THE TARRY-ESCOTT PROBLEM

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ABSTRACT

The numbers 1, 2, and 6 have the same sum and same sum of squares as 0, 4, 5. These two sets are solutions of degree 2 of the Tarry-Escott problem. This problem of finding sets of integers having equal sums of like powers has been investigated for at least two hundred years and we have presented most of the general results.

For any given k there exist solutions in integers of the system of equations $\sum_{i=1}^{s} a_i^j = \sum_{i=1}^{s} b_i^j \quad (j = 1, 2, \ldots, k)$ for $s \geqq k + 1$. If $s < k + 1$ any solution will be composed of a set and a permutation of the set; such solutions are called trivial. Many writers have attempted to provide non-trivial solutions for the optimum case where $s = k + 1$. These so called ideal solutions exist for all $k \leqq 9$ but no such solutions have been found for $k \geqq 10$. We have been interested in providing solutions where s is smaller than for previous known examples, and have generated such solutions using a digital computer. Some of our results also apply to an extension of the Tarry-Escott problem in view of a result concerning bounds for this problem.

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CHAPTER I

THE NATURE OF THE PROBLEM

1. Introduction

The Tarry-Escott problem in Diophantine Analysis is to find two sets of integers equal in number such that the integers in each set have the same sum, the same sum of squares, etc., up to and including the same sum of kth powers, i.e. we are to find solutions in integers of the system of equations

$$\sum_{i=1}^{s} a_i^j = \sum_{i=1}^{s} b_i^j \quad (j = 1, 2, \ldots, k) \qquad \text{(A)}$$

A solution of (A) is written $a_1, \ldots, a_s \overset{k}{=} b_1, \ldots, b_s$ and a set of integers $(a_1, \ldots, a_s\ ;\ b_1, \ldots, b_s)$ satisfying (A) will be referred to as a set of degree k. A solution of (A) in which the a's are merely a permutation of the b's will be called trivial; we are concerned with non-trivial solutions.

This problem has attracted the attention of number theorists since the time of Goldbach and Euler who noted (1750-51) that

$$a, b, c, a + b + c \overset{2}{=} a + b, a + c, b + c, 0.$$

Dickson [3, Chapter 24] has given a comprehensive summary of papers on the problem of sets of integers with equal sums of like powers, and it was at his suggestion in view of the contributions made to this problem by G. Tarry and E. B. Escott that the problem is referred to as the Tarry-Escott problem.

2. Existence of Solutions

General parametric solutions of (A) have been found for only a few values of k and s. Dickson [4, pp.52] has proved that every set $(a_1, a_2, a_3 ; b_1, b_2, b_3)$ of degree 2 is obtained by adding an arbitrary integer to each term of $AD, AG + BD, BG \overset{2}{=} AD + BG, BD, AG$ (where A,B,D,G can be formed using the proof of this theorem from the set $(a_i ; b_i)$). Dickson [4, pp.54-58] also gives general solutions of (A) for $s = 4$ and $k = 2$, and for $s = 4$ and $k = 3$.

There are numerous particular solutions in both parametric and numerical form:

$$2, 3, 7 \overset{2}{=} 1, 5, 6$$

$$a + c, b + c, 2a + 2b + c \overset{2}{=} c, 2a + b + c, a + 2b + c$$

$$0, 5, 5, 10 \overset{3}{=} 1, 2, 8, 9$$

$$\pm (23a + 57b), \pm (40a - 6b), \pm (17a - 63b)$$

$$\overset{5}{=} \pm (23a - 57b), \pm (40a + 6b), \pm (17a + 63b)$$

However in view of the following two theorems we can prove the existence of solutions of (A) for other values of s and k without depending on illustrations.

Theorem 1. If $a_1, \ldots, a_s \overset{k}{=} b_1, \ldots, b_s$ then $Ma_1 + K, \ldots, Ma_s + K \overset{k}{=} Mb_1 + K, \ldots, Mb_s + K$ where M, K are arbitrary integers.

This theorem is due to M. Frolov [5] and can be proved using the binomial theorem. The theorem allows us to operate on a set $(a_1, \ldots, a_s ; b_1, \ldots, b_s)$ according to the rules of elementary algebra. If one solution of (A) comes from another through the use of Theorem 1 the two solutions are said to be equivalent. We define distinct solutions as solutions that are not equivalent. From Theorem 1 it follows that for each solution there is an equivalent one where $\sum a_i = \sum b_i = 0$. This equivalent solution has been called the standard form by Escott. Thus in:

$$0, 11, 13, 22 \overset{3}{=} 1, 7, 18, 20$$

if we multiply by 2 in order to make the sum divisible by four, and then subtract one fourth of this new sum from each term, we have

$$-23, -1, 3, 21 \overset{3}{=} -21, -9, 13, 17$$

in standard form.

<u>Theorem 2.</u> If $(a_1, \ldots, a_s ; b_1, \ldots, b_s)$ is a set of degree k then for any integer d

$$(a_1, \ldots, a_s, b_1 + d, \ldots, b_s + d ; b_1, \ldots, b_s, a_1 + d, \ldots, a_s + d)$$

is a set of degree $k + 1$

This theorem is due to Tarry [8] and can also be proved using the binomial theorem. Theorem 2 allows us to build up a solution for (A) of any desired degree starting from any particular solution of (A). Moreover if we choose d to be the number which occurs most frequently among the differences $a_i - a_j$ and $b_i - b_j$ we are then able to remove a good many of the terms which occur on both sides of the resulting solution of degree $k + 1$. To illustrate the power of this theorem we present the following sequence [6, pp. 331].

$$0, 3 \stackrel{1}{=} 1, 2$$

d = 3

$$0, 4, 5 \stackrel{2}{=} 1, 2, 6$$

d = 5

$$0, 4, 7, 11 \stackrel{3}{=} 1, 2, 9, 10$$

d = 7

$$0, 4, 8, 16, 17 \stackrel{4}{=} 1, 2, 10, 14, 18$$

d = 8

$$0, 4, 9, 17, 22, 26 \stackrel{5}{=} 1, 2, 12, 14, 24, 25$$

d = 13

$$0, 4, 9, 15, 26, 27, 37, 38 \stackrel{6}{=} 1, 2, 12, 13, 24, 30, 35, 39$$

d = 11

$$0, 4, 9, 23, 27, 41, 46, 50 \stackrel{7}{=} 1, 2, 11, 20, 30, 39, 48, 49$$

3. Ideal solutions

A number of writers have been interested in finding the least value of s for which (A) will have solutions for any particular k. The following theorem due to Bastien [1, pp. 171-172] provides a lower bound for s, and we present a proof for the sake of completeness.

Theorem 3. If equations (A) have a non-trivial solution, then $s \geqq k + 1$

Proof. Suppose $s \leqq k$, then the sets $(a_i \,;\, b_i)$ have the same sums of powers from the first to the kth. and hence the same symmetric functions. Hence $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$ are roots of the same equation and the a's are merely a permutation of the b's.

Wright [9, pp. 261] has defined a function $N(k)$ as the least number N such that $a_1, \ldots, a_N \stackrel{k}{=} b_1, \ldots, b_N$ has non-trivial solutions.

Theorem 3 states that $N(k) \geqq k+1$ and Tarry [8] gave the first upper bound for $N(k)$ by showing that $N(k) \leqq 2^{k-1}$ (this result follows immediately from any solution of (A) with $s = 4$ and $k = 3$ together with Theorem 2). It has been conjectured that in fact $N(k) = k + 1$, and solutions of (A) with $s = k + 1$ have been called ideal solutions by Chernick [2, pp. 626] who proved that there exists an infinite number of distinct ideal solutions of (A) for every value of $k \leqq 7$. We have provided examples above of ideal solutions of (A) for $k = 1, 2, 3, 4, 5, 7$ and these together with the following three examples [6, pp. 332 and 338] give Theorem 4.

$$0, 18, 27, 58, 64, 89, 101 \overset{6}{=} 1, 13, 38, 44, 75, 84, 102$$

$$0, 24, 30, 83, 86, 133, 157, 181, 197 \overset{8}{=} 1, 17, 41, 65, 112, 115, 168, 174, 198$$

$$\pm 12, \pm 11881, \pm 20231, \pm 20885, \pm 23738 \overset{9}{=} \pm 436, \pm 11857, \pm 20449, \pm 20667, \pm 23750$$

<u>Theorem 4.</u> $N(k) = k + 1$ for all $k \leqq 9$

CHAPTER II

SOME RECENT RESULTS

1. Bounds for N(k)

The best upper bound so far for $N(k)$ is that due to Wright [9, pp.261] who proved

$$N(k) \leqq W(k) = \begin{cases} \frac{1}{2}\ (k^2+3) & k \text{ odd} \\ \frac{1}{2}\ (k^2+4) & k \text{ even} \end{cases}$$

In 1961 Melzak [7, pp.234] gave an exact expression for $N(k)$ when he proved that

$$N(k) = \frac{1}{2} \min_{P \in \Omega} S\left[P(x)(1-x)^{k+1}\right]$$

where Ω is the class of all polynomials whose coefficients are integers, not all zero, and

$$S\left[P\right] = \sum_{i=0}^{n} |a_i| \text{ for } P = P(x) = \sum_{i=0}^{n} a_i\, x^i$$

This expression does not allow one to compute $N(k)$, but with each estimate N_k for $N(k)$ it leads to solutions of (A) with $s = N_k$. Melzak found that relatively low bounds for $N(k)$ result from taking $P(x)$ of the form

$$P(x) = \left[\prod_{j=1}^{p} (1-x^j)^{\beta_j}\right] \left[\prod_{n=1}^{k} \sum_{j=0}^{n} x^j\right]$$

where p is a small positive integer and $\beta_j = 0$ or 1.

The bounds on $N(k)$ are then of the form

$$\frac{1}{2}\ S\left[Q(x) \prod_{j=1}^{k+1} (1-x^j)\right] \text{ with } Q(x) = \prod_{j=1}^{p} (1-x^j)^{\beta_j}.$$

In constructing his table of results Melzak used four multipliers $Q(x)$:

$$1, \quad 1 - x, \quad 1 - x^2, \quad (1 - x)(1 - x^2).$$

He selected the lowest estimate N_k for $N(k)$ and showed that

$$k + 1 < N_k < W(k) \qquad \text{for} \quad 2 \leqq k \leqq 29.$$

We have improved these results slightly and also extended them to all $k \leqq 85$.

We considered the following multipliers $Q(x)$:

$1 - x$	$(1 - x)(1 - x^2)$
$1 - x^2$	$(1 - x)(1 - x^3)$
$1 - x^3$	$(1 - x)(1 - x^4)$
$1 - x^4$	$(1 - x^2)(1 - x^3)$
$1 - x^5$	$(1 - x^2)(1 - x^4)$
$1 - x^6$	$(1 - x^3)(1 - x^4)$
$1 - x^7$	$(1 - x)(1 - x^2)(1 - x^3)$
$1 - x^8$	$(1 - x)(1 - x^2)(1 - x^4)$
$(1 - x^2)(1 - x^3)(1 - x^4)$	$(1 - x)(1 - x^3)(1 - x^4)$

$(1 - x)(1 - x^3)(1 - x^4)(1 - x^5)$ and $\prod_{j=1}^{n} (1 - x^n)$ for $n = 4, 5, 6, 7$.

For each $Q(x)$, the expression $\frac{1}{2} S \left[Q(x) \prod_{j=1}^{k+1} (1 - x^j) \right]$ was evaluated (using an I.B.M. 1620) for $1 \leqq k \leqq 30$. It was apparent from these results that the lowest estimates N_k were given when $Q(x) = \prod_{j=1}^{n} (1 - x^n)$, where n varies with k. The calculations were then continued for $31 \leqq k \leqq 85$ with $Q(x) = \prod_{j=1}^{n} (1 - x^n)$ where $1 \leqq n \leqq 7$.

Table I was formed by selecting the lowest estimate N_k for $2 \leqq k \leqq 85$ and inserting the value of n relevant to each k.

TABLE I

k	N_k	n	k	N_k	n
2	3	1	44	588	4
3	4	1	45	588	3
4	6	1	46	627	4
5	6	1	47	644	4
6	10	2	48	742	4
7	12	1	49	802	4
8	18	1	50	830	4
9	18	1	51	872	4
10	22	1	52	834	4
11	22	1	53	896	5
12	30	2	54	958	5
13	32	1	55	1072	5
14	41	1	56	1202	5
15	46	1	57	1206	4
16	58	1	58	1218	4
17	58	1	59	1248	5
18	68	2	60	1270	5
19	74	1	61	1376	5
20	88	2	62	1517	5
21	92	2	63	1464	5
22	119	2	64	1694	5
23	124	2	65	1750	5
24	118	2	66	1866	5
25	146	2	67	1902	5
26	159	2	68	1990	5
27	166	3	69	1994	5
28	196	2	70	2120	6
29	198	3	71	2224	6
30	207	2	72	2372	6
31	228	3	73	2618	6
32	274	2	74	2947	6
33	258	3	75	2906	6
34	305	3	76	2902	6
35	308	3	77	2822	6
36	344	3	78	2853	6
37	332	3	79	3150	7
38	381	3	80	3386	6
39	402	3	81	3604	7
40	472	3	82	3903	7
41	462	3	83	4136	7
42	525	4	84	4502	7
43	514	3	85	4547	7

FIGURE 1

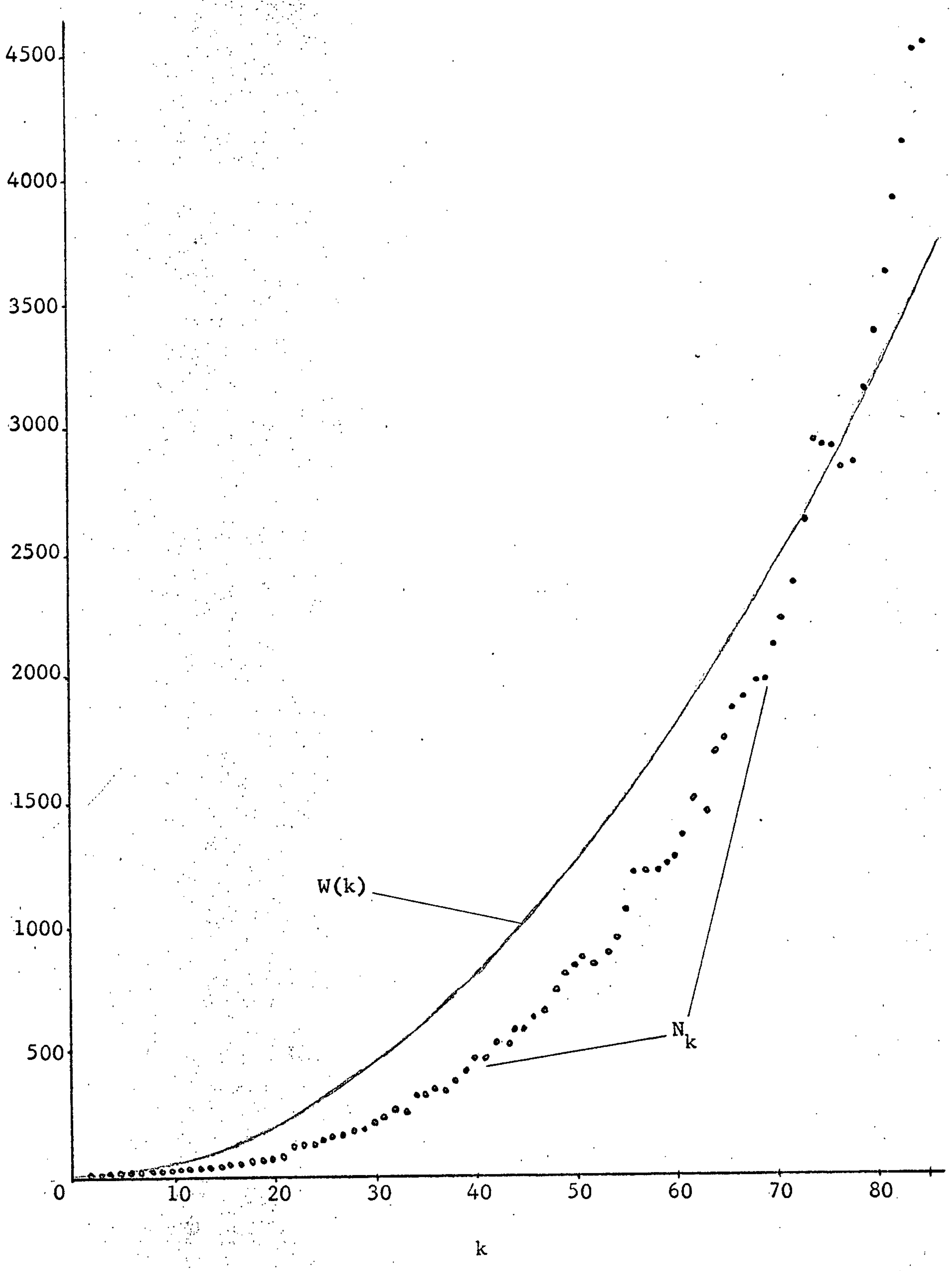
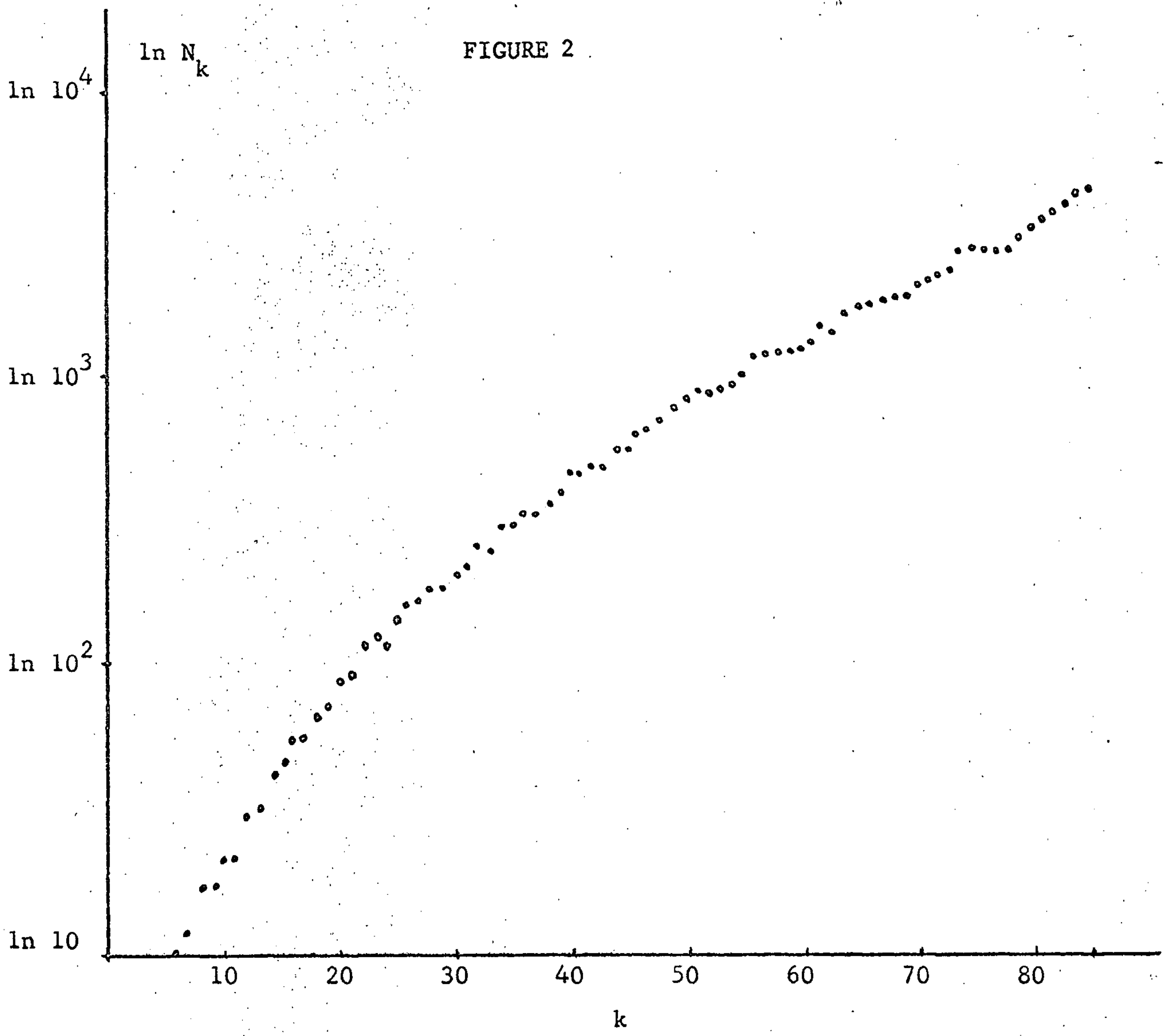

4500
4000
3500
3000
2500
2000
1500
1000
500
0
10
20
30
40
50
60
70
80
k
W(k)
N_k

Figure 1 is the graph of Table I together with the graph of $W(k)$. It is obvious from Figure 1 that while our upper bounds on $N(k)$ are lower than $W(k)$ for $2 \leqq k \leqq 73$, they soon become larger than $W(k)$. Hence if this method is to give further useful results new multipliers $P(x)$ are needed.

Following a suggested result of P. Erdös we attempted to fit $\exp (k^{1-c})$ for some $c < 1$ to the graph of Table I (see Figure 2). A reasonable approximation to this graph is given when $c = 0.52$.

FIGURE 2



2. Smallest solutions for certain degrees.

By use of Theorem 2 we attempted to obtain solutions of (A) for $k \geqq 10$ where the number of terms s is less than the estimates N_k given in Table I. An I.B.M. 1620 was programmed so that it would read a solution of (A) of any reasonable length and degree, and then calculate the difference d that occurs most frequently between any two terms from the same side of this given solution. It would then use d with Theorem 2 to produce a solution of (A) of the next higher degree, and continue in this manner.

By considering solutions of (A) of many different lengths and degrees we have found examples of solutions for $10 \leqq k \leqq 22$ where the number of terms s is less than those given in Table I. Table II gives the value of s corresponding to each value of k; the actual solutions of (A) may be found in the Appendix.

TABLE II

k	10	11	12	13	14	15	16	17	18	19	20	21	22
s	14	18	24	30	30	30	38	48	58	58	65	80	84

However the following weakness was discovered in the algorithm. It had been assumed that from any particular solution of (A) solutions of higher degree would be generated containing the least number of terms s, so long as the most frequent difference d was used at each step. This assumption was false.

When forming Table I the multiplier $(1 - x)$ was used with $\prod_{j=1}^{11} (1 - x^j)$ to produce a solution of (A) where $s = 22$ for $k = 11$.

This is equivalent to starting with the solution $0, 2 \overset{1}{=} 1, 1$ and using Theorem 2 with $d = 2,..11$. Using the algorithm with this solution the results shown in Table III were obtained.

TABLE III

k	d	s	$\frac{1}{2}\, S\left[(1 - x) \prod_{j=1}^{k+1} (1 - x^j)\right]$
1		2	2
	2		
2		3	3
	3		
3		4	4
	5		
4		6	6
	4		
5		6	6
	7		
6		8	11
	11		
7		10	12
	9		
8		14	18
	13		
9		14	18
	17		
10		18	22
	19		
11		24	22

Thus, by a more careful choice of d, the length of solutions can be decreased for $k = 6, 7, 8, 9, 10$. But for $k = 11$ this gives a solution of (A) where $s = 24$. This solution is longer than that obtained from a sequence of solutions which were constructed from values of d that were not always the most frequent.

3. Sequences of ideal solutions.

$I(m,n)$ was defined to be any sequence of ideal solutions of (A) of each degree from m to n inclusive, that is generated by Theorem 2. Then we indicate the proof of the following Theorem.

Theorem 5. $I(1,n)$ does not exist for $n \geqq 6$

Proof. We need only show that no ideal solution of (A) of degree 6 can be obtained by the use of Theorem 2 from any sequence of ideal solutions of consecutive degrees starting with degree 1.

All ideal solutions of (A) of degree 1 are equivalent to

$$0,\ a \overset{1}{=} b,\ c \qquad \text{(where } b \leqq c).$$

This gives

$$0,\ 2b + c,\ b + 2c \overset{2}{=} b,\ c,\ 2b + 2c \qquad (d = a = b + c)$$

$$0,\ b + c,\ 2c - b \overset{2}{=} b,\ c - b,\ 2c \qquad (d = c - b)$$

These give

$$0,\ b + 2c,\ 3b + c,\ 4b + 3c \overset{3}{=} b,\ c,\ 4b + 2c,\ 3b + 3c \qquad (d = 2b + c)$$

$$0,\ 2b + c,\ b + 3c,\ 3b + 4c \overset{3}{=} b,\ c,\ 3b + 3c,\ 2b + 4c \qquad (d = b + 2c)$$

$$0,\ 2c - b,\ 2b + c,\ b + 3c \overset{3}{=} b,\ c - b,\ 2b + 2c,\ 3c \qquad (d = b + c)$$

$$0,\ b + c,\ 3c - 2b,\ 4c - b \overset{3}{=} b,\ c - b,\ 3c,\ 4c - 2b \qquad (d = 2c - b)$$

$$0,\ b + c,\ 2c - 3b,\ 3c - 2b \overset{3}{=} b,\ 2c,\ c - 2b,\ 3c - 3b \qquad (d = c - 2b)$$

Now consider the solution

$$0,\ b + 2c,\ 3b + c,\ 4b + 3c \overset{3}{=} b,\ c,\ 4b + 2c,\ 3b + 3c$$

This will give an ideal solution of degree 4 only if the same difference occurs between three pairs of terms. The only numerical solutions that satisfy this condition are

$$0,\ 3,\ 4,\ 7 \overset{3}{=} 1,\ 1,\ 6,\ 6$$

$$0,\ 5,\ 10,\ 15 \overset{3}{=} 1,\ 3,\ 12,\ 14$$

$$0,\ 6,\ 7,\ 13 \overset{3}{=} 1,\ 3,\ 10,\ 12$$

$$0,\ 4,\ 7,\ 11 \overset{3}{=} 1,\ 2,\ 9,\ 10$$

$$0,\ 8,\ 9,\ 17 \overset{3}{=} 2,\ 3,\ 14,\ 15$$

These give

$$0,\ 6,\ 8,\ 17,\ 19 \overset{4}{=} 1,\ 3,\ 12,\ 14,\ 20$$

$$0,\ 4,\ 8,\ 16,\ 17 \overset{4}{=} 1,\ 2,\ 10,\ 14,\ 18$$

These give

$$0,\ 5,\ 6,\ 16,\ 17,\ 22 \overset{5}{=} 1,\ 2,\ 10,\ 12,\ 20,\ 21$$

$$0,\ 6,\ 8,\ 23,\ 25,\ 31 \overset{5}{=} 1,\ 3,\ 11,\ 20,\ 28,\ 30$$

$$0,\ 4,\ 9,\ 17,\ 22,\ 26 \overset{5}{=} 1,\ 2,\ 12,\ 14,\ 24,\ 25$$

None of these solutions will generate an ideal solution of degree 6. Theorem 5 follows after considering in a similar manner the remaining four ideal solutions of degree 3.

4. An extension of the problem.

In an extension of the Tarry-Escott problem the function M(k) has been defined as the least value of s such that (A) has a solution with

$$a_1^{k+1} + \ldots + a_s^{k+1} \neq b_1^{k+1} + \ldots + b_s^{k+1}$$

Clearly $M(k) \geqq N(k) \geqq k + 1$, while Theorem 3 and Theorem 4 prove that $M(k) = N(k) = k + 1$ for all $k \leqq 9$. Wright [9, pp.262] proved that $M(k) \leqq N(k^2)$, and was then [10, pp.48] able to prove that $M(k) < \frac{7k^4}{216}$.

The results obtained in Table I also apply to $M(k)$ in view of the following theorem.

<u>Theorem 6</u> $$M(k) = \frac{1}{2} \min_{P \in \Omega'} S\left[P(x)(1 - x)^{k+1}\right]$$

where Ω' is the class of all polynomials whose coefficients are integers, not all zero, and furthermore if $P \in \Omega'$ then $P(1) \neq 0$.

<u>Proof</u> For every $P \in \Omega'$, $P(x)(1 - x)^{k+1}$ generates a solution of (A) of degree k. Assume that this solution is also of degree $k+1$. Then it must be generated by $Q(x)(1 - x)^{k+2}$ for some $Q \in \Omega$.

Hence $P(x) = (1 - x)\, Q(x)$, which is false.

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APPENDIX I

DATA FOR TABLE II

(a) 0,3,7,10,23,35,50,53,56,81,82,93,96,97

$\overset{10}{=}$ 1,2,5,16,17,42,45,48,63,75,88,91,95,98.

Using $d = 3$ this generates a solution for $k = 11$ where $s = 18$,

which for $d = 4$ generates a solution for $k = 12$ where $s = 24$.

(b) We could not produce solutions for $k = 13$ and $k = 14$ where

$s < 30$ and hence used

0,5,7,19,21,21,25,32,46,47,48,50,53,74,75,78,79,100,103,105,106,
107,121,128,132,132,134,146,148,153.

$\overset{15}{=}$

1,2,13,15,15,27,29,30,40,44,51,55,56,65,76,77,88,97,98,102,109,
113,123,124,126,138,138,140,151,152.

Using $d = 25$ this generates a solution for $k = 16$ where $s = 38$

which for $d = 27$ generates a solution for $k = 17$ where $s = 48$

which for $d = 21$ generates a solution for $k = 18$ where $s = 58$

which for $d = 31$ generates a solution for $k = 19$ where $s = 58$

which for $d = 29$ generates a solution for $k = 20$ where $s = 65$.

(c) 1,6,8,9,20,23,32,43,44,45,49,57,60,66,68,69,79,80,84,92,101,102,
103,104,105,115,116,119,127,129,131,138,139,140,143,143,151,154,
155,163,166,174,175,178,186,186,189,190,191,198,200,202,210,213,
214,224,225,226,227,228,237,245,249,250,260,261,263,269,272,280,
284,285,286,297,306,309,320,321,323,328.

$\overset{21}{=}$

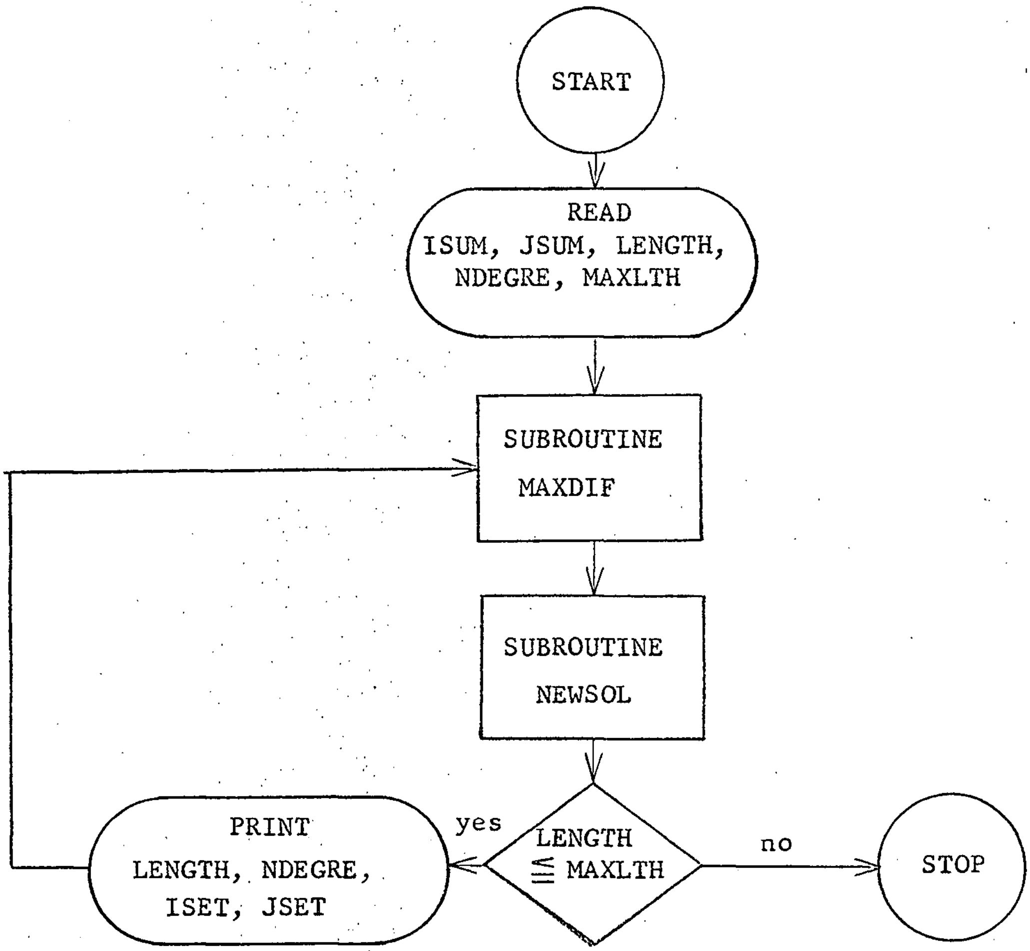
2,3,10,11,16,24,38,39,39,50,51,53,59,63,74,75,76,77,85,89,98,99,
100,111,112,113,120,122,124,125,132,133,134,136,144,148,156,159,
160,161,168,169,170,173,181,185,193,195,196,197,204,205,207,209,
216,217,218,229,230,231,240,244,252,253,254,255,266,270,276,278,
279,290,290,291,305,313,318,319,326,327.

Using $d = 35$ this generates a solution for $k = 22$ where $s = 84$.

APPENDIX II

GENERATING SOLUTIONS BY COMPUTER

FIGURE 3



LENGTH - number of terms on one side of a solution
NDEGRE - degree of the solution
ISET - one side of a solution
JSET - remaining side of the solution
MAXLTH - maximum number of terms acceptable on one side of a solution
MAXDIF - most frequent difference between pairs of terms from the same side of a solution

Figure 3 is a simple block diagram of the program used to generate solutions of (A) by Theorem 2 starting from any particular solution.

Subroutine MAXDIF determined the most frequent difference occuring between pairs of terms from the same side of a given solution. The program was routine but care has to be taken to ensure that for a solution such as $0, 3, 3 \overset{2}{=} 1, 1, 4$ the difference 3 occurs effectively twice and not four times.

One cannot program a computer to simply strike out terms that occur on both sides of a solution of (A), but subroutine NEWSOL generated solutions by Theorem 2 and disposed of common terms by use of an algebraic technique. The following example should indicate the method.

The solution $0, 3, 3 \overset{2}{=} 1, 1, 4$ is converted to the generating function

$$1 - 2x + 2x^3 - x^4$$

Using Theorem 2 with MAXDIF = 3 is equivalent to multiplying this generating function by $1 - x^3$.

Hence

$$\begin{array}{rrrrrrr}
1 & -2x & +2x^3 & -x^4 & & & \\
 & & 1 & -x^3 & & & \\
\hline
1 & -2x & +2x^3 & -x^4 & & & \\
 & & -x^3 & +2x^4 & & -2x^6 & +x^7 \\
\hline
1 & -2x & +x^3 & +x^4 & & -2x^6 & +x^7
\end{array}$$

which generates the solution

$$0, 3, 4, 7 \overset{3}{=} 1, 1, 6, 6$$

Subroutine NEWSOL appeared to be most efficient and we present it below.

```
      SUBROUTINE  NEWSOL (ISET, JSET, LENGTH, MAXDIF, NDEGRE)
      DIMENSION   NP(1500), IX(1500), ISET(200), JSET(200)
      IF(ISET(LENGTH)-JSET(LENGTH)) 30,30,31
30    NBIG=JSET(LENGTH)+1
      NBIG2=2*NBIG
      GO TO 22
31    NBIG=ISET(LENGTH)+1
      NBIG2=2*NBIG
22    DO 20 MX=1,NBIG2
20    NP(MX)=0
      DO 21 JIT=1,LENGTH
      ISET1=ISET(JIT)+1
      JSET1=JSET(JIT)+1
      NP(ISET1)=NP(ISET1)+1
21    NP(JSET1)=NP(JSET1)-1
      M2=0
      DO 41 NAT=1,MAXDIF
41    IX(NAT)=0
      LOW=MAXDIF+1
      NHIGH=NBIG+MAXDIF
      DO 79 K3=LOW,NHIGH
      K4=K3-MAXDIF
79    IX(K3)=NP(K4)
      DO 42 NIS=1,NHIGH
      NP(NIS)=NP(NIS)-IX(NIS)
      NPNIS=NP(NIS)
      M1=ABSF(NPNIS)+M2
42    M2=M1
      I8=0
      I9=0
      J8=0
      J9=0
      DO 43 IX1=1,NHIGH
      IF(NP(IX1)) 44,43,45
44    NP1=-NP(IX1)
      I8=I9+1
      I9=I9+NP1
      DO 54 L1=I8,I9
      MT=IX1-1
54    JSET(L1)=MT
      GO TO 43
45    NP2=NP(IX1)
      J8=J9+1
      J9=J9+NP2
      DO 55 L2=J8, J9
      MP=IX1-1
55    ISET(L2)=MP
43    CONTINUE
      LENGTH=I9
      NDEGRE=NDEGRE+1
      RETURN
      END
```