ON THE BOUNDARY OF SOME FUNCTION ALGEBRAS

by

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ABSTRACT

The aim of this thesis is to prove the existence of the Shilov boundary and the minimal boundary with respect to some function algebras and investigate their topological structures.

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SECTION 1 Introduction

Let X be a compact topological space. We shall denote by $C(X)$ the set of all real-valued continuous functions f on X. It is well known that $C(X)$ with the uniform (or sup-) norm:

$$\|f\| = \sup \{|f(x)| : x \in X\}$$

is a commutative Banach algebra with unity over the real field R.

We say that a set C of functions f defined on X is separating, or C separates the points of X, if for every pair of points $x \neq y$ in X, there is a function f in C such that $f(x) \neq f(y)$.

A point x_o in X is said to be a strong boundary point of X relative to C if for each neighborhood V of x, there is a function f in C such that $\|f\| = |f(x_o)|$ and for any $y \in X$ with $\|f\| = |f(y)|$ we have: $y \in V$.

A point x_o in X is said to be a peak point of X relative to C if there is a function f in C such that $\|f\| = |f(x_o)|$ and for every y in X such that $y \neq x_o$ we have: $|f(x_o)| > |f(y)|$.

Shilov proved that [7, p. 80] if A is a separating subalgebra of $C(X)$ then there is a smallest closed subset $\partial_A X$ of X (called the Shilov boundary for X with respect to A) having the property that every function in A attains its maximal absolute value at some point of $\partial_A X$.

In view of the above theorem, E. Bishop asked whether there existed a smallest subset $M_A X$ of X (called the minimal boundary of X with respect to A) such that every function in A attains its maximal absolute value at some point of $M_A X$. He answered this question in theorem 1 of [1]: If X is a compact metrisable Hausdorff space and if A is a separating Banach subalgebra of $C(X)$, then A has a minimal boundary $M_A X$ and $M_A X = P_A X$ where $P_A X$ is the set of all peak points relative to A.

We prove a more general result in theorem 4.11: If X is a compact Hausdorff space and A is a separating closed subalgebra of $C(X)$ which contains the constant function 1, then $S\partial_A X$, the set of all strong boundary points relative to A, is the minimal boundary for X relative to A provided that every point of $S\partial_A X$ is a G_α (i.e. an intersection of countably many open sets), and in this case $M_A X = S\partial_A X = P_A X$.

In [1], E. Bishop also gave two examples; one of them shows that the metrisability of the space X is necessary in theorem 1 of [1], the other shows that the minimal boundary is distinct from the shilov boundary for function algebras in general. We present these examples in detail in section 5.

Recently, Jozef Siciak [5] proved that the Shilov boundary exists for some function families more general than algebras, namely, for separating families C which are closed under either

multiplication or addition of functions of C. Also M. W. Grossman [6] proved the following result: Let X_1, X_2 be two compact Hausdorff spaces and H_1, H_2 be two separating linear subspaces of $C(X_1)$, $C(X_2)$ respectively such that both H_1 and H_2 contain the constant functions. Then a subfamily $H_1 + H_2$ of $C(X_1 \times X_2)$ can be defined in a natural way, for which $\partial_{H_1} X_1 \times \partial_{H_2} X_2 = \partial_{H_1+H_2}(X_1 \times X_2)$. We prove some similar results in section 6 and 7.

Finally, in section 8 we prove that if X, Y are two compact Hausdorff spaces, if T is a homeomorphism of X onto Y and if T' from $C(Y)$ into $C(X)$ is the mapping induced by T:

$$T'(g) = g \circ T \quad , \qquad g \in C(Y)$$

then

$$T[\partial_A X] = \partial_{T'^{-1}[A]} Y$$

where A is any subset of $C(X)$ for which $\partial_A X$ exists.

SECTION 2 The existence of Shilov boundary.

Let X be a compact topological space.

2.1 Definition. For any f in $C(X)$, the subset

$$S(f) = \{x : |f(x)| = \|f\|\}$$

of X is called the maximal set of f. Clearly $S(f)$ is closed and non-empty.

2.2 Definition. Let C be any non-empty subset of $C(X)$. Then a subset N of X is called a C-set if and only if

$$N \cap S(f) \neq \emptyset, \quad \text{for all } f \text{ in } C.$$

A C-set N which is closed in the topological space X is called a closed C-set.

Let $\mathcal{F}$ be the collection of all closed C-sets, partially ordered by inclusion. Then a set N in $\mathcal{F}$ which does not properly contain any other element in $\mathcal{F}$ is called a minimal closed C-set. If a set N in $\mathcal{F}$ has the much stronger minimal property of being contained in every element of $\mathcal{F}$ then it is called the C-Shilov boundary of X or simply Shilov boundary of X and is denoted by $\partial_C X$. It is obvious that $\partial_C X$ is uniquely determined if it exists.

2.3 Proposition. For any non-empty subset C of $C(X)$, there always exists a minimal closed C-set in $\mathcal{F}$.

Proof. $\mathcal{F}$ is not empty since X is in $\mathcal{F}$. Consider any linearly ordered subfamily $\mathcal{F}'$ of $\mathcal{F}$. Let

$$F = \cap\{N : N \text{ is in } \mathcal{F}'\} ,$$

which being an intersection of closed sets is closed.

We prove that F is a (closed) C-set.

For any f in C, if $f = 0$ then $S(f)$ is the whole space X, and clearly $F \subset S(f)$. If $f \neq 0$ then $S(f)$ which is a closed subset of the compact space X is therefore compact. Consider

$$\Omega = \{S(f) \cap N : N \text{ is in } \mathcal{F}'\} ,$$

each $S(f) \cap N \neq \emptyset$ and is closed in the set $S(f)$, and since $\mathcal{F}'$ has the finite intersection property, so does Ω. Hence

$$S(f) \cap F = \cap\Omega \neq \emptyset .$$

Thus F is a closed C-set and F is a lower bound for $\mathcal{F}'$. By Zorn's lemma, there is a maximal linearly ordered subfamily $\mathcal{F}_0$ of $\mathcal{F}$. Then $\cap \mathcal{F}_0$ is in $\mathcal{F}$. Indeed, it is a lower bound for $\mathcal{F}_0$ and therefore, a minimal element of $\mathcal{F}$ by the maximality of $\mathcal{F}_0$.

2.4 <u>Definition</u>. A <u>subset C of $C(X)$ is said to separate the points of X</u> if for any two distinct points x,y in X there is a function f in C such that $f(x) \neq f(y)$.

2.5 <u>Proposition</u>.

(i) If X is Hausdorff then $C(X)$ separates the points of X.

(ii) If there is a subset C of $C(X)$ which separates the points of X then X is Hausdroff.

Proof.

(i) For any x,y in X, $\{x\}$ and $\{y\}$ are closed subsets since X is Hausdorff. Let $x \neq y$. Since X is compact and Hausdorff, it is normal, and by Urysohn's lemma, there is a function f in $C(X)$ such that $f(x) = 1$, $f(y) = 0$. Hence $C(X)$ separates the points of X.

(ii) For any two distinct elements x,y in X, let f be a function in C with $f(x) \neq f(y)$. Let U,V be disjoint neighborhoods of $f(x)$ and $f(y)$ respectively. Then $f^{-1}(U)$, $f^{-1}(V)$ are disjoint neighborhoods of x,y respectively. Hence X is Hausdorff.

2.6 Lemma. Let C be a subset of $C(X)$ which separates the points of X. Then the weak topology τ' induced on X by C is identical with the original topology τ for X.

Proof. Since τ' is the smallest topology for X such that every function in C is continuous, $\tau' \subset \tau$. By proposition 2.5 (ii), τ' is a Hausdorff topology for X. Let F be a τ-closed subset of X. Since (X,τ) is compact, F is τ-compact. Let $\mathcal{F}$ be a τ'-open cover of F. Then $\mathcal{F}$ is a τ-open cover of F since $\tau' \subset \tau$. Hence there is a finite subcover of $\mathcal{F}$. This proves that F is τ'-compact, but a compact subset of a Hausdorff space is closed, so F is τ'-closed. Therefore $\tau \subset \tau'$. Consequently, $\tau = \tau'$.

We have seen in proposition 2.3 that minimal closed C-sets exist without further conditions on C. But for arbitrary C, the Shilov boundary does not exist in general. For instance, let C consist of all constant functions in $C(X)$ where X is a compact, Hausdorff space containing more than one point. Then every singleton $\{x\}$ of X is a minimal closed C-set. Thus the Shilov boundary of C does not exist. However, we have the following theorem:

2.7 <u>Theorem</u>. Let A be a subalgebra of $C(X)$ and suppose that A separates the points of X. Then the A-Shilov boundary exists.

<u>Proof</u>. In view of lemma 2.6, we may assume that the given topology for X is the A-topology, i.e. the topology induced by A. By proposition 2.3, there is a minimal closed A-set F in the collection $\mathcal{F}$ of all closed A-sets. We shall prove that $F = \partial_A X$ by showing that F is contained in every element of $\mathcal{F}$. To do this, it suffices to show that for any closed subset B of X, if B does not contain F, then B is not an A-set.

Accordingly, let B be any closed subset of X, which does not contain F. Then $F-B$ is not empty, we can choose an element a in $F-B$ which is contained in the open set $X-B$, so there exist $f_1, f_2, \ldots, f_k$ in A and a real number $\epsilon > 0$ such that

$$U = \{x : |f_i(x) - f_i(a)| < \epsilon \ , \quad i = 1,2,\ldots,k\}$$

is an open neighborhood of a and $U \cap B = \emptyset$, $F-U$ is a proper closed subset of F and since F is minimal in $\mathcal{F}$, $F-U$ is not an A-set. Therefore, there exists $g(\neq 0)$ in A with

$$\sup_{x \in F-U} |g(x)| < \|g\|$$

Let b be in $S(g)$. Since $F-U$ is closed in X , it is compact. There is an element c in $F-U$ such that

$$|g(c)| = \sup_{x \in F-U} |g(x)|$$

Thus

$$0 \leq |g(c)| \quad |g(b)|^{-1} < 1$$

and hence for a large integer n ,

$$(|g(c)| \quad |g(b)|^{-1})^n < (1 + \sum_{i=1}^{k} \sup_{x \in X} | f_i(x) - f_i(a)|)^{-1} \epsilon = \delta$$

so $$|g(c)|^n < \delta |g(b)|^n \tag{1}$$

and $$\delta |f_i(x) - f_i(a)| < \epsilon \quad \text{for all } x \text{ in } X \text{ and } i = 1,\ldots k \tag{2}$$

Consider the function $h = g^n (\neq 0)$ which is in A and

$$|g(b)|^n = \|h\|$$

$$|g(c)|^n = \sup_{x \in F-U} |h(x)|$$

From (1),

$$\sup_{x \in F-U} |h(x)| < \delta \|h\| \ .$$

If x is in U then $|f_i(x)-f_i(a)| < \epsilon$, $i = 1,2,\ldots,k$ so that

$$|(f_i h-f_i(a)h)(x)| = |f_i(x)-f_i(a)|\ |h(x)| < \epsilon\ \|h\|$$

If x is in $F-U$, then $|h(x)| < \delta\|h\|$, so that

$$\begin{aligned} |(f_i h-f_i(a)h)(x)| &< |f_i(x)-f_i(a)|\ \delta\|h\| \\ &< \epsilon\|h\| \qquad \text{by (2)} \end{aligned}$$

Hence, for all x in F , we have:

$$|(f_i h-f_i(a)h)(x)| < \epsilon\|h\| \ , \quad \text{for } i = 1,2,\ldots,k \ .$$

Since A is an algebra, $f_i h-f_i(a)h$ is in A and since F is an A-set, there is an element z in F such that

$$|(f_i h-f_i(a)h)(z)| = \|f_i h-f_i(a)h\|$$

so

$$\|f_i h-f_i(a)h\| < \epsilon\ \|h\| \ .$$

Let x' be in $S(h)$. Then for each $i = 1,2,\ldots,k$

$$\begin{aligned} |h(x')|\ \ |f_i(x')-f_i(a)| &= |(f_i h-f_i(a)h)(x')| \\ &\leq \|f_i h-f_i(a)h\| \\ &< \epsilon\ \|h\| \\ &= \epsilon\ |h(x')| \end{aligned}$$

Since $h \neq 0$, $|h(x')| \neq 0$. Hence

$$|f_i(x')-f_i(a)| < \epsilon \qquad i = 1,2,\ldots,k \ .$$

This shows that x' is in U and hence $S(h) \subset U$. But $U \cap B = \emptyset$ so $S(h) \cap B = \emptyset$. This completes the proof that B is not an A-set.

SECTION 3 Some special boundary points.

Let X be a compact Hausdorff space.

3.1 Theorem. Let C be a subfamily of $C(X)$ such that the Shilov boundary exists. Then for any x in X, x belongs to $\partial_C X$ iff for every neighborhood V of x, there exists a function f in C with $S(f) \subset V$.

Proof. Suppose x is in $\partial_C X$ and there exists an open neighborhood V of x such that V does not contain $S(f)$ for any f in C. Then

$$S(f) \cap (X-V) \neq \emptyset$$

for every f in C. Hence $X-V$ is a closed C-set and $X-V$ does not contain $\partial_C X$, since x is not in $X-V$. This contradicts the fact that every closed C-set contains $\partial_C X$.

Suppose every neighborhood V of x contains $S(f)$ for some f in C. Since $S(f) \cap \partial_C X \neq \emptyset$, $V \cap \partial_C X \neq \emptyset$. This means that every neighborhood of x intersects $\partial_C X$ and since $\partial_C X$ is closed, x is in $\partial_C X$.

This theorem tells us that the Shilov boundary of C consists of all points x with the property that, for every neighborhood V of x, there exists some f in C such that the maximal absolute value of f is assumed only inside V. But notice that this maximal value need not be assumed at x itself, for instance, let X be the closed interval $[0,1]$ with the indiscrete topology and let C be the family of all powers

$f_n = x^n$, $n = 1,2,\ldots$. Then $\partial_C X = X$. But every f_n assumes its absolute maximal value at 1 only.

3.2 <u>Definition</u>. Let C be a subfamily of $C(X)$ and x a point of X . We call x a <u>strong boundary point</u> of X relative to C if for each neighborhood V of x , there is a function f in C such that x is a point of $S(f)$ and $S(f) \subset V$.

Denote the set of all strong boundary points by $S\partial_C X$. From theorem 3.1 , $S\partial_C X \subset \partial_C X$ if $\partial_C X$ exists.

3.3 <u>Definition</u>. If $S(f) = \{x : f(x) = 1\}$ then we call $S(f)$ a <u>special maximal set</u>.

If $S(f)$, $S(g)$ are two (special) maximal sets then $S(f) \cap S(g)$ is either void or equal to the (special) maximal set $S(fg)$. Indeed, if $S(f) \cap S(g) \neq \emptyset$ then

$$x \in S(f) \cap S(g) \Rightarrow |f(x)| \geq |f(y)| , \quad |g(x)| \geq |g(y)|, \text{ for all } y \in X.$$
$$\Rightarrow |f(x)g(x)| \geq |f(y)g(y)| , \quad \text{for all } y \text{ in } X.$$
$$\Rightarrow x \in S(fg)$$

and

$$x \notin S(f) \cap S(g) \Rightarrow |f(x)| < |f(z)| \text{ or } |g(x)| < |g(z)| , \quad \text{for all } z \in S(f) \cap S(g)$$
$$\Rightarrow |f(x)g(x)| < |f(z)g(z)| , \quad z \in S(f) \cap S(g)$$
$$\Rightarrow x \notin S(fg) .$$

Hence the intersection of any finite collection of (special) maximal sets is either void or a (special) maximal set.

3.4 <u>Lemma.</u> Let f be a non-zero function in $C(X)$. Let $a \in S(f)$. Define $g = (1/2)(f/f(a) + 1)$. Then

$$S(f) \supset \{x : f(x) = f(a)\} = \{x : g(x) = 1\} = S(g)$$

<u>Proof.</u> $a \in S(g)$. Indeed

$$|g(a)| = \frac{1}{2} \left| f(a)/f(a) + 1 \right| = 1$$

and for any x in X, $|f(x)| \leq |f(a)|$ since $a \in S(f)$.

So $-1 \leq f(x)/f(a) \leq 1$. This implies

$$|g(x)| = \frac{1}{2} \left| f(x)/f(a) + 1 \right| \leq 1$$

Hence $a \in S(g)$ and $\|g\| = 1$.

Let $b \in S(g)$ then

$$|g(b)| = \frac{1}{2} \left| f(b)/f(a) + 1 \right| = 1 \qquad (1)$$

But $-1 \leq f(b)/f(a) \leq 1$, (1) is true only if $f(b)/f(a) = 1$ or $f(b) = f(a)$.

Therefore

$$S(g) \subset \{x : f(x) = f(a)\}.$$

On the other hand, if $f(x) = f(a)$ then

$$g(x) = (1/2)(f(x)/f(a) + 1) = 1. \text{ Thus}$$

$$\{x : f(x) = f(a)\} \subset \{x : g(x) = 1\}.$$

But

$$S(g) \supset \{x : g(x) = 1\}$$

We have:

$$S(g) \subset \{x : f(x) = f(a)\} \subset \{x : g(x) = 1\} \subset S(g) .$$

Moreover, $S(f) \supset \{x : f(x) = f(a)\}$. Hence

$$S(f) \supset \{x : f(x) = f(a)\} = \{x : g(x) = 1\} = S(g) .$$

3.5 Remark. It follows from lemma 3.4 that, given a subalgebra A of $C(X)$ such that A contains 1 , then for any f in A and any x_o in $S(f)$ there is a function g in A such that $S(g)$ is a special maximal set and $x_o \in S(g) \subset S(f)$.

3.6 Remark. (Kelley [9], Problem F (a) p. 163) For any topological space X , let Ω be a family of closed compact subsets of X such that $\cap\Omega$ is a subset of an open set U . Then there is a finite subfamily F of Ω such that $\cap F \subset U$.

3.7 Definition. Let A be a subfamily of $C(X)$. We call the intersection of an arbitrary collection of special maximal sets associated with elements of A simply a special set associated with A .

Since X is compact and every maximal set is closed, special sets are compact and the intersection of any family of special sets is again a special set.

3.8 Theorem. Let x be any point in X , and A a subalgebra of $C(X)$ which contains the function 1 . Then $\{x\}$ is

a special set associated with A iff x is a strong boundary point relative to A.

<u>Proof</u>. Suppose x is a strong boundary point of X relative to A. Then for any open neighborhood V of x there is an f in A with $x \in S(f) \subset V$. By remark 3.5, there exists g in A such that $S(g)$ is a special maximal set and $x \in S(g) \subset S(f) \subset V$.

If $S(g) = \{x\}$, there is nothing left to prove. Otherwise, for any y in $S(g)$ with $x \neq y$, since X is Hausdorff, there exist disjoint neighborhoods V_y, U_y of x and y respectively. Since x is a strong boundary point, by the above argument, there exists a special maximal set $S(g_y)$ where g_y is in A such that

$$x \in S(g_y) \subset V_y .$$

Since $V_y \cap U_y = \emptyset$, $y \notin S(g_y)$. Thus

$$\bigcap_{\substack{y \in S(g) \\ y \neq x}} (S(g) \cap S(g_y)) = \{x\} .$$

Hence $\{x\}$ is a special set associated with A.

Suppose

$$\{x\} = \bigcap_{g \in B} S(g)$$

for some $B \subset A$, and for each g in B, $S(g)$ is a special maximal set. Let V be any neighborhood of x. Then

$$\{x\} = \bigcap_{g \in B} S(g) \subset V ,$$

and each $S(g)$ is a closed compact subset of X. By remark 3.6 there exists a finite number of elements $g_1, g_2, \ldots g_n$ in B such that

$$x \in \bigcap_{i=1}^{n} S(g_i) \subset V .$$

Since

$$\bigcap_{i=1}^{n} S(g_i) = S(g_1 g_2 \ldots g_n)$$

and $g_1 g_2 \ldots g_n$ is in A, x is a strong boundary point of X relative to A.

3.9 <u>Definition</u>. Let A be a subfamily of $C(X)$. A point x in X is a <u>peak point of X relative to A</u> if there exists a function f in A with $S(f) = \{x\}$. Note that $f \neq 0$.

If A is a subalgebra, we may assume that $S(f)$ is a special maximal set i.e. $f(x) = 1$, since if $f(x) > 0$ then $g = f/\|f\|$ is in A and $S(g) = S(f)$ and $g(x) = 1$. If $f(x) < 0$, then $g = -f/\|f\|$ is in A and $S(g) = S(f)$ and $g(x) = 1$.

3.10 <u>Theorem</u>. Let A be a closed subalgebra of $C(X)$ and the function 1 belong to A. A point a in X is a peak point of X relative to A if and only if a is a strong boundary point of X relative to A and a is a G_δ-set i.e. $\{a\}$ is an intersection of countably many open sets.

<u>Proof.</u> Suppose a is a peak point. Let f be in A and $S(f) = \{a\}$, $f(a) = 1$. Then the sets

$$G_n = \{x \; ; \; 1 - (1/n) < |f(x)|\}, \quad n = 1,2,\ldots$$

are open and

$$\bigcap_{n=1}^{\infty} G_n = \{a\}$$

i.e. $\{a\}$ is a G_δ-set.

Since for every neighborhood V of a , $\{a\} = S(f) \subset V$, a is a strong boundary point.

On the other hand, let $\{a\} = \bigcap_{n=1}^{\infty} G_n$ where each G_n is open. If a is a strong boundary point then for each n , as shown in the proof of theorem 3.8, there is a special maximal set $S(f_n)$ (f_n is in A) which contains a and is contained in G_n .

Define

$$f = \sum_{n=1}^{\infty} f_n/2^n$$

which is in A since A is closed. We have:

$$\{a\} = \bigcap_{n=1}^{\infty} S(f_n) = \bigcap_{n=1}^{\infty} G_n$$

and

$$\begin{aligned} f(a) &= \sum_{n=1}^{\infty} f_n(a)/2^n \\ &= \sum_{n=1}^{\infty} 1/2^n \\ &= 1 . \end{aligned}$$

For x in X and $x \neq a$, there exists some k such that x is not in $S(f_k)$, so $|f_k(x)| < 1$. Hence $|f(x)| < 1$. Thus $S(f) = \{a\}$.

SECTION 4 The existence of the minimal boundary.

Let X be a compact Hausdorff space and C a subset of $C(X)$. Let $\mathcal{H}$ be the collection of all C-sets and $\mathfrak{F}$ the collection of all closed C-sets. In theorem 2.7, we have proved that if C is an algebra of continuous functions on X which separates the points of X then there is a smallest element in $\mathfrak{F}$ i.e. an element being contained in every other element in $\mathfrak{F}$. But this is not true for $\mathcal{H}$ with respect to an algebra C of continuous functions on X which separates the points of X in general. This will be shown by an example in the next section. However, if there does exist a smallest element in $\mathcal{H}$, we shall call such an element the (C-) minimal boundary of X, and denote it by $M_C X$.

4.1 Proposition. For any subset C of $C(X)$ let $P_C X$ be the set of all peak points of X relative to C. Then $P_C X$ is contained in every C-set.

Proof. Let N be any C-set and x be any peak point. Then there exists f in C with $S(f) = \{x\}$. But $S(f) \cap N \neq \emptyset$ hence x is in N. Thus $P_C X$ is contained in N.

It follows from proposition 4.1 that if $P_C X$ is a C-set then it is the C-minimal boundary. So, our aim now is to answer the question: "What conditions have to be imposed on C and X so that $P_C X$ is a C-set?" To answer this, we need some preliminary results.

4.2 <u>Definition</u>. Let C be a subset of $C(X)$ and F be a special set associated with C. Then F is called minimal if it is non-empty and does not properly contain non-empty special set.

4.3 <u>Theorem</u>. Let A be a subalgebra of $C(X)$ which contains 1. Then every maximal set contains a minimal special set.

<u>Proof</u>. For any f in A, because of remark 3.5, we may assume that $S(f)$ is a special maximal set. Let E be the collection of all non-empty special sets which are contained in $S(f)$. Since $S(f)$ is in E, $E \neq \emptyset$. Since $S(f)$ is compact and every special set is closed, and any non-empty linearly ordered subset E' of E has the finite intersection property, E' has a non-empty intersection, which is again a special set. Hence by Zorn's lemma, E has a minimal element. Thus $S(f)$ contains a minimal special set.

4.4 <u>Lemma</u>. Let A be a closed subalgebra of $C(X)$ which contains 1. If f is in A then $|f|$ is in A where $|f|(x) = |f(x)|$ for every x in X.

<u>Proof</u>. The binomial expansion for $(1-a)^{1/2}$ converges uniformly on the interval $[0,1]$. Thus the binomial expansion for

$$(1 - (1 - f^2/\|f\|^2))^{1/2}$$

converges uniformly on X to

$$(f^2/\|f\|^2)^{1/2} = |f| / \|f\| .$$

Therefore $|f| / \|f\|$ and hence $|f|$ is in A for every f in A and $f \neq 0$. If $f = 0$, then $|f| = f$. So f in A implies that $|f|$ is in A.

Among the continuous functions on X we define a partial ordering as follows:

$$f \leq g \quad \text{if} \quad f(x) \leq g(x) \qquad \forall\, x \in X .$$

Thus we have:

4.5 <u>Proposition</u>. Let C be a subset of $C(X)$. Then C is a <u>lattice</u> if and only if, the functions

$$(f \vee g)(x) = \max(f(x), g(x))$$

and

$$(f \wedge g)(x) = \min(f(x), g(x))$$

are in C for every pair f and g in C.

4.6 <u>Corollary</u>. Let A be a closed subalgebra of $C(X)$ which contains 1. Then A is a lattice.

<u>Proof</u>. This follows from lemma 4.4 and the identities:

$$f \vee g = (1/2)(f+g + |f-g|)$$
$$f \wedge g = (1/2)(f+g - |f-g|) .$$

4.7 <u>Theorem</u>. Let A be a closed subalgebra of $C(X)$ which contains 1 and let F be a non-empty special set associated with A. Let $A|_F$ be the subalgebra of $C(F)$ obtained by restriction of functions of A to F. For any proper subset F_o of F, if F_o is a special

maximal set of some element in $A|_F$ then it is a special set associated with A.

<u>Proof</u>. Let f be in A and denote the restriction of f to F by $f|_F$. Suppose $S(f|_F) = F_o$ is a special maximal set i.e.

$$F_o = \{x \in F : f(x) = 1\}$$

and $|f(x)| < 1$ for $x \in F - F_o$.

For each positive integer n, let

$$G_n = \{x \in X : |f(x)| < 1 + 1/2^n\} .$$

Then G_n is an open set in X and G_n contains F. Since F is a special set associated with A, we may write:

$$F = \cap\{S(h) : h \in B\}$$

where $B \subset A$ and each $S(h)$ is a special maximal set. Since X is compact, and each $S(h)$ is a compact closed set, by remark 3.6, there exists a finite number of functions $g_{1,n}, g_{2,n}, \ldots g_{k,n}$ in B such that

$$F \subset F_n = \bigcap_{i=1}^{k} S(g_{i,n}) \subset G_n .$$

Let $f_n = g_{1,n} \cdot g_{2,n} \cdot \ldots g_{k,n}$, which is in A. Then

$$S(f_n) = \bigcap_{i=1}^{k} S(g_{i,n}) = F_n$$

is a special maximal set of f_n.

For $x \notin G_n$, we have: $x \notin F_n$ so $|f_n(x)| < 1$, hence

$\max_{x \notin G_n} |f_n(x)| < 1$. Thus for a large integer k_n ,

$$|f_n(x)|^{k_n}|f(x)| < 1/2^n$$

for $x \notin G_n$.

Define $g = \sum_{n=1}^{\infty} f_n^{k_n} f/2^n$. Let x by any point in X .

<u>Case 1.</u> x is in G_n for every n . Then

$$|f(x)| < 1 + 1/2^n \quad \text{for every } n ,$$

so

$$|f(x) \leq 1 .$$

Hence

$$|g(x)| \leq \sum_{n=1}^{\infty} |(f_n^{k_n} f)(x)|(1/2^n)$$

$$\leq \sum_{n=1}^{\infty} 1/2^n \qquad (\text{since } \|f_n^{k_n}\| = 1)$$

$$= 1 .$$

<u>Case 2.</u> x is in some G_n , but not in all of them. Since

$$G_1 \supset G_2 \supset G_3 \ \ldots\ldots\ldots$$

there exists m such that $x \in G_{m-1}$ while $x \notin G_n$ for $n \geq m$.

Thus

$$|(f_n^{k_n} f)(x)| < 1 + (1/2^{m-1})$$

for $n = 1,2,3,\ldots,m-1$. And

$$|(f_n^{k_n} f)(x)| < 1/2^n < 1/2^{m-1}$$

for $n \geq m$. Therefore,

$$|g(x)| < (1 + (1/2^{m-1})) \sum_{n=1}^{m-1} 1/2^n + 1/2^{m-1} \sum_{n=m}^{\infty} 1/2^n$$

$$\leq (1 + (1/2^{m-1}))\,(1 - (1/2^{m-1}) + (1/2^{m-1})(1/2^{m-1})$$

$$= 1 .$$

<u>Case 3.</u> x is in $X-G_1$. Then $x \notin G_n$ for every n , so

$$|g(x)| \leq \sum_{n=1}^{\infty} (1/2^n)^2$$

$$< \sum_{n=1}^{\infty} 1/2^n$$

$$= 1 .$$

In all cases, the series defining $g(x)$ converges uniformly, and since A is closed, g is in A .

Next, for x in $\bigcap_{n=1}^{\infty} F_n \supset F$, $f_n(x) = 1$ for all n , so

$$g(x) = \sum_{n=1}^{\infty} f(x)/2^n$$

$$= f(x) .$$

It follows that $\|g\| = 1$ since $g(x) = f(x) = 1$ for x in F_o . Therefore

$$F_o = \{ x \text{ in } F: f(x) = 1\}$$

$$= \{ x \text{ in } F: g(x) = 1\}$$

$$= F \cap \{ x \text{ in } X: g(x) = 1\} .$$

Moreover,

$$F_o = F \cap \{ x \text{ in } X: |g(x)| = 1\} .$$

Since $x \in F \cap \{ y \text{ in } X : |g(y)| = 1\}$ implies x in $F_o \cup (F - F_o)$ and $|g(x)| = 1$, but $x \in F - F_o$ yields $|g(x)| = |f(x)| < 1$, this contradicts that $|g(x)| = 1$. Hence x is in F_o. On the other hand,

$$F_o \subset F \cap \{x \text{ in } X : |g(x)| = 1\}$$

is clear. Consequently,

$$\begin{aligned} F_o &= F \cap \{x \text{ in } X : |g(x)| = 1\} \\ &= F \cap S(|g|) \end{aligned}$$

where $S(|g|)$ is a special maximal set of $|g|$, for $\| |g| \| = \|g\| = 1$.

Since the intersection of special sets is again a special set, the proof will be completed if we show that $|g|$ is in A. This follows immediately from lemma 4.4.

4.8 <u>Theorem.</u> Let A be a closed subalgebra of $C(X)$ which contains 1 and separates the points of X. Then every minimal special set associated with A reduces to a single point.

<u>Proof.</u> Suppose F is a special set associated with A and contains more than one point. Let $x_o, y_o \in F$ and $x_o \neq y_o$. Since A separates the points of X there exists g in A such that $g(x_o) \neq g(y_o)$. In case that $x_o, y_o \in S(g)$ then $g(x_o) = -g(y_o)$. Take $f = g \wedge 0$. Then $f \in A$ and at most one of x_o, y_o can be in $S(f)$. So we may assume that at most one of x_o, y_o can be in $S(g)$.

Consider the maximal set $S(g|_F)$ which is a proper subset of F, since not both x_o, y_o can be in $S(g|_F)$. By remark 3.5, $S(g|_F)$ contains a special maximal set $S(h|_F)$ where $h|_F$ is in $A|_F$. By theorem 4.7, $S(h|_F)$ is a special set associated with A and F contains $S(h|_F)$ properly. Thus F is not a minimal special set. This proves the theorem.

4.9 <u>Corollary</u>. Let A be a closed subalgebra of $C(X)$ which contains 1 and separates the points of X. Then the set $S\partial_A X$ of all strong boundary points is identical with the union of all minimal special sets associated with A.

<u>Proof</u>. This follows from theorem 3.8 and corollary 4.8.

4.10 <u>Corollary</u>. Let A be a closed subalgebra of $C(X)$ which contains 1 and separates the points of X. Then the set $S\partial_A X$ is an A-set.

<u>Proof</u>. This follows from theorem 4.3 and corollary 4.9.

We are now able to answer the question stated above.

4.11 <u>Theorem</u>. Let A be a closed subalgebra of $C(X)$ which contains 1 and separates the points of X. Then $S\partial_A X$ is the minimal boundary for X relative to A if every point in $S\partial_A X$ is a G_δ in X. In this case

$$S\partial_A X = P_A X$$

where $P_A X$ is the set of all peak points of X relative to A.

<u>Proof</u>. If every point in $S\partial_A X$ is a G_δ by theorem 3.10,

$$S\partial_A X = P_A X .$$

By corollary 4.10, $P_A X$ is an A-set and hence is the minimal boundary for X relative to A by proposition 4.1 .

In particular we have:

4.12 <u>Theorem</u>. Let X be a compact metric space and A be a closed subalgebra of $C(X)$ which separates points of X and contains 1 . Then A has a minimal boundary $M_A X$ and

$$M_A X = P_A X .$$

<u>Proof</u>. Since X is a metric space, every point of X is a G_δ . By theorem 4.11, we have this theorem.

The following theorem and corollary describe the topological structure of the minimal boundary more precisely.

4.13 <u>Theorem</u>. Let X be a compact metric space and A a closed subalgebra of $C(X)$ which contains 1 and separates the points of X . For any x in X and each positive integer n , define

$$S_n(x) = \{y \text{ in } X : \rho(x,y) < 1/n\}$$

where ρ is the metric on X ,

$$D_n(x) = X - S_n(x) \quad ,$$

and

$$U_n = \{x \text{ in } X : \exists\ f \in A \text{ such that } \|f\| \leq 1\ ,\ |f(x)| > 3/4$$

$$\text{and } \sup_{y \in D_n(x)} |f(y)| < 1/4 \text{ if } D_n(x) \neq \emptyset\} .$$

Then $M_A X = \bigcap_{n=1}^{\infty} U_n$.

<u>Proof</u>. Let x be in $P_A X$. We can find f in A such that

$$S(f) = \{x\} \text{ and } \|f\| = 1 .$$

If $D_n(x) = \emptyset$ then x is in U_n . If $D_n(x) \neq \emptyset$, then

$$\sup_{y \in D_n(x)} |f(y)| = |f(y_o)|$$

for some y_o in $D_n(x)$ since $D_n(x)$ is compact. But x is not in $D_n(x)$ so $x \neq y_o$. Since $\|f\| = 1$ and $S(f) = \{x\}$,

$$1 = |f(x)| > |f(y)| \text{ for all } y \text{ in } X \text{ and } y \neq x.$$

Hence $1 = |f(x)| > |f(y_o)| = \sup_{y \in D_n(x)} |f(y)|$. Therefore, for some positive integer k_n, f^{k_n} is in A and

$$\sup_{y \in D_n(x)} |f^{k_n}(y)| < 1/4$$

and

$$|f^{k_n}(x)| = |f(x)|^{k_n} = 1 > 3/4 .$$

Thus x is in U_n . Since this is true for all n ,

$$P_A X \subseteq \bigcap_{n=1} U_n .$$

Let x belong to U_n for all n. For every neighborhood V of x, there exists some integer k such that $S_k(x) \subset V$. Since x is in U_k there exists f in A with

$$\|f\| \leq 1 \quad \text{and} \quad |f(x)| > 3/4$$

and either $\quad D_k(x) = \emptyset \quad$ or $\quad \sup_{y \in D_k(x)} |f(y)| < 1/4$.

Define

$$h = |f| \wedge 3/4$$

which is in A by corollary 4.6 and lemma 4.4 . We have:

$$\|h\| = 3/4 \quad .$$

If $D_k(x) = \emptyset$ then $S_k(x) = X = V$. Clearly x is in $S(h) \subset V$. If $D_k(x) \neq \emptyset$ then $\sup_{y \in D_k(x)} |f(y)| < 1/4$. So $\sup_{y \in D_k(x)} |h(y)| < 1/4$ and for any z in $S(h)$, $|h(z)| = 3/4$, Thus $z \notin D_k(x)$. Hence

$$x \in S(h) \subset S_k(x) \subset V \quad .$$

Therefore x is a strong boundary point and hence a peak point by theorem 4.11 . This shows that

$$\bigcap_{n=1}^{\infty} U_n \subset P_A X \quad .$$

Consequently,

$$P_A X = \bigcap_{n=1}^{\infty} U_n \quad .$$

By theorem 4.12 , we get:

$$M_A X = \bigcap_{n=1}^{\infty} U_n \quad .$$

4.14 <u>Corollary</u>. Let X, ρ , A be given as in theorem 4.13 .

Then $M_A X$ is an intersection of countably many open sets.

<u>Proof</u>. We need only show that the set U_n defined in theorem 4.13 , is open for each n .

For each n and for any f in A , define

$$G_n(f) = \{x : \text{either } D_n(x) = \emptyset \text{ or } \sup_{y \in D_n(x)} |f(y)| < 1/4\}$$

and $$\pi_n(f) = \{x : |f(x)| > 3/4\} \cap G_n(f) \quad .$$

We shall show that $\pi_n(f)$ is open.

Let x_o be any point in $G_n(f)$. If $D_n(x_o) = \emptyset$, then $S_n(x_o) = X$. The function $\rho(x_o, z)$, z in X , is continuous by Kelley [9] Theorem 8, p. 120, and since X is compact,

$$\sup_{z \in X} \rho(x_o, z) = \rho(x_o, z_o) < 1/n$$

for some point z_o in X .

Let $0 < 1/k < 1/n - \rho(x_o, z_o)$ for some integer k .

For a fixed but arbitrary point x' in $S_k(x_o)$ and any x in X ,

$$\begin{aligned} \rho(x',x) &\leq \rho(x',x_o) + \rho(x_o,x) \\ &\leq 1/k + \rho(x_o,z_o) \\ &< 1/n - \rho(x_o,z_o) + \rho(x_o,z_o) \\ &= 1/n \ . \end{aligned}$$

Thus $S_n(x') = X$ and hence $D_n(x') = \emptyset$ so x' is in $G_n(f)$. This shows that $S_k(x_o) \subset G_n(f)$ for some integer k.

If $D_n(x_o) \neq \emptyset$ then $\sup_{y \in D_n(x_o)} |f(y)| < 1/4$. Let $0 < \epsilon < 1/4 - \sup_{y \in S_n(x_o)} |f(y)|$. Since f is continuous on a compact metric space X, it is uniformly continuous on X. Therefore, there exists $\delta > 0$ such that

$$\rho(p,q) < \delta \text{ implies that } |f(p) - f(q)| < \epsilon \ .$$

Define

$$B' = \{z \text{ in } X : \rho(z, D_n(x_o)) \leq \delta'\}$$

$$B = \{z \text{ in } X : \rho(z, D_n(x_o)) < \delta\}$$

where $0 < \delta' < \delta$.

Since $\rho(z, D_n(x_o))$ is continuous on X, B is open and B' is closed and both B, B' contain $D_n(x_o)$. For any z in B', since $D_n(x_o)$ is closed and compact, there exists z_o in $D_n(x_o)$ such that

$$0 \leq \rho(z,z_o) = \rho(z,D_n(x_o) \leq \delta'$$

So $\rho(z,z_o) < \delta$. This implies that

$$\begin{aligned} |f(z)| - |f(z_o)| &\leq |f(z) - f(z_o)| \\ &< \epsilon \\ &< 1/4 - \sup_{y \in D_n(x_o)} |f(y)| \end{aligned}$$

Thus

$$\begin{aligned} |f(z)| &< 1/4 - \sup_{y \in D_n(x_o)} |f(y)| + |f(z_o)| \\ &\leq 1/4 \quad . \end{aligned}$$

Hence

$$\begin{aligned} \sup_{z \in B} |f(z)| &\leq \sup_{z \in B'} |f(z)| \\ &< 1/4 \quad . \end{aligned}$$

Since $B \supset D_n(x_o)$, $X-B \in S_n(x_o)$. There exists y_o in $X-B$ such that

$$\begin{aligned} \rho(x_o,y_o) &= \sup_{z \in X-B} \rho(x_o,z) \\ &< 1/n \qquad \text{(since } y_o \text{ is in } S_n(x_o)\text{)}. \end{aligned}$$

Choose some integer p such that $0 < 1/p < 1/n - \rho(x_o,y_o)$. For any x' in $S_p(x_o)$, there exists y' in $X-B$ such that

$$\rho(x',y') = \sup_{z \in X-B} \rho(x',z) \quad .$$

Now,

$$\begin{aligned}\rho(x', y') &\leq \rho(x', x_o) + \rho(x_o, y') \\ &\leq \rho(x', x_o) + \rho(x_o, y_o) \\ &< 1/n - \rho(x_o, y_o) + \rho(x_o, y_o) \\ &= 1/n .\end{aligned}$$

Hence $X-B \subset S_n(x')$. So $B \supset X-S_n(x') = D_n(x')$. Thus

$$\begin{aligned}\sup_{z \in D_n(x')} |f(z)| &\leq \sup_{z \in B} |f(z)| \\ &< 1/4 .\end{aligned}$$

This shows that x' is in $G_n(f)$ and hence $S_p(x_o) \subset G_n(f)$. This completes the proof that $G_n(f)$ is open.

Since $\{x \in X : |f(x)| > 3/4\}$ is open and $\pi_n(f)$ being the intersection of two open sets is open, therefore,

$$U_n = \cup \{\pi_n(f) : f \text{ in } A \text{ and } \|f\| = 1\}$$

which is a union of open sets, is open.

SECTION 5 Examples.

In this section, we shall give two examples. The first example shows that theorem 4.12 may fail to hold if the space X is not metrisable. The second one shows that the minimal boundary is distinct from the Shilov boundary for a function algebra in general.

5.1 Example. Let I be the unit interval $[0,1]$ with the usual topology. Let J be an uncountable set. Let X consist of all points $x = \{x_\alpha\}_{\alpha \in J}$ with x_α in $[0,1]$. Thus X is the Cartesian product of an uncountable number of closed intervals. Since $[0,1]$ is closed and bounded in R, it is compact. By Tychonoff's theorem on the product of compact sets, X is compact. But X is not metrisable. Indeed, since $[0,1]$ is not discrete, by theorem 6, p. 92 in Kelley [9], X does not satisfy the first axiom of countability and hence X is not metrisable, since every metric space satisfies the first axiom of countability.

Let A consist of all real-valued continuous functions f on X which have the property that there exists a countable subset N of J such that $f(x) = f(y)$ whenever x and y are points in X satisfying $x_\alpha = y_\alpha$ for all α in N. Note that N depends on f. Then

(i) A separates the points of X.

For each α in J, let P_α be the projection:

$$P_\alpha(x) = x_\alpha$$

for each x in X.

Then P_α is continuous by definition of the product topology. Take N to be $\{\alpha\}$. Then, if x,y are points in X such that $x_\alpha = y_\alpha$ then $P_\alpha(x) = x_\alpha = y_\alpha = P_\alpha(y)$. Hence P_α is in A for each α in J. Now let x,y be any pair of elements in X such that $x \neq y$. Then $x_\alpha \neq y_\alpha$ for at least one α in J. We have:

$$P_\alpha(x) = x_\alpha \neq y_\alpha = P_\alpha(y) .$$

Hence A separates the points of X.

(ii) A is a normed algebra.

Let a be any real number and f_i $(i = 1,2)$ be functions in A. Let N_i $(i = 1,2)$ be countable subsets of J such that $f_i(x) = f_i(y)$ whenever x,y are in X such that $x_\alpha = y_\alpha$ for every α in N_i. Then $N_1 \cup N_2$ is a countable subset of J and for any x and y in X such that $x_\alpha = y_\alpha$ for every α in $N_1 \cup N_2$ we have:

$$\begin{aligned}(f_1 - af_2)(x) &= f_1(x) - af_2(x) \\ &= f_1(y) - af_2(y) \\ &= (f_1 - af_2)(y)\end{aligned}$$

and

$$\begin{aligned}(f_1f_2)(x) &= f_1(x)\, f_2(x) \\ &= f_1(y)\, f_2(y) \\ &= (f_1f_2)(y) \quad .\end{aligned}$$

So $f_1 - af_2$ and f_1f_2 are in A . Since A is a subset of $C(X)$, A is a normed algebra with the sup-norm.

(iii) A is complete (and hence A is closed since in a complete metric space, a subset is complete iff it is closed).

Let $\{f_i\}_{i = 1,2,\ldots}$ be a Cauchy sequence in A and for each i , let N_i be a countable subset of J such that for any x,y in X , if $x_\alpha = y_\alpha$ for all α in N_i then $f_i(x) = f_i(y)$. Take N to be $\bigcup_{i=1}^{\infty} N_i$ which is a countable subset of J . For each x in X , $\{f_i(x)\}_{i = 1,2,\ldots}$ is a Cauchy sequence in R , so it has a limit, say $f(x)$. Then

$\{f_i\}$ converges to the function $f : x \to f(x)$ for all x in X. f is in $C(X)$ since $C(X)$ is complete. Moreover, for any x,y in X if $x_\alpha = y_\alpha$ for every α in N, then $x_\alpha = y_\alpha$ for each α in N_i and for all i. This implies that $f_i(x) = f_i(y)$ for all i. But $\{f_i(x)\}_{i = 1,2,\ldots}$ converges to $f(x)$ and $\{f_i(y)\}_{i = 1,2,\ldots}$ converges to $f(y)$ and since the space is Hausdorff, $f(x) = f(y)$. Hence f is in A. Thus A is complete.

We have seen that A and X satisfy all the conditions except the metrisability of X in theorem 4.12.

(iv) The minimal boundary of X relative to A does not exist.

Let $H_1 = \{x \text{ in } X : x_\alpha = 0 \text{ except for a countable set of } \alpha\}$

and $H_2 = \{x \text{ in } X : x_\alpha = 1 \text{ except for a countable set of } \alpha\}$.

Then $H_1 \cap H_2 = \emptyset$.

For any f in A, let x' belong to $S(f)$. Let N be the corresponding countable subset of J such that $f(x) = f(y)$ if $x_\alpha = y_\alpha$ for all α in N.

Pick the point y' in X such that $y'_\alpha = x'_\alpha$ for all α in N and $y'_\alpha = 0$ otherwise. Then $y' \in H_1$ and $f(x') = f(y')$. Hence $y' \in S(f)$. We have $S(f) \cap H_1 \neq \emptyset$ for all f in A.

Similarly,

$$S(f) \cap H_2 \neq \emptyset \text{ for all } f \text{ in } A .$$

This shows that H_1 and H_2 are A-sets. But $H_1 \cap H_2 = \emptyset$. Thus, the minimal boundary for X relative to A does not exist.

This completes the proof that the metrisability of the space X in theorem 4.12 is "necessary".

Suppose $\partial_A X$ exists. It is clear that for any A-set H satisfying $H \subset \partial_A X$, H is dense in $\partial_A X$. Moreover, the closure of $M_A X$ is equal to $\partial_A X$ if $M_A X$ exists. Since $\overline{M_A X}$ is a closed A-set we have for any closed A-set B : $M_A X \subset B$. This implies $\overline{M_A X} \subset B$. Hence $\overline{M_A X}$ is the smallest closed A-set i.e. $\overline{M_A X} = \partial_A X$. Hence the existence of the minimal boundary implies the existence of the Shilov boundary. But the converse is not true as is seen from example 5.1 and theorem 2.7.

5.2 <u>Example.</u> A function algebra whose minimal boundary is distinct from its Shilov boundary.

Let X be the subset $\{z : |z| = 1\}$ of the complex plane and let A be the set of all complex valued continuous functions f on X which have the property that there exists a continuous function f^* on $\{z : |z| \leq 1\}$ such that $f^*(z) = f(z)$ for all z in X and f^* is analytic on $\{z : |z| < 1\}$, and that $f^*(0) = f^*(1)$.

We recall that the set D of all complex-valued continuous functions on $\{z : |z| \leq 1\}$ and analytic on $\{z : |z| < 1\}$ is a Banach algebra over the field of complex numbers with respect to the sup-norm.

(i) A is a Banach algebra.

$A \neq \emptyset$ since A contains every constant function. Let f,g belong to A and α be any complex number. Let f^*,g^* be continuous extensions of f,g to $\{z : |z| \leq 1\}$ with f^*,g^* analytic on $\{z : |z| < 1\}$ and $f^*(0) = f^*(1)$, $g^*(0) = g^*(1)$. Then $f^* - \alpha g^*$, f^*g^* are the corresponding continuous extensions of $f - \alpha g$, fg respectively, i.e. $f - \alpha g$, fg are in A.

Let $\{f_i\}_{i = 1,2,\ldots}$ be a Cauchy sequence in A. Let $\{f_i^*\}_{i = 1.2,\ldots}$ be the corresponding sequence of continuous extension of $\{f_i\}_{i = 1,2,\ldots}$. For any pair of integers i and j, $f_i^* - f_j^*$ is a continuous

function on $\{z : |z| \leq 1\}$ and analytic on $\{z : |z| < 1\}$. By the maximum principle,

$$\begin{aligned} &\sup \{|f_i^*(z) - f_j^*(z)| : |z| \leq 1\} \\ &= \sup \{|f_i^*(z) - f_j^*(z)| : |z| = 1\} , \\ &= \sup \{|f_i(z) - f_j(z)| : |z| = 1\} , \\ &= \|f_i - f_j\| \end{aligned}$$

Hence $\{f_i^*\}_{i = 1,2,\ldots}$ is a Cauchy sequence in D. Since D is complete, f_i^* converges to a function h^* in D. Moreover, since $f_i^*(0) = f_i^*(1)$ for every i and $f_i^*(0)$ converges to $h^*(0)$, $f_i^*(1)$ converges to $h^*(1)$, $h^*(0) = h^*(1)$ and hence $h = h^*|_X$ is in A and h is the limit of $\{f_i\}_{i = 1,2,\ldots}$. Thus A is a Banach algebra.

(ii) A separates the points of X. Hence $M_A X$ exists by theorem 4.12.

Let α, β be any real numbers such that $0 \leq \alpha$, $\beta < 2\pi$, and $\beta \neq \alpha$, $\beta \neq 0$, then the function

$$f(z) = z(z-1)\,(z-e^{i\alpha})/(e^{i\beta} - e^{i\alpha})$$

is in A and $f(e^{i\alpha}) = 0 \neq f(e^{i\beta})$.

(iii) $\partial_A X = X$ (see [1])

(iv) $1 \notin M_A X$. Hence $M_A X \neq \partial_A X$.

According to theorem 4.12 $M_A X$ consists of all peak points, so we only need to show that 1 is not a

peak point. Suppose that there exists some f in A with $S(f) = \{1\}$. Then f is not a constant.

Let f^* be the corresponding extension of f, then $S(f^*)$ contains $\{0,1\}$. This contradicts the maximum principle. So 1 is not a peak point relative to A.

5.3 <u>Remark.</u> We saw in example 5.2 that $1 \in \partial_A X$, since by theorem 3.1 for every neighborhood U of 1, there exists a function f in A such that $S(f)$ is contained in U. But f does not assume its absolute maximal value at 1 if $f \neq$ constant.

SECTION 6 Some function families with boundaries.

In section 2, we proved existence of the Shilov boundary for separating algebras. In this section, our aim is to show that the Shilov boundary exists with respect to some function families more general than separating algebras, namely, for separating families C which are closed under either multiplication or addition of functions of C.

6.1 Proposition. Let X be a compact Hausdorff space. Let B be any non-empty subset of $C(X)$. Define

$$B' = \{g : g = cf \text{ for some real number } c \neq 0, \text{ and } f \text{ in } B\}$$

Then a subset N of X is a B-set iff N is a B'-set.

Proof. This follows immediately from the relation, $S(f) = S(cf)$ for f in B and $c \neq 0$.

From proposition 6.1, we have: N is a minimal closed B-set iff N is a mimimal closed B'-set and N is the B-Shilov boundary iff N is the B'-Shilov boundary.

6.2 Theorem. Let X be a compact metric space. Let A be a non-empty subset of $C(X)$ which satisfies the conditions:

(1) $f \in A$ implies that $f(x) \geq 0$ for all x in X,

(2) f,g in A imply that fg is in A,

(3) A separates the points of X.

Then the A-Shilov boundary exists.

<u>Proof</u>. Since a subset of X is the A-Shilov boundary iff it is the A'-Shilov boundary, we may assume that if f belongs to A and $c \neq 0$ then cf is in A. By virtue of proposition 2.3, we need only prove the uniqueness of the minimal closed A-set.

Suppose that F_1 and F_2 are two different minimal closed A-sets. Then $F_1 - F_2 \neq \emptyset$, otherwise, F_2 contains F_1 and F_2 would not be a minimal closed A-set. Take a point x_1^o in $F_1 - F_2$ and let

$$U_1(x_1^o) = \{x : \rho(x_1^o, x) < 1/2^{n_1}\}$$

for some large integer n_1 so that $U_1 \cap F_2 = \emptyset$.

Since F_1 is a minimal closed A-set, there exists f in A such that

$$S(f) \cap (F_1 - U_1) = \emptyset$$

otherwise,

$$S(f) \cap (F_1 - U_1) \neq \emptyset \text{ for all } f \text{ in } A,$$

implies that $(F_1 - U_1)$ is a closed A-set. But $F_1 - U_1$ is a proper closed subset of F_1. This contradicts the assumption that F_1 is a minimal closed A-set. Since $F_1 = (F_1 \cap U_1) \cup (F_1 - U_1)$ and F_1 is an A-set, there exists some point x_1' in $U_1 \cap F_1$ such that $\|f\| = f(x_1') > f(y)$ for all y in $F_1 - U_1$. So $f(y)/\|f\| < 1$ for y in $F_1 - U_1$. Define

$$f_1 = (f/\|f\|)^{k_1}$$

where k_1 is an integer so large that

$$f_1(y) < 1/4 \text{ for all } y \text{ in } F_1 - U_1 .$$

Note that

$$f_1 \in A \text{ and } \|f\| = f_1(x_1') = 1 , \quad x_1' \text{ is in } U_1 .$$

Since F_2 is an A-set, $S(f_1) \cap F_2 \neq \emptyset$. There exists some point y_1^o in F_2 such that $\|f_1\| = f_1(y_1^o)$. Since $F_2 \cap U_1 = \emptyset$, $y_1^o \notin U_1$ and since f_1 is continuous on a compact metric space, it is uniformly continuous on X . Thus there exists an integer m_1 so large that

$$\text{for } x \in V_1(y_1^o) = \{x \text{ in } X : \rho(y_1^o, x) < 1/2^{m_1}\}$$

$$f_1(x) > 3/4 \text{ and } \overline{V_1(y_1^o)} \cap \overline{U_1(x_1^o)} = \emptyset .$$

Since F_2 is a minimal closed A-set, apply the same argument for F_1 and U_1 to F_2 and V_1 , and there exists g in A such that

$$\|g\| = g(y_1') = 1 \text{ for some } y_1' \text{ in } V_1 \cap F_2$$

and

$$g(x) < 1/4 \text{ for } x \text{ in } F_2 - V_1 .$$

Now, let $h = f_1 g$ which is in A . Then for x

in $F_1 - U_1$, $h(x) = f_1(x)\ g(x)$

$$\leq f_1(x)\ \|g\|$$

$$= f_1(x)$$

$$< 1/4 \quad ,$$

for x in $F_2 - V_1$, $h(x) < \|f_1\|\ g(x)$

$$< 1/4$$

and

$$h(y_1') = f_1(y_1')\ g(y_1')$$

$$= f_1(y_1')$$

$$> 3/4 \quad ,$$

since y_1' belongs to V_1 .

Hence $\|h\| \geq 3/4$ and

$$h(x)/\|h\| < 1/3 \text{ for } x \text{ in } (F_1 - U_1) \cup (F_2 - V_1).$$

Let

$$h_1 = (h/\|h\|)^{k_2}$$

where k_2 is an integer so large that

$$h_1(x) < 1/4 \text{ for } x \text{ in } (F_1 - U_1) \cup (F_2 - V_1).$$

Note that: h_1 is in A and $\|h_1\| = 1$. Moreover, since $F_1 = (F_1 - U_1) \cup (F_1 \cap U_1)$, $F_2 = (F_2 - V_1) \cup (F_2 \cap V_1)$ and F_1 , F_2 are A-sets, there exist some points x_1 in $F_1 \cap U_1$ and y_1 in $F_2 \cap V_1$ such that $\|h_1\| = h_1(x_1) = h_1(y_1) = 1$.

Since h_1 is uniformly continuous on X, we can find an integer $n_2 > n_1$ so large that

$$U_2(x_1) = \{x : \rho(x_1,x) < 1/2^{n_2}\} \subset U_1$$

and

$$h_1(x) \geq 3/4 \quad \text{for} \quad x \quad \text{in} \quad U_2 \ .$$

Since F_1 is a minimal closed A-set, there exists f' in A such that

$$\|f'\| = 1 = f'(x_2^o) \quad \text{for some} \quad x_2^o \quad \text{in} \quad U_2 \cap F_1 \ ,$$

and

$$f'(x) < 1/4 \quad \text{for} \quad x \quad \text{in} \quad F_1 - U_2 \ .$$

The function $f_1 = f'h_1$ is in A and satisfies: for x in $F_1 - U_2$, $f_1(x) = f'(x)\, h_1(x)$

$$< f'(x)$$

$$< 1/4 \quad ,$$

for x in $F_2 - V_1$, $f_1(x) = f'(x)\, h_1(x)$

$$< \|f'\|\, h_1(x)$$

$$< h_1(x)$$

$$< 1/4 \quad ,$$

and $f_1(x_2^o) = f'(x_2^o)\, h_1(x_2^o) = h_1(x_2^o) \geq 3/4$, since $x \in U_2$. Hence $\|f_1\| \geq 3/4$. Since $F_2 = (F_2 - V_1) \cup (F_2 \cap V_1)$ and F_2 is an A-set, there exists some y_2^o in $F_2 \cap V_2$ such that $f_1(y_1^o) = \|f_1\|$. For a sufficiently large integer k_3, the function

$$f'' = (f_1 / \|f_1\|)^{k_3}$$

is in A and satisfies:

$$\|f''\| = f''(y_2^o) = 1$$

and

$$f''(x) < 1/4 \quad \text{for} \quad x \quad \text{in} \quad (F_1 - U_2) \cup (F_2 - V_1) .$$

Since f'' is uniformly continuous, we can find an integer $m_2 > m_1$, such that

$$V_2(y_2^o) = \{x : \rho(y_2^o, x) < 1/2^{m_2}\} \subset V_1$$

and $f''(y) > 3/4$, for y in V_2.

Since F_2 is a minimal closed A-set, there exists a function g' in A such that $\|g'\| = g'(x_2') = 1$ for some point x_2' in $V_2 \cap F_2$ and $g'(y) < 1/4$ for y in $F_2 - V_2$. The function $h' = f'' g'$ is in A and has the properties:

For x in $F_1 - U_2$, $h'(x) \leq f''(x) < 1/4$.

For x in $F_2 - V_2$, $h'(x) \leq g'(x) < 1/4$,

and $h'(x_2') = f''(x_2') g'(x_2') = f''(x_2') \geq 3/4$, since $x_2' \in V_2$.

Hence $\|h'\| \geq 3/4$ and $(h'(x) / \|h'\|) < 1/3$ for x in $(F_1 - U_2) \cup (F_2 - V_2)$. We define the function,

$$h_2 = (h' / \|h'\|)^{k_4}$$

where k_4 is an integer so large that

$$h_2(x) < 1/4 \quad \text{for} \quad x \quad \text{in} \quad (F_1 - U_2) \cup (F_2 - V_2)$$

Note that h_2 is in A and $\|h_2\| = h_2(x_2) = h_2(y_2) = 1$, for some points x_2 in U_2, and y_2 in V_2.

Continuing this procedure, we have:

(1) two sequences of points $\{x_i\}_{i=1,2,\ldots}$, $\{y_i\}_{i=1,2,\ldots}$,

(2) two sequences of neighborhoods $\{U_i(x_i)\}_{i=1,2,\ldots}$ $\{V_i(y_i)\}_{i=1,2,\ldots}$ such that $U_i \supset U_{i+1}$, $V_i \supset V_{i+1}$ and $\overline{U}_1 \cap \overline{V}_1 = \emptyset$, $i = 1,2,\ldots$

(3) a sequence of functions $\{h_i\}_{i=1,2,\ldots}$ in A such that for each i, $\|h_i\| = h_i(x_i) = h_i(y_i) = 1$ and

$$h_i(x) < 1/4 \text{ for } x \text{ in } (F_1-U_i) \cup (F_2-V_i) .$$

Claim (i). $\{x_i)_{i=1,2,\ldots}$, $\{y_i\}_{i=1,2,\ldots}$ are Cauchy sequences.

Given any $\epsilon > 0$, we can find k such that $\epsilon > 2/2^{n_k}$. Then for any $m, n \geq k$, we may assume $n \geq m$. Then x_m, x_n belong to U_m. We have:

$$\rho(x_m,x_n) \leq \text{ diameter of } U_m = 2/2^{n_m} < 2/2^{n_k} < \epsilon .$$

Hence $\{x_i\}_{i=1,2,\ldots}$ is a Cauchy sequence. Similarly, $\{y_i\}_{i=1,2,\ldots}$ is a Cauchy sequence.

Since X is a compact metric space, it is complete. Hence $\{x_i\}_{i=1,2,\ldots}$ converges to some point x_o in X and $\{y_i\}_{i=1,2,\ldots}$ converges to some point y_o in X.

Claim (ii). Given $\epsilon > 0$, $S(x_o,\epsilon) = \{x : \rho(x_o,x) < \epsilon\}$ contains some U_r.

Indeed, since $\{x_i\}_{i=1,2,\ldots}$ tends to x_o, there exists an integer N such that $n \geq N$ implies $\rho(x_o,x_n) < \epsilon/2$. Let $r \geq N$, such that $\epsilon/2 > 2/2^{n_r}$. Then for z in $U_r(x_r)$, $\rho(z,x_o) \leq \rho(x_o,x_r) + \rho(x_r,z)$

$$\begin{aligned} &\leq \epsilon/2 + \text{diameter of } U_r \\ &= \epsilon/2 + 2/2^{n_r} \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

Hence $U_r \subset S(x_o,\epsilon)$.

Similarly, given any $\epsilon' > 0$, $S(y_o,\epsilon') \supset V_{r'}$ for some r'.

Therefore, for any neighborhoods U, V of x_o, y_o respectively, we can find an integer r, such that

$$U_r \subset U \text{ and } V_r \subset V$$

Since $\{x_i : i=1,2,\ldots\} \subset U_1$, $\{y_i : i = 1,2,\ldots\} \subset V_1$ and $\overline{U}_1 \cap \overline{V}_1 = \emptyset$, $x_o \neq y_o$. There exists a function a in A such that $a(x_o) \neq a(y_o)$. We may assume that $0 \leq a(x_o) < a(y_o)$ Let $a(y_o) - a(x_o) = 3\epsilon$. By continuity of a, we can find two neighborhoods $U(x_o)$ and $V(y_o)$ such that

$$a[U(x_o)] \subset (a(x_o) - \epsilon\ ,\ \ a(x_o) + \epsilon)$$

and

$$a[V(y_o)] \subset (a(y_o) - \epsilon\ ,\ \ a(y_o) + \epsilon)$$

For $U(x_o)$, $V(y_o)$ we can find an integer r such that $U_r \subset U(x_o)$ and $V_r \subset V(x_o)$.

Let m be an integer so large that $(\|a\| / 4^m) < \epsilon$, then the function

$$b = ah_r^m$$

is in A and has the following properties:

For x in $F_1 - U_r$,

$$\begin{aligned} b(x) &= a(x)\ (h_r(x))^m \\ &< \|a\| / 4^m \\ &< \epsilon \\ &\leq a(x_o) + \epsilon \end{aligned}$$

For x in U_r ,

$$\begin{aligned} b(x) &= a(x)\ (h_r(x))^m \\ &< (a(x_o) + \epsilon) \end{aligned}$$

For the point y_r in V_r ,

$$\begin{aligned} b(y_r) &= a(y_r)\ (h_r(y_r))^m \\ &> a(y_o) - \epsilon \\ &> a(x_o) + \epsilon \end{aligned}$$

Hence for every x in $F_1 = (F_1 - U_r) \cup (F_1 \cap U_r)$, we have:

$$b(x) < a(x_o) + \epsilon < b(y_r)\ .$$

Hence $S(b) \cap F_1 = \emptyset$. Therefore F_1 is not an A-set. This contradicts that F_1 is a minimal closed A-set, and hence the minimal closed A-set is unique.

6.3 <u>Theorem</u>. Let X be a compact metric space. Let A be a subset of $C(X)$ such that:

(1) $f,g \in A \Rightarrow f + g \in A$,

(2) A separates the points of X ,

(3) $f \in A \Rightarrow f(x) \geq 0$ for all x in X .

Then the A-Shilov boundary exists.

<u>Proof</u>. Let $B = \{e^f : f \text{ in } A\}$. Since the relation ,

$$S(e^f) = S(f)$$

holds for every f in A , a subset of X is an A-set iff it is a B-set and hence a subset of X is the A-Shilov boundary iff it is the B-Shilov boundary. Since B satisfies the assumptions of theorem 6.2 , the A-Shilov boundary exists.

SECTION 7 Shilov boundary for the product of two compact metric spaces.

We recall that if X is a compact metric space and A a subset of $C(X)$ such that:

(1) $f \in A \Rightarrow f \geq 0$,

(2) $f, g \in A \Rightarrow f + g \in A$,

(3) $0 \in A$,

(4) A separates the points of X ,

then by theorem 6.3 , the Shilov boundary $\partial_A X$ exists.

Let X, Y be two compact metric spaces and $A \subset C(X)$, $B \subset C(Y)$ be such that conditions (1), (2), (3) and (4) hold. Let $X \times Y$ be the product topological space of X and Y . For each f in A and g in B , we define the real-valued function $[f + g]$ on $X \times Y$ as follows:

$$[f + g]\,(x,y) = f(x) + g(y)$$

for all (x,y) in $X \times Y$. Let

$$A + B = \{[f + g] : f \text{ in } A\,,\ g \text{ in } B\}\,.$$

For any (x,y), (x',y') in $X \times Y$ and any $[f + g]$ in $A + B$, we have:

$$\begin{aligned} &|[f + g]\,(x,y) - [f + g]\,(x',y')| \\ &\qquad = |f(x) + g(x) - f(x') - g(y')| \\ &\qquad \leq |f(x) - f(x')| + |g(y) - g(y')| \end{aligned}$$

From this inequality, it is evident that $[f + g]$ is continuous. Hence $A + B \subset C(X \times Y)$. Moreover, it is easy to see that $A + B$ satisfies conditions (1), (2), (3). Also (4) is satisfied, for let (x,y), (x',y') be any points in $X \times Y$ and $(x,y) \neq (x',y')$. Either $x \neq x'$ or $y \neq y'$, for definiteness we say $x \neq x'$. Since A separates the points of X, there exists f in A such that $f(x) \neq f(x')$, therefore, the function $[f + 0]$ is in $A + B$ and

$$[f + 0]\,(x,y) \neq [f + 0]\,(x',y') \quad .$$

Consequently, the Shilov boundary for $A + B$ exists. Furthermore, we have the following theorem.

7.1 <u>Theorem</u>. $\partial_{A+B}(X \times Y) = \partial_A X \times \partial_B Y$.

<u>Proof</u>. Let $[f + g] \in A + B$. Since

$$\begin{aligned}
\|[f+g]\| &= \sup\{|[f + g]\,(x,y)| : (x,y) \text{ in } X \times Y\} \\
&= \sup\{|f(x) + g(y)| : x \text{ in } X,\ y \text{ in } Y\} \\
&= \sup\{ f(x) + g(y) : x \text{ in } X,\ y \text{ in } Y\} \\
&= \sup\{ f(x) : x \text{ in } X\} + \sup\{g(y) : y \text{ in } Y\} \\
&= \sup\{|f(x)| : x \text{ in } X\} + \sup\{|g(y)| : y \text{ in } Y\} \\
&= \|f\| + \|g\| \quad ,
\end{aligned}$$

$$\begin{aligned}
S([f+g]) &= \{(x,y) : |[f + g]\,(x,y)| = \|f + g\|\} \\
&= S(f) \times S(g) \ .
\end{aligned}$$

Hence

$$\begin{aligned}
S([f + g]) &\cap \partial_A X \times \partial_B Y \\
&= S(f) \times S(g) \cap \partial_A X \times \partial_B Y \\
&= (S(f) \cap \partial_A X) \times (S(g) \cap \partial_B Y) \\
&\neq \emptyset
\end{aligned}$$

since $S(f) \cap \partial_A X \neq \emptyset$ and $S(g) \cap \partial_B Y \neq \emptyset$ by definition. Therefore $\partial_A X \times \partial_B Y$ is a closed $(A + B)$-set and hence $\partial_{A+B}(X \times Y)$, being the smallest of closed $(A + B)$-sets is contained in $\partial_A X \times \partial_B Y$.

On the other hand, let (a,b) be a fixed point in $\partial_A X \times \partial_B Y$ and U, V be any neighborhoods of a and b respectively. Then by theorem 3.1, there exists f in A and g in B such that $S(f) \subset U$ and $S(g) \subset V$. Hence if $(x,y) \notin U \times V$ then either $f(x) < \|f\|$ or $g(y) < \|g\|$, so

$$\begin{aligned} |[f + g]\,(x,y)| &= f(x) + g(y) \\ &\leq \|f\| + \|g\| \\ &= \|f + g\| \end{aligned}$$

i.e. $(x,y) \notin S(f + g)$. Hence $S([f + g]) \subset U \times V$. Now, the collection,

$$\{U \times V : U \text{ is a neighborhood of } a, \; V \text{ is a neighborhood of } b\}$$

is a base for the neighborhood system of (a,b). So, for any neighborhood W of (a,b) there exists $[f + g] \in A + B$ such that

$$S([f + g]) \subset W .$$

Hence (a,b) belongs to $\partial_{A+B}(X \times Y)$ by theorem 3.1. Thus

$$\partial_A X \times \partial_B Y \subset \partial_{A+B}(X \times Y)$$

and the proof is complete.

SECTION 8 Relation between boundaries of two compact Hausdorff spaces.

Let X and Y be two compact Hausdorff spaces. If T is a continuous mapping from X into Y, then T' will denote the induced mapping from $C(Y)$ into $C(X)$:

$$T'(g) = g \circ T \qquad , \quad g \in C(Y)$$

8.1 Proposition. (i) If T is onto then T' is an isomorphism of $C(Y)$ into $C(X)$.

(ii) If T is a homeomorphism onto then T' is an isomorphism from $C(Y)$ onto $C(X)$. Moreover, T' is an isometry.

Proof. It is evident that T' is a homomorphism.

(i) Suppose that $g \in C(Y)$ and $T'(g) = 0$. Then $g(T(x)) = 0$ for all x in X. Since T is onto, $g = 0$. Hence T' is an isomorphism (into).

(ii) By hypothesis, T^{-1} is a continuous mapping from Y onto X. For any f in $C(X)$, the function $f \circ T^{-1}$ is in $C(Y)$. Let $g = f \circ T^{-1}$, then clearly $T'(g) = f$, i.e. T' is onto. By (i) it is an isomorphism onto.

To see that T' is an isometry, let f,g be any two elements in $C(Y)$, then

$$\begin{aligned} \|T'(f)-T'(g)\| &= \|f \circ T - g \circ T\| \\ &= \sup_{x \in X} |f(T(x)) - g(T(x))| \end{aligned}$$

$$= \sup_{x \in X} |(f-g)\ (T(x))|$$

$$= \sup_{y \in Y} |(f-g)\ (y)| \quad \text{(since } T \text{ is onto)}$$

$$= \|f - g\|$$

Thus T' is an isometry.

8.2 <u>Theorem</u>. Let X, Y be two compact Hausdorff spaces and let T be a continuous mapping from X onto Y. Let T' be the induced mapping of T. Suppose A is a subset of $T'[C(Y)]$. Then

(i) If H is an A-set then $T[H]$ is a $T'^{-1}[A]$-set.

(ii) Every $T'^{-1}[A]$-set is of the form $T[H]$ where H is an A-set.

In addition, if T is one-one, then

(iii) The collection of all A-sets is in one-one correspondence with the collection of all $T'^{-1}[A]$-sets under the mapping : $H \to T[H]$.

<u>Proof</u>. (i) Suppose H is a subset of X and $T[H]$ is not a $T'^{-1}[A]$-set. Then there exists g in $T'^{-1}[A]$ such that $S(g) \cap T[H] = \emptyset$. For the mapping $T'(g) = g \circ T$ in A, we have: $S(g \circ T) = T^{-1}[S(g)]$. Indeed,

$$\begin{aligned} a \text{ in } S(g \circ T) &\Leftrightarrow |g(T(a))| = \sup\{|g(T(x))| : x \text{ in } X\} \\ &\quad = \sup\{|g(y)| : y \text{ in } Y\}, \text{ (since } T \text{ is onto)} \\ &\Leftrightarrow T(a) \text{ in } S(g) \\ &\Leftrightarrow a \text{ in } T^{-1}[S(g)] \end{aligned}$$

Now, $T^{-1}[S(g)] \cap H \subset T^{-1}[S(g) \cap T[H]] = T^{-1}[\emptyset] = \emptyset$. Thus $S(g \circ T) \cap H = \emptyset$. But $g \circ T$ is in A , so H is not an A-set.

(ii) Suppose G is a $T'^{-1}[A]$-set. Let $H = T^{-1}[G]$. Since T is onto , T' is one-one by proposition 8.1 (i) . Hence $A = T'[T'^{-1}[A]]$. For any f in A , we can find g in $T'^{-1}[A]$ such that $T'(g) = g \circ T = f$. We have seen in the proof of (i) that $S(f) = S(g \circ T) = T^{-1}[S(g)]$. Now g is in $T'^{-1}[A]$, so $S(g) \cap G \neq \emptyset$. This yields that $T^{-1}[S(g)] \cap T^{-1}[G] \neq \emptyset$ or $S(f) \cap H \neq \emptyset$. Thus $H = T^{-1}[G]$ is an A-set. We have $G = T[H]$.

(iii) Suppose that H and H' are two A-sets such that $H \neq H'$. Since T is one-one $T[H] \neq T'[H]$. By (i), (ii), we have (iii) .

As a corollary we have:

8.3 <u>Corollary</u>. If T maps X homeomorphically onto Y , then

$$T[\partial_A X] = \partial_{T'^{-1}[A]} Y$$

where $A \subset C(X)$ and $\partial_A X$ exists.

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