ON THE EQUATIONS OF MOTION OF MECHANICAL SYSTEMS SUBJECT TO NONLINEAR NONHOLONOMIC CONSTRAINTS

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ABSTRACT

The author has obtained the equations of motion for a nonlinear nonholonomic mechanical system in many a different form. The importance of these forms lies in their simplicity and novelty. Some of these forms are deduced from the principle of d'Alembert-Lagrange using the definition of virtual (possible) displacements due to N. G. Cetaev (Izv. Kazan, Fiz.-Mat Obsc.6 (1933), no. 3, 68-71), The others are obtained as a result of certain transformations. Moreover, these different forms of equations of motion are written either in terms of the generalised coordinates or in terms of nonlinear nonholonomic coordinates introduced by V. S. Novoselov (Leningrad. Gos Univ. Ucenye Zap. 217. Ser. Mat. Nauk 31 (1957), 50-83). These forms involve the energy of acceleration of the system or the kinetic energy or some new functions depending upon the kinetic energy of the system. Two of these new functions, denoted by R & K, can be identified to a certain approximation, with the energy of acceleration of the system and the Gaussian constraint, respectively.

An alternative proof is given to the fact that, if virtual displacements are defined in the sense of N. G. Cetaev, the two fundamental principles of analytical dynamics—the principle of d'Alembert-Lagrange and the principle of least constraint of Gauss—are consistent.

If the constraints are rheomonic but linear, a generalisation of the classical theorem of Poisson is obtained in terms of quasi-coordinates and the generalised Poisson's brackets introduced by V. V. Dobronravov (C. R. (Doklady) Akad. Sci. U.R.S.S. (N.S.) 44 (1944), 221-234).

The advantage of the various novel forms for the equations of motion is illustrated by solving a few problems.

A B S T R A C T

Suppose $q_1, q_2, \ldots, q_n$ are the generalised coordinates of a mechanical system moving with constraints expressed by r non-integrable equations ($r < n$)

(1) $$f_\alpha(t; q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) = 0 \qquad (\alpha = 1, 2, \ldots, r),$$

where the dots denote differentiation with respect to the time t, and f_α are nonlinear in the $\dot{q}$'s. The equations (1) are said to represent nonlinear nonholonomic constraints and the system moving with such constraints is called nonlinear nonholonomic.

From a purely analytical point of view, the author has obtained the equations of motion for a nonlinear nonholonomic mechanical system in many a different form. The importance of these forms lies in their simplicity and novelty. Some of these forms are deduced from the principle of d'Alembert-Lagrange using the definition of virtual (possible) displacements due to Četaev [11]. The others are obtained as a result of certain transformations. Moreover, these different forms of equations of motion are written either in terms of the generalised coordinates or in terms of nonlinear nonholonomic coordinates introduced by V.S. Novoselov [23].

These forms involve the energy of acceleration of the system or the kinetic energy or some new functions depending upon the kinetic energy of the system. Two of these new functions, denoted by R (Sec. 2.3) and K (Sec. 2.4), can be identified, to a certain approximation, with the energy of acceleration of the system and the Gaussian constraint, respectively.

An alternative proof (Sec.2.5) is given to the fact that, if virtual displacements are defined in the sense of N.G. Četaev [11] , the two fundamental principles of analytical dynamics - the principle of d'Alembert-Lagrange and the principle of least constraint of Gauss - are consistent.

If the constraints are rheonomic but linear, a generalisation of the classical theorem of Poisson is obtained in terms of quasi-coordinates and the generalised Poisson's brackets introduced by V.V. Dobronravov [17] .

The advantage of the various novel forms for the equations of motion is illustrated by solving a few problems.

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INTRODUCTION.

Let q_1, q_2, ..., q_n be the generalised coordinates of a mechanical system subject to constraints expressed by r nonintegrable equations of the type ($r < n$)

(1) $$f_\alpha(t;q_1,q_2,\dots,q_n;\dot{q}_1,\dot{q}_2,\dots,\dot{q}_n)=0 \quad (\alpha=1,2,\dots,r),$$

where the $\dot{q}$'s are the derivatives of the q's with respect to the time t and f_α are nonlinear in the $\dot{q}$'s. The equations (1) are said to represent nonlinear nonholonomic constraints. However, if the equations (1) reduce to nonintegrable Pfaffian equations, the constraints are referred to as linear non-holonomic.

The problem of mechanical systems moving with nonlinear nonholonomic constraints is an acute problem of analytical dynamics. The idea of such constraints originated with Appell [3;4;5], Delassus [12;15;16] and their contemporaries who, in an attempt to deduce the fundamental principles of analytical dynamics for such systems from the dynamics of systems moving with linear constraints, were confronted with two serious problems. First, the real existence of such constraints was not known. Secondly, considering such constraints, from a purely analytical point of view, the two fundamental principles of analytical dynamics - the principle of d'Alembert-Lagrange and the principle of least constraint of Gauss - appeared to be inconsistent. Though the first problem is still open, the second has been discussed by N.G. Četaev [11]. In 1933 he offered a new definition of a virtual (possible) displacement for such systems. As it should be expected, his definition embraces the usual definition of such displacements for systems which are holonomic or move with linear nonholonomic constraints.

In 1948 G.S. Pogosov [26] found the equations of motion for a nonlinear nonholonomic system in the form

$$(2) \qquad \frac{\partial S'}{\partial \ddot{q}_i} = Q_i - a_{\alpha i} Q_\alpha \qquad (\alpha = 1, 2, \dots, r;\ i = r+1, r+2, \dots, n),$$

where S' is the energy of acceleration of the system calculated on the basis of the equations of constraint (1), Q's are the generalised forces and $a_{\alpha i} = -\frac{\partial \dot{q}_\alpha}{\partial \dot{q}_i}$. These equations which are essentially Appell's equations were deduced from the principle of least constraint of Gauss by a long and complicated method.

In 1957 V.S. Novoselov [22;23;24] started a series of papers on nonlinear nonholonomic systems. One of his papers [22] contains a variety of results deduced from the equations of motion involving undetermined multipliers of Lagrange. Another paper [23] deals exhaustively with the various forms of the equations of motion in nonlinear quasi-coordinates or nonlinear nonholonomic coordinates. He obtains several important and interesting results. The quintessence of his researches is the generalisation of the classical results for linear nonholonomic systems to nonlinear nonholonomic systems.

The present thesis is concerned with nonlinear nonholonomic mechanical systems from a unified point of view. The starting point of these considerations is a synthesis of the differential principle of d'Alembert-Lagrange and an idea of P. Woronetz [29]. According to Woronetz we consider in the equations of constraint, a certain number of velocity-parameters, equal in number to the degrees of freedom of the system, as independent parameters. Throughout the discussion the indicial and summation conventions are used. A brief resume of the different aspects of the work is given below:

(i) The consistency of the principle of d'Alembert-Lagrange and the principle of Gauss, as proved by Četaev [11], demands that the former principle must lead to Appell's equations of motion. That it is so is shown by finding the equations of motion (Sec.2.2)

$$\frac{\partial S'}{\partial \ddot{q}_i} = Q'_i \qquad (i = r+1, r+2, \dots, n), \tag{3}$$

where $Q'_i = Q_i - a_{\alpha i} Q_\alpha$. The method applied is easier and more direct than that of Pogosov [26]. Furthermore, if S is the function S' for the corresponding holonomic system, it is shown that the equations (3) can be written in the symmetric form

$$\frac{\partial S}{\partial \ddot{q}_i} - Q_i = a_{\alpha i}\left(\frac{\partial S}{\partial \ddot{q}_\alpha} - Q_\alpha\right) \tag{4}$$

$$(\alpha = 1, 2, \dots, r;\ i = r+1, r+2, \dots, n).$$

Let T' be the kinetic energy of the system calculated by taking the constraints into account and let T'_0 be the function T' considered as a function of q's and t only. The equations of motion are then obtained in a new form (Sec. 2.3)

$$(5) \qquad \frac{\partial R'}{\partial \ddot{q}_i} = Q'_i \qquad (i = r+1, r+2, \ldots, n),$$

where $$R' = \frac{1}{2}\left(\ddot{T}' - 3\ddot{T}'_0\right).$$

It is also shown that R' coincides with S' as far as the terms in $\ddot{q}$ are concerned.

In Sec. 2.4 the equations of motion are transformed to the form

$$(6) \qquad \frac{\partial K'}{\partial \ddot{q}_i} = 0 \qquad (i = r+1, r+2, \ldots, n),$$

where $$K' = R' - Q_i \ddot{q}_i - Q_\alpha \ddot{q}_\alpha .$$

Later on (Sec. 2.5) the function K' is identified, to a certain approximation, with the Gaussian constraint. In the same section an alternative proof for the consistency of the principle of d'Alembert-Lagrange and the principle of Gauss is given.

If R and T are the functions R' and T' for the corresponding holonomic system, the identity (Sec.2.6)

$$\frac{\partial R}{\partial \ddot{q}_s} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} \qquad (s = 1, 2, \ldots, n)$$

yields the equations of motion

$$(7) \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i = a_{\alpha i}\left\{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha\right\}$$

$$(\alpha = 1, 2, \ldots, r;\ i = r+1, r+2, \ldots, n).$$

If in place of T the function T' is used, the equations are transformed into the equations (Sec.2.7)

(8) $$\frac{d}{dt}\frac{\partial T'}{\partial \dot{q}_i} - \frac{\partial T'}{\partial q_i} + \frac{\partial T_1}{\partial q_i} - \frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_i} + \frac{\partial R_1}{\partial \ddot{q}_i} = Q'_i$$
$$(i = r+1, r+2, \cdots, n),$$

where T_1 is what T becomes when considered as a function of the $\dot{q}$'s only and R_1 is R regarded as a function of the $\ddot{q}$'s only.

With the help of the identity

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial \dot{T}}{\partial \dot{q}_s} = -\frac{\partial T}{\partial q_s} \qquad (s = 1, 2, \cdots, n)$$

the equations are obtained in a novel form:

(9) $$2\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial \dot{T}}{\partial \dot{q}_i} - Q_i = a_{\alpha i}\left\{2\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial \dot{T}}{\partial \dot{q}_\alpha} - Q_\alpha\right\}$$
$$(\alpha = 1, 2, \cdots, r;\ i = r+1, r+2, \cdots, n).$$

Again, by virtue of the identity

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} + \frac{\partial \dot{T}}{\partial \dot{q}_s} = \frac{\partial \ddot{T}}{\partial \ddot{q}_s} \qquad (s = 1, 2, \ldots, n)$$

the equations of motion assume another novel form (Sec.2.9)

(10) $$2\frac{\partial \ddot{T}_0}{\partial \ddot{q}_i} - 3\frac{\partial \dot{T}_0}{\partial \dot{q}_i} = Q'_i \qquad (i = r+1, r+2, \ldots, n),$$

where $\dot{T}_0$ denotes $\dot{T}$ considered as a function of the $\dot{q}$'s only, and $\ddot{T}_0$ denotes $\ddot{T}$ regarded as a function of the $\ddot{q}$'s only.

A certain transformation discussed in Sec.2.6 allows the transition from the Eqs.(7) to equations in terms of Lagrange's multipliers. The converse problem is discussed in Sec.2.10.

In Sec.2.11 the Eqs.(7) are put in the form of determinants all of which are obtained according to a general scheme from an $(r+1)$xn matrix.

When the mechanical system is holonomic or moving with linear nonholonomic constraints many results of other authors, notably I.Cenov [7;8;9;10] and I.I.Metelicyn [21], follow as immediate corollaries from the results of this chapter.

(ii) Despite the fact that in nature no mechanical system has so far been discovered which moves with non-linear nonholonomic constraints, it is sometimes possible to write in an artificial manner the equations of linear constraints in a nonlinear form. Based on such considerations three well-known examples have been solved to support the general treatment of Chap.2. The classical methods of solving these examples depend on the equations of motion in terms of Lagrange's multipliers. The methods used in this thesis completely avoid the use of such multipliers.

(iii) Sec. 4.2 deals with the generalisation of the classical theorem of Poisson in terms of linear quasi-coordinates or linear nonholonomic coordinates so as to be applicable to systems moving with rheonomic constraints. To this end use has been made of the generalised Poisson's brackets introduced by V.V.Dobronravov [17] . If the constraints are scleronomic the result reduces to that of Dobronravov [17] established in 1944.

Introducing nonlinear nonholonomic coordinates in the manner of V.S.Novoselov [23], the author has obtained the general equations of Appell (Sec.4.3) in the form

$$\frac{\partial S'}{\partial \dot{\omega}_s} = Q'_s \qquad (s = 1, 2, \ldots, n) \tag{11}$$

for holonomic systems, and in the form

$$\frac{\partial S'}{\partial \dot{\omega}_i} = Q'_i \qquad (i = r+1, r+2, \ldots, n) \tag{12}$$

for nonlinear nonholonomic systems. The ω's denote the kinetic characteristics.

CHAPTER I

NOTATIONS AND DEFINITIONS

1.1 Consider a mechanical system consisting of N particles, and denote by x_ν one of the three rectangular coordinates of any one particle of mass m_ν. Further denote by X_ν the component of the resultant external force corresponding to x_ν.

If the mechanical system is free to move, the motion of the system will be governed by the Newtonian equations

$$m_{(\nu)}\ddot{x}_\nu = X_\nu , \qquad (\nu = 1,2,\ldots 3N), \tag{1.1.1}$$

where the dots denote differentiation with respect to the time t.

In writing the equations (1.1.1) as well as throughout our work we use the following

Notations:

(i) An index unrepeated implies a given range of values, and, when repeated in a single term, summation over that range.

(ii) As a derogation from this rule, an index within parenthesis, although repeated in a single term, will not be an index of summation.

With these notations the motion of a free mechanical system is completely determined by the equations

(1.1.1).

On the other hand, if the motion of the mechanical system is subject to some constraints expressed by $r < 3N$ equations of the type:

$$F_\alpha(t; x_1, x_2, \ldots, x_{3N}; \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_{3N}) \equiv F_\alpha(t; x_\nu; \dot{x}_\nu) = 0, \quad (\alpha = 1, 2, \cdots, r), \tag{1.1.2}$$

the equations (1.1.1) are no longer valid, and to obtain the equations of motion it is necessary to apply one of the two fundamental principles of analytical dynamics, either the principle of d'Alembert-Lagrange or the principle of least constraint of Gauss.

Although the principle of least constraint of Gauss is the most general principle, it is the formalism of the principle of d'Alembert-Lagrange which is mostly used in analytical mechanics and which we shall apply in most of our work.

To formulate the principle of least constraint of Gauss let us first define the term "constraint".

Definition 1. Let $\ddot{x}_\nu$ be the acceleration of the particle of mass m_ν in any kinemetically possible trajectory for which the coordinates and velocities at the instant considered are the same as in some actual trajectory. The constraint is defined by the function

$$G(\ddot{x}_\nu) = \frac{1}{2} m_\nu \left(\frac{X_\nu}{m_{(\nu)}} - \ddot{x}_\nu \right)^2 \qquad \nu = 1, 2, \ldots, 3N, \tag{1.1.3}$$

which is of the second degree in $\ddot{x}_\nu$.

The following is then the formulation of

The principle of least constraint of Gauss - Of all the trajectories consistent with the constraints (which are supposed to do no work), the actual trajectory is that which has the least constraint.

The other principle - the principle of d'Alembert-Lagrange - is the unification of the principle of d'Alembert and the principle of virtual displacements. This combined principle was given by Lagrange. That this is a differential principle can be seen from the formulation of

The principle of d'Alembert-Lagrange - For every system of virtual (possible) displacements δx_ν satisfying the conditions

(1.1.4) $$\frac{\partial F_\alpha}{\partial x_\nu}\,\delta x_\nu = 0,$$

the equation

(1.1.5) $$\left(m_{(\nu)}\ddot{x}_\nu - X_\nu\right)\delta x_\nu = 0, \qquad \nu = 1, 2, \ldots, 3N,$$

holds.

1.2. The mechanical system with which we deal is of the most general type. It may be subject to moving constraints, in which case it is rheonomic: if the constraints are fixed, i.e. independent of the time, it is scleronomic. The constraints may be defined by non-integrable equations in $\dot{x}_\nu$, in which case it is non-holonomic: otherwise holonomic. In the case of a non-holonomic system the constraints if defined by non-integrable Pffafian equations will be called linear: otherwise nonlinear. The rheonomic non-linear non-holonomic system is the most general, including the others as special or degenerate cases.

The equations (1.1.2), supposed to be independent, allow us to express the rectangular coordinates x_ν as functions of $n = 3N-r$ independent parameter $q_1, q_2, \dots, q_n$, called the generalised coordinates, and of the time t. Let the transformation equations be

$$x_\nu = x_\nu(t; q_1, q_2, \dots, q_n) \equiv x_\nu(t; q_s), \qquad s = 1, 2, \dots, n. \tag{1.2.1}$$

Differentiating (1.2.1) with respect to the time, we get

$$\dot{x}_\nu = \frac{\partial x_\nu}{\partial q_s}\dot{q}_s + \frac{\partial x_\nu}{\partial t}. \tag{1.2.2}$$

Substituting the value of x_ν and $\dot{x}_\nu$ from (1.2.1) and (1.2.2) in (1.1.2) we get the equations of constraint in the following form:

$$f_\alpha(t; q_1, q_2, \dots, q_n; \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n) \equiv f_\alpha(t; q_s; \dot{q}_s) = 0, \qquad \alpha = 1, 2, \dots, r. \tag{1.2.3}$$

Let us now assume that the equations (1.2.3) can be solved to obtain any r, say the first r, $\dot{q}$'s as functions of t, q's and the remaining $\dot{q}$'s. Then we shall have relations of the form:

$$\dot{q}_\alpha = \dot{q}_\alpha(t; q_1, q_2, \dots q_n; \dot{q}_{r+1}, \dot{q}_{r+2}, \dots, \dot{q}_n) \equiv \dot{q}_\alpha(t; q_s; \dot{q}_i) \tag{1.2.4}$$

$$(\alpha = 1, 2, \dots, r;\ i = r+1, r+2, \dots, n;\ s = 1, 2, \dots, n).$$

In view of (1.2.4) the relations (1.2.2) take the form:

(1.2.5) $$\dot{x}_\nu = \dot{x}'_\nu(t; q_s; \dot{q}_i) \equiv \dot{x}'_\nu .$$

1.3. So far as holonomic or linear non-holonomic systems are concerned, the principle of d'Alembert-Lagrange and the principle of least constraint of Gauss are found to be consistent. The question arises: "Can these principles be extended to nonlinear non-holonomic systems?" In an attempt to answer this question Appell [3;4;5] and Delassus [12;15;16] found that the principle of least constraint of Gauss could be extended whereas the principle of d'Alembert-Lagrange broke down. In other words, the two fundamental principles of analytical dynamic showed an inconsistency.

In 1933 N.G.Četaev [11] considered the problem of nonlinear non-holonomic systems. In order to remove the inconsistency between the two above-mentioned principles he proposed a new definition of a virtual (possible) displacement which can be expressed as follows:

<u>Definition 2.</u> δx_ν is said to be a virtual displacement consistent with the constraints (1.2.5) provided that the relations

(1.3.1) $$\delta x_\nu = \frac{\partial \dot{x}'_\nu}{\partial \dot{q}_i}\, \delta q_i \qquad (i = r+1, r+2, \dots, n;\ \nu = 1, 2, \dots, 3N)$$

hold, where δq_i are infinitely small arbitrary quantities.

The constraints for which the relations (1.3.1) hold are called <u>constraints of the type of Četaev</u>.

A salient feature of Definition 2 of a virtual displacement is the fact that it contains as a special case the usually given definition of a virtual displacement for a holonomic or linear non-holonomic system. Moreover, the existence of virtual displacements, satisfying the conditions (1.3.1) has been shown by Četaev [11] . He also proved the relation

$$d\dot{x}_\nu - \delta'\dot{x}_\nu = \frac{\partial \dot{x}'_\nu}{\partial \dot{q}_i}\left(d\dot{q}_i - \delta'\dot{q}_i\right). \tag{1.3.2}$$

In the above relations $d\dot{x}_\nu$, $d\dot{q}_i$ denote the change in $\dot{x}_\nu$, $\dot{q}_i$, respectively, along the actual motion during an interval of time dt and $\delta'\dot{x}_\nu$, $\delta'\dot{q}_i$ refer to the corresponding changes, during an interval of time $\delta' t = dt$, along any conceivable motion which is consistent with the imposed constraints. From (1.3.1) and (1.3.2) it follows that δx_ν, δq_i can be taken proportion to $d\dot{x}_\nu - \delta'\dot{x}_\nu$, $d\dot{q}_i$, $\delta'\dot{q}_i$, respectively.

By virtue of the relations (1.3.1) the constraints (1.1.2) impose the following conditions on the variations of the rectangular coordinates x_ν :

$$\frac{\partial F_\alpha}{\partial \dot{x}_\nu}\delta x_\nu = 0 \qquad (\alpha = 1,2,\dots,r;\ \nu = 1,2,\dots,3N). \tag{1.3.3}$$

The conditions (1.3.3) in the generalised coordinates assume the form:

$$\frac{\partial f_\alpha}{\partial \dot{q}_s}\delta q_s = 0 \qquad (\alpha = 1,2,\dots,r;\ s = 1,2,\dots,n). \tag{1.3.4}$$

That is, the relations (1.3.3) and (1.3.4) are equivalent.

As a consequence of Definition 2 of a virtual displacement it becomes necessary to restate

The principle of d'Alembert-Lagrange for constraints of the Četaev type:

In the case of ideal constraints for every system of virtual displacements δx_ν satisfying the conditions

$$\frac{\partial F_\alpha}{\partial \dot{x}_\nu} \delta x_\nu = 0,$$

the equation

$$(1.3.5) \qquad (m_{(\nu)} \ddot{x}_\nu - X_\nu) \delta x_\nu = 0 \qquad (\nu = 1, 2, \ldots, 3N)$$

holds.

CHAPTER 2

EQUATIONS OF MOTION AND THEIR TRANSFORMATIONS.

2.1. Some General Considerations.

In this chapter we shall derive the equations of motion in various forms. The mechanical system will be assumed to be subject to nonlinear non-holonomic constraints of the Četaev type. The derivation of the different forms of the equations of motion will be either centred around the application of the principle of d'Alembert-Lagrange, as given by the equation (1.3.5), or based on some transformations. Moreover, the equations will either involve the kinetic energy or the energy of acceleration or some function R or K to be defined later. Of the functions R and K the former will be shown to coincide, under certain conditions, with the energy of acceleration and the latter, under the same conditions, with the Gaussian constraint defined by (1.1.3)

2.2. The General Equations of Appell.

As shown by N.G. Četaev [11] , on the basis of Definition 2 of a virtual displacement, the two fundamental principles of analytical dynamics - the principle of d'Alembert-Lagrange and the principle of least constraint of Gauss - are consistent. It is, therefore, necessary that the application of either principle should lead to the same form of the equations of motion. We shall deduce the so-called equations of Appell from the principle of d'Alembert-Lagrange. These equations were first obtained by Appell [5], using the principle of least constraint of Gauss. However, the principle of d'Alembert-Lagrange failed to give them.

Let us consider a mechanical system whose position is characterized by n generalised coordinates $q_1, q_2, \ldots, q_n$, and assume that it moves under the most general type of nonlinear non-holomic constraints of the type of Četaev. Let these constraints be expressed by $r < n$ equations:

$$f_\alpha(t; q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) \equiv f_\alpha(t; q_s; \dot{q}_s) = 0 \tag{2.2.1}$$

$$(\alpha = 1, 2, \ldots, r;\ s = 1, 2, \ldots, n).$$

Further, let us suppose that the functional matrix

$$\left\| \frac{\partial f_\alpha}{\partial \dot{q}_s} \right\|$$

is of rank r. According to Woronetz [29] we can then choose, without loss of generality, the last n-r $\dot{q}_i$ $(i = r+1, r+2, \ldots, n)$ as independent parameters and solve the system of equations (2.2.1) with respect to $\dot{q}_\alpha$ $(\alpha = 1, 2, \ldots, r)$. Thus we obtain the following equations

$$\dot{q}_\alpha = \dot{q}_\alpha(t; q_1, q_2, \ldots, q_n; \dot{q}_{r+1}, \dot{q}_{r+2}, \ldots, \dot{q}_n) \equiv \dot{q}_\alpha(t; q_s; \dot{q}_i). \tag{2.2.2}$$

The equations of transformation from the set of rectangular coordinates (x_ν) to the set of (q_s) variables are

$$x_\nu = x_\nu(t; q_1, q_2, \ldots q_n) \equiv x_\nu(t; q_s), \tag{2.2.3}$$

$$(\nu = 1, 2, \ldots, 3N).$$

Differentiating the equations (2.2.3) with respect to t, we get

(2.2.4) $$\dot{x}_\nu = \frac{\partial x_\nu}{\partial q_s}\dot{q}_s + \frac{\partial x_\nu}{\partial t}.$$

Substituting from (2.2.2) in the equations (2.2.4) we find, by putting a dash to every function of the independent velocity-parameters:

(2.2.5) $$\dot{x}_\nu = \dot{x}'_\nu(t; q_1, q_2, \dots, q_n; \dot{q}_{r+1}, \dot{q}_{r+2}, \dots \dot{q}_n) \equiv \dot{x}'_\nu,$$
$$\ddot{x}_\nu = \frac{\partial \dot{x}'_\nu}{\partial \dot{q}_i}\ddot{q}_i + \psi'_\nu \qquad (i = r+1, r+2, \dots, n),$$

where ψ'_ν represents terms not containing $\ddot{q}_i$.

From (2.2.5) it follows that

(2.2.6) $$\frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_i} = \frac{\partial \dot{x}'_\nu}{\partial \dot{q}_i}.$$

According to the principle of d'Alembert-Lagrange we have

$$\left(m_{(\nu)}\ddot{x}_\nu - X_\nu\right)\delta x_\nu = 0,$$

where δx_ν satisfy the conditions (1.3.1). Hence we have

(2.2.7) $$\left(m_{(\nu)}\ddot{x}_\nu - X_\nu\right)\frac{\partial \dot{x}'_\nu}{\partial \dot{q}_i}\delta q_i = 0.$$

Since δq_i are independent, (2.2.7) leads to the relations:

(2.2.8) $$m_{(\nu)}\ddot{x}_\nu \frac{\partial \dot{x}'_\nu}{\partial \dot{q}_i} = X_\nu \frac{\partial \dot{x}'_\nu}{\partial \dot{q}_i} \qquad (i = r+1, r+2, \dots, n).$$

Introduce the function

$$S = \frac{1}{2} m_{(\nu)}\ddot{x}_\nu \ddot{x}_\nu,$$

called the energy of acceleration of the system, and substitute in S the expression for $\ddot{x}_\nu$ from (2.2.5). Then S transforms into S' which is a function of $\dot{q}_i, \ddot{q}_i$ $(i = r+1, r+2, \ldots, n)$, q_s $(s = 1, 2, \ldots, n)$ and t.

By virtue of (2.2.6) we obtain

$$\frac{\partial S'}{\partial \ddot{q}_i} = m_{(\nu)} \ddot{x}_\nu \frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_i} = m_{(\nu)} \ddot{x}_\nu \frac{\partial \dot{x}'_\nu}{\partial \dot{q}_i}. \tag{2.2.9}$$

If we put

$$Q'_i = X_\nu \frac{\partial \dot{x}'_\nu}{\partial \dot{q}_i}, \tag{2.2.10}$$

the equations (2.2.8), with the help of (2.2.9) and (2.2.10) reduce to the form

$$\frac{\partial S'}{\partial \ddot{q}_i} = Q'_i, \qquad (i = r+1, r+2, \ldots, n). \tag{2.2.11}$$

These are the general equations of Appell.

Corollary 1. In 1948 G.S. Pogosov [26] obtained the equations of motion for nonlinear non-holonomic constraints of the Četaev type, using the principle of least constraint of Gauss. These equations follow as an immediate corollary of the equations (2.2.11).

From the relations (2.2.4) we have

$$\frac{\partial \dot{x}'_\nu}{\partial \dot{q}_i} = \frac{\partial x_\nu}{\partial q_i} + \frac{\partial x_\nu}{\partial q_\alpha} \frac{\partial \dot{q}_\alpha}{\partial \dot{q}_i} \qquad (\alpha = 1, 2, \ldots, r;\ i = r+1, r+2, \ldots, n). \tag{2.2.12}$$

If we put

$$a_{\alpha i} = -\frac{\partial \dot{q}_\alpha}{\partial \dot{q}_i}, \tag{2.2.13}$$

we obtain from (2.2.6) and (2.2.12) the relations

$$\frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_i} = \frac{\partial \dot{x}'_\nu}{\partial \dot{q}_i} = \frac{\partial x_\nu}{\partial q_i} - a_{\alpha i}\frac{\partial x_\nu}{\partial q_\alpha}. \tag{2.2.14}$$

Putting

$$Q_\alpha = X_\nu \frac{\partial x_\nu}{\partial q_\alpha}, \quad Q_i = X_\nu \frac{\partial x_\nu}{\partial q_i}, \tag{2.2.15}$$

it follows from (2.2.10) and (2.2.14) that

$$Q'_i = Q_i - a_{\alpha i} Q_\alpha .$$

Hence the equations of Appell take the form

$$\frac{\partial S'}{\partial \ddot{q}_i} = Q_i - a_{\alpha i} Q_\alpha \tag{2.2.16}$$

$$(\alpha = 1,2,\ldots,r;\ i = r+1, r+2,\ldots,n).$$

These are the equations of motion obtained by Pogosov.

Corollary 2. It is possible to write the equations (2.2.16) in a symmetric form. To this end, all we have to do is to use the function S in place of S′ .

We have

$$\frac{\partial S'}{\partial \ddot{q}_i} = \frac{\partial S}{\partial \ddot{q}_i} - \frac{\partial S}{\partial \ddot{q}_\alpha}\frac{\partial \ddot{q}_\alpha}{\partial \ddot{q}_i}. \tag{2.2.17}$$

Also from (2.2.2) it follows, on differentiation with respect to t, that

$$\ddot{q}_\alpha = \frac{\partial \dot{q}_\alpha}{\partial \dot{q}_i}\ddot{q}_i + \text{terms not containing } \ddot{q}.$$

Hence by virtue of (2.2.13) we get

$$\frac{\partial \ddot{q}_\alpha}{\partial \ddot{q}_i} = \frac{\partial \dot{q}_\alpha}{\partial \dot{q}_i} = -a_{\alpha i}. \tag{2.2.18}$$

Finally the equations (2.2.16), in view of (2.2.17) and (2.2.18), assume the symmetric form

(2.2.19) $$\frac{\partial S}{\partial \ddot{q}_i} - Q_i = a_{\alpha i}\left(\frac{\partial S}{\partial \ddot{q}_\alpha} - Q_\alpha\right),$$

$$(\alpha = 1,2,\ldots,r;\ i = r+1, r+2,\ldots,n).$$

Corollary 3. Suppose we define r parameters $\lambda_1, \lambda_2, \ldots, \lambda_r$ by means of the relations

(2.2.20) $$\frac{\partial S}{\partial \ddot{q}_\alpha} - Q_\alpha = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_\alpha} \qquad (\alpha, \beta = 1,2,\ldots,r).$$

Then the equations (2.2.19) give

(2.2.21) $$\begin{aligned}\frac{\partial S}{\partial \ddot{q}_i} - Q_i &= a_{\alpha i}\lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_\alpha} \\ &= \lambda_\beta\, a_{\alpha i}\frac{\partial f_\beta}{\partial \dot{q}_\alpha}, \qquad (i = r+1, r+2,\ldots,n).\end{aligned}$$

But from the equations (2.2.1) we have

$$\frac{\partial f_\beta}{\partial \dot{q}_\alpha}\frac{\partial \dot{q}_\alpha}{\partial \dot{q}_i} + \frac{\partial f_\beta}{\partial \dot{q}_i} = 0,$$

or, by virtue of (2.2.13) we get

(2.2.22) $$\frac{\partial f_\beta}{\partial \dot{q}_i} = a_{\alpha i}\frac{\partial f_\beta}{\partial \dot{q}_\alpha}.$$

Using (2.2.22), the equations (2.2.21) become

(2.2.23) $$\frac{\partial S}{\partial \ddot{q}_i} - Q_i = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_i} \qquad (\beta = 1,2,\ldots,r;\ i = r+1, r+2,\ldots,n).$$

The n equations (2.2.20) together with (2.2.23) give the equations of motion in terms of Lagrange's multipliers.

2.3. A New Form for the Equations of Motion.

Here we again obtain the equations of motion for the mechanical system treated in the previous section. These equations, in place of involving S, the energy of acceleration of the system, will involve a new function R which depends on the kinetic energy of the system. Furthermore, we shall investigate the relationship between the functions S and R.

Suppose the position of the mechanical system is defined by n generalised coordinates $q_1, q_2, \ldots, q_n$, and let the nonlinear non-holonomic constraints of the type of Četaev, imposed on the mechanical system, be defined by $r < n$ equations of the form

$$f_\alpha(t; q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) = 0, \qquad (\alpha = 1, 2, \ldots, r). \tag{2.3.1}$$

If the functional matrix

$$\left\| \frac{\partial f_\alpha}{\partial \dot{q}_s} \right\| \qquad (s = 1, 2, \ldots, n)$$

is of rank r and the q's are suitably numbered, we have

$$\dot{q}_\alpha = \dot{q}_\alpha(t; q_1, q_2, \ldots, q_n; \dot{q}_{r+1}, \dot{q}_{r+2}, \ldots, \dot{q}_n) \equiv \dot{q}_\alpha(t; q_s; \dot{q}_i) \tag{2.3.2}$$

$$(\alpha = 1, 2, \ldots, r;\ i = r+1, r+2, \ldots, n;\ s = 1, 2, \ldots, n).$$

Let the cartesian coordinates, x_ν, in terms of the generalised coordinates be given by the following equations:

$$x_\nu = x_\nu(t; q_1, q_2, \ldots, q_n) \equiv x_\nu(t; q_s).$$

Differentiating these transformation equations thrice with respect to t, we get

$$
(2.3.3)\quad\left\{
\begin{aligned}
\dot{x}_\nu &= \frac{\partial x_\nu}{\partial q_i}\dot{q}_i + \frac{\partial x_\nu}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial x_\nu}{\partial t},\\
\ddot{x}_\nu &= \frac{\partial x_\nu}{\partial q_i}\ddot{q}_i + \frac{\partial x_\nu}{\partial q_\alpha}\ddot{q}_\alpha + \frac{\partial^2 x_\nu}{\partial q_i \partial q_j} + \frac{\partial^2 x_\nu}{\partial q_\alpha \partial q_\beta}\dot{q}_\alpha \dot{q}_\beta + 2\frac{\partial^2 x_\nu}{\partial q_\alpha \partial q_i}\dot{q}_\alpha \dot{q}_i\\
&\quad + 2\frac{\partial^2 x_\nu}{\partial t \partial q_i}\dot{q}_i + 2\frac{\partial^2 x_\nu}{\partial t \partial q_\alpha}\dot{q}_\alpha + \frac{\partial^2 x_\nu}{\partial t^2},\\
\dddot{x}_\nu &= 3\left\{\frac{\partial^2 x_\nu}{\partial t \partial q_i}\ddot{q}_i + \frac{\partial^2 x_\nu}{\partial t \partial q_\alpha}\ddot{q}_\alpha + \frac{\partial^2 x_\nu}{\partial q_i \partial q_j}\ddot{q}_i \dot{q}_j + \frac{\partial^2 x_\nu}{\partial q_\alpha \partial q_\beta}\ddot{q}_\alpha \dot{q}_\beta + \frac{\partial^2 x_\nu}{\partial q_\alpha \partial q_i}\ddot{q}_\alpha \dot{q}_i + \frac{\partial^2 x_\nu}{\partial q_\alpha \partial q_i}\ddot{q}_i \dot{q}_\alpha\right\} +\\
&\quad + \text{terms not containing } \ddot{q},\\
&(\alpha, \beta = 1, 2, \ldots, r;\ i, j = r+1, r+2, \ldots, n).
\end{aligned}\right.
$$

Hence, using the notation (2.2.13), we have

$$
(2.3.4)\qquad \frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_i} = \frac{\partial \dot{x}_\nu}{\partial \dot{q}_i} = \frac{\partial x_\nu}{\partial q_i} - a_{\alpha i}\frac{\partial x_\nu}{\partial q_\alpha}.
$$

Let T be the kinetic energy of the system which, with the help of the equations of constraint (2.3.2), is transformed to T' . Then we have the following results:

$$
T' = \frac{1}{2} m_{(\nu)} \dot{x}_\nu \dot{x}_\nu , \quad \dot{T}' = m_{(\nu)} \dot{x}_\nu \ddot{x}_\nu , \quad \ddot{T}' = m_{(\nu)} \ddot{x}_\nu \ddot{x}_\nu + m_{(\nu)} \dot{x}_\nu \dddot{x}_\nu .
$$

Using (2.3.3) and (2.3.4), we find

$$
\begin{aligned}
(2.3.5)\qquad \frac{\partial \ddot{T}'}{\partial \ddot{q}_i} &= 2 m_{(\nu)} \ddot{x}_\nu \frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_i} + m_{(\nu)} \dot{x}_\nu \frac{\partial \dddot{x}_\nu}{\partial \ddot{q}_i}\\
&= 2 m_{(\nu)} \ddot{x}_\nu \frac{\partial \dot{x}_\nu}{\partial \dot{q}_i} + 3 m_{(\nu)} \dot{x}_\nu \left\{\frac{\partial^2 x_\nu}{\partial t \partial q_i} + \frac{\partial^2 x_\nu}{\partial t \partial q_\alpha}\frac{\partial \ddot{q}_\alpha}{\partial \ddot{q}_i} + \right.\\
&\quad \left. + \frac{\partial^2 x_\nu}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 x_\nu}{\partial q_\alpha \partial q_\beta}\dot{q}_\beta \frac{\partial \ddot{q}_\alpha}{\partial \ddot{q}_i} + \frac{\partial^2 x_\nu}{\partial q_\alpha \partial q_i}\dot{q}_i \frac{\partial \ddot{q}_\alpha}{\partial \ddot{q}_i} + \frac{\partial^2 x_\nu}{\partial q_\alpha \partial q_i}\dot{q}_i\right\}.
\end{aligned}
$$

Let us now introduce a function T_o' , which is the function T' considered as a function of the q's and t only, i.e. for fixed values of the $\dot{q}$'s. In what follows we denote by $\dot{q}_{so}$ the fixed value of $\dot{q}_s$'s, $s = 1, 2, \ldots, n$. With this in view, corresponding to the expressions (2.3.3) we get the following expressions:

$$(2.3.6)\quad \begin{cases} \dot{x}_\nu = \dfrac{\partial x_\nu}{\partial q_i}\dot{q}_{io} + \dfrac{\partial x_\nu}{\partial q_\alpha}\dot{q}_{\alpha o} + \dfrac{\partial x_\nu}{\partial t}, \\ \ddot{x}_\nu = \dfrac{\partial^2 x_\nu}{\partial t \partial q_i}\dot{q}_i + \dfrac{\partial^2 x_\nu}{\partial t \partial q_\alpha}\dot{q}_\alpha + \dfrac{\partial^2 x_\nu}{\partial q_i \partial q_j}\dot{q}_{io}\dot{q}_j + \dfrac{\partial^2 x_\nu}{\partial q_\alpha \partial q_\beta}\dot{q}_{\alpha o}\dot{q}_\beta + \dfrac{\partial^2 x_\nu}{\partial q_\alpha \partial q_i}\dot{q}_{io}\dot{q}_\alpha + \\ \qquad + \dfrac{\partial^2 x_\nu}{\partial q_i \partial q_\alpha}\dot{q}_{\alpha o}\dot{q}_i + \dfrac{\partial^2 x_\nu}{\partial t^2}, \\ \dddot{x}_\nu = \left\{ \dfrac{\partial^2 x_\nu}{\partial t \partial q_i}\ddot{q}_i + \dfrac{\partial^2 x_\nu}{\partial t \partial q_\alpha}\ddot{q}_\alpha + \dfrac{\partial^2 x_\nu}{\partial q_i \partial q_j}\dot{q}_{io}\ddot{q}_j + \dfrac{\partial^2 x_\nu}{\partial q_\alpha \partial q_\beta}\dot{q}_{\alpha o}\ddot{q}_\beta + \right. \\ \qquad \left. + \dfrac{\partial^2 x_\nu}{\partial q_\alpha \partial q_i}\dot{q}_{io}\ddot{q}_\alpha + \dfrac{\partial^2 x_\nu}{\partial q_i \partial q_\alpha}\dot{q}_{\alpha o}\ddot{q}_i \right\} + \\ + \text{terms not containing } \ddot{q}. \end{cases}$$

Since the second of the relations (2.3.6) shows that

$$\frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_i} = 0,$$

it follows from the relations (2.3.6) that

$$(2.3.7)\qquad \frac{\partial \ddot{T}_o'}{\partial \ddot{q}_i} = 2 m_{(\nu)} \ddot{x}_\nu \frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_i} + m_{(\nu)} \dot{x}_\nu \frac{\partial \dddot{x}_\nu}{\partial \ddot{q}_i}$$

$$= m_{(\nu)} \dot{x}_\nu \left\{ \frac{\partial^2 x_\nu}{\partial t \partial q_i} + \frac{\partial^2 x_\nu}{\partial t \partial q_\alpha}\frac{\partial \ddot{q}_\alpha}{\partial \ddot{q}_i} + \frac{\partial^2 x_\nu}{\partial q_i \partial q_j}\dot{q}_{jo} + \frac{\partial^2 x_\nu}{\partial q_\beta \partial q_\alpha}\dot{q}_{\beta o}\frac{\partial \ddot{q}_\alpha}{\partial \ddot{q}_i} + \right.$$

$$\left. + \frac{\partial^2 x_\nu}{\partial q_\alpha \partial q_i}\dot{q}_{io}\frac{\partial \ddot{q}_\alpha}{\partial \ddot{q}_i} + \frac{\partial^2 x_\nu}{\partial q_i \partial q_\alpha}\dot{q}_{\alpha o} \right\}.$$

Because of (2.3.7) the relation (2.3.5) becomes

$$(2.3.8)\qquad \frac{\partial \ddot{T}'}{\partial \ddot{q}_i} = 2 m_{(\nu)} \ddot{x}_\nu \frac{\partial \dot{x}_\nu}{\partial \dot{q}_i} + 3 \frac{\partial \ddot{T}_o'}{\partial \ddot{q}_i}.$$

Let us define a function R' as follows:

$$R' = \frac{1}{2}\left(\ddot{T}' - 3\ddot{T}_o'\right).$$

The relation (2.3.8) then reduces to

(2.3.9) $$\frac{\partial R'}{\partial \ddot{q}_i} = m_{(\nu)} \ddot{x}_\nu \frac{\partial \dot{x}_\nu}{\partial \dot{q}_i} \qquad (i = r+1, r+2, \dots, n).$$

But from the principle of d'Alembert-Lagrange we have

$$\left(m_{(\nu)} \ddot{x}_\nu - X_\nu\right) \frac{\partial \dot{x}_\nu}{\partial \dot{q}_i} \delta q_i = 0$$

or, the independence of δq_i 's leads to the relations

(2.3.10) $$m_{(\nu)} \ddot{x}_\nu \frac{\partial \dot{x}_\nu}{\partial \dot{q}_i} = Q_i' \qquad (i = r+1, r+2, \dots, n),$$

where

$$Q_i' = X_\nu \frac{\partial \dot{x}_\nu}{\partial \dot{q}_i}.$$

From (2.3.9) and (2.3.10) it follows that

(2.3.11) $$\frac{\partial R'}{\partial \ddot{q}_i} = Q_i' , \qquad (i = r+1, r+2, \dots, n)$$

which are the required equations of motion.

Comparing the equations (2.2.11) and (2.3.11) we observe that both S' and R' satisfy the same equation. In other words, the function R' coincides with the function S' , the energy of acceleration of the mechanical system, as far as the terms in $\ddot{q}_i$ $(i = r+1, r+2, \dots, n)$ are concerned.

Further, let R denote the function R' without taking into consideration the equations of constraint (2.3.1), i.e. without changing the dependent $\dot{q}_\alpha$ into $\dot{q}_i$ and $\ddot{q}_\alpha$ into $\ddot{q}_i$. Then we have

$$\frac{\partial R'}{\partial \ddot{q}_i} = \frac{\partial R}{\partial \ddot{q}_i} - \frac{\partial R}{\partial \ddot{q}_\alpha}\frac{\partial \ddot{q}_\alpha}{\partial \ddot{q}_i}$$
$$(\alpha = 1,2,\ldots,r\,;\ i = r+1, r+2,\ldots,n).$$

In view of (2.2.18) the above relations become

(2.3.12) $$\frac{\partial R'}{\partial \ddot{q}_i} = \frac{\partial R}{\partial \ddot{q}_i} - a_{\alpha i}\frac{\partial R}{\partial \ddot{q}_\alpha} \qquad (\alpha = 1,2,\ldots,r;\ i = r+1, r+2,\ldots,n).$$

Also from (2.2.14) and (2.2.15) we have

(2.3.13) $$Q_i' = Q_i - a_{\alpha i}Q_\alpha.$$

By virtue of (2.3.12) and (2.3.13) the equations of motion (2.3.11) assume the symmetric form

(2.3.14) $$\frac{\partial R}{\partial \ddot{q}_i} - Q_i = a_{\alpha i}\left(\frac{\partial R}{\partial \ddot{q}_\alpha} - Q_\alpha\right)$$
$$(\alpha = 1,2,\ldots,r\,;\ i = r+1, r+2,\ldots,n).$$

Comparing the equations (2.3.14) with (2.2.19), we find that R and S both satisfy the same differential equations. Consequently the function R coincides with S as far as the terms in $\ddot{q}_s$ $(s = 1,2,\ldots,n)$ are concerned.

<u>Special Case.</u> Let the linear non-holonomic constraints be of the form

$$\dot{q}_\alpha = A_{\alpha i}\dot{q}_i + A_\alpha \qquad (\alpha = 1,2,\ldots,r;\ i = r+1, r+2,\ldots,n),$$

where $A_{\alpha i}$, A_α are functions of $q_1, q_2, \ldots, q_n$ and t.

Then

$$a_{\alpha i} = -\frac{\partial \dot{q}_\alpha}{\partial \dot{q}_i} = -A_{\alpha i} \qquad (\alpha = 1,2,\ldots,r;\ i = r+1, r+2,\ldots,n).$$

Accordingly the general equations of motion (2.3.11) by virtue of (2.3.13) reduce to the following ones:

$$\frac{\partial R'}{\partial \ddot{q}_i} = Q_i + A_{\alpha i} Q_\alpha \qquad (\alpha = 1,2,\ldots,r;\ i = r+1, r+2,\ldots,n). \tag{2.3.15}$$

The above equations for the linear non-holonomic systems were established by I.Cenov [9] .

2.4. A Transformation of the Equationsof Motion.

In the preceding section we obtained the equations of motion in the symmetric form.

$$\frac{\partial R}{\partial \ddot{q}_i} - Q_i = a_{\alpha i}\left(\frac{\partial R}{\partial \ddot{q}_\alpha} - Q_\alpha\right) \qquad (\alpha = 1,2,\ldots,r;\ i = r+1, r+2,\ldots,n). \tag{2.4.1}$$

Let us define a function K as follows:

$$K = R - Q_i \ddot{q}_i - Q_\alpha \ddot{q}_\alpha . \tag{2.4.2}$$

Then by virtue of (2.2.18) we obtain

$$\frac{\partial K}{\partial \ddot{q}_i} = \frac{\partial R}{\partial \ddot{q}_i} - a_{\alpha i}\frac{\partial R}{\partial \ddot{q}_\alpha} - Q_i + a_{\alpha i} Q_\alpha$$

and, consequently the equations (2.4.1) reduce to the form

$$\frac{\partial K}{\partial \ddot{q}_i} = 0 \qquad (i = r+1, r+2,\ldots,n). \tag{2.4.3}$$

Moreover, let K' denote the function K when the equations of constraint (2.2.1) are taken into consideration. Then K' will satisfy the equation

$$K' = R' - Q_i \ddot{q}_i - Q_\alpha \ddot{q}_\alpha , \tag{2.4.4}$$

where $\ddot{q}_\alpha$ is considered as a function of $\ddot{q}_i$ $(i = r+1, r+2,\ldots,n)$.

Hence we again have

$$\frac{\partial K'}{\partial \ddot{q}_i} = \frac{\partial R'}{\partial \ddot{q}_i} - Q_i + a_{\alpha i} Q_\alpha .$$

By virtue of (2.3.13) the equations of motion (2.3.11) then assume the form

$$\frac{\partial K'}{\partial \ddot{q}_i} = 0 \qquad (i = r+1, r+2, \dots, n). \tag{2.4.5}$$

The equations of motion in the form (2.4.5) show that the function K' assumes stationary values in the actual motion when compared to any conceivable motion (consistent with the constraints), obtained by varying $\ddot{q}_i$ in K'.

In the next section we shall prove that the function K' is actually a minimum along the actual motion of the mechanical system.

2.5. The Function K and the Gaussian Constraint.

In order to show that of all trajectories consistent with the constraints, the actual trajectory is that which has the least value of the function K', we shall first prove that, as far as terms in $\ddot{q}_i$ are concerned, the function K' coincides with the Gaussian constraint defined by the equation (1.1.3)

If G denotes the Gaussian constraint, we have

$$\begin{aligned} G &= \frac{1}{2} m_\nu \left(\frac{X_\nu}{m_{(\nu)}} - \ddot{x}_\nu \right)^2 \\ &= \frac{1}{2} m_{(\nu)} \ddot{x}_\nu \ddot{x}_\nu - X_\nu \ddot{x}_\nu + \text{terms not containing } \ddot{x}_\nu . \end{aligned}$$

The first term on the right-hand side is the energy of acceleration S' obtained by taking constraints into account. If in the second term we substitute for $\ddot{x}_\nu$ its expression from the second of the relations (2.3.3), we get

$$G = S' - X_\nu\left(\frac{\partial x_\nu}{\partial q_i}\ddot{q}_i + \frac{\partial x_\nu}{\partial q_\alpha}\ddot{q}_\alpha\right) + \text{terms not containing } \ddot{q}.$$

As remarked in Sec.2.3 the function R' coincides with S' as far as terms in $\ddot{q}$ are concerned. Therefore, we can write

$$G = R' - Q_i\ddot{q}_i - Q_\alpha\ddot{q}_\alpha + \text{terms not containing } \ddot{q}$$

$$= K' + \text{terms not containing } \ddot{q}.$$

Thus the truth of the assertion is proved.

Next, to show the minimum property of K', we only have to prove that this property holds also for G.

To establish this result, let $\ddot{x}_\nu$ be a typical component of acceleration in a trajectory under consideration (which is supposed to be kinemetically possible but is not necessarily the actual trajectory). Further, let $\ddot{x}_{\nu o}$ be the corresponding component of acceleration in the actual trajectory. We also assume that at the time t the coordinates, x_ν, and the velocities, $\dot{x}_\nu$, of the system are the same in the considered and the actual trajectory. Then, if $d\dot{x}_\nu$ is the change in $\dot{x}_\nu$ along the actual trajectory in an interval of time dt, and $\delta'\dot{x}_\nu$ is the change along the considered trajectory in an interval of time $\delta' t = dt$, we have

(2.5.1) $$\ddot{x}_\nu = \frac{\delta'\dot{x}_\nu}{\delta' t} \quad , \quad \ddot{x}_{\nu 0} = \frac{d\dot{x}_\nu}{dt} .$$

Now, according to equation (1.3.2) a small displacement of the system, δx_ν , which is proportional to $d\dot{x}_\nu - \delta'\dot{x}_\nu$, is consistent with the equations of constraint, i.e. it is a virtual displacement. Hence the principle of d'Alembert-Lagrange can be written in the form

$$\left(m_{(\nu)}\ddot{x}_\nu - X_\nu\right)\left(d\dot{x}_\nu - \delta'\dot{x}_\nu\right) = 0,$$

or, by virtue of (2.5.1), in the form

(2.5.2) $$\left(m_{(\nu)}\ddot{x}_\nu - X_\nu\right)\left(\ddot{x}_\nu - \ddot{x}_{\nu 0}\right) = 0,$$

or, finally in the form

$$\frac{1}{2} m_\nu \left(\frac{X_\nu}{m_{(\nu)}} - \ddot{x}_\nu\right)^2 - \frac{1}{2} m_{(\nu)} \left(\frac{X_\nu}{m_{(\nu)}} - \ddot{x}_{\nu 0}\right)^2 = \frac{1}{2} m_\nu \left(\ddot{x}_\nu - \ddot{x}_{\nu 0}\right)^2 .$$

Since the terms in the summation on the right-hand side are all positive, it follows that

$$\frac{1}{2} m_\nu \left(\frac{X_\nu}{m_{(\nu)}} - \ddot{x}_\nu\right)^2 > \frac{1}{2} m_\nu \left(\frac{X_\nu}{m_{(\nu)}} - \ddot{x}_{\nu 0}\right)^2 ,$$

which establishes the result.

Remark 1. For a linear non-holonomic system the mimimum property of the function K' was proved by I.Cenov [10].

Remark 2. As a consequence of the fact that the function K' coincides with the Gaussian constraint G as far as terms in $\ddot{q}$ are concerned, it follows that

(2.5.3) $$\frac{\partial G}{\partial \ddot{q}_i} = 0 \qquad \left(i = r+1, r+2, \ldots, n\right).$$

These equations establish the stationary property of G. That the stationary property automatically leads to a minimum has already been proved above. Since the equations (2.5.3) were deduced from the principle of d'Alembert-Lagrange, the compatibility of this principle with the principle of least constraint of Gauss is indirectly established.

The following is an alternative but interesting approach of deducing the principle of Gauss from the principle of d'Alembert-Lagrange.

From the equation (2.5.2) we have

$$(2.5.4) \qquad \left(m_{(\nu)}\ddot{x}_\nu - X_\nu\right)\left(\ddot{x}_\nu - \ddot{x}_{\nu 0}\right) = 0 .$$

But since $\ddot{x}_\nu - \ddot{x}_{\nu 0}$ represents the change in the acceleration causing a deviation in the trajectory of the particle, we can put

$$\delta \ddot{x}_\nu = \ddot{x}_\nu - \ddot{x}_{\nu 0} .$$

The equation (2.5.4) then reduces to

$$\left(m_{(\nu)}\ddot{x}_\nu - X_\nu\right)\delta \ddot{x}_\nu = 0 .$$

Since the forces applied are given and cannot be varied, the above equation may be rewritten as follows:

$$\left(X_\nu - m_{(\nu)}\ddot{x}_\nu\right)\delta\left(\frac{X_\nu - m_{(\nu)}\ddot{x}_\nu}{m_{(\nu)}}\right) = 0 .$$

This, however, means that

$$\delta\left[\frac{1}{2} m_\nu \left(\frac{X_\nu}{m_{(\nu)}} - \ddot{x}_\nu\right)^2\right] = 0 .$$

This again establishes the stationary property of the Gaussian constraint for the actual motion. To prove that it is actually a minimum we can proceed as before.

2.6. Lagrangian Form for the Equations of Motion.

In Sec. 2.3. we obtained the equations of motion for a nonlinear non-holonomic system in the form given by equations (2.3.14), i.e.

$$\frac{\partial R}{\partial \ddot{q}_i} - Q_i = a_{\alpha i}\left(\frac{\partial R}{\partial \ddot{q}_\alpha} - Q_\alpha\right) \qquad (\alpha = 1,2,\dots,r;\ i = r+1, r+2,\dots,n) \tag{2.6.1}$$

with

$$R = \tfrac{1}{2}\left(\ddot{T} - 3\ddot{T}_0\right),$$

where T is the kinetic energy of the system without taking into consideration the constraints imposed on the system, and T_0 is the value of T for fixed values of the generalised velocities $\dot{q}_s$ $(s = 1,2,\dots,n)$.

Here our aim is to transform the equations (2.6.1) so that they assume a form similar to Lagrange's equations of motion. To this end we first prove the following

Lemma: For R and T defined above, the identity

$$\frac{\partial R}{\partial \ddot{q}_s} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} \tag{2.6.2}$$

holds for $s = 1,2,\dots,n$.

Proof: We have

$$\left\{\begin{aligned} T &= \tfrac{1}{2} m_{(\nu)} \dot{x}_\nu \dot{x}_\nu, \\ \dot{T} &= m_{(\nu)} \dot{x}_\nu \ddot{x}_\nu, \\ \ddot{T} &= m_{(\nu)} \ddot{x}_\nu \ddot{x}_\nu + m_{(\nu)} \dot{x}_\nu \dddot{x}_\nu. \end{aligned}\right. \tag{2.6.3}$$

Moreover, since

$$x_\nu = x_\nu(t\,;q_s)$$

we get

$$(2.6.4)\quad \begin{cases} \dot{x}_\nu = \dfrac{\partial x_\nu}{\partial q_s}\dot{q}_s + \dfrac{\partial x_\nu}{\partial t}, \\ \ddot{x}_\nu = \dfrac{\partial x_\nu}{\partial q_s}\ddot{q}_s + \dfrac{\partial^2 x_\nu}{\partial q_s \partial q_\ell}\dot{q}_s\dot{q}_\ell + 2\dfrac{\partial^2 x_\nu}{\partial q_s \partial t}\dot{q}_s + \dfrac{\partial^2 x_\nu}{\partial t^2}, \\ \dddot{x}_\nu = 3\left\{\dfrac{\partial^2 x_\nu}{\partial q_s \partial t}\ddot{q}_s + \dfrac{\partial^2 x_\nu}{\partial q_s \partial q_\ell}\ddot{q}_s\dot{q}_\ell\right\} + \text{terms not containing } \ddot{q}, \\ \qquad (s, \ell = 1, 2, \ldots, n). \end{cases}$$

From the relations (2.6.3) and (2.6.4) it follows that

$$\frac{\partial \ddot{T}}{\partial \ddot{q}_s} = 2m_{(\nu)}\ddot{x}_\nu \frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_s} + m_{(\nu)}\dot{x}_\nu \frac{\partial \dddot{x}_\nu}{\partial \ddot{q}_s}$$

or,

$$(2.6.5)\qquad \frac{\partial \ddot{T}}{\partial \ddot{q}_s} = 2m_{(\nu)}\ddot{x}_\nu \frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_s} + 3m_{(\nu)}\dot{x}_\nu\left\{\frac{\partial^2 x_\nu}{\partial q_s \partial t} + \frac{\partial^2 x_\nu}{\partial q_s \partial q_\ell}\dot{q}_\ell\right\}.$$

Now if we denote the fixed value of $\dot{q}_s$ by $\dot{q}_{so}$, then

$$(2.6.6)\quad \begin{cases} \dot{x}_\nu = \dfrac{\partial x_\nu}{\partial q_s}\dot{q}_{so} + \dfrac{\partial x_\nu}{\partial t}, \\ \ddot{x}_\nu = \dfrac{\partial^2 x_\nu}{\partial q_s \partial t}\dot{q}_{so} + \dfrac{\partial^2 x_\nu}{\partial q_s \partial q_\ell}\dot{q}_{so}\dot{q}_\ell + \dfrac{\partial^2 x_\nu}{\partial t \partial q_s}\dot{q}_s + \dfrac{\partial^2 x_\nu}{\partial t^2}, \\ \dddot{x}_\nu = \dfrac{\partial^2 x_\nu}{\partial q_s \partial q_\ell}\dot{q}_{\ell o}\ddot{q}_s + \dfrac{\partial^2 x_\nu}{\partial t \partial q_s}\ddot{q}_s + \text{terms not containing } \ddot{q}, \end{cases}$$

where in the first term on the right-hand side of the last relation we have interchanged the repeated suffixes.

For these expressions of $\dot{x}_\nu$, $\ddot{x}_\nu$ and $\dddot{x}_\nu$ we have

$$T_0 = \frac{1}{2} m_{(\nu)} \dot{x}_\nu \dot{x}_\nu ,$$

$$\dot{T}_0 = m_{(\nu)} \dot{x}_\nu \ddot{x}_\nu ,$$

$$\ddot{T}_0 = m_{(\nu)} \ddot{x}_\nu \ddot{x}_\nu + m_{(\nu)} \dot{x}_\nu \dddot{x}_\nu .$$

Hence, in view of (2.6.6) we find

$$\frac{\partial \ddot{T}_0}{\partial \ddot{q}_s} = 2 m_{(\nu)} \ddot{x}_\nu \frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_s} + m_{(\nu)} \dot{x}_\nu \frac{\partial \dddot{x}_\nu}{\partial \ddot{q}_s}$$

or,

(2.6.7) $$\frac{\partial \ddot{T}_0}{\partial \ddot{q}_s} = m_{(\nu)} \dot{x}_\nu \left\{ \frac{\partial^2 x_\nu}{\partial q_s \partial t} + \frac{\partial^2 x_\nu}{\partial q_s \partial q_l} \dot{q}_{l0} \right\} .$$

Thus (2.6.5) with the help of (2.6.7) reduces to

$$\frac{\partial \ddot{T}}{\partial \ddot{q}_s} - 3 \frac{\partial \ddot{T}_0}{\partial \ddot{q}_s} = 2 m_{(\nu)} \ddot{x}_\nu \frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_s}$$

or,

(2.6.8) $$\frac{\partial R}{\partial \ddot{q}_s} = m_{(\nu)} \ddot{x}_\nu \frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_s} , \qquad (s = 1, 2, \cdots, n).$$

Also we have

$$\begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} &= \frac{d}{dt} \frac{\partial}{\partial \dot{q}_s} \left(\frac{1}{2} m_{(\nu)} \dot{x}_\nu \dot{x}_\nu \right) - \frac{\partial}{\partial q_s} \left(\frac{1}{2} m_{(\nu)} \dot{x}_\nu \dot{x}_\nu \right) \\ &= m_{(\nu)} \ddot{x}_\nu \frac{\partial \dot{x}_\nu}{\partial \dot{q}_s} + m_{(\nu)} \dot{x}_\nu \frac{d}{dt} \frac{\partial \dot{x}_\nu}{\partial \dot{q}_s} - m_{(\nu)} \dot{x}_\nu \frac{d}{dt} \frac{\partial x_\nu}{\partial q_s} . \end{aligned}$$

But from the first two relations of (2.6.4) it follows that

$$\frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_s} = \frac{\partial \dot{x}_\nu}{\partial \dot{q}_s} = \frac{\partial x_\nu}{\partial q_s} .$$

Hence we get

(2.6.9) $$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = m_{(\nu)} \ddot{x}_\nu \frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_s} , \qquad (s = 1, 2, \cdots, n).$$

By virtue of (2.6.8) and (2.6.9) the Lemma is established.

The above is an independent proof of the lemma which, of course, can be easily established if we recognise the fact that R coincides with S, the enrgy of acceleration, as far as the terms in $\ddot{q}_s$ are concerned, and make use of the well-known result

$$\frac{\partial S}{\partial \ddot{q}_s} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} \qquad (s = 1, 2, \cdots, n).$$

Let us now use the identity (2.6.2) to transform the equations of motion (2.6.1). This leads us to the following form of the equations of motion

$$(2.6.10) \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i = a_{\alpha i}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha\right)$$

$$(\alpha = 1, 2, \ldots, r;\ i = r+1, r+2, \cdots, n).$$

These are the Lagrangian equations of motion for the nonlinear non-holonomic systems.

Some Special Cases:

Case I. If the system is holonomic with n degrees of freedom, we have

$$a_{\alpha i} = 0 \qquad (\alpha = 1, 2, \ldots, r;\ i = r+1, r+2, \ldots, n).$$

The equations (2.6.10) then reduce to the usual form:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s \qquad (s = 1, 2, \cdots, n).$$

Case II. If the system is linear honholonomic, the constraints are given by non-integrable equations of the type:

$$\dot{q}_\alpha = A_{\alpha i}\dot{q}_i + A_\alpha \qquad (\alpha = 1, 2, \ldots, r;\ i = r+1, r+2, \ldots, n),$$

where $A_{\alpha i}$, A_α are functions of $q_1, q_2, \ldots, q_n$ and t.

In such a case we have

$$a_{\alpha i} = -\frac{\partial \dot{q}_\alpha}{\partial \dot{q}_i} = -A_{\alpha i}.$$

With these values of $a_{\alpha i}$ the equations (2.6.10) become

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i + A_{\alpha i}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha\right) = 0$$

$$(\alpha = 1, 2, \ldots, r;\ i = r+1, r+2, \ldots, n).$$

Case III. Let us define r parameter $\lambda_1, \lambda_2, \ldots, \lambda_r$ in the following manner:

(2.6.11) $$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_\alpha} \qquad (\alpha, \beta = 1, 2, \ldots, r),$$

where

$$f_\alpha(t; q_s; \dot{q}_s) = 0 \qquad (\alpha = 1, 2, \ldots, r;\ s = 1, 2, \ldots, n)$$

are the equations of constraint.

Then the equations (2.6.10) yield

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i = a_{\alpha i}\lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_\alpha}$$
$$= \lambda_\beta a_{\alpha i}\frac{\partial f_\beta}{\partial \dot{q}_\alpha}.$$

Using the relations (2.2.22), the above becomes

(2.6.12) $$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_i} \qquad (\beta = 1, 2, \ldots, r;\ i = r+1, r+2, \ldots, n).$$

The equations (2.6.11) together with (2.6.12) represent the equations of motion in terms of the Lagrangian multipliers.

2.7. Another Transformation for the Equations of Motion.

In the last section we considered the equations of motion in the so-called Lagrangian form, involving the kinetic energy T. It is assumed that in the expression for T no substitution has been made for the dependent velocities in terms of the independent ones, i.e. constraints have not been taken into account. In the present section our aim is to transform the above-mentioned equations by changing T into T', in which the dependent velocities have been expressed in terms of the independent ones.

Let us assume that the equations (2.6.10) can be thrown into the following form:

$$\frac{d}{dt}\frac{\partial T'}{\partial \dot q_i} - \frac{\partial T'}{\partial q_i} + D_i = Q_i - a_{\alpha i} Q_\alpha \qquad (\alpha = 1,2,\dots,r;\ i = r+1, r+2,\dots,n) \tag{2.7.1}$$

where D_i is a corrective term to be determined later.

By virtue of equations (2.6.10) we obtain from (2.7.1) the following expressions for D_i :

$$D_i = \frac{d}{dt}\frac{\partial T}{\partial \dot q_i} - \frac{\partial T}{\partial q_i} - a_{\alpha i}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q_\alpha} - \frac{\partial T}{\partial q_\alpha}\right) - \frac{d}{dt}\frac{\partial T'}{\partial \dot q_i} + \frac{\partial T'}{\partial q_i}. \tag{2.7.2}$$

But on using (2.2.18) we have

$$\frac{\partial T'}{\partial \dot q_i} = \frac{\partial T}{\partial \dot q_i} - a_{\alpha i}\frac{\partial T}{\partial \dot q_\alpha},$$

$$\frac{d}{dt}\frac{\partial T'}{\partial \dot q_i} = \frac{d}{dt}\frac{\partial T}{\partial \dot q_i} - a_{\alpha i}\frac{d}{dt}\frac{\partial T}{\partial \dot q_\alpha} - \dot a_{\alpha i}\frac{\partial T}{\partial \dot q_\alpha}, \tag{2.7.3}$$

and

$$\frac{\partial T'}{\partial q_i} = \frac{\partial T}{\partial q_i} + \frac{\partial T}{\partial \dot q_\alpha}\frac{\partial \dot q_\alpha}{\partial q_i} \qquad (\alpha = 1,2,\dots,r;\ i = r+1, r+2,\dots,n). \tag{2.7.4}$$

By virtue of (2.7.3) and (2.7.4) we get from (2.7.2) the following expression for D_i :

(2.7.5) $$D_i = a_{\alpha i}\frac{\partial T}{\partial q_\alpha} + \frac{\partial T}{\partial \dot{q}_\alpha}\left(\dot{a}_{\alpha i} + \frac{\partial \dot{q}_\alpha}{\partial q_i}\right)$$

$$(\alpha = 1,2,\dots,r\,;\ i = r+1, r+2,\dots,n).$$

Let us now regard T as a function of $\dot{q}_\alpha$ (α=1,2,..., r) only, and in the sequel denote it by T_1 . Then using (2.2.18), we have

$$\frac{\partial T_1}{\partial \dot{q}_i} = -a_{\alpha i}\frac{\partial T}{\partial \dot{q}_\alpha},$$

(2.7.6) $$\frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_i} = -a_{\alpha i}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \dot{a}_{\alpha i}\frac{\partial T}{\partial \dot{q}_\alpha},$$

and

(2.7.7) $$\frac{\partial T_1}{\partial q_i} = \frac{\partial T}{\partial \dot{q}_\alpha}\frac{\partial \dot{q}_\alpha}{\partial q_i}.$$

Using (2.6.2),(2.7.6) and (2.7.7), we get from (2.7.5) the following expression:

$$D_i = a_{\alpha i}\frac{\partial T}{\partial q_\alpha} + \frac{\partial T_1}{\partial q_i} - a_{\alpha i}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_i}$$

$$= \frac{\partial T_1}{\partial q_i} - \frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_i} - a_{\alpha i}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha}\right)$$

or,

(2.7.8) $$D_i = \frac{\partial T_1}{\partial q_i} - \frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_i} - a_{\alpha i}\frac{\partial R}{\partial \ddot{q}_\alpha}.$$

Finally let us consider R as a function of $\ddot{q}_\alpha$ (α =1,2,...,r) only and in the sequel denote it by R_1 .

Then by virtue of (2.2.18) we get

$$\frac{\partial R_1}{\partial \ddot{q}_i} = -a_{\alpha i}\frac{\partial R}{\partial \ddot{q}_\alpha}. \tag{2.7.9}$$

In view of (2.7.9) the final expression for D_i, given by (2.7.8), becomes

$$D_i = \frac{\partial T_1}{\partial q_i} - \frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_i} + \frac{\partial R_1}{\partial \ddot{q}_i}.$$

With this expression for D_i the equations (2.7.1) take the form

$$\frac{d}{dt}\frac{\partial T'}{\partial \dot{q}_i} - \frac{\partial T'}{\partial q_i} + \frac{\partial T_1}{\partial q_i} - \frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_i} + \frac{\partial R_1}{\partial \ddot{q}_i} = Q_i - a_{\alpha i}Q_\alpha, \tag{2.7.10}$$

$$(\alpha = 1, 2, \ldots, r;\ i = r+1, r+2, \ldots, n).$$

These are the required transformed equations.

In the case of linear nonholonomic systems the equations of motion in the form (2.7.10) were established by I. Cenov [8].

2.8. A Novel Form for the Equations of Motion.

Once again we shall transform the equations of motion (2.6.10) by means of an identity to be established below. This novel form of the equations of motion will include as a special case the result obtained by I. Cenov [7] for holonomic systems.

Let us first prove the following

Lemma: If T denotes the kinetic energy of a holonomic mechanical system, then the identity

$$(2.8.1) \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial \dot{T}}{\partial \dot{q}_s} = -\frac{\partial T}{\partial q_s}$$

<u>holds</u> <u>for</u> $s=1,2,\ldots,n$.

Proof: We have

$$(2.8.2) \quad \begin{cases} T = \frac{1}{2} m_{(\nu)} \dot{x}_\nu \dot{x}_\nu , \\ \dot{T} = m_{(\nu)} \dot{x}_\nu \ddot{x}_\nu . \end{cases}$$

Also we have for $\ell, s=1,2,\ldots,n$ the following expressions:

$$(2.8.3) \quad \begin{cases} x_\nu = x_\nu(t;q_s), \\ \dot{x}_\nu = \frac{\partial x_\nu}{\partial q_s}\dot{q}_s + \frac{\partial x_\nu}{\partial t}, \\ \ddot{x}_\nu = \frac{\partial x_\nu}{\partial q_s}\ddot{q}_s + \frac{\partial^2 x_\nu}{\partial q_s \partial q_\ell}\dot{q}_s\dot{q}_\ell + 2\frac{\partial^2 x_\nu}{\partial q_s \partial t}\dot{q}_s + \frac{\partial^2 x_\nu}{\partial t^2}. \end{cases}$$

Now in view of the relations (2.8.2) and (2.8.3) we find that

$$(2.8.4) \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} = m_{(\nu)}\dot{x}_\nu \frac{d}{dt}\frac{\partial \dot{x}_\nu}{\partial \dot{q}_s} + m_{(\nu)}\ddot{x}_\nu \frac{\partial \dot{x}_\nu}{\partial \dot{q}_s},$$

and

$$(2.8.5) \qquad \frac{\partial \dot{T}}{\partial \dot{q}_s} = m_{(\nu)}\dot{x}_\nu \frac{\partial \ddot{x}_\nu}{\partial \dot{q}_s} + m_{(\nu)}\ddot{x}_\nu \frac{\partial \dot{x}_\nu}{\partial \dot{q}_s}.$$

Since

$$\frac{\partial \dot{x}_\nu}{\partial \dot{q}_s} = \frac{\partial x_\nu}{\partial q_s},$$

from (2.8.4) and (2.8.5) we have

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial \dot{T}}{\partial \dot{q}_s} = m_{(\nu)}\dot{x}_\nu\left(\frac{d}{dt}\frac{\partial x_\nu}{\partial q_s} - \frac{\partial \ddot{x}_\nu}{\partial \dot{q}_s}\right).$$

By virtue of (2.8.3) the above becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial \dot{T}}{\partial \dot{q}_s} = m_{(\nu)}\dot{x}_\nu\left[\frac{\partial^2 x_\nu}{\partial q_s \partial q_\ell}\dot{q}_\ell + \frac{\partial^2 x_\nu}{\partial q_s \partial t} - 2\left\{\frac{\partial^2 x_\nu}{\partial q_s \partial q_\ell}\dot{q}_\ell + \frac{\partial^2 x_\nu}{\partial q_s \partial t}\right\}\right]$$

$$= -m_{(\nu)}\dot{x}_\nu\left[\frac{\partial^2 x_\nu}{\partial q_s \partial q_\ell}\dot{q}_\ell + \frac{\partial^2 x_\nu}{\partial q_s \partial t}\right]$$

$$= -m_{(\nu)}\dot{x}_\nu\frac{\partial \dot{x}_\nu}{\partial q_s}$$

$$= -\frac{\partial T}{\partial q_s}.$$

This proves the Lemma.

Taking into consideration the identity (2.8.1) we have

$$(2.8.6)\qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = 2\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial \dot{T}}{\partial \dot{q}_s} \qquad (s=1,2,\dots,n).$$

Hence the equations of motion (2.6.10) assume the form

$$(2.8.7)\qquad 2\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial \dot{T}}{\partial \dot{q}_i} - Q_i = a_{\alpha i}\left(2\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial \dot{T}}{\partial \dot{q}_\alpha} - Q_\alpha\right),$$

$$(\alpha=1,2,\dots,r;\ i=r+1,r+2,\dots,n).$$

Special Cases:

Case I. If the system is holonomic with n degrees of freedom, we have

$$a_{\alpha i} = 0 \qquad \text{for } \alpha=1,2,\dots,r;\ i=r+1,r+2,\dots,n.$$

In this case the equations (2.8.7) reduce to

$$2\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial \dot{T}}{\partial \dot{q}_s} = Q_s \qquad (s=1,2,\dots,n),$$

a result of I.Cenov [7].

Case II. If the system is linear nonholonomic, the constraints are given by non-integrable equations of the type:

$$\dot{q}_\alpha = A_{\alpha i}\dot{q}_i + A_\alpha \qquad (\alpha=1,2,\dots,r;\ i=r+1,r+2,\dots,n),$$

where $A_{\alpha i}$, A_α depend only on $q_1, q_2, \dots, q_n$ and t.

Hence

$$a_{\alpha i} = -\frac{\partial \dot{q}_\alpha}{\partial \dot{q}_i} = -A_{\alpha i}\,.$$

Consequently the equations of motion (2.8.7) take the form

$$2\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial \dot{T}}{\partial \dot{q}_i} - Q_i + A_{\alpha i}\left(2\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial \dot{T}}{\partial \dot{q}_\alpha} - Q_\alpha\right) = 0$$

$$(\alpha=1,2,\dots,r;\ i=r+1,r+2,\dots,n).$$

Case III. Let us define r parameters $\lambda_1, \lambda_2, \dots, \lambda_r$ by means of the equations

(2.8.8) $$2\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial \dot{T}}{\partial \dot{q}_\alpha} - Q_\alpha = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_\alpha} \qquad (\alpha,\beta=1,2,\dots,r),$$

where f_β are defined by the equations of constraint (2.2.1). Then from equations (2.3.8) we have

(2.8.9) $$2\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial \dot{T}}{\partial \dot{q}_i} - Q_i = a_{\alpha i}\lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_\alpha}, \qquad (i=r+1,r+2,\dots,n).$$

Changing the order of summation on the right-hand side of equations (2.8.9) and making use of the relations:

$$\frac{\partial f_\beta}{\partial \dot{q}_i} = a_{\alpha i}\frac{\partial f_\beta}{\partial \dot{q}_\alpha}$$

we can write the equations (2.8.9) in the following form:

$$(2.8.10) \qquad 2\frac{d}{dt}\frac{\partial T}{\partial \dot q_i} - \frac{\partial \dot T}{\partial \dot q_i} - Q_i = \lambda_\beta \frac{\partial f_\beta}{\partial \dot q_i} \qquad (\beta = 1,2,\ldots,r;\ i = r+1, r+2,\ldots,n).$$

The set of equations (2.8.8) together with (2.8.10) forms the equations of motion with r Lagrangian multipliers.

2.9. Another Novel Form for the Equations of Motion.

Here we shall transform the equations of motion (2.8.7) so that they assume a very simple and novel form. This interesting result will include as a special case a result obtained by I.Cenov [7] .

To obtain the equations of motion in the desired form let us first prove the following

Lemma: If T denotes the kinetic energy of a holonomic mechanical system, then the identity

$$(2.9.1) \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot q_s} + \frac{\partial \dot T}{\partial \dot q_s} = \frac{\partial \ddot T}{\partial \ddot q_s}$$

holds for $s=1,2,\ldots,n$.

Proof: We have

$$(2.9.2) \qquad \begin{cases} T = \frac{1}{2} m_{(\nu)} \dot x_\nu \dot x_\nu , \\ \dot T = m_{(\nu)} \dot x_\nu \ddot x_\nu , \\ \ddot T = m_{(\nu)} \dot x_\nu \dddot x_\nu + m_{(\nu)} \ddot x_\nu \ddot x_\nu , \end{cases}$$

where

$$(2.9.3) \qquad \begin{cases} x_\nu = x_\nu(t; q_s) \\ \dot x_\nu = \frac{\partial x_\nu}{\partial q_s}\dot q_s + \frac{\partial x_\nu}{\partial t} , \\ \ddot x_\nu = \frac{\partial x_\nu}{\partial q_s}\ddot q_s + \frac{\partial^2 x_\nu}{\partial q_s \partial q_l}\dot q_s \dot q_l + 2\frac{\partial^2 x_\nu}{\partial q_s \partial t}\dot q_s + \frac{\partial^2 x_\nu}{\partial t^2} , \\ \qquad (l, s = 1,2,\ldots,n). \end{cases}$$

By virtue of the relations (2.9.2) and (2.9.3) we have

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_s} + \frac{\partial \dot T}{\partial q_s} = \frac{d}{dt}\left(m_{(\nu)}\dot x_\nu \frac{\partial \dot x_\nu}{\partial \dot q_s}\right) + m_{(\nu)}\dot x_\nu \frac{\partial \ddot x_\nu}{\partial \dot q_s} + m_{(\nu)}\ddot x_\nu \frac{\partial \dot x_\nu}{\partial q_s}$$

$$= 2m_{(\nu)}\ddot x_\nu \frac{\partial \dot x_\nu}{\partial \dot q_s} + m_{(\nu)}\dot x_\nu \frac{d}{dt}\left(\frac{\partial \dot x_\nu}{\partial \dot q_s}\right) + m_{(\nu)}\dot x_\nu \frac{\partial \ddot x_\nu}{\partial \dot q_s}$$

$$= 2m_{(\nu)}\ddot x_\nu \frac{\partial \dot x_\nu}{\partial q_s} + m_{(\nu)}\dot x_\nu\left(\frac{\partial^2 x_\nu}{\partial q_s \partial q_l}\dot q_l + \frac{\partial^2 x_\nu}{\partial q_s \partial t}\right) + 2m_{(\nu)}\dot x_\nu\left(\frac{\partial^2 x_\nu}{\partial q_s \partial q_l}\dot q_l + \frac{\partial^2 x_\nu}{\partial q_s \partial t}\right)$$

$$= 2m_{(\nu)}\ddot x_\nu \frac{\partial \ddot x_\nu}{\partial \ddot q_s} + 3m_{(\nu)}\dot x_\nu\left(\frac{\partial^2 x_\nu}{\partial q_s \partial q_l}\dot q_l + \frac{\partial^2 x_\nu}{\partial q_s \partial t}\right), \qquad \text{(A)}$$

since

$$\frac{\partial \ddot x_\nu}{\partial \ddot q_s} = \frac{\partial \dot x_\nu}{\partial \dot q_s} = \frac{\partial x_\nu}{\partial q_s}.$$

Now differentiating the last relation of (2.9.3) we get

(2.9.4) $\dddot x_\nu = 3\left(\frac{\partial^2 x_\nu}{\partial q_s \partial t}\ddot q_s + \frac{\partial^2 x_\nu}{\partial q_s \partial q_l}\ddot q_s \dot q_l\right)$ + terms not containing $\ddot q$.

In view of the expressions for $\ddot T$ from (2.9.2) and $\dddot x$ from (2.9.4) we get

$$\frac{\partial \ddot T}{\partial \ddot q_s} = 2m_{(\nu)}\ddot x_\nu \frac{\partial \ddot x_\nu}{\partial \ddot q_s} + m_{(\nu)}\dot x_\nu \frac{\partial \dddot x_\nu}{\partial \ddot q_s}$$

$$= 2m_{(\nu)}\ddot x_\nu \frac{\partial \ddot x_\nu}{\partial \ddot q_s} + 3m_{(\nu)}\dot x_\nu\left(\frac{\partial^2 x_\nu}{\partial q_s \partial t} + \frac{\partial^2 x_\nu}{\partial q_s \partial q_l}\dot q_l\right) \qquad \text{(B)}.$$

On comparison of (A) and (B) the identity (2.9.1) follows.

In section 2.8 we obtained the equations of motion in the form

$$2\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial \dot{T}}{\partial \dot{q}_i} - Q_i = a_{\alpha i}\left(2\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial \dot{T}}{\partial \dot{q}_\alpha} - Q_\alpha\right)$$

$$(\alpha = 1, 2, \ldots, r;\ i = r+1, r+2, \ldots, n),$$

or,

$$(2.9.5)\quad 2\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - a_{\alpha i}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \left(\frac{\partial \dot{T}}{\partial \dot{q}_i} - a_{\alpha i}\frac{\partial \dot{T}}{\partial \dot{q}_\alpha}\right) = Q_i - a_{\alpha i}Q_\alpha = Q_i', \text{ say.}$$

With the help of the identity (2.9.1) this reduces to

$$(2.9.6)\quad 2\left(\frac{\partial \ddot{T}}{\partial \ddot{q}_i} - a_{\alpha i}\frac{\partial \ddot{T}}{\partial \ddot{q}_\alpha}\right) - 3\left(\frac{\partial \dot{T}}{\partial \dot{q}_i} - a_{\alpha i}\frac{\partial \dot{T}}{\partial \dot{q}_\alpha}\right) = Q_i' .$$

Now let $\dot{T}_o$ denote $\dot{T}$ considered as a function of $\dot{q}$'s only, and let $\ddot{T}_o$ denote $\ddot{T}$ regarded as a function of $\ddot{q}$'s only. Then we immediately have

$$(2.9.7)\quad \frac{\partial \dot{T}_o}{\partial \dot{q}_i} = \frac{\partial \dot{T}}{\partial \dot{q}_i} - a_{\alpha i}\frac{\partial \dot{T}}{\partial \dot{q}_\alpha},$$

and

$$(2.9.8)\quad \frac{\partial \ddot{T}_o}{\partial \ddot{q}_i} = \frac{\partial \ddot{T}}{\partial \ddot{q}_i} - a_{\alpha i}\frac{\partial \ddot{T}}{\partial \ddot{q}_\alpha}.$$

By virtue of (2.9.7) and (2.9.8) the equations (2.9.6) take the form

$$2\frac{\partial \ddot{T}_o}{\partial \ddot{q}_i} - 3\frac{\partial \dot{T}_o}{\partial \dot{q}_i} = Q_i' \qquad (i = r+1, r+2, \ldots, n),$$

which are the equations of motion in the desired form.

2.10. Transition from Equations of Motion with Lagrange's Multipliers to Equations Free from Them.

In several previous sections we derived the equations of motion for a nonlinear non-holonomic system in various forms which did not involve the undetermined multipliers of Lagrange. Later, however, by means of a certain transformation we obtained from them the equations of motion containing the said multipliers. Here we propose to consider the converse problem, i.e. the transition from the equations involving the undetermined multipliers of Lagrange to those free from them. In the case of a linear non-holonomic system such a problem was solved by I.I. Metelicyn [21] in 1934.

Let us start from the equations of motion

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_s} - \frac{\partial T}{\partial q_s} - Q_s = \lambda_\alpha \frac{\partial f_\alpha}{\partial \dot q_s} \tag{2.10.1}$$

$$(\alpha = 1,2,\ldots,r;\ s = 1,2,\ldots,n),$$

obtained in section 2.6, where

$$f_\alpha(t;\, q_s;\, \dot q_s) = 0 \qquad (\alpha = 1,2,\ldots,r) \tag{2.10.2}$$

are the equations defining the nonlinear constraints.

To obtain the equations (2.10.1) the following assumptions were made:

(i) The functional matrix

$$\left\| \frac{\partial f_\alpha}{\partial \dot q_s} \right\|$$

is of rank r.

(ii) The functional determinant

$$\partial(f_1, f_2, \ldots, f_r)/\partial(\dot q_1, \dot q_2, \ldots, \dot q_r) \neq 0. \tag{2.10.3}$$

Under these assumptions it is possible to apply the implicit function theorem in order to obtain the following expressions for the dependent velocities:

$$\dot{q}_\alpha = \dot{q}_\alpha(t; q_s; \dot{q}_i) \qquad (\alpha = 1, 2, \ldots, r;\ i = r+1, r+2, \ldots, n;\ s = 1, 2, \ldots, n).$$

On differentiating the relations (2.10.2) with respect to $\dot{q}_i$ we find

$$\frac{\partial f_\alpha}{\partial \dot{q}_\beta}\frac{\partial \dot{q}_\beta}{\partial \dot{q}_i} + \frac{\partial f_\alpha}{\partial \dot{q}_i} = 0 \qquad (\alpha, \beta = 1, 2, \ldots, r;\ i = r+1, r+2, \ldots, n). \tag{2.10.4}$$

In view of (2.10.3) we can solve the system of equations (2.10.4) for $\dfrac{\partial \dot{q}_\beta}{\partial \dot{q}_i}$, obtaining the following expressions:

$$\begin{aligned}
\frac{\partial \dot{q}_1}{\partial \dot{q}_i} &= -a_{1i} = -\frac{\partial(f_1, f_2, \ldots, f_r)/\partial(\dot{q}_i, \dot{q}_2, \ldots, \dot{q}_r)}{\partial(f_1, f_2, \ldots, f_r)/\partial(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_r)},\\
\frac{\partial \dot{q}_2}{\partial \dot{q}_i} &= -a_{2i} = -\frac{\partial(f_1, f_2, \ldots, f_r)/\partial(\dot{q}_1, \dot{q}_i, \ldots, \dot{q}_r)}{\partial(f_1, f_2, \ldots, f_r)/\partial(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_r)},\\
&\text{------------------------------}\\
\frac{\partial \dot{q}_r}{\partial \dot{q}_i} &= -a_{ri} = -\frac{\partial(f_1, f_2, \ldots, f_r)/\partial(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_i)}{\partial(f_1, f_2, \ldots, f_r)/\partial(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_r)}.
\end{aligned} \tag{2.10.5}$$

Denoting the left-hand side of the equations (2.10.1) by M_s, we can rewrite them in the form

$$\lambda_\alpha \frac{\partial f_\alpha}{\partial \dot{q}_s} = M_s \qquad (\alpha = 1, 2, \ldots, r;\ s = 1, 2, \ldots, n). \tag{2.10.6}$$

If we introduce the notation

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_r \end{pmatrix}$$

we can rewrite the equations (2.10.6) in the matrix form

$$
(2.10.7) \qquad \begin{pmatrix} \dfrac{\partial f_1}{\partial \dot{q}_1} & \dfrac{\partial f_2}{\partial \dot{q}_1} & \cdots & \dfrac{\partial f_r}{\partial \dot{q}_1} \\ \dfrac{\partial f_1}{\partial \dot{q}_2} & \dfrac{\partial f_2}{\partial \dot{q}_2} & \cdots & \dfrac{\partial f_r}{\partial \dot{q}_2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial f_1}{\partial \dot{q}_n} & \dfrac{\partial f_2}{\partial \dot{q}_n} & \cdots & \dfrac{\partial f_r}{\partial \dot{q}_n} \end{pmatrix} \lambda = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix}
$$

Let us now partition the matrices in (2.10.7) in the following manner:

$$
A = \begin{pmatrix} \dfrac{\partial f_1}{\partial \dot{q}_1} & \dfrac{\partial f_2}{\partial \dot{q}_1} & \cdots & \dfrac{\partial f_r}{\partial \dot{q}_1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial f_1}{\partial \dot{q}_r} & \dfrac{\partial f_2}{\partial \dot{q}_r} & \cdots & \dfrac{\partial f_r}{\partial \dot{q}_r} \end{pmatrix}, \qquad B = \begin{pmatrix} \dfrac{\partial f_1}{\partial \dot{q}_{r+1}} & \dfrac{\partial f_2}{\partial \dot{q}_{r+1}} & \cdots & \dfrac{\partial f_r}{\partial \dot{q}_{r+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial f_1}{\partial \dot{q}_n} & \dfrac{\partial f_2}{\partial \dot{q}_n} & \cdots & \dfrac{\partial f_r}{\partial \dot{q}_n} \end{pmatrix},
$$

$$
m_1 = \begin{pmatrix} M_1 \\ \vdots \\ M_r \end{pmatrix} \quad \text{and} \quad m_2 = \begin{pmatrix} M_{r+1} \\ \vdots \\ M_n \end{pmatrix}.
$$

Then the equation (2.10.7) is equivalent to the matrix equations

$$
(2.10.8) \qquad \begin{cases} A\lambda = m_1 , \\ B\lambda = m_2 . \end{cases}
$$

Taking into consideration (2.10.3) we can eliminate λ between the equations (2.10.8) to yield

(2.10.9) $$BA^{-1}m_1 = m_2 .$$

If $|A|$ denotes the determinant of A and $A_{\alpha\beta}$ is the cofactor of $\frac{\partial f_\alpha}{\partial \dot{q}_\beta}$ in $|A|$, we have

$$BA^{-1} = \frac{1}{|A|}\begin{pmatrix} \frac{\partial f_1}{\partial \dot{q}_{r+1}} & \cdots & \frac{\partial f_r}{\partial \dot{q}_{r+1}} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_1}{\partial \dot{q}_n} & \cdots & \frac{\partial f_r}{\partial \dot{q}_n} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ \cdots & \cdots & \cdots & \cdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{pmatrix}$$

$$= \frac{1}{|A|}\begin{pmatrix} \frac{\partial f_\alpha}{\partial \dot{q}_{r+1}} A_{\alpha 1} & \frac{\partial f_\alpha}{\partial \dot{q}_{r+1}} A_{\alpha 2} & \cdots & \frac{\partial f_\alpha}{\partial \dot{q}_{r+1}} A_{\alpha r} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_\alpha}{\partial \dot{q}_n} A_{\alpha 1} & \frac{\partial f_\alpha}{\partial \dot{q}_n} A_{\alpha 2} & \cdots & \frac{\partial f_\alpha}{\partial \dot{q}_n} A_{\alpha r} \end{pmatrix},$$

$$(\alpha = 1, 2, \ldots, r).$$

But for $\alpha, \beta = 1, 2, \ldots, r$ and $i = r+1, r+2, \ldots, n$ by virtue of (2.10.5) we have

$$\frac{\partial f_\alpha}{\partial \dot{q}_i} A_{\alpha\beta} = \frac{\partial f_1}{\partial \dot{q}_i} A_{1\beta} + \frac{\partial f_2}{\partial \dot{q}_i} A_{2\beta} + \cdots + \frac{\partial f_r}{\partial \dot{q}_i} A_{r\beta}$$

$$= \frac{\partial(f_1, f_2, \ldots, f_\beta, \ldots, f_r)}{\partial(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_i, \ldots, \dot{q}_r)}$$

$$= a_{\beta i}\, |A| .$$

Hence we get

(2.10.10) $$BA^{-1} = \begin{pmatrix} a_{1,r+1} & a_{2,r+1} & \cdots & a_{r,r+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{r,n} \end{pmatrix} .$$

Substituting from (2.10.10) into (2.10.9) we finally obtain

$$\begin{pmatrix} a_{1,r+1} & a_{2,r+1} & \dots & a_{r,r+1} \\ a_{1,r+2} & a_{2,r+2} & \dots & a_{r,r+2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \dots & a_{r,n} \end{pmatrix} m_1 = m_2 ,$$

or,

$$a_{\alpha i} M_\alpha = M_i \qquad (\alpha = 1, 2, \dots, r; i = r+1, r+2, \dots, n).$$

Writing out the full expressions for M_α and M_1 the last equations become

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_i} - \frac{\partial T}{\partial q_i} - Q_i = a_{\alpha i}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q_\alpha} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha\right)$$

$$(\alpha = 1, 2, \dots, r; i = r+1, r+2, \dots, n).$$

These equations as established previously are the equations of motion free from Lagrange's multipliers.

2.11. The Equations of Motion in the Form of Determinants.

Starting from the equations of motion

(2.11.1) $$\frac{d}{dt}\frac{\partial T}{\partial \dot q_i} - \frac{\partial T}{\partial q_i} - Q_\alpha = a_{\alpha i}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q_\alpha} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha\right)$$

$$(\alpha = 1, 2, \dots, r; i = r+1, r+2, \dots, n)$$

we propose to rewrite them in terms of determinants all of which can be obtained from a certain matrix according to a general scheme. For a linear nonholonomic system such a problem was solved by I.I.Metelicyn [21] .

If we put

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} - Q_s = M_s \qquad (s=1,2,\ldots,n) \tag{2.11.2}$$

the equations (2.11.1) take the following form:

$$M_i - a_{\alpha i} M_\alpha = 0 \qquad (\alpha=1,2,\ldots,r;\ i=r+1,r+2,\ldots,n). \tag{2.11.3}$$

Now let us consider the determinant

$$|A^{(i)}| = \begin{vmatrix} M_1 & M_2 & \ldots & M_r & M_i \\ \frac{\partial f_1}{\partial \dot{q}_1} & \frac{\partial f_1}{\partial \dot{q}_2} & \ldots & \frac{\partial f_1}{\partial \dot{q}_r} & \frac{\partial f_1}{\partial \dot{q}_i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_r}{\partial \dot{q}_1} & \frac{\partial f_r}{\partial \dot{q}_2} & \ldots & \frac{\partial f_r}{\partial \dot{q}_r} & \frac{\partial f_r}{\partial \dot{q}_i} \end{vmatrix} \tag{2.11.4}$$

If we denote by A_s the cofactor of M_s ($s=1,2,\ldots,n$) in $|A^{(i)}|$, we find, on expanding (2.11.4) in terms of the cofactors of the top row, that

$$M_\alpha A_\alpha + M_{(i)} A_i = |A^{(i)}| \qquad (\alpha=1,2,\ldots,r;\ i=r+1,r+2,\ldots,n). \tag{2.11.5}$$

But we easily find that

$$A_i = (-)^r \frac{\partial(f_1,f_2,\ldots,f_r)}{\partial(\dot{q}_1,\dot{q}_2,\ldots,\dot{q}_r)}\,, \tag{2.11.6}$$

$$\begin{aligned} A_\alpha &= (-)^{r-1} \frac{\partial(f_1,f_2,\ldots,f_\alpha,\ldots,f_r)}{\partial(\dot{q}_1,\dot{q}_2,\ldots,\dot{q}_i,\ldots,\dot{q}_r)} \\ &= (-)^{r-1} a_{\alpha i} \frac{\partial(f_1,f_2,\ldots,f_r)}{\partial(\dot{q}_1,\dot{q}_2,\ldots,\dot{q}_r)} \qquad (\alpha=1,2,\ldots,r)\,, \end{aligned} \tag{2.11.7}$$

where $a_{\alpha i}$ are given by (2.10.5)

With the help of (2.11.6) and (2.11.7) the expansions (2.11.5) become

$$(-)^{r-1} M_\alpha a_{\alpha i} \frac{\partial(f_1,f_2,\cdots,f_r)}{\partial(\dot q_1,\dot q_2,\cdots,\dot q_r)} + (-)^r M_i \frac{\partial(f_1,f_2,\cdots,f_r)}{\partial(\dot q_1,\dot q_2,\cdots,\dot q_r)} = |A^{(i)}|$$

or,

(2.11.8) $$a_{\alpha i} M_\alpha - M_i = (-)^{r-1} |A^{(i)}| \Big/ \frac{\partial(f_1,f_2,\cdots,f_r)}{\partial(\dot q_1,\dot q_2,\cdots,\dot q_r)} \qquad \begin{pmatrix}\alpha = 1,2,\dots,r; \\ i = r+1, r+2,\dots,n\end{pmatrix}.$$

Since

(2.11.9) $$\frac{\partial(f_1,f_2,\cdots,f_r)}{\partial(\dot q_1,\dot q_2,\cdots,\dot q_r)} \neq 0$$

the equations (2.11.8) by virtue of (2.11.3) yield

(2.11.10) $$|A^{(i)}| = 0 \qquad (i = r+1, r+2,\dots,n).$$

As a consequence of (2.11.9) and the fact that n - r determinants of the type (2.11.4) vanish, it follows that any determinant of order (r+1) obtained from the (r+1)xn matrix

(2.11.11) $$\begin{pmatrix} M_1 & M_2 & \dots & M_r & M_{r+1} & \dots & M_n \\ \frac{\partial f_1}{\partial \dot q_1} & \frac{\partial f_1}{\partial \dot q_2} & \dots & \frac{\partial f_1}{\partial \dot q_r} & \frac{\partial f_1}{\partial \dot q_{r+1}} & \dots & \frac{\partial f_1}{\partial \dot q_n} \\ \frac{\partial f_2}{\partial \dot q_1} & \frac{\partial f_2}{\partial \dot q_2} & \dots & \frac{\partial f_2}{\partial \dot q_r} & \frac{\partial f_2}{\partial \dot q_{r+1}} & \dots & \frac{\partial f_2}{\partial \dot q_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_r}{\partial \dot q_1} & \frac{\partial f_r}{\partial \dot q_2} & \dots & \frac{\partial f_r}{\partial \dot q_r} & \frac{\partial f_r}{\partial \dot q_{r+1}} & \dots & \frac{\partial f_r}{\partial \dot q_n} \end{pmatrix}$$

must also vanish.

Hence the equations of motion (2.11.1) can be found by equating to zero any determinant of order $r+1$ obtained from the matrix (2.11.11).

CHAPTER 3

APPLICATIONS

3.1. Some General Considerations.

There are not very many known examples of mechanical systems moving with nonlinear non-holonomic constraints. In 1911 Appell [4] gave an example of such constraints. However, nonlinear non-holonomic constraints can be reälised in problems concerning the regulation of the motion, or in other problems of technical interest where the constraints between the moving parts are realised by means of electromagnetic devices. It is expected that with technical development the use of nonlinear non-holonomic constraints will also increase.

The procedure for solving problems with nonlinear nonholonomic constraints is quite straightforward. To obtain the equations of motion one has only to write down T, the kinetic energy, and the external forces in terms of the generalised coordinates, and substitute them in one of the many forms of the equations established in the previous chapter.

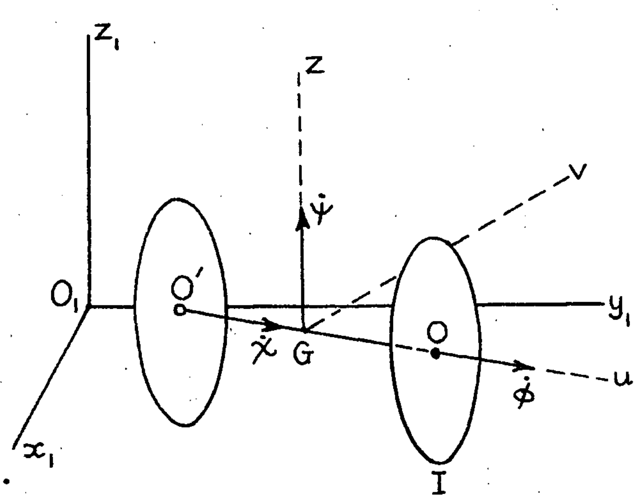
Let us now consider some examples of this procedure:

1. A system of two wheels and their axle moving on a horizontal plane.
2. A disc moving on a horizontal plane.
3. A heavy ball moving on a horizontal plane.

Despite the fact that the equations of constraint in all these examples are essentially linear nonholonomic, they can be artificially thrown into a nonlinear form. The purpose of doing so is two-fold. First, it provides us with examples of nonlinear non-holonomic constraints. Secondly, it serves to illustrate the general treatment of the theory developed in the previous chapter.

In view of the linearity of constraints the solutions of the above-mentioned examples are well-known, but the method depends on the use of the equations of motion in terms of Lagrange's undetermined multipliers. This, of course, requires the determination of these multipliers prior to the actual solution of the problem. But in the methods employed below we use the equations of motion established in Chap. 2, which are free from such multipliers. Consequently the calculations become simple.

3.2. Motion of a System of Two Wheels and Their Axle on a Horozontal Plane.



Let the axle be a homogeneous rod of length 2a and mass m_1, and the wheels be two homogeneous discs, each of radius a and mass m_2, which are fixed normally to the rod at the centres O and O' and free to turn about it.

Let $O_1 x_1 y_1 z_1$, be a

reference system fixed in space and let the wheels move on the plane $z=o$ (the wheel with centre O' having a contact without friction and that with centre O having a perfectly rough contact).

Suppose we introduce an intermediate trihedron $Guvz$ at the centre, G, of the rod with Gu along the rod, Gv horizontal and perpendicular to Gu, and Gz vertical. The parameters, characterizing the position of the system, are the coordinates (x_1, y_1) of the centre G, the angle ψ which Gu makes with O_1x_1, and the angles of rotation ϕ and χ of the two discs with centres O and O' respectively.

The well-known theorem of König, when applied first to the entire system, then to each disc, immediately gives for the kinetic energy, T, the following expression:

$$2T = (m_1+2m_2)(\dot{x}_1^2+\dot{y}_1^2) + \left(\frac{m_1}{3}+\frac{5m_2}{2}\right)a^2\dot{\psi}^2 + \frac{m_2 a^2}{2}\dot{\phi}^2 + \frac{m_2 a^2}{2}\dot{\chi}^2. \tag{3.2.1}$$

Since the forces of gravity do no work we can assume that there are no externally applied forces.

If I denotes the instantaneous point of contact, we shall express the kinematical condition of the absence of sliding at I by means of nonlinear (with respect to the velocities) differential equations. The absence of sliding demands that the velocity at the point I of the disc be zero. But this velocity is the resultant of the velocity of G and of $a(\dot{\psi}+\dot{\phi})$, parallel to Gv, due to the rotations $\dot{\psi}$ and $\dot{\phi}$. Hence we must have

$$(3.2.2)\quad \begin{cases} \dot{x}_1^2+\dot{y}_1^2 = a^2(\dot{\psi}+\dot{\phi})^2, \\[2ex] \dfrac{\dot{x}_1}{\dot{y}_1} = -\tan\psi. \end{cases}$$

These are the equations of constraint of which the first is nonlinear in velocities.

Solving the system (3.2.2) for $\dot{x}_1$ and $\dot{y}_1$ we get

$$(3.2.3)\quad \begin{cases} \dot{x}_1 = a(\dot{\psi}+\dot{\phi})\sin\psi, \\ \dot{y}_1 = -a(\dot{\psi}+\dot{\phi})\cos\psi. \end{cases}$$

Taking $\dot{\psi}$, $\dot{\phi}$ and $\dot{\chi}$ as the independent velocities, we have, in the notation of Sec.2.3,

$$2T' = (m_1+2m_2)a^2(\dot{\psi}+\dot{\phi})^2+\left(\frac{m_1}{3}+\frac{5m_2}{2}\right)a^2\dot{\psi}^2+\frac{m_2a^2}{2}\dot{\phi}^2+\frac{m_2a^2}{2}\dot{\chi}^2,$$

and

$$2T_0' = (m_1+2m_2)a^2(\dot{\psi}_0+\dot{\phi}_0)^2+\left(\frac{m_1}{3}+\frac{5m_2}{2}\right)a^2\dot{\psi}_0^2+\frac{m_2a^2}{2}\dot{\phi}_0^2+\frac{m_2a^2}{2}\dot{\chi}_0^2.$$

Hence we get

$$\dot{T}' = (m_1+2m_2)a^2(\dot{\psi}+\dot{\phi})(\ddot{\psi}+\ddot{\phi})+\left(\frac{m_1}{3}+\frac{5m_2}{2}\right)a^2\dot{\psi}\ddot{\psi}+\frac{m_2a^2}{2}\dot{\phi}\ddot{\phi}+\frac{m_2a^2}{2}\dot{\chi}\ddot{\chi},$$

$$\ddot{T}' = (m_1+2m_2)a^2(\ddot{\psi}+\ddot{\phi})^2+\left(\frac{m_1}{3}+\frac{5m_2}{2}\right)a^2\ddot{\psi}^2+\frac{m_2a^2}{2}\ddot{\phi}^2+\frac{m_2a^2}{2}\ddot{\chi}^2+\text{terms}$$

not containing the second derivatives,

$$\dot{T}_0' = \ddot{T}_0' = 0.$$

By virtue of the above expressions the function

$$R' = \frac{1}{2}\left(\ddot{T}'-3\ddot{T}_0'\right)$$

reduces to

$$R' = \frac{1}{2}\ddot{T}'$$

or, as far as the terms in second derivatives are concerned, to

$$2R' = (m_1+2m_2)a^2(\ddot{\psi}+\ddot{\phi})^2+\left(\frac{m_1}{3}+\frac{5m_2}{2}\right)a^2\ddot{\psi}^2+\frac{m_2a^2}{2}\ddot{\phi}^2+\frac{m_2a^2}{2}\ddot{\chi}^2.$$

Using the equations (2.3.11) we have for the equations of motion of the system considered:

$$(3.2.4)\qquad \begin{cases} (m_1+2m_2)(\ddot{\psi}+\ddot{\phi})+\left(\frac{m_1}{3}+\frac{5m_2}{2}\right)\ddot{\psi}=0, \\ (m_1+2m_2)(\ddot{\psi}+\ddot{\phi})+\left(\frac{m_1}{3}+\frac{5m_2}{2}\right)\ddot{\psi}=0, \\ \ddot{\chi}=0. \end{cases}$$

The equations (3.2.4) can be integrated, yielding the three first integrals:

$$(3.2.5)\qquad \begin{cases} (m_1+2m_2)(\dot{\psi}+\dot{\phi})+\left(\frac{m_1}{3}+\frac{5m_2}{2}\right)\dot{\psi}=(m_1+2m_2)(\dot{\psi}^\circ+\dot{\phi}^\circ)+\left(\frac{m_1}{3}+\frac{5m_2}{2}\right)\dot{\psi}^\circ, \\ (m_1+2m_2)(\dot{\psi}+\dot{\phi})+\frac{m_2}{2}\dot{\phi}=(m_1+2m_2)(\dot{\psi}^\circ+\dot{\phi}^\circ)+\frac{m_2}{2}\dot{\phi}^\circ, \\ \dot{\chi}=\dot{\chi}^\circ, \end{cases}$$

where $\dot{\psi}^\circ, \dot{\phi}^\circ, \dot{\chi}^\circ$ are the initial values of $\dot{\psi}, \dot{\phi}, \dot{\chi}$ respectively. The equations (3.2.5) are equivalent to

$$(3.2.6)\qquad \dot{\psi}=\dot{\psi}^\circ, \quad \dot{\phi}=\dot{\phi}^\circ \quad \text{and } \dot{\chi}=\dot{\chi}^\circ.$$

Integrating the above equations we get

$$(3.2.7)\qquad \psi=\dot{\psi}^\circ t, \quad \phi=\dot{\phi}^\circ t, \quad \chi=\dot{\chi}^\circ t.$$

By virtue of (3.2.6) and (3.2.7) we get from the equations (3.2.3):

$$(3.2.8)\qquad \begin{cases} \dot{x}_1 = a(\dot{\psi}^\circ+\dot{\phi}^\circ)\sin\dot{\psi}^\circ t, \\ \dot{y}_1 = -a(\dot{\psi}^\circ+\dot{\phi}^\circ)\cos\dot{\psi}^\circ t. \end{cases}$$

Integrating the equations (3.2.8) and suitably choosing the arbitrary constants, we get

$$x_1 = -\frac{a(\dot{\psi}^\circ+\dot{\phi}^\circ)}{\dot{\psi}^\circ}\cos\dot{\psi}^\circ t,$$
$$\dot{y}_1 = -\frac{a(\dot{\psi}^\circ+\dot{\phi}^\circ)}{\dot{\psi}^\circ}\sin\dot{\psi}^\circ t.$$

The last equations show that the trajectory of the centre G is a circle, of radius $a\left|\frac{\dot{\psi}^{\circ}+\dot{\phi}^{\circ}}{\dot{\psi}^{\circ}}\right|$, described with a uniform velocity.

3.3. <u>Motion of a Heavy Circular Disc on a Horizontal Plane.</u>

Let a circular disc, of unit mass and radius a, roll (without sliding) along a fixed horizontal plane $o_1 x_1 y_1$. Let the centre of inertia, G, of the disc be the centre of the figure and the central ellipsoid of inertia be an ellipsoid of revolution about Gz of the disc.

The parameters characterising the position of the disc are the Eulerian angles θ, ψ, ϕ and the coordinates x_1, y_1 of the point G, for which z_1 is obviously equal to a sine. If Guvz is an intermediate trihedron, the components p,q,r of the instantaneous rotation $\bar{\omega}$ of the disc along the axes of Guvz are given by the expressions:

(3.3.1) $$p=\dot{\theta}, \quad q=\dot{\psi}\sin\theta, \quad r=\dot{\phi}+\dot{\psi}\cos\theta,$$

whence we get

(3.3.2) $$\dot{\theta}=p, \quad \dot{\psi}\sin\theta=q, \quad \dot{\psi}\cos\theta=q\cot\theta.$$

If $\bar{V}_G$ is the velocity of the centre G, the velocity of the point of contact I is given by the expression

$$\bar{V}_G+\bar{\omega}\times\overline{GI}\,.$$

However, the kinematical condition of the absence of sliding demands that the velocity of I be zero. Hence we have

(3.3.3) $$\bar{V}_G+\bar{\omega}\times\overline{GI}=0.$$

Since the coordinates of I referred to $Guvz$ are (o,-a,o), the components of $\bar{\omega} \times \overline{GI}$ along the axes of Guvz are

$$-ar,\ 0,\ ap.$$

When projected along the fixed axes O_1x_1 and O_1y_1 they become

$$-ar\cos\psi + ap\sin\theta\sin\psi \quad \text{and} \quad -ar\sin\psi - ap\sin\theta\cos\psi.$$

Hence the relations (3.3.3) gives the equations of constraint in the form:

$$(3.3.4) \quad \begin{cases} \dot{x}_1^2 + \dot{y}_1^2 = a^2(r^2 + p^2\sin^2\theta), \\ \dfrac{\dot{y}_1}{\dot{x}_1} = \dfrac{r\sin\psi + p\sin\theta\cos\psi}{r\cos\psi - p\sin\theta\sin\psi} \end{cases},$$

the first of which is nonlinear in velocities.

Solving the equations (3.3.4) for x and y , we get

$$(3.3.5) \quad \begin{cases} \dot{x}_1 = -ar\cos\psi + ap\sin\theta\sin\psi, \\ \dot{y}_1 = -ar\sin\psi - ap\sin\theta\cos\psi. \end{cases}$$

Hence

$$(3.3.6) \begin{cases} \ddot{x}_1 = -a\dot{r}\cos\psi + ar\dot{\psi}\sin\psi + a\dot{p}\sin\theta\sin\psi + ap^2\cos\theta\sin\psi + ap\dot{\psi}\sin\theta\cos\psi, \\ \ddot{y}_1 = -a\dot{r}\sin\psi - ar\dot{\psi}\cos\psi - a\dot{p}\sin\theta\cos\psi - ap^2\cos\theta\cos\psi + ap\dot{\psi}\sin\theta\sin\psi, \end{cases}$$

where, as far as terms in the second derivatives are concerned, we have

$$(3.3.7) \quad \begin{cases} \dot{p} = \ddot{\theta}, & \ddot{p} = 0, \\ \dot{q} = \ddot{\psi}\sin\theta + p\dot{\psi}\cos\theta, & \ddot{q} = 2\ddot{\psi}p\cos\theta + \dot{\psi}\dot{p}\cos\theta, \\ \dot{r} = \ddot{\phi} + \ddot{\psi}\cos\theta - p\dot{\psi}\sin\theta, & \ddot{r} = -2\ddot{\psi}p\sin\theta - \dot{\psi}\dot{p}\sin\theta. \end{cases}$$

In the notation of Sec. 2.7 we have

$$2T = (A + a^2\cos^2\theta)p^2 + Aq^2 + Cr^2 + \dot{x}_1^2 + \dot{y}_1^2,$$

where A,C are the principal moments of inertia of the disc with respect to Gu and Gz respectively.

Hence

$$2T' = (A + a^2)p^2 + Aq^2 + (C + a^2)r^2, \tag{3.3.8}$$

$$2T_1 = \dot{x}_1^2 + \dot{y}_1^2 + \cdots. \tag{3.3.9}$$

If T_{10} denotes the expression of T_1 for fixed values of $\dot{x}_1$ and $\dot{y}_1$, then clearly we have

$$\ddot{T}_{10} = 0.$$

Consequently

$$R_1 = \frac{1}{2}\ddot{T}_1 = \frac{1}{2}(\ddot{x}_1^2 + \ddot{y}_1^2) + \text{terms not containing } \ddot{x}_1, \ddot{y}_1. \tag{3.3.10}$$

With the help of (3.3.7) and (3.3.8) we get

$$\left\{\begin{aligned} \frac{\partial T'}{\partial \theta} &= Aq\dot{\psi}\cos\theta - (C + a^2)r\dot{\psi}\sin\theta, \\ \frac{\partial T'}{\partial \psi} &= 0, \\ \frac{\partial T'}{\partial \phi} &= 0, \end{aligned}\right. \tag{3.3.11}$$

and

$$\left\{\begin{aligned} \frac{\partial T'}{\partial \dot{\theta}} &= (A + a^2)p, \\ \frac{\partial T'}{\partial \dot{\psi}} &= Aq\sin\theta + (C + a^2)r\cos\theta, \\ \frac{\partial T'}{\partial \dot{\phi}} &= (C + a^2)r. \end{aligned}\right. \tag{3.3.12}$$

Again, with the help of (3.3.5), (3.3.6) and (3.3.9) we find that

$$(3.3.13)\quad \begin{cases} \dfrac{\partial T_1}{\partial \theta} = -a^2 r \dot{\psi} \sin\theta + a^2 p^2 \cos\theta \sin\theta, \\ \dfrac{\partial T_1}{\partial \psi} = 0, \\ \dfrac{\partial T_1}{\partial \phi} = 0, \end{cases}$$

and

$$(3.3.14)\quad \begin{cases} \dfrac{\partial T_1}{\partial \dot{\theta}} = a^2 p \sin^2\theta, \\ \dfrac{\partial T_1}{\partial \dot{\psi}} = a^2 r \cos\theta, \\ \dfrac{\partial T_1}{\partial \dot{\phi}} = a^2 r. \end{cases}$$

Finally, from (3.3.6), (3.3.7) and (3.3.10) we find

$$\frac{\partial R_1}{\partial \ddot{\theta}} = a^2 \sin\theta \left(r\dot{\psi} + \dot{p} \sin\theta + p^2 \cos\theta \right),$$

$$\frac{\partial R_1}{\partial \ddot{\psi}} = -a^2 \cos\theta \left(p\dot{\psi} \sin\theta - \dot{r} \right),$$

$$\frac{\partial R_1}{\partial \ddot{\phi}} = -a^2 \left(p\dot{\psi} \sin\theta - \dot{r} \right).$$

The θ- equation, written with the help of (2.7.10) and simplified with the help of (3.3.2) gives

$$(3.3.15)\quad (A+a^2)\dot{p} - Aq^2 \cot\theta + (C+a^2) qr = -ga\cos\theta.$$

Similarly, the ψ- and ϕ - equations are

$$(3.3.16)\quad \frac{d}{dt}\left[Aq \sin\theta + (C+a^2) r\cos\theta \right] - \frac{d}{dt}\left(a^2 r \cos\theta \right) - a^2 \cos\theta \left(p\dot{\psi} \cos\theta - \dot{r} \right) = 0,$$

(3.3.17) $(C+a^2)\dot{r} - a^2 pq = 0.$

Simplifying (3.3.16) with the help of (3.3.17) we get

(3.3.18) $A\dot{q} + p(Aq\cot\theta - Cr) = 0.$

The equations (3.3.15), (3.3.17) and (3.3.18) are the well-known equations describing the motion of the disc.

3.4. Motion of a Heavy Ball on a Fixed Horizontal Plane.

Let a heavy non-homogeneous sphere, of centre O and radius a, roll and pivot without sliding on a horizontal plane $z_1=0$ of the fixed reference system $O_1x_1y_1z_1$. Let us also suppose that the centre O of the sphere is the centre of inertia and the central ellipsoid is an ellipsoid of revolution about a diameter Oz of the sphere, where $Oxyz$ is a trihedron rigidly connected with the sphere.

The parameters of the sphere are the coordinates (x_1, y_1, z_1) of the point O and the three Eulerian angles θ, ψ, ϕ. The condition that the sphere remains in contact with the plane $z_1=0$ gives

$$z_1 - a = 0.$$

The condition of contact without sliding demands that $\bar{V}_I = 0$, $\bar{V}_I$ being the velocity of the point I of the sphere which is in contact with the plane. But

$$\bar{V}_I = \bar{V}_o + \bar{\omega} \times \overline{OI},$$

where $\bar{V}_o$ is the velocity of O and $\bar{\omega}$ is the instantaneous rotation. The components of $\bar{V}_o$, $\bar{\omega}$ and $\overline{OI}$, along the axes fixed in space, are respectively (x_1, y_1, o), (p_1, q_1, r_1) and $(o, o, -a)$, where

$$(3.4.1)\quad \begin{cases} p_1 = \dot\phi \sin\theta \sin\psi + \dot\theta\cos\psi, \\ q_1 = -\dot\phi\sin\theta\cos\psi + \dot\theta\sin\psi, \\ r_1 = \dot\phi\cos\theta + \dot\psi. \end{cases}$$

The requirement of the absence of sliding demands that

$$(3.4.2)\quad \begin{cases} \dot x_1^2 + \dot y_1^2 = a^2(q_1^2 + p_1^2), \\ \dfrac{\dot y_1}{\dot x_1} = -\dfrac{p_1}{q_1}, \end{cases}$$

the first of which is nonlinear in velocities.

Taking into consideration the relations (3.4.1) the equations (3.4.2) are equivalent to

$$(3.4.3)\quad \begin{cases} \dot x_1 = a q_1 = a(\dot\theta\sin\psi - \dot\phi\sin\theta\cos\psi), \\ \dot y_1 = -a p_1 = -a(\dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi). \end{cases}$$

Differentiating the equations (3.4.3) with respect to the time, we get

$$(3.4.4)\quad \begin{cases} \ddot x_1 = a(\ddot\theta\sin\psi + \dot\theta\dot\psi\cos\psi - \ddot\phi\sin\theta\cos\psi - \dot\phi\dot\theta\cos\theta\cos\psi + \dot\phi\dot\psi\sin\theta\sin\psi), \\ \ddot y_1 = -a(\ddot\theta\cos\psi - \dot\theta\dot\psi\sin\psi + \ddot\phi\sin\theta\sin\psi + \dot\phi\dot\theta\cos\theta\sin\psi + \dot\phi\dot\psi\sin\theta\cos\psi). \end{cases}$$

The kinetic energy, in terms of the notations of Sec. 2.9, can be written as

$$(3.4.5)\quad 2T = m(\dot x_1^2 + \dot y_1^2) + A(\dot\theta^2 + \dot\psi^2\sin^2\theta) + C(\overset{\psi}{\dot\phi} + \overset{\phi}{\dot\psi}\cos\theta)^2,$$

where m is the mass of the sphere and A and C are the moments of inertia about Ou and Oz, $Ouvz$ being an intermediate trihedron.

From (3.4.5) we get on differentiation with respect to the time

$$(3.4.6) \qquad \dot{T} = m(\dot{x}_1\ddot{x}_1+\dot{y}_1\ddot{y}_1)+A(\dot{\theta}\ddot{\theta}+\dot{\psi}\ddot{\psi}\sin^2\theta+\frac{1}{2}\dot{\psi}^2\dot{\theta}\sin 2\theta)+ \\ +C(\dot{\phi}+\dot{\psi}\cos\theta)(\ddot{\phi}+\ddot{\psi}\cos\theta-\dot{\psi}\dot{\theta}\sin\theta),$$

$$(3.4.7) \qquad \ddot{T} = m(\ddot{x}_1^2+\ddot{y}_1^2)+A(\ddot{\theta}^2+\ddot{\psi}^2\sin^2\theta+2\ddot{\psi}\dot{\psi}\dot{\theta}\sin 2\theta+\frac{1}{2}\ddot{\theta}\dot{\psi}^2\sin 2\theta)+ \\ +C(\ddot{\phi}^2+\ddot{\psi}^2\cos^2\theta+2\ddot{\phi}\ddot{\psi}\cos\theta-2\ddot{\phi}\dot{\psi}\dot{\theta}\sin\theta-\ddot{\psi}\dot{\psi}\dot{\theta}\sin 2\theta)- \\ -C(\dot{\phi}+\dot{\psi}\cos\theta)(2\ddot{\psi}\dot{\theta}\sin\theta+\ddot{\theta}\dot{\psi}\sin\theta)+$$

+ terms not containing the second derivatives.

Using the notations of Sec.2.9 we have, with the help of (3.4.3), (3.4.4), (3.4.6), and (3.4.7), the following expressions:

$$2\frac{\partial\ddot{T}_0}{\partial\ddot{\theta}}-3\frac{\partial\dot{T}_0}{\partial\dot{\theta}}=(A+ma^2)\ddot{\theta}+(ma^2-C)\dot{\phi}\dot{\psi}\sin\theta-\frac{1}{2}(A+C)\dot{\psi}^2\sin 2\theta,$$

$$2\frac{\partial\ddot{T}_0}{\partial\ddot{\psi}}-3\frac{\partial\dot{T}_0}{\partial\dot{\psi}}=A(\ddot{\psi}\sin^2\theta+\dot{\psi}\dot{\theta}\sin 2\theta)+C(\ddot{\psi}\cos^2\theta+\ddot{\phi}\cos\theta-\dot{\phi}\dot{\theta}\sin\theta-\dot{\psi}\dot{\theta}\sin 2\theta),$$

$$2\frac{\partial\ddot{T}_0}{\partial\ddot{\phi}}-3\frac{\partial\dot{T}_0}{\partial\dot{\phi}}=-ma^2(\dot{\theta}\dot{\psi}-\ddot{\phi}\sin\theta-\dot{\phi}\dot{\theta}\cos\theta)\sin\theta+C(\ddot{\phi}+\ddot{\psi}\cos\theta-\dot{\psi}\dot{\theta}\sin\theta).$$

Hence the equations of motion are

$$(3.4.8) \qquad (A+ma^2)\ddot{\theta}+(ma^2-C)\dot{\phi}\dot{\psi}\sin\theta-\frac{1}{2}(A+C)\dot{\psi}^2\sin 2\theta=0,$$

$$(3.4.9) \qquad A(\ddot{\psi}\sin^2\theta+\dot{\psi}\dot{\theta}\sin 2\theta)+C(\ddot{\psi}\cos^2\theta+\ddot{\phi}\cos\theta-\dot{\phi}\dot{\theta}\sin\theta-\dot{\psi}\dot{\theta}\sin 2\theta)=0,$$

$$(3.4.10) \qquad -ma^2(\dot{\theta}\dot{\psi}-\ddot{\phi}\sin\theta-\dot{\phi}\dot{\theta}\cos\theta)\sin\theta+C(\ddot{\phi}+\ddot{\psi}\cos\theta-\dot{\psi}\dot{\theta}\sin\theta)=0.$$

Equation (3.4.10) can be written as

$$ma^2 \sin\theta \frac{d}{dt}(\dot\phi \sin\theta) + C\frac{d}{dt}(\dot\phi + \dot\psi\cos\theta) = ma^2\,\dot\theta\,\dot\psi \sin\theta,$$

or,

$$(3.4.11) \quad ma^2 \sin\theta \frac{d}{d\theta}(\dot\phi \sin\theta) + C\frac{d}{d\theta}(\dot\phi + \dot\psi\cos\theta) = ma^2\,\dot\psi \sin\theta.$$

Also the equation (3.4.9) can be put in the following form:

$$A\frac{d}{dt}(\dot\psi \sin^2\theta) + C\frac{d}{dt}(\dot\phi\cos\theta + \dot\psi\cos^2\theta) = 0,$$

for which a first integral is obviously

$$(3.4.12) \quad A(\dot\psi \sin^2\theta) + C(\dot\phi + \dot\psi\cos\theta)\cos\theta = \text{constant}.$$

We can ignore the equation (3.4.8) since in the case under consideration we have

$$2T' = (A + ma^2)\dot\theta^2 + A\dot\psi^2\sin^2\theta + ma^2\dot\phi^2\sin^2\theta + C(\dot\phi + \dot\psi\cos\theta)^2,$$

which, being a quadratic form in $\dot\theta$, $\dot\psi$, $\dot\phi$, leads to a first integral

$$(3.4.13) \quad T' = \text{constant}.$$

Hence the motion of the sphere is completely determined by the equations (3.4.11), (3.4.12) and (3.4.13).

CHAPTER 4

QUASI-COORDINATES OR NON-HOLONOMIC COORDINATES

4.1. Some General Considerations.

Let $q_1, q_2, \ldots, q_n$ be the generalised coordinates defining the position of a mechanical system. Following Hamel [18], let us introduce n parameters

$$\omega_1, \omega_2, \ldots, \omega_n,$$

called the kinetic characteristics, by means of the relations:

$$\omega_\ell = a_{\ell s} \dot{q}_s + a_\ell \qquad (\ell, s = 1, 2, \ldots, n), \tag{4.1.1}$$

where $a_{\ell s}$, a_ℓ are functions of $q_1, q_2, \ldots, q_n$ and t. In general the Pfaffian forms

$$a_{\ell s} \dot{q}_s + a_\ell$$

are non-integrable.

Assuming that the determinant of the coefficients $a_{\ell s}$ is non-zero, we can express $\dot{q}_s$ as linear functions of ω_ℓ. Let these functions be

$$\dot{q}_\ell = b_{\ell s} \omega_s + b_\ell \qquad (\ell, s = 1, 2, \ldots, n) \tag{4.1.2}$$

where

$$b_{\ell s} = \overset{(s\ell)}{a}, \quad b_\ell = -\sum_{s=1}^{n} \overset{(s\ell)}{a} a_s \qquad (\ell, s = 1, 2, \ldots, n), \tag{4.1.3}$$

$\| \overset{(s\ell)}{a} \|$ being the inverse of the matrix $\| a_{s\ell} \|$.

With each ω_ℓ we can associate a quantity $d\pi_\ell$, defined by

(4.1.4) $d\pi_l = \omega_l dt = a_{ls} dq_s + a_l dt \qquad (l, s = 1, 2, \ldots, n).$

The quantities $d\pi_l$ are called the differentials of the quasi-coordinates [28] or the differentials of non-holonomic coordinates π_l .

If the forms

$$a_{ls} dq_s + a_l dt$$

are exact differentials, π_l exist and are the true coordin the usual sense; otherwise π_l do not exist.

The variations, representing the virtual displacements, of q_s are given by

(4.1.5) $\delta\pi_l = a_{ls}\delta q_s \qquad (l, s = 1, 2, \ldots, n).$

In case of a holonomic system, with n degrees of freedom, all of $\delta\pi_l$ are independent. However, if the system is subject to linear non-holonomic constraints of the type:

$$A_{\alpha s}\dot{q}_s + A_\alpha = 0 \qquad (s = 1, 2, \ldots, n; \alpha = 1, 2, \ldots, r < n)$$

we can take

$$a_{\alpha s} = A_{\alpha s}, \ a_\alpha = A_\alpha$$

so that the equations of constraint become

$$\omega_\alpha = 0 \qquad (\alpha = 1, 2, \ldots, r).$$

Corresponding to ω_α we have

$$\delta\pi_\alpha = 0 \qquad (\alpha = 1, 2, \ldots, r).$$

The remaining $\delta\pi_i$ $(i = r+1, r+2, \ldots, n)$ are independent.

In 1957 V.S.Novoselov [23] generalised the definition of non-holonomic coordinates. His definition includes as a special case the above given definition of nonholonomic coordinates for holonomic or linear nonholonomic systems. It is well-suited in case where the constraints being nonlinear nonholonomic are of the type of Četaev.

Following the point of view of Novoselov, let us define the kinetic characteristics by the relations:

(4.1.6) $$\omega_\ell = \omega_\ell(t; q_s; \dot{q}_s) \qquad (\ell, s = 1, 2, \ldots, n),$$

where ω_ℓ are not necessarily linear functions of $\dot{q}_s$.

If the functional matrix

$$\left\| \frac{\partial \omega_\ell}{\partial \dot{q}_s} \right\|$$

is of rank r, we shall have

(4.1.7) $$\dot{q}_\ell = \dot{q}_\ell(t; q_s; \omega_s) \qquad (\ell, s = 1, 2, \ldots, n).$$

The variations, representing the virtual displacements, of q_ℓ are defined by

(4.1.8) $$\delta q_\ell = \frac{\partial \dot{q}_\ell}{\partial \omega_s} \delta \pi_s , \qquad (\ell, s = 1, 2, \ldots, n).$$

The $\delta \pi_s$ in (4.1.8) are called the differentials of nonholonomic coordinates .

For nonlinear non-holonomic systems subject to constraints of the type of Četaev and expressed by the equations

$$f_\alpha(t; q_s; \dot{q}_s) = 0 \qquad (\alpha = 1, 2, \ldots, r < n ; s = 1, 2, \ldots, n)$$

we take

$$\omega_\alpha = f_\alpha(t; q_s; \dot{q}_s) = 0.$$

The remaining ω_i $(i = r + 1, r + 2, \ldots, n)$ are arbitrary functions of the form (4.1.6). The $n - r$ independent $\delta \pi_i$

satisfy the relations

$$\delta q_{\nu_\ell} = \frac{\partial \dot{q}_\ell}{\partial \omega_i}\,\delta\pi_i \qquad (\ell = 1,2,\ldots,n;\ i = r+1, r+2,\ldots,n).$$

In case ω_ℓ in the equations (4.1.6) are linear functions of $\dot{q}_s$ we call $\delta\pi_\ell$ the differentials of linear nonholonomic coordinates; otherwise they are called non-linear nonholonomic coordinates.

4.2. Poisson's Theorem in Linear Non-holonomic Coordinates.

In 1944 the classical theorem of Poisson, dealing with the properties of integrals of canonical equations of dynamics, was extended by V.V.Dobronravov [17] to the case of canonical equations expressed in linear nonholonomic coordinates. This generalisation was achieved by assuming the so-called kinetic characteristics to be independent of the time. We propose to generalise his result by taking the kinetic characteristics to be time-dependent.

Consider a holonomic system for which the kinetic characteristics ω_ℓ and the corresponding differentials of linear non-holonomic coordinates $d\pi_\ell$ are given respectively by the equations (4.1.1) and (4.1.4)

For such systems G.Lampariello [20] in 1942 established the equations of Volterra-Hamel in the form

$$(4.2.1)\qquad \frac{d}{dt}\frac{\partial T^*}{\partial \omega_s} - \frac{\partial T^*}{\partial \pi_s} + \gamma^m_{s\ell}\,\omega_\ell\,\frac{\partial T^*}{\partial \omega_m} + \gamma^m_s\,\frac{\partial T^*}{\partial \omega_m} = Q^*_s$$

$$(\ell, m, s = 1,2,\ldots,n).$$

Here T^* is the kinetic energy of the mechanical system expressed as a function of the time t, the coordinates q_s, and the kinetic characteristics ω_s ; Q^*_s are the generalised forces corresponding to the linear nonholonomic coordinates π_s.

In the above equations the operator $\frac{\partial}{\partial \pi_s}$ means the relation

(4.2.2) $$\frac{\partial}{\partial \pi_s} \equiv b_{ks} \frac{\partial}{\partial q_k} \qquad (k,s = 1,2,\dots,n),$$

and the γ's are defined by the relations

(4.2.3) $$\gamma^m_{sl} = b_{ks} b_{ul} \left(\frac{\partial a_{mk}}{\partial q_u} - \frac{\partial a_{mu}}{\partial q_k} \right),$$

(4.2.4) $$\gamma^m_s = b_{ks} b_u \left(\frac{\partial a_{mk}}{\partial q_u} - \frac{\partial a_{mu}}{\partial q_k} \right) + b_{ks} \left(\frac{\partial a_{mk}}{\partial t} - \frac{\partial a_m}{\partial q_k} \right),$$

$$(k,l,m,s,u = 1,2,\dots,n),$$

where the a's and b's are given by the relations (4.1.1) and (4.1.3), respectively.

It is known that

(4.2.5) $$\gamma^m_{sl} + \gamma^m_{ls} \equiv 0 .$$

Let U be the potential function for the generalised forces Q^*_s and let

(4.2.6) $$p_s = \frac{\partial T^*}{\partial \omega_s} \qquad (s = 1,2,\dots,n).$$

If the generalised Hamiltonian function is defined by

(4.2.7) $$H(t; q_s; p_s) = p_s \omega_s - \left(T^* + U \right),$$

Lampariello [20] showed that the cononical equations of motion are

$$(4.2.8)\quad \begin{cases} \dot{p}_s = -\dfrac{\partial H}{\partial \pi_s} - \gamma^m_{sl}\, p_m \dfrac{\partial H}{\partial p_l} - \gamma^m_s\, p_m\,, \\[2ex] \dot{q}_s = b_{sl}\dfrac{\partial H}{\partial p_l} + b_s\,, \\ \qquad\qquad (l,m,s=1,2,\dots,n). \end{cases}$$

Next let us introduce, following Dobronravov [17], the generalised Poisson brackets denoted by double parentheses:

$$(4.2.9)\quad ((f_1,f_2)) = \frac{\partial f_1}{\partial \pi_s}\frac{\partial f_2}{\partial p_s} - \frac{\partial f_1}{\partial p_s}\frac{\partial f_2}{\partial \pi_s} + \gamma^m_{ls}\frac{\partial f_1}{\partial p_l}\frac{\partial f_2}{\partial p_s}\, p_m\,,$$
$$(l,m,s=1,2,\dots,n),$$

where $f_1 = f_1(t;q_s;p_s)$ and $f_2 = f_2(t;q_s;p_s)$.
It was also shown that these generalised brackets possess the properties of the usual Poisson brackets, namely:

$$(4.2.10)\quad ((f_1,f_2)) + ((f_2,f_1)) = 0,$$

$$(4.2.11)\quad ((f_1,((f_2,f_3)))) + ((f_2,((f_3,f_1)))) + ((f_3,((f_1,f_2)))) = 0,$$

where $f_3 = f_3(t;q_s;p_s)$.

In terms of the generalised Poisson brackets, we shall investigate the condition that $f(t;q_s;p_s) = \text{constant}$ be a first integral of the canonical equations of motion (4.2.8).

In fact, if we take the complete derivative of f

with respect to t and substitute in it for $\dot{p}_s$ and $\dot{q}_s$ the expressions obtained from the equations (4.2.8), we have

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial q_s}\left(b_{s\ell}\frac{\partial H}{\partial p_\ell}+b_s\right)+\frac{\partial f}{\partial p_s}\left(-\frac{\partial H}{\partial \pi_s}-\Upsilon^m_{s\ell}\,p_m\frac{\partial H}{\partial p_\ell}-\Upsilon^m_s\,p_m\right)\equiv 0$$

or, by virtue of (4.2.2) and (4.2.5), we have

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial \pi_\ell}\frac{\partial H}{\partial p_\ell}-\frac{\partial f}{\partial p_s}\frac{\partial H}{\partial \pi_s}+\Upsilon^m_{\ell s}\frac{\partial f}{\partial p_\ell}\frac{\partial H}{\partial p_s}p_m+b_s\frac{\partial f}{\partial q_s}-\Upsilon^m_s\,p_m\frac{\partial f}{\partial p_s}\equiv 0.$$

In the second term on the left-hand side we change the repeated suffix ℓ to s and make use of (4.2.9) to obtain the following:

$$\frac{\partial f}{\partial t}+((f,H))=\Upsilon^m_s\,p_m\frac{\partial f}{\partial p_s}-b_s\frac{\partial f}{\partial q_s}\qquad (m,s=1,2,\dots,n), \tag{4.2.12}$$

which is the condition we were looking for.

Finally, let us suppose that $f_1(t;q_s;p_s)=$ constant and $f_2(t;q_s;p_s)=$ constant are two first integrals of the canonical equations (4.2.8). Then according to (4.2.12) we have

$$\begin{cases}\dfrac{\partial f_1}{\partial t}+((f_1,H))=\Upsilon^m_s\,p_m\dfrac{\partial f_1}{\partial p_s}-b_s\dfrac{\partial f_1}{\partial q_s},\\[2em]\dfrac{\partial f_2}{\partial t}+((f_2,H))=\Upsilon^m_s\,p_m\dfrac{\partial f_2}{\partial p_s}-b_s\dfrac{\partial f_2}{\partial q_s},\qquad (m,s=1,2,\dots,n).\end{cases} \tag{4.2.13}$$

By virtue of (4.2.13) the identity (4.2.11) yields

$$\left(\left(f_1,-\frac{\partial f_2}{\partial t}+\Upsilon^m_s\,p_m\frac{\partial f_2}{\partial p_s}-b_s\frac{\partial f_2}{\partial q_s}\right)\right)+\left(\left(f_2,\frac{\partial f_1}{\partial t}-\Upsilon^m_s\,p_m\frac{\partial f_1}{\partial p_s}+b_s\frac{\partial f_1}{\partial q_s}\right)\right)+((H,((f_1,f_2))))\equiv 0,$$

or,

$$\left(\!\left(\left(\!\left(f_1,f_2\right)\!\right),H\right)\!\right)+\left(\!\left(f_1,\frac{\partial f_2}{\partial t}\right)\!\right)+\left(\!\left(\frac{\partial f_1}{\partial t},f_2\right)\!\right)=\left(\!\left(f_1,\gamma_s^m p_m\frac{\partial f_2}{\partial p_s}\right)\!\right)+\left(\!\left(\gamma_s^m p_m\frac{\partial f_1}{\partial p_s},f_2\right)\!\right)-$$

$$-\left(\!\left(f_1,b_s\frac{\partial f_2}{\partial q_s}\right)\!\right)-\left(\!\left(b_s\frac{\partial f_1}{\partial q_s},f_2\right)\!\right),$$

or,

$$\frac{\partial}{\partial t}\left(\!\left(f_1,f_2\right)\!\right)+\left(\!\left(\left(\!\left(f_1,f_2\right)\!\right),H\right)\!\right)=\gamma_s^m p_m\frac{\partial}{\partial p_s}\left(\!\left(f_1,f_2\right)\!\right)-b_s\frac{\partial}{\partial q_s}\left(\!\left(f_1,f_2\right)\!\right)+\frac{\partial f_2}{\partial p_s}\left(\!\left(f_1,\gamma_s^m p_m\right)\!\right)-$$

$$-\frac{\partial f_1}{\partial p_s}\left(\!\left(f_2,\gamma_s^m p_m\right)\!\right)-\frac{\partial f_2}{\partial q_s}\left(\!\left(f_1,b_s\right)\!\right)+\frac{\partial f_1}{\partial q_s}\left(\!\left(f_2,b_s\right)\!\right).$$

Hence

$$\left(\!\left(f_1,f_2\right)\!\right)+\int\left[\frac{\partial f_1}{\partial p_s}\left(\!\left(f_2,\gamma_s^m p_m\right)\!\right)-\frac{\partial f_2}{\partial p_s}\left(\!\left(f_1,\gamma_s^m p_m\right)\!\right)-\frac{\partial f_1}{\partial q_s}\left(\!\left(f_2,b_s\right)\!\right)+\frac{\partial f_2}{\partial q_s}\left(\!\left(f_1,b_s\right)\!\right)\right]dt=\text{const.} \tag{4.2.14}$$

is also a first integral of the canonical equations of motion (4.2.13).

Thus the theorem of Poisson can be stated as follows:

Let $f_1(t;q_s;p_s)=$ const. and $f_2(t;q_s;p_s)=$ const. be two integrals of the canonical equations of motion (4.2.13). If the expression on the left-hand side of (4.2.14) does not reduce to zero or a constant, and if moreover it is not expressible in terms of f_1, f_2, then the equation (4.2.14) constitutes a new integral of the system (4.2.13).

In particular, if the kinetic characteristics do not depend on the time, i.e. if $\gamma_s^m = 0$ for $s, m = 1, 2, \ldots, n$, the equation (4.2.14) reduces to

$$((f_1, f_2)) = \text{constant},$$

yielding a new integral of the system

$$\dot{p}_s = -\frac{\partial H}{\partial \pi_s} - \gamma_{s\ell}^m p_m \frac{\partial H}{\partial p_\ell},$$

$$\dot{q}_s = b_{s\ell} \frac{\partial H}{\partial p_\ell},$$

a result which was proved by Dobronravov [17].

In case of a linear non-holonomic system with $n - r$ degrees of freedom, the equations of constraint can be expressed by

$$(4.2.15) \qquad \omega_\alpha = 0 \qquad (\alpha = 1, 2, \ldots, r).$$

Hence the independent kinetic characteristics are $\omega_i (i = r + 1, r + 2, \ldots, n)$.

Consequently

$$(4.2.16) \qquad p_\alpha = 0 \qquad (\alpha = 1, 2, \ldots, r)$$

and the remaining $p_i (i = r + 1, r + 2, \ldots, n)$ are independent. Furthermore,

$$\frac{d\pi_\alpha}{dt} = 0 \qquad (\alpha = 1, 2, \ldots, r)$$

or $\pi_\alpha =$ constant ($\alpha = 1, 2, \ldots, r$). That is the linear non-holonomic systems, when referred to linear non-holonomic coordinates, assume a holonomic form. Taking into consideration the equations (4.2.15) and (4.2.16), the canonical equations of motion (4.2.8) still hold. As a consequence, the theorem of Poisson in its generalised form holds for linear non-holonomic systems provided that we take into account the relations (4.2.15) and (4.2.16).

4.3. Appell's Equations in Nonlinear Non-holonomic Cöordinates.

Let us consider a holonomic mechanical system with generalised coordinates $q_1, q_2, \ldots, q_n$. Following Novoselov [23], let us define the kinetic characteristics $\omega_1, \omega_2, \ldots, \omega_n$ by the relations

(4.3.1) $$\omega_l = \omega_l(t; q_s; \dot{q}_s) \qquad (l, s = 1, 2, \ldots, n)$$

which are nonlinear in the $\dot{q}$'s.

Assuming that the functional matrix

$$\left\| \frac{\partial \omega_l}{\partial \dot{q}_s} \right\|$$

has the rank r, we have

(4.3.2) $$\dot{q}_l = \dot{q}_l(t; q_s; \omega_s) \qquad (l, s = 1, 2, \ldots, n).$$

The variations of the q's are given by

(4.3.3) $$\delta q_l = \frac{\partial \dot{q}_l}{\partial \omega_s} \delta \pi_s \qquad (l, s = 1, 2, \ldots, n)$$

where $\delta \pi_s$ are the variations of nonlinear nonholonomic coordinates π_s.

According to the relation (4.3.1) the variations of the cartesian coordinates, x_ν, and the q's are related as follows

$$\delta x_\nu = \frac{\partial \dot{x}_\nu}{\partial \dot{q}_l} \delta q_l \qquad (\nu = 1, 2, \ldots, 3N;\ l = 1, 2, \ldots, n).$$

Or, by virtue of (4.3.3)

$$\delta x_\nu = \frac{\partial \dot{x}_\nu}{\partial \dot{q}_\ell} \frac{\partial \dot{q}_\ell}{\partial \omega_s} \delta \pi_s \qquad (\nu=1,2,\dots,3N;\ \ell,s=1,2,\dots,n)$$

i.e.

(4.3.4) $$\delta x_\nu = \frac{\partial \dot{x}'_\nu}{\partial \omega_s} \delta \pi_s \qquad (\nu=1,2,\dots,3N;\ s=1,2,\dots,n)$$

In (4.3.4) $\dot{x}'_\nu$ represents $\dot{x}_\nu$ after substituting for $\dot{q}_\ell$ from (4.3.2), i.e.

(4.3.5) $$\dot{x}'_\nu = \dot{x}_\nu\big(t;q_s;\dot{q}_s(t;q_\ell;\omega_\ell)\big).$$

Hence the principle of d'Alembert-Lagrange, expressed by the equation

$$\left(m_{(\nu)}\ddot{x}_\nu - X_\nu\right)\delta x_\nu = 0$$

takes the form

(4.3.6) $$\left(m_{(\nu)}\ddot{x}_\nu - X_\nu\right)\frac{\partial \dot{x}'_\nu}{\partial \omega_s}\delta \pi_s = 0 \qquad (\nu=1,2,\dots,3N;\ s=1,2,\dots,n)$$

Since the mechanical system is holonomic, $\delta\pi_s$ are independent. Therefore the equation (4.3.6) leads to the system of equations

(4.3.7) $$m_{(\nu)}\ddot{x}_\nu \frac{\partial \dot{x}'_\nu}{\partial \omega_s} = X_\nu \frac{\partial \dot{x}'_\nu}{\partial \omega_s} \qquad (\nu=1,2,\dots,3N;\ s=1,2,\dots,n).$$

Let us introduce the energy of acceleration, S, defined by

$$S = \frac{1}{2} m_{(\nu)}\ddot{x}_\nu \ddot{x}_\nu$$

and denote by S' the function S when $\dot{q}_s$ and $\ddot{q}_s$ are changed into ω_s and $\dot{\omega}_s$ by means of the equations (4.3.2). Also, from (4.3.5) we have

$$\ddot{x}_\nu = \frac{\partial \dot{x}'_\nu}{\partial \omega_s}\dot{\omega}_s + \text{terms not containing } \dot{\omega} ,$$

so that

$$\frac{\partial \ddot{x}_\nu}{\partial \dot{\omega}_s} = \frac{\partial \dot{x}'_\nu}{\partial \omega_s} \tag{4.3.8}$$

Now

$$\frac{\partial S'}{\partial \dot{\omega}_s} = m_{(\nu)}\ddot{x}_\nu \frac{\partial \ddot{x}_\nu}{\partial \dot{\omega}_s}$$

or, by virtue of (4.3.8)

$$\frac{\partial S'}{\partial \dot{\omega}_s} = m_{(\nu)}\ddot{x}_\nu \frac{\partial \dot{x}'_\nu}{\partial \omega_s} . \tag{4.3.9}$$

Further, let us put

$$Q'_s = X_\nu \frac{\partial \dot{x}'_\nu}{\partial \omega_s} , \qquad (s = 1, 2, \dots, n) . \tag{4.3.10}$$

As a consequence of (4.3.9) and (4.3.10) we can rewrite the equations (4.3.7) in the following form:

$$\frac{\partial S'}{\partial \dot{\omega}_s} = Q'_s \qquad (s = 1, 2, \dots, n) \tag{4.3.11}$$

which are the so-called equations of Appell.

In case of a nonlinear nonholonomic system we can take the non-linear constraints of the type of Četaev to be given by

$$\omega_\alpha = 0 \qquad (\alpha = 1, 2, \dots, r < n).$$

Hence the independent kinetic characteristics are ω_i (i = r + 1,

$r+2,\ldots,n$). As a consequence the relations (4.3.2) and (4.3.3) are replaced by

(4.3.12) $$\dot{q}_l = \dot{q}_l(t; q_s; \omega_i),$$

and

(4.3.13) $$\delta q_l = \frac{\partial \dot{q}_l}{\partial \omega_i} \delta \pi_i \qquad (l, s = 1, 2, \ldots, n;\ i = r+1, r+2, \ldots, n),$$

respectively, where $\delta\pi_i$ are the variations of the $n-r$ independent π_i.

Proceeding exactly as in the holonomic case we derive the equations of motion for the nonlinear non-holonomic system in the form

(4.3.14) $$\frac{\partial S'}{\partial \dot{\omega}_i} = Q'_i \qquad (i = r+1, r+2, \ldots, n)$$

where S' is a function of t, q_s, ω_i and $\dot{\omega}_i$, and

$$Q'_i = X_\nu \frac{\partial \dot{x}'_\nu}{\partial \omega_i}$$

The equations (4.3.14) are Appell's equations of motion for nonlinear non-holonomic systems.

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