ON THE STEINER PROBLEM

by

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ABSTRACT

The classical Steiner Problem may be stated: Given n points \(a_1, \ldots, a_n\) in the Euclidean plane, to construct the shortest tree(s) (i.e. undirected, connected, circuit free graph(s)) whose vertices include \(a_1, \ldots, a_n\).

The problem is generalised by considering sets in a metric space rather than points in \(E^2\) and also by minimising a more general graph function than length, thus yielding a large class of network minimisation problems which have a wide variety of practical applications.

The thesis is concerned with the following aspects of these problems.

1. Existence and uniqueness or multiplicity of solutions.
2. The structure of solutions and demonstration that minimising trees of various problems share common properties.
3. Solvability of problems by Euclidean constructions or by other geometrical methods.
TABLE OF CONTENTS

CHAPTER I Introduction 1

CHAPTER II Existence Theorem 5

CHAPTER III An Effective Algorithm for \( (S_n) \) in \( E^2 \) 8

CHAPTER IV \( (S_n) \) for Sets in \( E^2 \) 18

CHAPTER V \( (S_n) \) in Other Metric Spaces
1. Euclidean \( m \)-space 21
2. A Surface in \( E^3 \) 21
3. Plane Minkowski Metric Space 23

CHAPTER VI More Properties of \( (S_{n\alpha\beta\gamma}) \) 36

BIBLIOGRAPHY 41
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I INTRODUCTION

Our starting point is the well known elementary problem:

(S₃) Given 3 distinct points \( a_1, a_2, a_3 \) in \( \mathbb{E}^2 \) to find the point \( p \) which minimises the sum of distances \( p a_1 + p a_2 + p a_3 \). If triangle \( a_1 a_2 a_3 \) has an angle \( \geq 120^\circ \), then \( p \) is its vertex, otherwise \( p \) is the unique point at which the sides of the triangle subtend angles of \( 120^\circ \) and is called the Steiner point of the triangle. (S₃) may be generalised in many ways. For example

(Pₙ) Given \( n \) distinct points \( a_1, ..., a_n \) in \( \mathbb{E}^2 \) to find the point \( p \) which minimises the function \( \sum_{i=1}^{n} p a_i \).

(P₃) and (S₃) are identical.

Let \( b_1, ..., b_N \) be any set of distinct points in the plane. By a tree \( U \) on the vertices \( b_1, ..., b_N \) we mean any set consisting of some of the \( \binom{N}{2} \) closed straight segments \( b_i b_j \) with the property that any two vertices can be joined by a sequence of segments belonging to \( U \) in one and only one way. A segment \( b_i b_j \) in \( U \) is called a branch of \( U \), the length \( L(U) \) of \( U \) is the sum of the lengths of its branches and \( \{ b_i \} \) is the set of all vertices sending branches to the vertex \( b_i \). The valency of \( b_i \), written \( w(b_i) \), is the number of vertices in \( \{ b_i \} \).

We can now formulate further generalisations of (S₃):

(Sₙ) Given \( n \) distinct points \( a_1, ..., a_n \) in the plane (\( n \geq 3 \)), to construct the shortest tree(s) whose vertices include \( a_1, ..., a_n \) and any set of \( k \) additional plane points \( s_1, ..., s_k \) (\( k \geq 0 \)).

(Sₙαβγ) Given three non-negative real numbers \( \alpha, \beta, \gamma \) and \( n \) distinct points \( a_1, ..., a_n \) in the plane, to find an integer \( k \) and \( k \) additional points \( s_1, ..., s_k \), and to construct the tree(s) \( U \) on the vertices \( a_1, ..., a_n, s_1, ..., s_k \) so as to minimise the sum

\[
T = L(U) + \alpha \sum_{i=1}^{n} w(a_i) + \beta \sum_{i=1}^{k} w(s_i) + \gamma k.
\]
If $\alpha = \beta = \gamma = 0$, $(S_{n\alpha\beta\gamma})$ reduces to $(S_n)$. Suppose now that $\beta = 0$ and $\alpha > \gamma$ where $\gamma$ is sufficiently large. $T$ will then be smallest when each $w(a_i^j)$ has its minimum value 1 so that as few extra vertices as possible are adjoined. However, since a tree is connected, $w(a_i^j) = 1$ for each $i$ implies that $k \geq 1$. It follows, therefore, that when $\beta = 0$ and for suitable $\alpha, \gamma$, the minimising trees of $(S_{n\alpha\beta\gamma})$ will be precisely the minimum length trees among those having $w(a_i^j) = 1$ for each $i$ and $k = 1$ i.e. $(S_{n\alpha\beta\gamma})$ reduces to $(P_n)$. A similar argument shows that if $\max(\beta, \gamma) >> 1$ and $\alpha = 0$, then $(S_{n\alpha\beta\gamma})$ reduces to $(C_n)$

$(C_n)$ To connect $n$ distinct given points in the plane by the shortest trees whose vertices are these $n$ points.

$(C_n)$ is not a generalisation of $(S_3)$ and has the important property of being discrete i.e. the length is to be minimised over a finite set of trees, while $(S_n)$, $(P_n)$ and $(S_{n\alpha\beta\gamma})$ are not discrete, since the co-ordinates of the extra vertices $s_1, \ldots, s_k$ are continuously varying unknowns.

$(S_{n\alpha\beta\gamma})$ and its special cases may be extended still further by replacing the $n$ given points $a_1, \ldots, a_n$ with $n$ disjoint plane, compact, connected sets $A_1, \ldots, A_n$. The definition of a tree given above is still valid with the following minor modifications. A vertex is a set and by the "segment" $B_i B_j$ we mean a line of shortest distance joining the sets $B_i$ and $B_j$. Such extremals certainly exist by a standard continuity and compactness argument.

The final generalisation is to change the metric space in the formulation. In the definition of a tree "segment" is replaced by "geodesic". For example we may consider identical problems in $E^m$, on the surface of a sphere in $E^3$ or in Minkowski metric spaces $M^m$. We formulate our most general problem:

$(S_{n\alpha\beta\gamma})$ Let $M$ be a metric space with metric $\rho$ which has the following properties:
1. $M$ is finitely compact.
2. There exists a geodesic in $M$ joining each two points of $M$.
3. For all $a, b \in M$, $\rho(a, b)$ is equal to the length of a geodesic joining $a$ and $b$.

Given three non-negative real numbers $\alpha, \beta, \gamma$ and $n$ disjoint, compact, connected sets $A_1, \ldots, A_n$ in $M$, to find an integer $k$ and $k$ additional points $s_1, \ldots, s_k \in M$, and to construct the tree $U$ on the vertices $A_1, \ldots, A_n, s_1, \ldots, s_k$ so as to minimise the sum

$$T = L(U) + \alpha \sum_{i=1}^{n} w(A_i) + \beta \sum_{i=1}^{k} w(s_i) + \gamma k.$$ 

Conditions (2) and (3) give meaning to the idea of a minimum length tree in the space $M$ while (1) will be used to demonstrate the existence of such a tree.

This class of problems offers a wide variety of practical applications. The problems of joining geographical points, metropolitan areas, a set of lakes or sets of electrical terminals by minimum length systems of roads, railways, canals or connecting wire respectively are all examples of $(S_n)$ for points or sets in some surface in $E^3$ or in $E^3$ itself. The particular Minkowski metric space which has distance function $d(x_1, x_2) = |x_1 - x_2| + |y_1 - y_2|$ is called the Manhattan metric.

If there are $n$ stores in a network of city blocks to be supplied by separate trips in rotation from a central supply depot, the optimal position for the depot is a solution of $(P_n)$ in this Manhattan metric space. This has further applications in some printed circuit designs where terminals may be joined only by wires running in two perpendicular directions. Finally suppose we wish to minimise the cost of a communications network joining areas $A_1, \ldots, A_n$ in which there is a cost per unit length and also costs per terminal depending on the number of connections at the terminal, then the minimum cost networks will be solutions of $(S_{n, \alpha, \beta, \gamma})$ in some metric space and for some $\alpha, \beta, \gamma$. 
The problem \((S_n)\) dates back to Fermat and the generalisation \((S_n)\) in \(E^2\) is called the Steiner problem, and appears in the collected works of Steiner. In [1] there is a summary of the knowledge of \((S_n)\) in 1941 and some interesting solutions, found by stretching soap films between pegs and glass plates, are exhibited. Recently due to the diverse applications, there has been renewed interest in these problems. Principally Z.A. Melzak [2] showed that \((S_n)\) in \(E^2\) is solvable by a finite number of Euclidean constructions (i.e. ruler-compass constructions in the classical sense) and posed \((S_{n|\alpha|\beta})\). R.C. Prim [3] has given an algorithm for solving \([C_n]\). Other results include a solution of \((S_n)\) for \(n\) points in Manhattan metric space by Hanan [4], and a uniqueness theorem for \((P_n)\) by Palermo [5]. Other minor references are [6] and [7].

The Thesis is concerned with the following aspects of these problems:
(a) Multiplicity or uniqueness of solution (b) feasibility of constructing all solutions by Euclidean constructions or possibly by wider geometrical algorithms, (c) the structure of minimising trees and demonstration that minimising trees of various problems share common structures.

The first section proves the existence of all minimum points and minimising trees mentioned in the thesis. We then consider the problem \((S_n)\) for \(n\) points in \(E^2\) giving another proof of the theorem that it is solvable by a finite number of Euclidean constructions. The proof clearly explains the algorithm involved and exhibits the structure of minimising trees. The methods used here are then generalised to solve \((S_n)\) for \(n\) sets in \(E^2\). We continue with a demonstration that solutions of \((S_n)\) in other spaces namely \(E^D\), a general surface in \(E^3\) and Minkowski metric spaces share common properties. The final section discusses the problem \((S_{n|\alpha|\beta})\), showing that in Minkowski space it has a finite number of solutions and that in \(E^2\) it is not in general solvable by Euclidean constructions.
II - EXISTENCE THEOREM

In this section we prove that minimizing trees (points) exist for all problems mentioned in the thesis. A preliminary result shows that \((S_{n\alpha\beta})\) is equivalent to a finite number of minimum length problems.

Let \(A_1, \ldots, A_n\) be \(n\) disjoint, compact, connected sets in a metric space \(M\) having the three properties listed in section 1.

**Definition.** \(U\) is a \(\mu\)-tree on \(A = \{A_1, \ldots, A_n\}\) if \(U\) is a tree whose vertices are \(A_1, \ldots, A_n\) and the points \(s_1, \ldots, s_k\) which satisfies:

\[
\begin{align*}
(i) & \quad w(s_i) \geq 3, \quad i = 1, \ldots, k. \\
(ii) & \quad 0 \leq k \leq n-2.
\end{align*}
\]

**Lemma.** If a solution of \((S_{n\alpha\beta})\) exists, it is a \(\mu\)-tree on \(A\).

**Proof.** We shall call the extra vertices \(s_1, \ldots, s_k\), "\(s\)-points". \(w(s_i) \geq 2\) for each \(i\), otherwise we could decrease \(T\) by deleting \(s_i\). Now suppose \(w(s_i) = 2\) for some \(i\). Then the tree formed by replacing \(s_i\) by the single branch \(xy\), the two branches joining \(s_i\) has a smaller value of \(T\). Hence (i). The number of branches of \(U\) leading to \(s\)-points is \(\geq 3k/2\) from (i), the connectivity of \(U\) assures us that the number of branches from the \(A\)-sets is \(\geq n/2\) and a tree with \(n+k\) vertices has \(n+k-1\) branches. Therefore \((n+3k)/2 \leq n+k-1\), from which we deduce (ii).

**Definition.** By the association of a \(\mu\)-tree \(U\) on \(A\), we mean the integer \(k\) and the sets \(\{A_i\}\) and \(\{s_j\}\) (\(i = 1, \ldots, n\), \(j = 1, \ldots, k\)).

**Theorem 1.** The problem \((S_{n\alpha\beta})\) is reducible to a finite number of minimum length problems.

**Proof.** The relation "has the same association as" on the set of all \(\mu\)-trees on \(A\) is an equivalence relation. We show that the number of equivalence classes is finite. Suppose there are \(k\) additional vertices. Then we have \(n+k\) vertices on which to construct a tree, i.e. \(n+k-1\) pairs of vertices to be joined, must be selected from the possible \(\binom{n+k}{2}\) pairs.
Thus the number of associations with $k$ extra vertices is not greater than

$$g(n,k) = \binom{n+k}{2} \binom{n+k-1}{n+k-1}$$

and the total number of associations is not greater than

$$\sum_{k=0}^{n-2} g(n,k)$$

and hence is finite.

Let the equivalence classes be $C_1, \ldots, C_N$ and let $C$ be any one of these classes. The tree(s) which minimise $T$ in $C$ are precisely the tree(s) of minimum length of $C$ (if one exists), since the association common to all the tree(s) of $C$ fixes the other three terms of $T$ i.e. for all $U \in C$

$$\alpha \sum_{i=1}^{n} w(A_i) + \beta \sum_{j=1}^{k} w(s_j) + \gamma k \ 	ext{is constant.}$$

The minimising trees of $(S_n \alpha \beta \gamma)$ are a subset of the trees belonging to $C_1, \ldots, C_N$ by the Lemma. Therefore, each minimising tree of $(S_n \alpha \beta \gamma)$ is a solution to one of the following $N$ minimum length problems:

1. For $i = 1, \ldots, N$ to construct the tree(s) $U \in C_i$ which minimise $L(U)$.

Theorem 2. Let $C \in \{C_1, \ldots, C_N\}$. There exists a tree of minimum length in $C$.

Proof. The association of trees in $C$ stipulates which of the pairs $s_i A_j, s_i s_j, A_i A_j$ will be joined by geodesics as branches of trees in $C$. We exclude the case $k = 0$ for which the theorem is obvious. Let

$$[A_{i_1}, A_{i_2}, \ldots, A_{i_N}] = \{A_t : A_t \in A \text{ and } s_{i_t} A_t \text{ is a branch of trees in } C\}$$

and let $R_1, R_2$, sets of unordered pairs of integers be defined as follows:

$$R_1 = \{(i,j) : s_i s_j \text{ is a branch of trees in } C\}.$$

$$R_2 = \{(i,j) : A_i A_j \text{ is a branch of trees in } C\}.$$
Then the length of a tree in $C$ is

$$f(s_1, \ldots, s_k) = \sum_{i=1}^{k} \sum_{j=1}^{k} \rho(s_i, A_{ij}) + \sum_{(i,j) \in R_1} \rho(s_i, s_j) + \sum_{(i,j) \in R_2} \rho(A_i, A_j).$$

Suppose $L$ is the length of the shortest tree with vertices $A_1, \ldots, A_n$ only. Let

$$Z = \{ z : z \in M \text{ and } \min_i \rho(z, A_i) \leq L \}.$$

Then every $s$-point of a tree of shortest length in $C$ (if one exists) is in $Z$, for otherwise the length of the tree would necessarily be greater than $L$. Thus if $\{s_1, \ldots, s_k\}$ is a set of $s$-points of a minimum length tree in $C$, then $\{s_1, \ldots, s_k\}$ is an element of the cartesian product $Z^k$. Now since $Z$ is closed and bounded and $M$ is finitely compact, $Z$ is compact implying by the Tychonoff Theorem that $Z^k$ is compact. But $f(s_1, \ldots, s_k)$ is continuous on $Z^k$ and so has a minimum value on $Z^k$. Hence the theorem.

**Corollary.** There exist minimising trees of $(S_{n\alpha\beta\gamma})$ in $M$. For $(S_{n\alpha\beta\gamma})$ is equivalent to the finite set (1) of minimum length problems (Theorem 1.) each of which by the theorem has a solution.
III - AN EFFECTIVE ALGORITHM FOR \( (S_n) \) IN \( \mathbb{R}^2 \).

We begin by stating elementary constructions of the solution of \( (S_3) \) which was given in the introduction. These constructions are of fundamental importance in the development of the algorithm for \( (S_n) \) both for points in the plane and for \( n \) sets in the plane which will be discussed in the next chapter.

Let \( a_1, a_2, a_3 \) be the vertices of a plane triangle \( T \) with no angle \( \geq 120^\circ \) and \( a_{12}a_{23}a_{31} \) be the three third vertices of the equilateral triangles built outward on the sides of \( T \) (with obvious notation). Then \( p \), the minimum point of \( (S_3) \), lies at the unique point of intersection inside \( T \) of the circumscribing circles of the equilateral triangles. Again, if \( p' \) is the intersection of the line \( a_{12}a_3 \) with the circle through \( a_1a_2a_{12} \), each side of \( T \) subtends \( 120^\circ \) at \( p' \) which therefore coincides with \( p \).

It follows that each of the lines \( a_{12}a_3, a_{23}a_1, a_{31}a_2 \) passes through \( p \) giving a third method of construction.

![Fig. 1.](image-url)
Using the notation as in fig. 1. by the sine law

\[ \frac{a_1 p}{\sin \alpha} = \frac{a_2 p}{\sin(60-\alpha)} = \frac{a_{12} p}{\sin(60+\alpha)}. \]

\[ \Rightarrow a_1 p + a_2 p = a_{12} p \left\{ \frac{\sin \alpha + \sin(60-\alpha)}{\sin(60+\alpha)} \right\} = a_{12} p. \]

The addition of \( a_3 p \) to each side gives

\[ a_{12} a_3 = a_1 p + a_2 p + a_3 p. \]

Thus each of the three lengths \( a_{12} a_3, a_3 a_2, a_{23} a_1 \) equals the minimum value of the function \( a_1 x + a_2 x + a_3 x \). We remark that if \( T \) has an angle \( \geq 120^\circ \) none of the above constructions will produce a point inside \( T \).

Suppose \( A = \{a_1, \ldots, a_n\} \) is a set of \( n \) distinct points in the plane and that \( U \), a minimising tree of \( (S) \) for the set \( A \), has extra vertices \( s_1, \ldots, s_k \). Then

- **P1.** \( U \) is non-self intersecting, i.e. two branches of \( U \) do not intersect except at an end point.
- **P2.** \( w(s_i) = 3, \, i = 1, \ldots, k. \)
- **P3.** \( w(a_j) \leq 3, \, j = 1, \ldots, n. \)
- **P4.** \( 0 \leq k \leq n-2. \)
- **P5.** Each \( s_i \) is the Steiner point of the triangle formed by \( \{s_i\} \).

These properties are given in [1]. We indicate the proofs for completeness.

Suppose that two branches \( x_1 x_2, x_3 x_4 \) of \( U \) intersect at \( x \). Then one of the angles at \( x \) (say \( \angle x_1 x_2 x_3 \)) is less than \( 120^\circ \) and \( U \) may be shortened by replacing the segments \( x_1 x, x_3 x \) by \( x_1 p, x_3 p, xp \) where \( p \) is the solution of \( (S_3) \) for the triangle \( x_1 x_3 x \). Hence P1. An identical argument proves that \( w(x) \leq 3 \) for all vertices of \( U \). Therefore \( U \) has the property P3 and \( w(s_i) \leq 3 \) for each \( i \). However \( w(s_i) > 3 \) for otherwise there is no gain in introducing the additional vertex \( s_i \). Thus P2 is established and from this P5 is immediate. P4 is proved as in the Lemma on Page 5,
Definitions. A tree with vertices \( \{a_1, \ldots, a_n\} \) and \( \{s_1, \ldots, s_k\} \) (from this point these will be termed \( a \)-points and \( s \)-points respectively) has the property \( P_4 \) if \( k = n - 2 \).

A tree \( V \) is a subtree of a tree \( U \) if and only if (i) \( V \) is a tree and (ii) the set of geodesics of \( V \) is contained in the set of geodesics of \( U \).

A tree \( U \) is an \( S \)-tree on \( A \) if it has properties \( P_1, P_2, P_3, P_4, P_5 \).

A tree \( U \) is an \( S^* \)-tree on \( A \) if it has properties \( P_1, P_2, P_3, P_4, P_5 \).

The finite set \( \{A_1, A_2, \ldots, A_t, R\} \), \( t \geq 0 \) is a Division of the set \( A \) if and only if

1. Each \( A_i \subseteq A \), \( R \subseteq A \).
2. \( R \cap A_i = \emptyset \), \( i = 1, \ldots, t \).
3. No \( a_j \) can be an element of more than 3 of the sets \( A_i \).
4. Each \( A_i \) has 3 or more elements.
5. \( A_1 \cup A_2 \cup \ldots \cup A_t \cup R = A \).

Lemma 1. If \( U \) is any \( S \)-tree on \( A \) then for some division \( \mathcal{C} = \{A_1, A_2, \ldots, A_t, R\} \) of \( A \) there exist \( S \)-subtrees of \( U \) on \( A_i \) for \( i = 1, \ldots, t \).

Proof. If \( U \) contains no \( s \)-point, the required division is \( \{A\} \). If \( S \), the set of \( s \)-points of \( U \) is non-empty, we define the relation "o" on \( S \) as follows.

\( s_i \circ s_j \) iff the sequence of segments of \( U \) joining \( s_i \) to \( s_j \) contains no \( a \)-point of \( U \). The relation is an equivalence relation and therefore partitions \( S \) into mutually exclusive and exhaustive sets \( S_1, \ldots, S_t \) \( (t > 0) \).

Define \( A_i = \{a_j : a_j \in \{s_k\} \text{ for some } s_k \in S_i\} \), \( i = 1, \ldots, t \) and \( R = A - \bigcup_{i=1}^t A_i \). The set \( \{A_1, A_2, \ldots, A_t, R\} \) is a division of \( A \). It remains to show that there is an \( S \)-subtree of \( U \) on each \( A_i \). Let \( U_i \) be the subtree of \( U \) whose vertex set is \( A_i \cup S_i \), \( i = 1, \ldots, t \). (This is certainly a subtree of \( U \) by construction). \( U_i \) has the properties \( P_1, P_2, P_3 \) and \( P_5 \). We prove \( P_4 \). Let \( A_i \) contain \( p \) points, \( S_i \) \( q \) points and further suppose that \( S_i \) contains \( n_1, n_2, n_3 \) \( s \)-points which directly join 1, 2 and 3 other \( s \)-points respectively. Then
\[(2) \quad n_1 + n_2 + n_3 = q.\]

The number of branches of \(U_i\) connecting \(s\)-points is \((n_1 + 2n_2 + 3n_3)/2\). But by the defining property of \(S_i\) this number is \(q-1\). Hence

\[(3) \quad n_1 + 2n_2 + 3n_3 = 2(q-1).\]

The number of branches of \(U_i\) connecting an \(a\)-point to an \(s\)-point is \(2n_1 + n_2\) since the valency of each \(s\)-point is 3. Hence

\[(4) \quad 2n_1 + n_2 = p.\]

From equations (2), (3) and (4) we deduce \(q = p-2\). Therefore \(U_i\) satisfies \(\overline{P}_4\) and is an \(\overline{S}\)-subtree of \(U\). Hence the Lemma.

Incidentally one can also prove \(n_1 - n_3 = 2\) from (2), (3) and (4) which implies that \(n_1 \geq 2\). We note that the non-selfintersection property is not involved in the establishment of these equations and state that an \(S^*\)-tree on a set \(A\) has at least two \(s\)-points which directly join exactly one other \(s\)-point and two \(a\)-points. This fact will be used in the next Lemma.

We call the subtrees \(U_i\) \((i = 1, \ldots, t)\) the Components of \(U\) and suggest that the components of a minimising tree \(U\) may be considered as stability sets for \(U\) in the following sense. If one \(a\)-point \(a_i\) is perturbed by a sufficiently small amount, there is a minimising tree \(U'\) for the new set of \(n\) \(a\)-points which is identical to \(U\) except for a small perturbation of the components to which \(a_i\) belongs. If \(p, q\) are points in the plane we shall denote by \((pq)\) and \((qp)\) the third vertices of the equilateral triangle on \(pq\) as base, \((pq)\) being the point to the left of \(p\) looking from \(p\) along \(pq\).

The construction we now describe, a direct consequence of the solution of \((S_3)\), is crucial to the proof. Let \(U\) be an \(\overline{S}\)-tree on \(A = \{a_1, \ldots, a_n\}\) with \(s\)-points \(s_1, \ldots, s_k\) then there exists (see note following Lemma 1.) an \(s\)-point say \(s_1\) which is directly connected
to two a-points say \( a_1 \) and \( a_2 \) and a third vertex \( x \). In fact a portion of \( U \) appears as in fig. 2.

![Diagram](image)

**Fig. 2.**

Since \( s_1 \) is the Steiner point of \( \{ s_1 \} \), the line \( xs_1 \) produced passes through \( (a_2s_1) \) and the tree \( U' \) on \( A' = \{(a_2s_1), a_3, \ldots, a_n\} \) with s-points \( s_2, \ldots, s_k \) formed from \( U \) by replacing the branches \( a_1s_1, a_2s'_1, s_1x \) by the single branch \( (a_2s_1)x \) is an \( S' \)-tree on \( A' \) (the non-selfintersection property may have been contradicted). Further \( a_1s_1 + a_2s'_1 + s_1x = (a_2s_1)x \) therefore \( U, U' \) have equal lengths. The important point is that \( (a_2s_1) \) is constructed from the original a-points only. We call the above the "Equilateral construction."

We next define the term "Association" of an \( S \)-tree on a set \( A \). From \( U \), we form a tree \( U' \) and set \( A' \) as above. The construction is repeated forming a new tree \( U'' \) and set \( A'' \), \( U''' \) and \( A''' \) etc. until the set \( A^{(r)} \) contains only two points. (Actually \( r = \) the number of s-points in the original tree \( U \)). The two points of \( A^{(r)} \) can be expressed
in terms of the original a-points of U and the equilateral triangle bracketing notation defined above. This representation of $A(r)$ we call an "Association" of the tree U and the line joining the two points of $A(r)$ we call an "axis" of U. We give a simple example below. We note the following:

(i) The process is always possible since at every stage the tree $U^{(k)}$ is an $S^*$-tree on $A^{(k)}$ and hence has an s-point which directly joins two a-points, (in fact at least 2 such s-points by the note following Lemma 1).

(ii) It follows from (i) that every $S$-tree on A has an association and an axis (certainly not a unique association.)

(iii) At each stage length is preserved. i.e. The length $L(U) = \text{length of an axis of } U$.

(iv) No two $S$-trees on A have a common association.

Example. In Fig. 3, $U$ is an $S$-tree on $A = \{1,2,3,4,5\}$ with s-points $s_1, s_2, s_3$.

![Diagram of a tree with s-points and labels](image-url)
We "pair" the points 1 and 3 giving

\[ A' = \{(13), 2, 4, 5\} \text{ and } U' \text{ with branches } (13)s_3, s_3s_5, s_2s_3, 2s_2, 4s_2. \]

Next we pair 4 and 2

\[ A'' = \{(13), (42), 5\} \text{ and } U'' \text{ has branches } (42)s_3, (13)s_3, s_3s_5. \]

Finally we pair (13) and (42)

\[ A''' = [(42)(13)], 5 \text{ and } U''' \text{ has branch } ((42)(13))5. \]

The underlined portion i.e. \( A''' \) without the set parentheses is an association of \( U \). The length of \( U \) is the length of the branch of \( U''' \).

**Lemma 2.** If it is known that \( U \), an \( S \)-tree on \( A \) has a certain association \( \alpha \), we can construct \( U \) by a finite number of Euclidean constructions.

**Proof.** The Lemma is true for \( n = 3 \). Assume it is true for \( n = N \) and let \( A_{N+1} = \{a_1, \ldots, a_{N+1}\} \) be any plane set of \( N+1 \) points on which \( U \) is an \( S \)-tree with association \( \alpha \). Suppose the labelling of points in \( A_{N+1} \) is such that \( a_1, a_2 \) in this order in \( \alpha \) have no brackets or commas separating them. We now consider the set \( A_N = [(a_1a_2), a_3, \ldots, a_{N+1}] \). From the equilateral construction there exists \( U' \), an \( S \)-tree on \( A_N \) which has association \( \alpha \) except that \( (a_1a_2) \) is now regarded as a single point.

By the inductive hypothesis we can construct \( U' \) by a finite number of Euclidean constructions. Let \( (a_1a_2)x \), the branch of \( U' \) connecting \( (a_1a_2) \), be replaced by the branches \( a_1s, a_2s, sx \) where \( s \) is the point of intersection of the circle through \( (a_1a_2), a_1, a_2 \) with the line \( (a_1a_2)x \). The resulting tree \( U \), by the equilateral construction, is the (unique by note (iv) Page 13) \( S \)-tree on \( A_{N+1} \) with association \( \alpha \). Hence the Lemma by induction.

**Lemma 3.** The set of all minimum length \( S \)-trees on \( A = \{a_1, \ldots, a_n\} \) is finite and may be constructed by a finite number of Euclidean constructions.

**Proof.** Any two points formed by combining the elements of \( A \) by the above equilateral point bracketing notation we call an association of \( A \). Then the set of all associations of all \( S \)-trees on \( A \) is a subset of the finite
set $\mathcal{B}$ of all associations of $A$ (in fact a proper subset for $n > 3$).
If for each $b \in \mathcal{B}$ we perform the finite number of Euclidean constructions
(Lemma 2) that constructs the $\mathcal{S}$-tree on $A$ with association $b$ (if such a
tree exists) we will construct all the $\mathcal{S}$-trees on $A$. Hence the Lemma.

For the main theorem of this chapter we shall need to refer
to Prim's efficient algorithm for

\[ (C_n) : \text{Given } n \text{ compact, connected, disjoint sets } W_1, \ldots, W_n, \text{to connect } \]
\[ \text{them together by the shortest tree(s) whose vertices are exactly these } \]
\[ \text{sets.} \]
The method is as follows:
(1) Join $W_1$ to its nearest neighbor, say $W_2$,
(2) Replace $W_1$ and $W_2$ by their union.
(3) Repeat the same procedure for the new class of $(n-1)$ sets and keep
on repeating until only one set remains.

Actually, Prim's algorithm was originally intended for the case of points;
however, it works equally well for sets. Moreover if the nearest neighbor
of some $W_i$ can be connected to it by several segments of the same minimal
length, or if $W_i$ has several nearest neighbors, we perform the connection
in all possible ways and get then the set of all connecting trees of the
same minimal length.

**Theorem 3.** For every $n$, there exists a finite number of Euclidean
constructions yielding all the minimising trees of the problem $(S_n)$.
The minimising trees of $(S_n)$ are precisely the minimum length $S$-trees on
$A$, each of which has minimum length components (Lemma 1) on some division
of $A$. The following method, therefore yields all solutions of $(S_n)$.

1. From the finite set $\mathcal{C} = \{ \mathcal{C}_1, \ldots, \mathcal{C}_N \}$ of all divisions
   of $A$.
2. For each $\mathcal{C}_i = \{ A_{i1}, A_{i2}, \ldots, A_{it}, R_i \}$, we construct by
   a finite sequence of Euclidean constructions, the finite
   set $C_{ij}$ of all minimum length $\mathcal{S}$-trees on $A_{ij}$ ($j = 1, \ldots, t_i$).
   (Lemma 3.)
3. If $C_{ij} = \emptyset$ for any $j = 1, \ldots, t_i$, (there may not exist an
   $\mathcal{S}$-tree on an arbitrary set of points), we reject the
division $\mathcal{C}_i$. If each $C_{ij} \neq \emptyset$, we call $\mathcal{C}_i$ "admissible".
4. For each admissible division $\mathcal{O}_i$ we now form the finite set

$$ \Gamma_i = \{(U_{i1}, U_{i2}, \ldots, U_{it_i}) : U_{ij} \in C_{ij}\} .$$

5. For each element of $\Gamma_i$ we connect the $U_{ij}$ to each other and the residual set $R_i$ in optimal way(s) so that the resulting tree(s) on $A$ have minimum length. ($\mathcal{S}$-trees on non-disjoint sets $A_{ip}, A_{iq}$ are automatically joined).

Prim's algorithm may be used to effect the joining. We note that the application of this algorithm must not contradict any of the properties P1-P5 on the resulting tree could not have minimum length e.g. every connection must be a segment joining two $a$-points. The number of optimal joinings is certainly finite. We thus obtain a finite set of trees on $A$ from which we select the set $V_i$ with minimum length. ($V_i = \emptyset$ if $\mathcal{O}_i$ is not admissible). Then the solutions of $(S_n)$ are precisely the minimum length trees of the set $\bigcup_{i=1}^{N} V_i$ which has been constructed by a finite sequence of Euclidean constructions. This completes the proof of the theorem.

We conclude this chapter with a diagram of an $S$-tree on $A = \{a_1, \ldots, a_{11}\}$ which has two component $\mathcal{S}$-trees $U_1, U_2$ on sets $A_1 = \{a_1, a_2, a_3, a_4\}$ and $A_2 = \{a_5, a_6, a_7, a_8, a_9\}$ respectively. The residual set $R$ is $\{a_{10}, a_{11}\}$ and the corresponding division of $A$ is $\mathcal{O} = \{A_1, A_2, R\}$. 

Fig. 4.
IV - (S^n) for sets in E^2.

We now discuss the extension of the techniques of Chapter III to the problem (S^n) for \( n \) compact, connected, disjoint sets \( A_1, \ldots, A_n \) in the plane. Let \( U \) be a minimising tree of (S^n) which has additional vertices \( s_1, \ldots, s_k \). An end point of a branch of \( U \) is either a point in \( B(A_i) \) or a vertex \( s_i \). The following properties of \( U \) are simply deducible by the same methods as their counterparts in Chapter III.

Q1. Two branches may not intersect except at an end point.
Q2. If two branches share an end point, they subtend there an angle \( \geq 120^\circ \).
Q3. For each \( s_i \) (\( i = 1, \ldots, k \)), \( w(s_i) = 3 \).
Q4. Each \( s_i \) is the Steiner point of \( \{ s_i \} \), where \( \{ s_i \} \) is the set of three end points of branches joining \( s_i \).
Q5. \( 0 \leq k \leq n-2 \).

**Definition.** Let \( A, B \) be compact, connected sets in \( E^2 \) and let \( (ab) \) be defined as in Chapter III. We define the equilateral sum \( (AB) \) of \( A \) and \( B \) by

\[
(AB) = \{(ab) : a \in A, b \in B\}.
\]

\( (AB) \) is compact and connected. \( (AB) \neq (BA) \). If \( A \) is a point, \( (AB) \) and \( B \) are congruent under a rotation of \( 60^\circ \) about \( A \). If \( c \) is a boundary point of \( (AB) \) then \( c = (ab) \) where \( a, b \) are boundary points of \( A, B \) respectively.

Distributive laws hold. If \( A = C \cup D \) then

\[
(AB) = (CB) \cup (DB)
\]

and similarly for \( B = E \cup F \). The following properties hold for \( (AB) \) whenever \( A \) and \( B \) have them: convexity, arcwise-connectedness, being the smooth boundary of a region, being a simple polygon. Let \( d(X,Y) \) denote the distance between two compact, disjoint sets \( X, Y \).

**Lemma.** (Generalised equilateral construction).

Let \( A_1, A_2, A_3 \) be three compact, connected, disjoint sets in \( E^2 \). Suppose that a minimum length tree \( U \) connecting these sets consists of three
straight segments $a_1 v, a_2 v, a_3 v$, $(a_i \in B(A^i))$, meeting at an additional vertex $v$ and let the rotation $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_1$ be counter-clockwise about $v$.

Then (i) $a_1(a_2 a_3), a_2(a_3 a_1), a_3(a_1 a_2)$ intersect in $v$.

(ii) $a_1(a_2 a_3) = d(A_1, (A_2 A_3))$

$= a_2(a_3 a_1) = d(A_2, (A_3 A_1))$

$= a_3(a_1 a_2) = d(A_3, (A_1 A_2)) = L(U)$.

i.e. $a_1, a_2, a_3$ are selected in $A_1, A_2, A_3$ so that $a_1(a_2 a_3)$ is a shortest segment connecting $A_1$ and $(A_2 A_3)$ etc.

**Proof.** $v$ is the Steiner point of $a_1, a_2, a_3$ and therefore by the equilateral triangle solution of $(S_3)$ in Chapter III, $a_1(a_2 a_3), a_2(a_3 a_1), a_3(a_1 a_2)$ intersect at $v$ and the length of each of these segments is equal to $L(U)$.

Hence $U$ is a minimising tree if and only if $a_1(a_2 a_3)$ (or similarly $a_2(a_3 a_1), a_3(a_1 a_2)$) attains its minimum value i.e. $a_1(a_2 a_3) = d(A_1, (A_2 A_3))$ as required.

The problem $(S_n)$ for a class $A$ of $n$ sets $[A_1, ..., A_n]$ can now be solved from the properties Q1-Q5 and the generalised equilateral construction by exactly the same sequence of steps which solved $(S_n)$ for points using P1-P5 and the equilateral construction. We form the finite set of all divisions of the class $A$. A division of $A$ has the form $[\alpha_1, \alpha_2, ..., \alpha_r]$ where each $\alpha_i$ and $r$ is a subclass of $A$. We find the minimum length $S$-trees on each $\alpha_i$ of each division and join together these components optimally using Prim's method. The components are constructed using the association and axis technique as before. The higher equilateral sums e.g. $((A_2 A_1)(A_6 A_5))$, are unambiguously defined, and by an axis e.g. $[((A_2 A_1)(A_6 A_5)), (A_4 A_3)]$ we understand the straight segment joining points in $X = ((A_2 A_1) (A_6 A_5))$ and in $Y = (A_4 A_3)$ for which the minimum $d(X,Y)$ is attained. We note the following:

**Theorem 4.** Let $A = \{A_1, ..., A_n\}$ be a class of $n$ simple polygons. Then minimum length trees on $A$ can be found using a finite sequence of Euclidean constructions.
To prove this it suffices to observe that:

(a) the equilateral sum of two polygons is a polygon and hence constructible by Euclidean means.

(b) the closest distance between two polygons can be found by Euclidean means.

Fig. 5 shows that there may be an infinite number of minimum length trees connecting a family of polygons. \( A = \{A_1, A_2, A_3\} \) where \( A_1, A_2 \) are equilateral triangles and \( A_3 \) is a single point. It is easily verified, using the generalised equilateral construction, that if \( a_1, a_2 \) are any pair of points of \( X_1, Y_1, X_2, Y_2 \) which are symmetrically placed with respect to \( A_3 \) then the network \( U \) has minimum length.

\[ A = \{A, A^*, A^\} \]

\[ a_1, a_2 \]

\[ X_1, Y_1, X_2, Y_2 \]

\[ A_1, A_2, A_3 \]

**Fig. 5**

Theorem 4 leads immediately to

**Theorem 5.** Let \( A = \{A_1, \ldots, A_n\} \) be \( n \) sets and suppose that each \( A_i \) is arbitrarily well approximable by simple polygons. Then minimum length trees on \( A \) can be found by a finite sequence of Euclidean constructions to within arbitrary accuracy.

For if \( A_1 \) and \( A_2 \) are approximated by the polygons \( P_1 \) and \( P_2 \), then \( (A_1A_2) \) is approximated by \( (P_1P_2) \).
V - $(S_n)$ IN OTHER METRIC SPACES

1. Euclidean m-Space.

Theorem 6. The minimising trees of $(S_n)$ for $n$ points in $E^m$ ($m \geq 3$) have the properties P1-P5 listed in Chapter III.

Proof. We show that each vertex of a minimising tree $U$ which has $a$-points $a_1, \ldots, a_n$ and $s$-points $s_1, \ldots, s_k$ has valency $\leq 3$. For suppose $U$ has a vertex $x$ and branches $xx_i$ along the directions of the unit vectors $u_i$, $(i=1, \ldots, 4)$. Then one of the angles at $x$ (say $\angle x_1xx_2$) is less than $120^\circ$ since

$$u_1^2 + 2 \sum_{i \neq j} u_i \cdot u_j < 4 + 2 \cdot 6(-\frac{1}{2}) = -2,$$

which is impossible. It follows that $xx_1, xx_2$ is not the minimum length network connecting $xx_1x_2$ contrary to assumption. The rest of the proof is identical to that given for the properties P1-P5 in $E^2$.

2. A Surface in $E^3$.

Let $D$ be a surface in $E^3$ free from singularities of any kind. It will be shown that minimising trees of $(S_n)$ in $D$ have properties identical to those for $E^m$. We first prove two results which show that the $120^\circ$ property of additional vertices holds in $D$.

Suppose $A$, $B$, $C$ are distinct points in $D$ and $P \notin \{A,B,C\}$ minimises the sum $\rho(P,A) + \rho(P,B) + \rho(P,C)$, we prove that the angles at $P$ between the geodesics $PA$, $PB$, $PC$ are each $120^\circ$. Let $\rho(P,A) = a$, $\rho(P,B) = b$ and $\rho(P,C) = c$. Consider the geodesic ellipse $E$ (= the locus of points $Z$ such that $\rho(Z,A) + \rho(Z,B) = a + b$) and the geodesic circle $\rho(Z,C) = c$. These closed curves touch at $P$, for otherwise there would be a point $Y$ interior to both curves such that $\rho(Y,A) + \rho(Y,B) < a + b$ and $\rho(Y,C) < c$ contradicting the minimum property of $P$. Since geodesic PC meets $\rho(Z,C) = c$ orthogonally, geodesic PC meets $E$ orthogonally. By a result of classical Differential Geometry [8] page 120, $E$ bisects the angle between the geodesic parallels $\rho(Z,A) = a$ and $\rho(Z,B) = b$ and therefore, since the geodesics $AP$, $BP$ meet these circles orthogonally, the angles $\alpha$ and $\beta$ of Fig. 6. are equal.
Similarly by considering the geodesic ellipse $\rho(Z,A) + \rho(Z,C) = a + c$ we prove $\angle APB = \angle BPC$ and this together with (5) proves the result.

Secondly we show that if ABC is a geodesic triangle on D with the angle at A less than $120^\circ$, then A does not minimise the sum $\rho(Z,A) + \rho(Z,B) + \rho(Z,C)$. Let V be an $\varepsilon$-neighborhood of A sufficiently small so that for all $r, s \in V$ there is only one geodesic joining them. Let $B(V)$ intersect the geodesics AB, AC in X and Y. Consider the following 1-1 mapping of the geodesic triangle AXY onto the tangent plane at A. For Q in the geodesic triangle AXY with $\rho(A,Q) = q$, the corresponding point Q' is the point on the tangent line to the geodesic AQ at A such that AQ' = q. Since the angle A of the plane triangle AX'Y' is less than $120^\circ$, there exists P' in the tangent plane such that $AP' + XP' + YP' < AX' + AY'$ and furthermore the difference is proportional to $\varepsilon$ i.e. $AX' + AY' - (AP' + XP' + YP') = \varepsilon k_1$ for some $k_1$. 

Fig. 6.

.". (5) $\angle APC = \angle BPC$. 

\[ \frac{a}{\angle APB} = \frac{c}{\angle BPC} \] 

Similarly by considering the geodesic ellipse $\rho(Z,A) + \rho(Z,C) = a + c$ we prove $\angle APB = \angle BPC$ and this together with (5) proves the result.

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Fig. 6.
If $\varepsilon$ is sufficiently small, for all $Q_1, Q_2 \in V$, $|Q_1 \cdot Q_2 - \rho(Q_1, Q_2)| < L \varepsilon^2$ where $L$ is constant. Hence if $P \in V$ corresponds to $P'$ in the tangent plane, 
\[(A'P' + X'P' + Y'P') - \{ \rho(A, P) + \rho(X, P) + \rho(Y, P) \} < k_2 \varepsilon^2 \text{ for some } k_2.\]

\[\therefore \{ \rho(A, X) + \rho(A, Y) \} - \{ \rho(A, P) + \rho(X, P) + \rho(Y, P) \} \]
\[= (AX' + AY') - (AP' + X'P' + Y'P') \]
\[+ (AP' + X'P' + Y'P') - \{ \rho(A, P) + \rho(X, P) + \rho(Y, P) \} \]
\[> k_1 \varepsilon^2 - k_2 \varepsilon^2 > 0 \text{ if } \varepsilon \text{ is sufficiently small.} \]

\[\therefore \rho(A, X) + \rho(A, Y) > \rho(A, P) + \rho(X, P) + \rho(Y, P).\]

If we now add $\rho(X, B) + \rho(X, C)$ to each side and apply the triangle inequality on the right we obtain

\[\rho(A, B) + \rho(A, C) > \rho(A, P) + \rho(B, P) + \rho(C, P)\]
showing that $A$ does not minimise $\rho(Z, A) + \rho(Z, B) + \rho(Z, C)$ as required.

Using these 120° properties and proofs identical to those for P1-P5 (Chapter III), we deduce that $U$, a minimising tree of $S_n$ in $D$ with extra vertices $s_1, \ldots, s_k$, has the properties P1, P2, P3, P4 and the following analog of P5: P'5: For each $i = 1, \ldots, k$ if $[s_i]$ contains points $p_i, q_i, r_i$ then each of the angles at $s_i$ between the geodesics $p_is_i, q_is_i, r_is_i$ is 120°.


Let $\Sigma$ be a centrally symmetric convex surface in $E^m$ with centre $0$. The $m$-dimensional Minkowski metric space $M^m$ associated with $\Sigma$ is obtained by defining the distance $\rho(x, y)$ for $x, y \in E^m$ as follows. If $x = y$, $\rho(x, y) = 0$. If $x \neq y$ let the ray with initial point 0 which is parallel to $xy$ meet $\Sigma$ at $P$. Then $\rho(x, y) = xy/OP$, where $xy$ and $OP$ are usual Euclidean distances. $M^m$ is a metric space satisfying the three conditions of the introduction and having the following properties ([9] page 21), $x$ and $y$ are considered as $m$-dimensional vectors. For all $x, y \in M^m$.
(i) $\rho(x,y) = \rho(0,x-y)$ and more generally, any translation is an isometry.

(ii) The triangle inequality is strict provided that $\Sigma$ is strictly convex and the three points involved are non-collinear.

(iii) For $\Sigma$ strictly convex

$$\rho(0, x+y) \leq \rho(0,x) + \rho(0,y)$$

and this inequality is strict unless $0,x,y$ are collinear with $x,y$ lying on the same side of 0.

Small results in the necessary preliminary theory of this section will be termed propositions and the principal results are Theorems 7 and 8. For the rest of the section $M^2$ will mean a plane Minkowski metric space, the defining curve of which is strictly convex.

**Proposition 1.** Given $n$ distinct non-collinear points $a_1, \ldots, a_n$ in $M^2$. There exists a unique point $z$ minimising the function

$$f(z) = \sum_{i=1}^{n} \rho(z,a_i).$$

**Proof.** The existence of a minimum point was proved in Chapter II. Suppose, contrary to the proposition, that $f(z)$ has minima $z_1$ and $z_2$. Then

$$f\left(\frac{z_1 + z_2}{2}\right) = \sum_{i=1}^{n} \rho\left(\frac{z_1 + z_2}{2}, a_i\right)$$

$$= \sum_{i=1}^{n} \rho\left(\frac{z_1 + z_2}{2} - a_i, 0\right)$$

$$= \sum_{i=1}^{n} \rho\left(\frac{1}{2}(z_1 - a_i) + \frac{1}{2}(z_2 - a_i), 0\right)$$

$$\leq \frac{1}{2} \sum_{i=1}^{n} \rho(z_1 - a_i, 0) + \frac{1}{2} \sum_{i=1}^{n} \rho(z_2 - a_i, 0) \text{ (triangle law)}$$

$z_1, z_2$ and some $a_i$ are non-collinear since we are given that not all the $a_i$ are collinear. Hence the inequality (6) is strict proving that
(7) \[ f\left(\frac{z_1 + z_2}{2}\right) < \frac{1}{2} f(z_1) + \frac{1}{2} f(z_2) = \lambda/2 + \lambda/2 = \lambda. \]

(7) contradicts the assumption that \( \lambda \) is the minimum and also shows that \( f(z) \) is strictly convex.

The next few results concern the unique point \( P \) which minimises 
\[ \rho (P,A) + \rho (P,B) + \rho (P,C) \]
where \( A, B, C \) are distinct points of \( M^2 \). We shall use the following abbreviations:

\[ P = \min \rho \{ABC\} \] for the above point, \( Q(ABC) \) for the sum \( \rho (Q,A) + \rho (Q,B) + \rho (Q,C) \) and \( AB \) for \( \rho (A,B) \).

We omit the proof of the following

**Proposition 2.** Let \( P = \min \rho \{ABC\} \) then \( P \) lies within or on the triangle \( ABC \).

**Proposition 3.** Let \( P = \min \rho \{ABC\} \) and suppose \( l \not\in \{A,B,C\} \). Then if \( A' \) is any point of the line \( PA \) between \( P \) and \( A \), \( P = \min \rho \{A'BC\} \).

**Proof.** Suppose the contrary and \( \min \rho \{A'BC\} = Q \neq P \). Then
\[ A'Q + BQ + CQ < A'P + BP + CP. \]

Applying the triangle inequality on the left we obtain \( Q(ABC) < P(ABC) \), contradicting the hypothesis \( P = \min \rho \{ABC\} \).

**Proposition 4.** Let \( P = \min \rho \{APB\} \). Then for all \( A',B' \) on \( PA, PB \) on the same side of \( P \) as \( A,B \) respectively \( P = \min \rho \{A'PB'\} \).

**Proof.** Case (i) Let \( A',B' \) be between \( A \) and \( P \), \( B \) and \( P \) respectively and suppose the contrary. Then there exists \( Q \) such that
\[ A'Q + B'Q + PQ < A'P + B'P. \]

Applying the triangle inequality to the terms bracketed on the left gives \( Q(APB) < P(APB) \) contradicting the minimum property of \( P \).
Case (ii) If conditions of Case (i) are not satisfied, draw $A''B''$ parallel to $A'B'$ with $A''$, $B''$ satisfying these conditions (see Fig. 7). Then by Case (i) $P = \min \rho \{A''PB''\}$ and by similar triangles $P = \min \rho \{A'B'P\}$ as required.

![Fig. 7.](image)

**Proposition 5.** Let $\min \rho \{ABC\} = B$ and $A'$ be any point on the line $AC$ but not in the closed segment $AC$. Then $B = \min \rho \{A'BC\}$.

**Proof** Take points $R, S, T$ on $BA'$, $BA$, $BC$ respectively such that $BR = BS = BT$. By hypothesis and Proposition 4, $\min \rho \{BST\} = B$. Suppose the contrary of the Proposition i.e. $\min \rho \{A'BC\} \neq B$. This implies by Proposition 4 that

![Fig. 8.](image)
\[
\min \rho \{BRT\} = P \neq B.
\]

There are two cases to consider:

Case (i) $P$ lies on or to the left of $BS$ (see fig. 8a). Let $PT$ meet $BS$ at $X$, (internally since the unit ball $\Sigma$ of the metric is strictly convex). Then

\[BX + SX + TX = BS + TX = BR + TX \leq BR + TP \leq BP + RP + TP \text{ by the triangle inequality.}
\]

\[< BR + BT \text{ by Equation (8).}
\]

\[= BS + BT.
\]

i.e. $X \{BST\} < B\{BST\}$. Contradiction since $B = \min \rho \{BST\}$.

Case (ii) $P$ lies to the right of $BS$ (fig. 8b). Let $PR$ meet $BS$ at $X$. Now $P = \min \rho \{BRT\}$ implies by Proposition 3 that $P = \min \rho \{BXT\}$.

Using the uniqueness property, $B \neq \min \rho \{BXT\}$ and therefore by Proposition 4. $B \neq \min \rho \{BST\}$. Contradiction. Cases (i) and (ii) together prove the proposition.

**Definition.** Let $PA_1, PA_2, PB_1, PB_2$ be four lines from $P$ meeting a line $ST$ not through $P$ in $A_1, A_2, B_1, B_2$ respectively such that $B_1$ and $B_2$ are contained in the closed segment $A_1A_2$. We say that $\triangle A_1PA_2$ contains $\triangle B_1PB_2$ properly if $B_1$ or $B_2$ or both is in the open segment $A_1A_2$.

**Proposition 6.** (i) If $P = \min \rho \{B_1PB_2\}$ and $\triangle A_1PA_2$ contains $\triangle B_1PB_2$ then $P = \min \rho \{A_1PA_2\}$

(ii) If $B = \min \rho \{ABC\}$ and $P$ is any point within or on triangle $ABC$ except on $AC$, then $P = \min \rho \{APC\}$.

**Proof** (i) is immediate from the above definition and Proposition 5. To prove (ii) draw parallels to $AB, BC$ through $P$ and let these meet $AC$ in $X$ and $Y$. Since $\triangle XPY$ is a translation of $\triangle ABC$, $P = \min \rho \{FXY\}$. Then by part (i), $P = \min \rho \{APC\}$ as asserted.
Proposition 7. If $C = \min \rho(ABC)$ there exists $A'$ between $A$ and $B$ on the line $AB$ such that for all $X$ on this line between $A'$ and $B$, $\min \rho(XBC) \neq C$.

Proof Let the metric ball centre $B$ through $C$ cut $AB$ at $A'$. $A'$ is between $A$ and $B$ since $AB$ is the "longest" (in the sense of the metric) side. Then for all $X$ between $A'$ and $B$, $XB < A'B$.

$\therefore XC + XB < XC + A'B = XC + BC$.

(by construction $A'B = BC$).

i.e. $\forall X \in \{XBC\} < C(XBC)$ hence $C \notin \min \rho(\{XBC\})$

Definition $\triangle ABC$ is a critical angle if and only if

(i) $\min \rho(ABC) = B$

and (ii) $\min \rho(A'BC') \neq B$ for any $A'$, $C'$ such that

$\triangle A'BC'$ is properly contained in $\triangle ABC$.

Propositions 4 and 7 show that this definition is meaningful and critical angles exist. For the Euclidean metric, critical angles are $120^\circ$ angles. In Minkowski spaces critical angles will vary in their Euclidean magnitude.

Proposition 8. (i) $C = \min \rho(ABC)$ if and only if $\triangle ACB$ contains a critical angle.

(ii) In any triangle exactly one angle or no angle contains a critical angle.

(iii) The angle vertically opposite a critical angle is itself critical.

Proof (i) follows from the definition, (ii) is immediate from the definition and the uniqueness of the minimum point. (iii) is proved using similar triangles. We now prove our principal result on the minimizing point of a triangle:

Theorem 7. For any triangle $ABC$ in $M^2$ exactly one of the following occurs:

Either (i) Exactly one angle (say $\triangle BAC$) contains a critical angle and $A = \min \rho(ABC)$

or (ii) There exists a unique point $P$ at which the sides of the triangle subtend critical angles and $P = \min \rho(ABC)$. 
Proof (i) Suppose $\angle BAC$ contains a critical angle. Then no other angle of $ABC$ contains a critical angle and $A = \min \rho_{ABC}$. (Proposition 8)

(ii) Suppose no angle of $ABC$ contains a critical angle, then $P = \min \rho_{ABC}$ is not at a vertex. Now assume that $\angle APB$ does not contain a critical angle. Then

\begin{equation}
\min \rho_{APB} = X \neq P.
\end{equation}

\begin{equation}
CX \leq PX + PC. \quad \text{(triangle inequality)}
\end{equation}

\[ AX + BX + CX \leq AX + BX + PX + PC < AP + BP + PC. \quad \text{by (9)}. \]

\[ \text{i.e. } X_{\{ABC\}} < P_{\{ABC\}} \text{ contradicting the minimum property of } P. \]

Therefore $\angle APB$ (and similarly $\angle APC, \angle BPC$) contains a critical angle.

We continue the proof by showing that there exists only one point at which each angle subtended by a side of the triangle contains a critical angle. Suppose the contrary, then there exist two such points namely $Q$ and $P = \min \rho_{ABC}$. We consider two separate cases:

Case 1. Assume $P$ is strictly inside triangle $BQC$ and $AP$ meets $QC$ in $X$. Then since $Q = \min \rho_{QBC}$, Proposition 6 implies that $X = \min \rho_{XBC}$.

But $P = \min \rho_{ABC}$. Therefore, by Proposition 3, $P = \min \rho_{XBC}$ which contradicts the uniqueness of $\min \rho_{XBC}$. Proofs for $P$ strictly inside $AQC$ or $AQB$ are similar.

Case 2. Assume $P$ lies in the open segments $QA$. $\angle AQB$ contains a critical angle, therefore $\angle APB$ properly contains a critical angle and there exists a point $R$ on the open segment $CP$ such that $\angle ARB$ contains a critical angle. $\angle s ARC, BRC$ also contain critical angles (Proposition 6), hence each angle subtended at $R$ by a side of $\triangle ABC$ contains a critical angle. We now apply Case 1 using $R$ instead of $Q$ and obtain a similar contradiction.

To complete the proof of Theorem 7 we have only to show that if the angles subtended at a point $P$ by the sides of triangle $ABC$, contain critical angles then these angles are exactly critical angles. Suppose the contrary and $\angle APB$ properly contains a critical angle. Then by
Propositions 5 and 7 there is a point D on AB such that for all A' in the closed segment AD, the angles subtended at P by A'B, BC, A'C each contain a critical angle and hence for all such A', \( P = \min \rho [A'BC] \) which is impossible with a strictly convex unit ball. Hence the Theorem.

**Proposition 9.** If of the three angles subtended by AB, AC, CA at P, two are critical, then the third is critical.

**Proof.** Suppose \( \triangle APC, \triangle BPC \) are critical and \( \triangle APB \) does not contain a critical angle. Then \( \min \rho \{APB\} = X \neq P \) and by Theorem 7 the angles PXB, PXA, AXB are critical. Draw parallels to AX, BX through P and produce XP. We see that one of the critical angles APC, BPC (in Fig. 9 it is \( \triangle APC \)) properly contains a critical angle which is impossible. Therefore \( \triangle APB \) contains a critical angle and by Theorem 7 it is a critical angle.

![Fig 9.](image-url)
Corollary 1. If of 3 angles at a point, 2 contain critical angles, one of them properly, then the third angle does not contain a critical angle.

Corollary 2. Supplementary angles cannot both contain critical angles.

Proof Suppose the contrary and let \( \triangle AOB, \triangle BOC \) be supplementary angles each containing a critical angle (Fig. 10). Take any point \( A' \) within \( \triangle AOB' \) as shown. Then \( \triangle A'OB \) properly contains the critical angle \( AOB \) and \( \triangle BOC \) contains a critical angle by hypothesis. Hence by Corollary 1, \( \triangle A'OC \) does not contain a critical angle. But the critical angle \( B'OC \) (Proposition 8 iii) is contained in \( \triangle A'OC \). Contradiction.

![Diagram](image.png)

Fig. 10.

We digress and state the following fact which will be used in the next section. Suppose \( A, B, C \) are distinct points in an \( m \)-dimensional Minkowski Metric Space with strictly convex defining surface \( S \) and let the plane \( \Pi \) defined by \( A, B, C \) meet \( S \) in the curve \( \Sigma \). Then the above theory holds in the plane Minkowski space defined on \( \Pi \) by \( \Sigma \) i.e. we can apply Propositions 1-9 and Theorem 7 to three points \( A, B, C \) in an \( m \)-dimensional Minkowski Metric Space.
Theorem 8. Let $U$ be a minimising tree of $(S_n)$ in $M^2$ with additional vertices $s_1, \ldots, s_k$. Then $U$ has the properties P1-P4 of section 3 and the following analog of P5:

P"5. For each $i = 1, \ldots, k$, $s_i = \min_p \rho(xyz)$ where $x, y, z$ are the points of $\{s_i\}$ and each angle $xs_iy, ys_iz, zs_ix$ is a critical angle.

Proof Suppose branches $x_1y_1, x_2y_2$ intersected at $p$ (not a vertex of $U$) then some angle at $P$ say $x_1px_2$ does not contain a critical angle. (Proposition 9, Corollary 2). Therefore $\min_p \rho(x_1px_2) = z \neq p$ and a replacement of $x_1p, x_2p$ by the three lines $x_1z, x_2z, pz$ shortens the assumed minimising tree which proves P1. Similarly, no vertex $x$ of $U$ has $w(x) > 3$. Thus $U$ satisfies P3 and $w(s_i) \leq 3$ for all $i = 1, \ldots, k$. There is no gain in introducing additional vertices with valency $< 3$. Hence P2. The proof of P4 is identical to that given for $(S_n)$ in $E^2$ and P"5 is the result of theorem 7.

It is shown in [4] that for $(S_n)$ in the Manhattan metric, where the defining curve is not strictly convex, the property $w(x) \leq 3$ for each vertex $x$ of a minimising tree does not hold.

The equilateral construction of the Steiner point of a triangle in $E^2$ enabled us to solve the problem $(S_n)$. Accordingly one is lead to search for a generalisation of this construction for $P = \min \rho(ABC)$ in $M^2$ when no angle of $\triangle ABC$ contains a critical angle. The following two conjectures were made:

(i) Let $(AB)$ be the third vertex of the triangle built outward on $AB$ whose exterior angles are critical angles. Then the line $(AB)C$ passes through $P$.

(ii) A triangle $XYZ$ in $M^2$ is $\rho$ - equilateral if $\rho(X,Y) = \rho(Y,Z) = \rho(Z,X)$. Let $(AB)$ be the third vertex of the $\rho$ - equilateral triangle built outward on $AB$. Then $(AB)C$ passes through $P$.

We note that in $E^2$ (i) and (ii) are equivalent and may be used to construct the Steiner point. The counter examples below show that in $M^2$
the conjectures are not equivalent and that neither is true in general. The examples will use the metric with unit ball $|x|^p + |y|^p = 1$, which is strictly convex if $p > 1$. Then $\rho((x_1,y_1), (x_2,y_2)) = (|x_2-x_1|^p + |y_2-y_1|^p)^{1/p}$. Let $A = (0,1), B = (1,0), C = (-1,0)$.

Symmetry and uniqueness insist that $P = \min_p \rho(A,B,C)$ lies on the $y$-axis and by elementary methods, the sum $\rho(A,P) + \rho(B,P) + \rho(C,P)$ takes its minimum value at the point $P(0,\lambda)$ where

$$\lambda = \left(\frac{1}{2p/p-1-1}\right)^{1/p}.$$

Recall that if $A$ is moved to any point $A'$ on PA on the same side of $P$ then $P = \min \rho(A'BC)$.

**Example 1.** To show that a triangle whose exterior angles are critical, is not necessarily $\rho$-equilateral.

Using the special case given above, the three angles at $P$ in fig. 11(a) are critical angles since $P = \min \rho(A,B,C)$. AP, BP, CP have slopes $\infty$, $\lambda$, $-\lambda$. The sides of triangle ORX shown in fig. 11(b) have identical slopes and hence the exterior angles of this triangle are critical angles e.g. $\triangle ORQ$ is a translation of the critical angle BPC. We show that triangle ORX is not $\rho$-equilateral.

![Fig. 11](image-url)
\[ \rho(PX) = \left\{ \left( \frac{1}{2} \right)^p + \left( \frac{1}{2^\lambda} \right)^p \right\}^{1/p} = \frac{1}{2} \left( 1 + \frac{1}{\lambda^p} \right)^{1/p} \]

\[ = \frac{1}{2} \left( 1 + \left( \frac{p}{p-1} - 1 \right)^{1/p} \right) = \frac{2}{2-p}/(p-1) \]

But \( \rho(OX) = 1 \). Hence triangle is \( \rho \)-equilateral if and only if \( \frac{2}{2-p}/(p-1) = 1 \).

i.e. if and only if \( p = 2 \) and the metric is Euclidean.

**Example 2.** To show conjecture (i) is false.

In Fig. 12 \( P = \min \rho[ABC] \); the exterior angles \( DAC', EC'A \) of triangle \( ABC' \) are translations followed by a rotation through 90° of the angles \( APC, APB \). Since each such transformation is an isometry for this metric, the angles \( DAC', EC'A \) are critical angles and hence triangle \( ABC' \) has its exterior angles critical. We show that \( CC' \) does not pass through \( P \).
CC' has equation $3\lambda y = 1 - x$ and passes through $(0,\lambda)$ if and only if $3\lambda^2 = 1$ or $\lambda = \frac{1}{\sqrt{3}}$ i.e. CC' passes through P if and only if $p = 2$ and the metric is Euclidean.

Example 3. To show conjecture (ii) is false.

In Fig. 13 A, B' have co-ordinates $(0, (2^p-1)^{1/p})$, $(2, (2^p-1)^{1/p})$. Then it is easily verified that triangle AB'C is $\rho$-equilateral with sides of length 2. We show that BB' does not in general pass through $P = \min \rho \{ABC\}$. BB' has equation $y = \frac{1}{3} (2^p-1)^{1/p} (x + 1)$ hence has y-intercept $(2^p-1)^{1/p}/3$. \text{ '}. BB' passes through $P$ if and only if

$$
(2^p-1)^{1/p}/3 = (2^{p-1}-1)^{1/p}
$$

or

$$
(2^p-1)(2^{p-1}-1) = 3^p.
$$

We see that this is satisfied for the Euclidean case $p = 2$ but is certainly false for any integer $p > 2$. 

\text{Fig. 13}
VI - MORE PROPERTIES OF \((S_{mB})\)

In this chapter we prove two results for \((S_{mB})\), concerning finiteness of solution in Minkowski Space and non-constructibility of solutions in \(E^2\) by Euclidean constructions.

The following theorem uses the notation of Theorem 2. Chapter II.

**Theorem 9.** If \(M\) is a Minkowski Metric space \(M_m\) for which the defining surface \(\Sigma\) is strictly convex, then there is a unique tree of minimum length in \(C \in \{C_1, \ldots, C_N\}\).

**Proof** Suppose the contrary and \(f\) has minima, value \(\ell\), at \(\{s_1, \ldots, s_k\}\) and \(\{t_1, \ldots, t_k\}\) where \(t_i \neq s_i\) for some \(i\). Consider the set \(\{\frac{s_1 + t_1}{2}, \ldots, \frac{s_k + t_k}{2}\}\).

Using Properties (i) - (iii) of Minkowski Spaces Chapter V.

\[
\rho\left(\frac{s_i + t_i}{2}, a_{ij}\right) = \rho\left(0, \frac{s_i - a_{ij}}{2} + \frac{t_i - a_{ij}}{2}\right)
\]

\[
\leq \rho\left(0, \frac{s_i - a_{ij}}{2}\right) + \rho\left(0, \frac{t_i - a_{ij}}{2}\right)
\]

\[
= \frac{1}{2} \rho\left(0, s_i - a_{ij}\right) + \frac{1}{2} \rho\left(0, t_i - a_{ij}\right)
\]

\[
= \frac{1}{2} \rho\left(s_i, a_{ij}\right) + \frac{1}{2} \rho\left(t_i, a_{ij}\right).
\]

This inequality is strict unless \(a_{ij}, s_i, t_i\) are collinear with \(s_i, t_i\) on the same side of \(a_{ij}\).

\[
\therefore \frac{k}{2} \sum_{i=1}^{k} \lambda_i \rho\left(\frac{s_i + t_i}{2}, a_{ij}\right) \leq \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{\lambda_i} \left(\rho\left(s_i, a_{ij}\right) + \rho\left(t_i, a_{ij}\right)\right)
\]

and the inequality is strict unless for each \(i = 1, \ldots, k\), \(a_{i1}, a_{i2}, \ldots, a_{i\lambda_i}, s_i, t_i\) are collinear with \(s_i, t_i\) occupying suitable positions on the line. Such a situation cannot occur in a minimum length tree of \(C\).

Assuming \(n > 2\) (otherwise the problem is trivial), there exists \(i\) for
which \( \lambda_i \geq 2 \) and \( s_i \) joins only one other \( s \)-point. If \( \lambda_i > 2 \) a simple application of the triangle inequality proves that the assumed tree could not be minimum length in \( C \) and the case \( \lambda_i = 2 \) is disposed of using Proposition 9, Corollary 2. Thus we can conclude

\[
\frac{1}{k} \sum_{i=1}^{k} \frac{\lambda_i}{\sum_{j=1}^{\lambda_i}} \rho \left( \frac{s_i + t_i}{2}, a_{ij} \right) < \frac{1}{2} \sum_{i=1}^{k} \frac{\lambda_i}{\sum_{j=1}^{\lambda_i}} \rho \left( s_i, a_{ij} \right) + \frac{1}{2} \sum_{i=1}^{k} \frac{\lambda_i}{\sum_{j=1}^{\lambda_i}} \rho \left( t_i, a_{ij} \right).
\]

By a similar use of the properties of \( M^m \) we can show

\[
\sum_{(i,j) \in R_1} \rho \left( \frac{s_i + t_i}{2}, \frac{s_j + t_j}{2} \right) \leq \frac{1}{2} \sum_{(i,j) \in R_1} \rho \left( s_i, s_j \right) + \frac{1}{2} \sum_{(i,j) \in R_1} \rho \left( t_i, t_j \right)
\]

Adding this to (10) and \( \sum_{(i,j) \in R_2} \rho \left( a_i, a_j \right) \) to both sides we obtain

\[
f \left( \frac{s_1 + t_1}{2}, \ldots, \frac{s_k + t_k}{2} \right) < \frac{1}{2} f(s_1, \ldots, s_k) + \frac{1}{2} f(t_1, \ldots, t_k) = \ell
\]

which contradicts the minimum property of \( \ell \).

**Corollary** In \( M^m \), \( (S_n^\alpha\beta\gamma) \) and \( (S_n^\alpha\beta) \) have a finite number of minimising trees.

**Proof** Immediate from Theorems 1, 2 and 9.

Similar proofs to those given here may be used to establish identical results when the function to be minimised is

\[
F \left( L(U), \sum_{i=1}^{n} w(a_i), \sum_{j=1}^{k} w(s_j), k \right)
\]

where \( F \) is any positive function which is strictly increasing in each of its four variables.

It was demonstrated in Chapter 1 that \( (S_n^\alpha\beta\gamma) \) reduces to \( (P_n) \) for suitable values of the constants \( \alpha, \beta, \gamma \). We show now that in general \( (P_n) \) in \( E^2 \) (and hence \( (S_n^\alpha\beta\gamma) \) ) is not solvable by Euclidean constructions.

We use \( n = 5 \) for our example since \( (P_3) \) is solvable by Euclidean constructions (by our equilateral construction). The solution of \( (P_4) \) is the intersection of the diagonals if the configuration is convex and the vertex interior to
the convex hull otherwise. We take 5 points \( A_i \) \( (i = 1, \ldots, 5) \) symmetrically placed with respect to the x-axis as shown in fig. 14.

\[
\sum_{i=1}^{5} PA_i = x + 2\sqrt{1+x^2} + 2\sqrt{b^2+(a-x)^2}.
\]

Minimising this function by elementary methods, we find that the co-ordinate \( x \) of \( P \) satisfies an eighth degree polynomial equation \( f(x) = 0 \) whose coefficients are polynomials in \( a \) and \( b \).

We show that for suitable integers \( a, b \), \( f(x) \) is irreducible over the rationals and \( f(x) = 0 \) has Galois group over the rationals which does not have order \( 2^k \) where \( k \) is a positive integer. Therefore \( x \) is not an element belonging to an extension field of the rationals of degree \( 2^k \) and hence the segment \( OP \) is not constructible by Euclidean constructions (See [10] page 185), i.e. for suitable choices of the five points \( (P_5) \) is not solvable by Euclidean constructions.
The leading coefficient of \( f(x) \) is 15. In order to use the theory of [10] page 190-191, we need to work with a monic polynomial and therefore make the transformation \( x = y/15 \) and multiply the equation through by \( 15^7 \) thus obtaining equation \( g(y) = 0 \) where \( g(y) \) is monic. We note that such a transformation does not affect reducibility over the rationals or the Galois group of the equation. The coefficients of \( g(y) \) are:

\[
\begin{align*}
y^8 & : 1 \\
y^7 & : -60a \\
y^6 & : 15 (90a^2 + 22b^2 + 22) \\
y^5 & : -15^2 (88a + 60a^3 + 44ab^2) \\
y^4 & : 15^3 (42 + 15a^4 + 154a^2) \\
y^3 & : -15^4 (88a^3 + 120ab^2 - 36a) \\
y^2 & : 15^5 (22a^4 + 60a^2b^2 + 6b^4 + 6b^2 - 54a^2) \\
y & : 15^6 . 12a . (3a^2 - b^2) \\
1 & : -15^7 (b^2 - 3a^2)^2 .
\end{align*}
\]

We take \( a = 10^b = 3 \) and notice that the coefficients of \( y^8 \) and the constant are odd while the rest of the coefficients are even, so that Eisenstein's irreducibility criterion using the prime 2 shows that \( g(y) \) is irreducible over the rationals and hence \( g(y) = 0 \) has no multiple root. Let \( \mathbb{Q}/(7) \) be the field of residue classes of integers modulo 7. Over this field \( g(y) = 0 \) reduces to the equation

\[
\begin{align*}
g_7(y) &= y^8 + 2y^7 + y^6 + y^5 + 4y^4 + y^3 + 4y^2 + 4y + 5 = 0 .
\end{align*}
\]

The greatest common divisor of \( g_7(y) \) and its derivative over \( \mathbb{Q}/(7) \) in 1, hence \( g_7(y) = 0 \) has no multiple root. \( g_7(y) \) has the following factorisation mod 7:

\[
(11) \quad (y^3 + 5y^2 + 4y + 2)(y^5 + 4y^4 + 5y^3 + 4y + 6)
\]

and the cubic factor is irreducible mod 7. Therefore the Galois group of the equation \( g(y) = 0 \) contains a permutation \( \alpha \) whose representation as a product of disjoint cycles contains a cycle of order 3. \( \alpha \) does not have order \( 2^k \) for any positive integer \( k \), hence the Galois group does not have
order $2^k$ and the proof is complete. We state the result formally:

**Theorem 10.** $(P_n)$ and $(S_{n\alpha \beta \gamma})$ in $E^2$ are not, in general, solvable by Euclidean constructions.

Finally, we show how the above example was found. 7 is the smallest prime which can be used. (It is clear from the coefficients that $g(y)$ has multiple roots modulo 2, 3 and 5). One therefore searches for values of $a$ and $b$ such that $g_7(y) = 0$ has no multiple root and an irreducible cubic, quintic or sextic factor mod 7. The arithmetic being mod 7, it suffices to consider $a,b = 0,\ldots,6$. In fact since $b$ only occurs in the coefficients as $b^2$, we need only take $b = 0,1,2,3$ as $b^2 \equiv (7-b)^2 \pmod{7}$. The 28 polynomials were tested for an irreducible cubic factor using a digital computer. The method used was to divide each polynomial by the finite number of irreducible cubics over $\mathbb{F}_7$ and investigate the remainder. The values $a = 3, b = 3$ yielded the factorisation (11). However Eisenstein's criterion works conveniently for $g(y)$ with prime 2 if $a$ is even and $b$ is odd so we take $a = 10, b = 3$. The polynomial obtained has the same reduction (11) mod 7.

This method of reduction mod $p$ in conjunction with a computing machine can provide much information on the structure of Galois groups of equations.
BIBLIOGRAPHY


