RADICALS IN NEAR-RINGS

by

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ABSTRACT

An algebraic system which satisfies all the ring axioms with the possible exceptions of commutativity of addition and the right distributive law is called a near-ring. This thesis is intended as a survey of radicals in near-rings, and an organization of the theory which has been developed to date.

Because of the absence of the right distributive law, the zero element of a near-ring need not annihilate the near-ring from the left. If we impose the condition that \( 0 \cdot p = 0 \) for all elements \( p \) of a near-ring \( P \), then we call \( P \) a C-ring. This condition is ensured if we demand that the near-ring \( P \) be generated, as an additive group, by a set \( S \) of elements of \( P \) such that

\[
(p_1 + p_2)s = p_1s + p_2s
\]

for all \( p_1, p_2 \) in \( P \), and \( s \) in \( S \). In this case, \( P \) is said to be distributively generated by \( S \).

The work is divided into three main sections; the first deals with general near-rings, the second with C-rings, and the third with distributively generated near-rings.

Appendix I gives a proof of a vital result for distributively generated near-rings, due to Laxton [11]; appendix II introduces a little used radical due to Deskins [6]; appendix III is included as a concrete example of a near-ring and its theory, due to Berman and Silverman [2].
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Let $G$ be an additive group* and $T(G)$ the set of all transformations of $G$. We define an addition and multiplication on the elements of $T(G)$ as follows:

\[
\begin{align*}
g(t_1 + t_2) &= gt_1 + gt_2 & \text{for all } g \in G, \ t_1, t_2 \in T(G) \\
g(t_1 \cdot t_2) &= (gt_1)t_2
\end{align*}
\]

Clearly, $T(G)$ forms a group under addition and a semigroup under multiplication, and the multiplication is left distributive over addition. i.e.: $t_1(t_2 + t_3) = t_1t_2 + t_1t_3$. $T(G)$ is not, however, an abelian group under addition unless $G$ is, and multiplication is not right distributive over addition. (For example, if $G$ is the additive group of the integers, and $t_1 = t_2 = \text{the identity transformation}$, $t_3$ the transformation such that $zt_3 = 1$ for all $z \in G$, then $(t_1 + t_2)t_3 = t_3$, but $t_1t_3 + t_2t_3 = t_3 + t_3 \neq t_3$).

This model gives rise to the following:

1.1 **DEFINITION** Let $P$ be a non-empty set with an addition (+) and a multiplication (\cdot) defined on it. $P$ is a near-ring if and only if:

1. $P$ is a group under addition
2. $P$ is a semigroup under multiplication
3. Multiplication is left distributive over addition.

As usual, we denote by 0 the additive identity of $P$. We define $P^+$ to be the additive group of $P$, and $P^*$ the multiplicative semigroup.

The absence of the right distributive law produces certain anomalies in the arithmetic of near-rings. For instance, the zero of a near-ring $P$ annihilates all the elements of $P$ from the right,
but not necessarily from the left. If a near-ring contains an identity 1, its additive inverse \(-1\) may not commute with all the elements of \(P\).

1.2 DEFINITION Let \(P\) be a near-ring, \(S \subseteq P\).

(i) \(S\) is a sub-near-ring of \(P\) if and only if \(S\) is a near-ring relative to \(+\) and \(\cdot\).

(ii) the set \(a + S = \{a + s | s \in S\}\) is called the left coset of \(P\) modulo \(S\) determined by \(a\).

(iii) \(S\) is (left) right invariant in \(P\) if and only if \((aS \subseteq S)\) \(Sa \subseteq S\) for all \(a \in P\).

(iv) A sub-near-ring \(Q\) of \(P\) is normal in \(P\) if and only if \(Q^+\) is a normal subgroup of \(P^+\).

(v) \(S\) is invariant in \(P\) if and only if \((a + s_1)(b + s_2) - ab \in S\) for all \(a, b \in P, s_1, s_2 \in S\).

(vi) \(S\) is an ideal of \(P\) if and only if \(S\) is an invariant normal sub-near-ring of \(P\).

1.3 DEFINITION An element \(a \in P\) is right distributive if and only if \((b + c)a = ba + ca\) for all \(b, c \in P\). \(P\) is right distributive if and only if every element of \(P\) is right distributive.

1.4 PROPOSITION (i) If a near-ring \(P\) is right distributive, and every element of \(P\) is a product of two elements, (e.g. if \(P\) has an identity) then \(P\) is a ring. (ii) If \(P\) is a near-ring with commutative multiplication, and every element of \(P\) is a product of two elements, (e.g. \(P\) has an identity) then \(P\) is a ring.

PROOF Let \(a = a_1a_2, b = b_1b_2 \in P\). Then we have

\[(a_1 + b_1)(b_2 + a_2) = (a_1 + b_1)b_2 + (a_1 + b_1)a_2 = a_1b_2 + b + a + b_1a_2.\]

Also, \((a_1 + b_1)(b_2 + a_2) = a_1(b_2 + a_2) + b_1(b_2 + a_2) = a_1b_2 + a + b + b_1a_2\), so that \(a_1b_2 + b + a + b_1a_2 = a_1b_2 + a + b + b_1a_2\), and part (i) is proven. Part (ii) is proven in a similar way.
1.5 **DEFINITION** An element $a \in P$ is anti-right distributive if and only if $(b+c)a = ca + ba$ for all $b, c \in P$. An element $a \in P$ is weakly right distributive if and only if $a$ is a finite sum of right distributive and anti-right distributive elements.

1.6 **DEFINITION** An $R$-ring is a near-ring in which all the elements are weakly right distributive.

1.7 **PROPOSITION** If addition is commutative in a near-ring $P$, and $P$ is an $R$-ring, then $P$ is a ring.

1.8 **PROPOSITION** The left cosets of $P$ modulo an ideal $I$ form a near-ring (called the difference near-ring of $P$ modulo $I$ and written $P-I$) relative to the compositions $(a+I) + (b+I) = (a+b)+I$, and $(a+I)(b+I) = ab + I$.

This is a direct consequence of the definition of invariance, and, in fact, the whole motivation for that definition. Thus ideals of near-rings are reasonable generalizations of ideals of rings.

1.9 **DEFINITION** A homomorphism of a near-ring $P$ is a mapping $\phi$ of $P$ into a near-ring $Q$ such that for all $a, b \in P$

$$ (a+b)\phi = a\phi + b\phi, \text{ and } (ab)\phi = a\phi \cdot b\phi. $$

An isomorphism is a 1:1 homomorphism, and we write "$P \cong Q$" for "$P$ is isomorphic to $Q$".

1.10 **PROPOSITION** The mapping $\phi: a \mapsto a+I$ of $P$ onto $P-I$ is a homomorphism (called the natural homomorphism) of $P$ onto $P-I$.

1.11 **PROPOSITION** A subset $S$ of $P$ is an ideal of $P$ if and only if $S$ is the kernel of some homomorphism $\phi$ of $P$ into near-ring $Q$.

1.12 **PROPOSITION** Every homomorphic image of a near-ring $P$ is isomorphic to the difference near-ring $P-I$ where $I$ is the kernel.
of the homomorphism.

1.13 **Proposition** I is an ideal of a near-ring P if and only if I is left invariant normal sub-near-ring of P and \((a+i)b - ab \in I\) for all \(a, b, i \in P\).

We proceed towards a suitable definition of left ideals, right ideals, and left and right modules.

1.14 **Definition** An additive group \(G\) (not necessarily abelian) with elements of \(P\) as right [left] operators is called a right [left] \(P\)-group if and only if, for all \(g \in G, p_1, p_2 \in P\), \(g(p_1 \cdot p_2) = (gp_1)p_2\) \[p_1(p_2g) = (p_1p_2)g\]

1.15 **Definition** A group homomorphism \(\phi\) of a right [left] \(P\)-group \(G\) into a right [left] \(P\)-group \(H\) is called a right [left]-\(P\)-homomorphism if and only if \((g\phi)p = (gp)\phi\) \([p(\phi g) = \phi(pg)]\) for all \(g \in G, p \in P\).

Notice that if operating by \(p\) is defined to be multiplication by \(p\), then \(P^+\) is both a right and left \(P\)-group.

1.16 **Definition** The additive subgroups of \(P^+\) which are also right [left] \(P\)-groups are called right [left] \(P\)-modules.

The reader is cautioned that this definition differs considerably from that of ring modules.

These are simply the right [left] invariant subgroups of \(P^+\).

1.17 **Definition** A subset \(S\) of \(P\) is a right [left] ideal of \(P\) if and only if there is a right [left] \(P\)-homomorphism \(\phi\) of \(P^+\) onto some right [left] \(P\)-group \(G\), such that \(S = \ker(\phi)\).

1.18 **Proposition** A subset \(S\) of \(P\) is a right [left] ideal of \(P\) if and only if \(S\) is a normal subgroup of \(P^+\) and for all \(s \in S, p_1, p_2 \in P\), \((s+p_1)p_2 - p_1p_2 \in S\) \([p_1s \in S]\).

**Proof** Let \(S\) be a right ideal of \(P\). Then \(S\) is the kernel of a right \(P\)-homomorphism \(\phi : P^+ \to G\), where \(G\) is some right \(P\)-group.
Let $s \in S$, $p_1, p_2 \in P$.

Then 
\[
((s+p_1)p_2 - p_1p_2) \phi = ((s+p_1)p_2) \phi - (p_1p_2) \phi \\
= (s+p_1) \phi \cdot p_2 - p_1 \phi \cdot p_2 \\
= (0 + p_1 \phi) p_2 - p_1 \phi \cdot p_2 = 0
\]

Thus we have $(s+p_1)p_2 - p_1p_2 \in S$, and since $S$ is the kernel of a homomorphism of $P^+$, $S$ is a normal subgroup of $P^+$.

Conversely, suppose $S$ is a normal subgroup of $P^+$ such that 
\[
(s+p_1)p_2 - p_1p_2 \in S \text{ for all } s \in S, \ p_1, p_2 \in P.
\]

Consider $\phi$, the natural homomorphism of $P^+$ onto $P^+ - S$. If we define $(p+S)p'_1 = pp'_1 + S$ for all $p, p'_1 \in P$, then we need only check independence of representative to show that $P^+ - S$ is a right $P$-group.

Let $p+S = p' + S$. Then $p' = s + p$ for some $s \in S$. Thus $(p' + S)p'_1 = p'p_1 + S = (s + p)p_1 + S$. But $(s+p)p_1 - pp_1 \in S$ for all $p, p_1 \in P$, $s \in S$.

So we have $(p' + S)p'_1 = (s + p)p_1 + S = pp_1 + S = (p + S)p'_1$, and $P^+ - S$ is a right $P$-group.

$S$ is the kernel of $\phi$, and $\phi$ is trivially a right $P$-homomorphism, so that $S$ is a right ideal.

The proof for left ideals is quite similar and is omitted.

1.19 **PROPOSITION** A subset $S$ of $P$ is an ideal if and only if $S$ is both a left and a right ideal of $P$.

This is immediate from propositions 1.13 and 1.18.

1.20 **PROPOSITION** The group sum of two ideals [right ideals, left ideals] is an ideal [right ideal, left ideal].

**PROOF** Let $R_1$ and $R_2$ be right ideals of a near-ring $P$. Let $r_1 \in R_1$ and $r_2 \in R_2$ be any two elements. Then we have $(p+(r_1 + r_2)p' - pp' = (p+(r_1 + r_2))p' - (p+r_1)p' + (p+r_1)p' - pp'$. Since 
\[
((p+r_1)+r_2)p' - (p+r_1)p' \in R_2, \text{ and } (p+r_1)p' - pp' \in R_1, \text{ we have}
\]
\[(p + (r_1 + r_2))p' - pp' \in R_2^+ R_1 = R_1 + R_2, \text{ so that } R_1 + R_2 \text{ is a right ideal of } P.\]

1.21 **DEFINITION** The ideal generated by a subset \(A\) of a near-ring \(P\) is the intersection of all the ideals of \(P\) which contain \(A\), and is written \((A)\).

1.22 **DEFINITION** The product of two ideals \(A\) and \(B\) of \(P\) is the ideal generated by the set of all products \(ab\) with \(a \in A\), \(b \in B\).

### II PRIME IDEALS AND NIL RADICALS

2.1 **DEFINITION** \(I\) is a prime ideal of \(P\) if and only if for any ideals \(A\) and \(B\) of \(P\) such that \(A \not\subseteq I\) and \(B \not\subseteq I\), we have \(AB \not\subseteq I\).

2.2 **PROPOSITION** The following four conditions are equivalent:

1. \(I\) is a prime ideal of \(P\)
2. if \(a, b, k \in I\), then \((a)(b) \not\in I\)
3. if \(a, b, k \in I\), then there is \(a' \in (a), b' \in (b)\) such that \(a'b' \not\in I\).
4. if \(A\) and \(B\) are ideals of \(P\) which properly contain \(I\), then \(AB\) is a non-zero ideal of \(P = P - I\).

2.3 **DEFINITION** A set \(M\) of elements of \(P\) is an \(m\)-system if and only if for \(a, b \in M\), there is \(a' \in (a), b' \in (b)\) such that \(a'b' \in M\).

(Notice that the empty set is an \(m\)-system).

From proposition 2.2 it is clear that an ideal is prime if and only if its complement is an \(m\)-system.

2.4 **DEFINITION** A set \(S\) of elements of \(P\) is called an \(s\)-system if and only if \(S\) contains a set \(S^*\) with \(S^* S^* \subseteq S^*\) such that for all \(p \in S\), we have \((p) \cap S^* \neq \emptyset\). Again, \(\emptyset\) is clearly an \(s\)-system. We write \(S(S^*)\) for the \(s\)-system \(S\) with kernel \(S^*\).

Clearly, any \(s\)-system is an \(m\)-system.
2.5 **DEFINITION** An ideal \( I \) of \( P \) is an \( s \)-prime ideal if and only if its complement in \( P \) is an \( s \)-system.

By the remark preceding the definition, an \( s \)-prime ideal is clearly a prime ideal.

2.6 **THEOREM** Let \( M[S(S^*)] \) be an \( m \)-system \( [s \)-system\] \) in \( P \), and \( A \) an ideal of \( P \) which does not meet \( M[S(S^*)] \). Then \( A \) is contained in an ideal \( I^* \) which is maximal with respect to excluding \( M[S(S^*)] \).

The ideal \( I^* \) is a prime \([s \)-prime\] ideal of \( P \).

**PROOF** In both cases, the existence of \( I^* \) is assured by Zorn's lemma.

(1) Let \( I^* \) be an ideal containing \( A \), \( I^* \) maximal with respect to excluding \( M \). We show that \( I^* \) is prime. Let \( B, C \) be ideals of \( P \) such that \( B \not\subseteq I^* \), \( C \not\subseteq I^* \). Then by maximality of \( I^* \), there is \( b \) in \( B \cap M \), \( c \) in \( C \cap M \), and \((b)(c) \subseteq BC \). Now since \( M \) is an \( m \)-system there is \( b_1 \in (b), c_1 \in (c) \) such that \( b_1c_1 \in M \). But \( b_1c_1 \in (b)(c) \subseteq BC \). Thus \( BC \not\subseteq I^* \), so that \( I^* \) is a prime ideal.

(2) Let \( I^* \) be maximal with respect to excluding \( S(S^*) \).

We show that \( I^* \) is \( s \)-prime. Let \( S_1 = P \setminus I \) \( (\text{set theoretic difference}) \), and \( S^*_1 = \{p | p \equiv s^*(1) \text{ for some } s^* \in S^* \} \). Clearly \( S^*_1 \cap I^* = \emptyset \), and if \( p_1, p_2 \in S^*_1 \), then \( p_1 \equiv s^*_1 (I^*) \), \( p_2 \equiv s^*_2 (I^*) \), \( s^*_1, s^*_2 \in S^* \), then since \((p_1-s^*_1)(p_2-s^*_2) \in (I^*)\), we have

\[ (-p_1 + (p_1 - s^*_1))(-p_2 + (p_2 - s^*_2)) - (-p_1)(-p_2) \in I^* \quad (\text{cf. def. 2(vi)}) \]

so that \((-s^*_1)(s^*_2) = p_1p_2 \in I^* \), whence \( p_1p_2 = s^*_1s^*_2 \quad (I^*) \)

and we have \( p_1p_2 \in S^*_1 \).

Let \( a \in S_1 \). We show that \((a) \cap S^*_1 \neq \emptyset \). Since \( I^* \) is maximal, and \( a \not\subseteq I^* \), we have \((I^*, a) \cap S^* \neq \emptyset \). Thus \((I^*, a) \cap S^* \neq \emptyset \): we choose \( s^* \in (I^*, a) \cap S^* \).
But \((I^*,a) \subseteq I^* + (a)\), so that \(s^*\) can be written as \(i^* + a^*\), \(i^* \in I^*\), \(a^* \in (a)\). We have then
\[ s^* = i^* + a^* \]
so that \(a^* = s^*(I^*)\). Thus \(s^* \in S^*_I \cap (a)\), and the proof is complete.

2.7 **DEFINITION** Let \(I\) be an ideal of \(P\),
\[ S(I) = \{ p \in P | \text{if } S \text{ is an } s\text{-system and } p \in S, \text{ then } S \cap I \neq \emptyset \} \]
\[ M(I) = \{ p \in P | \text{if } M \text{ is an } m\text{-system and } p \in M, \text{ then } M \cap I \neq \emptyset \} \]
We call \(S(I)\) the upper radical of \(I\), \(M(I)\) the lower radical of \(I\).

2.8 **THEOREM** The upper [lower] radical of an ideal \(I\) of \(P\) is the intersection of all the \(s\)-prime [prime] ideals of \(P\) which contain \(I\).

**PROOF** First we prove the theorem for the upper radical.
Clearly \(I \subseteq S(I)\). Let \(J\) be an \(s\)-prime ideal, \(I \subseteq J\), and \(s\) an element of \(S(I)\). If \(s \notin J\), then \(s\) is in \(P \sim J\), which is an \(s\)-system. Thus \(P \sim J\) must intersect \(I\), but \(J \trianglerighteq I\), so that \((P \sim J) \cap I = \emptyset\). We conclude that \(s \notin J\). We have shown that any \(s\)-prime ideal which contains \(I\) must also contain \(S(I)\), and hence that \(S(I)\) is contained in the intersection of these \(s\)-prime ideals.

Conversely, let \(p \in P \sim S(I)\). Then there is an \(s\)-system \(S\) which contains \(p\) but does not intersect \(I\). By the preceding theorem, there is an \(s\)-prime ideal containing \(I\) which does not meet \(S\). That is, \(p \notin S\). We have shown that any element in the intersection of the \(s\)-prime ideals containing \(I\), must also be in \(S(I)\).

The proof for the lower radical is exactly analogous to the proof for the upper radical.

**COROLLARY 1** The upper and lower radicals of an ideal are ideals.

**COROLLARY 2** For any ideal \(I\) of \(P\), we have \(M(I) \subseteq S(I)\).

2.9 **THEOREM** The upper radical of an ideal \(I\) of \(P\) is a nil ideal modulo \(I\). Also,
\[ S(I) = \{ p \mid (p) \text{ is a nil ideal modulo } I \} \]

**PROOF** If \( a \in S(I) \), the \( S \)-system \( \{a, a^2, a^3, \ldots \} \) contains \( a \), and must therefore intersect \( I \). Thus \( S(I) \) is a nil ideal modulo \( I \), and every element of \( S(I) \) generates a nil ideal modulo \( I \).

Conversely, if \( a \notin S(I) \), then there is an \( S \)-system \( S(S^*) \) which contains \( a \) and does not meet \( I \). Thus there is an element \( a^* \in (a) \cap S^* \), and since \( S^* \) is a multiplicative system, we see that \( a^{*n} \in S^* \) for all \( n \), and that consequently \( (a) \) is not a nil ideal modulo \( I \).

2.10 **COROLLARY 1** \( M(I) \) is a nil ideal modulo \( I \).

2.11 **COROLLARY 2** The sum of any collection of ideals which are nil ideals modulo an ideal \( I \) is a nil ideal modulo \( I \).

2.12 **DEFINITION** A prime ideal \( J \) of \( P \) is a minimal prime ideal belonging to the ideal \( I \) if and only if \( I \subseteq J \) and there are no prime ideals \( J' \) properly contained in \( J \) with \( I \subseteq J' \).

2.13 **LEMMA** The intersection of a descending chain of prime ideals is a prime ideal.

**PROOF** Let \( J \) be the intersection of a descending chain \( \Sigma \) of prime ideals. Suppose \( a, b \notin J \). Then there is \( J_1 \in \Sigma \) with \( a, b \notin J_1 \). Thus \( (a)(b) \neq 0 (J_1) \), and hence \( (a)(b) \neq 0 (J) \).

2.14 **THEOREM** The lower radical \( M(I) \) of an ideal \( I \) of \( P \) is the intersection of all the minimal prime ideals belonging to \( I \).

**PROOF** By Zorn's lemma, and lemma 2.13, we know that any prime ideal containing \( I \) contains a minimal prime ideal belonging to \( I \). The theorem follows from Theorem 2.8.

2.15 **DEFINITION** An \( S \)-prime ideal \( I \) is a quasi-minimal \( S \)-prime ideal belonging to the ideal \( I \) if and only if \( I \subseteq J \), and there is a kernel \( S^* \) for the \( S \)-system \( P \sim J \) such that if \( S^*_1 \) is any multiplicative system properly containing \( S^* \), then \( S^*_1 \) meets \( I \).
2.16 **PROPOSITION** Let I be an ideal of P, \( S(S^*) \) an s-system which does not meet I, and \( S_1^* \) a maximal multiplicative system containing \( S^* \) which does not meet I. If \( S_1 = \{ x \mid (x) \cap S_1^* \neq \emptyset \} \), then \( S_1(S_1^*) \) is an s-system which does not meet I, and \( P - S \) is an s-prime ideal which contains I.

**PROOF** Clearly \( S_1(S_1^*) \) is an s-system, and does not meet I. Now suppose \( J' \) is a maximal s-prime ideal which contains I and does not meet \( S_1 \). (cf. Theorem 2.6). Then \( S_2 = P - J' \) is an s-system which contains \( S_1 \). Also, if we define \( S_2^* = \{ x \in P \mid x \equiv s_1^* (I) \text{ for some } s_1 \in S_1^* \} \), as in Theorem 2.6, then \( S_2^* \) is a kernel of \( S_2 \), and \( S_2^* \supseteq S_1^* \). But \( S_2^* \) does not meet I, and hence by maximality of \( S_1^* \), we have \( S_2^* = S_1^* \). Now by construction of \( S_1 \), we have \( S_2 = S_1 \).

2.17 **THEOREM** Let I be an ideal of P, J an s-prime ideal containing I. Then J contains a quasi-minimal s-prime ideal belonging to I, and hence \( S(I) \) is the intersection of all the quasi-maximal s-prime ideals belonging to I.

**PROOF** Immediate from Proposition 2.16 and Theorem 2.8

2.18 **DEFINITION** An ideal I of a near-ring P is a nil radical of P if and only if:

(i) I is a nil ideal and

(ii) if J is an ideal of P such that \( J \not\subseteq I \), then J is not nilpotent modulo I.

2.19 **THEOREM** Both \( S(0) \) and \( M(0) \) are nil radicals of P.

**PROOF** We have already shown that \( S(0) \) and \( M(0) \) are nil ideals. We show that an ideal of P not contained in \( S(0) \) is not nilpotent modulo \( S(0) \) by showing that if \( p \not\in S(0) \), then \( (p) \) is not nilpotent modulo \( S(0) \). If \( p \not\in S(0) \), then there is a prime (even an s-prime) ideal J of P such that \( p \not\in J \). Thus \( (p) \) is not nilpotent.
modulo \( J \), and since \( J \supseteq S(0) \), \((p)\) is not nilpotent modulo \( S(0) \).
The other part of the theorem is proven in exactly the same way.

Since \( S(0) \) is a nil radical of \( P \), and contains all the nil ideals of \( P \), we define the upper nil radical of \( P \):

2.20 **DEFINITION** The upper nil radical of a near-ring \( P \) is defined to be \( S(0) \).

2.21 **LEMMA** Let \( I \) be an ideal of \( P \), \( \bar{P} = P - I \), and \( a \in P \). Then \((\bar{a}) = \bar{(a)}\).

**PROOF** Let \( \gamma : P \rightarrow \bar{P} \) be the natural homomorphism, \( \gamma a = \bar{a} \).
Then \( \bar{(a)} = \gamma (a) \) is an ideal containing \( \bar{a} \), so that \( \bar{(a)} \supseteq \bar{(a)} \). Also, \( \gamma^{-1}(\bar{a}) \) is an ideal containing \( a \), so that \( \gamma^{-1}(\bar{a}) \supseteq (a) \), whence we have \( \gamma \gamma^{-1}(\bar{a}) \supseteq \gamma (a) \), and thus \( \bar{(a)} \supseteq \bar{(a)} \). This completes the proof.

2.22 **DEFINITION** If \( 0 \) is an s-prime [prime] ideal of \( P \), we call \( P \) an s-prime [prime] near-ring.

2.23 **LEMMA** Let \( P \) be a near-ring, \( I \) an ideal of \( P \), \( \bar{P} = P - I \). Then \( \bar{P} \) is an s-prime near-ring if and only if \( I \) is an s-prime ideal of \( P \).

**PROOF** Let \( \bar{P} \) be an s-prime near-ring, \( Q(Q^*) = \bar{P} \cong \{0\} \). Let \( S = P - I \), \( S^* \) the pre-image of \( Q^* \) under the natural homomorphism.

Then \( S^* \) is clearly a multiplicative system, and since \( \emptyset \notin Q^* \), we have \( S^* \cap I = \emptyset \). That is, \( S^* \subseteq S \). Now if \( a \in P - I \), we have \( \bar{a} \neq \emptyset \), so that \( \bar{a} \notin Q \), and hence \( (\bar{a}) \cap Q^* \neq \emptyset \). Let \( \bar{a}^* \in (\bar{a}) \cap Q^* \). Then since \( \bar{a}^* \in Q^* \), we have \( a^* \in S^* \). Since \( \bar{a}^* \in (\bar{a}) = (a) \), we have \( a^* + 1 \in (a) \) so that \( a^* + i = a^*_i \in (a) \). Since \( \bar{a}^*_i = a^*_i = a^*_i \), we have \( a^*_i \in S^* \). Thus \( a^*_i \in (a) \cap S^* \), and \( I \) is an s-prime ideal of \( P \).

Now if \( I \) is an s-prime ideal of \( P \), \( P - I = S(S^*) \), \( Q^* = S^* \), \( Q = P - \emptyset \).
Then if \( \bar{a} \notin \emptyset \), \( \bar{a} \neq \emptyset \), we have \( a \notin I \), so that \( (a) \cap S^* \neq \emptyset \). Thus \((\bar{a}) \cap S^* \neq \emptyset \), \((\bar{a}) \cap Q^* \neq \emptyset \), completing the proof.

2.24 **COROLLARY** Let \( P \), \( P' \) be near-rings, \( I \) an ideal of \( P \) and \( I' \) an ideal of \( P' \) such that \((P - I) \cong (P' - I') \).

Then if \( I \) is an s-prime ideal of \( P \),...
then $I'$ is an s-prime ideal of $P'$.

**Proof** Let $\phi: P-I \cong P'-I'$, $S(S^*) = (P-I) \sim \{0\}$, $Q = (P'-I') \sim \{0\}$, $Q^* = S^* \phi$. Let $\bar{a} \in Q$. Then $(\bar{a}) \cap S^* \neq \emptyset$ so that $(\bar{a}) \phi \cap S^* \phi \neq \emptyset$ and thus $(\bar{a}) \phi \cap Q^* \neq \emptyset$. Since $Q(Q^*)$ is an s-system, we conclude by the lemma that $I'$ is an s-prime ideal of $P'$.

2.25 **Lemma** Let $I$ be an ideal of a near-ring $P$, $\bar{P} = P-I$. Then $I$ is a prime ideal of $P$ iff $\bar{P}$ is a prime near-ring.

**Proof** Suppose $I$ is a prime ideal of $P$, and $A, B$ are non-zero ideals of $P$. Then $A, B$ are ideals of $P$ which properly contain $I$. Thus we have

$$AB \notin I, \quad \bar{AB} \neq \bar{0}$$

$$\bar{AB} = (\{ab | a \in A, b \in B\})$$

$$= (\{\bar{a}\bar{b} | a \in \bar{A}, b \in \bar{B}\}) \quad \text{(by an extension of 2.21)}$$

$$= (\{\bar{a}\bar{b} | a \in \bar{A}, b \in \bar{B}\}) = \bar{AB}$$

whence $\bar{A} \cdot \bar{B} \neq \bar{0}$, and $\bar{P}$ is thus a prime near-ring.

Now if $\bar{P}$ is a prime near-ring, and $A, B$ are ideals of $P$ not contained in $I$, then $\bar{AB} \neq \bar{0}$, so that $\bar{AB} \neq \bar{0}$, and we have $AB \notin I$. Thus $I$ is a prime ideal.

As in 10.29, we can derive:

2.26 **Corollary** If $P, P'$ are near-rings, $I$ a prime ideal of $P$ and $I'$ an ideal of $P'$ such that $P-I \cong P'-I'$, then $I'$ is a prime ideal of $P'$.

We show next that $L = M(0)$ is the intersection of all the nil radicals of $P$, and thus can be called the lower nil radical of $P$.

2.27 **Theorem** Let $I$ be an ideal of $P$ such that if $J \notin I$ then $J$ is not nilpotent modulo $I$. Then $I$ is the intersection of all the prime ideals containing $I$.

**Proof** Let $a$ be an element of $P$ which is not in $I$. We show that there is a prime ideal of $P$ which contains $I$ but not $a$. This follows
from Theorem 2.6 once we have shown that \( a \) is contained in an \( m \)-system which does not meet \( I \).

Since \( a \not\equiv 0 \) (I), we have \((a)^2 \not\equiv 0 \) (I). Thus there are elements \( a_1, a_2 \in (a) \) such that \( a_1 a_2 \not\equiv 0 \) (I). By the same reasoning, there are elements \( a', a_2' \in (a_1 a_2) \) such that \( a_1' a_2' \not\equiv 0 \) (I). We obtain the sequence

\[
\begin{align*}
& a, a_1 a_2, a_1' a_2', \\
& (a) \supseteq (a_1 a_2) \supseteq (a_1' a_2') \supseteq \ldots
\end{align*}
\]

and none of these ideals is contained in \( I \). The set \( \{ a, a_1 a_2, a_1' a_2', \ldots \} \) is clearly an \( m \)-system which does not meet \( I \).

2.28 **Theorem** Let \( U \) be the upper nil radical of \( P \), and \( L \) the lower nil radical. Then the difference near-ring \( P-U \) has zero upper nil radical, and the difference near-ring \( P-L \) has zero lower nil radical

**Proof** Let \( \bar{a} \) be in the upper nil radical of \( P-U \). Then \( \bar{a} \) is contained in all the \( s \)-prime ideals of \( P-U \). If \( \bar{a} \not\equiv \emptyset \), then \( a \not\equiv 0 \) (U), and hence \( a \) is not contained in some \( s \)-prime ideal \( J \) of \( P \). Since \( J \supseteq U \), we have \((P-J) \equiv (P-U)-(J-U) \) and by Corollary 2.24, \( J-U \) is an \( s \)-prime ideal of \( P-U \). Moreover, \( J-U \) does not contain \( \bar{a} \) since \( a \not\equiv 0 \) (J). Since we assume that \( \bar{a} \) is contained in all the \( s \)-prime ideals of \( P-J \), we have a contradiction. Thus we must have \( \bar{a}=\emptyset \), which completes the proof. The proof for the lower nil radical is almost identical, and is omitted.

2.29 **Theorem** A near-ring \( P \) of which the upper radical [lower radical] is zero is isomorphic to a subdirect sum of \( s \)-prime [prime] near-rings.

The proof of this statement is identical to the proof of the corresponding theorem in ring theory. It depends upon the fact that an ideal is the kernel of a homomorphism, which is true of both rings and near-rings.
III  C-RINGS:  THE RADICAL  \( J(P) \)

3.1 **DEFINITION**  A right \( P \)-group \( V \) is called a right representation space of \( P \), or simply a right \( P \)-space, if and only if \( v(p_1 + p_2) = vp_1 + vp_2 \) for all \( v \in V \), \( p_1, p_2 \in P \).

3.2 **DEFINITION**  A right \( P \)-space \( V \) is proper if and only if \( VP \neq 0 \).

3.3 **DEFINITION**  An irreducible right \( P \)-space is a proper right \( P \)-space with no non-trivial \( P \)-subspaces.

We point out that the right modules of \( P \) are right \( P \)-spaces.

3.4 **DEFINITION**  Let \( V \) be a right \( P \)-space, and \( v \) an element of \( V \).

Then \( A_v = \{ p \in P | vp = 0 \text{ for all } v \in V \} \)

3.5 **PROPOSITION**  If \( V \) is a right \( P \)-space and \( v \in V \), then \( A_v \) is an ideal in \( P \), and \( A_v \) a right ideal in \( P \).

The proof is trivial from the definitions of ideal and right ideal.

3.6 **DEFINITION**  A near-ring \( P \) is a C-ring if and only if \( 0 \cdot p = 0 \) for all \( p \in P \).

3.7 **PROPOSITION**  Any ideal of a C-ring \( P \) is both left and right invariant in \( P \).

**PROOF**  Left invariance is already proven. Let \( K \) be an ideal of a C-ring \( P \). Then \( K \) is invariant in \( P \), so that we have

\[(k_1 + p_1)(k_2 + p_2) - p_1p_2 \in K \text{ for all } k_1, k_2 \in K, p_1, p_2 \in P. \]

Since \( K \) is a normal additive subgroup of \( P \), this is equivalent to:

\[(p_1 + k_1)(p_2 + k_2) - p_1p_2 \in K \text{ for all } \ldots. \]

In particular,

\[(k_1 + 0)(0 + p_2) - 0p_2 \in K \text{ for all } k_1 \in K, p_2 \in P. \text{ i.e. } kp \in K \text{ for all } k \in K, p \in P. \]

For the remainder of this paper we restrict our attention to C-rings. Any exception to this will be explicitly noted.
3.8 **THEOREM** Let $P$ be a near-ring, $V$ a right $P$-space, $\phi$ a right $P$-homomorphism of $P^+$ onto $V$ with $\ker(\phi) = K$. If we define $(p+K)p' = pp'+K$ for all $p, p' \in P$, then $P^+ - K$ is a right $P$-space and the mapping

$$\gamma: p + K \to p\phi$$

is a right $P$-isomorphism of $P^+ - K$ onto $V$. The proof is clear.

3.9 **DEFINITION** If $V, V'$ are right $P$-spaces, then we write $V \cong V'$ for "$V$ is right $P$-isomorphic to $V'$".

3.10 **THEOREM** Let $\eta$ be a right $P$-homomorphism of $P^+$ onto the right $P$-space $V$, with $\ker(\eta) = K$. Let $R$ be a right ideal of $P$ containing $K$, and $R' = R\eta$. Then if we define $(v+R')p = vp + R'$ for all $v \in V, p \in P$, then $V-R'$ is a right $P$-space. If $\phi$ is the natural homomorphism of $V$ onto $V-R'$, then $\eta \phi$ is a right $P$-homomorphism of $P^+$ onto $V-R'$, and by theorem 3.8, $P^+ - R$ is right $P$-isomorphic to $V-R'$.

**PROOF** We show first that the product $(v+R')p = vp+R'$ is well defined. (i.e. independent of representative). Suppose $v + R' = v' + R'$, $v, v' \in V$. Then $v' = v + r'$, for some $r' \in R'$. Now

$$(v+R')p' = ((v+r')+R')p'$$

for all $p' \in P$, if and only if $(v+r')p' - vp' \in R'$ for all $p' \in P$. But $\eta$ maps $P^+$ onto $V$, and $R' = R\eta$.

Thus $(v+r')p' - vp' = (p \eta + r \eta)p' - p \eta p' = (p+r)p', p \in P, r \in R$.

$$= (p+r)p' \eta = pp' \eta = ((p+r)p' - pp') \eta \in R$$

since $R$ is a right ideal of $P$. Thus $(v+r')p' - vp' \in R = R'$ for all $p' \in P$, and the product is well defined.

It is clear now that $\eta \phi$ is a right $P$-homomorphism of $P^+$. V-R', and $\ker(\eta \phi) = R$, so that the rest of the theorem follows.

3.11 **COROLLARY** Let $\eta$ be the natural homomorphism of $P^+$ onto $P^+ - K^+$, where $K$ is a right ideal of $P$. Then if $R$ is a right ideal
of $P$ containing $K$, we have as a special case of the theorem

$$P^+ - R^+ \cong (P^+ - K^+) - (R^+ - K^+)$$

3.12 **THEOREM** Let $R_1$ and $R_2$ be right ideals of $P$. If we define (as usual) \(((r_1 + r_2) + R_2^+)p = (r_1 + r_2)p + R_2^+\), then \((R_1^+ + R_2^+) - R_2^+\) is a right $P$-space, and the mapping $\gamma : r_1 \rightarrow r_1 + R_2^+$ is a right $P$-homomorphism of $R_1^+$ onto \((R_1^+ + R_2^+) - R_2^+\) with $\ker(\gamma) = R_1^+ \cap R_2^+$. By theorem 3.8 we have $R_1^+ - (R_1^+ \cap R_2^+) \cong (R_1^+ + R_2^+) - R_2^+$ under the mapping

$$\phi : r_1^+ (R_1^+ \cap R_2^+) \rightarrow r_1^+ R_2^+.$$

**PROOF** We show that the product \(((r_1 + r_2) + R_2)p = (r_1 + r_2)p + R_2\) is well defined. Since $R_1$ is a right ideal, it is also a right $P$-module, so that $r, p \in R_1$ for all $r \in R_1$, $p \in P$. Since $R_2$ is a right ideal, we have $(r_1 + r_2)p = r_1p + r_2p$ for $r_1 \in R_1$, $r_2 \in R_2$, $p \in P$. Hence \((r_1 + r_2)p = (r_1 + r_2)p - r_1p + r_1p \in R_1 + R_2\). (We have used the normality of $R_1^+$, $R_2^+$ as subgroups of $P$.)

Suppose now that \((r_1 + r_2) + R_2 = r_1' + r_2' + R_2\). Then \(r_1 + r_2 = r_1' + r_2' + r_2''\).

\((r_1' + r_2')p + R_2^+ = (r_1' + r_2' + r_2'')p + R_2\) for all $p \in P$

if and only if \((r_1' + r_2' + r_2'')p - (r_1' + r_2')p \in R_2^+$.

This follows from the right ideal property of $R_2$. The rest of the proof is clear.

3.13 **LEMMA** (Zassenhaus) Let $R_i$, $R'_i$ be right ideals of $P$ for $i = 1, 2$, such that $R'_i \subseteq R_i$. Then \((R_1^+ \cap R_2^+) + R_1^+\) is a normal subgroup of \((R_1^+ \cap R_2^+) + R_2^+\), and the corresponding difference groups (with the usual product definition) are right $P$-isomorphic.

**PROOF** For convenience we write $R_i$, $R'_i$ in place of $R_1^+$, $R_2^+$. We show first that \((R_1^+ \cap R_2^+) + R_1^+\) is a normal subgroup of \((R_1^+ \cap R_2^+) + R_1^+\).
Let $x \in R_1 \cap R_2$, $y \in R_1 \cap R'_2$, $z, t \in R'_1$

Since $-x + y + x \in R_1 \cap R'_2$, and $-x + z + x \in R'_1$, we have

$$-x + y + x - z + x \in (R_1 \cap R'_2) + R'_1,$$ so that

$$-x + (y + z) + x \in (R_1 \cap R'_2) + R'_1,$$ and thus

$$-x + (R_1 \cap R'_2) + R'_1 + x \subseteq (R_1 \cap R'_2) + R'_1 \quad (I)$$

Since $-t + y + t = -t + (y + t - y) + y \in R'_1 + (R_1 \cap R'_2) = (R_1 \cap R'_2) + R'_1$, we have

$$-t + (y + z) + t = -t + y + t - t + z + t \in (R_1 \cap R'_2) + R'_1,$$ so that

$$-t + (R_1 \cap R'_2) + R'_1 + t \subseteq (R_1 \cap R'_2) + R'_1 \quad (II)$$

From (I) and (II) we conclude $-t + x + (R_1 \cap R'_2) + R'_1 + x + t$ is contained in $(R_1 \cap R'_2) + R'_1$.

That is, $(R_1 \cap R'_2) + R'_1$ is normal in $(R_1 \cap R'_2) + R'_1$.

Now $(R_1 \cap R'_2) + R'_1$ is a right ideal of $P$, and $R_1 \cap R_2$ is a right ideal of $P$. Thus recalling theorem 3.12,

$$((R_1 \cap R'_2) + R'_1) \cap (R_1 \cap R_2)$$ is a normal subgroup of $(R_1 \cap R_2)$ and

$$(R_1 \cap R_2) - [(((R_1 \cap R'_2) + R'_1) \cap (R_1 \cap R_2)) \overset{P}{\approx} \left[ (R_1 \cap R_2) + (R_1 \cap R'_2) + R'_1 \right] - [(R_1 \cap R'_2) + R'_1]$$

$$= [(R_1 \cap R_2) + R'_1] - [(R_1 \cap R'_2) + R'_1] \quad (III)$$

Also $((R_1 \cap R'_2) + R'_1) \cap (R_1 \cap R_2) = ((R_1 \cap R'_2) + R'_1) \cap R_2 \quad (IV)$

and if $p \in (R_1 \cap R'_2) + R'_1$, then $p = y + z$ for some $y \in (R_1 \cap R'_2)$, $z \in R'_1$

if $p$ is also in $R_2$, then $z = y^{-1}yz \in R_2$, so that $z \in R_2 \cap R'_1$, and thus

$$y + z \in (R_1 \cap R'_2) + (R'_1 \cap R_2),$$ so that

$$((R_1 \cap R'_2) + R'_1) \cap R_2 \subseteq (R_1 \cap R'_2) + (R'_1 \cap R_2).$$
The reverse inclusion is clear, and we have
\[
((R_1 \cap R_2') + R_1') \cap R_2 = (R_1 \cap R_2') + (R_1' \cap R_2)
\] (V)

From (III), (IV), and (V) we have
\[
(P_{R_1 \cap R_2}) - [(R_1 \cap R_2') + (R_1' \cap R_2)] \equiv [(R_1 \cap R_2') + R_1'] - [(R_1 \cap R_2') + R_1] \quad (VI)
\]
and by symmetry
\[
(R_1 \cap R_2) - [(R_1 \cap R_2') + (R_1' \cap R_2)] \equiv [(R_1 \cap R_2') + R_1'] - [(R_2 \cap R_1') + R_2'] \quad (VII)
\]
The lemma follows now from (VI) and (VII).

3.14 **DEFINITION** Let \( P = R_1 \supseteq R_2 \supseteq ... \supseteq R_{s+1} = 0 \) be a sequence of right ideals of a near-ring \( P \). Such a sequence is called a normal series for \( P \), and the difference groups \( R_1 - R_2, R_2 - R_3, ..., R_s - R_{s+1} = R_s \) are called the factors of the normal series.

The two normal series
\[
P = R_1 \supseteq R_2 \supseteq ... \supseteq R_{s+1} = 0
\]
\[
P = Q_1 \supseteq Q_2 \supseteq ... \supseteq Q_{s+1} = 0
\]
are said to be equivalent if it is possible to set up a one to one correspondence between the factors of the two series such that the paired factors are right \( P \)-isomorphic.

We say that a normal series \( S \) is a refinement of a normal series \( T \) if every right ideal in \( T \) occurs in \( S \).

3.15 **THEOREM** (Schreier) Any two normal series for a near-ring \( P \) have equivalent refinements.

**PROOF** Let two normal series of \( P \) be given by
\[
P = R_1 \supseteq R_2 \supseteq ... \supseteq R_{s+1} = 0 \quad (VIII)
\]
\[
P = Q_1 \supseteq Q_2 \supseteq ... \supseteq Q_{s+1} = 0 \quad (IX)
\]
Set \( R_{ik} = (R_i \cap Q_k) + R_{i+1} \), \( k = 1, 2, ..., t+1 \)
\( Q_{ki} = (R_i \cap Q_k) + Q_{k+1} \), \( i = 1, 2, ..., s+1 \)
then \( P = R_1 \supseteq R_2 \supseteq \cdots \supseteq R_{t+1} = 0 \) is a refinement of (VIII) and

\[
P = Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_{s+1} = 0
\]

is a refinement of (IX).

By the lemma, we see that \( R_{ik} - R_{ik+1} \equiv Q_{ki} - Q_{k,i+1} \).

3.16 **DEFINITION** A normal series \( P = R_1 \supseteq \cdots \supseteq R_{s+1} = 0 \) is a composition series if every \( R_i \) is a maximal right ideal in \( R_{i-1} \).

3.17 **THEOREM** (Jordan Hölder) Any two composition series for a near-ring are equivalent.

**PROOF** By Schreier's theorem, the composition series have equivalent refinements, but it is clear from the definition of composition series that any refinement must have the same non-zero factors as the original series. In the one to one correspondence between the factors of the refinements, the zero factors are paired. Hence the non-zero factors are also paired. Since these are the same as the factors of the original series, the two composition series are equivalent.

3.18 **DEFINITION** For any near-ring \( P \), we define \( J(P) \), the radical of \( P \) to be the intersection of all the ideals of \( P \) which are annihilators of irreducible right \( P \)-spaces, and \( J(P) = P \) if \( P \) has no irreducible right \( P \)-spaces.

Notice that \( J(P) \) is an ideal of \( P \), since it is the intersection of a family of ideals.

3.19 **THEOREM** If \( P \) is a ring, then each irreducible right \( P \)-space \( V \) is commutative, and the mapping \( \phi : v \rightarrow \nu p \) is an endomorphism of \( V \) for each \( p \in P \). Thus \( V \) is an irreducible right \( P \)-module in the notation of [8].
PROOF Let \( V \) be an irreducible right \( P \)-space, and \( P \) a ring. Since \( VP \neq 0 \), there is an element \( \omega \in V \) such that \( \omega P \neq 0 \). Since \( V \) is irreducible, and \( \omega P \) is a non-zero sub-space of \( V \), we have \( \omega P = V \). Thus, the mapping \( \phi: P^+ \to V \), such that \( p\phi = \omega p \), is a homomorphism of \( P^+ \) onto \( V \), and \( \ker (\phi) = A_\omega \).

We have then, \( V \simeq P^+ - A_\omega \). Since \( P \) is a ring, \( P^+ - A_\omega \) is abelian and consequently \( V \) is abelian. To show that \( V \) is a right \( P \)-module, we need only show that \( (v_1 + v_2)p = v_1p + v_2p \) for all \( v_1, v_2 \in V, p \in P \).

We simply write \( v_1 = \omega p_1, v_2 = \omega p_2 \) since \( V = \omega P \). Then \( (v_1 + v_2)p = (\omega p_1 + \omega p_2)p = \omega (p_1 + p_2)p \) and since \( P \) is a ring, we have \( \omega (p_1 + p_2)p = \omega p_1 p + \omega p_2 p = v_1 p + v_2 p \).

It is an immediate consequence of this theorem that the radical \( J(P) \) of a near-ring \( P \) is precisely the Jacobson radical of \( P \) if \( P \) is a ring. (ref. [8]).

3.20 THEOREM The radical of the difference near-ring \( P-J(P) \) is the zero ideal.

PROOF Let \( V \) be an irreducible right \( P \)-space. If we define \( v(p + J(P)) = vp \) for all \( v \in V, p \in P \), then \( V \) is a right \( (P-J(P)) \)-space, and clearly irreducible. The annihilator of \( V \) as a right \( (P-J(P)) \)-space is \( A_v + J(P) = \{ p + J(P) \mid vp = 0 \text{ for all } v \in V \} \).
The intersection of all these annihilator ideals is $J(P)$, so we conclude that the radical of $P-J(P)$ is contained in $J(P)$ and is therefore the zero ideal.

3.21 **DEFINITION** A near-ring $P$ is semisimple if and only if $J(P) = 0$.

3.22 **DEFINITION** A near-ring $P$ is a D.C.C. near-ring if and only if $P$ satisfies the descending chain condition on right modules.

3.23 **THEOREM** Let $P$ be a D.C.C. near-ring. Then $P$ is semisimple if and only if $P$ has no non-zero nilpotent right modules.

**PROOF** Let $P$ be a semisimple D.C.C. near-ring and assume that $N$ is a non-zero nilpotent right $P$-module. Without loss of generality we can assume that $N$ is a minimal right $P$-module, and hence, that $N^2 = 0$ (since $N^2 \neq N$). Since $J(P) = 0$, and $J(P)$ is the intersection of all the annihilators of irreducible right $P$-spaces, we can conclude that $N$ is not contained in this intersection, and therefore $VN \neq 0$ for some irreducible right $P$-space $V$.

Now there is $\omega \in V$ such that $\omega N \neq 0$, and since $\omega N$ is a right $P$-space contained in $V$, we have $\omega N = V$. Thus $(\omega N)N \neq 0$. But $(\omega N)N = \omega(N \cdot N) = 0$. Thus contradiction proves necessity of the condition.

Conversely, if $J(P) \neq 0$, then $J(P)$ is a right ideal (also a left ideal) and thus a right module, by proposition 3.7. Therefore $J(P)$ contains a minimal right module $M$, which is also an irreducible right $P$-space, and $M^2 \leq M \cdot J(P) = 0$. Therefore $M$ is a nilpotent right module of $P$.

The following work, up to and including theorem 3.42 is due primarily to Blackett [4]. The radical $J(P)$ and theorem 3.23 connecting $J(P)$ with Blackett's work are both given by G. Betsch [3].
3.24 **Theorem** Let $P$ be a semisimple, D.C.C. near-ring. Then any non-zero right module $M$ of $P$ contains an idempotent element $e$, and if $M$ is a minimal non-zero right module of $P$, then $M$ contains an idempotent element $e$ such that $eP = eM = M$.

**Proof** Since $P$ is a D.C.C. near-ring, $M$ contains a minimal right $P$-module $N$, and the first part of the theorem follows from the second part.

Let $M$ be a minimal right $P$-module. Then $M^2 \neq 0$, by theorem 3.23 and therefore $MP \neq 0$. Thus $M$ is an irreducible right $P$-space.

Since $M^2 \neq 0$, there is an element $m_0 \in M$ such that $m_0M \neq 0$. Let $\tau$ be the mapping $m \rightarrow m_0m$. Then $\tau$ is a right $P$-homomorphism of $M$ such that $\tau(M) \neq 0$.

The kernel of $\tau$ is a right ideal, and hence a right module, so that $\ker(\tau) = 0$, by the minimality of $M$. Thus $\tau$ is a right $P$-isomorphism of $M$ onto $m_0M$. Again by minimality of $M$, we have $m_0M = M$, and $\tau : M \cong M$. Now there is $e \in M$ such that $m_0e = m_0$, so that $m_0e^2 = m_0e = m_0$, $e^2 = e\tau$, and $e^2 = e$.

Thus $eM \subseteq eP \subseteq MP \subseteq M$, and since $e^2 = e \neq 0$, we have $eM \neq 0$, so that $eM = M$.

3.25 **Theorem** A minimal non-zero right ideal $R$ of a D.C.C. semisimple near-ring $P$ is an irreducible right $P$-space.

**Proof** If $R$ is a right ideal of $P$, then $R$ is a right $P$-space (actually, a right $P$-module). If $M$ is a minimal non-zero right $P$-module contained in $R$, then $M^2 \neq 0$, and thus $m_0M \neq 0$ for suitable $m_0 \in M$. Since $m_0M$ is a right $P$-module contained in $M$, we have $m_0M = M$, so that the mapping $\tau : n \rightarrow m_0n$ is a right $P$-homomorphism of $N^+$ onto $M$. $\ker(\tau) = A_{m_0}$ is a right ideal, and since $0 \neq m_0M \subseteq m_0R$, we have $A_{m_0} \cap R = 0$. This means that $\tau|R$ is a right $P$-isomorphism of $R$ onto $M$. 
The three conditions: \( T : R \rightarrow M \) (1:1 and onto) \( T : M \rightarrow M \) (1:1 and onto) 

\[ M \subseteq R \]

together imply that \( M = R \), so that \( R \) is an irreducible right \( P \)-space.

3.26 **Theorem** A non-zero right module \( M \) of a semisimple D.C.C. near-ring \( P \) is a finite direct sum of minimal non-zero right modules contained in it.

**Proof** By the descending chain condition, \( M \) contains a minimal non-zero right module \( M_0 \), and by theorem 3.24, \( M_0 \) contains an idempotent element \( e_0 \) such that \( e_0 P = e_0 M_0 = M_0 \). If \( m \in M \), \( m = e_0 m + (-e_0 m + m) \).

We notice that \( e_0 m \in M_0 \), and \( (-e_0 m + m) \in A_{e_0} \), so that \( M \subseteq M_0 + A_{e_0} \).

Also, \( M_0 \cap A_{e_0} = 0 \), since \( e_0 m = m \) for all \( m \in M_0 \). If \( M = M_0 \) we are finished; if \( M \neq M_0 \), then \( M \cap A_{e_0} \neq 0 \), that is, \( M \) properly contains a non-zero submodule which is the intersection of \( M \) with a right ideal of \( P \). Similarly, \( M \cap A_{e_0} \) is a minimal non-zero submodule or it properly contains a smaller non-zero submodule of the form \( (M \cap A_{e_1}) \cap R \), where \( R \) is a right ideal of \( P \). Since \( A_{e_0} \cap R \) is a right ideal of \( P \), \( M \cap (A_{e_0} \cap R) \) is the intersection of \( M \) with a right ideal of \( P \). After a finite number of steps we arrive at \( M_1 \), a minimal non-zero submodule of \( M \) which is the intersection of \( M \) with a right ideal of \( P \). Since right ideals of \( P \) are normal subgroups of \( P^+ \), \( M_1 \) is a normal subgroup of \( P^+ \), and also of \( M \). Let \( e_1 \in M_1 \) such that \( e_1 P = e_1 M_1 \), and write \( M = M_1 \oplus (M \cap A_{e_1}) \), since both \( M_1 \) and \( M \cap A_{e_1} \) are normal subgroups of \( M \). Repeating this process, with \( M \) replaced by \( (M \cap A_{e_1}) \) and succeeding groups, we get \( M = M_1 \oplus \ldots \oplus M_s \).

The descending chain condition assures that the sum is finite.
3.27 **COROLLARY** A non-zero right ideal \( R \) of a semisimple D.C.C. near-ring \( P \) is a direct sum of minimal non-zero right ideals contained in it.

3.28 **THEOREM** Every right ideal \( R \) of a semisimple D.C.C. near-ring \( P \) contains an idempotent element \( e \) such that \( eR = R \).

**PROOF** Recalling the proof of theorem 3.20, we write
\[
R = e_1R \oplus \ldots \oplus e_pR \text{ where each } e_iR \text{ is a minimal right ideal, and } e_1R \oplus \ldots \oplus e_pR \text{ is contained in } A_{e_{i-1}}.
\]

We show first that
\[
(e_1 + \ldots + e_p)R = e_1R \oplus \ldots \oplus e_pR = R
\]
Assume as an induction hypothesis, that
\[
(e_1 + \ldots + e_s)R = e_1R \oplus \ldots \oplus e_sR.
\]
Since each of the \( e_iR \) is a right ideal,
\[
(e_1 + \ldots + e_s + e_{s+1})R = (e_1 + \ldots + e_s)r \in e_{s+1}R
\]
and
\[
(e_1 + \ldots + e_s + e_{s+1})r - e_{s+1}r \in e_1R \oplus \ldots \oplus e_sR.
\]
Therefore \((e_1 + \ldots + e_s + e_{s+1})r - (e_1 + \ldots + e_s)r - e_{s+1}r \) is in \((e_1R \oplus \ldots \oplus e_sR) \cap e_{s+1}R\), and this intersection contains only 0.

Thus
\[
(e_1 + \ldots + e_s + e_{s+1})r = (e_1 + \ldots + e_s)r + e_{s+1}r.
\]
Repeating this process, we get
\[
(e_1 + \ldots + e_s + e_{s+1})r = e_1r + \ldots + e_{s+1}r
\]
for all \( r \in R \). Replacing \( r \) by \( e_{s+1}r \), we have
\[
(e_1 + \ldots + e_{s+1})e_{s+1}r = e_{s+1}r,
\]
by the assumption that \( e_1R \oplus \ldots \oplus e_pR \subseteq A_{e_{i-1}}\).

Therefore \((e_1 + \ldots + e_{s+1})R\) contains \( e_{s+1}R \). Also, we have
\[
(e_1 + \ldots + e_{s+1})(r - e_{s+1}r) = (e_1 + \ldots + e_{s+1})r - e_{s+1}r = e_1r + \ldots + e_{s}r = (e_1 + \ldots + e_{s})r
\]
so that \((e_1 + \ldots + e_{s+1})R\) contains \((e_1 + \ldots + e_s)R\), and hence

\[(e_1 + \ldots + e_{s+1})R \supseteq e_1 R \oplus \ldots \oplus e_{s+1}R.
\]

The inclusion in the other direction is clear from (*) above.

From the foregoing it follows that the mapping \(T : n \rightarrow (e_1 + \ldots + e_p)n\) is a right \(P\)-homomorphism of \(R\) onto itself. Now if \(r = r_s + \ldots + r_p\) is a non-zero element of \(R\), \(r_s \in e_i R\), and \(r_s \neq 0\), then

\[(e_1 + \ldots + e_p)r = (e_1 + \ldots + e_p)(r_s + \ldots + r_p)\]

\[= 0 + e_s (r_s + \ldots + r_p) + \ldots + e_p (r_s + \ldots + r_p)\]

\[= e_s r_s + e_{s+1} (r_s + \ldots + r_p) + \ldots + e_p (r_s + \ldots + r_p).
\]

Since \(e_s r_s = r_s \neq 0\), and

\[-e_s r_s + (e_1 + \ldots + e_p)r = e_{s+1} (r_s + \ldots + r_p) + \ldots + e_p (r_s + \ldots + r_p)\]

we have

\[-e_s r_s + (e_1 + \ldots + e_p)r \in e_{s+1}R \oplus \ldots \oplus e_p R.
\]

Since \(e_s R \cap (e_{s+1} R \oplus \ldots \oplus e_p R) = 0\), we see that \((e_1 + \ldots + e_p)r \neq 0\).

Thus \(\ker (T) = 0\), so that \(T\) is an automorphism of \(R\). Letting \(e\) be the element of \(R\) such that \((e_1 + \ldots + e_p)e = e_1 + \ldots + e_p\), we have \(e^2 = e\), and \(er = e \tau r = (e_1 + \ldots + e_p)r = r\tau\), so that \(er = r\) for all \(r \in R\), proving the lemma.

3.29 **COROLLARY** Every D.C.C. semisimple near-ring \(P\) has a left unit.

**PROOF** Let \(T : R \rightarrow (e_1 + \ldots + e_p)n\) be as defined in the proof of theorem 3.28 and let \(r \in R\). Then \((er)T = (er)T = (e_1 + \ldots + e_p)r = r\tau\). Thus \(er = r\) for any \(r \in R\). But \(P\) is a right ideal of itself, and contains a corresponding idempotent \(e\). This idempotent is a left unit.

3.30 **COROLLARY** If \(R = R_1 \oplus \ldots \oplus R_p\) is a direct sum decomposition of a right ideal \(R\) into smaller right ideals \(R_i\), idempotents \(e_i\) can be selected from the respective \(R_i\) so that \(e_i e_j = 0\) for \(i \neq j\).
PROOF

R contains an idempotent e which is also a left unit of R. This idempotent e has a decomposition \( e = e_1 + \ldots + e_p \), \( e_i \in R_i \). For \( r_i \in R_i \), we have

\[
1 = e_i r_i = (e_1 + \ldots + e_p) r_i = e_1 r_i + \ldots + e_p r_i.
\]

But the sum \( R_1 \oplus \ldots \oplus R_p \) is direct, so the representation \( r_i = 0 + \ldots + r_i + \ldots + 0 \) is unique. Thus we have \( e_j r_i = 0 \) for \( j \neq i \), and \( e_i r_i = r_i \).

3.31 COROLLARY

The right ideals of a D.C.C. semisimple near-ring satisfy the ascending chain condition.

PROOF

Suppose \( R_1 \subseteq R_2 \subseteq \ldots \) is an ascending chain of right ideals of a D.C.C. semi-simple near-ring \( R \). Let \( R = \bigcup_i R_i \). Select idempotents \( e_i \in R_i \) such that \( e_i R_i = R_i \). Then \( R_{i+1} = R_1 \oplus (R_{i+1} \cap A_{e_i}) \) as in proof of theorem 3.20.

Notice now

\[
R_2 = R_1 \oplus (R_2 \cap A_{e_1})
\]

\[
R_3 = R_2 \oplus (R_3 \cap A_{e_2}) = R_1 \oplus (R_2 \cap A_{e_1}) \oplus (R_3 \cap A_{e_2})
\]

\[
\vdots
\]

\[
R_n = R_1 \oplus [(R_2 \cap A_{e_1}) \oplus \ldots \oplus (R_n \cap A_{e_{n-1}})].
\]

Thus \( R = R_1 \oplus [(R_2 \cap A_{e_1}) \oplus (R_3 \cap A_{e_2}) \oplus \ldots] \)

If we define \( R'_1 = (R_{1+1} \cap A_{e_1}) \oplus (R_{1+2} \cap A_{e_{1+1}}) \oplus \ldots \) for all \( i \), then \( R = R_1 \oplus R'_1 \) for all \( i \).

Now by the descending chain condition, we have

\[
R'_1 \supseteq R'_2 \supseteq \ldots \supseteq R'_n = R'_{n+1} = \ldots \quad \text{for some } n.
\]

If \( R'_n = R'_{n+1} \), then \( R_{n+1} \cap A_{e_n} = 0 \), and the mapping \( \mathcal{T} : R_{n+1} \rightarrow R_{n+1} \) such that \( r_{n+1} \mathcal{T} = e_n r_{n+1} \) is 1:1 and the image of \( \mathcal{T} \) is \( R_n \).

However, \( \mathcal{T} \mid R_n \) is also one-to-one and onto \( R_n \). Thus \( R_{n+1} = R_n \).
3.32 **THEOREM** If $P$ is a D.C.C. semisimple near-ring and $V$ an irreducible right $P$-space, then $V$ is right $P$-isomorphic to a minimal non-zero right ideal $R$ of $P$. In particular $V$ is right $P$-isomorphic to any minimal non-zero right ideal $R$ of $P$ for which $VR \neq 0$.

**PROOF** By definition, irreducible right $P$-spaces are proper, so that $VP \neq 0$. By corollary 3.27 applied to $P$, $VR \neq 0$ for some minimal non-zero right ideal $R$ of $P$. If $v \in V$ such that $vR \neq 0$, the mapping $T : r \mapsto vr$ is a right $P$-homomorphism of $R$ into $V$. Since $V$ is irreducible, $T$ is onto, and since $R$ is minimal, $T$ is 1:1. Thus $R$ is right $P$-isomorphic to $V$.

3.33 **THEOREM** A D.C.C. semisimple near-ring $P$ has only a finite number of non-$P$-isomorphic types of irreducible $P$-spaces. The number of types is the same as the number of minimal non-zero right ideals which are non-$P$-isomorphic.

**PROOF** Theorem 3.25 and theorem 3.32 show that the number of types is the same as the number of types of minimal non-zero right ideals.

If $R$ is a minimal non-zero right ideal of $P$, then the sequence $P \supseteq R \supseteq 0$ can be refined to a composition series*, and $R$ will be one of the factors of this series, since $R$ is minimal. By the Jordan-Hölder theorem, we see that the number of types of minimal non-zero right ideals must be finite.

3.34 **DEFINITION** A near-ring $P$ is simple if and only if:

(I) it has no proper non-zero two-sided ideals and

(II) it has no non-zero right modules which are annihilated from the right by the whole near-ring.

* since we have both chain conditions on right ideals.
3.35 **THEOREM** If $P$ is a D.C.C. simple near-ring, then $P$ is semisimple and has only one type of irreducible right $P$-space.

**PROOF** If $M$ is a non-zero nilpotent right $P$-module, then there is an integer $k$ such that $M^k = 0 \neq M^{k-1}$. Thus $A_M \supseteq M^{k-1}$, and $A_M$ is a two-sided ideal. We conclude that $A_M = P$, and thus $MP = 0$, contradicting the simplicity of $P$. Thus $P$ is semisimple.

Since $P^+$ is a right module, and $P$ has D.C.C. on right modules, $P$ must contain a minimal right module. Thus $P$ has at least one type of irreducible right $P$-space. If $V_1$ and $V_2$ are irreducible right $P$-spaces, neither can be annihilated by a minimal right ideal of $P$, since $A_{V_1} = A_{V_2} = 0$. Thus by theorem 3.32, if $R$ is any minimal non-zero right ideal of $P$, then $V_1 \cong R \cong V_2$.

3.36 **THEOREM** A D.C.C. semisimple near-ring $P$ with only one type of irreducible right $P$-space is simple.

**PROOF** There can be no non-zero right $P$-modules annihilated by all of $P$, since such a module would be trivially nilpotent. If $I$ is a proper two-sided ideal of $P$, there is at least one minimal right ideal not contained in $I$, because $P$ is a direct sum of some of its right ideals. If $R$ is such a right ideal, then $RI \cap I = 0$ (minimality of $R$), and $RI \subseteq R$, $RI \subseteq I$ ($R$ is a right module, $I$ a left ideal). Thus $RI \subseteq RI \cap I = 0$.

Now if $R'$ is a minimal right ideal contained in $I$, then $R = R'$, since they are both irreducible right $P$-spaces. Also, since $RR' = 0$, we have $R'R' = 0$, by the isomorphism, and therefore $R'$ is nilpotent. This contradiction shows that $P$ is simple.

3.37 **DEFINITION** Let $P$ be a D.C.C. semisimple near-ring. Then we can choose non-$P$-isomorphic irreducible right $P$-spaces $V_1, \ldots, V_s$ such that any other irreducible right $P$-space is right $P$-isomorphic to one of the $V_i$. 
Having chosen the $V_i$, we can define a two-sided ideal $A_i^0$ corresponding to $V_i$ such that

$$A_i^0 = \bigcap_{j \neq i} A_j$$

(Ref. definition 3.4)

3.38 **Lemma** A minimal right module of a D.C.C. semisimple near-ring $P$ is contained in $A_i^0$ if and only if it is right $P$-isomorphic to $V_i$.

**Proof** Let $M$ be a minimal right $P$-module which is right $P$-isomorphic to $V_j$, $j \neq i$. Then $MA_j^0 = 0$ since $A_j^0 \subseteq A_j$. Therefore $M$ is not contained in $A_i^0$, or we would have $M^2 = 0$, contradicting the assumption that $P$ is semisimple.

If $M$ is a minimal right $P$-module and $M \cong V_i$, then $V_j M = 0$ for all $j \neq i$, or we would have $V_j \cong V_i$ for some $j \neq i$, and $V_i \cong V_j$, a contradiction. Thus $M \subseteq A_j$ for all $j \neq i$, and consequently $M \subseteq A_i^0$.

3.39 **Theorem** $A_i^0$ is the sum of the minimal right $P$-modules which are right $P$-isomorphic to $V_i$.

**Proof** Immediate from theorem 3.26 and lemma 3.38.

3.40 **Lemma** Let $P$ be a D.C.C. semisimple near-ring, and let $A_i^0$ be as defined above. Then $A_i^0 \cap \sum_{j \neq i} A_j^0 = (\sum_{j \neq i} A_j^0)A_i^0 = 0$.

**Proof** $A_i^0 \cap \sum_{j \neq i} A_j^0 = \cap_k A_k^0$, since if $p \in A_i^0$, and for all $i,\cap_k A_k^0 \subseteq A_i^0$, so that $p \in \sum_{j \neq i} A_j^0$, and $\cap_k A_k^0 \subseteq A_i^0 \cap \sum_{j \neq i} A_j^0$.

Conversely, if $p \in A_i^0$, and $p \in \sum_{j \neq i} A_j^0$, then $p \in \cap_k A_k^0$, and

$$p \in \cap_k A_k^0 + \ldots + \cap_{k\neq i-1} A_k^0 + \cap_{k\neq i-1} A_k^0 + \ldots + \cap_{k\neq s} A_k^0,$$

but each of these summands is a subset of $A_k^0$, therefore

$$p \in \cap_k A_k^0 \cap A_i^0 = \cap_k A_k^0,$$

so that $\cap_k A_k^0 = A_i^0 \cap \sum_{j \neq i} A_j^0$.
But $\bigcap_{k} A_{V} = 0$, since otherwise it would contain a minimal right P-module $R$ (which is an irreducible right P-space) and $V_{k} \cong R$ for some $k$. Now since $V_{k} \cap R = 0$, we have $RR = 0$ by the isomorphism, which contradicts the semisimplicity of $P$.

Therefore $A_{i} \cap \sum_{j \neq i} A_{j} = 0$.

Since $A_{i}^{0}$ and $\sum_{j \neq i} A_{j}^{0}$ are two-sided ideals, we have:

$$(\sum_{j \neq i} A_{j}^{0})A_{i}^{0} \subseteq A_{i}^{0} \cap \sum_{j \neq i} A_{j}^{0} = 0.$$  

3.41 **THEOREM** A D.C.C. semisimple near-ring $P$ is the direct sum of the two-sided ideals $A_{i}^{0}$.

**PROOF** By theorem 3.26, $P$ is the direct sum of minimal right $P$-modules, and by lemma 3.38 each of these minimal right $P$-modules is contained in some $A_{i}^{0}$. $P$ is therefore the sum of the ideals $A_{i}^{0}$, and the sum is direct, by lemma 3.40.

3.42 **THEOREM** Let $P$ be a D.C.C. semisimple near-ring, $A_{i}^{0}$ as above. Then $A_{i}^{0}$ is a D.C.C. simple near-ring.

**PROOF** If $M_{i}$ is a right $A_{i}^{0}$ module,

$$M_{i}P = M_{i}(A_{1}^{0} \oplus \ldots \oplus A_{s}^{0})$$

$$= M_{i}A_{1}^{0} + \ldots + M_{i}A_{s}^{0}$$

$$= M_{i}A_{i}^{0}, \text{ since } M_{i} \subseteq A_{i}^{0} \text{ and } A_{i}^{0}A_{j}^{0} = 0$$

$$\subseteq M_{i}$$

Thus $M_{i}$ is a right $P$-module, and we have immediately that $A_{i}^{0}$ is semisimple, D.C.C., since $P$ is. Any minimal right $A_{i}^{0}$-module is a minimal right $P$-module contained in $A_{i}^{0}$, and is therefore (lemma 2) right $P$-isomorphic to $V_{i}$. If we apply lemma 3.38 now to $A_{i}^{0}$, we see that all the minimal right $A_{i}^{0}$-modules are right $P$-isomorphic to $V_{i}$, and hence $A_{i}^{0}$ has only one type of irreducible right $A_{i}^{0}$-space. The theorem follows now from theorem 3.36.
IV FURTHER CHARACTERIZATIONS OF THE RADICAL

4.1 DEFINITION A right $P$-space $V$ is called faithful if and only if $A_V = 0$.

4.2 DEFINITION A near-ring $P$ with at least one faithful irreducible right $P$-space is called a primitive near-ring.

4.3 DEFINITION An ideal $I$ of a near-ring $P$ is called a primitive ideal if $P-I$ is a primitive near-ring.

Clearly, a primitive near-ring is semisimple, by the definition of the radical.

4.4 THEOREM The radical $J$ of a near-ring $P$ is the intersection of all the primitive ideals of $P$.

PROOF Let $V$ be an irreducible right $P$-space, $I = A_V$. Then $V$ is an irreducible right $(P-I)$-space if we define $v(p+I) = vp$ for all $v \in V$, $p \in P$. Now $V$ is clearly a faithful right $(P-I)$-space, so that $(P-I)$ is a primitive near-ring, and $I$ is thus a primitive ideal.

Now if $I$ is a primitive ideal of $P$, and $V$ a faithful irreducible right $(P-I)$-space, then $V$ is a right $P$-space if we set $vp = v(p+I)$ for all $v \in V$, $p \in P$. Now $A_V = I$, and $V$ is still irreducible.

We have shown that an ideal $I$ of $P$ is primitive if and only if $I = A_V$ for some irreducible right $P$-space $V$. The theorem is immediate now, by definition of the radical.

4.5 DEFINITION A right ideal $K \neq P$ is called regular if and only if there is an element $e \in P$ such that $p-ep \in K$ for all $p \in P$.

Betsch [3] calls these ideals "modular" following Jacobson's definition for ideals of rings. However, this is in conflict with Laxton's definition of "maximal modular right ideals" (Definition 4.8).
4.6 **THEOREM** Let $V$ be an irreducible right $P$-space. Then $V$ contains an element $\omega$ such that $\omega P = V$ and $A_\omega$ is a regular maximal right ideal.

Conversely, $K$ is a regular maximal right ideal, then there is an irreducible right $P$-space $V$, and an element $\omega \in V$ such that $\omega P = V$, and $K = A_\omega$.

**PROOF** Since $V$ is irreducible, $VP \neq 0$, and therefore $\omega P \neq 0$ for some $\omega \in V$. But $\omega P$ is a right $P$-space contained in $V$. Thus $\omega P = V$, and there is $v \in V$ such that $w = \omega$. We have, then, $wen = wn$ for all $n \in P$ so that $\omega (en-n) = 0$ for all $n \in P$. Since $A_\omega \neq P$, $A_\omega$ is a regular right ideal of $P$. $P^+ - A_\omega$ is right $P$-isomorphic to $\omega P = V$, so that $P^+ - A_\omega$ is an irreducible right $P$-space. $A_\omega$ is therefore a maximal right ideal of $P$.

Now let $K \neq P$ be a regular maximal right ideal of $P$ and $p-ep \in K$ for all $p \in P$. Then $P^+ - K$ forms an irreducible right $P$-space if we define $(p+K)p' = pp' + K$ for all $p, p' \in P$. ($(P^+ - K)$ is proper, since if $(P^+ - K)P = 0$, then $PP \subseteq K$, $ep \in K$ for all $p \in P$, $(p-ep) + ep \in K$ for all $p \in P$ and we have $P = K$, a contradiction.)

Now $(e+K)p = ep+K = p-ep + ep + K = p+K$ for all $p \in P$, so that $(e+K)p = P^+ - K$, and $A_{e+K} = \{ p \in P \mid ep \in K \}$.

But if $ep \in K$, then $p-ep + ep \in K$, and $p \in K$, and conversely. Thus we have $A_{e+K} = K$, and the theorem is proven.

4.7 **THEOREM** Let $P$ be a near-ring, $J$ the radical of $P$. Then $J$ is the intersection of all the regular maximal right ideals of $P$.

**PROOF** Let $V$ be an irreducible right $P$-space. Then $A_V$ is the intersection of the right ideals $A_\omega$, $\omega \in V$. If $\omega P = 0$, then $A_\omega = P$. If $\omega P \neq 0$, then $\omega P = V$ ($V$ is irreducible).

Thus $A_V = \bigcap \{ A_\omega \mid \omega P = V \}$. By the previous theorem, these are
precisely the regular maximal right ideals of $P$.

4.8 **DEFINITION** A right ideal of a near-ring $P$ is a maximal modular right ideal if and only if it is a maximal right $P$-module.

4.9 **THEOREM** Let $P$ be a near-ring with identity. Then $P$ contains a maximal modular right ideal if and only if the radical $J$ of $P$ is not equal to $P$.

**PROOF** Suppose $P$ has a maximal modular right ideal $M$. Then the right $P$-space $P^+ - M$ is proper (under the usual definition of product) since $(P^+ - M)P = 0$ if and only if $P^2 \subseteq M$, but $P$ has an identity. Thus $J \neq P$, by the definition of the radical.

Now if $P$ has an irreducible right $P$-space $V$ (i.e. $J \neq P$) then since $V \neq 0$, there is $v \neq 0 \in V$ such that $vP \neq 0$. Since $V$ is irreducible, we have $vP = V$, and the right $P$-homomorphism of $P^+$ onto $V$ given by $p \mapsto vp$ for all $p \in P$, shows that $V \cong P^+ - A_v$. Thus $P^+ - A_v$ is an irreducible right $P$-space, and hence $A_v$ is a maximal modular right ideal of $P$.

4.10 **THEOREM** Let $P$ be a near-ring with identity such that $J$, the radical of $P$ is not equal to $P$. Then $J$ is the intersection of all the maximal modular right ideals of $P$.

**PROOF** By the previous theorem, $P$ has a maximal modular right ideal $M$. Let $B(M) = \{x \in P | Px \subseteq M\}$. We show first that $B(M)$ is an ideal of $P$. Let $b \in B(M)$. Then $Pb \subseteq M$ so that $P(pb) = (Pp)b \subseteq Pb \subseteq M$, and we have $pB(M) \subseteq B(M)$ for all $p \in P$. Now for $p, p' \in P$, $(p+b)p' - pp' \in B(M)$ if and only if for $p'' \in P$, we have $p''((p+b)p' - pp') \in M$ but $p''((p+b)p' - pp') = (p''p - p''b)p' - p''pp'$, and since $p''b \in B(P)$, we have $P(p''b) \subseteq M$, and since $P$ has an identity element, we have $p''b \in M$. 


Now by the right ideal property of $M$, we see that $(p''p - p''b)p' - p''pp' \in M$. Thus $B(M)$ is an ideal of $P$, and is contained in $M$, since $P$ has an identity element.

Now $P^+ - M$ is a faithful irreducible right $(P - B(M))$-space (usual product definition), so that $P - B(M)$ is a primitive near-ring, and $B(M)$ is a primitive ideal.

We have: \[ \cap \{ \text{maximal modular right ideals } M \text{ of } P \} \supseteq \cap \{ B(M) \mid M \text{ is a maximal modular right ideal of } P \} \supseteq \cap \{ \text{primitive ideals of } P \} = J \]

Now suppose $B$ is a primitive ideal of $P$, and $V$ a faithful irreducible right $(P-B)$-space. The $V$ is an irreducible right $P$-space (under the usual definition of product) and $B = A_V$. Moreover, $B = A_V = \bigcap_{v \in V} A_v$, and each of these $A_v$ is a maximal modular right ideal of $P$ (recalling the proof of the previous theorem.)

Thus $J = \cap \{ B \mid B \text{ is a primitive ideal of } P \} = \cap \{ \cap A_v \mid v \in V, v \neq 0 \} \supseteq \cap \{ \text{maximal modular right ideals of } P \}$

and the theorem is proven.
V DISTRIBUTIVELY GENERATED NEAR-RINGS

5.1 **DEFINITION** A near-ring $P$ is a distributively generated near-ring (dg near-ring) if and only if there is a subset $S$ of $P$ such that all the elements of $S$ are right distributive and the additive group $P^+$ is generated by $S$.

5.2 **PROPOSITION** Homomorphic images of dg near-rings are dg near-rings.

5.3 **LEMMA** Let $P$ be a near-ring distributively generated by a subset $S$. Let $A$ be the set of all $n$-tuples $[x_1, \ldots, x_n]$ with $x_i \in P$. Let $B$ be a subset of $A$ satisfying the conditions:

1. if $x_i \in S$, or $x_i = 0$ for $i = 1, \ldots, n$, then $[x_1, \ldots, x_n] \in B$.
2. if $[-x_1, \ldots, -x_n] \in B$, then $[x_1, \ldots, x_n] \in B$.
3. if $[x_1, \ldots, x_n], [y_1, \ldots, y_n] \in B$, then $[x_1 + y_1, \ldots, x_n + y_n] \in B$.

Then $B = A$.

**PROOF** We define the length of an element $p \in P$ so that $\ell(p) = 0$ if $p = 0$, and $\ell(p)$ is the length of the shortest representation of $p$ as a sum of elements of $S$ is $p \neq 0$. The proof is by induction on $\ell(x^i)$.

Let $P^k$ be the statement: "If $\ell(x^i) \leq k$ for $i = 1, \ldots, n$, then $[x_1, \ldots, x_n] \in B$". Then $P^0$ is immediate from condition (1). To prove $P^1$, we assume that $\ell(x^i) \leq 1$ for $i = 1, \ldots, n$ and form the two new $n$-tuples $[y_1, \ldots, y_n]$ and $[z_1, \ldots, z_n]$.

If $x_i \in S$, or $x_i \neq 0$, then let $y_i = x_i$, and $z_i = 0$.

If $-x_i \in S$, $x_i \neq 0$, then let $y_i = 0$, and $z_i = x_i$.

Now $[y_1, \ldots, y_n]$ and $[z_1, \ldots, z_n] \in B$, since $y_i \in S$ or $y_i = 0$ for $i = 1, \ldots, n$, and $[-z_1, \ldots, -z_n] \in B$ for the same reason.

Thus $[y_1 + z_1, \ldots, y_n + z_n] \in B$ so that $[x_1, \ldots, x_n] \in B$. 
This proves $P_1$. Now we assume $P_K$ holds and let $[x_1, \ldots, x_n]$ be an element of $A$ with $l(x) \leq K + 1$ for all $i = 1, \ldots, n$. We break $x_i$ into two parts $y_i$ and $z_i$, as in the proof of $R_1$, where $l(y_i) \leq 1$, $l(x_i) \leq K$. The theorem follows easily.

It may be shown now that the restriction, "$0 \cdot p = 0$ for all $p \in P$" is superfluous when $P$ is a dg near-ring.

5.4 **PROPOSITION** If $P$ is a dg near-ring, then $0 \cdot p = 0$ for all $p \in P$.

**PROOF** We use lemma 5.3, with $n = 1$ and $B = \{ p | 0 \cdot p = 0 \}$. Then $0 \in B$, since $0 \cdot 0 = 0$, and if $a, b \in B$, then $0(a+b) = 0 \cdot a + 0 \cdot b = 0 + 0 = 0$, so that $a+b \in B$. If $P$ is distributively generated by $S$, and $p \in S$, then $0 \cdot p = (0+0)p = 0p+0p$, and we have $0 \cdot p = 0$, so that $p \in B$. Using lemma 5.3, we see that $B = P$. That is $0 \cdot p = 0$ for all $p \in P$.

The following three propositions are also easily proven using lemma 5.3.

5.5 **PROPOSITION** If a near-ring $P$ is distributively generated by a set $S$ and $S$ is a multiplicative semigroup with identity $e$, then $e$ is the identity of $P$.

5.6 **PROPOSITION** If $A$ is an additive subgroup of $P$, then $A$ is a right $P$ module if and only if $A$ is a right $S$ module, where $P$ is distributively generated by $S$. (i.e. $AS \subseteq A$).

5.7 **PROPOSITION** If $P$ is distributively generated by $S$, and $V$ is an additive group such that $VS \subseteq V$, $v(s_1 + s_2) = vs_1 + vs_2$ for all $s_1, s_2 \in S$, $v \in V$, and $(vs_1)s_2 = v(s_1 s_2)$ for $v \in V$, $s_1, s_2 \in S$ (i.e. $V$ is a right $S$-space), then $V$ is a right $P$-space.
5.8 **Lemma** Let \( P \) be a dg near-ring, and \( I \) a normal subgroup of \( P^+ \). Then
\[ I \text{ is an ideal of } P \text{ if and only if } P \text{ is both right and left invariant.} \]
\[ I \text{ is a right ideal if and only if } I \text{ is right invariant.} \]

**Proof** Since \( P \) is a C-ring, any ideal of \( P \) is both right and left invariant. If \( I \) is a normal subgroup of \( P^+ \), and \( PI \subseteq I, IP \subseteq I \), then we show that \( (p_1 + i_1)(p_2 + i_2) - p_1p_2 \in I \) for all \( p_1, p_2 \in P \), \( i_1, i_2 \in I \). Let \( p_2 = p_2' + \cdots + p_2^n \), where \( p_2^i \) is right distributive for \( i = 1, \ldots, n \). Then
\[
(p_1 + i_1)(p_2 + i_2) - p_1p_2 = (p_1 + i_1)p_2 + (p_1 + i_1)i_2 - p_1p_2
\]
\[
= (p_1 + i_1)p_2 + i_3 - p_1p_2, \quad i_3 \in I
\]
\[
= (p_1 + i_1)(p_2' + \cdots + p_2^n) + i_3 - p_1p_2
\]
\[
= (p_1 + i_1)p_2' + \cdots + (p_1 + i_1)p_2^n + i_3 - p_1p_2
\]
\[
= (p_1p_2' + i_1' + \cdots + p_1p_2^n + i_n' + i_3 - p_1p_2, \quad i_k \in I
\]
\[
= p_1(p_2' + \cdots + p_2^n) + i_4 - p_1p_2, \quad i_4 \in I
\]
\[
= p_1p_2 + i_4 - p_1p_2
\]
\[
= i_5 \in I.
\]

The proof of the second part is quite similar.

For the remainder of this chapter, we restrict our attention to dg near-rings containing an identity element, which we denote by \( e \). In view of proposition 5.4, we shall make use of earlier results obtained for C-rings. (cf. definition).

5.9 **Lemma** Let \( R \) be a right ideal of \( P \), and \( Q \) a right \( P \)-module. Then for all \( r \in R, q \in Q, p \in P \), we have \( (q+r)p = qp + r' \), where \( r' \in R \).

**Proof** By lemma 5.3. Let \( B = p|(q+r)p = qp + r', q \in Q, r, r' \in R \). Then \( 0 \in B \), and if \( P \) is distributively generated by \( S \), then \( S \subseteq B \).
If \( b, b' \in B \), then we have
\[
(q+r)(b + b') = (q+r)b + (q+r)b' \\
= qb + r' + qb' + r'' \\
= qb + qb' = r'_1 + r'' \\
= q(b+b') + r''
\]
so that \( b + b' \in B \). If \( b \in B \), then clearly \(-b \in B\). Thus \( B = \mathbb{P} \), and the proof is complete.

5.10 **Theorem** If \( \mathbb{P} \) is a dg near-ring and \( J \) is the radical of \( \mathbb{P} \), then \( J \) contains all the nilpotent right \( \mathbb{P} \)-modules of \( \mathbb{P} \).

**Proof** If \( J = \mathbb{P} \) then the theorem is trivially true. Suppose \( J \neq \mathbb{P} \). Then \( \mathbb{P} \) has a maximal modular right ideal \( M \) (ref. theorem 4.9). Let \( Q \) be a nilpotent right \( \mathbb{P} \)-module such that \( Q^n = 0 \), and suppose \( Q \not\subseteq M \). Since \( M \) is a maximal right \( \mathbb{P} \)-module, we have \( Q + M = \mathbb{P} \), so that any element \( p \in \mathbb{P} \) can be written as \( p = q + m \), \( q \in Q \), \( m \in M \). Thus there are elements \( q \in Q \) and \( m \in M \) such that \( e = q + m \). Let \( q_1 q_2 \ldots q_{p-1} \in Q^{p-1} \). Then by lemma 5.9
\[
(q+m)(q_1q_2\ldots q_{p-1}) = qq_1q_2\ldots q_{p-1} + m', \quad m' \in M.
\]
Thus \( (q+m)(q_1q_2\ldots q_{p-1}) = 0 + m', \quad m' \in M \), and we have \( q_1q_2\ldots q_{p-1} \in M \), so that \( Q^{p-1} \subseteq M \). Now let \( q_1q_2\ldots q_{p-2} \in Q^{p-2} \).

Again by lemma 5.9
\[
(q+m)(q_1\ldots q_{p-2}) = qq_1\ldots q_{p-2} + m', \quad m' \in M
\]
so that \( q_1\ldots q_{p-2} \in M \), and we have \( Q^{p-2} \subseteq M \).

We can clearly continue this process to show that \( Q \subseteq M \), which contradicts the earlier assumption. Thus if \( Q \) is a nilpotent right \( \mathbb{P} \)-module, and \( M \) is a maximal modular right ideal, then \( Q \subseteq M \). Thus \( Q \) is contained in the intersection of the maximal modular right ideals, as required. (ref. theorem 4.10).
5.11 **Theorem** If $P$ satisfies the descending chain condition on right $P$-modules, then the radical $J$ of $P$ is not equal to $P$.

**Proof** If $P$ is a D.C.C. near-ring, then $P$ contains minimal right $P$-modules. These minimal right $P$-modules are irreducible right $P$-spaces (the existence of the identity element ensures that they are proper (def. 3.2)). Since none of these is annihilated by the whole near-ring, the radical, which is defined to be the intersection of their annihilators, is not the whole near-ring.

5.12 **Theorem** If $P$ is a D.C.C., near-ring, then the radical $J$ of $P$ is the intersection of the maximal ideals of $P$.

**Proof** We show that $M$ is a primitive ideal of $P$ if and only if $M$ is a maximal ideal of $P$, and use theorem 4.4.

Let $M$ be a primitive ideal of $P$. Then $P-M$ is a primitive dg near-ring, and satisfies the D.C.C. on right modules. Thus, by Appendix 1, $P-M$ is a simple dg near-ring, so that $P-M$ has no non-zero ideals. $M$ is thus a maximal ideal of $P$.

Now if $M$ is a maximal ideal of $P$, then $P-M$ is a simple dg near-ring with D.C.C. on right modules. Thus $P-M$ is a primitive dg near-ring and $M$ is a primitive ideal (ref. appendix 1).

5.13 **Theorem** The radical $J$ of a D.C.C. dg near-ring $P$ contains all the nilpotent left $P$-modules.

**Proof** (We have already shown that $J$ contains all the nilpotent right $P$-modules.) We show that if $L$ is a nilpotent left $P$-module, then $L$ is contained in every maximal ideal of $P$. Theorem 5.12 completes the proof.

Consider $m(L)$, the set of all finite sums of elements of the form $x+\ell p-x$, $\ell \in L$, $x, p \in P$. $m(L)$ is the smallest normal subgroup of $P^+$.
containing \( L \). We show that \( m(L) \) is an ideal, and hence the smallest ideal containing \( L \).

Clearly \( P \cdot m(L) \subseteq m(L) \). Now let \( p' \in P \), \( p' = s_1 + \ldots + s_n \), where \( s_i \) is right distributive for \( i = 1, \ldots, n \).

Now \( \left( \sum (x + \ell p - x) \right) p' = \left( \sum (x + \ell p - x) \right) (s_1 + \ldots + s_n) \)
\[ = \sum (x + \ell p - x)s_1 + \ldots + \sum (x + \ell p - x)s_n \]
\[ = \sum (xs_1 + \ell p s_1 - xs_1) + \ldots + \sum (xs_n + \ell p s_n - xs_n) \]
\[ = \sum (x' + \ell p' - x') \in m(L). \]

Thus by lemma 5.8, \( m(L) \) is an ideal, and since \( e \in P \), we have \( L \subseteq m(L) \).

Now suppose \( m(L) \) is not contained in some maximal ideal \( M \) of \( P \). \( (L \subseteq M \text{ if and only if } m(L) \subseteq M) \). Then \( P = m(L) + M \), and \( e = y + m, y \in m(L), m \in M \). If \( L^p = 0 \neq L^{p-1} \), then for any \( \ell_1 \ldots \ell_{p-1} \in L^{p-1} \), we have
\[ \ell_1 \ldots \ell_{p-1} = \ell_1 \ldots \ell_{p-1}(y + m) = \ell_1 \ldots \ell_{p-1}y + \ell_1 \ldots \ell_{p-1}m = \ell_1 \ldots \ell_{p-1}y \]
\[ \text{(M)} \]
but \( y = \sum (x + \ell p - x) \), \( x, p \in P \), \( \ell \in L \), so that
\[ \ell_1 \ldots \ell_{p-1}y = \ell_1 \ldots \ell_{p-1} \]
\[ = [(\ell_1 \ldots \ell_m x + \ell_1 \ldots \ell_{p-1} \ell p - \ell_1 \ldots \ell_{p-1} x)] = 0 \]
Thus we have \( L^{p-1} \subseteq M \). Repeating the process we show that \( L^{p-2} \subseteq M \), and finally that \( L \subseteq M \). Thus \( m(L) \subseteq M \), contradicting our original assumption.

5.14 **COROLLARY** If \( P \) contains a non-zero nilpotent left \( P \)-module, then \( P \) contains a non-zero nilpotent right \( P \)-module. \( (P \) is D.C.C., dg near-ring). 

**PROOF** Immediate from theorem 5.13 and theorem 4.7.
5.15 **Theorem** If \( N \) is a nil right \( P \)-module, then \( N \) is nilpotent.  
(\( P \) is D.C.C. dg near-ring).

**Proof** Let \( N \) be a nil right \( P \)-module. Then
\[
N \supseteq N^2 \supseteq \ldots \supseteq N^k = N^{k+1} = \ldots
\]
by D.C.C. Suppose \( N^k \neq 0 \). Then
\[
N^k N^k = N^{2k} = N^k
\]
so that \((N^k)^2 \neq 0\). Thus there is a right \( P \)-module contained in \( N^k \), say \( R \), such that \( RN^k \neq 0 \), and there is a minimal such \( R \). Let \( r \in R \) be an element such that \( rN^k \neq 0 \). Then \( rN^k \subseteq R \), and \( rN^k N^k = rN^k \neq 0 \), so that by the minimality of \( R \), we have \( rN^k = R \). Thus there is \( n \in N^k \) such that \( rn = r \). Now \( rn = rN^2 = \ldots = rN^s = 0 \), since \( N \) is nil. Now we have \( r = 0 \) and \( rN^k \neq 0 \). This contradiction proves the theorem.

5.16 **Definition** Let \( P \) be a D.C.C. dg near-ring. By theorem 4.9 and theorem 5.11, \( P \) contains maximal right ideals. The intersection of the maximal right ideals of \( P \) is called the quasi-radical of \( P \), and is denoted by \( \mathcal{Z}(P) \).

5.17 **Theorem** If the radical \( J \) of \( P \) is nilpotent, then \( J = \mathcal{Z}(P) \).  
(\( P \) is a D.C.C., dg near-ring).

**Proof** Suppose \( J \) is nilpotent, and \( J \not\subseteq R \), a maximal right ideal of \( P \). Then \( P = J + R \). Thus \( e = j + r \), where \( j \in J \), \( r \in R \). Now if \( J^k = 0 \neq J^{k-1} \), then for any \( j_1 \ldots j_{k-1} \in J^{k-1} \), we have
\[
J_1 \ldots J_{k-1} = J_1 \ldots J_{k-1}(j + r) = J_1 \ldots J_{k-1}r \in R.
\]
Thus \( J^{k-1} \subseteq R \). Continuing (as in the proof of theorem 5.13, we have \( J \subseteq R \), a contradiction, so that \( J \) is contained in \( \mathcal{Z}(P) \). Now by theorem 4.10, \( J \) is the intersection of the maximal modular right ideals, which contains the intersection of all the maximal right ideals. Thus \( J = \mathcal{Z}(P) \).
5.18 **DEFINITION** Let \( p \in P \), be an element of a D.C.C. dg near-ring \( P \). We define \( N_p \) to be the smallest normal subgroup of \( P^+ \) which contains the additive group generated by all elements of the form \( x - px, x \in P \).

That is,
\[
N_p = \left\{ \text{finite sums of elements of the form } y + x - px - y; \quad x, y \in P \right\}
\]

5.19 **Proposition** \( N_p \) is the minimal right ideal of \( P \) containing all elements of the form \( x - px \).

**Proof** \( N_p \) is a normal subgroup of \( P^+ \), and using lemma 5.3, one can easily show that \( N_p P \subseteq N_p \). The proof follows, by lemma 5.8.

5.20 **Definition** For any element \( p \in P \) (\( P \) a D.C.C. dg near-ring) \( p \) is right quasi-regular (rqr) if and only if \( N_p^p = P \).

5.21 **Definition** A right \( P \)-module is quasi-regular (qr) if and only if all of its elements are rqr.

5.22 **Proposition** \( p \) is rqr if and only if \( p \in N_p \).

**Proof** If \( p \in N_p \), then \( xp \in N_p \) for all \( x \in P \), since \( N_z P \subseteq N_z \). Also, \( x - xp \in N_p \), so that \( x \in N_p \) for all \( x \in P \). The necessity is trivial.

5.23 **Theorem** Let \( P \) be a D.C.C. dg near-ring. Then \( \mathcal{Z}(P) \) is quasi-regular.

**Proof** Let \( z \in \mathcal{Z}(P) \), and suppose \( N_z \neq P \). Then the class of all right ideals of \( P \) which contain the right ideal \( N_z \) and not the identity \( e \) of \( P \) is non-empty. Thus by Zorn's lemma, \( N_z \) is contained in a maximal right ideal which does not contain the element \( e \). If \( R \) is such a maximal ideal, then we have \( \mathcal{Z}(P) \subseteq R, z \in \mathcal{Z}(P) \), so that \( z \in R \). Also \( p - zp \in N_z \subseteq R \) for all \( p \in P \). Thus \( p \in R \) for all \( p \in P \), and we have \( r \in R \). This contradiction shows that any right ideal which contains \( N_z \) also contains \( e \),
and in particular $e \in N_z$ so that $N_z = P$.

5.24 **Theorem** If $P$ is a D.C.C. dg near-ring, then $\mathcal{Z}(P)$ is nilpotent.

**Proof** Let $\mathcal{Z}(P) = Q$. Then $Q \supseteq Q^2 \supseteq \ldots \supseteq Q^n = Q^{n+1} = \ldots$ by the D.C.C. on light $P$-modules. Let $Q^n = R$, and suppose $R \neq 0$. Then as in the proof of theorem 5.15, we can find $z, \ell \in R$, $\ell \neq 0$ such that $\ell z = \ell$. Since $z \in R \subseteq \mathcal{Z}(P)$, we have $N_z = P$ (ref. theorem 5.23). Thus $z \in N_z$, so that $z = \sum(y_1 + x_1 - zx_1 - y_1)$, and $\ell z = \sum(\ell y_1 + \ell x_1 - \ell zx_1 - \ell y_1)$

\[
\ell z = \sum_{i=1}^{n} (\ell y_1 + \ell x_1 - \ell x_1 - \ell y_1) = 0.
\]

This contradiction shows that $\mathcal{Z}(P)$ is nilpotent.

Combining theorems 5.17 and 5.24, we have:

5.25 **Theorem** If $P$ is a D.C.C. dg near-ring then the radical $J$ of $P$ is nilpotent if and only if $J = \mathcal{Z}(P)$.

5.26 **Theorem** Let $P$ be a D.C.C. dg near-ring. Then $J$, the radical of $P$, is nilpotent if and only if all the maximal right ideals of $P$ are maximal modular right ideals.

**Proof** If all the maximal right ideals are maximal modular right ideals, then $J = \mathcal{Z}(P)$, by theorems 4.10 and 5.11, so that $J$ is nilpotent by theorem 5.24.

Now if $J$ is nilpotent, then $P-J$ is D.C.C., so by theorem 3.26, corollary 1, we have that any right ideal of $P-J$ is a direct sum of minimal right ideals contained in it. Also $P-J$ itself is a direct sum of minimal right ideals, and any right module of $P-J$ is a direct sum of minimal right $(P-J)$-modules which are also minimal right ideals. From this it is clear that any maximal right ideal of $P-J$ is a maximal right $(P-J)$-module. Since $J$ is nilpotent, it is contained in all the maximal right ideals of $P$. (cf. theorem 5.17). Since these maximal right ideals are mapped onto maximal modular right ideals of $P-J$ under the natural
homomorphism, they are themselves maximal modular right ideals. Thus all the maximal right ideals of $P$ are maximal modular right ideals.

5.27 **Theorem** If $P$ is a near-ring (D.C.C. dg) with nilpotent radical, and $P-Q$ is any difference near-ring, then $P-Q$ has a nilpotent radical.

**Proof** $P-Q$ is a homomorphic image of $P$, so that a maximal right ideal of $P-Q$ is the image of a maximal modular right ideal of $P$, and is hence a maximal modular right ideal of $P-Q$. Theorem 5.26 completes the proof.

5.28 **Theorem** Let $P$ be a D.C.C. dg near-ring. Then $\mathcal{Z}(P)$ is a nilpotent right ideal of $P$ containing all the nilpotent right ideals of $P$. Thus $P$ has no non-zero nilpotent right ideals if and only if $\mathcal{Z}(P) = 0$.

**Proof** Let $N$ be a nilpotent right ideal of $P$ which is not contained in some maximal right ideal $R$ of $P$. Then $N+R = P$. Suppose $N^k = 0 \neq N^{k-1}$, and let $n \in N$, $r \in R$ be such that $n+r = e$, the identity of $P$. Then for any $n_1 \cdots n_{k-1} \in N^{k-1}$, we have

$$n_1 \cdots n_{k-1} = (n+r)n_1 \cdots n_{k-1} = 0 + r', \quad r' \in R.$$ 

Thus $N^{k-1} \subseteq R$. Continuing the process, we can show that $N \subseteq R$. This contradiction proves the theorem.

Immediately, from theorem 5.28, it follows that

5.29 **Theorem** The sum of two nilpotent ideals is a nilpotent ideal. The sum of two nilpotent right ideals is a nilpotent ideal.

5.30 **Definition** An element $p \in P$ is B-quasi-regular (written Bqr) if and only if there is an element $p' \in P$ such that $(e-p)p' = e$. A right $P$-module is Bqr if and only if each of its elements is Bqr.

5.31 **Theorem** If $B$ is a Bqr right $P$-module, then $B$ is contained in the radical $J$ of $P$. ($P$ is D.C.C. dg.)
**PROOF** Suppose $B \not\subseteq J$. Then by theorem 4.10 and theorem 5.11, $J$ is the intersection of the maximal modular right ideals of $P$, so that there is a maximal modular right ideal $R$ such that $B \not\subseteq R$.

Now $B + R = P$, by the maximality of $R$, so that $e = b + r$, for some $b \in B$, $r \in R$. Now $B$ is $Bqr$, so there is an element $p \in P$ such that $e = (e - b)p = r'p$ where $r' = b + r - b \in R$. Thus $e = r'p$, so that $e \in R$, and $R = P$, a contradiction. This completes the proof.

Let $M$ be a right $P$-module, $M \neq P$. Since $P$ contains an identity, it follows by Zorn's lemma that $M$ is contained in a maximal right $P$-module. In particular, $P$ contains maximal right $P$-modules. We have accordingly

5.32 **DEFINITION** The radical module of a near-ring $P$ is defined to be the intersection of all the maximal right $P$-modules. ($P$ is D.C.C. dg near-ring).

5.33 **THEOREM** The radical module $A$ of a near-ring $P$ is a $Bqr$ right module containing all the $Bqr$ right ideals of $P$ ($P$ is D.C.C. dg near-ring).

**PROOF** Let $a$ be an element of $A$. Then $(e - a)P = P$, since if $(e - a)P$ is a proper right $P$-module, then $(e - a)P$ is contained in a maximal right $P$-module $B$. Thus $e = (e - a) + a \in B$, a contradiction ($a \in B$ for any maximal right $P$-module $B$, since $a \in A$). Since $(e - a)P = P$ we have $(e - a)p' = e$ for some $p' \in P$. It follows that $A$ is $Bqr$.

Now suppose $A'$ is a non-zero $Bqr$ right ideal of $P$. If $A' \not\subseteq A$ then there is a maximal right $P$-module $M$ such that $A' \not\subseteq M$. Thus $P = M + A'$. Let $e = m + a'$, $m \in M$, $a' \in A'$. Then since $A'$ is $Bqr$, there is $p \in P$ such that $(e - a')p = e$. Thus $e = (m + a' - a')p = mp \in M$, a contradiction. (If $e \in M$, then $M = P$.)
5.34 **COROLLARY** The group sum of two Bqr right ideals of P is a Bqr ideal of P.

5.35 **THEOREM** Let P be a D.C.C. dg near-ring. If C is a Bqr right P-module, then C is nilpotent.

**PROOF** Let C be a non-zero Bqr right P-module. Let $C_n$ denote the right P-module generated by finite products of n elements of C. We have $C \supseteq C_1 \supseteq C_2 \supseteq \ldots$, a decreasing chain of right P-modules.

Let $k$ be an integer such that $C_k = C_{k+1} = \ldots$. Let $B = C_k$, and suppose $B \neq 0$. Let $B \circ B$ denote the subgroup of $P^+$ generated by elements of the form $b_1b_2$, where $b_1, b_2 \in B$. It may easily be shown that $B \circ B$ is a right P-module, since P is a dg near-ring. Also $B \supseteq B \circ B \supseteq C_k C_k = \left\{ a\, a' \mid a, a' \in C_k \right\}$. Thus $B \supseteq B \circ B \supseteq C_{2k} = C_k = B$, so that $B \circ B = B \neq \{0\}$. Since $B \circ B \neq \{0\}$, there is a minimal right P-module $D \subseteq B$ such that $D \circ B \neq \{0\}$, and an element $d \in D$ such that $dB \neq 0$. Since $(dB) \circ B = d(B \circ B) = dB$, we have $dB \circ B \neq \{0\}$, $dB \subseteq D$, so that $dB = D$. Let $b \in B$ be such that $db = d$. Then there is $p \in P$ such that $(e - b)p = e$, whence we have $d(e - b)p = d, (de - db)p = d, 0 \cdot p = d, d = 0$, a contradiction. The theorem is thus proven.

5.36 **COROLLARY** Let P be a D.C.C. dg near-ring. The radical $J$ of P is nilpotent if it is Bqr.

5.37 **THEOREM** Let P be a D.C.C. dg near-ring. The quasi-radical $Q$ of P is a Bqr right ideal of P if and only if the radical module $A$ contains all the nilpotent right ideals of P.

**PROOF** Assume $A$ contains all the nilpotent right ideals of P. Since $Q$ is a nilpotent right ideal (theorem 5.24) $Q \subseteq A$, hence $Q$ is Bqr,
If Q is Bar, then Q ≤ A, by theorem 5.33, and by theorem 5.28, Q contains all the nilpotent right ideals of P. The theorem follows.

5.38 **COROLLARY** Let P be a D.C.C. dg near-ring. The quasi-radical Q of P is Bqr if and only if every nilpotent right ideal of P is Bqr.
REFERENCES


APPENDIX 1

In the proof of theorem 5.12, we require the following result:

Let \( P \) be a D.C.C. dg near-ring. Then \( P \) is primitive if and only if \( P \) is simple.

The work up to and including the proof of this result is extracted from [11].

6.1 **DEFINITION** Let \( P \) be a dg near-ring, distributively generated by \( S \), and \( V \) a right \( P \)-space. Then \( V \) is a right \( (P,S) \)-space if and only if for all \( s \in S, v_1, v_2 \in V \), we have \((v_1 + v_2)s = v_1s + v_2s\).

As in chapter 5, we restrict our attention to dg near-rings containing an identity \( e \). We also require that if \( P \) is distributively generated by \( S \), then \( e \in S \), and if \( V \) is a right \( P \)-space, then \( ve = v \) for all \( v \in V \).

6.2 **DEFINITION** The set of all right \( P \)-endomorphisms of a right \( P \)-space \( V \) (All endomorphisms \( \phi \) of \( V \) such that \( v\phi = (vp)\phi \) for all \( v \in V, p \in P \)) is called the centralizer of \( V \) and is written \( S_v^{(1)} \) or simple \( S_v^{(1)} \) when no confusion can result.

6.3 **PROPOSITION** Let \( P \) be a dg near-ring, and \( V \) an irreducible right \( P \)-space. Then for any \( v \in V \), \( A_v \) is a maximal modular right ideal, \( P - A_v \cong V \). (ref. def. 3.4).

**PROOF** The mapping \( \phi: x \rightarrow vx, x \in P \), is a right \( P \)-homomorphism of \( P^+ \) onto \( V \), with kernel \( A_v \). Thus \( P^+ - A_v \cong V \), so that \( P^+ - A_v \) is irreducible, and hence \( A_v \) is a maximal right \( P \)-module, and a right ideal (ref. proposition 3.5)
6.4 **THEOREM** Let $P$ be a dg near-ring, $V$ an irreducible right $P$-space, and $v_1, v_2 \in V$. Then $A_{v_1} = A_{v_2}$ if and only if there is $\phi \in S(1)$ such that $v_1 \phi = v_2$.

**PROOF** Suppose $A_{v_1} = A_{v_2}$. Consider the mapping $\phi: v_1 x \rightarrow v_2 x$, $x \in P$. If $v_2 x = 0$, then $x \in A_{v_2} = A_{v_1}$ so that $v_1 x = 0$. Thus $\phi$ is a one to one mapping of $v_1 P$ onto $v_2 P$. Since $v_1 P \subseteq V$, and $v_1 P$ is a right $P$-space, we have $v_1 P = V$ or $v_1 P = 0$. If $v_1 P = 0$ then $v_1 = 0$ (since $e \in P$) and $v_2 = 0$, so that $\phi: v_1 \rightarrow v_2$. If $v_1 P = V$, then $\phi: v_1 e \rightarrow v_2 e$ so that $v_1 \phi = v_2$.

Now suppose $\phi \in S(1)$ is such that $v_1 \phi = v_2$. If $x \in A_{v_1}$, then $v_2 x = v_1 \phi x = v_1 x \phi = 0 \phi = 0$, so that $x \in A_{v_2}$. Thus $A_{v_1} \subseteq A_{v_2}$, and by maximality of $A_{v_1}$, we have $A_{v_1} = A_{v_2}$.

6.5 **DEFINITION** Let $V$ be an irreducible right $P$-space. ($P$ a dg near-ring). Then for any elements $v_1, v_2 \in V$, we write $v_1 \sim v_2$ if and only if there is an endomorphism $\phi \in S(1)$ such that $v_1 \phi = v_2$. Equivalently, we write $v_1 \sim v_2$ if and only if $A_{v_1} = A_{v_2}$.

The relation "\~" is clearly an equivalence relation, and we say $v_1$ is equivalent to $v_2$ if and only if $v_1 \sim v_2$.

This relation partitions $V$ into equivalence classes, such that the class containing zero contains only zero. We denote these equivalence classes by $C_0, C_k$, where $C_0 = 0$, and $k$ runs over some index set $K$.

We observe that $C_k = \{ v \phi \mid \phi \in S(1), \phi \neq 0 \}$ and $v$ is some element of $C_k$, and $A_{C_k} = A_v$ for any $v \in C_k$. This yields immediately

6.6 **PROPOSITION** $A_{C_k} = A_{C_L}$ if and only if $C_k = C_L$. 
6.7 **Lemma** If addition is commutative in a dg near-ring \( P \), then \( P \) is a ring.

**Proof** Only the right distributive law is in question, and this is easily demonstrated by letting \( a, b, c \), be elements of \( P \), \( c = s_1 + \ldots + s_m \), where \( s_i \) is in a distributively generating subset \( S \) of \( P \). Then

\[
(a+b)c = (a+b)(s_1 + \ldots + s_n) = (a+b)s_1 + \ldots + (a+b)s_n
\]

\[
= as_1 + bs_1 + \ldots + as_n + bs_n
\]

\[
= a(s_1 + \ldots + s_n) + b(s_1 + \ldots + s_n)
\]

\[
= ac + bc.
\]

6.8 **Lemma** If \( P \) is a dg near-ring which is not a ring, and if \( V \) is a faithful irreducible right \( P \)-space, then \( V \) is not an abelian group.

**Proof** If \( P \) is not a ring, then \( P^+ \) is not abelian. Let \( x, y \in P^+ \) be elements such that \( x+y \neq y+x \). Then since \( V \) is faithful, there is an element \( v \in V \) such that \( v(x+y-x-y) \neq 0 \). Thus \( vx + vy \neq vy + vx \), and \( V \) is non-abelian.

6.9 **Theorem** Let \( P \) be a primitive dg near-ring which is not a ring, \( V \) an irreducible faithful right \( P \)-space. We define the statement \( P_n \) for all \( n \geq 1 \).

\( P_n \): If \( N \) is the union of \( n \) equivalence classes \( C_k \) in \( V \), \( C_0 \notin N \), then \( P^+ - A_N \) is a direct sum of \( n \) copies of \( V \). If also \( C_i \notin N \), \( C_i \neq 0 \), then \( A_{C_i} \supseteq A_N \), and \( A_{C_i} + A_N = P \).

Then \( C_n \) holds for all \( n \geq 1 \).

**Proof** By induction on \( n \): \( P_1 \) is immediate from proposition 6.3 and the fact that \( A_{C_k} \) is a maximal module for any \( C_k \), and \( A_{C_k} = A_{C_i} \) if and only if \( C_k = C_i \). Assume now that \( P_n \) holds for all \( n < m \).
Let \( N = \bigcup_{k=1}^{m} C_k \); \( N' = \bigcup_{k=1}^{m-1} C_k \), \( C_k \neq C_o \), \( k = 1, \ldots, m \)

\[ C_k \neq C_L \), \( k \neq L \)

By the induction hypothesis, \( A_{C_m} \supseteq A_{N'} \), and \( A_{C_m} + A_{N'} = P \).

Thus

\[ P^+ - A_N = P^+ - (A_{N'} \cap A_{C_m}) \]

and

\[ P^+ - A_N = (A_{C_m} \oplus A_{N'}) - A_N = (A_{C_m} - A_N) \oplus (A_{N'} - A_N). \]

Thus \( v \in C \) and \( v \in C \), we have \( v \cap x = 0 \) and \( v \cap x = 0 \), since otherwise we would have \( v P = v P = 0 \), and thus \( C = C = C = C \). Since \( v \cap A_N \neq 0 \neq v \cap A_N \), we have \( v \cap A_N = v \cap A_N = V \).

Consider the mapping \( \phi : v \mapsto v \cdot x \) for all \( x \in A_{N'} \). If \( v \cdot x = 0 \), then \( x \in A_{N'} \cap A_{C_m} = A_N \subseteq A_{C_L} \), so that \( v \cdot x = 0 \), and \( \phi \) is uniquely defined.

Since \( A_{N'} \) is a right \( P \)-module, \( \phi \) is a right \( P \)-endomorphism of \( V \) onto itself. We show that \( v \cdot \phi = v \cdot \phi \) so that \( C = C \).

Since \( A_{N'} \) is a normal subgroup of \( P^+ \), for any \( x \in A_{N'} \), \( y \in P^+ \)

we have \( y + x - y \in A_{N'} \), so that \( [v_m (y+x-y)] \phi = v_1 (y+x-y) = v_1 y + v_1 x - v_1 y \).
Since \( \phi \) is a right \( P \)-homomorphism, we also have
\[
[v_m(y+x-y)]\phi = [v_m y + v_m x - v_m y] \phi = (v_m y)\phi + v_i x - (v_m y)\phi.
\]
whence
\[
v_i y + v_i x - v_i y = (v_m y)\phi + v_i x - (v_m y)\phi
\]
so that
\[
[-v_i y + (v_m y)\phi] + v_i x = v_i x + [-v_i y + (v_m y)\phi]
\]
for all \( x \in A_N \), \( y \in P \). Since \( V = v_1 A_N \), it follows that all finite sums of elements of the form \([-v_i y + (v-y)\phi]\) commute with all elements of \( V \).

Now if \( P \) is distributively generated by some subset \( S \), then since for any \( s \in S \),
\[
[-v_i y + (v_m y)\phi]s = -v_i ys + (v_m y)\phi s = -v_i ys + (v_m ys) \phi,
\]
an element of the same form, it follows easily that the subgroup \( \Gamma \) generated by all elements of the form \([-v_i y + (v_m y)\phi]\) is a right \( P \)-subspace of \( V \), and is contained in the center of \( V \).

Since \( P \) is not a ring, \( V \) is non-abelian (ref. lemma 6.8). Thus the center of \( V \) is a proper subgroup of \( V \), and since \( V \) is irreducible, and \( \Gamma \) is a proper subspace, we have \( \Gamma = 0 \).

Thus \( v_i y = (v_m y)\phi \) for all \( y \in P \), so that \( v_i e = (v_m e)\phi \), and hence \( v_1 = v_m \phi \).

6.10 PROPOSITION If \( P \) is a primitive dg near-ring which is not a ring, and \( P \) satisfies the descending chain condition on right ideals, then any faithful irreducible right \( P \)-space \( V \) has only a finite number of equivalence classes \( C_k \).
PROOF Suppose $C_0$, $C_1$, \ldots, $C_i$ are equivalence classes of $V$ for all positive integers $i$. Then define $N_1 = C_1$; $N_2 = C_1 \cup C_2$, \ldots, $N_i = N_{i-1} \cup C_i$. Let $A_k = A_{N_k}$. Then $A_1 \supseteq A_2 \supseteq \ldots$, and if $A_n = A_{n+1}$, then

$$\bigcap_{i=1}^{n} A_{C_i} = \bigcap_{i=1}^{n+1} A_{C_i}, \quad \text{and} \quad A_{C_{n+1}} \text{ contains } A_n.$$ 

But by theorem 6.9, if $C_{n+1} \notin N_n$, then $A_{C_{n+1}} \notin A_n$.

Thus $A_1 \supseteq A_2 \supseteq A_3 \ldots$, a strictly descending chain of right ideals. This contradiction proves the proposition.

6.11 DEFINITION Let $P$ be a dg near-ring, not a ring, and $V$ a faithful irreducible right $P$-space. Then $V$ is finitely partitioned by the centralizer if and only if $V$ has only a finite number of distinct equivalence classes $C_k$.

We remark that in view of theorem 6.9, if $V$ is partitioned into $n+1$ distinct equivalence classes, then $P$ is a direct sum of $n$ copies of $V$.

6.12 DEFINITION Let $P$ and $V$ be as in definition 6.11. If $V$ is finitely partitioned by $S^{(1)}$ into $C_0$, $C_1$, \ldots, $C_n$, then we define

$$L_i = \bigcap_{j=1}^{n} A_{C_j}, \quad \text{for } i = 1, 2, \ldots, n.$$ 

That is, $L_i$ is the set of all elements $p \in P$ such that $C_j p = 0$ for all $j \neq i$. The $L_i$ are proper right ideals of $P$, and non-zero, since by theorem 6.9 $L_i \notin A_{C_i}$.

6.13 THEOREM If $P$ is a primitive dg near-ring which is not a ring, and if $P$ satisfies the descending chain condition for right ideals
(or if the faithful irreducible right P-space $V$ is finitely partitioned by the centralizer) then $P$ can be written as $P = L_1 + \ldots + L_n$, where each $L_i$ is a right ideal which is a minimal right $P$-module, and right $P$-isomorphic to $V$.

**Proof** Let $v \in C_i \neq C_0$. The mapping $\phi: x \mapsto vx$ for all $x \in L_i$ is a right $P$-homomorphism of $L_i$ into $V$. Now if $x \in L_i$ is an element such that $vx = 0$, then $v'x = 0$ for all $v' \in C_i$ (ref. definition 6.5). Thus, by definition of $L_i$, $x$ annihilates all of $V$, so that $x = 0$ (since $V$ is faithful). Thus $\phi$ is an isomorphism of $L_i$ onto $vL_i = V$ (since $V$ is irreducible). Thus $L_i$ is a minimal right $P$-module.

Now $P$ is a direct sum of $n$ copies of $V$, and contains the $n$ distinct right ideals $L_i$, which are right $P$-isomorphic to $V$. But if $x$ is in $L_i \cap (\sum_{j \neq i} L_j)$, then $vx = 0$ for all $v \in V$. Thus $x = 0$, and the representation $P = L_1 \oplus \ldots \oplus L_n$ follows.

For the remainder of this section, we require that $P$ be primitive, and $V$ an irreducible faithful right $P$-space.

**6.14 Lemma** Let $x_i \in L_i$, $i = 1, \ldots, n$, and $y \in P$. Then $(x_1 + \ldots + x_n)y = x_1y + \ldots + x_ny$.

**Proof** We show first that for $x_i \in L_i$, $x_j \in L_j$, $j \neq i$, we have $x_1 + x_j = x_j + x_i$.

Setting $p_1 = x_i + x_j$, $p_2 = x_j + x_i$, we have

$$p_1 - p_2 = x_1 + x_j - x_1 - x_j \in L_j \quad (\text{since } L_j \text{ is normal in } P^+)$$

$$-(p_1 - p_2) = p_2 - p_1 = x_j + x_i - x_j - x_i \in L_1.$$
Thus since $L_i$ and $L_j$ are subgroups of $P^+$, we have

$$p_1 - p_2 \in L_i \cap L_j = 0, \text{ so that } p_1 = p_2.$$

Now if we write $y = s_1 + \ldots + s_k$, where $s_i \in S$, $i = 1, \ldots, k$ and $S$ is a distributively generating subset of $P$, we expand the expression $(x_1 + \ldots + x_n)y$ using the left distributive law first, and the lemma follows easily.

We point out that if $V$ is finitely partitioned by the centralizer, then we can write the elements of $V$ as $v_i$, where $i$ is in some index set $I$. We can further require that $I$ contains the natural numbers $0, 1, \ldots, n$, and that $v_i \in C_i$ for $i = 0, \ldots, n$. In particular, this means that $v_0 = 0$.

6.15 **DEFINITION** Let $\phi_{ke}$ be the element of $S^{(1)}$ such that

$$v_e \phi_{ke} = v_k \text{ if } v_e \sim v_k$$

$$v_e \phi_{ke} = 0 \text{ if } v_e \not\sim v_k$$

If $v_i \in C_i$, then $v_i L_i \neq 0$, since if $v_i x = 0$ for some $x \in L_i$, then $x \in L_i \cap A_{v_i} = L_i \cap A_{C_i} = 0$. Thus $v_i L_i = V$, so that the mapping $\phi: x \rightarrow v_i x, (x \in L_i)$, is a right $P$-isomorphism of $L_i$ onto $V$.

6.16 **LEMMA** For $i = 1, \ldots, n$, $L_i$ consists of the elements $e_j$ such that $v_i e_j = v_j$ for all $j \in I$. The elements $e_j$ are uniquely defined for all $i = 1, \ldots, n$, $j \in I$, by the preceding argument.

6.17 **THEOREM** $L_i$ consists of those elements $e_j$ such that $v_k e_{ji} = v_j \phi_{ki}$ for all $v_k \in V$. (cf. definition 6.15).

**PROOF** If $v_k \not\sim v_i$ (as in lemma 6.16) or $v_k = 0$, then $v_k L_i = 0$, and $v_k e_{ji} = 0$. Since $\phi_{ki} = 0$ in either case, we have $v_k e_{ji} = v_j \phi_{ki}$.
If \( v_k \sim v_i \), then there is an endomorphism \( \phi \in \mathcal{S}(1) \) such that \( v_i \phi = v_k \). In particular, \( v_1 k_1 = v_k \). Since \( v_1 e_{j1} = v_{j} \), we have

\[
v_j k_1 = v_i e_{ji} k_1 = v_i k_1 e_{ji} = v_k e_{ji}.
\]

The proof is complete, by lemma 6.16.

6.18 DEFINITION For \( i = 1, \ldots, n \), let \( e_{ii} \) be given as in lemma 6.16. Then we define \( e_1 = e_{i1} \). By lemma 6.16 these are uniquely defined.

6.19 PROPOSITION The identity element \( e \) of \( P \) can be decomposed as follows:

\[
e = e_1 + e_2 + \ldots + e_n, \quad e_1 \text{ as in definition 6.18}.
\]

PROOF Let \( v_k \in V \) be any non-zero element of \( V \). Then \( v_k \in C_i \) for some \( i, \ 1 \leq i \leq n \). We have

\[
v_k (e_1 + \ldots + e_n) = v_k e_1 = v_k e_{i1} = v_1 k_1 = v_k e_k
\]

and

\[
0(e_1 + \ldots + e_n) = 0 = 0e.
\]

Thus \( (e_1 + \ldots + e_n) \) and \( e \) act identically upon \( V \), so that

\[
(e_1 + \ldots + e_n) - e \in A_v = 0.
\]

Using lemma 6.14, we see that for \( y_1 \in L_1 \), we have

\[
y_1 = ey_1 = (e_1 + \ldots + e_n)y_1 = e_1 y_1 + \ldots + e_n y_1 = e_1 y_1,
\]

so that \( e_1 \) is a left identity of \( L_1 \).

In particular \( e_1^2 = e_1, \ e_1 e_i = 0 \) for \( i \neq j \). Thus \( L_1 = e_1 L_1 = e_1 P \) for all \( i = 1, \ldots, n \).

6.20 THEOREM A dg near-ring \( P \) satisfying the descending chain condition on right ideals, and possessing a minimal right \( P \)-module is primitive if and only if it is simple.

PROOF Since the theorem is known for rings, we can assume that \( P \) is not a ring.
Let $P$ be a primitive dg near-ring with dcc on right ideals, $V$ a faithful irreducible right $P$-space. Let $N$ be a non-zero ideal of $P$, $x \neq 0$ an element of $N$. Then $x = x_1 + \ldots + x_n$, $x_i \in L_i$ for $i = 1, \ldots, n$, and since at least one of the $x_i$ is non-zero, we assume that $x_1 \neq 0$. Then $x = x_1 e_1 + \ldots + x_n e_n$, so that $xe_1 = x_1 e_1^2 + 0 + \ldots + 0 = x_1 e_1$ is a non-zero element of $N$, and thus $x_1 = x_1 e_1 \in N$. Since $L_1$ is a minimal right $P$-module (cf. theorem 6.13) and $L_1 \cap N \subseteq L_1$, we have $L_1 \cap N = L_1$, or $L_1 \subseteq N$.

Now there is an element $e_{12} \in L_2$ such that $v_2 e_{12} = v_1$ and $v \cdot e_{12} = v \cdot e_{12}$ for all $v \in V$. ($v_1, v_2$ as in lemma 6.16). But we have

$v \cdot e_{12} = v \cdot e_{12} = v \cdot e_{12} = v \cdot e_{12}$

for any $v \in V$.

Now $v_2 e_{12} e_{11} = v_2 e_{12} = v_1 \neq 0$, so that $e_{12} e_{11} \neq 0$, and hence $e_{12} L_1 \neq 0$. Since $e_{12} L_1 \subseteq L_2$, and $L_2$ is minimal, we have $e_{12} L_1 = L_2$, so that $e_{12} L_1 \subseteq e_{12} N \subseteq N$, and we have $L_2 \subseteq N$. Similarly we can show that $L_i \subseteq N$ for all $i = 1, \ldots, n$, so that $P = L_1 + \ldots + L_n \subseteq N$.

Thus $P$ is simple.

Now let $P$ be a simple dg near-ring satisfying the dcc on right ideals, and possessing a minimal right $P$-module $R$. Then $R$ is an irreducible right $P$-space, and since $A_R$ cannot be all of $P$ ($e \in P$), and $A_R$ is an ideal, we have $A_R = 0$, so that $R$ is faithful, and $P$ is primitive.
APPENDIX 2

The following radical, defined by W.E. Deskins [6], has seen very little use in the literature to date. It is included for the sake of completeness. All the near-rings \( P \) considered in this appendix are assumed to be D.C.C., and to satisfy the condition \( 0 \cdot p = 0 \) for all \( p \in P \).

7.1 **DEFINITION** \( S(P) = \cap \{ I | I \) is the annihilator of some minimal right \( P \)-module.} \)

7.2 **PROPOSITION** \( P \) is a semisimple near-ring if and only if \( S(P) = 0 \).

**PROOF** Since \( J(P) \) is contained in \( S(P) \), by definition of \( J(P) \), we have immediately if \( S(P) = 0 \), then \( J(P) = 0 \) and \( P \) is semisimple.

If \( P \) is semisimple, and \( S(P) \neq 0 \), then \( S(P) \) contains a minimal right \( P \)-module \( R \neq 0 \). Now \( R \subseteq S(P) \subseteq \text{annihilator of } R \), so that \( RR = 0 \). This is a contradiction by theorem 3.23, so that if \( P \) is semisimple, then \( S(P) = 0 \).

7.3 **PROPOSITION** The near-ring \( P - S(P) \) is semisimple.

**PROOF** Let \( P_1 = P - S(P) \), and suppose \( Q_1 \) is a minimal nilpotent right \( P_1 \)-module. Let \( Q \) be the pre-image of \( Q_1 \) under the natural homomorphism. Then \( Q \) is a right \( P \)-module, and \( Q^2 \subseteq S(P) \). If \( Q \notin S(P) \), then there is a minimal right \( P \)-module \( M_1 \) such that \( M_1 \subseteq Q \neq 0 \), since if \( M_1 \subseteq Q \neq 0 \) for all minimal right \( P \)-modules \( M_1 \), then \( Q \subseteq S(P) \). Let \( m \in M_1 \) be such that \( mQ = M_1 \) (we use minimality of \( M_1 \)). Then \( M_1 \subseteq Q = mQQ \subseteq mS(P) = 0 \), since \( M_1 S(P) = 0 \). This contradiction completes the proof.
7.4 Definition Let $K$ be the set of all ideals of $P$ which contain no non-zero idempotent elements, but which contain at least one non-zero element which annihilates all idempotents of $P$ from the left. Let $T$ be the sum of all the ideals of $K$. If $K = \emptyset$, let $T = 0$. Let $P_1 = P - T$. $\bar{P} = P_1 - S(P_1)$. Let $R$ be the kernel of the homomorphism of $P$ onto $\bar{P}$. We call $R$ the D-radical of $P$, and denote it by $D(P)$.

7.5 Theorem The near-ring $P - D(P)$ is semisimple. $P$ is semisimple if and only if $D(P) = 0$.

Proof Let $Q$ be a nilpotent ideal of $P$. Then under the homomorphism $\gamma : P \to \bar{P}$ (as in def. 7.4) the image $\overline{Q}$ of $Q$ is a nilpotent ideal of $\bar{P}$. Since $\bar{P}$ is semisimple (cf. proposition 7.3), $\overline{Q} = 0$, so that $Q$ is contained in the kernel of $\gamma = D(P)$. Thus $D(P)$ contains all the nilpotent ideals of $P$, so that $P - D(P)$ is semisimple.

Now if $P$ is semisimple, then by theorem 3.28, every ideal of $P$ contains an idempotent, so that $K = \emptyset$, $T = 0$, $P_1 = P$, and $P = P - S(P)$. If $P$ is semisimple, then $S(P) = 0$, so that $\bar{P} = P$, and the homomorphism $\gamma : P \to \bar{P}$ is the identity map. Thus $D(P) = 0$.

It follows from the first part of the theorem that if $D(P) = 0$, then $P$ is semisimple.
APPENDIX 3

8.1 **DEFINITION** Let $G$ be an additive group, $T(G)$ the set of all transformations of $G$. $T_0(G)$ is defined to be the set of all elements $t$ of $T(G)$ such that $0 \cdot t = t \cdot 0$.

We have pointed out in chapter 1 that $T(G)$ is a near-ring. $T_0(G)$ is clearly a sub-near-ring. In this appendix we show that both $T(G)$ and $T_0(G)$ are simple near-rings for any additive group $G$.

8.2 **LEMMA** If $I$ is an ideal of $T(G)$, then $I T_0(G) \subseteq I$.

**PROOF** Let $p \in I$. Then $pt_0 = (0+p)t_0 = 0t_0 \in I$ for any $t_0 \in T_0(G)$.

8.3 **LEMMA** The only sub-near-ring of $T(G)$ which properly contains $T_0(G)$ is $T(G)$ itself.

**PROOF** Let $T_0(G) \subseteq P \subseteq T(G)$, and let $p \in P$, $p \notin T_0(G)$. Let $z_a \in T(G)$ be a transformation such that $gz_a = a$ for all $g, a \in G$. Then $z_o$ is the zero element of $T(G)$, and $z_o (p-z_{op}) = z_{op} - z_{op} = z_o = (p-z_{op}) z_o$, so that $p - z_{op} \in T_0(G)$. Since $p \notin T_0(G)$, we have $z_{op} \neq p z_o$; that is, $z_{op} \neq z_o$, or $0p \neq 0$. Let $q \in T(G)$ be such that $q: 0 \rightarrow 0$ and $q: g' \rightarrow g$ whenever $g' \neq 0$, $g' \in G$. Then $q \in T_0(G)$, so that $z_{op} q \in P$, that is $z_{op} q \in P$. But $z_{op} q = z_g$. We have now $z_g \in P$ for all $g \in G$. Thus for any $t \in T_0(G)$,

$$t = (t - z_{ot}) + z_{ot} \in P,$$

so that $P = T(G)$.

8.4 **DEFINITION** If $t$ is a transformation of $G$, then $r(t)$, the rank of $t$ is defined to be the cardinality of the image of $t$.

8.5 **LEMMA** Any non-zero ideal $I$ of $T_0(G)$ contains all the elements of $T_0(G)$ of rank 2.
PROOF Let $t \in T_0(G)$ be an element of rank 2, and suppose $t(G) = \{0, g\}$. (Since $0t = 0$, we know that $0 \in t(G)$.) Let $G_1 = t^{-1}(g)$, $G_2 = t^{-1}(0)$. Let $v$ be a non-zero element of some non-zero ideal $I$ of $T_0(G)$. Let $g_1, g_2 \in G$ be two elements such that $v: g_1 \to g_2$, and $g_2 \neq 0$. Let $w \in T_0(G)$ be such that $w: G_1 \to g_1$, $w: G_2 \to 0$. Let $x \in T_0(G)$ be such that $x: 0 \to 0$, $x: g^* \to g$ for all $g^* \neq 0$. Then $t = wx \in I$, since $v \in I$ (ref. lemma 8.2).

8.6 LEMMA If $G$ is finite then $T_0(G)$ is simple.

PROOF Let $I$ be an ideal of $T_0(G)$, and suppose $I$ contains all elements of $T_0(G)$ of rank $\leq k$. (Notice that $I$ contains all elements of rank 2, by the preceding lemma.) Let $t \in T_0(G)$ be an element of rank $k+1$, $t(G) = \{g_0, g_1, \ldots, g_k\}$, where $g_0 = 0$. Let $G_i = t^{-1}(g_i)$ for $i = 0, \ldots, k$. Let $v \in T_0(G)$ be such that $g_i = g_i$ for all $g \in G_i$, $i = 0, \ldots, k-1$, and $g_i = 0$ for $g \in G_k$. Let $w \in T_0(G)$ be such that $gw = 0$ for $g \in G_1$, $i = 0, \ldots, k-1$, $gw = g_k$ for $g \in G_k$. Then $r(v) = k$, $r(w) = 2$. Thus $v \in I$, $w \in I$ and hence $t = v+w \in I$. The lemma follows, by induction.

8.7 LEMMA Let $G$ have infinite cardinality, and let $h$ be any non-zero element of $G$. Then there is a maximal set $A \subseteq G$ such that $A \cap (A + h) = \emptyset$ and $A \cup (A + h) \cup (A - h) = G$. Since $A$, $A+h$ and $A-h$ have the same cardinality, they must have the same cardinality as $G$.

PROOF Consider the collection $\mathcal{S}$ of subsets of $G$: $\mathcal{S} = \{S| (S+h) \cap S = \emptyset\}$ Since $\{0\} \in \mathcal{S}$, $\mathcal{S}$ is not empty. If $S_1 \subseteq S_2 \subseteq \ldots \subseteq S_t \subseteq \ldots$ is an ascending chain of such sets, then $S' = \bigcup_t S_t$ is such a set, since if $s' = s''+h$ for some $s', s'' \in S'$, then there is $S_t$ such that $s', s'' \in S_t$, so that $s' \neq s''+h$. Thus by Zorn's lemma, $\mathcal{S}$ contains a maximal element $A$. 


If $G = A \cup (A+h)$, then the sets $A$ and $A+h$ have the same cardinality as $G$. If not, then let $k \in G$ be such that $k \notin A$, $k \notin A+h$. If $k+h \notin A$, then $[A,k]$ is disjoint from $[A,k]+h$, contradicting the maximality of $A$. Thus $k+h \in A$, so that $k \in A-h$. We have then, $G = A \cup (A+h) \cup (A-h)$, and since the sets $A$, $A+h$ and $A-h$ have the same cardinality, the lemma is proven.

8.8 **Lemma** If an ideal $I$ of $T^o(G)$ contains a transformation of rank $m$, then $I$ contains all transformations of rank $\leq m$.

**Proof** Partition $G$ into disjoint sets $G_x$, where $x$ ranges over an index set $X$ of cardinality $m$, and $0 \in G_x$. For $x \in X$, let $g_x'$ be an element of $G$, and suppose $g_x' = 0$. Let $v$ be an element of rank $m$ in $I$, with image of $v = \{g_x' | x \in X\}$ and since $0v = 0$, let $g_x'' = 0$. Let $g_x'' \in G$ be such that $g_x''v = g_x'$ for all $x \in X$, $g_x'' = 0$. Define $a \in T^o(G)$ such that $ga = g_x'$ for all $g \in G_x$. Let $b \in T^o(G)$ be such that $g_x'b = g_x'$ for all $x \in X$. Then $g avb = g_x''vb = g_x'b = g_x'$.

Since the set of all $g_x$, $x \in X$ is an arbitrary image of cardinality $\leq m$, and the partitioning of $G$ into the disjoint sets $G_x$ was arbitrary, we have shown that any transformation of rank $\leq m$ is in $I$, since $avb$ is in $I$. (cf. lemma 8.2)

8.9 **Theorem** If $G$ has infinite cardinality, then $T^o(G)$ is simple.

**Proof** Define $d_h \in T^o(G)$ such that $gd_h = h$ for all $g \in G$, $g \neq 0$, and $0d_h = 0$. Then any ideal $I$ of $T^o(G)$ contains $d_h$, since $r(d_h) = 2$ (ref. lemma 8.5). Define $c \in T^o(G)$ such that $gc = g$ for all $g \in A$, $gc = 0$ for all $g \notin A$, where $A$ is as in lemma 8.7. Then if $t = (1 + d_h)c - c$, where $1$ is the identity map, we have $t \in I$, since $I$ is a right ideal. Now $gt = (g+h)c - gc = -g$ for all $g \in A$. Thus the
rank of \( t = \text{cardinality of } A = \text{cardinality of } G \), so that \( I \) contains all transformations of rank less than or equal to cardinality of \( G \). Thus \( I = T_0(G) \).

8.10 **Lemma** \( T(G) \) is simple

**Proof** If \( G \) has order 2, then we can check the theorem directly. Suppose order of \( G \) is greater than 2, and let \( I \) be a non-zero ideal of \( T(G) \). If there is \( c \in I \cap T_0(G) \), such that \( c \neq 0 \), then \( T_0(G) \subseteq I \), since \( I \cap T_0(G) = T_0(G) \). Also, since \( I \) is a left ideal, \( zgc \in I \) for all \( g \in G \), (\( z_g \) as in proof of lemma 2)

Let \( g \in G \) be such that \( gc \neq 0 \). Then \( zgc = zgc \notin T_0(G) \), since \( 0zgc \neq 0 \). Thus \( zgc \in I \), \( zgc \notin T_0(G) \), so that \( I \) is a sub-near-ring of \( T(G) \) properly containing \( T_0(G) \). By lemma 8.3, \( I = T(G) \).

If \( I \cap T_0(G) = 0 \), then consider \( b \neq 0 \), \( b \in I \), and \( g \in G \) such that \( gb \neq 0 \), \( gb = g_1 \). Then \( zgb = zg_1 \in I \), since \( I \) is a left ideal. Let \( c \in T_0(G) \) be such that \( g_1c = 0 \), \( (g_2 + g_1)c \neq g_2c \) for some \( g_2 \in G \), \( g_2 \neq g_1 \), \( g_2 \neq 0 \). (this is possible, since order of \( G \) is \( > 2 \).)

Let \( d = (1 + z_{g_1})c - c \in I \). Then \( d \in T_0(G) \), and \( g_2d = (g_2 + g_1)c - g_2c \neq 0 \) Thus \( d \neq 0 \), and \( d \in T_0(G) \), and \( g_2d = (g_2 + g_1)c - g_2c \neq 0 \). Thus \( d \neq 0 \), and \( d \in I \cap T_0(G) \). This contradiction completes the proof.

We summarize our results in

8.11 **Theorem** For any group \( G \), the near-rings \( T(G) \) and \( T_0(G) \) are simple.