THE PERMANENT OF A CERTAIN MATRIX

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ABSTRACT

The purpose of this thesis is to attempt to evaluate the permanent function of a $n \times n$ complex matrix with entries $a_{ij} = \theta^{ij}$, θ being a primitive n^{th} root of unity.

If this matrix is denoted by A_n then its permanent function is given by

$$\text{per } A_n = \sum_{\sigma \in S_n} \prod_i^n a_{i\sigma(i)}$$
$$= \sum_{\sigma \in S_n} \theta^{\sum_i i\sigma(i)}.$$

In this thesis the following results are proved. Per A_n is always an integer; with per $A_n \equiv 0 \mod n$. If n is even per $A_n = 0$. For n odd however, the problem is in general not resolved. It is shown that if $n = p^2$ with p a prime, that per $A_n \equiv 0 \mod p^4$ and that for any prime n, per A_n can be narrowed down to be one of a restricted class of numbers.

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I   INTRODUCTION

The purpose of this thesis is to attempt to evaluate
the permanent function of an n×n complex matrix with entries
$a_{ij} = \theta^{ij}$, θ being a primitive n^{th} root of unity.

(a) Permanents.

For a matrix A of order n with entries in some field,
the permanent function of A is defined by

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i}^{n} a_{i\sigma(i)},$$

where S_n is the symmetric group of order n. This scalar
function of the matrix A appears frequently in combinatorial
problems concerning enumerations [1], and has the following
formal properties. The per(A) remains invariant under arbitrary
permutations of the rows and columns of A and is also invariant
under transposition; i.e., per(A) = per(A^T). If α is a
scalar from the underlying field of A, then the multiplication
of a row or column of A by α replaces per(A) by $\alpha \cdot$per(A).

The similarity of the above definition to that of a
determinant function of a matrix suggests the possibility of a
computational procedure for per(A) analogous to the well
known theory for det(A). Certain determinantal laws such as

the "Laplace Expansion Theorem", and the "Binet-Cauchy Theorem" for example have analogies for the permanent function, but the most useful property, that determinants are invariant under addition of a multiple of a row (or a column) to another row (or column), has no counterpart. This alone unfortunately invalidated the analogy for the permanent of the basic multiplicative relation, det(AB) = (det A)(det B) as well as the fact that det(A) can be expressed in terms of the characteristic roots λ_i of A (namely det(A) = $\prod_{i=1}^{n}\lambda_i$).

It is these properties that are not possessed by the permanent function that allow the determinants of most matrices to be easily evaluated, and the lack of them greatly inhibits the computation of per(A) and, in fact, make it an extremely difficult problem. Many matrices have easily evaluated determinants and undetermined permanents.

Efforts to relate permanents to other more tractable matrix scalar functions, to overcome this inherit computational difficulty, have not been overly successful. In fact, it has been shown for determinants, that no uniform affixing of $\pm$ signs to the elements of a matrix can convert the permanent into the determinant [2] as well as that there is no linear operation on matrices T : A→T(A) such that per T(A) = det(A) for all A [3]. In fact, it is even a difficult problem to establish if per(A) $\geq$ det(A) or vice versa, for special classes of matrices.

Much work has been done recently [4] to establish bounds and inequality relations for the permanent function by presenting the permanent as an inner product in the symmetric class of completely symmetric tensors (and using the Cauchy-Schwarz inequality). The best result that can be offered at the present time regarding the permanent of a product is the following inequality obtained from this approach.

$$\left| per(AB) \right|^2 \leq (per(AA^*))(per(B^*B)) \text{ where } A^* = \overline{A^T},$$

the transposed conjugate of A. (For unitary matrix U this inequality trivially gives the result $\left| per\ U \right| \leq 1$). The useful inequality $\left| per\ A \right| \leq \left(\sum_{i=1}^{n} w_i / n \right)^{\frac{1}{2}}$ where w_i (for all i = 1, 2, . . . , n) are the characteristic roots of A^*A, and the Binet-Cauchy relation mentioned earlier, are examples of results obtained by this technique.

To evaluate per(A) directly is a formidable task if n is large, even for a high speed computer, since this computation involves n! permutations. (On an IBM 7040 this would take almost a day if n = 11 and 5 months if n = 13.) Various methods have been developed to avoid this evaluation of n! permutations in the computation of per(A): The best known is the following formula due to RYSER [5].

Let B be a n-square matrix and let B_r denote a matrix obtained from B by replacing some r columns of B by zeros. Let S(X) be the product of the row sums of the matrix X.

Then

$$\text{per (B)} = S(B) - \sum S(B_1) + \sum S(B_2) - \ldots (-1)^{n-1}\sum S(B_{n-1}),$$

where $S(B_r)$ denotes the sum over all $\binom{n}{r}$ replacements of r of the columns by columns of zeros.

(b) The Matrix A_n.

A_n is defined to be the n-square matrix over the complex field with general ij^{th} entry θ^{ij}, where θ is a primitive n^{th} root of unity:

$$\text{i.e. } A_n = \begin{vmatrix} \theta^1 & \theta^2 & \theta^3 & . & . & \theta^{n-1} & 1 \\ \theta^2 & \theta^4 & \theta^6 & . & . & \theta^{n-2} & 1 \\ \theta^3 & \theta^6 & \theta^9 & . & . & \theta^{n-3} & 1 \\ . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \theta^{n-2} & \theta^{n-4} & \theta^{n-6} & . & . & \theta^2 & 1 \\ \theta^{n-1} & \theta^{n-2} & \theta^{n-3} & . & . & \theta^1 & 1 \\ 1 & 1 & 1 & . & . & 1 & 1 \end{vmatrix}.$$

It is readily seen that A_n is independent of the choice of a primitive n^{th} root of unity.

This matrix, which is met frequently in problems concerning extreme values of hermitian forms, also occurs very naturally in the study of circulants, which are matrices of the type:

$$C = \begin{vmatrix} c_1 & c_2 & \cdot & \cdot & \cdot & c_{n-1} & c_n \\ c_n & c_1 & c_2 & \cdot & \cdot & c_{n-2} & c_{n-1} \\ \cdot & \cdot & \cdot & & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \\ \cdot & & & & \cdot & \cdot & \cdot \\ c_2 & c_3 & \cdot & \cdot & \cdot & c_n & c_1 \end{vmatrix}$$

All circulant matrices of order n have in common the set of orthonormal eigenvectors $\left\{ 1/\sqrt{n} \ U_k : k = 1, 2, \ldots, n \right\}$ where

$$U_k = \begin{vmatrix} \theta^{1k} \\ \theta^{2k} \\ \cdot \\ \cdot \\ \theta^{nk} \end{vmatrix} \quad \text{for } \theta \text{ a primitive } n^{th} \text{ root of unity.}$$

The eigenvalues of a circulant are $\lambda_k = \sum_{j=1} c_j \theta^{k(j-1)}$, k=1,...,n. Let $U = (U_1, U_2, \ldots, U_n)/\sqrt{n}$. Then the matrix U is the unitary matrix which transforms all circulant matrices to the diagonal matrix of their eigenvalues [6],

$$\text{i.e., } U^* C \ U = \begin{vmatrix} \lambda_1 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \lambda_n \end{vmatrix}, \text{ where C is a circulant.}$$

(Note that U is just the matrix $A_n/\sqrt{n}$.)

By the basic multiplicative law for determinants and since the determinant of a unitary matrix has absolute value 1,

$$\det C = \prod_{k=1}^{n} \lambda_k;$$

i.e., $\det C = \prod^{n} \left(\sum_{j=1}^{n} c_j \theta^{k(j-1)} \right)$, and the determinant of a circulant is thus easily obtained. To determine the permanent function for circulants on the other hand, poses a very difficult problem which is still unresolved.

In attempting to resolve this problem by applying the "Binet-Cauchy Theorem" we obtain

$$\text{per } C = \sum_w 1/\mu(w) \quad \text{per } U^*\left[1,2,\ldots,n \mid w\right] \text{per } U\left[w \mid 1,2,\ldots,n\right] \prod \lambda_{w_t}$$

$$= \sum_w 1/\mu(w) \, \lambda_w \left| \text{per } U\left[1,2,\ldots,n \mid w\right] \right|^2,$$

where $w = \left\{ w_1, w_2, \ldots, w_n \right\}$, $1 \leq w_1, \ldots, \leq w_n \leq n$;
$\lambda_w = \lambda_{w_1}, \lambda_{w_2}, \ldots, \lambda_{w_n}$; and $\mu(w)$ is the product of the multiplicities of the distinct integers appearing in the sequence w. This leads to the study of per (U) or since $U = A_n/\sqrt{n}$ to the study of per A_n.

This matrix A_n is indeed interesting in its own right.

A_n is normal (i.e., $A_n A_n^* = A_n^* A_n$).
$A_n/\sqrt{n}$ is unitary (i.e., $A_n^{-1}/\sqrt{n} = A_n^*/\sqrt{n}$).
A_n can also be put in equivalent forms by elementary row and column operations such that the permanent remains invariant.

In one equivalent form it is a Vandermonde matrix.
i.e. a matrix of the type,

$$B = (r_i^{j-1}) = \begin{vmatrix} 1 & r_1^1 & r_1^2 & . & . & r_1^{n-1} \\ 1 & r_2^1 & r_2^2 & . & . & r_2^{n-1} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & r_n^1 & r_n^2 & r_n^3 & . & r_n^{n-1} \end{vmatrix}.$$

Since per $A_n = \text{per}(\theta^{ij})$ by definition,

$$= (\theta^1.\theta^2.....\theta^{n-1})\text{per}(\theta^{i(j-1)}), \text{ by taking } \theta^i \text{ out}$$

of ith row of $A_n \ \forall \ i.$

$$= \theta^{n(n+1)/2} \text{ per}(\theta^{i(j-1)}).$$

$$\text{per } A_n = \begin{cases} \text{per}(\theta^{i(j-1)}) & \text{if } n \text{ odd}, \\ - \text{per}(\theta^{i(j-1)}) & \text{if } n \text{ even}, \end{cases}$$

where $(\theta^{i(j-1)})$ is a Vandermonde matrix.

In another equivalent form A_n is a circulant. If $(\theta^{(i+j)^2})$
denotes the matrix with general ij^{th} entry $\theta^{(i+j)^2}$, then
$(\theta^{(i+j)^2})$ is a circulant and is easily shown to be equivalent
to A_n, with per $A_n = \text{per}(\theta^{(i+j)^2}).$

The determinants of all the matrices mentioned above
are readily found but almost nothing can be said about the
permanent of any of them.

(c)   The Permanent of A_n.

Per(A_n) can be computed by hand without too much difficulty for n = 2, 3, 4, 5 to give;

$$per(A_2) = 0 .$$
$$per(A_3) = -3,$$
$$per(A_4) = 0 .$$
$$per(A_5) = -5.$$

One would perhaps conjecture from these values that per(A_n) equals 0 for n even and equals -n for n odd. The former proves to be correct. The latter is destroyed by the result for n = 7:

$$per(A_7) = - 3 \cdot 5 \cdot 7 = -105.$$

This result and those for n = 9, 11, 13 were obtained by use of the University of British Columbia's IBM 7040 computer.

$$per(A_9) = +3^4 = +81.$$
$$per(A_{11}) = 3 \cdot 5 \cdot 11 \cdot 41 = 6765.$$
$$per(A_{13}) = 11 \cdot 13 \cdot 1229 = 175747.$$

These results which were also communicated by Professor D.H. Lehmer of the Department of Mathematics, University of Santa Barbara, California, do not appear to present any simple pattern.

As we shall subsequently see, the permanent of A_n is always an integer and is always divisable by n. In particular if $n=p^2$, where p is a prime, then per(A_n) is divisable by p^4. By using any of a variety of permanent

inequalities $[7]$ we can show $-\sqrt{n^n} \le \operatorname{per}(A_n) \le \sqrt{n^n}$ or equivalently that $-\left|\det(A_n)\right| \le \operatorname{per}(A_n) \le \left|\det(A_n)\right|$ (since $(\det(A_n))^2 = n^2$). For n even, $\operatorname{per}(A_n)$ is determined (=0) but for n odd the problem remains generally unresolved. In Section IV of this thesis we investigate the case of n prime, $n > 3$; the best result obtained, is that the permanent of A_n can be narrowed down to be one of a restricted set of integers.

Since
$$\operatorname{per}(A_n) = \sum_{\sigma \in S_n} \theta^{\sum_i^n i\sigma(i)}$$
$$= \sum_{k=0}^{n-1} a_k \theta^k.$$

where a_k denotes the number of permutations $\sigma \in S_n$ s.t. $\sum_{i=1}^n i\sigma(i) \equiv K \pmod n$, we can attempt to find $\operatorname{per}(A_n)$ by resolving the alternate number theory problem, of finding the number of permutations σ in the symmetric group S_n,

such that $\sum_{i=1}^n i\sigma(i) \equiv k \pmod n$, for $k=0,1,2,\ldots,n-1$.
It is easily seen that if n is a prime these two problems are equivalent. In the proof of the results that follow, almost no linear algebra is used; the approach generally being to find the permanent of A_n by solving this second problem. This involves for the most part a study of the symmetric group S_n.

II   PRELIMINARY THEOREMS

The following lemma shows that per A_n is independent of the choice of the primitive n^{th} root of unity.

Lemma 2.1.   If $A_n = (\theta^{ij})$ is n×n, where θ is a primitive n^{th} root of unity, and k is a positive integer such that $(k,n) = 1$, then

$$per(\theta^{ij}) = per(\theta^{kij}).$$

Proof: Since $(k,n) = 1$, $kj(\text{mod } n) = \sigma(j)$ where σ is a permutation in S_n. Thus

$$per(\theta^{kij}) = per(\theta^{i\sigma(j)}).$$

The matrix $(\theta^{i\sigma(j)})$, however, is just the matrix (θ^{ij}) with its columns permuted according to the permutation σ. Since the permanent of a matrix is invariant under permutations of rows or columns,

$$per(\theta^{kij}) = per(\theta^{ij}).$$

Q.E.D.

The permanent of A_n is obviously a polynomial in θ:

$$per\ A_n = per(\theta^{ij})$$

$$= \sum_{\sigma \in S_n} \prod_{i}^{n} \theta^{i\sigma(i)}$$

$$= \sum_{\sigma \in S_n} \theta^{\sum_{i}^{n} i\sigma(i)},$$

Hence

$$\text{per}(\theta^{ij}) = a_0 + a_1\theta + \ldots + a_{n-1}\theta^{n-1}, \tag{1}$$

where a_j is the number of permutations $\sigma \in S_n$ such that

$$\sum_{i=1}^{n} i\sigma(i) \equiv j \pmod{n}. \tag{2}$$

We are thus presented with an interesting problem in number theory, which to the best of our knowledge, has not been solved.

Not all of the coefficients in (1) can be distinct. More precisely, we have

Theorem 2.2. If $(k,n) = 1$, then in the representation

$$\text{per } A_n = a_0 + a_1\theta + \ldots + a_{n-1}\theta^{n-1},$$

$$a_i = a_{ki \pmod n}, \quad \text{for } i = 1, \ldots, n-1.$$

Proof:

$$\text{per}(\theta^{kij}) = \text{per}((\theta^k)^{ij})$$
$$= a_0 + a_1\theta^k + \ldots + \theta^{k(n-1)}.$$

By Lemma 2.1, $\text{per}(\theta^{kij}) = \text{per}(\theta^{ij})$,

therefore $\displaystyle\sum_{i=1}^{n-1} a_i \theta^{ki} = \sum_{i=1}^{n-1} a_i \theta^{i}.$

Since $\theta, \theta^2, \ldots, \theta^{n-1}$ are linearly independent over the real numbers,

$$a_i = a_{ki \pmod n}; \quad i = 1, \ldots, n-1.$$

Q.E.D.

<u>Corollary 2.3</u>. If n is a prime, n > 2, then in the representation

$$\text{per } A_n = a_o + a_1\theta + \ldots\ldots + a_{n-1}\theta^{n-1}, \qquad (1)$$

$$a_1 = a_2 = \ldots\ldots = a_{n-1} \cdot$$

<u>Proof</u>: If n is prime, (k,n) = 1 for all $1 \leq k \leq n-1$. Thus by Theorem 2.2,

$$a_k = a_1, \quad \text{for } 1 \leq k \leq n-1.$$

<div align="right">Q.E.D.</div>

In Theorem 2.2 we showed a relationship between the a_i for $1 \leq i \leq n-1$. The question whether a_o can be related to any of the other a_i, is answered if n is even in the next Theorem. We first require the following lemma.

<u>Lemma 2.4</u>. If σ is any permutation $\in S_n$ where n is even, and ρ is the full cycle permutation; $\rho = (1, 2, 3,\ldots\ldots, n)$, then

$$\sum_{i=1}^{n} i(\sigma\rho)(i) \equiv \sum_{i=1}^{n} i\sigma(i) + n/2 \pmod{n}.$$

<u>Proof</u>: $\sum_{i=1}^{n} i(\sigma\rho)(i) = \sum_{i=1}^{n} i\left[(\sigma(i) + 1)\pmod{n}\right],$

since $(\sigma\rho)(i) = (\sigma(i) + 1)\pmod{n}$.

$$\therefore \sum_{i=1}^{n} i(\sigma\rho)(i) \equiv \sum_{i=1}^{n} i\sigma(i) + \sum_{i=1}^{n} i \pmod{n}$$

$$= \sum_{i=1}^{n} i\sigma(i) + (n+1)n/2 \pmod{n}$$

$$= \sum_{i=1}^{n} i\sigma(i) + n/2 \pmod{n}, \quad \text{since n is even.}$$

<div align="right">Q.E.D.</div>

Theorem 2.5. If n is even, $a_i = a_{(i+n/2) \bmod n}$

for $i = 0, 1, \ldots, n-1$.

Proof: Let k be any natural number from 1 to n.

Consider the set

$$X_k = \left\{ \sigma \in S_n \;\middle|\; \sum_{i=1}^{n} i\sigma(i) \equiv k \pmod{n} \right\}; \quad k = 0, \ldots, n-1.$$

If $\rho = (1, 2, 3, \ldots, n)$ we have by the previous lemma

$$\sum_{i=1}^{n} i(\sigma\rho)(i) \equiv \sum_{i=1}^{n} i\sigma(i) + n/2 \pmod{n} \quad \forall \sigma \in X_k$$

$$= k + n/2 \pmod{n}.$$

Since for $\sigma, \tau \in S_n$ with $\sigma \neq \tau$, $\quad \sigma\rho \neq \tau\rho$

Therefore $\left| X_{(k+n/2) \bmod n} \right| \geq \left| X_k \right|$ where $\left| X_k \right|$ denotes the number of elements in X_k.

Similarily if we consider $\sigma \in X_{(k+n/2) \bmod n}$

we obtain

$$\left| X_{((k+n/2)+n/2) \bmod n} \right| \geq \left| X_{(k+n/2) \bmod n} \right|.$$

$$\text{i.e., } \left| X_k \right| \geq \left| X_{(k+n/2) \bmod n} \right|.$$

Therefore we have that the number of permutations $\sigma \in S_n$

that give $\sum_{i=1}^{n} i\sigma(i) \equiv k \pmod{n}$ is equal to the number

that give $\sum_{i=1}^{n} i\tau(i) \equiv k + n/2 \pmod{n}$.

i.e., $a_i = a_{(i+n/2) \bmod n}$ for $i = 0, \ldots, n-1$.

Q.E.D.

Observe that $S_n = \bigcup_{k=0}^{n-1} X_k$.

Theorem 2.2 gives the relation $\left|X_j\right| = \left|X_{jk(\bmod\ n)}\right|$ where $(k,n) = 1$. We would now like to look at the parity of the permutations in X_j.

Let X_j^+ and X_j^- denote the sets of even and odd permutations in X_j respectively, with $\left|X_j^+\right| = a_j^+$, $\left|X_j^-\right| = a_j^-$ where $a_j^+ + a_j^- = a_j$.

The analogy to Theorem 2.2 with X_j^+ or X_j^- replacing X_j is not generally true. For the case $k = n-1$ we can state the following.

Theorem 2.6. (a) If n is odd, $(n-1)/2$ even, or if n and $(n-2)/2$ are both even, then

$$a_j^+ = a_{j(n-1)\bmod\ n}^+, \text{ and } a_j^- = a_{j(n-1)\bmod\ n}^-, \quad j=0,1,\ldots,n.$$

(b) If n is even, $(n-2)/2$ odd, or if n and $(n-1)/2$ are both odd, then

$$a_j^+ = a_{j(n-1)\bmod\ n}^- \quad \text{for } j=0,1,\ldots,n-1.$$

Proof: Let $\sigma \in S_n$ be an arbitrary permutation and let

$$\gamma = \begin{pmatrix} 1, & 2, & 3, & \cdot & \cdot & \cdot & ,(n-1), & n \\ (n-1), & (n-2), & \cdot & \cdot & \cdot & \cdot & \cdot & 1, & n \end{pmatrix}$$

$$\sigma\gamma = \begin{pmatrix} 1, & 2, & \cdot & \cdot & n \\ \sigma(1), & \sigma(2), & \cdot & \cdot & \sigma(n) \end{pmatrix}\begin{pmatrix} 1, & 2, & \cdot & n \\ (n-1), & (n-2), & \cdot & n \end{pmatrix}$$

$$= \begin{pmatrix} 1, & 2, & \cdot & \cdot & \cdot & n \\ n-\sigma(1), & n-\sigma(2), & \cdot & \cdot & \cdot & n-\sigma(n) \end{pmatrix}$$

i.e., $(\sigma\gamma)(1) = n - \sigma(1)$.

Thus

$$\sum_{i=1}^{n} i(\sigma\gamma)(i) = \sum_{i=1}^{n} i(n - \sigma(i))$$

$$= (n-1)\sum_{i=1}^{n} i\sigma(i)(\bmod\ n),$$

i.e., If $\sigma \in X_j$ then $\sigma\gamma \in X_{j(n-1) \bmod n}$. This gives a one-to-one mapping of X_j onto $X_{j(n-1)\bmod n}$. (Since if $\sigma\gamma = \tau\gamma$ then $\sigma = \tau$.) It is readily checked that γ is an even permutation if n is odd, $(n-1)/2$ even, or if n, $(n-2)/2$ are both even. Similarly γ is an odd permutation if n is even, $(n-2)/2$ odd, or if n and $(n-1)/2$ are both odd.

Finally since the product of two even permutations is even, the product of an odd and an even is an odd permutation, etc., the result is proved.

Q.E.D.

Theorem 2.7. If n is even, $(n-2)/2$ odd, or if n and $(n-1)/2$ are both odd, then X_o has an equal number of even and odd permutations.

Proof: Consider any $\sigma \in X_o$ with σ even. As in the previous theorem, if

$$\gamma = \begin{pmatrix} 1 & , & 2 & , & 3 & ; & \cdot & \cdot & ; & 1 & ; & n \\ (n-1) & , & (n-2) & , & \cdot & \cdot & \cdot & & & & & \cdot \end{pmatrix},$$

then

$$\sum_{i=1}^{n} i(\sigma\gamma)(i) \equiv (n-1)\sum_{i=1}^{n} i\sigma(i) \pmod{n}$$

$$= 0, \quad \text{since} \quad \sum_{i=1}^{n} i\sigma(i) \pmod{n} \equiv 0.$$

$\therefore \quad (\sigma\gamma) \in X_o.$

Since γ under the conditions of this theorem is an odd permutation, there is thus a one-to-one mapping of the even permutations of X_o onto the odd permutations of X_o.

Q.E.D.

<u>Theorem 2.8.</u> If $r \mid n$, say $r \cdot n_1 = n$

and if $R = \left\{ x \mid x \equiv kr \pmod{n} \quad (k,n)=1, \quad 1 \le k < n \right\}$,

then $S = \sum_{x \in R} \theta^x$ is an integer.

<u>Proof:</u>

(a) Let r_1, r_2,...., $r_m = 1$ be an ordered listing of all the divisors of n, with $r_i > r_{i+1}$, $i = 1,...,m-1$.

Let

$$R_1' = \left\{ 1r_1, \; 2r_1, \; \ldots , \; (n_1-1)r_1 \right\} \quad \text{where } n_1 \cdot r_1 = n,$$

$$R_2' = \left\{ 1r_2, \; 2r_2, \; \ldots , \; (n_2-1)r_2 \right\} \quad n_2 \cdot r_2 = n,$$

$$= \qquad \qquad \qquad =$$

$$R_m' = \left\{ 1 , \; 2 , \; \ldots , \quad ,n-1 \right\} \quad n \cdot 1 = n,$$

and define

$$R_1 = R_1'$$

$$R_2 = R_2' - (R_2' \cap R_1)$$

$$=$$

$$R_m = R_m' - \bigcup_{i=1}^{m-1} (R_m' \cap R_i).$$

It is easily seen that all R_j are disjoint, and that

$$R_j = \left\{ x \mid x \equiv kr_j \pmod{n}, (k,n)=1, \quad 1 \le k < n \right\} \quad j = 1,2,..,m.$$

(b) To show $S_j = \sum_{x \in R_j} \theta^x$ is an integer $\forall$ $j = 1,...,m$ we proceed by induction as follows.

(1) $\quad S_1 = \sum_{i=1}^{n_1-1} \theta^{i r_1} = -1,$ an integer

(2) Assume that all S_i are integers for $i = 1, \ldots, k-1$

(3) To show S_k is an integer, consider

$$S_k' = \sum_{x \in R_k'} \theta^x = \sum_{i=1}^{n_k-1} \theta^{r_k i} = -1 .$$

By definition $R_k = R_k' - \bigcup_{i=1}^{k-1}(R_k' \cap R_i) .$

If r_k / r_i then $R_k' \cap R_i = R_i$ by construction of R_i and R_k'.

If $r_k \nmid r_i$ then $R_k' \cap R_i = 0,$

since say $x \in R_k' \cap R_i.$

Let r_{ki} be the L.C.M. of r_k and r^i.

$\therefore r_{ki} / x$ and also r_{ki} / n

Since r_{ki} is higher in the list $r_1, r_2, \ldots, r_n$

than r_i, this implies $x \in R_{ki}$, which gives a contradiction,

since R_j are disjoint.

We can now write S_k as follows.

$$S_k = \sum_{x \in R_k'} \theta^x - \sum_i \left[\sum_{x \in R_i} \theta^x \right]$$

where the summation over i,
is for all i,

$$= S_k' - \sum_i S_i$$

s.t. $R_i \cap R_k' \neq \emptyset.$

$\therefore S_k = (-1) - \sum_i (\text{integer}) ,$ since the S_i are integers by

the induction hypothesis.

<div align="right">Q.E.D.</div>

III SOME GENERAL RESULTS ON PER A_n.

Theorem 3.1. If n is even, per $A_n = 0$.

Proof:
$$Per(A_n) = a_o\theta^0 + a_1\theta^1 + \ldots + a_{n-1}\theta^{n-1} \qquad (1)$$

$$= \left(a_o\theta^0 + a_{n/2}\theta^{n/2}\right) + \left(a_1\theta^1 + a_{(n/2+1)}\theta^{(n/2+1)}\right) +$$

$$\ldots + \left(a_{(n/2-1)}\theta^{(n/2-1)} + a_{n-1}\theta^{n-1}\right).$$

By Theorem 2.5

$$a_i = a_{(i+n/2)}, \qquad \text{for } i = 0,\ldots,n-1.$$

Therefore
$$Per(A_n) = a_o(\theta^0 + \theta^{n/2}) + a_1(\theta^1 + \theta^{(n/2+1)})$$

$$\ldots + a_{(n/2-1)}(\theta^{(n/2-1)} + \theta^{n-1}).$$

But
$$\theta^k + \theta^{(n/2+k)} = \theta^k + \theta^{n/2} \cdot \theta^k = \theta^k + (-1)\theta^k$$

$$= 0 \quad \text{for any } k.$$

Thus the result is proved.

<div align="right">Q.E.D.</div>

Theorem 3.2.   Per A_n is an integer.

Proof: It is easily seen that we can write

$$\text{per } A_n = a_o \theta^o + \sum_i \left[\sum_{x \in R_i} a_x \theta^x \right] \tag{3}$$

where $R_i = \left\{ x \,\middle|\, x \equiv k r_i \pmod{n} \quad (k,n)=1, \quad 1 \le k < n \right\}$

and the summation $\sum_i$ is over all i, such that $r_i \,\middle|\, n$.

(Note that R_i does not contain n; hence $a_o \theta^o$ term in (3).)

Since by Theorem 2.2, $a_i = a_{ik \pmod n}$ for all $i = 1, \ldots, n$
and for $1 \le k < n,$ $(k,n)=1,$

therefore $\text{per } A_n = a_o + \sum_i \left[a_{r_i} \sum_{x \in R_i} \theta^x \right].$ $\tag{4}$

But by Theorem 2.8 $\sum_{x \in R_i} \theta^x$ is an integer.

Therefore per A_n is an integer.

<div align="right">Q.E.D.</div>

Theorem 3.3.   Per $A_n = 0 \pmod{n}$

Proof: If n is even the theorem is obviously true since
per $A_n = 0.$
The theorem is also true if n is odd, but we are not yet
in the position to prove this. It will follow from
Theorem 4.2 that n/a_i for $i = 0,1,2,\ldots,n-1.$
This fact together with Theorem 2.8 gives the result.

<div align="right">Q.E.D.</div>

<u>Theorem 3.4.</u> Let per $A_n = a_o + a_1\theta^1 + \ldots + a_{n-1}\theta^{n-1}$,

If n is an odd prime, then

$$\text{per } A_n = a_o - a_1.$$

<u>Proof:</u> By Corollary 2.3

$$a_1 = a_2 = \ldots = a_{n-1}.$$

Therefore per $A_n = a_o + a_1(\theta^1 + \theta^2 + \ldots + \theta^{n-1})$

$$= a_o + a_1(-1)$$

$$= a_o - a_1.$$

Q.E.D.

<u>Theorem 3.5.</u> Let per $A_n = a_o + a_1\theta^1 + \ldots + a_{n-1}\theta^{n-1}$,

where n is the square of a prime p. Then

$$\text{per } A_n = a_o - a_p.$$

<u>Proof:</u> Let Y_n be the set of residues mod n which are

relatively prime to n. Then by Theorem 2.2

$$a_i = a_{ik(\text{mod } n)}, \quad \text{for } k \in Y_n.$$

For $n = p^2$, the only numbers less than n not relatively

prime to n are p, 2p, $\ldots$, (p-1)p.

It follows that

$$\text{per } A_n = a_o + a_1\sum_{k \in Y_n}\theta^k + a_p\sum_{j=1}^{p-1}\theta^{jp}.$$

Now $$\sum_{i}^{n-1}\theta^1 = -1,$$

and $$\sum_{j=1}^{p-1}\theta^{jp} = -1.$$

Subtracting, $$\sum_{k \in Y_n}\theta^k = 0.$$

Thus per $A_n = a_o + a_1(0) + a_p(-1)$

$$= a_o - a_p.$$

Q.E.D.

IV   PER A_n IF n IS A PRIME > 3.

The aim in this section is to show that if n prime, n > 3, then the permanent of A_n is limited to be one of a restricted class of odd integers. Most of the preliminary Theorems used here are stated and proved in greater generality than is needed for the above result.

Let σ denote an arbitrary permutation in S_n, and ρ the full cycle permutation $(1, 2, 3, \ldots, n)$

i.e. $\rho = \begin{pmatrix} 1, & 2, & 3, & \ldots, & n \\ 2, & 3, & 4, & \ldots, & 1 \end{pmatrix}$.

It is easy to ckeck that

$$\rho^2 = \begin{pmatrix} 1, & 2, & 3, & \ldots, & n \\ 2, & 3, & 4, & \ldots, & 1 \end{pmatrix} \begin{pmatrix} 1, & 2, & 3, & \ldots, & n \\ 2, & 3, & 4, & \ldots, & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1, & 2, & 3, & \ldots, & n \\ 3, & 4, & 5, & \ldots, & 2 \end{pmatrix},$$

and in general that

$$\rho^k = \begin{pmatrix} 1 & , & 2 & , & 3 & , & \ldots, & n \\ \rho^k_{(1)}, & & \rho^k_{(2)}, & & \rho^k_{(3)}, & & \ldots, & \rho^k_{(n)} \end{pmatrix},$$

where $\rho^k_{(i)} = (k+i) \bmod n$. If

$$\sigma = \begin{pmatrix} 1 & , & 2 & , & 3 & , & \ldots, & n \\ \sigma(1) & , & \sigma(2), & & \sigma(3), & & \ldots, & \sigma(n) \end{pmatrix},$$

then

$$\rho\sigma = \begin{pmatrix} 1 & , & 2 & , & 3 & , & \cdots & , & n \\ 2 & , & 3 & , & \cdots & \cdots & , & 1 \end{pmatrix} \begin{pmatrix} 1 & , & 2 & , & 3 & , & \cdots & , & n \\ \sigma(1), & \sigma(2), & \sigma(3), & & \cdots & \cdots & , & \sigma(n) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & , & 2 & , & 3 & , & \cdots & , & n \\ \sigma(2), & \sigma(3), & & \cdots & \cdots & , & \sigma(1) \end{pmatrix},$$

and in general

$$\rho^k\sigma = \begin{pmatrix} 1 & , & 2 & , & 3 & , & \cdots & , & n \\ (\rho^k\sigma)(1), & (\rho^k\sigma)(2), & (\rho^k\sigma)(3), & & \cdots & , & (\rho^k\sigma)(n) \end{pmatrix}$$

where $\rho^k\sigma(i) \equiv \sigma[(i+k)(\bmod\ n)]$.

By giving k, values from 1 to n, we obtain the set
of n permutations;

$$\rho\sigma, \ \rho^2\sigma, \ \cdots, \rho^n\sigma \qquad \text{(with } \rho^n\sigma = \sigma\text{)}, \tag{5}$$

which are all obviously distinct (just looking at the image
of 1 under each permutation shows them all to be different).

Similarly the permutations obtained by multiplying
on the right by powers of ρ, are of the general form

$$\sigma\rho^k = \begin{pmatrix} 1 & , & 2 & , & 3 & , & \cdots & , & n \\ (\rho^k\sigma)(1), & (\rho^k\sigma)(2), & (\rho^k\sigma)(3), & & \cdots & , & (\rho^k\sigma)(n) \end{pmatrix},$$

where $(\sigma\rho^k)(i) \equiv [\sigma(i) + k]\ (\bmod\ n)$.
This again gives us n distinct permutations

$$\sigma\rho, \ \sigma\rho^2, \ \cdots, \sigma\rho^n \qquad \text{(with } \sigma\rho^n = \sigma \text{ since } \rho^n = 1) \tag{6}$$

which are not necessarily all equal to those in the first set.

The fact that $\sum_{i=1}^{n} i(\rho^k\sigma\rho^\ell)(i) \equiv \left[\sum_{i=1}^{n} i\sigma(i)\right] (\bmod\ n)$ for n odd
and for all k and ℓ, is proved in the following two theorems.

All this leads to the fact, that with any $\sigma \in S_n$
but $\sigma \notin G_n$, where G_n is a certain subgroup of S_n,

we can associate a block of n^2 distinct permutations (viz. the permutations $(\rho^k \sigma \rho^\ell)$ where $k, \ell = 1, 2, \ldots, n$) such that if σ and τ belong to such a block, then $\sum_{i=1}^{n} i\sigma(i) \equiv \left[\sum_{i=1}^{n} i\tau(i)\right] (\text{mod } n)$

Theorem 4.1. If $\sigma \in S_n$ and $\rho = (1, 2, \ldots, n)$ a full cycle permutation, then

$$\sum_{i=1}^{n} i(\sigma \rho^k)(i) \equiv \left[\sum_{i=1}^{n} i(\rho^k \sigma)(i)\right] (\text{mod } n), \quad k = 1, 2, \ldots, n.$$

Proof:

$$\sum_{i=1}^{n} i(\sigma \rho^k)(i) - \sum_{i=1}^{n} i(\rho^k \sigma)(i)$$

$$= \sum_{i=1}^{n} i\left([\sigma(i) + k](\text{mod } n)\right) - \sum_{i=1}^{n} i\sigma\left[(i + k)(\text{mod } n)\right]$$

$$= \left[\sum_{i=1}^{n} i\sigma(i) + k\sum_{i=1}^{n} i - \left(\sum_{i=1}^{n} i\sigma(i) - k\sum_{i=1}^{n} i\right)\right] (\text{mod } n)$$

$$= 2k\sum_{i=1}^{n} i (\text{mod } n)$$

$$= 2k(n+1)n/2 \ (\text{mod } n)$$

$$= 0 (\text{mod } n).$$

Therefore $\sum_{i=1}^{n} i(\sigma \rho^k)(i) \equiv \left[\sum_{i=1}^{n} i(\rho^k \sigma)(i)\right] (\text{mod } n)$.

Q.E.D.

Theorem 4.2. If $\sigma \in S_n$ and $\rho = (1,2,\dots,n)$ a full cycle permutation $\in S_n$ then

(a) if n <u>odd</u> $\displaystyle\sum_{i=1}^{n} i(\sigma\rho^k)(i) \equiv \sum_{i=1}^{n} i\sigma(i) \pmod{n}$ for any k ,

(b) if n <u>even</u> $\displaystyle\sum_{i=1}^{n} i(\sigma\rho^k)(i) \equiv \sum_{i=1}^{n} i\sigma(i) \pmod{n}$ for k even,

     $\displaystyle\sum_{i=1}^{n} i(\sigma\rho^k)(i) \equiv \sum_{i=1}^{n} i\sigma(i) \pm n/2 \pmod{n}$ k odd.

<u>Proof</u>:

$$\sum_{i=1}^{n} i(\sigma\rho^k)(i) \equiv \sum_{i=1}^{n} i\Big[(\sigma(i) + k) \bmod n\Big] \bmod n$$

$$= \sum_{i=1}^{n} i\sigma(i)\bmod n + k\sum_{i=1}^{n} i \bmod n$$

$$= \sum_{i=1}^{n} i\sigma(i)\bmod n + k(n+1)n/2 \bmod n .$$

(a) If n is <u>odd</u>, $k(n+1)/2$ is an integer.

Therefore $\displaystyle\sum_{i=1}^{n} i(\sigma\rho^k)(i) \equiv \sum_{i=1}^{n} i\sigma(i)\pmod{n}$ for all k.

(b) If n is <u>even</u>, $k(n+1)n/2$ is an integer,

$$\sum_{i=1}^{n} i(\sigma\rho^k)(i) \equiv \sum_{i=1}^{n} i\sigma(i)\pmod{n}, \quad \text{if k even,}$$

and $\displaystyle\sum_{i=1}^{n} i\sigma(i) \pm n/2 \bmod n$ if k odd.

                                                  Q.E.D.

Corollary 4.3. If n odd, then

$$\sum_{i=1}^{n} i(\rho^k\sigma\rho^l)(i) \equiv \sum_{i=1}^{n} i\sigma(i) \bmod n \quad \text{for all k, l.}$$

<u>Proof</u>: (From previous two theorems).

For any b, $0 \le b \le n-1$, and $m \in Y_n$, we can now define a function τ:

$$\tau(i) \equiv b + (i-1)m \pmod{n}, \qquad i = 1, 2, \ldots, n. \qquad \text{(a)}$$

where $1 \le \tau(i) \le n$. is a permutation of $\{1, 2, \ldots, n\}$ since $\tau(i) = \tau(j)$ implies $im \equiv jm \pmod{n}$, which in turn implies $i \equiv j \pmod{n}$ since $m \in Y_n$. Moreover, distinct values of b and m give rise to distinct permutations. For, suppose

$$\tau(i) \equiv b_1 + (i-1)m_1 \pmod{n} \quad \text{and}$$

$$\sigma(i) \equiv b_2 + (i-1)m_2 \pmod{n}.$$

Then $\tau = \sigma$ implies that $b_1 = \tau(1) = \sigma(1) = b_2$ and $b_1 + m_1 = \tau(2) = (b_2 + m_2)$ since $\tau(2) = \sigma(2)$. We define G_n to be the set of all permutations τ that are of the form (a).

Theorem 4.4. G_n is a subgroup of S_n containing $n \cdot \phi(n)$ elements, where $\phi(n)$ is the Euler Function.

Proof: From the above discussion the permutation τ in (a) can be specified in $n \cdot \phi(n)$ ways, corresponding to n choices for b and $\phi(n)$ choices for m.

To show that G_n is a group it is necessary only to show that $\sigma, \tau \in G_n$ implies $\sigma \tau^{-1} \in G_n$. Suppose

$$\sigma(i) \equiv b_1 + (i-1)m_1 \pmod{n} \quad \text{and}$$

$$\tau(i) \equiv b_2 + (i-1)m_2 \pmod{n}.$$

Since $(m_2, n) = 1$, there exists m_3 such that

$$m_2 m_3 \equiv 1 \pmod{n}.$$

It is readily checked that

$$\tau^{-1}(i) \equiv (b_3 + (i-1)m_3)(\bmod n),$$

where $b_3 \equiv m_3(1-b_2) + 1 \ (\bmod n)$.

Then $(\sigma\tau)(i) \equiv \tau^{-1}(\sigma(i))$

$$\equiv b_3 + \left[b_1 + (i-1)m_1 - 1\right]m_3$$

$$\equiv (b_3 + b_1 m_3 - m_3) + (i-1)m_1 m_3 \ (\bmod n).$$

Thus $\sigma\tau^{-1} \in G_n$.

<div align="right">Q.E.D.</div>

<u>Theorem 4.5</u>. If n is odd and $(n,3) = 1$, then

$$\sum_{i=1}^{n} i\sigma(i) \equiv 0(\bmod n) \quad \text{for all } \sigma \in G_n.$$

<u>Proof</u>: Let $\sigma(i) \equiv (b + (i-1)m)(\bmod n)$.

$$\text{Then } \sum_{i=1}^{n} i\sigma(i) \equiv b\sum_{i=1}^{n} i + m\sum_{i=1}^{n} i^2 - m\sum_{i=1}^{n} i$$

$$\equiv b \ n(n+1)/2 + m \ n(n+1)(2n+1)/6 - m \ n(n+1)/2$$

$$\equiv 0 \ (\bmod n).$$

<div align="right">Q.E.D.</div>

<u>Corollary 4.6</u>. If n is an odd prime, $n \neq 3$, then

$$\sum_{i=1}^{n} i\sigma(i) \equiv 0 \ (\bmod n) \quad \text{for all } \sigma \in G_n.$$

Thus in case n is a prime, $n > 3$ we have obtained a subgroup of S_n each element σ of which has the property

$$\sum_{i=1}^{n} i\sigma(i) \equiv 0(\bmod n).$$

There may however be other $\sigma \in S_n$ with this property.

We now proceed to show that the remaining portion of S_n consists of disjoint sets K_j each containing n^2 elements, each of which give rise to the same residue class.

<u>Theorem 4.7</u>. Let n be a prime > 3; and let $H_n = S_n - G_n$. Then H_n is the union of disjoint complexes K_j, $j = 1,\ldots,r$ such that

    (I) each K_j contains n^2 elements, which all

        have the same parity, and

    (II) for each j there exists c_j such that

$$\sum_{i=1}^{n} i\sigma(i) \equiv c_j \pmod{n} \quad \text{for all } \sigma \in K_j.$$

<u>Proof</u>: Let $\sigma \in H_n$; and let $\rho = (1,2,3,\ldots,n)$

Define $K_\sigma = \{\rho^{\ell}\sigma\rho^{m}; \ell,m = 1,2,\ldots,n\}$.

The n^2 permutations $\rho^{\ell}\sigma\rho^{m}$ are distinct, for suppose $\rho^{\ell_1}\sigma\rho^{m_1} = \rho^{\ell_2}\sigma\rho^{m_2}$; then, $\sigma(i) = \rho^{(\ell_2-\ell_1)}\sigma\rho^{(m_2-m)}(i)$ for $i = 1,2,\ldots,n$.

Specifically, for each i,

$$\sigma(i) \equiv \sigma(i+\ell_2-\ell_1) + m_2-m_1 \pmod{n} \text{ or}$$

$$\sigma(i+\ell_2-\ell_1) \equiv \sigma(i) + m_1-m_2 \pmod{n}. \tag{7}$$

Since n is prime, if $\ell_2 - \ell_1 \neq 0$, there exists k such that $(\ell_2-\ell_1)k \equiv 1 \pmod{n}$.

Then $\sigma(i+1) \equiv \sigma(i + k(\ell_2-\ell_1))$

$$\equiv \sigma(i) + k(m_1-m_2) \pmod{n}.$$

This means that $\sigma \in G_n$, contradicting our assumption that $\sigma \in H_n$. Therefore $\ell_2 - \ell_1 = 0$; it follows from (7)

that $m_1 - m_2 = 0$ also. Thus K_σ has n^2 distinct elements.
By Corollary 4.3 we have

$$\sum_{i=1}^{n} i (\rho^k \sigma \rho^\ell)(i) \equiv \sum_{i=1}^{n} i \sigma(i) \pmod{n}$$

for all $k, \ell = 1, 2, \ldots, n$. Thus for any particular
complex K_τ,

$$\sum_{i=1}^{n} i \sigma(i) \pmod{n} \qquad \text{gives the same residue for}$$

all permutations $\sigma \in K_\tau$.

We have

$$S_n = G_n \cup \left(\bigcup_{\sigma \in H_n} K_\sigma \right).$$

We now show that for two complexes K_σ, K_τ, either

$$K_\sigma \cap K_\tau = \emptyset \qquad \text{or} \qquad K_\sigma = K_\tau .$$

If $K_\sigma \cap K_\tau \neq \emptyset$, then $\rho^{\ell_1} \sigma \rho^{m_1} = \rho^{\ell_2} \tau \rho^{m_2}$
for some ℓ_1, m_1, ℓ_2, m_2. It follows that

$$\sigma = \rho^{\ell_2 - \ell_1} \tau \rho^{m_2 - m_1} = \rho^{\ell_3} \tau \rho^{m_3}, \qquad \text{where}$$

$\ell_3 = \ell_2 - \ell_1 \pmod{n}$, $m_3 \equiv m_2 - m_1 \pmod{n}$, and
$1 \leq \ell_3, m_3 \leq n$. Thus $\sigma \in K_\tau$, and $\rho^\ell \sigma \rho^m \in K_\tau$
for all ℓ, m, $1 \leq \ell, m \leq n$. Therefore $K_\sigma \subseteq K_\tau$
Since K_σ and K_τ each have n^2 elements,

$$K_\sigma = K_\tau .$$

Therefore complexes are either disjoint or equal.

To show all permutations in a complex K_τ, have the same parity; we note that for n odd,

$$\rho = (1,\ 2,\dots,\ n) = (1,2)(1,3),\dots,(1,n) \quad \text{is a product}$$

of an even number of transpositions. Also, since a product of even permutations is even and the product of an even and an odd permutation is odd, therefore $\sigma \in K_\tau$ is even or odd according if τ is even or odd.

Thus the n^2 elements in the complex have the same parity.

<div align="right">Q.E.D.</div>

<u>Note</u>: If r is the number of disjoint K_σ's,

$$S_n = G_n \cup K_1 \cup \ .\ .\ .\ . \cup K_r,$$

Considering the number of elements in each of these sets, we get

$$n! = n(n-1) + r\, n^2 \qquad (n\ \text{prime}).$$

Thus $(n-1)! = n - 1 + rn,$

or $(n-1)! = -1 (\text{mod } n).$

This is just Wilson's Theorem!

<u>Theorem 4.8</u>. If n is a prime, $n > 3$, and if a_k is the number of permutations in S_n such that

$$\sum_{i=1}^{n} i\sigma(i) \equiv k(\text{mod } n), \quad \text{then}$$

(I) $n(n-1)$ divides a_o,

and (II) n^2 divides a_k if $k = 1,2,\dots,n-1.$

<u>Proof</u>: By Theorem 4.7,

$$S_n = G_n \cup K_1, \ldots \cup K_s;$$ (8)

and if

$$\sum_{i=1}^{n} i\sigma(i) \equiv k \neq 0, \text{ then } \sigma \in \text{ some } K;$$

It follows that those σ such that $\sum_{i=1}^{n} i\sigma(i) \equiv k(\bmod n)$ completely occupy a certain number, r_k, of $K_j's$. Since each K_j has n^2 elements, $a_k = r_k \cdot n^2$; this proves (II). By Corollary 3, $a_1 = a_2 = , \ldots, = a_{n-1}.$ Hence, all r_k are equal, $1 \leq k \leq n-1$; say $r_k = r$. Then $a_k = rn^2.$ Now $a_0 + a_1 + , \ldots, + a_{n-1} = n!.$ Therefore $a_0 + (n-1)rn^2 = n!$.

$$a_0 = n(n-1)\left[(n-2)! - rn\right];$$

and (I) is proved.

<div align="right">Q.E.D.</div>

<u>Theorem 4.9</u>. For n a prime, n > 3, there exists non-negative integers y and z such that

$$\text{per } A_n = n\left[(n-1) + y(n-1)n - zn\right].$$

<u>Proof</u>: From Theorem 3.4,

$$\text{per } A_n = a_0 - a_1.$$

Now a_0 is the number of $\sigma \in S_n$ such that $\sum_{i=1}^{n} i\sigma(i) \equiv 0.$ Such σ occupy G_n and a certain number of the K_j's in the decomposition (8).

Thus $\quad a_0 = n(n-1) + wn^2.$

Since $n-1$ divides a_0, it also divides w.

Therefore $\quad a_0 = n(n-1) + y(n-1)n^2,\quad$ for some non-negative

integer y.

Since n^2 divides a_1,

therefore $\quad a_1 = zn^2$

for some non-negative integer z.

Therefore $\quad$ per $A_n = n(n-1) + y(n-1)n^2 - zn^2$

$$= n\left[(n-1) + y(n-1) - z\right].$$

$\qquad\qquad\qquad\qquad\qquad$ Q.E.D.

Theorem 4.10. For n a prime > 3, there exists a non-negative

integer y such that

$$\text{per } A_n = n^2 - n(n-2)! + n^3y.$$

Proof: From the discussion in Theorem 4.9,

$$a_0 = n(n-1) + yn^2(n-1),$$

$$\text{and} \quad a_k = zn^2, \quad k = 1, 2, \ldots, n-1.$$

Since $\quad n! = \displaystyle\sum_{i=0}^{n-1} a_i,$

$$n! = n(n-1) + yn^2(n-1) + (n-1)zn^2,$$

Hence $\quad zn = (n-2)! - 1 - yn,$

$\therefore \quad$ per $A_n = a_0 - a_1$

$$= n(n-1) + yn^2(n-1) + (yn + 1 - (n-2)!)n$$

$$= n^2 + n^3y - n(n-2)!.$$

$\qquad\qquad\qquad\qquad\qquad$ Q.E.D.

Remark: Computed values for the x and y above will be

indicated in Section VI.

This result does not answer the question as to the value of A_n, even for n a prime, but it does restrict considerably the possible values.

We note that $A_n/\sqrt{n}$ is a unitary matrix. By a result of Marcus and Newman [7]

$$\left|\text{per } A_n/\sqrt{n}\right| \le 1,$$

which implies

$$\left|\text{per } A_n\right| \le n^n.$$

Thus if n is a prime, then per A_n is restricted to the values of $n^2 + n^3 y - n(n-2)!$ that lie between $-n^n$ and $+n^n$, where y is a positive integer.

V THE CASE OF $n = p^2$, WHERE p IS A PRIME

In this section we shall obtain some results for the square of a prime somewhat similar to those obtained in Section IV for a prime. Let $n=p^2$, where p is a prime. The group G_n, which was useful in Section IV is not large enough for our purposes here. We shall form a larger group F_n consisting of permutations of the following type.

Let $\bar{\sigma}$ be a permutation of $\{1,2,\ldots,p\}$. Define:

$$\sigma(i + cp) \equiv \bar{\sigma}(i) + (a_i + cs)p \pmod{p^2} \qquad (20)$$

for $i = 1,2,\ldots,p$ and $c = 0,1,\ldots,p-1$; where $\{a_i\}$ and s are constants such that $0 \leq a_i \leq p-1$, $1 \leq s \leq p-1$. We show that each such function $\propto$ is indeed a permutation of $\{1,2,\ldots,p^2\}$. Suppose $\sigma(j) = \sigma(k)$, where, say, $j = i_1 + c_1 p$ and $k = i_2 + c_2 p$. Then

$$\bar{\sigma}(i_1) + (a_{i_1} + c_1 s)p \equiv \bar{\sigma}(i_2) + (a_{i_2} + c_2 s)p \pmod{p^2}, \qquad (21)$$

and $\qquad \bar{\sigma}(i_1) \equiv \sigma(i_2) \bmod p$.

By the definition of $\bar{\sigma}$, it follows that $i_1 = i_2$, $a_{i_1} = a_{i_2}$, and, from (21), $c_1 s \equiv c_2 s \pmod{p}$.

Hence $c_1 = c_2$, and finally $j = k$. Thus $\sigma(j) = \sigma(k)$ implies that $j = k$, so that σ is a permutation.

<u>Theorem 5.1.</u> The set F_n of permutations in S_n of the form (20) form a subgroup of S_n of order $p^p \cdot p! \cdot (p-1)$. Every permutation τ in S_n with the property

$$\tau(1+p) \equiv \tau(1) + kp \pmod{p^2}, \quad \text{for a fixed } k, \ 1 \le k \le p-1,$$

is a member of F_n.

<u>Proof:</u> Suppose

$$\sigma(1+cp) \equiv \bar{\sigma}(1) + (a_i+cs)p \pmod{p^2}. \quad (20)$$

Then σ^{-1} is given by

$$\sigma^{-1}(1+cp) \equiv \bar{\sigma}^{-1}(1) + (b_i+ct)p \pmod{p^2},$$

where $0 \le b_i \le p-1$, t is such that $1 \le t \le p-1$, $st \equiv 1 \pmod p$, and $1 \le b_i \le p-1$, $b_i \equiv -a_i t \pmod p$. For,

$$\sigma\sigma^{-1}(1+cp) \equiv \sigma^{-1}\left[\bar{\sigma}(1) + (a_i+cs)p\right] \pmod n$$

$$\equiv \bar{\sigma}\bar{\sigma}^{-1}(1) + \left[b_i + (a_i+cs)t\right]p \pmod n$$

$$\equiv 1 + \left[-a_i t + a_i t + c\right]p \pmod n$$

$$\equiv 1 + cp \pmod n.$$

Thus if $\sigma \in F_n$, $\sigma^{-1} \in F_n$.

Similarly, if μ is defined by

$$\mu(1+cp) \equiv \bar{\mu}(1) + (\alpha_i+cu)p \pmod n.$$

then $\sigma\mu(1+cp) \equiv \mu\left[\bar{\sigma}(1) + (a_i+cs)p\right] \pmod n$

$$\equiv \bar{\sigma}\bar{\mu}(1) + \left[\alpha_i + (a_i+cs)u\right]p \pmod n$$

$$\equiv \bar{\sigma}\bar{\mu}(1) + \left[\alpha_i + a_i u + csu\right]p \pmod n$$

$$\equiv \bar{\sigma}\bar{\mu}(1) + \left[\gamma_i + cw\right]p \pmod n,$$

where $0 \le \gamma_i \le p-1$, and $1 \le w \le p-1$. Hence $\sigma\tau \in F_n$, and F_n is a group.

To obtain the number of elements in F_n, we note
that there are $p!$ ways of choosing $\bar{\sigma}$; p ways of choosing
each a_i, $i=1,2,\ldots,p$; and $p-1$ ways of choosing s; yielding
$p! \cdot p^p \cdot (p-1)$ ways of choosing σ. These choices are
distinct; for, suppose

$$\sigma(i+cp) \equiv \bar{\sigma}(i) + (a_i+cs)p \pmod{p^2}$$

and $\quad \mu(i+cp) \equiv \bar{\mu}(i) + (b_i+ct)p \pmod{p^2}$

define two members σ and τ of F_n. If $\bar{\sigma} \neq \bar{\mu}$, there exists
j, $1 \leq j \leq p$, such that $\bar{\sigma}(j) \neq \bar{\mu}(j)$; then $\sigma(j) \neq \mu(j)$.
If $\bar{\sigma} = \bar{\mu}$, $a_i \neq b_i$ for some i, then $\sigma(i) \neq \mu(i)$.
If $\bar{\sigma} = \bar{\mu}$, all $a_i = b_i$, and $s \neq t$, then
$\sigma(i+p) \neq \mu(i+p)$. Thus there are $p! \cdot p^p \cdot (p-1)$ distinct
elements in F_n.

Now suppose that τ is a member of S_n such that
$$\tau(i+p) \equiv \tau(i) + kp \pmod{p^2}, \quad 1 \leq k \leq p-1,$$
for all $i=1,\ldots,n$. Then $\tau(i)$ can be written
$$\tau(i) = d_i + a_i p, \quad i=1,2,\ldots,p,$$
where $1 \leq d_i \leq p$, and $0 \leq a_i \leq p-1$. We show first that
$\{d_i\}$ is a permutation of $\{1,2,\ldots,p\}$.
Choose r such that $rk \equiv 1 \pmod{p}$. If
$d_i = d_j$ for $1 \leq j \leq p$, then

$$\tau\left[i + (a_j-a_i)rp\right] \equiv \tau(i) + (a_j-a_i)rkp \pmod{n}$$
$$\equiv d_i + a_i p + (a_j-a_i)p \pmod{n}$$
$$\equiv d_j + a_j p \pmod{n}$$
$$\equiv \tau(j) \pmod{n}.$$

Since τ is a permutation, $i + (a_j - a_i)rp \equiv j \pmod{n}$, which implies $i \equiv j \pmod{p}$, and finally that $i = j$.

We can now write $d_i = \bar{\tau}(i)$, where $\bar{\tau} \in S_p$; and

$$\tau(i) = \bar{\tau}(i) + a_i p.$$

Since $\tau(i+p) = \tau(i) + kp$,

we have $\tau(i+cp) = \bar{\tau}(i) + (a_i + ck)p$.

τ is therefore a member of F_n. This completes the proof of Theorem 5.1.

Q.E.D.

We next examine the residue class of

$$\sum_{i=1}^{n} i\sigma(i) \quad \text{for } \sigma \in F_n.$$

<u>Theorem 5.2.</u> If $\sigma \in F_n$, then

$$\sum_{i=1}^{n} i\sigma(i) \equiv p\sum_{i=1}^{p} i\bar{\sigma}(i) \pmod{n},$$

where $\bar{\sigma}$ is the permutation of $1, 2, \ldots, p$, given in (26).

The number of permutations σ in F_n such that

$$\sum_{i=1}^{n} i\sigma(i) \equiv jp \pmod{n}, \quad 0 \le j \le p-1, \text{ is equal to}$$

$p^p \cdot (p-1)$ times the number of permutations $\bar{\sigma} \in S_p$ such that

$$\sum_{i=1}^{p} i\bar{\sigma}(i) \equiv j \pmod{p}.$$

<u>Proof:</u> $\displaystyle\sum_{i=1}^{n} i\sigma(i) \equiv \sum_{i=1}^{p} i\left[\bar{\sigma}(i) + a_i p\right] + \sum_{i=1}^{p} (i+p)\left[\bar{\sigma}(i) + (a_i + s)p\right]$

$$+ \sum_{i=1}^{p} (i+2p)\left[\bar{\sigma}(i) + (a_i + 2s)p\right] + \ldots$$

$$+ \sum_{i=1}^{p} \left[i + (p-1)p\right]\left[\bar{\sigma}(i) + \{a_i + (p-1)s\}p\right]$$

$$\equiv p \sum_{i=1}^{p} i\bar{\sigma}(i) + p \sum_{i=1}^{p} i \; a_1 p + \sum_{i=1}^{p} i \left[s + 2s + \ldots + (p-1)s \right] p$$

$$+ \sum_{i=1}^{p} i\bar{\sigma}(i) \left[p + 2p + \ldots + (p-1)p \right]$$

$$\equiv p \sum_{i=1}^{p} i\bar{\sigma}(i) + p(p+1)/2 \left[s + 2s + \ldots + (p-1)s \right] p$$

$$+ p(p+1)/2 \left[p + 2p + \ldots + (p-1)p \right]$$

$$\equiv p \sum_{i=1}^{p} i\bar{\sigma}(i) \quad (\text{mod } n).$$

The latter statement in the theorem follows immediately from Theorem 5.1.

Q.E.D.

As in the previous section, we now consider for each $\sigma \in S_n - F_n$, the set

$$K_\sigma = \left\{ \rho^\ell \sigma \rho^m \;\middle|\; \ell, m = 0, 1, \ldots, n-1 \right\}.$$

(see page 27).

Theorem 5.3. If $\sigma \in S_n - F_n$, then $\left| K_\sigma \right| = n^2$.

For all elements $\tau \in K_\sigma$,

$$\sum_{i=1}^{n} i\tau(i) \equiv \sum_{i=1}^{n} i\sigma(i) \quad (\text{mod } n).$$

Proof: As ℓ, n run through $0, 1, \ldots, n-1$, n^2 permutations $\rho^\ell \sigma \rho^m$ are produced. It is required to show that they are different. It is sufficient to show that $\rho^\ell \sigma \rho^m = \sigma$ implies $\ell = m = 0$. If $\rho^\ell \sigma \rho^m = \sigma$, then $\sigma(i+\ell) + m \equiv \sigma(i)$ (mod n).

First, $\ell = 0$ implies $m = 0$. If $(\ell, n) = 1$, choose r so that $r\ell \equiv 1 \pmod{n}$. Then

$$\sigma(i + r\ell) \equiv \sigma(i) - r\,m \quad (\bmod\ n),$$

or $\quad \sigma(i + 1) \equiv \sigma(i) - r\,m \quad (\bmod\ n).$

Then $\quad r\,m \neq 0 \ (\bmod\ n),$ and

$$\sigma(i + p) \equiv \sigma(i) - r\,m\,p \quad (\bmod\ n).$$

By Theorem 5.1, $\sigma \in F_n$, a contradiction. The remaining alternative for ℓ is $\ell = cp$, $1 \leq c \leq p-1$. In this event

$$\sigma(i + p\ell) = \sigma(i + cp^2) \equiv \sigma(i) \ (\bmod\ n),$$

and hence $\quad \sigma(i) \equiv \sigma(i) - pm \quad (\bmod\ n).$

It follows that $m = 0 \ (\bmod\ p).$ Moreover $m \neq 0 \ (\bmod\ n),$ since, in that case,

$$\sigma(i + cp) \equiv \sigma(i) \quad (\bmod\ n),$$

which is impossible. Thus $m = kp$, $1 \leq k \leq p-1$.

By Theorem 5.1, $\sigma \in F_n$, a contradiction.

Thus $\left| K_\sigma \right| = n^2$.

Let $\tau = \rho^\ell \sigma \rho^m$; $\quad 0 \leq \ell, m \leq n-1$.

Then

$$\sum_i i\tau(i) = \sum_{i=1}^{n} i\, \rho^\ell \sigma \rho^m(i)$$

$$= \sum_{i=1}^{n} i\left[\sigma(i+\ell) + m\right]$$

$$= \sum_{i=1}^{n} (i+\ell)\,\sigma(i+\ell) - \sum_{i=1}^{n} \ell\,\sigma(i+\ell) + \sum_{i=1}^{n} i\,m$$

$$= \sum_{i=1}^{n} i\sigma(i) - \ell n(n+1)/2 + m\, n(n+1)/2$$

$$\equiv \sum_{i=1}^{n} i\sigma(i) \quad (\bmod\ n).$$
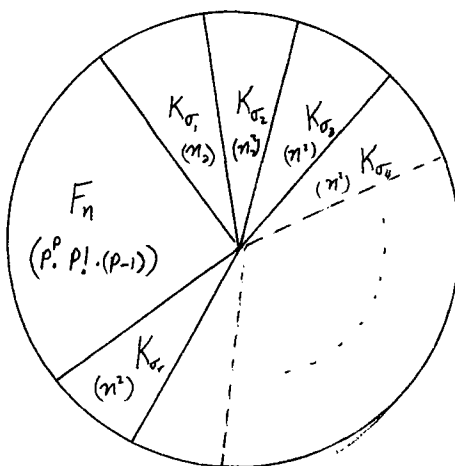
<div align="right">Q.E.D.</div>

<u>Theorem 5.4.</u> For $\sigma, \tau \notin F_n$, either $K_\sigma = K_\tau$, or $K_\sigma \cap K_\tau = \phi$.

<u>Proof</u>: If $\mu \in K_\sigma \cap K_\tau$, then $\mu = \rho^{l_1} \sigma \rho^{m_1} = \rho^{l_2} \tau \rho^{m_2}$;

which implies $\tau = \rho^{l_1 - l_2} \sigma \rho^{m_1 - m_2}$, and hence $\tau \in K_\sigma$.

It follows that $K_\tau \subseteq K_\sigma$.

Since $|K_\sigma| = |K_\tau|$, $K_\sigma = K_\tau$.

Thus S_n is partitioned as follows:

where the number of elements in the various subsets are

shown in brackets.

We are now in a position to get some information

on per A_n. Recall that

$$\text{per } A_n = a_0 + a_1 \theta + \ldots + a_{n-1} \theta^{n-1},$$

where a_j is the number of $\sigma \in S_n$ such that

$\sum_{i=1}^{n} i\sigma(i) = j \pmod{n}$. In fact, for $n = p^2$, by Theorem 3.5,

$$\text{per } A_n = a_0 - a_p.$$

By Theorem 5.2, the number of σ's in F_n for which

$\sum_{i=1}^{n} i\sigma(i) \equiv 0 \pmod{n}$ is $p^p \cdot (p-1)$ times the number

of $\bar{\sigma} \in S_p$ such that $\sum_{i=1}^{n} i\bar{\sigma}(i) \equiv 0 \pmod{p}$. By Theorem 4.8 the latter is $p(p-1) + k_1 p^2$ for some integer k_1.

There may, in addition be $\sigma \notin F_n$ such that $\sum_{i=1}^{n} i\sigma(i) \equiv 0$. The number of these will be a multiple k_2 of $n^2 = p^4$, by Theorems 5.3 and 5.4. Thus

$$a_o = p^{p+1}(p-1)^2 + k_1 p^{p+2}(p-1) + k_2 p^4.$$

Similarly, each σ in F_n such that $\sum_{i=1}^{n} i\sigma(i) \equiv p$ corresponds to a $\bar{\sigma}$ in S_p such that $\sum_{i=1}^{p} i\bar{\sigma}(i) \equiv 1 \pmod{p}$.

The number of the latter is $k_3 p^2$, by Theorem 4.8

Thus the number of $\sigma \in F_n$ such that $\sum_{i=11}^{n} i\sigma(i) \equiv p \pmod{n}$ is $k_3 p^{p+2}(p-1)$. Outside F_n there will be a multiple k_4 of p^4 further permutations of this type. We have then

$$a_p = k_3 p^{p+2}(p-1) = k_4 p^4;$$

and as a result

$$\text{per } A_n = p^{p+1}(p-1)^2 + (k_1-k_3)p^{p+2}(p-1) + (k_2-k_4)p^4.$$

Setting $k_1 + k_3 + 1 = k$, and $k_2 + k_4 = \ell$, we have

<u>Theorem 5.5.</u> For $n=p^2$, $p \neq 3$,

$$\text{per } A_n = p^{p+1}(p-1)(kp-1) + \ell p^4,$$

where k, ℓ are positive integers. In particular

$$\text{per } A_n \equiv 0 \pmod{p^4}.$$

VI COMPUTER RESULTS

(a) <u>Computer Programme</u>.

Per A_n was calculated for the odd primes, $n = 5, 7, 11, 13$, on an IBM 7040 computer. This was done using the following well known formula due to Ryser [5].

Let B be a n-square matrix and let B_r denote a matrix obtained from B by replacing some r columns of B by zeros. Let S(X) be the product of the row sums of the matrix X. Then

$$\text{per } B = S(B) - \sum S(B_1) + \sum S(B_2) - \ldots (-1)^{n-1} \sum S(B_{n-1}) \qquad (9)$$

where $\sum S(B_r)$ denotes the sum over all $\binom{n}{r}$ replacements of the r columns by columns of zeros.

If $B = A_n$, where n is odd, and if $A_{n(r)}$ denotes the matrix obtained by replacing r columns of A_n by columns of zeros, then it is easily checked that

$$(1/r)S(A_{n(r)}) = (1/(n-r))S(A_{n(n-r)}) .$$

If $S'(A_{n(r)})$ denotes the product of the first n-1 row sums of $A_{n(r)}$, then since the nth row of A_n consists of 1's only, we have

$$S(A_{n(r)}) = r \cdot S'(A_{n(r)}) .$$

Thus we can rewrite (9) as

$$\text{per } A_n = -1\sum S'(A_{n(1)}) + 2\sum S'(A_{n(2)}) - 3\sum S'(A_{n(3)})$$

$$\ldots + (-1)^{n-1}(n-1)\sum S(A_{n(n-1)}).$$

Therefore

$$\text{per } A_n = (n-2)\sum S'(A_{n(1)}) - (n-4)\sum S'A_{n(2)} +$$

$$+ (n-6)\sum S'A_{n(3)} + \ldots + (-1)^{n-1}\sum S'A_{n(n-1)/2}. \qquad (10)$$

Thus to evaluate per A_n if n is odd, it is only necessary to compute the product of the first n-1 row sums of all possible matrices $A_{n(r)}$ for $r = 1,\ldots,(n-1)/2$. That is for $\binom{n}{1}+\binom{n}{2}+ \ldots + \binom{n}{(n-1)/2}$ matrices in all.

The programme used to obtain values for per A_n, for n = 5, 7, 11, and 13, was based on the equation (10) in the above paragraph.

It consisted essentially of a combination generator, which generated all possible combinations of n things r at a time, for r = 1, 2,...,(k-1)/2. From these the matrices $A_{n(r)}$ were formed for each value of r, and $\sum S'A_{n(r)}$ evaluated. By equation (10), per A_n was thus obtained.

This programme was originally designed to compute and print out $S'A_{n(r)}$ for all r, in a particular order; with the hope of observing patterns which would allow a conjecture for per A_n to be obtained from equation (10).

Due to the excessive checking involved in the
ordering of the combinations, this programme proved to be
very inefficient if used to calculate per A_n for $n > 13$.
For $n = 17$, no result could be obtained in two hours computing
time. Whereas, for $n = 5, 7, 11, 13$, per A_n was computed
in less than five minutes. With an efficient combination
generator, A_{17} and A_{19} should be readily obtained by the
IBM 7040.

(b) Results.

For n an odd prime, $n > 3$ we have the following
computer results.

$$per\ A_5 = -5$$
$$per\ A_7 = -3 \cdot 5 \cdot 7$$
$$per\ A_{11} = 3 \cdot 5 \cdot 11 \cdot 41$$
$$per\ A_{13} = 11 \cdot 13 \cdot 1229 .$$

Theorem 4.9 gives $Per\ A_n = n\left[(n-1) + y(n-1)n - zn\right]$,
or (Theorem 4.10) $Per\ A_n = n^2 - n(n-2)! + n^3 y.$
Thus for $n = 5, 7, 11,$ and 13, we can list the values of y and z.

n	y	z
5	0	1
7	2	15
11	3004	29985
13	36274	2834249

```
C       COMPUTING PERMANENT OF A(N) BY RYSERS METHOD.
C
C
        COMPLEX X1,TH(17),PROD,RSUM,THETA(17,17),EN
        INTEGER SP(17), PER
        DIMENSION KOL(1431,8)
        DIMENSION MOL(17)
        DIMENSION MOT(17)
        PRINT 1
      1 FORMAT(34H COLUMNS     PRODUCT OF ROWSUMS                    )
        GO TO 112
     79 PER=0
        SP(1)=1
        SP(2) = (N-1)/2
        DO 113 I=1,MO
    113 PER=PER+(N-2*I)*N*SP(I)*(-1)**I
        WRITE(6,111) PER
    112 READ(5,2)N
      2 FORMAT(I2)
        PRINT 19,N
     19    FORMAT(1X,I2 )
     61 MO=(N-1)/2
        M=3
        GO TO 51
     38 M=M+1
     51 M1=M-1
        M2=M-2
        IF(M.GT.MO) GO TO 79
        T=N
        X1=CMPLX(0.,2.*3.1415927/T)
        N1=N-1
        DO 40 I=1,N1
        EN=FLOAT(I)
     40 TH(I)=CEXP(EN*X1)
        TH(N)=(1.0,0.0)
        DO 10 I=1,N
        DO 43 JA=1,N
        KA=I*JA
     44 IF(KA.LE.N) GO TO 43
        KA=KA-N
        GO TO 44
     43 THETA(I,JA)=TH(KA)
```

```
10 CONTINUE
   SP(M)=0
   NZERO=0
   K=1
   NO=1
   NI=1
   DO 3 I=1,M1
   NO=NO*(N-I)
   NI=NI*(I+1)
 3 NO=NO/NI
18 DO 5 I=1,M1
 5 KOL(1,I)=1
   KOL(1,M)=N+1-M
   DO 81 I=1,M
81 MOL(I)=KOL(1,I)
   DO 13 I=1,M
13 MOT(I)=I
   GO TO 55
11 K=K+1
   K1=K-1
 6 DO 7 I=1,M
 7 KOL(K,I)=MOL(I)
   KOL(K,M1)=MOL(M1)+1
   KOL(K,M)=MOL(M)-1
C
   DO 53 I=1,M
53 MOL(I)=KOL(K,I)
C
   L=1
37 MP=0
   J=1
C
62 DO 8 I=1,M
 8 IF(KOL(K,I).NE.KOL(J,I)) GO TO 20
   GO TO 300
20 IF(J.EQ.K1) GO TO 34
33 J=J+1
   GO TO 62
34 MP=MP+1
   J=1
   KM=KOL(K,1)
35 DO 36 I=1,M1
```

```fortran
         I1=I+1
    36 KOL(K,I)=KOL(K,I1)
         KOL(K,M)=KM
         IF(MP.EQ.M) GO TO 200
         GO TO 62
   300 IF(MOL(M).LE.1) GO TO 24
         GO TO 6
C
   200 MOT(1)=1
         DO 93 I=2,M
    93 MOT(I)=MOT(I-1)+MOL(I-1)
    55 PROD=(1.0,0.0)/FLOAT(M)
         DO 16 I=1,N
         RSUM=(0.0,0.0)
         DO 17 LA=1,M
         JA=MOT(LA)
    17 RSUM=RSUM+THETA(I,JA)
    16 PROD=PROD*RSUM
         SP(M)=SP(M)+INT(.5+REAL(PROD))
         WRITE(6,52) PROD,(MOT(LA),LA=1,M)
    52 FORMAT(2F10.2,4X,20I3)
         IF(K.EQ.1) GO TO 11
   201 K=K+1
         K1=K-1
         IF(K.GT.NO) GO TO 39
         GO TO 300
    39 WRITE(6,111) SP(M)
   111 FORMAT(60X,I10)
         GO TO 38
    24 L=L+1
         ML1=M-L+1
         ML=M-L
         DO 25 I=1,M
    25 KOL(K,I)=MOL(I)
         KOL(K,ML)=MOL(ML)+1
         IF(L.EQ.2) GO TO 41
         DO 26 I=ML1,M1
    26 KOL(K,I)=1
    41 KOL(K,M1)=1
         KOL(K,M)=0
         DO 30 I=1,M1
    30 KOL(K,M)=KOL(K,M)+KOL(K,I)
```

```
      KOL(K,M)=N-KOL(K,I)
      GO TO 14
14 IF(KOL(K,M).LT.1) GO TO 24
      DO 42 I=1,M
42 MOL(I)=KOL(K,I)
      GO TO 37
21    CALL EXIT
      END
$ENTRY
```

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