STABILITY IN THE LARGE OF AUTONOMOUS SYSTEMS

OF

TWO DIFFERENTIAL EQUATIONS

by

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STABILITY IN THE LARGE OF AUTONOMOUS SYSTEMS

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ABSTRACT

The object of this dissertation is to discuss the stability in the large of the trivial solution for systems of two differential equations using qualitative methods (of course in combination with the construction of Lyapunov function). The right hand sides of these systems do not contain the time t explicitly.

First of all we discuss the system of the type

$$\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = f(\alpha), \quad \alpha = cx - dy, \quad c \neq 0 \tag{1}$$

These equations occur in automatic regulation. Using qualitative methods we determine sufficient conditions in order that the trivial solution of system (1) be asymptotically stable in the large. In this connection we note that a theorem proved by Ershov (Prikl. Mat. Meh. 18(1954), 381-383) is wrong. We then solve the problem of Aǐzerman for the systems of two equations, namely, for the systems

$$\frac{dx}{dt} = f(x) + ay, \quad \frac{dy}{dt} = bx + cy \tag{2}$$

and

$$\frac{dx}{dt} = ax + f(y), \quad \frac{dy}{dt} = bx + xy \tag{3}$$

In the case of system (2) we give a new proof of a theorem which asserts that if $c^2 + ab \neq 0$, then the trivial solution is asymptotically stable in the large under the generalized Hurwitz conditions. The theorem was first proved by Erugin. For system (3) Malkin showed that the trivial solution is asymptotically stable in the large under the conditions:

$$a + c < 0, \quad (acy - bf(y))\, y > 0 \quad \text{for } y \neq 0 \text{ and} \tag{4}$$

$$\int_0^y (acy - bf(y))\, dy \to +\infty \quad \text{as } |y| \to +\infty \tag{5}$$

We prove a similar theorem without the requirement of (5). We also discuss the stability in the large of the systems

$$\frac{dx}{dt} = ax + f_1(y), \quad \frac{dy}{dt} = f_2(x) + cy$$

$$\frac{dx}{dt} = f_1(x) = f_2(y), \quad \frac{dy}{dt} = bx + cy$$

We consider again the system of the type (1) under assumptions as indicated by Ershov (Prikl. Mat. Meh. 17(1953), 61-72) who has discussed various cases where the stability in the large holds. Not agreeing fully with the proofs of these theorems we give our own proofs. Finally we discuss the stability in the large of the systems

$$\frac{dx}{dt} = h_1(y)\, x + ay, \quad \frac{dy}{dt} = h_2(x)\, x + by$$

$$\frac{dx}{dt} = xh_1(y) + ay, \quad \frac{dy}{dt} = bx + h_2(x)\, y \tag{6}$$

under suitable assumptions. As a sample case we prove that if $ab > 0$, then the trivial solution of system (6) is asymptotically stable in the large under conditions:

$$h_1(y) + h_2(x) < 0, \quad h_1(y)\, h_2(x) - ab > 0 \quad \text{for } x \neq 0, \; y \neq 0.$$

ABSTRACT

The object of this dissertation is to discuss the stability in the large of the trivial solution for systems of two differential equations using qualitative methods (of course in combination with the construction of Lyapunov' function). The right hand sides of these systems do not contain the time t explicitly.

First of all we discuss (Sec. 2) the system of the type

$$\frac{dx}{dt} = F(x,y)$$
$$\frac{dy}{dt} = f(\sigma), \quad \sigma = cx - dy \tag{1}$$

These equations occur in automatic regulation. Using qualitative methods we determine sufficient conditions in order that the trivial solution of system (1) be asymptotically stable in the large. In this connection we note that a theorem proved by Ershov [7] is wrong (Sec. 2). We then solve the problem of Aĭzerman for the systems of two equations (Sec. 3), namely, for the systems

$$\frac{dx}{dt} = f(x) + ay$$
$$\frac{dy}{dt} = bx + cy \tag{2}$$

and

$$\frac{dx}{dt} = ax + f(y)$$
$$\frac{dy}{dt} = bx + cy \tag{3}$$

In the case of system (2) we give a new proof of a theorem which asserts that if $c^2 + ab \neq 0$, then the trivial solution is asymptotically stable in the large under the generalized Hurwitz conditions. The theorem was first proved by Erugin [8]. For system (3) Malkin showed that the trivial solution is asymptotically stable in the large under the conditions $a + c < 0$, $(acy - bf(y))\, y > 0$ for $y \neq 0$ and

$$\int_0^y (acy - bf(y))\, dy \to +\infty \quad \text{as } |y| \to +\infty$$

We prove a similar theorem without the requirement of

$$\int_0^y (acy - bf(y))\, dy \to +\infty \text{ as } |y| \to +\infty$$

We also discuss (Sec. 4) the stability in the large of the systems

$$\frac{dx}{dt} = ax + f_1(y)\,, \quad \frac{dy}{dt} = f_2(x) + cy$$
$$\frac{dx}{dt} = f_1(x) + f_2(y)\,, \frac{dy}{dt} = bx + cy$$

We consider (Sec. 5) again the system of the type (1) but under assumptions as indicated by Ershov [6] who has discussed various cases where the asymptotic stability in the large holds. Not agreeing fully with the proofs of these theorems we give our own proofs. Finally we discuss (Sec. 6 and 7) the stability in the large of the systems

$$\frac{dx}{dt} = h_1(y)\,x + ay \quad , \quad \frac{dy}{dt} = h_2(x)\,x + by$$

$$\frac{dx}{dt} = xh_1(y) + ay \quad , \quad \frac{dy}{dt} = bx + h_2(x)\,y \qquad (4)$$

under suitable assumptions. As a sample case we prove that if $ab > o$, then the trivial solution of system (4) is asymptotically stable in the large under conditions

$$h_1(y) + h_2(x) < o \quad , \quad h_1(y)\,h_2(x) - ab > o, \text{for } x \neq o, y \neq o$$

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INTRODUCTION

The investigation of integral curves in the large using geometrical or qualitative methods for a system of two differential equations (i.e., on a plane) was started by Poincaré, and continued by many authors during the last eighty years. In 1950 N.P. Erugin ([8], [9]), formulated a general theorem of qualitative nature for the stability in the large. We have made frequent use of this theorem in our work.

The main purpose of the thesis is to study the stability in the large of systems of two differential equations. This problem is solved sometimes by constructing Lyapunov functions, sometimes on the basis of qualitative methods and sometimes by the combination of qualitative methods and the construction of Lyapunov functions. It should be noted that in solving the problem of stability by these methods we do not have to find either particular or general solutions of the differential equation.

In Section 1, we review the concepts of stability in the sense of Lyapunov and asymptotic stabilities in the small and in the large and give criteria for stability in the large based on the construction of Lyapunov functions and on qualitative methods. Towards the end of the section we construct a Lyapunov function for the equation of the second order

$$\frac{d^2x}{dt^2} + \Phi\left(\frac{dx}{dt}\right) g(x) + f(x)\psi\left(\frac{dx}{dt}\right) = 0$$

and give sufficient conditions which ensure the stability in the large of the trivial solution of the above equation.

In Section 2, we discuss the stability in the large of the following system of differential equations

$$\frac{dx}{dt} = F(x,y)$$

$$\frac{dy}{dt} = f(\sigma)\ , \text{ where } \quad \sigma = cx - dy,$$

using qualitative methods in combination with the construction of Lyapunov function and obtain certain sufficient conditions which guarantee the stability in the large.

In Section 3, we discuss the famous problem of Aĭzerman for the systems of two equations.

In Section 4, a sort of generalization of problem of Aĭzerman is discussed.

In Section 5, we discuss again the system of equations of Section 2 but under different assumptions and establish the stability in the large using qualitative methods only.

In Sections 6 and 7, we discuss the stability in the large of

$$\frac{dx}{dt} = h_1(y)\, x + ay$$

$$\frac{dy}{dt} = h_2(x)\, x + ay$$

and

$$\frac{dx}{dt} = xh_1(y) + ay$$

$$\frac{dy}{dt} = bx + h_2(x)\, y$$

mostly by qualitative methods. It may be noted that Gu,Čao-hao [16] has considered a similar problem. He has discussed the stability of

$$\frac{dx}{dt} = xh_1(y) + \Phi(y)$$

$$\frac{dy}{dt} = ax + f(y)$$

by constructing a Lyapunov function.

1. Some Basic Theorems On Stability. Let us consider a system of differential equations

$$\frac{dx_i}{dt} = X_i\,(x_1,\ x_2,\ \ldots, x_n, t) \tag{1.1}$$

of the perturbed motion. It is assumed that

$$X_i(o,\ o,\ \ldots,\ t) = o \qquad (i = 1,\ \ldots, n)$$

and the right hand sides X_i of (1.1) are continuous functions with respect to all their arguments and satisfy the condition of uniqueness of solutions of the system (1.1) in the region

$$-\infty < x_i < +\infty \quad , \quad t \geqslant o \tag{1.2}$$

If we denote the totalities $(x_1,\ x_2,\ \ldots,\ x_n)$ and $(X_1,\ \ldots,\ X_n)$ by x and X(x, t) respectively, each being (n x 1)matrix, then the system (1.1) is written in the form

$$\dot{x} = X(x,t) \tag{1.3}$$

Since it is assumed that X(o,t) = o, equation (1.3) admits the trivial solution $x(t) \equiv o$. The motion corresponding to this solution is called unperturbed motion and motions corresponding to all other solutions are known as perturbed motions.

Definition 1. The trivial solution x(t) = o is called stable in the sense of Lyapunov if, given a small $\epsilon > o$, there exists a $\delta(\epsilon, t_o)$ such that, for all perturbed motions x(t) for which $|x(t_o)| < \delta$ holds, the inequality $|x(t)| < \epsilon$ is satisfied for $t \geqslant t_o \geqslant o$.

Definition 2. If the trivial solution is stable in the above sense and every perturbed motion sufficiently close to it is such that $\lim_{t \to \infty} |x(t)| = o$, then we say that the trivial solution $x(t) = o$ is asymptotically stable in the small or in the sense of Lyapunov.

Definition 3. If however $|x(t)| \to o$ as $t \to \infty$, no matter what the point (x_o, t_o) may be, then the unperturbed motion is said to be asymptotically stable in the large.

Let us discuss in more detail the implications of the above definitions. By saying that the trivial solution is stable in the sense of Lyapunov, we understand that any perturbed motion started near $x = o$ possesses two properties (i) it is defined for all $t \geq t_o \geq o$ and (ii) satisfies the inequality $|x(t)| < \epsilon$ for the same values of t as in (i). The first property is not explicitly stated in the definition though it is always understood. Erugin [11] showed that the boundedness of solutions implies the existence of solutions for all $t \geq t_o \geq o$ when the right hand sides of (1.1) are defined and continuous in the region (1.2). Examples can be given where the solutions are bounded even if they are not defined for all $t \geq t_o \geq o$. This can happen, for example, in the case where the right hand sides of differential equations (1.1) are defined for all x and t but are not bounded for all $t > o$ (Erugin [12]).

We now turn to the definition of asymptotic stability according to Lyapunov. This concept includes in itself two properties of the solutions of system (1.1). One is that of stability in the sense of Lyapunov and the other is $\lim_{t \to +\infty} |x(t)| = o$.

There are cases where the solutions of the system (1.1) possess the second property but the trivial solution may not be stable in the sense of Lyapunov. One such class of a system of differential equations has been given by N.N. Krasovskiĭ [18].

Other types of stabilities, e.g. uniform stability, uniform asymptotic stability are also found in the literature. These types of stabilities have been considered by J.L. Massera [23], I.G. Malkin [22], N.N. Krasovskiĭ [20], and others.

For solving the problems relating to the stability of the trivial solution the methods are divided into two groups. In the first group we include all the methods in which either particular or general solutions of the equations of the perturbed motion are determined. In the second group the problem of stability is made to depend on a function $V(x,t)$ satisfying certain properties. As the above classification was done by Lyapunov we call the two methods the Lyapunov's first and second methods.

Let $V(x,t)$ denote any scalar function of x,t, continuous and having continuous partial derivatives of the first order in a domain $|x| \leq \delta$, $t \geq 0$, where $V(0,t) = 0$.

Definition 4. The function $V(x,t)$ is called semi definite in a domain if it assumes values of the same sign in that domain (the value zero is also allowed).

Definition 5. A function $W(x)$ independent of t is said to be positive definite in a domain if $W(x) > 0$ for all $x \neq 0$ and $W(0) = 0$.

<u>Definition 6.</u> We shall say that $V(x,t)$ is <u>positive definite</u> in a domain, if there exists a positive definite continuous function $W(x)$ such that $V(x,t) \geq W(x)$ in the domain of definition.

<u>Definition 7.</u> We shall say that V admits of an <u>infinitely small upper bound</u> if given $\epsilon > o$, there exists a $\delta > o$ such that $|V(x,t)| \leq \epsilon$ for $t \geq o$ whenever $|x| \leq \delta$.

<u>Definition 8.</u> If for every $M > o$, there exists a number $N > o$ such that for $|x| \geq N$, $t \geq o$ follows $|V(x,t)| > M$, then $V(x,t)$ is said to be <u>infinitely large</u>.

<u>Definition 9.</u> A definite function $V(x,t)$, the total derivative of which with respect to time in view of the perturbed equations is either a semi definite function of a sign opposite to that of $V(x,t)$ or is identically equal to zero, is called a <u>Lyapunov function.</u>

Lyapunov proved the following classical result on asymptotic stability.

<u>Theorem 1.1.</u> <u>If for the differential equations of perturbed motions there exists a Lyapunov function, possessing a definite derivative, and admitting of an infinitely small upper bound, then the unperturbed motion is asymptotically stable.</u>

It may be noted that this theorem is not reversible. A simple example to this effect has been given by J.L. Massera [23].

The following theorem an asymptotic stability in the large can be proved in the same way as is proved a theorem on asymptotic stability in the sense of Lyapunov by J.L. Massera [23].

Theorem 1.2. If there exists an infinitely large positive definite function $V(x,t)$ which possesses an infinitely small upper bound and which is such that its total time derivative is negative definite, then the solution $x = o$ is asymptotically stable in the large.

The inversion of this theorem has not been proved so far in its quite generality. Only in some particular cases this has been done.

1.2. In this Section we consider the following system of differential equations

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n) \quad (i = 1, \ldots, n) \qquad (1.2.1)$$

where the right hand sides are continuously differentiable functions of the variables $x_1, \ldots, x_n$ in the region $-\infty < x_i < +\infty$, $i = 1, 2, \ldots, n$. Furthermore

$$X_i(o,o, \ldots, o) = o \quad (i = 1,2,\ldots,n).$$

The theorem corresponding to Theorem 1.1 is the following

Theorem 1.2.1. If there exists for the system (1.2.1) a positive definite function $V(x_1, \ldots x_n)$ for which $\frac{dV}{dt}$ is negative definite, then the solution $x = o$ is asymptotically stable in the sense of Lyapunov.

This theorem is reversible and we have

Theorem 1.2.2. If the trivial solution of system (1.2.1) is asymptotically stable according to Lyapunov, then a positive definite V-function exists such that $\frac{dV}{dt}$ is negative definite.

The above two theorems show that the V-functions characterize the asymptotic stability of the zero solution in the sense of Lyapunov.

We give a simple example to show how Theorem 1.2.1 is applied to the problems concerning the asymptotic stability in the small. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= y - x^3 \\ \frac{dy}{dt} &= -x - y^3 \end{aligned} \tag{1.2.2}$$

the characteristic roots of the system of first approximation are $\pm i$, i.e., the real parts of the roots are zeros. Hence Lyapunov's first method cannot be applied. Let us take the following as V-function

$$V(x,y) = x^2 + y^2$$

Clearly, this function is positive definite. Its time derivative by virtue of (1.2.2) is given by

$$\frac{dV}{dt} = -2(x^4 + y^4)$$

which is obviously negative definite. Hence the trivial solution $x = o$ of system (1.2.2) is asymptotically stable in the small. Note here that we did not have to find either general or particular solutions of the system in order to decide the stability problem.

We now show by an example that the V-functions which guarantee the asymptotic stability in the small are not good enough for the establishment of the stability in the large of the trivial solution. Consider the system of two equations

$$\frac{dx}{dt} = y - \phi(x) \quad , \quad \frac{dy}{dt} = -\phi(x) \qquad (1.2.3)$$

where $\phi(o) = o$, $x\phi(x) > o$ for $x \neq o$.

Following Malkin [21] the V-function for this system can be taken as
$$V(x,y) = \tfrac{1}{2}y^2 + \int_o^x \phi(x)\,dx$$
Its total time derivative in view of (1.2.3) is given by
$$\frac{dV}{dt} = -(\phi(x))^2$$
Clearly, $V(x,y)$ is positive definite and $\frac{dV}{dt} < o$ for $x \neq o$ and $\frac{dV}{dt} = o$ for $x = o$. It can be shown that in this case asymptotic stability in the small holds even if $\frac{dV}{dt}$ is not negative definite. Now if $\int_o^x \phi(x)\,dx \not\longrightarrow +\infty$ as $|x| \longrightarrow +\infty$, then it is possible (Pliss [25]) that the stability in the large may not hold, i.e., we can show that there exist trajectories going to infinity for $t \longrightarrow +\infty$.

The above example shows that it becomes necessary to put an extra condition on the V-functions in order to realize asymptotic stability in the large. The V-function should be such that $V(x_1, x_2, \ldots, x_n) \leq C$, $C > o$, defines a bounded region containing the origin for all C. Because then we can be sure of the solutions being bounded and defined for all $t \geq t_o \geq o$, no matter what the initial point may be. Our purpose is served if we impose on the V-function an additional requirement of being infinitely large, since it is known (Erugin [14]) that $V(x)$ possessing the property $V(x) > o$ for $x \neq o$ and $V(o) = o$ does not define the region $V(x) \leqslant C$, $C > o$ which is always bounded.

The following theorems on stability in the large are due to E.A. Barbashin and N.N. Krasovskiĭ [2].

Theorem 1.2.3. If there exists an infinitely large positive definite function $V(x)$, the total time derivative $\frac{dV}{dt}$ of which by virtue of the perturbed equations is negative definite, then the trivial solution of (1.2.1) is asymptotically stable for arbitrary initial disturbances.

Theorem 1.2.4. If the trivial solution $x(t) = o$ is asymptotically stable in the large, then there exists a continuously differentiable infinitely large positive definite function $V(x)$ having negative definite derivative with respect to time provided that all solutions can be continued to the interval $-\infty < t \leqslant o$.

It was pointed out by Erugin [10], inverting a theorem of Wintner [30], that not every system of type (1.2.1) possesses the property that all its solutions be continuable on the whole interval $-\infty < t < +\infty$. It has been shown ([4], [31]) that the requirement of continuation of solutions on the negative t-axis in Theorem 1.2.4 is not essential.

An immediate generalization of Theorem 1.2.3 is the following.

Theorem 1.2.5. Let there exist an infinitely large positive definite function $V(x)$ and a set M such that

$$\frac{dV}{dt} < o \quad \text{outside } M; \frac{dV}{dt} \leqslant o \quad \text{on } M$$

Let the set M possess the property that an arbitrary intersection of the sets $V = C$ ($C \neq 0$) and M does not contain the positive half trajectory of the system (1.2.1) then the trivial solution $x = 0$ of system (1.2.1) is asymptotically stable for arbitrary initial disturbances.

As an example of Theorem 1.2.5 we consider the differential equation

$$\frac{d^2x}{dt^2} + \Phi\left(\frac{dx}{dt}\right) g(x) + f(x)\,\psi\left(\frac{dx}{dt}\right) = 0 . \qquad (1.2.4)$$

This can be thrown into the form

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\Phi(y)\, g(x) - f(x)\,\psi(y) \qquad (1.2.5)$$

We assume that

$$\begin{aligned} &x f(x) > 0 \text{ for } x \neq 0,\ f(0) = 0;\ g(x) > 0, \\ &\psi(y) > 0 \text{ and } y\,\Phi(y) > 0 \quad \text{for } y \neq 0,\ \Phi(0) = 0 \end{aligned} \qquad (1.2.6)$$

Furthermore, it is assumed that the right hand sides of (1.2.5) satisfy the conditions guaranteeing the existence and uniqueness of solutions of (1.2.5).

We construct the following Lyapunov function for the system (1.2.5)

$$V(x,y) = \int_0^x f(x)\, dx + \int_0^y \frac{y}{\psi(y)}\, dy$$

Clearly, $V(x,y)$ is positive definite. Let us compute its total time derivative in view of equations (1.2.5).

$$\frac{dV}{dt} = f(x)\,y + \frac{y}{\psi(y)}\Big[-\Phi(y)\,g(x) - f(x)\psi(y)\Big]$$
$$= -y\,\frac{\Phi(y)}{\psi(y)}\,g(x) < o \quad \text{for } y \neq o$$
$$= o \qquad \text{for } y = o$$

It is easy to see that $y = o$ does not contain a positive half trajectory of the system (1.2.5) except the origin. If we now assume that

$$\int_o^x f(x)\,dx \longrightarrow \infty \quad \text{for } |x| \to \infty \;;\; \int_o^y \frac{y}{\psi(y)}\,dy \to \infty \text{for } |y| \to \infty \tag{1.2.7}$$

then $V(x,y)$ is infinitely large. Thus we prove the following

Theorem 1.2.6. *If the conditions (1.2.6) and (1.2.7) are satisfied then the trivial solution of (1.2.4) is asymptotically stable in the large.*

It may be remarked here that the construction of suitable Lyapunov functions is possible in a very small number of examples (see [3], [5], [19], [24], [27], and [29]).

In 1950 Erugin [8] proved the following theorem for the system of two equations, i.e., for

$$\begin{aligned} \frac{dx}{dt} &= P(x,y) \\ \frac{dy}{dt} &= Q(x,y) \end{aligned} \tag{1.2.8}$$

Theorem 1.2.7. (Erugin) *We assume that*

(i) *the point (o,o) is the only point of equilibrium,*

(ii) *the unperturbed motion $x = o = y$ is asymptotically stable and consequently any motion started in a certain region (ϵ)*

$$x^2 + y^2 \leq \epsilon \tag{1.2.9}$$

possesses the property $x(t) \to 0$, $y(t) \to 0$

as $t \to \infty$, (1.2.10)

(iii) a straight line $L(0, \infty)$ going to infinity from the point $(0,0)$ is intersected by the motions in one direction only for $t \to \infty$,

(iv) the motions having bounded polar angles are bounded,

(v) there are no periodic motions;

then all the motions possess the property (1.2.10).

The above theorem has been generalized to the case of a system of n equations by V.A. Pliss [26].

2. Stability in the large of the system $\frac{dx}{dt} = F(x,y)$, $\frac{dy}{dt} = f(\sigma)$.

In this section we shall consider the system of equations

$$\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= f(\sigma), \qquad \sigma = cx - dy \end{aligned} \tag{2.1}$$

where c and d are constants, $c \neq 0$; the functions $F(x,y)$, $f(\sigma)$ are continuous and $F(0,0) = 0$, $f(0) = 0$. Besides, the fulfilment of conditions of uniqueness of the solution $x = 0 = y$ is assumed.

The above system was considered in the works of Ershov ([6], [7]) and Krasovskiĭ [19]. Following Krasovskiĭ we transform the system (2.1) to the following form (2.2) by the change of dependent variables expressed by the relations

$$\begin{aligned} \sigma &= cx - dy \\ y &= y \end{aligned}$$

The above transformation is non-singular because c is not assumed equal to zero. We then have

$$\frac{d\sigma}{dt} = \Phi(\sigma, y)$$
$$\frac{dy}{dt} = f(\sigma) \tag{2.2}$$

where $\Phi(\sigma, y) = c\,F\left(\frac{\sigma + dy}{c}, y\right) - d\,f(\sigma)$.

Krasovskiĭ constructed the following Lyapunov function for the system (2.2)

$$V(\sigma, y) = \int_0^{\sigma} f(\sigma)\,d\sigma - \int_0^{y} \Phi(o, y)\,dy$$

and using Theorem 1.2.5 proved the following theorem:

<u>Theorem 2.1.</u> <u>If the conditions</u>

$$\sigma f(\sigma) > o \qquad \text{for } \sigma \neq o \tag{2.3}$$

$$y\,\Phi(o, y) < o \qquad \text{for } y \neq o \tag{2.4}$$

$$\sigma\left[\Phi(\sigma, y) - \Phi(o, y)\right] < o \qquad \text{for } \sigma \neq o \tag{2.5}$$

<u>and</u> $$\left|\int_0^{\pm\infty} f(\sigma)\,d\sigma\right| = \infty, \quad \left|\int_0^{\pm\infty} \Phi(o, y)\,dy\right| = \infty \tag{2.6}$$

<u>are satisfied then the trivial solution x = o = y of system (2.1) is asymptotically stable in the large</u>.

It may be remarked that conditions (2.3) and (2.4) can be replaced by the following conditions:

$$\sigma f(\sigma) < o \qquad \text{for } \sigma \neq o \tag{2.3^1}$$

$$y\,\Phi(o, y) > o \qquad \text{for } y \neq o \tag{2.4^1}$$

Ershov [7] claimed that Theorem 2.1 holds without the requirement of conditions (2.6). In fact, he stated the following theorem:

Theorem 2.2. If conditions (2.3), (2.4) and (2.5) are satisfied for the system (2.1), then the trivial solution of system (2.1) is asymptotically stable in the large.

The following example shows that conditions (2.6) cannot be removed in general.

Example. Consider the system of equations

$$\begin{aligned} \frac{dx}{dt} &= -y - f(x) \\ \frac{dy}{dt} &= f(x) \end{aligned} \qquad (2.7)$$

where $f(x)$ is defined as below

$$\begin{aligned} f(x) &= \frac{e^{-2x}}{1 + e^{-x}} && \text{for } x \geq 1 \\ &= \frac{e^{-2}}{1 + e^{-1}}\, x && \text{for } x < 1 \end{aligned}$$

Obviously, $xf(x) > 0$ for $x \neq 0$ and $f(0) = 0$

$$y\,\phi(0,y) = y(-y - f(0)) = -y^2 < 0 \text{ for } y \neq 0$$

$$\sigma\left[\phi(\sigma,y) - \phi(0,y)\right] = x(-y - f(x) + y) = -x\,f(x) < 0$$

$$\text{for } x \neq 0$$

Moreover, it is not difficult to show that $f(x)$ is continuous and satisfies the Lipschitz condition. Thus all the conditions of Theorem 2.2 are satisfied. We show that the trivial solution of this system is asymptotically stable in the sense of Lyapunov but not in the large. The stability in the small follows from the following Lyapunov function

$$V(x,y) = \int_0^x f(x)\,dx + \tfrac{1}{2}y^2.$$

We now show that there exist trajectories going to infinity for $t \longrightarrow +\infty$.

It is easy to verify that $y = -e^{-x}$ is a particular integral of the system on the interval $1 \leq x < \infty$ passing through the point $(1, -e^{-1})$ at $t = o$. We show that along the curve $y = -e^{-x}$, $\frac{dx}{dt} > o$, i.e., x increases with the increase of time.

$$\frac{dx}{dt} = -y - f(x) = +e^{-x} - \frac{e^{-2x}}{1 + e^{-x}} = \frac{e^{-x}}{1 + e^{-x}} > o \text{ for } x \geqslant 1$$

We integrate $\frac{dx}{dt} = \frac{e^{-x}}{1+e^{-x}}$ along the trajectory $y = -e^{-x}$ and have

$$\int_1^x \frac{1 + e^{-x}}{e^{-x}}\, dx = \int_o^t dt \quad \text{or} \quad e^x + x \Big|_1^x = t \Big|_o^t$$

$$\text{or} \quad x + e^x - e - 1 = t$$

From the last equation it follows that as $t \longrightarrow +\infty$, $x \longrightarrow +\infty$, i.e., the positive half trajectory $y = -e^{-x}$ of the system (2.7) tends to infinity as $t \longrightarrow +\infty$. Hence it follows that the trivial solution is not asymptotically stable in the large.

<u>2.2.</u> Let us consider the system

$$\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= f(\sigma), \quad \sigma = cx - dy, \; c \neq o \end{aligned} \tag{2.2.1}$$

under the conditions:

$$F(o,o) = o, \quad f(o) = o \tag{2.2.2}$$

$$\sigma f(\sigma) > o \quad \text{for } \sigma \neq o \tag{2.2.3}$$

$$\sigma\left[\Phi(\sigma,y) - \Phi(o,y)\right] < o \text{ for } \sigma \neq o \tag{2.2.4}$$

$$y\,\Phi(o,y) < o \qquad \text{for } y \neq o \tag{2.2.5}$$

where
$$\Phi(\sigma,y) = c\,F\left(\frac{\sigma + dy}{c}, y\right) - d\,f(\sigma) \tag{2.2.6}$$

Besides, the fulfilment of conditions of uniqueness of the solution $x = o = y$ is assumed. As before we reduce the system (2.2.1) to the following system

$$\begin{aligned} \frac{d\sigma}{dt} &= \Phi(\sigma,y) \\ \frac{dy}{dt} &= f(\sigma) \end{aligned} \tag{2.2.7}$$

We consider the positions of the curves represented by the right-hand sides of (2.2.7) on the (σ,y) plane, i.e., of $\Phi(\sigma,y) = o$ and $f(\sigma) = o$. Since $\sigma f(\sigma) > o$ for $\sigma \neq o$ and $f(o) = o$, $f(\sigma) = o$ only when $\sigma = o$, i.e., $f(\sigma) = o$ represents the y-axis. We now turn to the curve represented by $\Phi(\sigma,y) = o$. We observe that $\Phi(o,o) = o$ and on the σ-axis $\Phi(\sigma,y) > o$ for $\sigma < o$, $\Phi(\sigma,y) < o$ for $\sigma > o$; on the y-axis $\Phi(\sigma,y) > o$ for $y < o$, $\Phi(\sigma,y) < o$ for $y > o$. From these facts it follows that $\Phi(\sigma,y) < o$, $\sigma > o$, $y > o$ and $\Phi(\sigma,y) > o$ for $\sigma < o$, $y < o$. In deriving these conclusions we have made use of the conditions (2.2.4) and (2.2.5). Thus it follows that $\Phi(\sigma,y)$ changes sign in the second and fourth quadrants and hence the curve $\Phi(\sigma,y) = o$ lies in the second and fourth quadrants. It is easy to see that

$\dfrac{d\sigma}{dt} = \Phi(\sigma,y) > o$ for the points lying to the left of the curve $\Phi(\sigma,y) = o$

$\dfrac{d\sigma}{dt} = \Phi(\sigma,y) < o$. for the points lying to the right of the curve $\Phi(\sigma,y) = o$

$\frac{dy}{dt} = f(\sigma) > o$ for the points lying to the right of y-axis

$\frac{dy}{dt} = f(\sigma) < o$ for the points lying to the left of y-axis

The curve $\Phi(\sigma,y) = o$ and the co-ordinate axes decompose the plane (σ,y) into six regions. $y(t)$ is maximum on the y-axis for $y > o$ and minimum for $y < o$; $\sigma(t)$ is maximum on the curve $\Phi(\sigma,y) = o$ for $\sigma > o$ and minimum for $\sigma < o$. The direction of motion is indicated in fig.1. We introduce polar co-ordinates $\sigma = r\cos\theta$, $y = r\sin\theta$, then

$$\dot{r} = \dot{\sigma}\cos\theta + \dot{y}\sin\theta$$

and
$$r\dot{\theta} = -\dot{\sigma}\sin\theta + \dot{y}\cos$$

The signs of $\dot{r}$ and $\dot{\theta}$ in different regions are given as below:

(1,4) $\dot{r}$ maybe $\gtreqless o$, $\dot{\theta} > o$ (2,5) $\dot{r}$ may be $\gtreqless o$, $\dot{\theta} > o$

(3,6) $\dot{r} < o$, $\dot{\theta}$ may be $\gtreqless o$

The Lyapunov function for the system (2.27)is

$$V(\sigma,y) = \int_o^{\sigma} f(\sigma)\, d\sigma - \int_o^{y} \Phi(o,y)\, dy$$

Its total time derivative in view of equations (2.2.7) is

$$\dot{V} = f(\sigma)\left[\Phi(\sigma,y) - \Phi(o,y)\right] < o \quad \text{for } \sigma \neq o$$
$$= o \quad \text{for } \sigma = o$$

Obviously, $V(\sigma,y)$ is positive definite and $\dot{V}$ is of negative sign.

Now $\frac{d\sigma}{dt} = \Phi(o,y)$ for $\sigma = o$ and it is different from zero unless $y = o$.

This means that $\sigma = o$ does not contain any other positive half trajectory of the system (2.2.7) than the origin. Whence follows the asymptotic stability in the sense of Lyapunov of the trivial solution of system (2.2.7). Thus we have shown

(i) the point (o,o) is the only point of equilibrium,

(ii) the unperturbed motion is asymptotically stable according to Lyapunov,

(iii) there are no periodic motions, since for (2.2.7) is constructed a Lyapunov function.

Since the σ positive half axis is intersected by the motions in one direction only, it can be taken for the straight line $L(o,\infty)$ appearing in Theorem 1.2.7. Thus all the conditions of Theorem 1.2.7 are satisfied except the fourth, i.e., the motions having bounded polar angles are bounded. We now indicate what additional conditions are to be imposed in order to realize the fourth condition of Theorem 1.2.7.

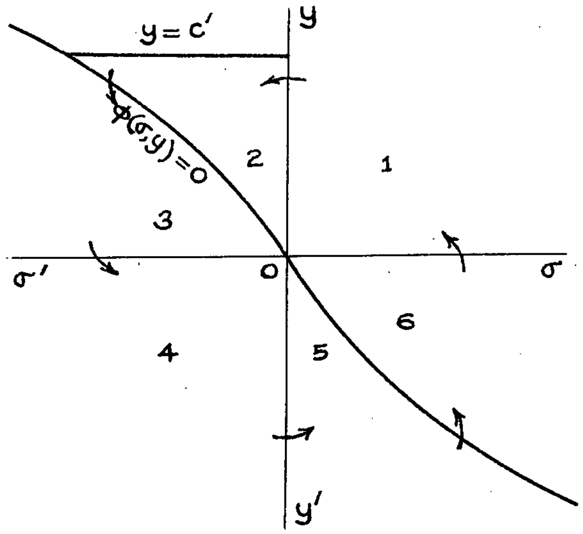

FIG. 1

Let a motion $M(t)$ start in a region, say (6). Any motion $M(t)$ started in the region (6) or entering this region either tends to the origin or goes out of this region and enters the region (1). This follows from the fact that $\dot{r} < o$ in this region. After entering the region (1) the motion $M(t)$ either crosses the y-axis or tends to infinity along the y-axis, but it cannot go to the origin since $\dot{\theta} > o$. We are thus led to impose the following

<u>Condition A.</u> We assume that $\Phi(\sigma, y)$ satisfies such conditions called A that the motion entering or starting in the regions (1) and (4) leaves these regions with the increase of time.

This condition guarantees that there are no motions with bounded polar angles in the regions (1) and (4). We impose another condition called B.

<u>Condition B.</u> We assume that $\Phi(\sigma, c') = o$ has a solution for all c'.

If Condition A is satisfied then the motion $M(t)$ enters the region (2). We now show that the motion $M(t)$ leaves the region (2) with the increase of time if Condition B is satisfied. In fact if $\Phi(\sigma, c') = o$ has a solution for all c', i.e., if $y = c'$ intersects the curve $\Phi(\sigma, y) = o$, the motion after entering the region (2) can neither cross the line $y = c'$, since $\frac{dy}{dt} < o$ in this region nor it can enter the origin because $\dot{\theta} > o$ in region (2). Therefore the motion must intersect the curve $\Phi(\sigma, y) = o$. Similar reasoning can be carried out in the regions (3), (4) and (5). The above analysis shows that there are no motions with bounded polar angles in the regions (1), (2), (4) and (5) and that the motions with bounded polar angles can occur only in the regions (3) and (6) and as proved above they are bounded. Thus any motion with bounded polar angle is bounded.

The Condition B was imposed with a view to ensure that in the regions (2) and (5) there are no motions with bounded polar angles. This can also be achieved if we assume that in these regions $\dot{\theta} > \epsilon > 0$. We call this condition as Condition C.

We now collect these results in the following:

Theorem 2.2.1. If Conditions (2.2.2)—(2.2.5) are satisfied and if either Conditions A and B or Conditions A and C are satisfied, then the trivial solution of (2.2.1) is asymptotically stable in the large.

It may be remarked that Conditions B and C are independent, i.e., if Condition B holds then Condition C may or may not hold or vice versa. In the linear case, i.e., when the right hand sides of system (2.2.1) are linear, however, both hold.

Let us consider a few examples now.

Example 1. Consider the system

$$\frac{dx}{dt} = -y + e^{-x} - 1$$
$$\frac{dy}{dt} = x \tag{2.2.8}$$

Obviously, this system satisfies the conditions (2.2.2)—(2.2.5). We verify that in this case Condition A is satisfied. Let us for the sake of definiteness assume that the motion M(t) is in the region (1). We assume that the Condition A is not satisfied. Then the motion M(t) goes to infinity as $t \longrightarrow \infty$ and during this $y(t) \longrightarrow \infty$. From the first equation of (2.2.8) follows that $\frac{dx}{dt}$ becomes infinitely large but negative which means that M(t) cannot remain in the first quadrant and hence our assumption is not true. It is not difficult to

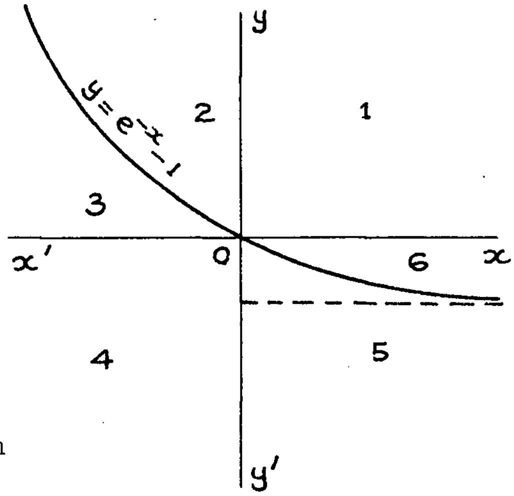

FIG. 2

verify that the straight line $y = c$ does not intersect the curve

$$- y + \bar{e}^{x} - 1 = o \quad \text{for all } c \text{ (see fig. 2)}$$

Hence Condition B is not satisfied. However Condition C holds. In fact $\quad r\,\dot{\theta} = -\dot{x}\sin\theta + \dot{y}\cos\theta = -\sin\theta\,(-y+\bar{e}^{x}-1) + x\cos\theta$

and $\quad \dot{\theta} = -\frac{\sin\theta}{r}(-y + \bar{e}^{x}-1) + \cos^{2}\theta > \epsilon > o$

in the regions (2) and (5). This establishes the asymptotic stability in the large of the trivial solution of (2.2.8).

<u>Example 2</u>. Consider the equation

$$\frac{d^2y}{dt^2} + \phi\left(\frac{dy}{dt}\right) g(y) + f(y) = o \qquad (2.2.9)$$

This equation can be thrown into the system

$$\frac{dx}{dt} = -\phi(x)\,g(y) - f(y)$$

$$\frac{dy}{dt} = x$$

by writing x for $\frac{dy}{dt}$

We assume that the following conditions are fulfilled:

$$y\,f(y) > o,\ f(o) = o \quad \text{and}$$

either $\quad h_1(x) > o$ for $x \neq o$ and $g(y) > o$ for all y

or $\quad h_1(x) < o$ for $x \neq o$ and $g(y) < o$ for all y

where $\quad h_1(x) = \dfrac{\Phi(x)}{x}$ for $x \neq o$

The Lyapunov function for this system is

$$V(x,y) = \int_o^x x\,dx + \int_o^y f(y)dy$$

Then

$$\dot{V} = x(-\Phi(x)\,g(y) - f(y)) + f(y)x$$
$$= -x\,\Phi(x)\,g(y)$$

It is not difficult to see that in this case Condition C of Theorem 2.2.1 is satisfied. Thus if Condition A is also satisfied then the trivial solution of (2.2.9) is asymptotically stable in the large.

<u>Example 3.</u> Consider the system

$$\frac{dx}{dt} = \Phi(x) + f_2(y) \tag{2.2.10}$$

$$\frac{dy}{dt} = f_1(x)$$

For the system (2.2.10) the V function is

$$V(x,y) = \int_o^x f_1(x)\,dx - \int_o^y f_2(y)\,dy$$

Then

$$\dot{V}(x,y) = f_1(x)\,\Phi(x)$$

We subject (2.2.10) to the following conditions:

$$\Phi(o) = o\ ,\ h_3(x) < o \text{ for } x \neq o\ ,\ \text{where } h_3(x) = \frac{\Phi(x)}{x} \text{ for } x \neq o \tag{2.2.11}$$

$f_1(o) = o, xf_1(x) > o$ for $x \neq o$, $f_2(o) = o$ and $yf_2(y) < o$ for $y \neq o$

The trivial solution of (2.2.10) is thus asymptotically stable in the large if in addition to (2.2.11) the Conditions A and B or A and C are satisfied.

Example 4. Consider the equation of the second order

$$\frac{d^2y}{dt^2} + f\left(\frac{dy}{dt},y\right)\frac{dy}{dt} + g(y) = o \qquad (2.2.12)$$

We write it in the form of a system of equations

$$\frac{dx}{dt} = -f(x,y)\,x - g(y)$$

$$\frac{dy}{dt} = x$$

We assume that

$$g(o) = o,\ y\,g(y) > o \text{ for } y \neq o \quad \text{and}$$

$$f(x,y) > o \quad \text{for } x \neq o$$

It is easy to verify that asymptotic stability in the large holds if Condition A is satisfied.

3. The Problem of Aĭzerman. In 1949 Aĭzerman [1] proposed the following problem.

Let there be given a system of linear differential equations

$$\frac{dx_1}{dt} = \sum_{j=1}^{n} a_{1j}x_j + ax_k\,,\quad \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}\,x_j \qquad (3.1)$$

$$(i = 2,3,\ldots,n)$$

Suppose that for the given constants a_{ij} $(i = 1,\ldots,n; j=1,\ldots,n)$ and for an arbitrary value of 'a' from the interval $\alpha < a < \beta$ all the roots of the characteristic equation of (3.1) have negative real parts. Let ax_k be replaced by $f(x_k)$ in (3.1). We then have

$$\begin{aligned} \frac{dx_1}{dt} &= \sum_{j=1}^{n} a_{1j} x_j + f(x_k) \\ \frac{dx_i}{dt} &= \sum_{j=1}^{n} a_{ij} x_j \qquad (i = 2, \ldots, n) \end{aligned} \tag{3.2}$$

It is required to find out whether the trivial solution $x_1 = x_2 = \cdots = x_n = o$ of system (3.2) is asymptotically stable in the large or not, for arbitrary choice of the continuous function $f(x_k)$, which reduces to zero for $x_k = o$ and which satisfies the inequality

$$\alpha x_k^2 < x_k f(x_k) < \beta x_k^2 \quad \text{for } x_k \neq o \tag{3.3}$$

The answer to the above problem is in the negative (Pliss [25]). The interest in the problem is revived if we slightly change the above problem and ask ourselves the following questions. For what values of a_{ij} the answer to the above problem is in the affirmative and for what values in the negative. If the trivial solution is not asymptotically stable in the large under the generalized Hurwitz conditions (i.e., the condition (3.3) and $f(o) = o$), what additional assumptions should be made on $f(x_k)$ so that the trivial solution becomes asymptotically stable in the large. We shall discuss here the case when $n = 2$. The case $n = 3$, $k = 2$ has been completely solved by Pliss [28]. First of all we take $n = 2 = k$, i.e., we consider the system

$$\begin{aligned} \frac{dx}{dt} &= ax + f(y) \\ \frac{dy}{dt} &= bx + cy \end{aligned} \tag{3.4}$$

under the assumptions that

$$f(o) = o \ , \ a + c < o \ , \quad ac - bh(y) > o, y \neq o \tag{3.5}$$

where $$h(y) = \frac{f(y)}{y} \quad \text{for} \quad y \neq o$$

Besides, the uniqueness of the trivial solution is assumed. System (3.4) was considered by N.P. Erugin [9] and I.G. Malkin[21].

Malkin [21] showed that the trivial solution of system (3.4) is asymptotically stable in the large if for sufficiently large values of $|y|$ the inequality $ac - b\,h(y) > \epsilon$ holds. This condition can be relaxed to the condition that

$$\int_o^y (ac - bh(y)\ y \ \ dy \longrightarrow +\infty \qquad \text{as } |y| \to +\infty \tag{3.6}$$

We show that asymptotic stability in the large of the trivial solution of (3.4) holds without the requirement of condition (3.6). For $b = o$, from (3.5) follows that $a < o$, $c < o$ and an immediate integration of the system (3.4) shows that the equilibrium is asymptotically stable for arbitrary initial disturbances and for an arbitrary choice of the function $f(y)$.

Let $b \neq o$. We introduce new variables defined by

$$x' = bx - ay$$

$$y' = y$$

then $$\frac{dx'}{dt} = b\,\frac{dx}{dt} - a\,\frac{dy}{dt} = b(ax+f(y)) - a(bx+cy) = -y'(ac-bh(y'))$$

$$\frac{dy'}{dt} = \frac{dy}{dt} = bx + cy = x' + ay' + cy' = x' + (a + c)\ y'$$

Thus the system (3.4) reduces to the following:

$$\frac{dx}{dt} = -y(ac - b\,h(y)) \tag{3.7}$$

$$\frac{dy}{dt} = x + (a + c)\,y$$

We represent the curves $-y(ac - bh(y)) = o$ and $x+(a+c)\,y=o$ obtained by putting the right hand sides of (3.7) equal to zero on the (x,y) plane. The first of these represents the straight line $y = o$ and the second the straight line $y = -\frac{x}{a + c}$. It is easy to see that

$\frac{dx}{dt} = -y(ac - bh(y)) > o$ below the x-axis

$\frac{dx}{dt} = -y(ac - bh(y)) < o$ above the x-axis

$\frac{dy}{dt} = x + (a + c)\,y > o$ below the straight line $x + (a + c)\,y = o$

$\frac{dy}{dt} = x + (a + c)y < o$ above the straight line $x + (a + c)\,y = o$

The direction of motion is indicated in fig. 3. The straight line $x + (a + c)\,y = o$ and the co-ordinate axes divide the plane (x,y) into six regions. We introduce polar co-ordinates $x = r\cos\phi$, $y = r\sin\phi$. Then $\dot{r} = \dot{x}\cos\phi + \dot{y}\sin\phi$ and $r\dot{\phi} = -\dot{x}\sin\phi + \dot{y}\cos\phi$. It is not difficult to verify that in different regions the signs of $\dot{r}$ and $\dot{\phi}$ are given as below :

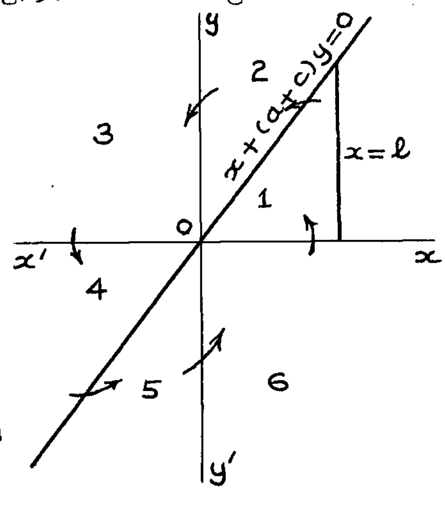

FIG.3

(1,4) $\dot{r}$ may be $\gtrless o$, $\dot{\phi} > o$

(2,5) $\dot{r} < o$, $\dot{\Phi}$ may be $\gtrless o$ (3,6) $\dot{r}$ may be $\gtrless o$, $\dot{\Phi} > o$.

The motion started in the region (6) must cross the axis of x. This follows from the inequality

$$\frac{dy}{dt} = x + (a + c)\, y \geq d > o$$

in the region (6). After entering the region (1) it cannot remain there and must cross the straight line $x + (a + c)\, y = o$ with the increase of time since $x = \ell$ intersects the straight line $x + (a + c)\, y = o$ for all ℓ. The Lyapunov function for the system (3.7) is easily seen to be

$$2V = x^2 + 2\int_o^y (ac-bh(y))\, y\, dy$$

Repeating the same argument as in Theorem 2.2.1, we arrive at the following

<u>Theorem 3.1.</u> <u>The trivial solution of system (3.4) is asymptotically stable in the large under Conditions (3.5).</u>

It may be noted that we may or may not have $\dot{\Phi} > \epsilon > o$ in the regions (1) and (4). Before studying the case $n = 2$, $k = 1$ we discuss the system

$$\begin{aligned} \frac{dx}{dt} &= y - F(x) \\ \frac{dy}{dt} &= - g(x) \end{aligned} \qquad (3.8)$$

This system is a particular case of the system (2.2.1). Since it is quite an important system, we discuss it independently. The conditions to which (3.8) is subjected are

$$\begin{aligned} x\, F(x) > o\,, \quad x\, g(x) > o \quad \text{for } x \neq o \\ F(o) = o = g(o) \end{aligned} \qquad (3.9)$$

The V-function in this case is

$$2V = y^2 + 2\int_0^x g(x)\,dx \tag{3.10}$$

and $\frac{dV}{dt} = y(-g(x)) + g(x)(y - F(x)) = -g(x)\,F(x) < 0$ for $x \neq 0$

$= 0$ for $x = 0$

The straight line $x = 0$ does not contain a positive half trajectory of the system (3.8) except the origin. In fact $\frac{dx}{dt} = y$ for $x = 0$. Hence it follows that the trivial solution is asymptotically stable in the small. Since there exists a Lyapunov function V for the system, there cannot be any limit cycle. Also we note that the origin is the only point of equilibrium. It is not difficult to see in virtue of the Conditions (3.9) that

$\frac{dx}{dt} = y - F(x) > 0$ to the left of the curve $y = F(x)$

$\frac{dx}{dt} = y - F(x) < 0$ to the right of the curve $y = F(x)$

$\frac{dy}{dt} = -g(x) > 0$ to the left of the y-axis

$\frac{dy}{dt} = -g(x) < 0$ to the right of the y-axis

The curve $y = F(x)$ and the co-ordinate axes divide the plane (x,y) into six regions. The direction of motion is represented in fig.4. We introduce polar co-ordinates $x = r\cos\phi$, $y = r\sin\phi$. Then

$$\dot{r} = \dot{x}\cos\phi + \dot{y}\sin\phi$$

$$r\dot{\phi} = -\dot{x}\sin\phi + \dot{y}\cos\phi$$

The signs of $\dot{r}$ and $\dot{\phi}$ in the regions are given as below :

(1,4) $\dot{r}$ may be $\gtrless 0$, $\dot{\phi} < 0$ (2,5) $\dot{r} < 0$, $\dot{\phi}$ maybe $\gtrless 0$

(3,6) $\dot{r}$ may be $\gtrless 0$, $\dot{\phi} < 0$.

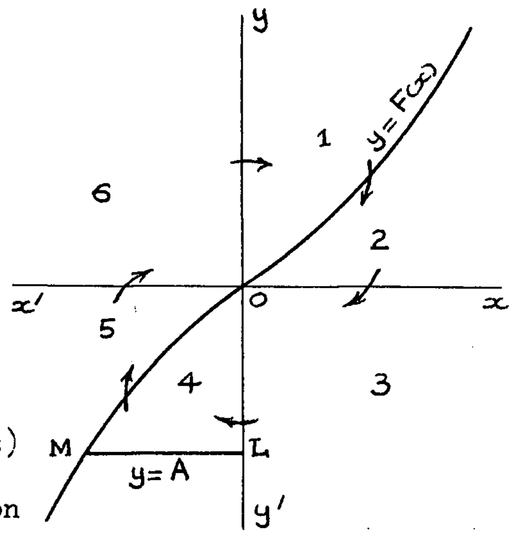

FIG.4

Let us follow the motion $M(t)$ after it intersects the negative half y-axis. Let us assume that $y = A$ intersects the curve $y = F(x)$ for all A, i.e., $F(x) = A$ has a solution for all A, then since in the region (4) $\dot{\varphi} < o$ and y is increasing the motion $M(t)$ cannot remain in the bounded region oLM. It definitely cannot cross LM and oL. Therefore it must cross the curve $y = F(x)$ and enter the region (5). In the region (5) it either enters the origin or the region (6). This follows from the fact that $\dot{r} < o$ in this region. Since $\frac{dx}{dt} = -F(x) + y \geqslant d > o$ in the region (6), the motion must leave this region and enter the region (1). Similar argument holds for the regions (1), (2) and (3). This shows that there are no motions with bounded polar angles in the regions (1), (3), (4) and (6). The motions with bounded polar angles can only occur in the regions (2) and (5), and as proved above they are bounded. Thus it follows that all motions with bounded polar angles are bounded. For the straight line $L(o,\infty)$ can be taken the positive half y-axis. Thus all the conditions of Theorem 1.2.7 are satisfied and we have the following theorem:

<u>Theorem 3.2.</u> <u>If the Conditions (3.9) are satisfied and $F(x) = A$ has a solution for all A, then the trivial solution of (3.8) is</u>

<u>asymptotically stable in the large</u>.

We remark that by using Theorem 1.2.5 the stability in the large holds if

$$\int_0^x g(x)\,dx \to \infty \text{ as } |x| \to \infty$$

This is required in order to make the V function in (3.10) infinitely large.

The widely discussed equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0 \qquad (3.11)$$

can be dealt in the same way as the system (3.8), since (3.11) can be transformed to

$$\frac{dx}{dt} = y - F(x)$$

$$\frac{dy}{dt} = -g(x) \quad \text{where } F(x) = \int_0^x f(x)\,dx$$

We shall now discuss the Aĭzerman problem for the case $n = 2$, $k = 1$, i.e., the system

$$\begin{aligned} \frac{dx}{dt} &= f(x) + ay \\ \frac{dy}{dt} &= bx + cy \end{aligned} \qquad (3.12)$$

under the conditions:

$$\begin{gathered} f(0) = 0\ ;\ c + h(x) < 0\ ;\ c\,h(x) - ab > 0 \\ \text{for } x \neq 0 \end{gathered} \qquad (3.13)$$

where $\qquad h(x) = \frac{f(x)}{x}, \quad x \neq 0$

This system is discussed in the works of Erugin ([8], [13]) and Malkin [21] . It was proved by Erugin that if $c^2 + ab \neq 0$, then the trivial solution of (3.12) is asymptotically stable in the large under conditions (3.13). The proof by Erugin of this theorem is quite lengthy, so we give here a short proof of the theorem. It will become clear from our proof why the stability in the large does not hold under conditions (3.13) in the case $c^2 + ab = o$. Erugin also showed that for the stability in the large in the case $c^2 + ab = o$, it is sufficient that

$$\begin{aligned} &\varlimsup_{x\to\infty}\left[\int_o^x (c\, f(x) - abx)\, dx + c\, f(x) - abx\right] = +\infty \\ &\varlimsup_{x\to -\infty}\left[\int_o^x (c\, f(x) - abx)\, dx - c\, f(x) + abx\right] = +\infty \end{aligned} \tag{3.14}$$

Krasovskiĭ [17] showed that conditions (3.14) are necessary as well as sufficient. The question of the region of stability in the case where the stability in the large does not hold is discussed by Pliss [25].

We assume that $c^2 + ab \neq o$. Let $a = o$, then immediate integration of the system yields the stability in the large of the trivial solution $x = o = y$ of system (3.12). Let now $a \neq o$. We introduce new dependent variables

$$x' = x$$

$$y' = ay - cx$$

Then
$$\frac{dx'}{dt} = \frac{dx}{dt} = f(x') + y' + cx'$$

$$\frac{dy'}{dt} = a\frac{dy}{dt} - c\frac{dx}{dt} = a(bx + cx) - c\,(f(x) + ay)$$

$$= abx' - c\, f(x')$$

The system (3.12) is reduced to the form

$$\frac{dx}{dt} = f(x) + cx + y$$

$$\frac{dy}{dt} = -x(c\,h(x) - ab)$$

Comparing it with system (3.8) we have

$$F(x) = -x(c + h(x)) \quad \text{and} \quad g(x) = x(c\,h(x) - ab)$$

The curve $y + cx + f(x) = o$ and the co-ordinate axes divide the (x,y) plane into six regions. The direction of motion is represented in fig. 5. We consider the following cases.

Case 1 $c < o$. Consider the straight line

$$y = -\frac{c^2 + ab}{c}\,x \qquad (3.15)$$

If $c^2 + ab > o$, then the straight line (3.15) lies in the first and third quadrants. Let us see how it is situated on the (x,y) plane with respect to the

curve $y + cx + f(x) = o$ (3.16)

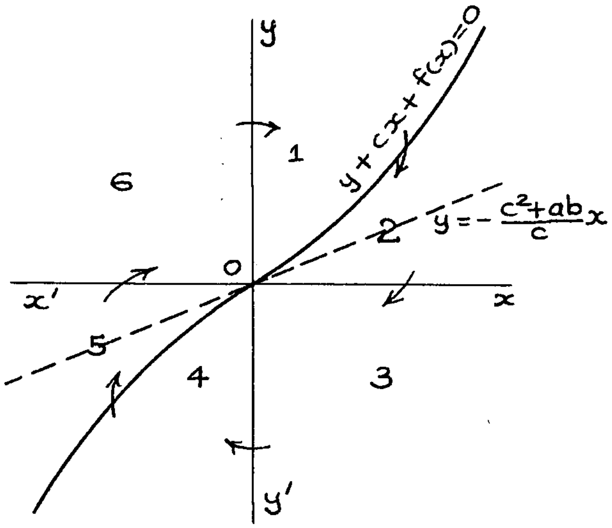

FIG. 5

Let $y^{(1)}$ and $y^{(2)}$ denote the ordinates of (3.16) and (3.15) respectively, then

$$y^{(1)} - y^{(2)} = -cx - f(x) + \frac{c^2 + ab}{c}\,x$$

$$= \frac{-c^2x - cf(x) + (c^2 + ab)\,x}{c} = \frac{-x(c\,h(x) - ab)}{c} \gtrless o$$

according as $x \gtrless o$,

i.e., the curve (3.16) lies above the straight line (3.15) in the first quadrant and below in the third quadrant. . Obviously, the straight lines $y = A$ and $y = -\frac{c^2 + ab}{c} x$ intersect, whence follows that $y = A$ and the curve $y = -cx - f(x)$ intersect for all A. Hence the trivial solution of (3.12) is asymptotically stable in the large in this case according to Theorem 3.2.

If $c^2 + ab < 0$, then the straight line $y = -\frac{c^2 + ab}{c} x$ lies in the second and fourth quadrants and no such conclusion as above can be drawn. However, we can prove that in this case $\dot{\Phi} < -\epsilon < 0$ in the regions (1) and (4) which ensures the stability in the large.

According to Conditions (3.13)

$$c + h(x) < 0, \quad c\,h(x) - ab > 0 \quad \text{for } x \neq 0$$

i.e. $$h(x) < -c, \; h(x) < \frac{ab}{c}$$

but since $c^2 + ab < 0$, the above inequalities are satisfied if we take $h(x) < -c$. $f(x)$ can then be written as

$$f(x) = -cx - \mathcal{L}(x) \quad \text{where } \mathcal{L}(x) \gtrless 0 \text{ according as}$$

$$x \gtrless 0$$

Let us calculate $\dot{\Phi}$:

$$\begin{aligned} r\,\dot{\Phi} &= -\sin\Phi(f(x) + cx + y) - \cos\Phi\,(ch(x) - ab)\,x \\ &= -\sin\Phi\,(y - \mathcal{L}(x)) - \cos\Phi\left[c(-cx - \mathcal{L}(x)) - abx\right] \\ &= -\sin\Phi(y - \mathcal{L}(x)) + \cos\Phi\left[(c^2 + ab)\,x + c\mathcal{L}(x)\right] \end{aligned}$$

$$\dot{\Phi} = -\frac{\sin\Phi}{r}(y - \mathcal{L}(x)) + \frac{\cos\Phi}{r}\left[(c^2 + ab)\,x + c\mathcal{L}(x)\right] < -\epsilon < 0$$

in the regions (1) and (4).

<u>Case 11</u> $c > o$. In this case $c^2 + ab < o$, because from Conditions (3.13)

$$\frac{ab}{c} < h(x) < -c$$

Let $h(x)$ be taken as $h(x) = -c - \mathcal{L}(x)$

Then
$$\frac{ab}{c} < -c - \mathcal{L}(x) < -c$$

$$c + \frac{ab}{c} < -\mathcal{L}(x) < o$$

$$-\frac{c^2 + ab}{c} > \mathcal{L}(x) > o$$

or
$$o < \mathcal{L}(x) < -\frac{c^2 + ab}{c}$$

Let us consider the region (1). If $\dot{\phi} \nless -\epsilon < o$, then it means that $\underset{x \to \infty}{\underline{\lim}} \frac{-1}{x}\left[(c^2 + ab)\, x + cx\,\mathcal{L}(x)\right] = o$, i.e., $\overline{\lim_{x \to \infty}}\, x\,\mathcal{L}(x) = \infty$,

whence it follows that the straight line $y = A$ intersects the curve

$$y = -cx - f(x) \text{ for all } A$$

In fact, $-A = cx + f(x) = cx + (-cx - x\mathcal{L}(x)) = -x\mathcal{L}(x)$

$$\text{therefore } x\,\mathcal{L}(x) = A$$

If $x\,\mathcal{L}(x) = A$ does not have a solution for all A, then

$$\overline{\lim_{x \to \infty}}\, x\mathcal{L}(x) = \text{finite} = D$$

$$\underset{x \to \infty}{\underline{\lim}} -\left[(c^2 + ab)x + cx\,\mathcal{L}(x)\right] \geqslant \underset{x \to \infty}{\underline{\lim}} \left[-(c^2 + ab)\, x\right] + \underset{x \to \infty}{\underline{\lim}} (-c\, x\mathcal{L}(x))$$

$$= \underset{x \to \infty}{\underline{\lim}} \left[-x(c^2 + ab)\right] - \overline{\lim_{x \to \infty}} (cx\,\mathcal{L}(x)),$$

whence follows that $\dot{\phi} < -\epsilon < o$.

<u>Case 111</u> $c = o$. The System (3.12) reduces to

$$\frac{dx}{dt} = f(x) + y$$

$$\frac{dy}{dt} = abx$$

and the Conditions (3.13) reduce to $h(x) < o$, $ab < o$.

For this system we take the following Lyapunov function

$$2V = -abx^2 + y^2$$

the derivative of which is

$$\frac{dV}{dt} = -ab\, x^2 h(x)$$

This V function satisfies all the conditions of Theorem 1.2.5. Combining all these results we have the following theorem:

Theorem 3.3. If for the System (3.12) $c^2 + ab \neq o$, then the trivial solution is asymptotically stable in the large under Conditions (3.13).

In the case $c^2 + ab = o$, we remark that $c < o$ and from the proof of stability in the Case 1, we find that the straight line $y = -\frac{c^2 + ab}{c}x$ coincides with the x-axis and hence we can neither say that the straight line $y = A$ intersects the curve $y + cx + f(x) = o$ for all A nor $\dot{\phi} < -\epsilon < o$ in the regions (1) and (4).

4. <u>A generalization of the problem of Aĭzerman.</u> In this section we discuss the system of differential equations given by

$$\begin{aligned} \frac{dx}{dt} &= ax + f_1(y) \\ \frac{dy}{dt} &= f_2(x) + cy \end{aligned} \tag{4.1}$$

under the conditions:

$$a + c < o, \ ac - h_1(y)\, h_2(x) > o \text{ for } x \neq o, \ y \neq o, \ f_1(o) = o = f_2(o) = o, \tag{4.2}$$

where $h_1(y)$ and $h_2(x)$ are defined by $f_1(y) = y\, h_1(y)$ and $f_2(x) = xh_2(x)$.

The above system was first discussed by Krasovskiĭ [18] who obtained certain theorems regarding the stability in the large of the trivial solution $x = o = y$ of (4.1). We prove here the following theorem:

<u>Theorem 4.1.</u> <u>If either $h_1(y) > o$ for $y \neq o$ and $h_2(x) > o$ for $x \neq o$ or $h_1(y) < o$ for $y \neq o$, $h_2(x) < o$ for $x \neq o$, then the trivial solution of (4.1) is asymptotically stable in the large under Conditions (4.2)</u>

<u>Proof</u>. We assume $h_1(y) > o$, $y \neq o$, and $h_2(x) > o$, $x \neq o$. From Conditions (4.2) follows that $a < o$, $c < o$. We consider the curves

$$ax + f_1(y) = o \text{ and } f_2(x) + cy = o \tag{4.3}$$

Since $h_1(y) > o$, $y \neq o$ and $a < o$, the curve $ax + f_1(y) = o$ lies in the first and third quadrants. The same is true for the curve $f_2(x) + cy = o$. We further note that the curve $ax + f_1(y) = o$ lies above the curve $f_2(x) + cy = o$ in the first quadrant and below in the third quadrant. In fact, if y_1 and y_2 denote the ordinates of the curves in (4.3) respectively, then

$$y_1 - y_2 = \frac{-ax}{h_1(y_1)} + \frac{x\, h_2(x)}{c} = -x\,\frac{ac - h_1(y)\, h_2(x)}{c\, h_1(y)} \gtrless 0 \text{ according as } x \gtrless 0$$

It is easy to see that

$$\frac{dx}{dt} = ax + f_1(y) > 0 \quad \text{to the left of the curve } ax + f_1(y) = 0$$

$$\frac{dx}{dt} = ax + f_1(y) < 0 \quad \text{to the right of the curve } ax + f_1(y) = 0$$

$$\frac{dy}{dt} = f_2(x) + cy < 0 \quad \text{to the left of the curve } cy + f_2(x) = 0$$

$$\frac{dy}{dt} = f_2(x) + cy > 0 \quad \text{to the right of the curve } cy + f_2(x) = 0$$

The function $x(t)$ is maximum on the curve $ax + f_1(y) = 0$ for $y > 0$ and minimum for $y < 0$; and the functions $y(t)$ is maximum on the curve $f_2(x) + cy = 0$ for $y > 0$ and minimum for $y < 0$. The curves and the direction of motion are shown in fig. 6. As before we introduce polar co-ordinates $x = r\cos\phi$, $y = r\sin\phi$. Then

$\dot r = \dot x \cos\phi + \dot y \sin\phi$

$r\dot\phi = -\dot x \sin\phi + \dot y \cos\phi$

The signs of $\dot r$ and $\dot\phi$ in different regions are (1,5) $\dot r$ maybe $\gtrless 0$, $\dot\phi < 0$

(2,6) $\dot r < 0$, $\dot\phi$ maybe $\gtrless 0$

(3,7) $\dot r$ maybe $\gtrless 0$, $\dot\phi > 0$

(4,8) $\dot r < 0$, $\dot\phi$ maybe $\gtrless 0$

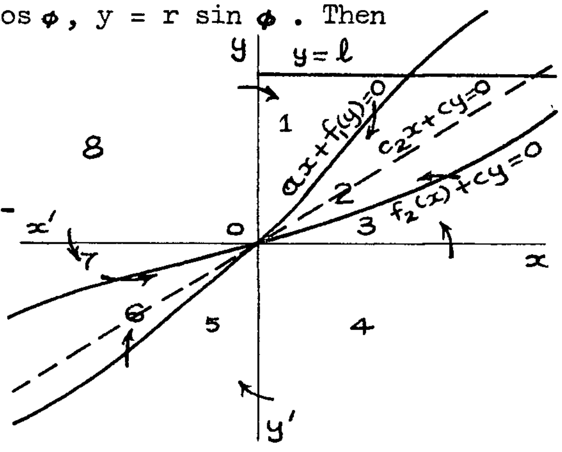

FIG. 6

From Theorem 2.1 of Erugin's work [8] it follows that there is at least one integral curve going to the origin, the only point of equilibrium, in each of the regions (4) and (8). Other motions started in the regions (4) or (8) either go to the origin or enter the regions (3), (5) or (1), (8) (since $\dot{r} < o$, $\dot{\phi}$ maybe $\gtrless o$ in (4) and (8)). Let us suppose that the motion enters the region (1). We show that the motion leaves this region with the increase of time. Let c_2 be the least upper bound of $h_2(x)$. c_2 is finite otherwise $ac - h_1(y)\, h_2(x) > o$ will not hold. We consider the straight line $c_2x + cy = o$. It is easy to verify that this straight line lies above the curve $f_2(x) + cy = o$ and below the curve $ax + f_1(y) = o$. in the first quadrant. In the third quadrant the positions are reversed. We consider the region bounded by $y = \ell$, the curve $ax + f_1(y) = o$ and the positive half y-axis. The motion cannot intersect the straight line $y = \ell$, since $\frac{dy}{dt} < o$. It cannot go to the origin, since $\dot{\phi} < o$. Therefore it must intersect the curve $ax + f_1(y) = o$ and enter the region (2) where it goes to the origin with the increase of time. For region (3) we take the straight line $x = \ell'$ and similarly show that the motion crosses the curve $f_2(x) + cy = o$ and enters the region (2) where it goes to the origin as $t \to +\infty$. Similar reasonings hold for the regions (5) and (7) which completes the proof of Theorem 4.1.

In the same way we can prove the following theorem for the system

$$\begin{aligned} \frac{dx}{dt} &= f_1(x) + f_2(y) \\ \frac{dy}{dt} &= bx + cy \end{aligned} \tag{4.4}$$

under the conditions:

$h_1(x) + c < o,\ x \neq o,\ ch_1(x) - h_2(y)\, b > o,\ x \neq o,\ y \neq o,\ f_1(o)=f_2(o)=o,$

where $\qquad h_1(x) = \dfrac{f_1(x)}{x},\ x \neq o,\ h_2(y) = \dfrac{f_2(y)}{y},\ y \neq o \qquad (4.5)$

Theorem 4.2. If $b\, h_2(y) > o$ for $y \neq o$ and the Conditions (4.5) are satisfied, then the trivial solution of system (4.4) is asymptotically stable in the large.

5. The stability in the large of the system $\frac{dx}{dt} = F(x,y)$, $\frac{dy}{dt} = f(\sigma)$ using qualitative methods.

In this section we consider the system (2.1). The assumptions under which we will be working will be different from those of Section 2. B.A. Ershov [6] discussed this system and obtained certain theorems regarding the stability in the large of the trivial solution. We here show that his results are correct but the proofs are wrong.

Let us consider the system

$$\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= f(\sigma), \quad \text{where } \sigma = c_1 x - d_1 y \end{aligned} \tag{5.1}$$

and c_1, d_1 are positive constants. We shall assume that $F(x,y)$ is a continuous function, having first order partial derivatives with respect to x and y, for all values of x, y. Further we assume that

$$\frac{\partial F}{\partial y} < 0, \quad F(0,0) = 0 \tag{5.2}$$

We shall discuss three cases (i) $\frac{\partial F}{\partial x} < 0$, (ii) $\frac{\partial F}{\partial x} = 0$, (iii) $\frac{\partial F}{\partial x} > 0$.

The continuous function $f(\sigma)$, appearing in the right hand side of the second equation of (5.1) is subjected to the following conditions:

$$\sigma f(\sigma) > 0 \quad \text{for } \sigma \neq 0 \tag{5.3}$$

$$f(0) = 0 \tag{5.4}$$

$$\frac{\partial f}{\partial \sigma} \geqslant 0 \quad \text{for all } \sigma \tag{5.5}$$

In the condition (5.5) $\frac{\partial f}{\partial \sigma}$ is not taken to be identically zero.

The equations (5.1) can be written as

$$\frac{dx}{dt} = \left(\frac{\partial F}{\partial x}\right)_{x=o,y=o} x + \left(\frac{\partial F}{\partial y}\right)_{x=o,y=o} y + \chi(x,y)$$

$$\frac{dy}{dt} = c_1\left(\frac{\partial f}{\partial \sigma}\right)_{x=o,y=o} x - d_1\left(\frac{\partial f}{\partial \sigma}\right)_{x=o,y=o} y + \psi(x,y) \tag{5.6}$$

where

$$\chi(x,y) = F(x,y) - \left(\frac{\partial F}{\partial x}\right)_{x=o,y=o} x - \left(\frac{\partial F}{\partial y}\right)_{x=o,y=o} y$$

$$\psi(x,y) = f(\sigma) - c_1\left(\frac{\partial f}{\partial \sigma}\right)_{x=o=y} x + d_1\left(\frac{\partial f}{\partial \sigma}\right)_{x=o=y} y$$

We write for simplicity

$$\left|\left(\frac{\partial F}{\partial x}\right)_{x=o=y}\right| = a, \quad \left|\left(\frac{\partial F}{\partial y}\right)_{x=o=y}\right| = b$$

$$c_1\left(\frac{\partial f}{\partial \sigma}\right)_{x=o,y=o} = c, \quad \left(d_1 \frac{\partial f}{\partial \sigma}\right)_{x=o=y} = d$$

The system (5.6) can then be written in the form

$$\frac{dx}{dt} = -Nax - by + \chi(x,y)$$

$$\frac{dy}{dt} = cx - dy + \psi(x,y) \tag{5.7}$$

The three cases thus correspond to the values 1, o, -1 of N in (5.7) respectively.

<u>Case 1</u> $\frac{\partial F}{\partial x} < o$. From equations (5.7) we have for N = 1

$$\frac{dx}{dt} = -ax - by + \chi(x,y)$$

$$\frac{dy}{dt} = cx - dy + \psi(x,y) \tag{5.8}$$

The equations of the first approximation are

$$\frac{dx}{dt} = - ax - by$$
$$\frac{dy}{dt} = cx - dy \tag{5.9}$$

The characteristic equation of (5.9) is

$$\begin{vmatrix} \lambda + a & -b \\ c & \lambda + d \end{vmatrix} = o$$

or
$$\lambda^2 + (a + d)\lambda + ad + bc = o \tag{5.10}$$

The roots of this equation have negative real parts, since

$$a + d > o \quad \text{and} \quad ad + bc > o$$

Since the roots of (5.10) have negative real parts, it follows that the trivial solution $x = o = y$ of (5.8) is asymptotically stable according to Lyapunov. The absence of periodic solutions is easily seen by using the criterion of Bendixson. In fact,

$$\frac{\partial F}{\partial x} + \frac{\partial f}{\partial y} = \frac{\partial F}{\partial x} - d_1 \frac{\partial f}{\partial \sigma} < o$$

in view of condition (5.5) and the fact that $\frac{\partial F}{\partial x} < o$, $d_1 > o$.

We represent the curves $F(x,y) = o$ and $f(\sigma) = o$ on the (x,y) plane. By virtue of Conditions (5.3) and (5.4) $f(\sigma) = o$ represents the straight line $\sigma = o$. Since c_1 and d_1 are positive constants, $\sigma = o$ is situated in the first and third quadrants. The curve $F(x,y) = o$ passes through the origin since $F(o,o) = o$. It is situated in the second and fourth quadrants since the slope of $F(x,y)=o$,

i.e.,
$$\frac{dy}{dx} = \frac{\partial F/\partial x}{-\partial F/\partial y} \quad \text{is negative.}$$

These considerations show that (o,o) is the only point of equilibrium of (5.8). We have shown so far that (i) the origin is the only point of equilibrium, (ii) it is asymptotically stable in the sense of Lyapunov, (iii) there exist no periodic solutions. We shall now show that there exists a straight line $L(o, \infty)$ which is intersected by the motions in one direction only and all motions with bounded polar angles are bounded. With this thing in view we examine the directions of motions (see fig. 7). It is easy to see that

$\frac{dx}{dt} = F(x,y) > o$ for the points, lying to the left of the curve $F(x,y) = o$

$\frac{dx}{dt} = F(x,y) < o$ for the points, lying to the right of the curve $F(x,y) = o$

$\frac{dy}{dt} = f(\sigma) > o$ for the points, lying below the straight line $\sigma = o$

$\frac{dy}{dt} = f(\sigma) < o$ for the points, lying above the straight line $\sigma = o$

The function $y(t)$ attains maximum for $y > o$ and minimum for $y < o$ on the straight line $f(\sigma) = o$; $x(t)$ attains maximum for $x > o$ and minimum for $x < o$ on the curve $F(x,y) = o$. The straight line $f(\sigma) = o$, the curve $F(x,y) = o$ and the co-ordinate axes divide the (x,y) plane into eight regions. We introduce

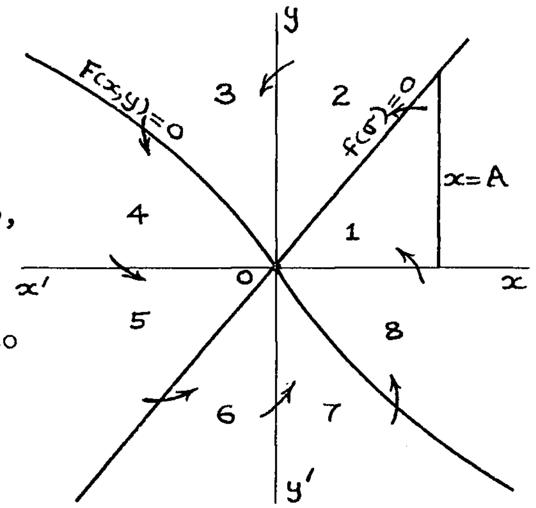

FIG.7

polar co-ordinates

$$x = r\cos\phi\,,\quad y = r\sin\phi\,.$$

Then $\quad \dot r = \dot x\cos\phi + \dot y\sin\phi$ and

$$r\dot\phi = -\dot x\sin\phi + \dot y\cos\phi$$

the signs of $\dot r$ and $\dot\phi$ in different regions are as follows :

(1,5) $\dot r$ may be $\gtrless$ o, $\dot\phi > o$ (2,6) $\dot r < o$, $\dot\phi$ may be $\gtrless$ o

(3,7) $\dot r$ may be $\gtrless$ o, $\dot\phi > o$ (4,8) $\dot r < o$, $\dot\phi$ may be $\gtrless$ o.

Ershov [6] argued that, since $\dot\phi > o$ in the regions (1), (3), (5) and (7) there cannot be any motion with bounded polar angle ϕ in these regions and any motion falling in these regions or starting there has to get out of these regions after intersecting either the straight line $f(\sigma) = o$ or the curve $F(x,y) = o$. To us this reasoning is doubtful, since $\dot\phi > o$ is not sufficient to guarantee that there cannot be any motion with bounded polar angle in these regions. However, the above assertion remains true if we could show that $\dot\phi > \epsilon > o$ in these regions. In fact, if $\dot\phi > \epsilon$, then integrating we have $\phi - \phi_o > \epsilon\,(t - t_o)$, whence follows that as t increases, ϕ increases and hence there will be an instant of time when the motion leaves these regions.

Let a motion M(t) after intersecting the negative half y-axis enter the region (7). We show that motion M(t) cannot remain in this region and consequently it will enter the region (8). To show this we write

$$\begin{aligned}\dot\phi &= \frac{1}{r}\left[-\dot x\sin\phi + \dot y\cos\phi\right] \\ &= \frac{1}{r}(-F(x,y)\sin\phi + f(\sigma)\cos\phi) > o \text{ in region (7)}\end{aligned} \tag{5.11}$$

If $M(t)$ does not cross the curve $F(x,y) = o$, then $x(t)$ becomes infinite since $x(t)$ and $y(t)$ are increasing and $\dot{\phi} > o$. For sufficiently large x, with the increase of x, σ increases. Since it is assumed that $\frac{\partial f}{\partial \sigma} \geqslant o$, $f(\sigma)$ is non decreasing and therefore from (5.11) it follows that $\dot{\phi}$ can always be taken greater than $\epsilon > o$ and hence with the increase of time the motion must leave the region (7), which contradicts our assumption. After entering the region (8) it either goes to the origin as $t \longrightarrow +\infty$ or intersects the x-axis and enter the region (1). This follows from the fact that in this region $\dot{r} < o$ and $\dot{\phi}$ may be $\gtrless o$. The motion cannot remain in the region (1) and must enter the region (2). To see this, consider the region bounded by the straight line $\sigma = o$, $x = A$ and the x-axis. Since in the region (1) x decreases, the motion cannot intersect the line $x = A$. It cannot go to the origin, since $\dot{\phi} > o$ and therefore it must necessarily go out of the region (1) and enter the region (2). Here, i.e., in region (2), since $\dot{r} < o$ the motion either goes to the point of equilibrium or enters the region (3). Similar reasoning holds for the rest of the regions. All this shows that any motion with bounded polar angle is bounded. For the straight line $L(o,\infty)$ we can take the positive half y-axis. Thus all the conditions of Theorem 1.2.7 are satisfied and we have the following theorem:

<u>Theorem 5.1.</u> <u>Let $\frac{\partial F}{\partial x} < o$. Then under conditions (5.2) - (5.5) the trivial solution of (5.1) is asymptotically stable in the large.</u>

<u>Case II</u> $\frac{\partial F}{\partial x} = o$. The equations (5.7) in this case take the form

$$\frac{dx}{dt} = -by + \gamma(y)$$
$$\frac{dy}{dt} = cx - dy + \psi(x,y) \qquad (5.12)$$

The equations of first approximation are

$$\frac{dx}{dt} = -by$$
$$\frac{dy}{dt} = cx - dy \qquad (5.13)$$

The characteristic equation of (5.13) is given by

$$\begin{vmatrix} \lambda & -b \\ c & \lambda + d \end{vmatrix} = o \qquad \text{or} \quad \lambda^2 + d\lambda + bc = o$$

The real parts of the roots of this equation are negative, since $d > o$ and $bc > o$, whence follows the asymptotic stability of the trivial solution of system (5.12) in the sense of Lyapunov.

The curve $F(y) = o$ represents the straight line $y = o$ and $f(\sigma) = o$ represents the straight line $\sigma = o$. The origin is the only common point of $f(\sigma) = o$. and $F(y) = o$. It is easy to see that

$\frac{dx}{dt} = F(y) > o$ for the points, lying below the x-axis

$\frac{dx}{dt} = F(y) < o$ for the points, lying above the x-axis

$\frac{dy}{dt} = f(\sigma) > o$ for the points, below the straight line $f(\sigma) = o$

$\frac{dy}{dt} = f(\sigma) < o$ for the points, above the straight line $f(\sigma) = o$

The function $y(t)$ is maximum for $y > o$ and minimum for $y < o$ for the points on $f(\sigma) = o$ and $x(t)$ is maximum for $x > o$ and minimum for x $<$ o for the points on the x-axis. The curves $F(y) = o$, $f(\sigma) = o$ and the axes of co-ordinates divide the plane into six regions. The direction of motion is represented in fig. 8. The signs of $\dot{r}$ and $\dot{\Phi}$ in different regions are given as below:

(1,4) $\dot{r}$ may be $\gtrless o$, $\dot{\Phi} > o$

(2,5) $\dot{r} < o$, $\dot{\Phi}$ may be $\gtrless o$

(3,6) $\dot{r}$ may be $\gtrless o$, $\dot{\Phi} > o$

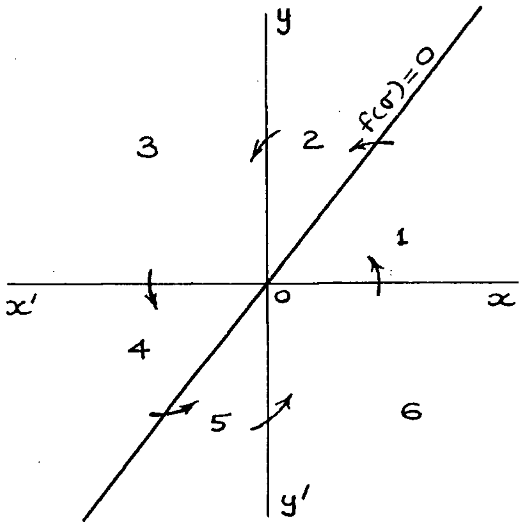

FIG.8

In the region (6) it is easy to see that $\frac{dy}{dt} \geqslant \ell > o$, whence it follows that there exists an instant t when the motion intersects the x-axis. As in Case I it can be shown that the motion M(t) intersects the straight line $f(\sigma) = o$ and enters the region (2) with the increase of time. In region (2) either the motion goes to the origin with the increase of time or enters the region (3). Similar arguments can be applied for the regions (3),(4) and (5). Thus we have shown that (i) (o,o) is the only point of equilibrium,(ii) it is asymptotically stable in the small,(iii) any motion with bounded polar angle is bounded. To show that there exists no periodic motions, we use the criterion of Bendixson

$$\frac{\partial F}{\partial x} + \frac{\partial f}{\partial y} = o - d_1 \frac{\partial f}{\partial \sigma} \leqslant o \quad \text{(not identically equal to zero)}$$

For the straight line $L(o,\infty)$ we can take the positive half x-axis. Thus all the conditions of Theorem 1.2.7 are satisfied and we have the following theorem:

Theorem 5.2. Let $\frac{\partial F}{\partial x} = o$. Then under conditions (5.2) - (5.5) the trivial solution of system (5.1) is asymptotically stable in the large.

Case III $\frac{\partial F}{\partial x} > o$. The equations (5.7) in this case take the form

$$\begin{aligned} \frac{dx}{dt} &= ax - by + \gamma(x,y) \\ \frac{dy}{dt} &= cx - dy + \psi(x,y) \end{aligned} \tag{5.14}$$

The characteristic equation of the first approximation is

$$\begin{vmatrix} \lambda - a & -b \\ c & \lambda + d \end{vmatrix} = o$$

$$\text{or } \lambda^2 + (d - a)\lambda + bc - ad = o \tag{5.15}$$

The roots of (5.15) will have negative real parts under the following conditions:

$$d - a > o \tag{5.16}$$

$$bc - da > o \tag{5.17}$$

If the conditions (5.16) and (5.17) are satisfied then the trivial solution $x = o = y$ of (5.14) is asymptotically stable in the small. We now represent the curves $F(x,y) = o$ and the straight line $f(\sigma) = o$ on the (x,y) plane. Since $\frac{\partial F}{\partial x} > o$, the curve $F(x,y) = o$ is situated in the first and third quadrants and so is the straight line given by $f(\sigma) = o$. Since we are interested in having a unique point of equilibrium we will have to impose an extra condition:

Condition 1. The curve $F(x,y) = o$ is situated between the x-axis and the straight line $f(\sigma) = o$.

Obviously condition 1 implies the condition (5.17). It is not difficult to see that

$\frac{dx}{dt} = F(x,y) > o$ for the points lying to the right of the curve $F(x,y) = o$

$\frac{dx}{dt} = F(x,y) < o$ for the points lying to the left of the curve $F(x,y) = o$

$\frac{dy}{dt} = f(\sigma) > o$ for the points below the straight line $f(\sigma) = o$

$\frac{dy}{dt} = f(\sigma) < o$ for the points above the straight line $f(\sigma) = o$

The function $y(t)$ is maximum for $y > o$ and minimum for $y < o$ for the points on the straight line $f(\sigma) = o$ and $x(t)$ attains maximum for $x > o$ and minimum for $x < o$ for the points lying on the curve $F(x,y) = o$, The direction of motion is shown in fig. 9. The curve $F(x,y) = o$, the straight line $f(\sigma) = o$ and the axes of co-ordinates divide the plane (x,y) into eight regions. As before we introduce polar co-ordinates

$$x = r\cos\phi\ ,\ y = r\sin\phi.$$

Then $\dot{r} = \dot{x}\cos\phi + \dot{y}\sin\phi$

$$r\dot{\phi} = -\dot{x}\sin\phi + \dot{y}\cos\phi$$

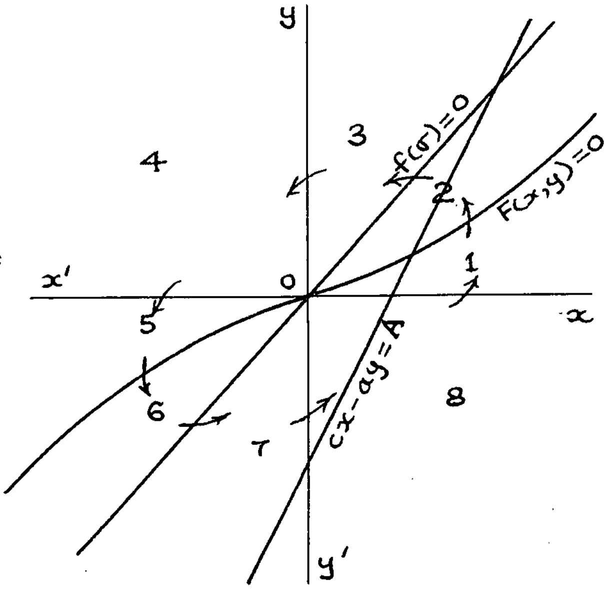

FIG.9

The signs of $\dot{r}$ and $\dot{\phi}$ in different regions are given as

(1,5) $\dot{r} > o$, $\dot{\phi}$ may be $\gtrless o$ (2,6) $\dot{r}$ may be $\gtrless o$, $\dot{\phi} > o$

(3,7) $\dot{r} < o$, $\dot{\phi}$ may be $\gtrless o$ (4,8) $\dot{r}$ may be $\gtrless o$, $\dot{\phi} > o$.

In the regions (1) and (5) we see that $\dot{r} > o$ and $\dot{\phi}$ may be $\gtrless o$. First of all we need a condition that makes $\dot{\phi} > o$ in these regions. This is necessary in order to ensure that there is no motion with bounded polar angle in the regions (1,5). Our purpose is served if for example we assume that

$$\begin{aligned} &\chi(x,y) < o, \quad \psi(x,y) > o \quad \text{in the region (1)} \\ &\chi(x,y) > o, \quad \psi(x,y) < o \quad \text{in the region (5)} \end{aligned} \qquad (5.18)$$

Next we require that there be no periodic motions and for that we must have

$$\frac{\partial F}{\partial x} - d_1 \frac{\partial f}{\partial \sigma} \leq o \qquad (5.19)$$

according to Bendixson criterion.

Let a motion M(t) enter the region (8) after intersecting the negative half y-axis. It is easy to see as in case 2 that the motion M(t) crosses the x-axis and enters the region (1). We now show that it cannot remain in the region (1) and must enter the region (2). This is done as follows. We consider the rate of change of $(cx - ay)$ along the motion M(t), i.e.,

$$\frac{d}{dt}(cx - ay) = acx - bcy + c\,\chi(x,y) - acx + ady - a\,\psi(x,y)$$

$$= y(ad - bc) + c\,\chi(x,y) - a\,\psi(x,y) < o \qquad (5.20)$$

in the region (1) in view of (5.17) and (5.18).

We consider the straight lines $cx - ay = A$ (A positive) and $cx - dy = o$ and find their point of intersection $\left(\frac{Ad}{c}\frac{1}{d-a}, \frac{A}{d-a}\right)$. It lies in the first quadrant since A, c, d, d - a are all positive quantities. Since $F(x,y) = o$ lies always below the straight line $f(\sigma) = o$, the straight line $cx - ay = A$ intersects the curve $F(x,y) = o$ for all A. Thus the motion M(t) entering the region (1) must cross the curve $F(x,y) = o$, because it cannot go to the origin since $\dot{\Phi} > o$ and cannot cross the straight line $cx - ay = A$ due to (5.20). After intersecting the curve $F(x,y) = o$ it cannot even remain in the region (2) because of the same reason. Thus the motion enters the region (3) where it either tends to the origin with the increase of time or goes out of this region and enters the region (4). Similar argument holds for the regions (4), (5), (6) and (7). This shows that there cannot be any motion with bounded polar angle in the regions (1), (2), (4), (5), (6) and (8). The motion with bounded polar angle can only occur in regions (3) and (7), where it is bounded, since $\dot{r} < o$. Thus any motion with bounded polar angle is bounded. For the straight line $L(o, \infty)$ we can take the positive half x-axis. Hence all the requirement of Theorem 1.2.7 are satisfied and we have the following theorem:

Theorem 5.3. Let $\frac{\partial F}{\partial x} > o$. Then under conditions (5.2) - (5.5), (5.16), (5.18), (5.19) and Condition 1, the trivial solution $x=o=y$ of system (5.1) is asymptotically stable in the large.

In 1954 Gu, Čao hao [16] discussed the stability of the trivial solution in the large of the following system

$$\frac{dx}{dt} = x\, h_1(y) + \Phi(y)$$

$$\frac{dy}{dt} = x\, h_2(y) + f(y)$$

by constructing a Lyapunov function. In the next two sections we shall discuss the stability in the large of

$$\frac{dx}{dt} = h_1(y)\,x + ay$$

$$\frac{dy}{dt} = h_2(x)\,x + by$$

and

$$\frac{dx}{dt} = xh_1(y) + ay$$

$$\frac{dy}{dt} = bx + h_2(x)\,y$$

mostly by qualitative methods.

6. The stability in the large of $\frac{dx}{dt} = h_1(y)\,x + ay$, $\frac{dy}{dt} = h_2(x)\,x + by$.

Let us consider the system

$$\frac{dx}{dt} = h_1(y)\,x + ay$$
$$\frac{dy}{dt} = h_2(x)\,x + by \qquad (6.1)$$

We assume that the right hand sides of the system (6.1) satisfy the conditions guaranteeing the existence and uniqueness of every solution.

6a. Let us assume that

$$ah_2(x) < o,\ x \neq o,\ b < o \quad \text{and} \quad h_1(y) < o,\ y \neq o$$

For the sake of definiteness we take $a < o$, $h_2(x) > o$, $x \neq o$. We represent the right hand sides of (6.1) on the (x,y) plane. The ordinates of the graphs of the curves are given by

$$y_1 = -\frac{h_1(y_1)}{a}\,x \gtrless o \quad \text{according as } x \lessgtr o$$
$$y_2 = -\frac{h_2(x)}{b}\,x \gtrless o \quad \text{according as } x \gtrless o \qquad (6.2)$$

This shows that the curves $h_1(y)\,x + ay = o$ and $h_2(x)\,x + by = o$ lie in the second, fourth and first, third quadrants respectively and consequently the origin is the only point of equilibrium. It is not difficult to see that

$$\frac{dx}{dt} = h_1(y)\,x + ay < o \quad \text{to the right of the curve } h_1(y)x + ay = o$$

$$\frac{dx}{dt} = h_1(y)\,x + ay > o \quad \text{to the left of the curve } h_1(y)x + ay = o$$

$$\frac{dy}{dt} = h_2(x)x + by < o \quad \text{to the left of the curve } h_2(x)x + by = o$$

$$\frac{dy}{dt} = h_2(x)x + by > o \quad \text{to the right of the curve } h_2(x)x + by = o$$

The function $y(t)$ attains maximum on the curve $h_2(x)x + by = o$ for $y > o$ and minimum for $y < o$; $x(t)$ is minimum on the curve $h_1(y)x + ay = o$ for $y > o$ and maximum for $y < o$. The direction of motion and the curves obtained by putting the right hand sides of (6.1) equal to zero are represented in fig.10. The curves and the co-ordinate axes divide the plane into eight regions. Using polar co-ordinates we see that the signs of $\dot{r}$ and $\dot{\phi}$ in these regions are

(1,5) $\dot{r}$ may be $\gtrless o$, $\dot{\phi} > o$

(2,6) $\dot{r} < o$, $\dot{\phi}$ may be $\gtrless o$

(3,7) $\dot{r}$ may be $\gtrless o$, $\dot{\phi} > o$

(4,8) $\dot{r} < o$, $\dot{\phi}$ may be $\gtrless o$

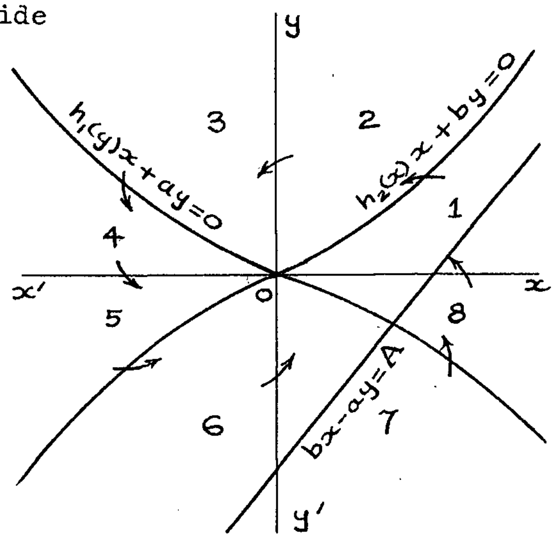

FIG.10

The trivial solution is easily shown to be asymptotically stable in the sense of Lyapunov. In fact, let a V-function be defined by

$$2V = 2\int_0^x x\, h_2(x)\, dx - ay^2$$

Then

$$\dot{V} = x\, h_2(x)(h_1(y)x + ay) - ay\,(x\, h_2(x) + by)$$

$$= x^2 h_1(y)\, h_2(x) - aby^2 < o \quad y \neq o$$

$$= o \quad \text{possibly for } y = o$$

Obviously, V is a positive definite function. The derivative $\dot{V}$ is negative for $y \neq o$ and possibly zero for $y = o$. Since $y = o$ does not contain a positive half-trajectory of the system (6.1) except $x=o=y$, the trivial solution is asymptotically stable according to Lyapunov. Since $b + h_1(y) < o$, there are no periodic solutions according to the criterion of Bendixson. This is obvious from the fact that for the system is constructed a Lyapunov function. For the straight line $L(o, \infty)$ appearing in Theorem 1.2.7 we take the positive half y-axis. We now show that the motions with bounded polar angles are bounded. We consider first the regions (2,4,6,8). In these regions any motion with bounded polar angle is bounded, since $\dot{r} < o$. Next we consider the regions (1,3,5,7). Here we show that there are no motions with bounded polar angles, i.e., any motion starting or entering these regions must leave them with the increase of time. For this we write

$$\dot{\Phi} = \frac{1}{r}\Big[- \sin \Phi \, (h_1(y)\, x + ay) + \cos \Phi \, (h_2(x)\, x + by)\Big]$$

Let us suppose that the motion started in the region (1) does not cross the curve $h_2(x)\, x + by = o$ then, since x is decreasing and y is increasing in (1), y becomes infinitely large and consequently $\dot{\Phi} > \epsilon > o$ and hence with the increase of time the motion leaves the region (1). This contradicts our assumption. The contradiction shows that the motion must leave the region (1) and enter the region (2). The same reasoning holds for the region (5). Next we consider the rate of change of the quantity $(bx - ay)$ along the trajectories of (6.1)

$$\frac{d}{dt}(bx - ay) = b(h_1(y)x + ay) - a(h_2(x)x + by)$$
$$= x(bh_1(y) - ah_2(x)) \gtrless o \text{ according as } x \gtrless o \qquad (6.3)$$

Consider the straight line

$$bx - ay = A \qquad (6.4)$$

The straight line (6.4) intersects the curve $xh_1(y) + ay = o$ for all A. Let A be negative, then the straight line (6.4) intersects the curve $h_1(y)x + ay = o$ in the fourth quadrant. We consider the region bounded by the negative half y-axis, the straight line $bx - ay = A$ and the curve $h_1(y)x + ay = o$. The motion entering this region must cross the curve $h_1(y)x + ay = o$, since it cannot cross the straight line $bx - ay = A$ because of (6.3) and cannot enter the origin, since $\dot{\Phi} > o$ in this region. Similar argument holds in region (3). The above analysis shows that all motions with bounded polar angles are bounded and we have the following theorem:

Theorem 6.1. If $ah_2(x) < o$ for $x \neq o$, $b < o$ and $h_1(y) < o$ for $y \neq o$, then the trivial solution of (6.1) is asymptotically stable in the large.

6b. We shall now discuss the case when $ah_2(x) > o$ and the conditions

$$b + h_1(y) < o, \; bh_1(y) - ah_2(x) > o, \; x \neq o, \; y \neq o \qquad (6.5)$$

are satisfied. From (6.5) it follows that if $ah_2(x) > o$ we must necessarily have $b < o$, $h_1(y) < o$ for $y \neq o$. We assume $a > o$, $h_2(x) > o$, $x \neq o$. From (6.2) we have

$$y_1 - y_2 = - x \frac{h_1(y_1)}{a} + x \frac{h_2(x)}{b}$$
$$= - x \frac{bh_1(y_1) - ah_2(x)}{ab} \gtrless o \text{ according as } x \gtrless o \qquad (6.6)$$

Since y_1 and y_2 are positive or negative according as x is positive or negative, the two curves

$$xh_2(x) + by = o$$
$$xh_1(y) + ay = o \qquad (6.7)$$

are situated in the first and third quadrants and such that the curve $xh_1(y) + ay = o$ is above the curve $xh_2(x) + by = o$ in the first quadrant and below in the third quadrant (because of (6.6)). We further note that

$\frac{dx}{dt} = h_1(y)x + ay > o$ to the left of the curve $xh_1(y) + ay = o$

$\frac{dx}{dt} = h_1(y)x + ay < o$ to the right of the curve $xh_1(y) + ay = o$

$\frac{dy}{dt} = h_2(x)x + by < o$ to the left of the curve $xh_2(x) + by = o$

$\frac{dy}{dt} = h_2(x)x + by > o$ to the right of the curve $xh_2(x) + by = o$

The function $x(t)$ is maximum on the curve $xh_1(y) + ay = o$ for $y > o$ and minimum for $y < o$; $y(t)$ is maximum on the curve $xh_2(x) + by = o$ for $y > o$ and minimum for $y < o$. The curves and the co-ordinate axes divide the (x,y) plane into eight regions. The direction of motion and the curves are shown in fig.11. As before we introduce polar co-ordinates and notice that the signs of $\dot{r}$ and $\dot{\phi}$ in different regions are as follows:

(1,5) $\dot{r}$ may be $\gtreqless o$, $\dot{\phi} < o$ (2,6) $\dot{r} < o$, $\dot{\phi}$ may be $\gtreqless o$

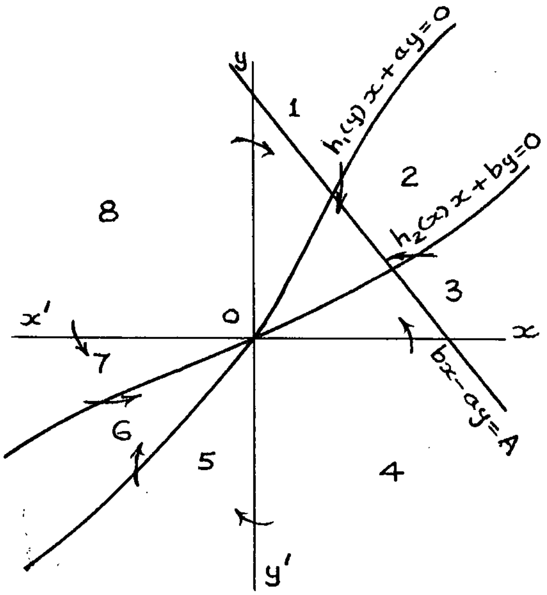

FIG.11

(3,7) $\dot{r}$ may be $\gtrless$ o, $\dot{\phi} >$ o

(4,8) $\dot{r} <$ o, $\dot{\phi}$ may be $\gtrless$ o

In each of the regions (4) and (8) there will be at least one integral curve going to the origin, the only point of equilibrium. This follows from Theorem 2.1 of Erugin's work [8] . Other motions started in the regions (4) and (8) either go to the origin or enter the regions (3,5) or (1,7) (since $\dot{r} <$ o, $\dot{\phi}$ may be $\gtrless$ o in the regions (4,8)). We now show that motions entering the regions (1), (3), (5) and (7) must leave these regions and enter the regions (2) or (6). To show this we consider the rate of change of the quantity (bx - ay),i.e.,

$$\frac{d}{dt}(bx - ay) = x(bh_1(y) - ah_2(x)) \gtrless o \text{ according as } x \gtrless o \qquad (6.8)$$

The straight line bx - ay = A (A negative) has positive intercepts with the axes of co-ordinates and hence intersects the curves (6.7) in the first quadrant. Consider the region bounded by the straight line bx - ay = A (-ve), the curve $xh_1(y) + ay = o$ and the positive half y-axis. The motion M(t) entering this region cannot cross the line bx - ay = A (A negative) because of (6.8) and cannot go to the origin,since $\dot{\phi} <$ o and hence must leave this region with the increase of time. Similar argument holds for the regions (3), (5) and (7). The motions after entering the regions (2) or (6) tend to the origin as $t \to \infty$. Thus we have proved the following theorem:

Theorem 6.2. If $ah_2(x) > o$ for $x \neq o$, then the trivial solution of (6.1) is asymptotically stable in the large under conditions (6.5).

6c. We now assume that either $b > o$ or $h_1(y) > o$ for $y \neq o$ and the conditions (6.5) are satisfied. For the sake of definiteness we let $b > o$. Then from (6.5) it follows that $h_1(y) < o$ and $ah_2(x) < o$. We let $a < o$, $h_2(x) > o$ for $x \neq o$. The ordinates of the curves in (6.7) are

$$y_1 = - x \frac{h_1(y_1)}{a} \gtrless o \quad \text{according as } x \lessgtr o$$

$$y_2 = - x \frac{h_2(x)}{b} \gtrless o \quad \text{according as } x \lessgtr o \qquad (6.9)$$

We compute the difference between the ordinates

$$y_1 - y_2 = - x \frac{h_1(y_1)}{a} + x \frac{h_2(x)}{b}$$

$$= - x \frac{bh_1(y_1) - ah_2(x)}{ab} \gtrless o \text{ according as } x \gtrless o \qquad (6.10)$$

From (6.9) and (6.10) it follows that the two curves lie in the second and fourth quadrants and the curve $xh_1(y) + ay = o$ lies above the curve $xh_2(x) + by = o$ in the fourth quadrant and below in the second quadrant. It is not difficult to verify that

$\frac{dx}{dt} = h_1(y)\, x + ay > o$ to the left of the curve $h_1(y)\, x + ay = o$

$\frac{dx}{dt} = h_1(y)\, x+ay < o$ to the right of the curve $h_1(y)\, x + ay = o$

$\frac{dy}{dt} = h_2(x)\, x + by > o$ to the right of the curve $h_2(x)\, x + by = o$

$\frac{dy}{dt} = h_2(x)\, x + by < o$ to the left of the curve $h_2(x)\, x + by = o$

The function $x(t)$ is minimum on the curve $xh_1(y) + ay = o$

for $y > o$ and maximum for $y < o$, and $y(t)$ is maximum on the curve $xh_2(x) + by = o$ for $y > o$ and minimum for $y < o$. The curves (6.7) and the co-ordinate axes divide the plane into eight regions. The curves and the direction of motion are shown in fig. 12.

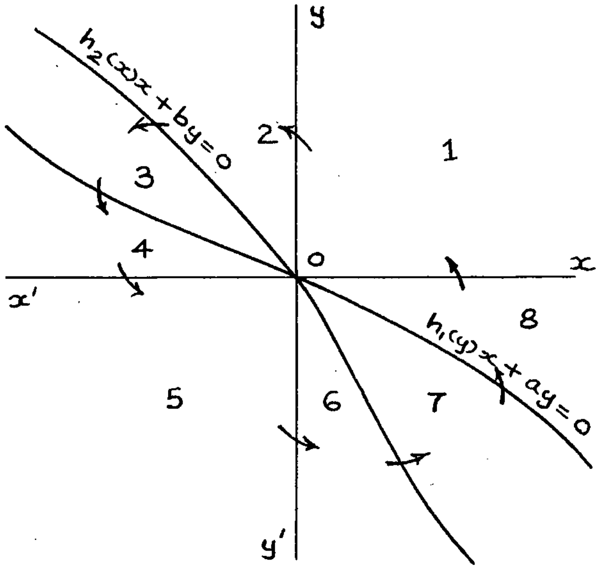

FIG. 12

The signs of $\dot{r}$ and $\dot{\Phi}$ in different regions are

(1,5) $\dot{r}$ may be $\gtrless o$, $\dot{\Phi} > o$

(2,6) $\dot{r} > o$, $\dot{\Phi}$ may be $\gtrless o$

(3,7) $\dot{r}$ may be $\gtrless o$, $\dot{\Phi} > o$

(4,8) $\dot{r} < o$, $\dot{\Phi}$ may be $\gtrless o$

Let us compute $\dot{\Phi}$ in the regions (2) and (6) and see whether it is positive or not. Let c_1 be the greatest lower bound of the function $h_1(y)$. c_1 is finite otherwise $bh_1(y) - ah_2(x) > o$ for $x \neq o$, $y \neq o$ is violated. We then have
$$c_1 \leqslant h_1(y) < -b$$

From the inequality $bh_1(y) - ah_2(x) > o$ we have $bc_1 - ah_2(x) \geqslant o$. The equality sign is admitted if $h_1(y)$ does not attain its greatest lower bound otherwise strict inequality holds. Let us suppose that c_1 is attainted, then

$$c_1 \leqslant h_1(y) < -b \tag{6.11}$$

and
$$h_2(x) > \frac{bc_1}{a}$$

If we let $h_1(y) = c_1 + \ell_1(y)$, then $o \leqslant \ell_1(y) < -(b + c_1)$

from (6.11). For $h_2(x)$ we can take

$$h_2(x) = \frac{bc_1}{a} + \ell_2(x), \text{ where } \ell_2(x) > 0$$

Now

$$\begin{aligned} r\dot{\phi} &= -\dot{x}\sin\phi + \dot{y}\cos\phi \\ &= -\sin\phi\,(xh_1(y) + ay) + \cos\phi\,(h_2(x)\,x + by) \\ &= -\sin\phi\left[x(c_1 + \ell_1(y)) + ay\right] + \cos\phi\left[\left(\frac{bc_1}{a} + \ell_2(x)\right)x + by\right] \end{aligned}$$

Then

$$\begin{aligned} \dot{\phi} &= -a\sin^2\phi + \frac{bc_1}{a}\cos^2\phi + (b - c_1)\sin\phi\cos\phi \\ &\qquad - \ell_1(y)\sin\phi\cos\phi + \ell_2(x)\cos^2\phi \\ &= -a\cos^2\phi\left[\tan^2\phi - \frac{b-c_1}{a}\tan\phi - \frac{bc_1}{a^2}\right] - \ell_1(y)\sin\phi\cos\phi \\ &\qquad + \ell_2(x)\cos^2\phi \quad (\phi \neq \text{multiple of } \pi/2) \\ &= -a\cos^2\phi\,(\tan\phi - \tfrac{b}{a})(\tan\phi + \tfrac{c_1}{a}) - \ell_1(y)\sin\phi\cos\phi \\ &\qquad + \ell_2(x)\cos^2\phi \qquad (6.12) \end{aligned}$$

The expression $(\tan\phi - \frac{b}{a})(\tan\phi + \frac{c_1}{a})$ can change its sign only while passing through the values $\frac{b}{a}$ and $-\frac{c_1}{a}$. The value $\frac{b}{a} > -\frac{c_1}{a}$ since $b + c_1 < 0$ and $a < 0$. Since $\frac{b}{a} > -\frac{c_1}{a}$, the straight line with slope $\frac{b}{a}$ lies above the straight line with slope $-\frac{c_1}{a}$ for $x > 0$ and below for $x < 0$. The straight lines $y = \frac{b}{a}x$ and $y = -\frac{c_1}{a}x$ lie in the second and fourth quadrants. We show that they do not lie in the regions (2) and (6). For this we have only to show that the straight line $y = -\frac{c_1}{a}x$ lies below the curve $xh_2(x) + by = 0$

in the second quadrant and above in the fourth quadrant. In fact, this is so, because

$$- x \frac{h_2(x)}{b} + \frac{c_1}{a} x = \frac{x}{b}\left(\frac{bc_1}{a} - h_2(x)\right) \gtrless o \quad \text{according as } x \lessgtr o$$

Thus we see from (6.12) that $\dot{\Phi}$ keeps the same sign throughout the regions (2) and (6) and which is easily seen to be positive.

According to conditions (6.5), the origin is the only point of equilibrium and there are no periodic motions. Let a motion M(t) start in region (1). The motion M(t) must enter the region (2) with the increase of time, otherwise y becomes infinite and then from $\frac{dx}{dt} = h_1(y)\,x + ay$ it follows that the motion M(t) cannot remain in the first quadrant. We now show that it cannot remain in the region (2) for all time. For this we consider the straight lines

$$bx - ay = A \quad \text{and} \quad ay + c_1 x = o$$

Their point of intersection is given by $\left(\frac{A}{b+c_1}, -\frac{c_1 A}{a(b+c_1)}\right)$. This lies in the second or fourth quadrant according as A is positive or negative. We consider the region bounded by the straight line $bx - ay = A(+ve)$, the positive half y-axis and the straight line $ay + c_1 x = o$. Since $\frac{d}{dt}(bx-ay) < o$ in the second quadrant and $\dot{\Phi} > o$ in (2), it follows that the motion enters the region (3). Here, in the region (3), since $\dot{\Phi} > o$ and the curve $xh_1(y) + ay = o$ lies above the straight line $bx - ay = o$ and $\dot{y} < o$, the motion enters the region (4) with increase of time. The motion after entering the region (4) either tends to the origin or enters the region (5), since $\dot{r} < o$ in region (4). Similar reasonings hold for the regions (5), (6), (7) and (8). The above analysis shows that any motion with bounded polar angle is bounded. For the straight line $L(o, \infty)$ appearing in Theorem 1.2.7 we can take positive half y-axis. Now if we assume that the trivial solution is asymptotically stable in the sense of Lyapunov, then we have proved the following theorem:

<u>Theorem 6.3.</u> <u>If either $b > o$ or $h_1(y) > o$, $y \neq o$ and conditions (6.5) are satisfied, then the trivial solution of system (6.1) is asymptotically stable in the large provided it is asymptotically stable in the small (i.e., according to Lyapunov).</u>

The requirement that the trivial solution be stable in the small can be realized, if we take, for example,

$$h_2(x) = \frac{bc_1}{a} + \mathcal{L}_2(x) = \frac{bc_1}{a} + m + \mathcal{L}_2'(x), \text{ where } m > o \text{ and } \mathcal{L}_2'(x)$$

does not contain any constant term and $h_1(y) = c + \mathcal{L}_1(y)$, where $\mathcal{L}_1(y)$ does not contain any constant term. The equations of first approximation can be written as

$$\frac{dx}{dt} = c_1 x + ay$$

$$\frac{dy}{dt} = \left(\frac{bc_1}{a} + m\right)x + by$$

The characteristic equation is

$$\begin{vmatrix} \lambda - c_1 & a \\ \frac{bc_1}{a} + m & \lambda - b \end{vmatrix} = o$$

or

$$\lambda^2 - (b + c_1)\lambda - am = o$$

The roots of this equation have negative real parts, since

$$b + c_1 < o \quad \text{and} \quad -am > o,$$

which ensures the stability in the small and which in turn ensures the stability in the large.

The origin is a node or a focus according as $\Delta = (b + c_1)^2 + 4am \gtrless o$. In case $\Delta > o$, the directions along which the motions tend to the origin are given by

$$au^2 + (c_1 - b)\, u - \left(\frac{bc_1}{a} + m \right) = o$$

This is obtained by putting $\dot{\Phi} = o$ in the expression

$$- r \; \dot{\Phi} = \sin \Phi \,(c_1 x + ay) - \cos \Phi \;\left(\left(\frac{bc_1}{a} + m\right) x + by\right)$$

and by writing $u = \tan \Phi$, where $\tan \Phi = \frac{y}{x}$. The two directions are called critical directions. A critical direction is called singular if it satisfies the equation

$$c_1 + au = o ,$$

otherwise it is an ordinary critical direction. Along the ordinary critical direction either enters an infinite number of trajectories or only one integral curve. The question whether along a particular ordinary direction enters a finite number of trajectories or only one can be decided by using Frommer' criterion [15].

On similar lines it is easy to prove the following theorem:

Theorem 6.4. If $b = o$, then the trivial solution of (6.1) is asymptotically stable in the large under conditions (6.5)

7. The stability in the large of $\frac{dx}{dt} = xh_1(y) + ay$, $\frac{dy}{dt} = bx + h_2(x)y$.

Let us consider the system

$$\begin{aligned} \frac{dx}{dt} &= xh_1(y) + ay \\ \frac{dy}{dt} &= bx + h_2(x)\, y \end{aligned} \tag{7.1}$$

We assume that the right hand sides of (7.1) satisfy conditions which guarantee the existence and uniqueness of every solution.

We first prove the following theorem:

Theorem 7.1. If $ab < o$ and $h_1(y) \leq o$, $y \neq o$; $h_2(x) \leq o$, $x \neq o$ (at least in one of these conditions the strict inequality is satisfied), then the trivial solution of (7.1) is asymptotically stable in the large.

Proof. We assume $a < o$, $b > o$. A Lyapunov function for (7.1) under the conditions of the theorem is

$$2V(x,y) = bx^2 - ay^2$$

Its total time derivative in view of (7.1) is

$$\begin{aligned} \dot{V} &= bx(xh_1(y) + ay) - ay\,(bx + h_2(x)\, y) \\ &= bx^2\, h_1(y) - ay^2\, h_2(x) < o \quad \text{for } x \neq o, \quad y \neq o \\ &\qquad\qquad = o \text{ possibly on } x = o \text{ or } y = o \end{aligned}$$

Obviously, $V(x,y)$ is an infinitely large positive definite function and $x = o$ or $y = o$ does not contain any positive half trajectory of (7.1) except the origin. Hence all the conditions of Theorem 1.2.5 are satisfied which proves the above theorem.

Next we consider the system (7.1) under conditions

$$h_1(y) + h_2(x) < o, \; h_1(y)\, h_2(x) - ab > o \tag{7.2}$$
$$\text{for } x \neq o \quad y \neq o$$

The following theorem can be proved in the same way as Theorem 4.1.

Theorem 7.2. If $ab > o$, then the trivial solution of (7.1) is asymptotically stable in the large under conditions (7.2).

The proof of Theorem 7.3 goes on similar lines as the proof of Theorem 6.3.

Theorem 7.3. If either $h_1(y) > o$ or $h_2(x) > o$ and conditions (7.2) are satisfied, then the trivial solution of system (7.1) is asymptotically stable in the large provided it is asymptotically stable in the small.

8. Remarks. There are some questions remaining to be answered in connection with the results we have obtained.

8.1. The first question arises in connection with Theorem 2.2.1. We showed that stability in the large holds if in addition to (2.2.2) - (2.2.5) either conditions A and B or A and C are satisfied. Is it not possible to derive necessary and sufficient conditions? In the author's view it is most unlikely. The second question which then naturally arises is this : what special form should $F(x,y)$ and $f(\sigma)$ have in order that the conditions could be necessary as well as sufficient? Having found this, the problem of boundaries and regions of stability could be discussed in those cases where the stability in the large does not hold.

8.2. We discussed the stability of the trivial solution of systems (6.1) and (7.1) using qualitative methods. We were not able to construct Lyapunov functions for the two systems. Is it possible to construct a Lyapunov function for the system

$$\frac{dx}{dt} = x\, h_1(y) + ay$$

$$\frac{dy}{dt} = f(x) + h_2(x)\, y$$

of which (6.1) and (7.1) are particular cases under suitable conditions on the right hand sides of the above system?

It is the author's aim to investigate these questions in the future.

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