \( i = 1, 2, \ldots, n \). Let \( \pi(f_1, \ldots, f_n) \)

be the projection from \( E^* \) into \( E \) \( = E^n \).

Then \( N = \sigma^*[X] \cap \pi(f_1, \ldots, f_n)[E^n \sim clh[X \sim G]] \) is an open set in \( \sigma^*[X] \) containing the point \( \sigma^*(p) \), since

\[
\pi(f_1, \ldots, f_n)(\sigma^*(p)) = (f_1(p), \ldots, f_n(p)) = h(p) \notin clh[X \sim G].
\]

Furthermore, if \( q \in N \) then \( q = \sigma^*(x) \) for some \( x \) in \( X \), and

\[
\pi(f_1, \ldots, f_n)(q) = \pi(f_1, \ldots, f_n)(\sigma^*(x)) = (f_1(x), \ldots, f_n(x)) = h(x) \notin clh[X \sim G].
\]

Thus \( x \in G \), and hence \( \sigma^*(x) = q \in \sigma^*[G] \). Therefore, \( \sigma^*(p) \in N \subseteq \sigma^*[G] \), so \( \sigma^*[G] \) is open in \( \sigma^*[X] \).

This proves that \( \sigma^* \) is an open mapping. Consequently, \( \sigma^* \) is a homeomorphism.

(b) Since \( \tau_f \cdot \sigma^*[X] = f[X] \) for every \( f \) in \( C^*(X,E) \) and \( f[X] \) is bounded in \( E \), by Proposition 8.7, \( \sigma^*[X] \) is bounded in \( P^* \).

(c) Since \( \sigma^*[X] \) is bounded in \( P^* \), its closure \( cl\sigma^*[X] \) is also bounded in \( P^* \) by Proposition 8.5.
For each $g$ in $C^*(\sigma^*[X], E)$, $g \circ \sigma^* \in C^*(X, E)$. Thus $\pi_{g \circ \sigma^*} \in C^*(P^*, E)$ and $\pi_{g \circ \sigma^*}|_{\sigma^*[X]} = g$. Let $h = \pi_{g \circ \sigma^*}|_{\text{cl}\sigma^*[X]}$. Then $h$ is the continuous extension of $g$ from $\sigma^*[X]$ to its closure $\text{cl}\sigma^*[X]$, and $h[\text{cl}\sigma^*[X]] \subset \text{cl}h[\sigma^*[X]] = \text{cl}g[\sigma^*[X]]$ which being the closure of the bounded set $g[\sigma^*[X]]$ in $E$ is bounded. Thus $h \in C^*(\text{cl}\sigma^*[X], E)$. This proves (c).

9.9 Theorem. Let $E$ be a topological ring and $X$ a topological space. Then the following statements are equivalent.

(a) $X$ is $E^*$-completely regular.
(b) $X$ is homeomorphic to a bounded subset of $E^\alpha$ for some cardinal number $\alpha$.
(c) $C^*(X, E)$ determines the topology of $X$.

Proof. (a) $\rightarrow$ (b). This is a consequence of Lemma 9.8 by taking $\alpha = \text{card} \; C^*(X, E)$.

(b) $\rightarrow$ (c). By the hypothesis of (b), we may regard $X$ as a bounded subset of $E^\alpha$ for some $\alpha$. Hence the topology of $X$ is induced by all the projections $\pi_i \; (i \leq \alpha)$ on $X$. Since $X$ is bounded in $E^\alpha$, by Proposition 8.7, $\pi_i|_X \in C^*(X, E)$ for all $i \leq \alpha$. Hence the topology for $X$ determined by $\{\pi_i|_X: i \leq \alpha\}$ is contained in the topology induced by $C^*(X, E)$, and the latter is the smallest topology for $X$ in which every member of $C^*(X, E)$ is continuous. Thus the topology of $X$ is determined by $C^*(X, E)$. 


(c) - (a). Suppose $A$ is a closed subset of $X$ and $y \in X \sim A$. Since $C^*(X,E)$ determines the topology of $X$, there are functions $f_1, f_2, \ldots, f_n$ in $C^*(X,E)$ such that

$$y \in \bigcap_{i=1}^n f_i^{-1}[U_i] \quad \text{and} \quad \bigcap_{i=1}^n f_i^{-1}[U_i] \cap A = \emptyset$$

for some open sets $U_i (1 \leq i \leq n)$ in $E$. Let $f$ be the evaluation map from $X$ into $E^{f_1, f_2, \ldots, f_n} = E^n$. Since $f_i \in C^*(X,F) \quad (i \leq n)$, by Lemma 9.6 (a), $f \in C^*(X,E^n)$. Since $\pi_{f_i} \circ f(y) = f_i(y) \in U_i$ for $i \leq n$, $f(y) \in \bigcap_{i=1}^n \pi_{f_i}^{-1}[U_i]$. But $\bigcap_{i=1}^n \pi_{f_i}^{-1}[U_i] \cap f[A] = \emptyset$, otherwise there exists $a \in A$ such that $f_i(a) \in U_i$ for $i \leq n$. Hence $a \in \bigcap_{i=1}^n f_i^{-1}[U_i]$. This contradicts $\bigcap_{i=1}^n f_i^{-1}[U_i] \cap A = \emptyset$. Therefore $f(y) \notin \text{cl}f[A]$. This proves that $X$ is $E^*$-completely regular.

The following two corollaries are consequences of Propositions 8.3 (a), 8.7 and Theorem 9.9.

9.10 Corollary. Any subspace of an $E^*$-completely regular space is $E^*$-completely regular.


We have seen that in the study of the ring of continuous functions from a space $X$ into a topological ring $E$, there is no need to deal with spaces that are not
E-completely regular. In the same way, the following theorem says that to study the ring of bounded continuous functions from a space $X$ into a topological ring, we need only to deal with spaces that are $E^*$-completely regular.

9.12 Theorem. Let $E$ be a topological ring. Given any topological space $X$, there exists an $E^*$-completely regular space $Y$ and a continuous function $\tau$ from $X$ onto $Y$ such that the induced mapping $\tau': g \mapsto g \circ \tau$ is an isomorphism from the ring $C^*(Y,E)$ onto the ring $C^*(X,E)$.

Proof. We write $x \equiv x'$ for $x, x'$ in $X$ whenever $f(x) = f(x')$ for all $f$ in $C^*(X,E)$. It is easy to see that $\equiv$ is an equivalence relation. Let $Y$ be the set of all equivalence classes. We define a mapping $\tau$ from $X$ onto $Y$ as follows: $\tau x$ is the equivalence class that contains $x$.

With each $f$ in $C^*(X,E)$, associate a function $g$ in $E^Y$ as follows: $g(y)$ is the common value of $f(x)$ at every point $x \in y$. Thus, $f = g \circ \tau$. Let $C'$ denote the family of all such functions $g$, i.e., $g \in C'$ if and only if $g \circ \tau \in C^*(X,E)$. Now endow $Y$ with the weak topology induced by $C'$. By definition, every function in $C'$ is continuous on $Y$, i.e., $C' \subseteq C(Y,E)$. The continuity of $\tau$ now follows from Lemma 0.2 (a).

For any $g \in C'$, $g \circ \tau \in C^*(X,E)$. Hence $g[Y] = (g \circ \tau)[X]$ is bounded in $E$, so $C' \subseteq C^*(Y,E)$. For any $h$ in $C^*(Y,E)$, since $\tau$ is continuous, $h \circ \tau \in C^*(X,E)$. But
this says that $h \in C'$. Therefore, $C' = C^*(Y,E)$, and it is clear that the mapping $g \rightarrow g \circ \tau$ is an isomorphism from $C^*(Y,E)$ onto $C^*(X,E)$.

It is evident that if $y$ and $y'$ are distinct points of $Y$, then there exists $g \in C'$ such that $g(y) \neq g'(y)$. Since $E$ is a Hausdorff space, this implies that $Y$ is a Hausdorff space. Hence $Y$ is $E^*$-completely regular, by Theorem 9.9 (c).

§10. $E^*$-Compact Spaces and $E^*$-Compactifications of an $E^*$-Completely Regular Space.

10.1 Definition. Let $E$ be a topological ring. By an $E^*$-compact space, we mean an $E^*$-completely regular space $X$ such that there does not exist any other space $Y$ which contains $X$ as a proper dense $C^*(Y,E)$-embedded subset.

10.2 Remark. (1) $E^*$-compactness is a topological invariant.

(2) Any compact, $E^*$-completely regular space is $E^*$-compact.

10.3 Lemma. An arbitrary closed and bounded subset $F$ of $E^\alpha$ where $E$ is a topological ring, is $E^*$-compact.

Proof. By Theorem 9.9, $F$ is $E^*$-completely regular. Suppose $F$ is not $E^*$-compact, and let $\hat{F}$ be a space which contains $F$ as a dense subset such that each $f$ in $C^*(F,E)$ has an extension $\hat{F}$ in $C^*(\hat{F},E)$. Since $F$ is bounded in $E^\alpha$, $\pi_1[F]$ is bounded in $E$ for every projection $\pi_1$. Thus,
\[ \pi_i \in C^*(F,E) \ (i \leq \alpha). \] Therefore \( \pi_i \) has an extension \( \overline{\pi}_i \in C^*(\overline{F},E) \ (i \leq \alpha). \) Define a mapping \( h: x \to (\overline{\pi}_i(x))_{i \leq \alpha} \) from \( \overline{F} \) into \( E^\alpha. \) Then \( h \) is the identity mapping on \( F \) and \( h[\overline{F}] = h[\text{cl}_F[F]] \subseteq \text{cl}_E h[F] = \text{cl}_E \alpha F = F. \) Hence \( h \in C(\overline{F}, F) \) and by Lemma 0.7, \( h[F \sim F] \subset F \sim h[F] = \emptyset. \) Thus \( \overline{F} = F. \) This proves that \( F \) is \( E^* \)-compact.

10.4 Theorem. Let \( X \) be an \( E^* \)-completely regular space. Then the following statements are equivalent.

(a) \( X \) is \( E^* \)-compact.

(b) The evaluation map \( \sigma^* \) from \( X \) into \( E^C(X,E) \) is a homeomorphism, and \( \sigma^*[X] \) is a closed and bounded subset of \( E^C(X,E). \)

(c) \( X \) is homeomorphic with a closed and bounded subset of \( E^\alpha \) for some cardinal number \( \alpha. \)

Proof. (a) \( \to \) (b). Suppose that \( X \) is \( E^* \)-compact. By definition, it is \( E^* \)-completely regular. It follows from Lemma 9.8 that \( X \) is homeomorphic to the bounded subset \( \sigma^*[X] \) of \( P^* = E^C(X,E) \) under the evaluation map \( \sigma^* \) from \( X \) into \( P^*. \) Identify \( X \) with \( \sigma^*[X]. \) Suppose \( \sigma^*[X] \) is not closed in \( P^*. \) Again by Lemma 9.8, \( \text{cl}_{P^*} \sigma^*[X] \) contains \( \sigma^*[X] \) as a proper dense \( C^*(\text{cl}_{P^*} \sigma^*[X],E) \)-embedded subset. This contradicts that \( X \) is \( E^* \)-compact. Thus \( \sigma^*[X] \) is closed and bounded in \( P^*. \)
(b) \to (c). Take \( \alpha = \text{card } C^*(X,E) \).

(c) \to (a). Suppose \( X \) is homeomorphic to some closed and bounded subset \( F \) of \( E^\alpha \) for some \( \alpha \). By Lemma 10.3, \( F \) is \( E^\ast \)-compact. Therefore \( X \) is \( E^\ast \)-compact.

Proposition 8.3, 8.7 together with Theorem 10.4 yield the following corollary.

10.5 Corollary. (a) Any closed subset of an \( E^\ast \)-compact space is \( E^\ast \)-compact.

(b) The union of a finite number of \( E^\ast \)-compact subsets of an \( E^\ast \)-completely regular space is \( E^\ast \)-compact.

(c) An arbitrary product of \( E^\ast \)-compact spaces is \( E^\ast \)-compact.

10.6 Definition. Suppose \( E \) is a topological ring. By an \( E^\ast \)-compactification of a space \( X \), we mean an \( E^\ast \)-compact space \( Y \) which contains \( X \) as a dense subset.

10.7 Lemma. Suppose \( X \) is dense in \( T \) and \( E \) is a topological ring. Then (a) and (b) are equivalent:

(a) \( X \) is \( C^*(T,E) \)-embedded.

(b) \( X \) is \( C(T,Y) \)-embedded for any \( E^\ast \)-compact space \( Y \).

Proof. (a) \( \to \) (b). Suppose \( Y \) is an \( E^\ast \)-compact space. Because of Theorem 10.4, we can regard \( Y \) as a closed and bounded subset of \( E^\alpha \) for some \( \alpha \). For each \( g \) in \( C(X,Y) \), let \( g_i \) be the \( i \)-th coordinate function of \( g \), i.e.,

\( g_i = \pi_i \circ g \) (\( i \leq \alpha \)) where \( \pi_i \) is the projection from \( E^\alpha \).
into the i-th coordinate space \( E \). Since \( g[X] \subseteq Y \) and 
\( Y \) is bounded in \( E^\alpha \), \( g_i \in C^*(X, E) \) by Lemma 9.6 (a). By 
hypothesis, \( g_i \) has an extension \( \overline{g}_i \) in \( C^*(T, E) \). Let \( \overline{g} \) 
be the function from \( T \) into \( E^\alpha \) whose i-th coordinate 
function is \( \overline{g}_i \). Again by Lemma 9.6 (a), \( \overline{g} \in C^*(T, E^\alpha) \), and 
\( \overline{g}[T] = \overline{g}[clX] \subseteq cl\overline{g}[X] = clg[X] \subseteq clY = Y \). Thus, \( \overline{g} \in C(T, Y) \), 
and clearly \( \overline{g} \) is the extension of \( g \).

(b) \rightarrow (a). For any \( g \) in \( C^*(X, E) \), \( g[X] \) is a 
bounded subset in \( E \). By Proposition 8.5, \( clg[X] \) is closed 
and bounded in \( E \) and hence \( clg[X] \) is \( E^* \)-compact by 
Lemma 10.3. Since \( g \in C(X, clg[X]) \), by hypothesis, it has 
an extension \( \overline{g} \) in \( C(T, clg[X]) \). Therefore \( \overline{g} \in C(T, E) \) is 
an extension of \( g \).

10.8 Theorem. Suppose \( E \) is a topological ring. Then every 
\( E^* \)-completely regular space \( X \) has an \( E^* \)-compactification 
\( v^*_EX \) with the following property (*) : if \( Y \) is any \( E^* \) 
compact space then each function \( f \) in \( C(X, Y) \) admits an 
extension \( \overline{f} \) in \( C(v^*_EX, Y) \). Furthermore, the space \( v^*_EX \) is 
quently determined by \( X \) in the sense that if an \( E^* \) 
compactification \( T \) of \( X \) satisfies (*), then there exists 
a homeomorphism from \( v^*_EX \) onto \( T \) which leaves \( X \) pointwise 
fixed.

Proof. Suppose \( X \) is \( E^* \)-completely regular. Then take 
\( v^*_EX = cl_{P^*} \sigma^*[X] \) where \( P^* = E^*C(X, E) \) and \( \sigma^* \) is the
evaluation map from $X$ into $P^*$. By Lemma 9.8 and Lemma 10.3, we can identify $X$ with $\sigma^*[X]$. Then $\nu^*_E X$ is an $E^*$-compactification of $X$, and $X$ is $C^*(\nu^*_E X, E)$-embedded in $\nu^*_E X$. By Lemma 10.7, $\nu^*_E X$ has property ($\ast$).

Suppose that $T$ is an $E^*$-compactification of $X$ with property ($\ast$). Then the identity mapping on $X$ has continuous extensions $\tau$ from $\nu^*_E X$ into $T$ and $\sigma$ from $T$ into $\nu^*_E X$. By Lemma 0.8, $\tau$ is a homeomorphism from $\nu^*_E X$ onto $T$. Clearly $\tau|_X$ is the identity map on $X$.

10.9 Remark. For any $E^*$-completely regular space $X$, the space $\nu^*_E X$ is characterized as an $E^*$-compactification of $X$ in which $X$ is $C^*(\nu^*_E X, E)$-embedded.

10.10 Theorem. Let $S \subset X$, where $X$ is $E^*$-completely regular. If $S$ is $C^*(X, E)$-embedded then $\text{cl}_{\nu^*_E X} S = \nu^*_E S$.

Proof. The set $\text{cl}_{\nu^*_E X} S$ being a closed subset of the $E^*$-compact space $\nu^*_E X$ is $E^*$-compact. Therefore, $\text{cl}_{\nu^*_E X} S$ is an $E^*$-compactification of $S$ in which $S$ is $C^*(\text{cl}_{\nu^*_E X} S, E)$-embedded. Thus, $\text{cl}_{\nu^*_E X} S = \nu^*_E S$.

10.11 Theorem. An arbitrary intersection of $E^*$-compact subspaces of a given $E^*$-completely regular space is $E^*$-compact.
Proof. Let \( \{ Y_\alpha \}_\alpha \) be a family of \( E^* \)-compact subspaces of an \( E^* \)-completely regular space \( Y \), and let \( X = \cap Y_\alpha \). For each \( \alpha \), the identity mapping \( \tau \) from \( X \) into \( Y \) has a continuous extension from \( \nu_{E^*}X \) into the \( E^* \)-compact space \( Y_\alpha \) (Theorem 10.8). As \( \tau \) can have only one continuous extension from \( \nu_{E^*}X \) into \( Y \), these extensions all coincide; hence this common extension \( \overline{\tau} \) carries \( \nu_{E^*}X \) into \( \cap Y_\alpha \), i.e., into \( X \). By Lemma 0.7, \( \overline{\tau}[\nu_{E^*}X \sim X] \subset X \sim \overline{\tau}[X] = \emptyset \).

Hence \( \nu_{E^*}X = X \), so \( X \) is \( E^* \)-compact.

10.12 Theorem. Let \( \tau \) be a continuous function from an \( E^* \)-compact space \( X \) into an \( E^* \)-completely regular space \( Y \). Then the total preimage of each \( E^* \)-compact subset of \( Y \) is \( E^* \)-compact.

Proof. Let \( F \) be an \( E^* \)-compact subset of \( Y \), and let \( S = \tau^{-1}[F] \). Because \( X \) is \( E^* \)-compact, the identity map \( \sigma \) on \( S \) has a continuous extension to a mapping \( \sigma^*: \nu_{E^*}S \rightarrow X \) (Theorem 10.8). Also, \( \tau|_S \) has a continuous extension \( (\tau|_S)^*: \nu_{E^*}S \rightarrow F \). Since \( S \) is dense in \( \nu_{E^*}S \), both these extensions are determined by their values on \( S \). Now, \( \tau|_S = \tau \circ \sigma \), and therefore \( (\tau|_S)^* = (\tau \circ \sigma)^* = \tau \circ \sigma^* \).

By Lemma 0.7, \( \sigma^*[\nu_{E^*}S \sim S] \subset X \sim S \), so that \( (\tau \circ \sigma^*)[\nu_{E^*}S \sim S] \subset Y \sim \tau[S] = Y \sim F \), whereas \( (\tau|_S)^*[\nu_{E^*}S \sim S] \subset F \). Therefore, \( \nu_{E^*}S \sim S = \emptyset \), or \( \nu_{E^*}S = S \). Hence \( S = \tau^{-1}[F] \) is \( E^* \)-compact.
10.13 Theorem. Suppose that $E$ is a topological ring. Then given any space $X$, there exists an $E^*$-compact space $W$, which contains a continuous image of $X$ as a dense, $C^*(W,E)$-embedded subset. Moreover, $C^*(W,E)$ is isomorphic with $C^*(X,E)$.

Proof. By Theorem 9.12, there exists an $E^*$-completely regular space $Y$ and a continuous function $\tau$ from $X$ onto $Y$ such that $\tau^*: g \to g \circ \tau$ is an isomorphism from $C^*(Y,E)$ onto $C^*(X,E)$. Let $W = \overline{\nu Y}$. Then $Y$ is a dense $C^*(W,E)$-embedded subset of $W$, and $h \to h|_Y$ is an isomorphism from $C^*(W,E)$ onto $C^*(Y,E)$. Therefore, $h \to h \circ \tau$ is an isomorphism from $C^*(W,E)$ onto $C^*(X,E)$.

We shall now consider the case where $E$ is the space $R$ of all real numbers.

10.14 Lemma. A space $X$ is compact if, and only if it is $R^*$-compact where $R$ is the topological ring of all real numbers.

Proof. Suppose $X$ is a compact (Hausdorff) space. Then $X$ is normal, and hence it is $R^*$-completely regular. Since a compact subset of a Hausdorff space is always closed, $X$ cannot be a proper dense subset of any other space. Therefore, $X$ is $R^*$-compact.

Conversely, suppose $X$ is $R^*$-compact. By Theorem 10.4, we can regard $X$ as a closed and bounded subset of $R^\alpha$ for some cardinal number $\alpha$. Therefore by Proposition 8.7,
\[ \pi_i[X] \] is bounded in \( R \) for all the projections \( \pi_i \) \( (i \leq \alpha) \).

But boundedness of a subset in the topological ring \( R \) is equivalent to the boundedness of the subset with respect to the usual metric on \( R \). Hence, for each \( i \) \( (i \leq \alpha) \), choose a closed interval \( I_i \) in \( R \) such that \( \pi_i[X] \subseteq I_i \) \( (i \leq \alpha) \). By Tychonoff's product theorem, \( \prod_{i \leq \alpha} I_i \) is compact, and we see that \( X \) is a closed subset of the compact space \( \prod_{i \leq \alpha} I_i \). Therefore, \( X \) is compact.

By Theorem 10.4 and Lemma 10.14, we have:

10.15 Theorem. A subset of \( R^\alpha \) where \( \alpha \) is any cardinal number is compact if and only if it is closed and bounded. (Bounded in the topological ring \( R^\alpha \) in the sense of Definition 8.1).

10.16 Remark. Since every metric space satisfies the first axiom of countability, by [6, p. 92] \( R^\alpha \) is not metrizable for any uncountable cardinal \( \alpha \). Thus, the concept of boundedness with respect to a metric cannot apply to \( R^\alpha \) for uncountable cardinal \( \alpha \). But for the space \( R^n \) where \( n \) is a finite positive integer, we know that being a bounded subset of the topological ring \( R^n \) is equivalent to being a bounded subset with respect to the usual metric on the Euclidean \( n \)-space \( R^n \). Therefore, we have as a special case the classical theorem of Heine-Borel-Lebesgue [6, p. 114]: A subset of Euclidean \( n \)-space is compact if and only if it is closed and bounded.
10.17 Remark. In view of Lemma 10.14, being a compactification of a completely regular space $X$ is equivalent to being an $R^*$-compactification of $X$, and the bounded continuous functions from $X$ into $R$ coincide with the usual bounded continuous real-valued functions. Thus $\mathbb{R}^*X$ is just the Stone-Cech compactification $\beta X$ of the completely regular space $X$. Therefore, Theorem 10.8 is a generalization of the Stone-Cech compactification theorem.

10.18 Corollary. Suppose $E$ is any topological ring. Then any $E^*$-compact space is $E$-compact. The converse is not true.

Proof. It is an immediate consequence of Theorem 2.6 (c) and Theorem 10.4. The converse is not true. For instance, take $X = E = R$. Then $X$ is $R$-compact but not $R^*$-compact i.e., $R$ is (Hewitt) real-compact but not compact.

10.19 Proposition. Suppose $E$ is a topological ring. If $X$ is $E$-compact as well as $E$-pseudocompact then $X$ is $E^*$-compact.

Proof. Since $X$ is $E$-pseudocompact, $C(X,E) = C^*(X,E)$.
Since $X$ is $E$-compact, there does not exist any space $Y$ which contains $X$ as a proper dense subset such that every $f$ in $C(X,E) = C^*(X,E)$ has an extension in $C(Y,E)$. Thus $X$ is $E^*$-compact.

10.20 Problem. Is the converse of Proposition 10.19 true? i.e., Is it true that: $X$ is $E^*$-compact implies $X$ is $E$-pseudocompact?
§11. Embedding $v_E^*X$ as a Subspace of $v_E^{*}X$.

Suppose $E$ is a topological ring and $X$ is an $E^*$-completely regular space. Evidently, $X$ is also an $E$-completely regular space. Therefore both of the spaces $v_E^X$ and $v_E^{*}X$ exists. But the constructions of $v_E^X$ and $v_E^{*}X$ fail to emphasize one essential property of these spaces, namely, that $v_E^X$ can be embedded in $v_E^{*}X$. To derive this result, we observe that for any $f$ in $C(X,E)$, $\text{cl}_E[f[X]]$ is an $E$-compact space (Lemma 2.3) and since $f \in C(X,v_E^*)$, $f$ has an extension $\overline{f}$ in $C(v_E^*,v_E^*)$. By Corollary 10.18 $v_E^{*}X$ is $E$-compact. Thus $\overline{f}[\text{cl}_E[f[X]]]$ is an $E$-compact subspace of $v_E^{*}X$ by Theorem 3.9. By Theorem 3.8, $B = \bigcap_{f \in C(X,E)} \overline{f}[\text{cl}_E[f[X]]]$ is an $E$-compact space. Moreover it is an $E$-compactification of $X$ in which $X$ is $C(B,E)$-embedded. Thus, we have the following theorem:

11.1 Theorem. $v_E^X = \bigcap_{f \in C(X,E)} \overline{f}[\text{cl}_E[f[X]]] \subset v_E^{*}X$, where $\overline{f} \in C(v_E^*,v_E^*)$ is the extension of $f$.

Since $v_E^X$ is a subspace of the $E^*$-completely regular space $v_E^{*}X$, $v_E^X$ is $E^*$-completely regular. Therefore, $v_E^{*}(v_E^X) = D$ exists and it is an $E^*$-compactification of $X$ in which $X$ is $C^*(D,E)$-embedded. Hence $v_E^{*}(v_E^X) = v_E^{*}X$.

Additional insight of the embedding is provided by the-
11.2 Lemma. Suppose $X, Y$ are $E^*$-completely regular spaces and $\theta$ is a continuous function from $X$ into $Y$. Let $\theta'$ be the mapping: $g \rightarrow g \circ \theta$ from $C^*(Y,E)$ into $C^*(X,E)$. Then $\theta'$ is onto implies $\theta$ is a homeomorphism (into).

Proof. Suppose $\theta(x_1) = \theta(x_2)$ for some $x_1, x_2$ in $X$. Then $(\theta'g)(x_1) = g(\theta x_1) = g(\theta x_2) = (\theta'g)(x_2)$ for all $g$ in $C^*(Y,E)$. But $\theta'$ is onto, so $f(x_1) = f(x_2)$ for all $f \in C^*(X,E)$. Since $X$ is $E^*$-completely regular, $C^*(X,E)$ separates points of $X$. Thus $x_1 = x_2$. Hence $\theta$ is one-one.

To see that $\theta\leftarrow: \theta[X] \rightarrow X$ is continuous, we observe that the basic closed sets of $Y$ are of the form $g^{-}[F]$, where $F$ is closed in $E$ and $g \in C^*(Y,E)$, since $C^*(Y,E)$ determines the topology of $Y$. Let $f = g \circ \theta$. Then $f \in C^*(X,E)$ and $f^{-}[F] = \theta^{-}[g^{-}[F] \cap \theta[X]]$, which is closed in $X$. Hence $\theta\leftarrow$ is continuous. Thus, $\theta$ is a homeomorphism.

By Corollary 3.6, $\nu^*_E X = \text{c}l_{P'}\sigma[X]$ where $\sigma$ is the evaluation map from $X$ into $P = E^C(X,E)$. By Theorem 10.8, $\nu^*_E X = \text{c}l_{P^*}\sigma^*[X]$ where $\sigma^*$ is the evaluation map from $X$ into $P^* = E^{C^*}(X,E)$. Let $\tau$ denote the restriction to $\sigma[X]$ of the projection from $P$ onto $P^*$. Clearly, $\tau$ is a continuous mapping from $\sigma[X]$ onto $\sigma^*[X]$ and $\tau \circ \sigma = \sigma^*$. The latter says that $\tau = \sigma^* \circ \sigma\leftarrow$. Since $\sigma\leftarrow$, $\sigma^*$ are...
homeomorphisms, so is τ. Since \( \text{cl}_{p*}\sigma^*[X] \) is \( E^* \)-compact it is \( E \)-compact by Corollary 10.18. By Theorem 1.4 (b) and Lemma 3.2, \( \tau \) has a continuous extension \( \overline{\tau} \) from \( \text{cl}_{p}\sigma[X] \) into \( \text{cl}_{p*}\sigma^*[X] \).

11.3 Theorem. \( \overline{\tau} \) is a homeomorphism from \( \text{cl}_{p}\sigma[X] \) into \( \text{cl}_{p*}\sigma^*[X] \).

Proof. Given \( f \) in \( C^*(\text{cl}_{p}\sigma[X],E) \), we are going to find \( g \) in \( C^*(\text{cl}_{p*}\sigma^*[X],E) \) such that \( \overline{\tau}'(g) = g \circ \overline{\tau} = f \). Since \( \tau \) maps \( \sigma[X] \) homeomorphically onto \( \sigma^*[X] \), we can define a function \( h: \sigma^*[X] \to E \) by \( h(\tau y) = f(y) \), \( y \in \sigma[X] \). Then \( h = f \circ \tau \) is in \( C^*(\sigma^*[X],E) \), since \( \tau \) is continuous and \( f \) is a bounded continuous function. Since \( \sigma^*[X] \) is \( C^*(\text{cl}_{p}\sigma[X],E) \)-embedded, \( h \) has an extension \( g \) in \( C^*(\text{cl}_{p*}\sigma^*[X],E) \). But \( h \circ \tau \in C^*(\sigma[X],E) \), and \( \sigma[X] \) is \( C^*(\text{cl}_{p}\sigma[X],E) \)-embedded, so \( h \circ \tau \) has an extension \( (h \circ \tau)^{-} \) in \( C^*(\text{cl}_{p}\sigma[X],E) \). Since \( h \circ \tau = f|_{\sigma[X]} \), \( (h \circ \tau)^{-} = f \). Since \( (g \circ \overline{\tau})|_{\sigma[X]} = h \circ \tau, g \circ \overline{\tau} = (h \circ \tau)^{-} \). Hence \( \overline{\tau}'(g) = g \circ \overline{\tau} = f \), i.e., \( \overline{\tau}' \) is a mapping from \( C^*(\text{cl}_{p*}\sigma^*[X],E) \) onto \( C^*(\text{cl}_{p}\sigma[X],E) \). By Lemma 11.2, \( \overline{\tau} \) is a homeomorphism.

§12. Characterization of the Space \( X \) by its Function Ring \( C^*(X,E) \).

Suppose \( E \) is a topological ring and \( X \) an \( E^* \)-completely regular space. Then \( \nu^*_E X = \text{cl}_{p*}\sigma^*[X] \), where
σ* is the evaluation map from X into P* = E^*(X,E).

Clearly, for each x ∈ X, σ*(x) is an E-homomorphism from C^*(X,E) into E. Denote by H*(X,E) the set of all E-homomorphisms from C^*(X,E) into E. Then σ*[X] ⊂ H*(X,E) ⊂ P*.

12.1 Lemma. H*(X,E) is closed in P*.

Proof. For any fixed f and g in C^*(X,E), let

\[ A(f,g) = \{ p ∈ P^* : p(f) + p(g) = p(f+g) \} \]

A(f,g) ≠ ∅ because A(f,g) ⊃ σ*[X]. We shall show that A(f,g) is closed in P*.

Suppose q ∉ A(f,g). Then q(f) + q(g) ≠ q(f+g). Since E is Hausdorff, there exist disjoint neighborhoods K and W of q(f) + q(g) and q(f+g) respectively. Since \( (x,y) \mapsto x+y \) is a continuous mapping from EXE into E, there exist neighborhoods U of q(f) and V of q(g) such that U + V ⊂ K. Let \( π_i \) be the projection from P* into the i-th coordinate space E. Then the set \( π_f^{-1}[U] \cap π_g^{-1}[V] \cap π_{f+g}^{-1}[W] \) is a neighborhood of q, and it is disjoint from A(f,g). Indeed, for any p in \( π_f^{-1}[U] \cap π_g^{-1}[V] \cap π_{f+g}^{-1}[W] \), \( p(f) + p(g) ∈ U + V ⊂ K \) and \( p(f+g) ∈ W \). But \( K \cap W = ∅ \), so \( p(f) + p(g) ≠ p(f+g) \) i.e., \( p ∉ A(f,g) \). Therefore, A(f,g) is a closed subset of P*.

Similarly, we can prove that the set \( M(f,g) = \{ p ∈ P^* : p(f)p(g) = p(fg) \} \) is closed in P*. Also, for each e ∈ E, the set \( π_e^{-1}(e) = \{ p ∈ P^* : p(e) = e \} \) is closed.
We observe that
\[ H^*(X,E) = (\cap \{A(f,g) \cap M(f,g) \colon f,g \in C^*(X,E)\}) \cap \{ \cap \pi_{\leq e} \}. \]

Therefore, \( H^*(X,E) \) being an intersection of closed sets is closed.

As a sequel, \( \text{cl}_p \sigma[X] \subset H^*(X,E) \). It is natural to ask: when does \( \text{cl}_p \sigma[X] = H^*(X,E) \)?

First, we shall consider some special cases; namely, when \( E \) is the space \( R \) of all real numbers, the space \( Q \) of all rationals, the space \( C \) of all complex numbers or the space of all real quaternions. In the first two cases, for each \( x \) in \( E \), let \( x^* = x \). In the last two cases, for each \( x \) in \( E \), let \( x^* \) denote the conjugate of \( x \) in \( E \); and \( \|x\| = (xx^*)^{1/2} \) is the usual norm on \( E \). Then \( x - x^* \) is a homeomorphism from \( E \) into itself. Furthermore, it carries a bounded subset of \( E \) into a bounded set. To see this, let \( B \) be a bounded subset in \( E \) and \( U \) be any \( 0 \)-neighborhood in \( E \). Then the set \( U^* = \{u^* : u \in U\} \) is a \( 0 \)-neighborhood. There exists a \( 0 \)-neighborhood \( V \) in \( E \) such that \( B \cdot V \subset U^* \) and \( V \cdot B \subset U^* \). Then, we have \( B^* \cdot V^* \cup V^* \cdot B^* \subset (U^*)^* = U \). Hence \( B^* = \{b^* : b \in B\} \) is bounded, since \( V^* \) is a \( 0 \)-neighborhood. Therefore, we have the following lemma.

12.2 Lemma. For each \( f \) in \( C^*(X,E) \), the function \( f^* \) defined by \( f^*(x) = f(x)^* \) is in \( C^*(X,E) \).
We shall call \( f^* \) the conjugate function of \( f \).

12.3 Lemma. Let \( U \) be any \( 0 \)-neighborhood. Then \((E \sim U)^{-1} = \{ b^{-1} \in E \colon b \not\in U \}\) is bounded in \( E \).

Proof. Let \( \{ x \in E \colon \|x\| < \varepsilon \} \subseteq U \). For any \( b \not\in U \), \( \|b\| > \varepsilon \).

Hence \( \|b^{-1}\| = \frac{1}{\|b\|} \leq \frac{1}{\varepsilon} \). We observe that a set which is bounded with respect to the norm implies it is bounded in the topological ring. Therefore, \((E \sim U)^{-1}\) is bounded.

12.4 Theorem. Let \( E \) be any one of the following spaces: \( Q, R, C \) or the real quaternions. For any \( E^* \)-completely regular space \( X \), \( \text{cl}_{P^*E^*}[X] = H^*(X,E) \).

Proof. The basic neighborhood of a point \( p \) in \( H^*(X,E) \) is of the form

\[
\bigcap_{k=1}^{n} \{ q \in H^*(X,E) : \|q(f_k) - p(f_k)\| < \varepsilon \},
\]

where \( \varepsilon > 0 \) and \( f_k \in C^*(X,E) \).

Since \( f_k - p(f_k) \in C^*(X,E) \), its conjugate function \((f_k - p(f_k))^* \in C^*(X,E) \). Let \( h = \sum_{k=1}^{n} (f_k - p(f_k))(f_k - p(f_k))^* \).

Clearly, \( h \in C^*(X,E) \). Since \( p \) is an \( E \)-homomorphism, \( p(h) = 0 \). Thus \( h \in \text{Ker} \ p \). But \( p \) is not a zero-homomorphism, so \( \text{Ker} \ p \) is a proper ideal of \( C^*(X,E) \). We claim that \( h[X] \cap U \neq \emptyset \), where \( U = \{ b \in E : \|b\| < \varepsilon^2 \} \). For otherwise, \( h[X] \subseteq E \sim U \), and since \( b \sim b^{-1} (b \neq 0) \) is continuous on
E, the function \( h^{-1} \) defined by \( h^{-1}(x) = (h(x))^{-1} \), \((x \in X)\) is continuous. By Lemma 12.3, \( h^{-1} \in C^*(X,E) \). Then \( h^{-1} \cdot h \in \text{Ker } p \) and \( h^{-1} \cdot h = \frac{1}{h^2} \), which contradicts that \( \text{Ker } p \subseteq C^*(X,E) \). Hence \( h[X] \cap U \neq \emptyset \). There exists \( x \in X \) such that \( h(x) \in U \), i.e.,

\[
\|h(x)\| = \left\| \sum_{k=1}^{n} (f_k(x) - p(f_k))(f_k(x) - p(f_k))^* \right\| 
\]

\[
= \sum_{k=1}^{n} \|f_k(x) - p(f_k)\|^2 < \epsilon^2
\]

Thus, \( \|f_k(x) - p(f_k)\|^2 < \epsilon^2 \), or \( \|f_k(x) - p(f_k)\| < \epsilon \) for \( k = 1, 2, \ldots, n \). But \( \sigma^*(x)(f_k) = f_k(x) \), so \( \|\sigma^*(x)(f_k) - p(f_k)\| < \epsilon \) for \( k = 1, 2, \ldots, n \). Therefore, \( \sigma^*(x) \) is in the given basic neighborhood of \( p \) in \( H^*(X,E) \). Hence \( \text{cl}_{p*\sigma^*}[X] = H^*(X,E) \).

12.5 Definition. \( E \) is a normed ring if

(1) \( E \) is a ring.
(2) \( E \) is a normed space.

12.6 Theorem. Suppose that \( E \) is a normed division ring with unity \( 1 \) and \( E \) has the following properties:

(a) \( b - b^{-1} \) is continuous for \( b \neq 0 \) in \( E \).
(b) \( \|a \cdot b\| = \|a\| \cdot \|b\| \), \((a, b \in E)\). (observe that \( b \rightarrow \|1\| = 1 \))
(c) \( \| \sum_{i=1}^{n} b_i \|^2 \geq \|b_j^2\| \) \( j = 1, 2, \ldots, n \). \((b_i \in E)\).

Then \( \text{cl}_{p*\sigma^*}[X] = H^*(X,E) \) for any \( E^* \)-completely regular
Proof. The basic neighborhood of a point \( p \) in \( H^*(X,\mathcal{E}) \) is of the form,

\[
\bigcap_{k=1}^{n} \{ q \in H^*(X,\mathcal{E}) : \| q(f_k) - p(f_k) \| < \varepsilon \},
\]

where \( \varepsilon > 0 \) and \( f_k \in C^*(X,\mathcal{E}) \). Denote \( h = \sum_{k=1}^{n} (f_k - p(f_k))^2 \).

Clearly \( h \in C^*(X,\mathcal{E}) \). As \( p \) is an \( \mathcal{E} \)-homomorphism, \( p(h) = 0 \), so \( h \in \text{Ker } p \). But \( p \) is not a \( \mathcal{C} \)-homomorphism, so \( \text{Ker } p \subsetneq C^*(X,\mathcal{E}) \).

We claim that \( h[X] \cap U \neq \emptyset \), where

\[
U = \{ b \in \mathcal{E} : \| b \| < \varepsilon^2 \}.
\]

For otherwise, \( h[X] \subset \mathcal{E} \sim U \). We show that \( (\mathcal{E} \sim U)^{-1} = \{ b^{-1} : b \not\in U \} \) is bounded. Given any \( b \not\in U \), \( \| b \| > \varepsilon^2 \), and \( 1 = \| 1 \| = \| b b^{-1} \| = \| b \| \| b^{-1} \| \), so \( \| b^{-1} \| = \frac{1}{\| b \|} \leq \frac{1}{\varepsilon^2} \). Therefore \( (\mathcal{E} \sim U)^{-1} \) is bounded with respect to the norm, and hence it is bounded. Define a function \( h^{-1} \) as follows: \( h^{-1}(x) = h(x)^{-1} \), \( x \in X \). Then \( h^{-1} \in C^*(X,\mathcal{E}) \), and \( h \cdot h^{-1} = 1 \in \text{Ker } p \), which contradicts that \( \text{Ker } p \subsetneq C^*(X,\mathcal{E}) \). Hence \( h[X] \cap U \neq \emptyset \). There exists \( x \in X \) such that \( h(x) \in U \), i.e., \( \| h(x) \| = \| \sum_{k=1}^{n} (f_k(x) - p(f_k))^2 \| < \varepsilon^2 \).

By (b) and (c) \( \| f_k(x) - p(f_k) \|^2 = \| (f_k(x) - p(f_k))^2 \| < \varepsilon^2 \). Hence \( \| f_k(x) - p(f_k) \| < \varepsilon \), \( k = 1, 2, \ldots, n \). But...
\[ \sigma^*(x)(f_k) = f_k(x), \text{ so } \| \sigma^*(x)(f_k) - p(f_k) \| < \varepsilon \text{ for } k = 1, 2, \ldots, n. \] Therefore, \( \sigma^*(x) \) belongs to the basic neighborhood of \( p \) in \( H^*(X, E) \). Hence, \( \text{cl}_{p^*}\sigma^*[X] = H^*(X, E) \).

**12.7 Remark.** If the condition (b) is replaced by (b'): \( \|a^n\| = \|a\|^n \) for all integers \( n, (a \in E) \), then Theorem 12.6 still holds. Because (b') implies \( \|b^{-1}\| = \frac{1}{\|b\|} \) for \( b \neq 0 \), so \( (E \sim U)^{-1} \) is bounded, where \( U = \{ b \in E : \|b\| < \varepsilon^2 \} \). (b') also implies \( \|a^2\| = \|a\|^2 \) for all \( a \) in \( E \). Therefore, the proof given in 12.6 is applicable.

**12.8 Definition.** A topological ring \( E \) is said to be an \( H^* \)-topological ring if \( H^*(X, E) = \text{cl}_{p^*}\sigma^*[X] \) for any \( E^* \)-completely regular space \( X \).

In the rest of this section, \( E \) will denote an \( H^* \)-topological ring.

**12.9 Theorem.** Suppose \( X \) is an \( E^* \)-completely regular space. Then \( X \) is \( E^* \)-compact if, and only if, every \( E \)-homomorphism \( \Phi \) from \( C^*(X, E) \) into \( E \) can be written in the form \( \Phi(f) = f(x) \) for every \( f \) in \( C^*(X, E) \), where \( x \) is a unique fixed point in \( X \).

**Proof.** By Theorem 10.4, the space \( X \) is \( E^* \)-compact if, and only if, \( X \) is homeomorphic with \( \sigma^*[X] \subset P^* = E^*(X, E) \) under the evaluation map \( \sigma^* \), and \( \sigma^*[X] \) is closed and
bounded in $P^*$. And since $E$ is an $H^*$-topological ring, $X$ is $E^*$-compact if, and only if, $\sigma^*[X] = H^*(X,E)$. But $\sigma^*[X] = H^*(X,E)$ means that, given any $\theta$ in $H^*(X,E)$, there exists a point $x$ in $X$, such that $\sigma^*(x) = \theta$. The point $x$ is uniquely determined by $\theta$, since $\sigma^*$ is a homeomorphism. Thus, $\theta(f) = (\sigma^*x)(f) = f(x)$ for every $f$ in $C^*(X,E)$.

12.10 Corollary. Suppose that $X$ is an $E^*$-completely regular space. Then, for each $\theta$ in $H^*(X,E)$, there exists a unique point $x$ in $\nu^*_EX$ such that $\theta(f) = \overline{f}(x)$ for all $f$ in $C^*(X,E)$, where $\overline{f}$ is the extension of $f$ in $C^*(\nu^*_EX,E)$.

Proof. Given $\theta$ in $H^*(X,E)$, we can define a mapping $\overline{\theta}$ from $C^*(\nu^*_EX,E)$ into $E$ by $\overline{\theta} (\overline{f}) = \theta(f)$ for every $\overline{f}$ in $C^*(\nu^*_EX,E)$. Clearly, $\overline{\theta}$ belongs to $H^*(\nu^*_EX,E)$. Since $\nu^*_EX$ is $E^*$-compact, by Theorem 12.9, there exists a unique point $x$ in $\nu^*_EX$ such that $\overline{\theta} (\overline{f}) = \overline{f}(x)$ for all $\overline{f}$ in $C^*(\nu^*_EX,E)$. Thus, $\theta(f) = \overline{f}(x)$ for all $f$ in $C^*(X,E)$.

12.11 Theorem. Let $t$ be an $E$-homomorphism from $C^*(Y,E)$ into $C^*(X,E)$. If $Y$ is $E^*$-completely regular, then there exists a unique continuous mapping $\tau$ from $X$ into $\nu^*_EY$ such that $t(g) = \overline{g} \circ \tau$ for every $g$ in $C^*(Y,E)$, where $\overline{g}$ is the extension of $g$ in $C^*(\nu^*_EY,E)$.

Proof. For each $x$ in $X$, the mapping $g \mapsto (tg)(x)$ is an $E$-homomorphism from $C^*(Y,E)$ into $E$. By Corollary 12.10, there exists a unique point $\tau x$ in $\nu^*_EY$ such that
(tg)(x) = \overline{g}(\tau x) \text{ for all } g \text{ in } C^*(Y,E). \text{ The mapping } \tau \text{ from } X \text{ into } v_E^Y \text{ thus defined, evidently satisfies} \\
tg = \overline{g} \circ \tau \text{ for all } g \text{ in } C^*(Y,E). \text{ Since } tg \text{ is continuous,} \\
and C^*(v_E^Y,E) \text{ determines the topology of } v_E^Y, \text{ by Lemma 0.2,} \\
\tau \text{ is continuous. The uniqueness of } \tau \text{ follows from the} \\
fact that } C^*(Y,E) \text{ separates the points of } Y. \\

12.12 \text{ Remark. In Theorem 12.11, if } Y \text{ is } E^*-\text{compact, then} \\
v_E^*Y = Y. \text{ Therefore, } \tau \text{ is a continuous mapping from } X \text{ into} \\
Y \text{ such that } tg = g \circ \tau \text{ for all } g \text{ in } C^*(Y,E). \\

12.13 \text{ Theorem. Suppose that } X \text{ and } Y \text{ are } E^*-\text{compact spaces.} \\
\text{Then the ring } C^*(X,E) \text{ is } E^*\text{-isomorphic with the ring } C^*(Y,E) \\
(\text{i.e., } C^*(X,E) \text{ and } C^*(Y,E) \text{ are isomorphic under an } E^*\text{-isomorphism}) \text{ if, and only if,} \\
X \text{ and } Y \text{ are homeomorphic.} \\

\text{Proof. Suppose } \tau \text{ is an } E^*\text{-isomorphism from } C^*(Y,E) \text{ onto} \\
C^*(X,E). \text{ Then, there exist continuous functions } \tau \text{ from } X \text{ into } Y, \text{ and } \sigma \text{ from } Y \text{ into } X \text{ such that:} \\
(tg)(x) = g(\tau x) \text{ for each } x \text{ in } X \text{ and } g \text{ in } C^*(Y,E); \text{ and} \\
(t \leftarrow f)(y) = f(\sigma y), \text{ for each } y \text{ in } Y \text{ and } f \text{ in } C^*(X,E). \text{ We have:} \\
g(y) = (t \leftarrow (tg))(y) = (tg)(\sigma y) = g(\tau(\sigma y)) \text{ for every } g \text{ in } C^*(Y,E) \text{ and every } y \text{ in } Y. \text{ Since } C^*(Y,E) \\
\text{separates the points of } Y, (\tau \circ \sigma)(y) = y \text{ for all } y \text{ in } Y. \text{ Similarly,} \\
(\sigma \circ \tau)(x) = x \text{ for all } x \text{ in } X. \\
\tau \text{ is one-one: for if } \tau x_1 = \tau x_2 \text{ for } x_1, x_2 \text{ in} \\
X, \text{ then } x = \sigma(\tau x_1) = \sigma(\tau x_2) = x_2.
\( \tau \text{ is onto: given } y \text{ in } Y, \text{ then } \sigma(y) \text{ is in } X \text{ and } \tau(\sigma y) = y. \)

\( \tau \) has a continuous inverse, namely, the mapping \( \sigma \). Hence \( \tau \) is a homeomorphism from \( X \) onto \( Y \).

Conversely, if \( \tau \) is a homeomorphism from \( X \) onto \( Y \), then the induced mapping \( \tau^*: g \to g \circ \tau \) is evidently an E-isomorphism from \( C^*(Y,E) \) onto \( C^*(X,E) \).

**12.14 Remark.** If \( X \) is an \( E^* \)-completely regular space, but not an \( E^* \)-compact space, then \( X \) and \( \nu^*X \) are not homeomorphic. But \( C^*(X,E) \) is \( E \)-isomorphic with \( C^*(\nu^*X,E) \) under the mapping: \( f \to \overline{\tilde{f}} \), where \( f \in C^*(X,E) \) and \( \overline{\tilde{f}} \) is the extension of \( f \) in \( C^*(\nu^*X,E) \). Therefore, the class of all \( E^* \)-compact spaces is a maximal class of spaces for which Theorem 12.13 holds.

**12.15 Remark.** Suppose \( X \) is \( E^* \)-compact. Then every \( E \)-isomorphism from the ring \( C^*(X,E) \) onto itself is the induced mapping of a unique homeomorphism from \( X \) onto itself. The correspondence establishes an anti-isomorphism from the group of all \( E \)-automorphism on \( C^*(X,E) \), onto the group of all homeomorphisms on \( X \). To see this, suppose \( t_i \) (\( i = 1, 2 \)) are \( E \)-automorphisms on \( C^*(X,E) \), and \( \tau_i \) (\( i = 1, 2 \)) are homeomorphisms on \( X \), such that \( t_i(g) = g \circ \tau_i \) (\( g \in C^*(X,E) \)). Then, \( (t_1 \cdot t_2)(g) = t_1(t_2g) = t_1(g \circ \tau_2) = (g \circ \tau_2) \cdot \tau_1 = g \circ (\tau_2 \cdot \tau_1) \). Hence \( t_1 \cdot t_2 \) corresponds with \( \tau_2 \cdot \tau_1 \).
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