ALGEBRAIC PROPERTIES OF CERTAIN RINGS OF CONTINUOUS FUNCTIONS

by

LI PI SU

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Department of Mathematics

The University of British Columbia
Vancouver 8, Canada

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We study the relations between algebraic properties of certain rings of functions and topological properties of the spaces on which the functions are defined.

We begin by considering the relation between ideals of rings of functions and $z$-filters. Let $\mathcal{C}^m(X)$ be the ring of all $m$-times differentiable functions on a $C^m$-differentiable $n$-manifold $X$, $L_c(X)$ the ring of all $L_c$-functions on a metric space $X$, and $\mathcal{O}(X)$ the ring of all analytic functions on a subset $X$ of the complex plane.

It is proved that two $m$-(resp. $L_c$-) realcompact spaces $X$ and $Y$ are $C^m$-diffeomorphic (resp. $L_c$-homeomorphic) iff $\mathcal{C}^m(X)$ and $\mathcal{C}^m(Y)$ (resp. $L_c(X)$ and $L_c(Y)$) are ring isomorphic.

Again, if $X$ and $Y$ are $m$-(resp. $L_c$-) realcompact spaces, then $X$ can be $\mathcal{C}^m$-(resp. $L_c$-) embedded as an open [resp. closed] subset in $Y$ iff $\mathcal{C}^m(X)$ (resp. $L_c(X)$) is a $\delta G$-[resp. $\delta F$-] homomorphic image of $\mathcal{C}^m(Y)$ (resp. $L_c(Y)$).

The subrings of $\mathcal{C}^m$ (resp. $L_c$) which determine the $C^m$-diffeomorphism (resp. $L_c$-homeomorphism) of the spaces are studied.

We also establish a representation for a transformation, more general than homomorphism, from a ring of $C^m$-differentiable functions to another ring of $C^m$-differentiable functions.

Finally, we show that, for arbitrary subsets $X$ and $Y$
of the complex plane, if there is a ring isomorphism from $\mathcal{O}(X)$ onto $\mathcal{O}(Y)$ which is the identity on the constant functions, then $X$ and $Y$ are conformally equivalent.
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INTRODUCTION

Let $C^m(X)$ and $C^m(Y)$ be the rings of $C^m$-differentiable functions defined on the $C^m$-differentiable $n_1(n_2)$-manifolds $X$, and $Y$, respectively, $L(X_1)$ and $L(Y_1)$ the rings of Lipschitzian functions defined on metric spaces $X_1$ and $Y_1$, respectively, $L_c(X_1)$ and $L_c(Y_1)$ the rings of $L_c$-functions defined on metric spaces $X_1$ and $Y_1$, respectively (see (10.1)), and $\mathcal{O}(X_2)$ and $\mathcal{O}(Y_2)$ the ring of analytic functions defined on domains of the complex plane (or Riemann surfaces) $X_2$ and $Y_2$, respectively, where $0 \leq m < \infty$ if the specific case is not mentioned.

During the last twenty years, the relations between the algebraic properties of $C^m(X)$ and $C^m(Y)$, $L(X_1)$ and $L(Y_1)$, and $\mathcal{O}(X_2)$ and $\mathcal{O}(Y_2)$ and the topological properties of $X$ and $Y$, $X_1$ and $Y_1$ and $X_2$ and $Y_2$, respectively, have been investigated. Hewitt (1948) [8] showed that two realcompact (or $Q$-) spaces $X'$ and $Y'$ (see §7 [8]) are homeomorphic iff $C(X')$ and $C(Y')$ (that is, when $m = 0$) are isomorphic (Theorem 57 [8], Theorem (8.3) [7]) by means of the Structure space (Stone topology) (4.9) [7]). Myers (1954) [16] proved that two compact $C^m$-differentiable $n$-manifolds $X$ and $Y$ ($1 \leq m < \infty$) provided with a Riemannian metric tensor of class $C^{m-1}$ are $C^m$-diffeomorphic iff $C^m(X)$ and $C^m(Y)$ are isomorphic. Pursell (1955) [20] established a stronger result: two $C^m$-differentiable $n$-manifolds $X$ and $Y$ with neighborhood-finite covering of coordinate neighborhoods are $C^m$-diffeomorphic iff $C^m(X)$ and $C^m(Y)$ are isomorphic, where
$1 \leq m \leq \infty$. Nakai (1959) [17], showed again by using the Riemannian metric tensor a stronger result than Myers' with $1 \leq m \leq \infty$.

Later, in 1960, in "Rings of Continuous Functions" [7] Gillman and Jerison gave a systematic study of the ring $C(X)$ on an arbitrary topological space $X$. They study the relations between algebraic properties of $C(X)$ and topological properties of $X$ by examining the special features of the family of zero-sets (1.6) of an ideal of functions. The method, used in the book, will play the most important role of this work.

For $L(X)$, Sherbert (1963) [26] has shown that two compact metric spaces $X_1$ and $Y_1$ are $L$-homeomorphic (14.2) iff $L(X)$ and $L(Y)$ are isomorphic (Theorem 5.1 [26]).

It also has been known for some time that the conformal structure of a domain in the complex plane is determined the algebraic structure of certain rings of analytic functions on it. Bers (1948) [3] proved that if $X_2$ and $Y_2$ are two plane domains, then $\mathcal{O}(X_2)$ and $\mathcal{O}(Y_2)$ are isomorphic iff $X_2$ and $Y_2$ are conformally equivalent. Rudin (1955) [25] and Royden (1956) [22] have extended this theorem to the case in which $X_2$ and $Y_2$ are arbitrary open Riemann surfaces.

Rudin (1955) [25] also showed that if $X_2$ and $Y_2$ are two plane domains with no AB-removable points, then they are conformally equivalent if the rings $\mathcal{O}^*(X_2)$ and $\mathcal{O}^*(Y_2)$ of bounded analytic functions on them are isomorphic. (*)

(*) Bers and Rudin do not assume a priori that the complex constants are preserved under the given isomorphism. Royden, however, has this a priori assumption on the complex constants.
Later, Ozawa and Mizumoto (1959) [18] proved that when \( X_2 \) and \( Y_2 \) are two plane domains whose complementary sets have positive capacity, respectively, if there exists a direct ring isomorphism \( \varphi \) of \( \log^+ (X_2) \) onto \( \log^+ (Y_2) \) such that \( \varphi(c) = c \) for every complex constant \( c \), then \( X_2 \) and \( Y_2 \) are conformally equivalent.

In other aspects, we are also interested in the representation of the transformations of rings of certain kinds of functions. Whittaker (1961) [29] and Kohls (1962) [12] gave a representation for the transformations of rings of continuous functions on different classes of topological spaces.

In 1965, Magill [14] has obtained the algebraic conditions relating \( C(X) \) and \( C(Y) \) which are both necessary and sufficient for embedding \( Y' \) in \( X' \), where \( X' \) and \( Y' \) are two realcompact spaces.

The primary aim of this thesis is to utilize the method of Gillman and Jerison for investigating the algebraic properties of \( C^m(X) \) (§§ 1–4 and § 6), and \( L_c(X) \) or \( L(X) \) (§§ 10–14), and how they are related with the topological properties of \( X \) and \( X_1 \), respectively. Secondly, we generalize Magill’s results to the rings \( C^m(X) \) (§ 7) and \( L_c(X_1) \) (§ 15). The other objective of this work is to establish a representation for the transformations of rings of \( C^m \)-differentiable functions (§ 8). In §§ 9 and 16, we also establish that if \( X \) and \( Y \) (or \( X_1 \) and \( Y_1 \)) are connected, then that \( C^m(X) \) and \( C^m(Y) \) (or \( L_c(X_1) \) and \( L_c(Y_1) \)) are isomorphic, implies \( X \) and \( Y \) (or \( X_1 \) and \( Y_1 \)) are \( C^m \)-diffeomorphic (\( L_c \)-homeomorphic). The subrings of \( C^m(X) \), \( C^m(Y) \) (or \( L(X_1) \) and \( L(Y_1) \)) which
can determine the $C^m$-diffeomorphism (or $L$-homeomorphism) are studied. We also give some algebraic properties of $C$ not applicable in $C^m$ in §9.

Finally, we discuss the ring of analytic functions defined on an arbitrary subset of the complex plane (§17).
PART I
RINGS OF DIFFERENTIABLE FUNCTIONS

§1. Rings of differentiable functions and ideals.

In order to study the relations between algebraic properties of \( C^m(X) \) and topological properties of \( X \), we shall examine the special features of the family of zero-sets (see (1.6)) of an ideal of \( C^m(X) \), where \( m \) will be a fixed non-negative integer or \( \infty \). Hereafter we will always refer to \( m \) as an arbitrary integer such that \( 0 \leq m \leq \infty \). As we have learned in Rings of continuous functions [7], such a family possesses properties analogous to those of a filter (see (1.8) to (1.15)). This fact will play an important role in the development of this work.

(1.1) Notation: We shall write all equations involving \( n \) variables as if there were a single variable present. For instance, we write

\[
f_0(x) \quad \text{for} \quad f_0(x_1, \ldots, x_n),
\]

\[
D_k f(x') \quad \text{for} \quad \frac{\partial^{k_1 + \ldots + k_n}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} f(x'_1, \ldots, x'_n),
\]

\[
(k) \quad \text{for} \quad \binom{k_1}{l_1} \ldots \binom{k_n}{l_n},
\]

etc. For any \( n \)-fold subscript \( k \), we also let

\[
\sigma_k = k_1 + \ldots + k_n, \quad \text{and} \quad \sigma_{k+l} = \sigma_k + \sigma_l.
\]

By \( d(x,y) \) we shall mean the distance between \( x \) and \( y \).
Note that \( f_k(x') = \sum_{\sigma_t \leq m-k} \frac{f_{k+t}(x)}{t!} (x' - x)^t + R_k(x',x) \) is short for
\[
f_{k_1 \ldots k_n}(x_1', \ldots, x_n') = \sum_{l_1^1 + \ldots + l_n^1 \leq m-(k_1^1 + \ldots + k_n^1)} \frac{f_{k_1^1 + \ldots + k_n^1}(x_1', \ldots, x_n')}{l_1^1! \ldots l_n^1!} (x_1' - x_1)^{l_1^1} \ldots (x_n' - x_n)^{l_n^1} + R_{k_1 \ldots k_n}(x_1', \ldots, x_n'; x_1, \ldots, x_n)
\]

(1.2) Definition: Let \( f(x) = f_0(x) \) be defined in the subset \( A \) of \( \mathbb{R}^n \). We say \( f(x) \) is of class \( C^m \) (will simply say \( C^m \)) in terms of the function \( f_{k_1 \ldots k_n}(x_1, \ldots, x_n) \), if the functions \( f_{k_1 \ldots k_n}(x_1, \ldots, x_n) \) are defined in \( A \) for all \( n \)-fold subscripts \( k_i \) with \( k_1 + \ldots + k_n \leq m \), and
\[
f_k(x') = \sum_{\sigma_t \leq m-k} \frac{f_{k+t}(x)}{t!} (x' - x)^t + R_k(x',x)
\]
for each \( f_k(x) \) \( (\sigma_k \leq m) \), where \( R_k(x',x) \) has the following property. For any point \( x^o \) in \( A \) and any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that if \( x \) and \( x' \) are any two points of \( A \) with \( d(x^o,x) < \delta \) and \( d(x',x^o) < \delta \), then
\[
|R_k(x';x)| \leq d(x,x')^{m-\sigma_k} \varepsilon.
\]
Remarks: (1) If \( m = 0 \), then (1-1) and (1-2) state merely that \( f(x) \) is continuous.

(2) For any isolated point \( x^o \), and any \( \varepsilon > 0 \), we may choose \( \delta > 0 \) so small that the only points \( x \) and \( x' \) satisfying \( d(x,x^o) < \delta \) and \( d(x',x^o) < \delta \) is \( x^o \) itself. Hence (1-1) becomes \( f_k(x') = f_k(x') + R_k(x';x') \) so \( R_k(x';x) = 0 \).
Thus (1-1) and (1-2) are satisfied even at the isolated points.
(3) Moreover, if \( A \) is an open set, then \( f(x) \) is \( C^m \) in the ordinary sense, and the \( f_k(x) \) are the partial derivatives of \( f(x) \) (see [30] §3). The converse is true, by Taylor's Theorem.

(1.3) Definition: Let \( X \) be any subset of \( E^n \). \( C^m(X) = \{ f : f \) is a function which is \( C^m \) in \( X \} \). \( C^m(X) = \{ f \in C^m(X) : f \) is bounded \}.

(1.4) Proposition: For any \( f \) and \( g \in C^m(X) \), we denote the pointwise addition, subtraction and multiplication of \( f \) and \( g \) by \( f+g \), \( f-g \), and \( f \cdot g \), respectively. Then \( C^m(X) \) is a commutative ring with unity element, denoted by \( u \), (\( \emptyset \) is the zero element) and \( C^m(X) \) is a commutative subring with \( u \) of \( C^m(X) \). Proof is obvious. (see Theorem 4 [31]).

(1.5) Definition: If \( f \in C^m(X) \) \( (f \in C^m(X)) \) has a multiplicative inverse in \( C^m(X) \) \( (C^m(X)) \) is said to be a unit in \( C^m(X) \). \( (C^m(X)) \).

(1.6) Definition: If \( f \in C^m(X) \), then \( Z(f) = \{ x : X : f(x) = 0 \} \) is said to be the zero-set of \( f \). Let \( Z(X) = \{ Z(f) : f \in C^m(X) \} \).

Remarks: (1) Note that it is clear that \( f \in C^m(X) \) is a unit iff \( Z(f) = \emptyset \) (see Theorem 4 [31]). Likewise, if \( f \) is a unit of \( C^m(X) \), then \( Z(f) = \emptyset \). But the converse need not hold, as the multiplicative inverse \( f^{-1} \) of \( f \) in \( C^m(X) \) may not be a bounded function. For example: Let \( f(x) = e^{-x^2} \). Then \( f \in C^m(E^1) \) and \( Z(f) = \emptyset \). But \( f^{-1}(x) = e^{+x^2} \), \( f^{-1} \in C^m(E^1) \) - \( C^m(E^1) \).

(2) We shall show that every closed subset of \( E^n \) is
a zero-set of \( C^m \)-differentiable function.

(1.7) Definition: A nonempty subfamily \( \mathfrak{F} \) of \( Z(X) \) is said to be a \( \mathfrak{z} \)-filter on \( X \), if it satisfied the following conditions.

(i) \( \emptyset \notin \mathfrak{F} \);

(ii) If \( Z_1, Z_2 \in \mathfrak{F} \), then \( Z_1 \cap Z_2 \in \mathfrak{F} \); and

(iii) If \( Z \in \mathfrak{F} \) and \( Z' \in Z(X) \) such that \( Z' \supseteq Z \), then \( Z' \in \mathfrak{F} \).

(1.8) Proposition: If \( I \) is a proper ideal in \( C^m(X) \), then the family \( Z[I] = \{ Z(f) : f \in I \} \) is a \( \mathfrak{z} \)-filter on \( X \).

Proof: (1) Since \( I \) contains no unit, \( \emptyset \notin Z[I] \).

(ii) Let \( Z_1, Z_2 \in Z[I] \), and \( f_1, f_2 \in I \) such that \( Z_1 = Z(f_1) \), \( Z_2 = Z(f_2) \). Since \( I \) is an ideal, \( f_1^2 + f_2^2 \in I \) and \( Z(f_1) \cap Z(f_2) = Z(f_1^2 + f_2^2) \in Z[I] \).

That is, \( Z_1 \cap Z_2 \in Z[I] \).

(iii) Let \( Z \in Z[I] \), \( Z' \in Z(X) \), \( f \in I \), and \( f' \in C^m(X) \) such that \( Z = Z(f) \) and \( Z' = Z(f') \). Since \( I \) is an ideal, \( f f' \in I \). However, \( Z' \supseteq Z \) we have \( Z' = Z' \cup Z = Z(f') \cup Z(f) = Z(f f') \in Z[I] \). Q.E.D.

Remark: The analogue of Prop. (1.8) with \( C^m(X) \) in place of \( C^m(X) \) is false, in general. For example, let us consider \( C^m(\mathbb{E}^1) \) and let \( f(x) = \frac{1}{1+x^2} \). Then \( f \in C^m(\mathbb{E}^1) \). Set \( I = (f) \), the ideal generated by \( f \) in \( C^m(\mathbb{E}^1) \). Then, it is clear that \( Z[I] \) satisfies (ii) and (iii). However, \( Z(f) = \emptyset \notin Z[I] \).

Note that "ideal" always means proper ideal, unless the contrary is mentioned.
Proposition: If $\mathcal{F}$ is a $z$-filter on $X$, then the family $Z^{-1}[\mathcal{F}] = \{f \in C^m(X) : Z(f) \in \mathcal{F}\}$ is an ideal in $C^m(X)$.

Proof: (1) Let $J = Z^{-1}[\mathcal{F}]$. Since $\emptyset \notin \mathcal{F}$, $J$ must contain no unit so that $J$ is a proper subset of $C^m(X)$. Let $f, g \in J$, and $h \in C^m(X)$. Then $Z(f + g) \supseteq Z(f) \cap Z(g) \in \mathcal{F}$. By property (ii) of $z$-filter $Z(f + g) \in \mathcal{F}$. That is, $f + g \in J = Z^{-1}[\mathcal{F}]$.

(2) $Z[hf] = Z(h) \cup Z(f) \supseteq Z(f) \in \mathcal{F}$. By property (iii) of $z$-filter again $Z(hf) \in \mathcal{F}$. Thus $hf \in J = Z^{-1}[\mathcal{F}]$. Hence $Z^{-1}[\mathcal{F}]$ is a proper ideal in $C^m(X)$.

Remarks: (1) For $f, g \in C^m(X)$, the ideal $(f, g)$ generated by $f$ and $g$ is proper iff $Z(f)$ meets $Z(g)$. Equivalently $f^2 + g^2$ is not a unit of $C^m(X)$.

(2) $Z[Z^{-1}[\mathcal{F}]] = \mathcal{F}$. For, let $I = Z^{-1}[\mathcal{F}] = \{f \in C^m(X) : Z(f) \in \mathcal{F}\}$. Then $Z[Z^{-1}[\mathcal{F}]] = Z[I] = \{Z(f) : f \in I\} = \{Z(f) : Z(f) \in \mathcal{F}\} = \mathcal{F}$.

Remark (2) shows that every $z$-filter is of the form $Z[I]$ for some ideal $I$ in $C^m(X)$.

(3) It is clear that $Z^{-1}[Z[I]] \supset I$. The inclusion may be proper. For instance, (a) we consider $C^m(E^1)$, where $m$ is any positive integer. Evidently $i \in C^m(E^1)$, where $i(x) = x$ for all $x \in E^1$. If $I = (i)$, then $I$ consists of all functions $f$ in $C^m(E^1)$ such that $f(x) = x \cdot g(x)$ for some $g \in C^m(E^1)$ so that every function in $I$ vanishes at 0. Hence every zero-set in $Z[I]$ contains the point 0. As a matter of fact, since $Z[I]$ is a $z$-filter that includes the set $\{0\}$, it must be the family of all zero-sets containing 0.
Let $M_0 = Z^{-1}[Z[I]]$. Evidently it consists of all functions in $C^m(E^1)$ that vanish at 0. Hence $M_0$ certainly contains $I$. However, $M_0 \not\supset I$, for $i^{3m+1} \in M_0$, and if $i^{\frac{3m+1}{2}} \in I$, then $i^{\frac{3m+1}{2}} = g \cdot i$ for some $g \in C^m(E^1)$. But, then $g = i^{\frac{3m-2}{2}} \notin C^m(E^1)$. Hence $i^{\frac{3m+1}{2}} \notin M_0 - I$.

Note that here $M_0$ is a maximal ideal. For, if $f \notin M_0$, then $Z(f) \cap Z(i) = \emptyset$. From Remark (1) $(1,f) = C^m(E^1)$. Since $(M_0,f) \supseteq (1,f)$, $(M_0,f) = C^m(E^1)$. In other words, $M_0$ is maximal.

(b) In case $m = \infty$, let us consider $C^m(E^1)$, and let $f_1(x) = e^{-\frac{1}{x^2}}$ for all $x \in E^1$.

Then $f_1 \in C^m(E^1)$, and $f_1(x) \neq 0$ for $x \in E^1$ and $x \neq 0$.

Let $I = (f_1)$. This consists of all functions $f$ in $C^m(E^1)$ such that $f(x) = f_1(x)g(x)$ for some $g \in C^m(E^1)$ so that every function in $I$ vanishes at 0. Hence every zero-set in $Z[I]$ contains the point 0. As a matter of fact, since $Z[I]$ is a z-filter that includes the set $\{0\}$, it must be the family of all zero-sets containing 0. Now, $M_0 = Z^{-1}[Z[I]]$ evidently consists of all functions in $C^m(E^1)$ that vanish at 0. Hence $M_0$ certainly contains $I$.

Moreover, $i \in M$. If $i \notin I$, then $x = g(x)f_1(x)$ for some $g \in C^m(E^1)$. Consequently, $g(x) = x/f_1(x)$, if $x \neq 0$ (as $f_1(x) = 0$ only when $x = 0$), and $\lim_{x \to 0^-} g(x) = -\infty$, $\lim_{x \to 0^+} g(x) = +\infty$. Hence $g$ cannot be continuous at the point 0. This is a contradiction.
In general, $x^n \notin M_0 - I$, for all positive integers $n$, hence for any $f \in C^m(E^1)$ which is analytic at the origin without a constant term is in $M_0 - I$.

Here, $Z[I] = Z[M_0]$ in spite of the fact that $M_0$ contains $I$ properly. Besides, $M_0$ is a maximal ideal in $C^m(E^1)$. For, if $f \notin M_0$, then $Z(f)$ is disjoint from $Z(f_1)$ and so $(f_1, f) = C^m(E^1)$. Hence $(M_0, f) \supseteq (f_1, f) = C^m(E^1)$.

(1.10) Definition: A $z$-ultrafilter on $X$ is a maximal $z$-filter.

Note that every subfamily of $Z(X)$ with the finite intersection property, by Zorn's Lemma, is contained in some $z$-ultrafilter on $X$.

(1.11) Proposition: If $M$ is a maximal ideal in $C^m(X)$, then $Z[M]$ is a $z$-ultrafilter on $X$.

Proof: It is obvious that if $I_1, I_2$ are two arbitrary ideals in $C^m(X)$ such that $I_1 \subseteq I_2$, then $Z[I_1] \subseteq Z[I_2]$. Also, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two arbitrary $z$-filters in $Z(X)$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $Z^{-1}[\mathcal{F}_1] \subseteq Z^{-1}[\mathcal{F}_2]$. Hence the result follows immediately from Propositions (1.8) and (1.9).

(1.12) Proposition: If $\mathcal{F}$ is a $z$-ultrafilter on $X$, then $Z^{-1}[\mathcal{F}]$ is a maximal ideal in $C^m(X)$.

Proof is similar to that of Prop. (1.11).

It follows from Propositions (1.11) and (1.12) that the mapping $Z$ is one-one from the set of all maximal ideals in $C^m(X)$ onto the set of all $z$-ultrafilters on $X$.
(1.13) **Proposition:** Let $M$ be a maximal ideal in $\mathcal{C}^m(X)$. If $Z(f)$ meets every member of $Z[M]$, then $f \in M$.

**Proof:** Suppose that $f \notin M$. Then, since $M$ is a maximal ideal, $(M,f) = \mathcal{C}^m(X)$. We know that the unity $\mathcal{U} \in \mathcal{C}^m(X) = (M,f)$. Hence $\mathcal{U} = h + g \cdot f$ for some $h \in M$ and $g \in \mathcal{C}^m(X)$. Hence $Z(h + g \cdot f) = Z(h)$. But $\mathcal{U} = Z(h + g \cdot f) \supset Z(h)$ and $Z(h + g \cdot f) = Z(h) \cap [Z(g) \cup Z(f)] = [Z(h) \cap Z(g)] \cup [Z(h) \cap Z(f)]$. By hypothesis $Z(h) \cap Z(f) \neq \emptyset$. Thus, we have $\mathcal{U} \supset [Z(h) \cap Z(g)] \cup [Z(h) \cap Z(f)]$. This is impossible. Hence $f \in M$.

(1.14) **Proposition:** Let $\mathcal{A}$ be a z-ultrafilter on $X$. If a zero-set $Z$ meets every member of $\mathcal{A}$, then $Z \in \mathcal{A}$.

**Proof:** Since $Z$ meets every member of $\mathcal{A}$, $\mathcal{A} \cup \{Z\}$ has the finite intersection property. By Zorn's Lemma, $\mathcal{A} \cup \{Z\}$ generates a z-ultrafilter $\mathcal{A}'$ which contains $\mathcal{A}$. But $\mathcal{A}$ is a z-ultrafilter, $\mathcal{A}' = \mathcal{A}$. Hence $Z \in \mathcal{A}' = \mathcal{A}$.

(1.15) **Definition:** An ideal $I$ in $\mathcal{C}^m(X)$ is said to be a z-ideal if $Z(f) \in Z[I]$ implies $f \in I$. That is, $I = Z^{-1}[Z[I]]$.

It is obvious that every maximal ideal is a z-ideal, while the ideals $I = (1)$ and $I_1 = (f_1)$, given in Remark (3) of Prop. (1.9) are not z-ideals.

§2. **Zero-sets and m-completely Regular Spaces.**

In this section, we will show that every closed subset of $E^n$ is a zero-set.

(2.1) **Lemma:** Let $f_n(x) = xe^{-nx}$ for each $x \in [0, \infty)$, where
m is a fixed positive integer. Then \( \{f_n : n \in \mathbb{N}\} \) converges uniformly to 0 on \([0, \infty)\), where \( \mathbb{N} \) always denotes the set of all positive integers.

Proof: \( f_n'(x) = mx^{m-1} - nx^m = x^m - nx^m(m - nx) \).

Set \( f_n'(x) = 0 \). We have \( x = 0 \) or \( x = \frac{m}{n} \).

\[
\begin{align*}
\frac{f_n''(x)}{n} &= (m - 1)x^{m-2}e^{-nx}(m - nx) - nx^{m-1}e^{-nx}(m - nx) - \frac{nx^{m-1}e^{-nx}x^m - \frac{nx^{m-1}e^{-nx}(m^2 - m - 2mn + n^2x^2)}{n^m}}{n^{m-2}e^{-m}} < 0.
\end{align*}
\]

Hence \( f_n(x) \) has its maximum at \( x = \frac{m}{n} \) with value \( \left(\frac{m}{n}\right)^m e^{-m} \).

Now, for any given \( \varepsilon > 0 \), take \( N_0 \) to be the positive integer such that \( N_0 \geq m/e \cdot \varepsilon \). Then \( \left(\frac{m}{n}\right)^m e^{-m} \leq m^m \cdot \frac{1}{m^m/e \cdot \varepsilon} \cdot e^{-m} = \varepsilon \), whenever \( n \geq N_0 \). That is \( |f_n| < \varepsilon \) whenever \( n \geq N_0 \). Hence \( \{g_n : n \in \mathbb{N}\} \) converges uniformly to 0.

Q.E.D.

Let \( \text{cl}_{E^n} B_r(a) = \{x \in E^n : \|x - a\| \leq r\} \) and \( \text{cl}_{E^n} B_r'(b) = \{x \in E^n : \|x - b\| \leq r'\} \), where \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n), x = (x_1, \ldots, x_n) \), and \( r/r' = k \). Then, it is evident that \( \text{cl}_{E^n} B_r(a) \) is \( C^\infty \)-diffeomorphic with \( \text{cl}_{E^n} B_r'(b) \) under \( \phi(x) = (\phi_1(x), \ldots, \phi_n(x)) \), where \( \phi_i(x) = \frac{1}{k}(x_i - a_i) + b_i \), \( 1 \leq i \leq n \).

Now, if \( f \) is a function defined on \( \text{cl}_{E^n} B_r(a) \), then \( g(x) = f(k(x_1 - b_1) + a_1, \ldots, k(x_n - b_n) + a_n) = f \cdot \phi^{-1}(x) \) is a function defined on \( \text{cl}_{E^n} B_r'(b) \), where \( \phi^{-1}(x) = (\phi_1(x), \ldots, \phi_n(x)) \), with \( \phi_i(x) = k(x_i - b_i) + a_i \), \( 1 \leq i \leq n \). Moreover,
if \( f \in C^\infty \), then \( g \in C^\infty \), and if \( |D_1 f(x)| \) is bounded by \( M_{g1} \), then \( |D_1 g(x)| \) is bounded by \( k^m M_{g1} \), for an arbitrary positive integer \( m \), where \( D_1 f(x), D_1 g(x) \) and \( \sigma_1 \) are as shown in (1.1).

(2.2) Theorem: Let \( F \) be an arbitrary closed subset of \( E^n \). Then \( F \) is a \( C^\infty \)-zero-set, i.e. there is \( f \in C^\infty(E^n) \) such that \( Z(f) = F \).

Proof: We know that \( E^n - F = \bigcup_{i=1}^\infty B_{r_i}(a^i) \), where \( B_{r_i}(a^i) \) denotes the open ball with radius \( r_i \neq 0 \), and center \( a^i = (a^i_1, \ldots, a^i_n) \). Let \( r_{i+1} = r_i/r_{i+1} \) for \( i \geq 1 \). Then \( r_{i+1} > 0 \). Choose \( 0 < r_0 < r_1 \). We now first define

\[
\varphi(t) = \begin{cases} 
\frac{1}{t^2} & t > 0 \\
0 & t \leq 0
\end{cases}
\]

It is clear that \( \varphi \in C^\infty(E^1) \). Next define

\[
g(s) = \frac{\varphi(r_1-s)}{\varphi(r_1-s) + \varphi(s-r_0)} \quad \text{Then} \quad g \in C^\infty(E^1)
\]

Finally, let

\[
f_1(x) = g(\|x - a^1\|) \quad \text{Then} \quad f_1 \in C^\infty(E^n) \quad \text{and}
\]

\[
f_1(x) = \begin{cases} 
1 & \text{for } x \in \text{cl}_{E^n} B_{r_0}(a^1) \\
0 & \text{for } x \in E^n - B_{r_1}(a^1) \\
0 < f_1(x) < 1 & \text{elsewhere}
\end{cases}
\]

Let \( f_2(x) = f_1(t_2(x_1 - a^2_1) + a^1_1, \ldots, t_2(x_n - a^2_n) + a^1_n) \).

Then we have

\[
f_1(x) = \begin{cases} 
1 & \text{for } x \in \text{cl}_{E^n} B_{r_0}/t_2(a^2) \\
0 & \text{for } x \in E^n - B_{r_2}(a^2) \\
0 < f_2(x) < 1 & \text{elsewhere}, \text{ and } f_2 \in C^\infty(E^n)
\end{cases}
\]


In general, we define
\[ f_1(x) = f_{1-l}(t_1(x_1 - a_1^l) + a_1^{1-l}, \ldots, t_1(x_n - a_n) + a_n^{1-l}) \]
for \( i = 2, 3, \ldots \). Then, we have
\[ f_1(x) = \begin{cases} 1 & \text{for } x \in \text{ct}_{E^n} B_{r_0}(t_2 \ldots t_1)(a^1) \\ 0 & \text{for } x \in E^n - B_{r_1}(a^1) \\ 0 < f_1(x) < 1 & \text{elsewhere} \end{cases} \]

By choice of \([f_n]\), we know that \( D_k f_1 \) is continuous for all \( k \in \mathbb{N} \), and \( D_k f_1 \) vanishes except on \( \text{ct}_{E^n} B_{r_1}(a^1) \) which is compact. Thus \( D_k f_1 \) is bounded on \( \text{ct}_{E^n} B_{r_1}(a^1) \) so is bounded on \( E^n \). Let \( |D_k f_1| \leq M_{gk} \) for \( gk \in \mathbb{N} \). Then, we have \( |D_k f_2| \leq t_j^g M_{gk} \). In general, \( |D_k f_1| \leq t_j^g \ldots t_1^g M_{gk} \).

Let \( f(x) = \sum_{i=1}^S \frac{f_i(x)}{C_i} \), where \( C_i = e^{\frac{j}{2} \int t_i j} \), for \( i \in \mathbb{N} \).

We will show that \( f \in C^\omega(E^n) \). By Theorem (7.17) [23], we then have to show that for each \( gk \in \mathbb{N} \), \( \sum_{i=1}^S \frac{D_k f_1(x)}{C_i} \) converges uniformly. However,
\[ \sum_{i=1}^S \frac{|D_k f_1(x)|}{C_i} \leq \sum_{i=1}^S \frac{1}{j} \frac{t_j^g M_{gk}}{C_i} = M_{gk} \sum_{i=1}^S \frac{1}{j} \frac{t_j^g}{C_i} \]

Thus we only have to show that \( \sum_{i=1}^S \frac{1}{j} \frac{t_j^g}{C_i} \) converges. Use the "Ratio test", \( \frac{\sum_{j=1}^{i+1} t_j^g / C_i + 1}{\sum_{j=1}^{i+1} t_j^g / C_i} = \frac{t_j^g / e(i+1) t_{i+1}}{e(i+1) t_{i+1}} \). By virtue of Lemma (2.1), \( \lim_{i \to \infty} t_j^g / e(i+1) t_{i+1} = 0 \). By corollary ([5], P.108),
$\sum_{i=1}^{\infty} \frac{1}{c_i} \frac{t_i^{ck}}{C_i}$ is convergent. Therefore $\sum_{i=1}^{\infty} \frac{D_k f_i(x)}{C_i}$ converges uniformly for each $ck \in \mathbb{N}$, so that $f \in C^\infty(\mathbb{R}^n)$.

Moreover, $Z(f) = \bigcap_{i=1}^{\infty} Z(f_i)$, as each function $f_i$ is a non-negative function. However $Z(f_i) = \mathbb{R}^n = B_{r_i}(a^i)$. Hence $Z(f) = \mathbb{R}$.

Q.E.D.

(2.3) Definition: Let $X$ be a topological space. $X$ is said to be $m$-completely regular if for every closed subset $F$ of $X$ and $x \in X - F$, there exists a function $f \in C^m(X)$ such that $f(x) = 1$ and $f(F) = \{0\}$.

(2.4) Theorem: A topological space is $m$-completely regular iff the family $Z(X) = \{Z(f) : f \in C^m(X)\}$ is a base for the closed subsets of $X$.

Proof: Necessity: Suppose that $X$ is $m$-completely regular. For any closed subset $F$ of $X$ and $x \in X - F$, there exists an $f \in C^m(X)$ such that $f(x) = 1$ and $f(F) = \{0\}$. Hence $Z(f) \supset F$ and $x \notin Z(f)$. Consequently $Z(X)$ is a base for the closed subsets of $X$. (See [7]).

Sufficiency: Suppose that $Z(X)$ is a base for the closed subsets of $X$. For each closed subset $F$ of $X$ and $x \in X - F$, there is $g \in C^m(X)$ such that $Z(g) \supset F$ and $x \notin Z(g)$. Let $g(x) = r$. Then $r \neq 0$. Let $f = g \cdot r^{-1}$, where $r^{-1}$ is the constant function of value $r^{-1}$. Then $f \in C^m(X)$ as $r^{-1}$ and $g \in C^m(X)$. Moreover, $f(x) = 1$ and $f(F) = \{0\}$.

Q.E.D.
(2.5) Definition: A topological space is said to be m-normal if for any disjoint closed subsets $F_1$ and $F_2$, there is an $f \in C^m(X)$ such that $f(F_1) = \{0\}$ and $f(F_2) = \{1\}$.

Having proved that every closed subset of $E^n$ is a zero-set, we can show that every subset of $E^n$ is m-completely regular as well as m-normal.

(2.6) Proposition: Every subset $X$ of $E^n$ is m-normal.

Proof: Let $F_1$ and $F_2$ be any two disjoint closed subsets of $X$. By property of the relative topology, we have two closed subsets $F'_1$ and $F'_2$ of $E^n$ such that $F'_i \cap X = F_i$, $i = 1, 2$. By Theorem (2.2), there are $f_1, f_2 \in C^m(E^n)$ such that $Z(f_i) = F'_i$. Let $f_i|X = g_i$. Then, $g_i \in C^m(X)$ and $Z(g_i) = Z(f_i) \cap X = F'_i \cap X = F_i$ so that $Z(g_1) \cap Z(g_2) = \phi$. Now, set $f = \frac{g_1}{g_1 + g_2}$. Since $Z(g_1) \cap Z(g_2) = \phi$, $f$ is well-defined and $f \in C^m(X)$. Moreover $Z(f) = Z(g_1)$ and $f(Z(g_2)) = \{1\}$. Hence $f(F_1) = \{0\}$, and $f(F_2) = \{1\}$. Q.E.D.

(2.7) Corollary: Every subset $X$ of $E^n$ is m-completely regular.

Proof: Since $X$ is Hausdorff and m-normal, $X$ is m-completely regular. Q.E.D.

§3 Fixed, free ideals and compact spaces.

In this section we shall see the characterization of fixed maximal ideals of $C^m(X)$ (see (3.3)) and how they are
related to a compact space (see (3.11)).

(3.1) Definition: Let $I$ be any ideal in $C^m(X)$ or $C^{m*}(X)$. Then $I$ is said to be a fixed ideal if $\cap Z[I]$ is not empty; and $I$ is said to be a free ideal if $\cap Z[I]$ is empty.

Remarks: (1) In view of the definition of free ideal, we know that $I$ is free iff for each $x \in X$, there is a function in $I$ that does not vanish at $x$.

(2) If $I$ is a fixed ideal in $C^m(X)$, then the set $S = \cap Z[I]$ is not empty, and the set $I' = \{f \in C^m(X) : f[S] = [0]\}$ is a fixed ideal containing $I$. Hence a fixed maximal ideal must be of this form. Furthermore, since $I'$ can be enlarged by making $S$ smaller, the only candidates for fixed maximal ideals are the ideals $I'$ for which $S$ contains just one point.

The corresponding statements hold for $C^{m*}(X)$.

(3) The ideal $I'$ mentioned above evidently contains the $z$-ideal $Z^{-1}[Z[I]]$. In general, the two are not the same; the set $S = \cap Z[I]$ need not be a member of $Z[I]$, even if $S$ is a zero-set. To see this, let us consider a space $X \subseteq E^1$, where $X$ contains at least one point which is not isolated, say $a$. Let $O_a$ denote the set of all $f \in C^m(X)$ for which $Z(f)$ is a neighborhood of $a$. $O_a$ is then a $z$-ideal. We claim that $\cap Z[O_a] = \{a\}$. Suppose $b \in \cap Z[O_a]$ and $b \neq a$. Since $X$ is $m$-completely regular there is an $f \in C^m(X)$ such that $Z(f)$ is a neighborhood of $a$ but $b \notin Z(f)$. That is, $f \in O_a$ so that $b \notin \cap Z[O_a]$ which is a contradiction. Hence $\cap Z[O_a] = \{a\}$. Now, since $\{a\}$ is a
closed set, by Theorem (2.2), it is a zero-set. However, "a" is not an isolated point it cannot be a neighborhood of itself. Hence \{a\} is not a member of \( \mathbb{Z}[0_a] \).

Note that since \( \mathbb{N}[I] \) is a closed subset in \( X \), by (2.2), it is a zero-set. (Compare [7] P.55).

(3.2) Notation: If \( I \) is an ideal in \( C^m(X) \), \( I(f) \) will denote the residue class of \( f \), \( f+I \); \( r \) will denote the constant function on \( X \) of value \( r \) for all \( r \in \mathbb{R} \), the field of real numbers.

(3.3) Theorem: (1) The fixed maximal ideals in \( C^m(X) \) are precisely the sets \( M_p = \{ f \in C^m(X) : f(p) = 0 \} \).

(2) The ideals \( M_p \) are distinct for distinct \( p \).

(3) For each \( p \), \( C^m(X)/M_p \) is isomorphic with the real field \( \mathbb{R} \). In fact, the mapping \( M_p(f) \rightarrow f(p) \) is the unique isomorphism of \( C^m(X)/M_p \) onto \( \mathbb{R} \), where \( p \in X \).

Proof: (1) Consider the mapping \( \phi \) from \( C^m(X) \) to \( \mathbb{R} \), defined by \( \phi(f) = f(p) \). Evidently it is a homomorphism, and its kernel is \( M_p \). Since for each \( r \in \mathbb{R} \), \( \phi(p) = r \), so that \( \phi \) is onto the field \( \mathbb{R} \). Hence its kernel \( M_p \) is maximal. On the other hand, if \( M \) is any fixed ideal in \( C^m(X) \), there exists a point \( p \in \mathbb{N}[M] \). Clearly, \( M \) is contained in \( M_p \) which has just been shown to be a maximal ideal. Hence if \( M \) is maximal, then we must have \( M = M_p \).

(2) If \( p \neq p' \), by the \( m \)-completely regularity of \( X \), there is \( f \in C^m(X) \) such that \( f(p') = 1 \) and \( f(p) = 0 \).
(Prop. (2.3)). Hence \( f \in M_p \) but \( f \notin M_p \). That is, \( M_p \neq M_p \).

(3) Form the proof of (1), \( M_p \) is the kernel of a homomorphism of \( \mathbb{C}^m(X) \) to \( \mathbb{R} \), so that \( \mathbb{C}^m(X)/M_p \) is isomorphic with \( \mathbb{R} \). The uniqueness of the isomorphism follows from the fact that the only non-zero automorphism of \( \mathbb{R} \) is the identity [7, (0.23)].

(3.4) Theorem: (1) The fixed maximal ideals in \( \mathbb{C}^m_*(X) \) are precisely the sets \( M_* = \{ f \in \mathbb{C}^m_*(X) : f(p) = 0 \} \).

(2) The ideals \( M_* \) are distinct for distinct \( p \).

(3) For each \( p \), \( \mathbb{C}^m_*(X)/M_* \) is isomorphic with the real field \( \mathbb{R} \). In fact, the mapping \( M_*^*(f) \rightarrow f(p) \) is the unique isomorphism of \( \mathbb{C}^m_*(X)/M_* \) onto \( \mathbb{R} \), where \( p \in X \).

Proof is identical with that of Theorem (3.3) except for the notation.

(3.5) Proposition: If \( X \) is compact, then every ideal \( I \) in \( \mathbb{C}^m_*(X) \) is fixed.

Proof: We know that \( Z[I] \) is a family of closed subset with finite intersection property in the compact space \( X \). Hence \( \cap Z[I] \neq \emptyset \).

(3.6) Proposition: If \( X \) is a compact space, then the correspondence \( p \rightarrow M_p \) is one-one from \( X \) onto the set of all maximal ideals in \( \mathbb{C}^m(X) \).

Proof: In view of Prop. (3.5), every ideal is fixed. By Prop. (3.3), each fixed maximal ideal in \( \mathbb{C}^m(X) \) is of the form...
Mp for some p ∈ X, and Mp = Mp′ iff p = p′. Hence the assertion holds.

Q.E.D.

We now shift our emphasis from ideal to z-filter. (see (1.8) to (1.12)).

(3.7) Definition: Let J be a z-filter in Z(X). Then J is said to be a free or fixed z-filter according as ∩J = ∅ or ∅. The following property is the immediate consequence of (3.7).

(3.8) Proposition: Every ideal of C(X) is fixed iff every z-filter is fixed.

(3.9) Lemma: A zero-set Z ∈ Z(X) is compact iff it belongs to no free z-filter.

Proof: Necessity: Suppose Z is compact and Z belongs to a z-filter J. Set J′ = {Z ∩ F : F ∈ J}. Then

(i) ∅ ∉ J′, since Z and F are in J, Z ∩ F ≠ ∅;
(ii) (Z ∩ F) ∩ (Z ∩ F′) = Z ∩ (F ∩ F′) ∈ J′ as F ∩ F′ ∈ J. Thus J′ is a family of closed sets in Z with the finite intersection property. It follows that ∩J′ ≠ ∅. But ∩J = ∩J′.

That is, J is fixed.

Sufficiency: Let § be any family of closed subsets of Z with the finite intersection property. Since Z is closed in X, the members of § are closed in X. Let J be the collection of all elements in Z(X) each of which contains
a finite intersection of members of $\mathcal{F}$. Evidently, $\mathcal{F}$ is a z-filter and $Z$ is in $\mathcal{F}$. (Since $Z \in \mathcal{F}$). By hypothesis, $\cap \mathcal{F} \neq \emptyset$. We know that every closed set is a zero-set and vice versa. Hence $\cap \mathcal{F} = \cap \mathcal{F} \neq \emptyset$. Therefore, $Z$ is compact.

(3.10) Proposition: Let $\mathcal{A}$ be a z-ultrafilter on $X$, and each of its members be noncompact. Then $\mathcal{A}$ is free.

Proof: Suppose that $p_0 \in \cap \mathcal{A}$. Then $\{p_0\}$ is a closed set so is a zero-set. Now, $\{p_0\}$ meets every member of $\mathcal{A}$ at $p_0$, by Prop. (1.14), $\{p_0\} \in \mathcal{A}$. This is a contradiction (3.9). Hence $\mathcal{A}$ is free. Q.E.D.

Remark: Nevertheless, it is not true that if every member of a z-filter $\mathcal{F}$, is noncompact, then $\mathcal{F}$ is free. For instance, consider $C^0(E^1)$, and let $f_0(x) = \sin x$. Then $f_0 \in C^0(E^1)$, but $Z(f_0) \neq \emptyset$ is not compact (as $Z(f_0)$ is not bounded).

Let $I = (f_0)$, the principal ideal in $C^0(E^1)$ generated by $f_0$. Hence every function $f \in I$ can be written as $f(x) = f_0(x)g(x)$ for some $g \in C^0(E^1)$. Then $Z(f)$ contains $Z(f_0)$ so that $Z(f)$ is not bounded. Thus $Z(f)$ is not compact for any $f \in I$. However, $\cap Z[I] = Z(f_0) \neq \emptyset$. In general, for any $f \in C^0(E^1)$ such that $Z(f)$ is not bounded, the z-filter $Z[I]$, where $I = (f)$, is fixed in spite of the fact that each member of it is noncompact.

(3.11) Theorem: In $X \subset E^n$, the following are equivalent:

1. $X$ is compact.
2. Every ideal in $C^0(X)$ is fixed, i.e. every z-filter is fixed.
(2*) Every ideal in $C^m(X)$ is fixed.

(3) Every maximal ideal in $C^m(X)$ is fixed, i.e. every $z$-ultrafilter is fixed.

(3*) Every maximal ideal in $C^m(X)$ is fixed.

Proof: That (1) is equivalent with (2) follows from Lemma (3.9) with $Z = X$ and the fact $X$ belongs to every $z$-filter in $X$.

Likewise, (1) implies (2*) because when $X$ is compact $C^m(X) = C^m(X)$. Now, suppose that (2*) is true, and $\mathfrak{F}$ is any family of closed subsets of $X$ with finite intersection property.

Let $A = \{f \in C^m(X) : 0 \leq f \leq 1$ and $Z(f) \supseteq F$, for some $F \in \mathfrak{F}\}$, and $\mathfrak{S} = \{Z(f) : f \in A\}$. Then, it is clear that $\cap \mathfrak{S} = \Omega \mathfrak{S}$. Let $I = (A)$, the ideal in $C^m(X)$ generated by $A$. By hypothesis, $\cap Z[I] \neq \emptyset$. But $Z[I] \supseteq \mathfrak{S}$, $\mathfrak{S} \supseteq \cap Z[I] \neq \emptyset$. Hence $\cap \mathfrak{S} \neq \emptyset$. In other words $X$ is compact.

This shows that (2*) implies (1). Hence (1) is equivalent with (2*). Consequently (2) and (2*) are equivalent.

Finally, (2) is equivalent to (3) and (2*) with (3*), because every free ideal is contained in a free maximal ideal. Q.E.D.

§4 Real Ideals, $m$-realcompact Spaces.

In 1948, E. Hewitt defined real maximal ideals and realcompact spaces ($Q$-spaces) (see [8], §7 and [7], Ch. 5). He also contributed many interesting properties about real maximal ideals and realcompact spaces ([8] and [7]). Unfortunately, those properties can not be carried to the rings of $C^m$-differentiable functions, since they are not lattice-ordered rings. (See [7] (0.19) and Ch. 5). Indeed, if $f \in C^m(X)$ is not nonnegative or nonpositive, then $|f| \notin C^m(X)$. 
Recently (1964), R. Bkouche has shown that every paracompact Hausdorff differentiable n-manifold is m-realcompact (see (4.2) and [4] Theorem 2). Here, we will show that every closed subset of $E^n$ is m-realcompact. Moreover, we know that every $C^m$-differentiable n-manifold with countable basis can be $C^m$-embedded in a closed subset of $E^{2n+1}$ (cor. 1.32 [16]).

We know that every residue class field of $C^m(X)$ or $C^{m*}(X)$ module a maximal ideal contains a canonical copy of the real field $\mathbb{R}$: the set of images of the constant functions under the canonical homomorphism. For, let $M$ be a maximal ideal in $C^m(X)$, and $r_1 \neq r_2$ be arbitrary constant functions. If $M(r_1) = M(r_2)$, then $M(r_1 - r_2) = M(r_1) - M(r_2) = 0$. That is $r_1 - r_2 \in M$. This is impossible for $r_2 - r_1$ is a unit. Hence $M(r_1) \neq M(r_2)$, and that the set $K = \{M(r) : r \in \mathbb{R}\}$ forms a field is clear. We shall identify this subfield with $\mathbb{R}$. Thus, in Theorems (3.3) and (3.4), we can write $M_p(f) = f(p)$ for all $f \in C^m(X)$, and $M^{*}_p(f) = f(p)$ for all $f \in C^{m*}(X)$.

(4.1) Definition: Let $M$ be any maximal ideal in $C^m(X)$ (or $C^{m*}(X)$). Then $M$ is said to be a real ideal if the canonical copy of $\mathbb{R}$ is the entire field $C^m(X)/M$ (respectively $C^{m*}(X)/M$), and $M$ is said to be a hyper-real ideal if the canonical copy of $\mathbb{R}$ is not the entire field $C^m(X)/M$ (respectively $C^{m*}(X)/M$).

Remark: By Theorems (3.3) and (3.4), every fixed maximal ideal in $C^m(X)$ or $C^{m*}(X)$ is real.
(4.2) Definition: A topological space $X$ is said to be m-realcompact, if every real maximal ideal in $C^m(X)$ is fixed.

It is clear that if $X$ is compact, then $X$ is m-realcompact.

We will show next that every closed subset of $E^n$ is m-realcompact.

(4.3) Lemma: An ideal in $C^m(X)$ is free iff for every compact subset $A$ of $X$ there exists an $f \in I$ having no zero in $A$.

Proof: Necessity: Suppose $I$ is free and $A$ is an arbitrary compact subset of $X$. If for each $f \in I$, $Z(f) \cap A \neq \emptyset$, then the family $\mathcal{F} = \{F = Z(f) \cap A : \text{for some } f \in I\}$ is a family of closed sets in $A$ having the finite intersection property. Indeed, for each $F_1, F_2 \in \mathcal{F}$, we have $F_i = Z(f_i) \cap A$ for some $f_i \in I$, $i = 1,2$. Then $F_1 \cap F_2 = (Z(f_1) \cap A) \cap Z(f_2) \cap A = (Z(f_1) \cap Z(f_2)) \cap A = Z(f_1^2 + f_2^2) \cap A \neq \emptyset$, as $f_1^2 + f_2^2 \in I$. Thus $\bigcap \mathcal{F} \neq \emptyset$. Hence $\bigcap Z[I] = \bigcap Z(f) \neq \emptyset$.

Sufficiency: Suppose for every compact set $A \subset X$ there is an $f \in I$ such that $Z(f) \cap A = \emptyset$. Then, for any $[x]$ which is compact, there is an $f \in I$ such that $Z(f) \cap [x] = \emptyset$. That is, $x \notin Z(f)$. Hence $\bigcap Z[I] = \emptyset$.

Q.E.D.

(4.4) Proposition: Let $X$ be a closed (unbounded) subspace of $E^n$. Then $X$ is m-realcompact.
Proof: Suppose that $M$ is a free maximal ideal and $C^m(X)/M$ is the real field $\mathbb{R}$. Let $g(x) = \frac{1}{\|x\|^2 + 1}$. Then that $g \in C^m(X)$ and $g$ is a unit is clear. Hence $g \not\in M$. That is, $M(g) \neq 0$. For any positive number $r$ and a sufficiently small number $\varepsilon > 0$, $g < r - \varepsilon$ for all but a compact subset of $E^n$, say $A_\varepsilon$. Then $B_\varepsilon = A_\varepsilon \cap X$ is compact in $X$ as $X$ is closed. Let $A' = cl_x(X - B_\varepsilon)$ which is closed in $X$ so is closed in $E^n$. Thus, there is an $f \in C^m(E^n) \subset C^m(X)$ such that $Z(f) = A'$. We will show that $Z(f) \subseteq Z$ for some $Z \in Z[M]$. It is enough to show that $Z(f) \supset Z$ for some $Z \in Z[M]$. However, we know that $B_\varepsilon$ is compact in $X$. By Lemma (4.35), there is $f_1 \in M$ such that $Z(f_1) \cap B_\varepsilon = \emptyset$. In other words, $Z(f_1) \subseteq X - B_\varepsilon \subseteq cl_x(X - B_\varepsilon) = Z(f)$. Hence $Z(f) \in Z[M]$. Therefore, $g < r - \varepsilon$ on the zero-set $Z(f)$, and $r - g \geq \varepsilon$.

Let $h_1 = (f - g)^2$ on $Z(f)$. Then $h_1$ is $C^m$ on $Z(f)$ which is closed in $E^n$. By Whitney's Analytic Extension Theorem (see [30]), we have a $C^m$ extension $h$. That is $h|Z(f) = h_1$. Hence $h^2 = r - g$ on $Z(f)$. Therefore, $h^2 = r - g$ (mod $M$). In other words $M(h^2) = M(r - g) = M(r) - M(g) = r - M(g)$. But, since $C^m(X)/M$ is real $M(h^2) = (M(h))^2 \geq 0$, so we have $M(g) \leq r$. Since $r$ is any positive number, $M(g)$ is infinitely small. This is a contradiction. Q.E.D.

§5 The Long Line.

We now will give an example to show that a non-paracompact space may not be an $m$-realcompact space.
Let $W$ be the set of all ordinal members less than the first uncountable ordinal, and $\{I_\alpha : \alpha \in W\}$ be a collection of open intervals indexed by the set $W$, that is, $I_\alpha$ is an open interval paired with an element $\alpha$ of $W$. We insert the open interval $I_\alpha$ between $\alpha$ and $\alpha + 1$, where $\alpha + 1$ denotes the immediate successor of $\alpha$ (and $\alpha - 1$ will denote the immediate predecessor of $\alpha$ if $\alpha$ is not a limit ordinal), and $L$ denote the union of $W$ and all the inserted open intervals. Now, we order $L$ by the following five conditions. Let $x$ and $y$ be any points of $L$. Then $x < y$ if

1. $x$ and $y$ are in $W$ and $x < y$ in $W$,
2. $x$ is in $W$, and $y$ is in an interval $I_\alpha$, and $x = \alpha$ or $x < \alpha$ in $W$,
3. $x$ is in interval $I_\alpha$, and $y$ is in $W$, and $\alpha < y$ in $W$,
4. $x$ is in interval $I_\alpha$, and $y$ is in an interval $I_\beta$, and $\alpha < \beta$ in $W$,
5. $x$ and $y$ are both in the same interval $I_\alpha$ and $x < y$ in $I_\alpha$.

We then topologize $L$ by means of the order topology and the resulting space is the "long line". [9].

We shall show now five properties of the long line.

(5.1) Proposition: $L$ satisfies the first axiom of countability

(5.2) Proposition: $L$ is a Hausdorff space.

The proofs of these two propositions are straightforward
from the definition of the long line.

(5.3) Definition: An isotone mapping is a mapping which is order preserving. An isotone-homeomorphism \( f \) is a mapping which is a homeomorphism and both \( f \) and its inverse \( f^{-1} \) are isotone. And two ordered spaces are called isotone-homeomorphic if there is an isotone-homeomorphism from one onto the other.

(5.4) Proposition: For each \( \alpha \in L, \alpha \neq 1, [1,\alpha] \) is isotone-homeomorphic to the unit interval \([0,1]\), consequently each point of \( L \), not the first element \( 1 \), has an open neighborhood which is homeomorphic to an open interval.

Proof: We will show this proposition by transfinite induction.

We know that \([1,2]\) in \( L \) is isotone-homeomorphic to \([0,1]\) in \( E^1 \). Now, assume that \([1,\beta]\) in \( L \) is isotone-homeomorphic to \([0,1]\) for each \( \beta < \alpha \). We have to show that \([1,\alpha]\) is also isotone-homeomorphic to \([0,1]\). Case 1, if \( \alpha \) is a non-limit ordinal, then by the induction hypothesis \([1,\alpha-1]\) is isotone homeomorphic to \([0,1]\). However, we know that \([\alpha-1,\alpha]\) and \([0,1]\), \([0,1]\) and \([0,\frac{1}{2}]\), and \([0,1]\) and \([\frac{1}{2},1]\) are isotone-homeomorphic. Thus, \([1,\alpha]\) is isotone-homeomorphic to \([0,1]\). Case 2, if \( \alpha \) is a limit ordinal, then there exists a sequence \( \{\alpha_n\} \) which converges to \( \alpha \). First of all, we shall show that there exists a sequence of functions \( \{f_n\} \) such that \( f_n \) is an isotone-homeomorphism of \([1,\alpha_n]\) onto \([0,\frac{n}{n+1}]\), and \( f_n([1,\alpha_{n-1}]) = f_{n-1} \), for all \( n \in \mathbb{N} \).
By our induction hypothesis, there exists \( g_n \) such that \( g_n \) is an isotone-homeomorphism of \([1,a_n]\) onto \([1,0]\). Let \( e_n \) map \([0,1]\) to \([0, \frac{n}{n+1}]\) such that \( e_n(x) = \frac{n}{n+1} x \) for each \( n \in \mathbb{N} \). Evidently \( e_n \) is an isotone-homeomorphism of \([0,1]\) onto \([0, \frac{n}{n+1}]\). Let \( h_n \) be a mapping defined as follows: \( h_n = e_n \circ g_n \) on \([1,a_n]\) for each \( n \in \mathbb{N} \). Then \( h_n \) is an isotone-homeomorphism of \([1,a_n]\) onto \([0, \frac{n}{n+1}]\). Now, let \( f_1(x) = h_1(x) \) for \( x \in [0,a_1] \) and \( f_2(x) = g_2 \circ h_2(x) \) for each \( x \in [a_1,a_2] \), where \( g_2 \) is defined as follows:

\[
g_2(y) = \frac{1}{2} + \frac{1}{6} \left( \frac{y - h_2(a_1)}{\frac{n-1}{n+1} - h_2(a_1)} \right)
\]

for \( y \in [h_2(a_1), h_2(a_2)] = [h_2(a_1), 2/3] \). Since \( g_2 \) is a linear transformation, it is an isotone-homeomorphism of \([h_2(a_1), 2/3]\) onto \([\frac{1}{2}, \frac{2}{3}]\).

Therefore \( f_2 \) is an isotone-homeomorphism of \([0,a_2]\) onto \([0,2/3]\) such that \( f_2|[0,a_1] = f_1 \). And so on. In general, for each \( n \), let \( f_n(x) = f_{n-1}(x) \) for \( x \in [0,a_{n-1}] \), \( f_n(x) = g_n \circ h_n(x) \) for \( x \in [a_{n-1}, a_n] \), where \( g_n(y) = \frac{n-1}{n} + \frac{n}{n+1} - \frac{n-1}{n} \frac{y - h_n(a_{n-1})}{n+1} - h_n(a_{n-1}) \)

is an isotone-homeomorphism of \([h_n(a_{n-1}), \frac{n}{n+1}]\) onto \([\frac{n-1}{n}, \frac{n}{n+1}]\). Thus, \( f_n \) is an isotone-homeomorphism of \([1,a_n]\) onto \([0,\frac{n}{n+1}]\) such that \( f_n|[1,a_{n-1}] = f_{n-1} \). Therefore, we have obtained the required sequence of functions \( \{f_n\} \). Next, we shall show that the sequence \( \{f_n\} \) has a limit \( f \), say, such that \( f \) is an isotone-homeomorphism of \([1,a]\) onto \([0,1]\). Define \( f(x) = \lim_{n \to \infty} f_n(x) \) for each \( x \in [1,a] \), and \( f(a) = 1 \). Then the function \( f \) is well-
defined. Indeed, for each \( x \in [l, a) \), there is an \( n_0 \) such that \( n > n_0 \) implies \( x \leq \alpha_n \) so that \( n > n_0 \) implies \( f_n(x) \) is constant, and hence the limit always exists. Then the following five properties follow immediately from the definition of \( f \) and the choice of \( \{f_n\} \):

1. \( f \) is one-one.
2. \( f^{-1} \) is well-defined. (By (1)).
3. \( f(\alpha_n) = f_n(\alpha_n) = \frac{n}{n+1} \) for each \( n \), so that
   \[
   \lim_{n \to \infty} f(\alpha_n) = 1,
   \]
4. \( f \) and \( f^{-1} \) are isotone mappings,
5. and finally
6. \( f \) and \( f^{-1} \) are continuous at each point of \([l, a)\) and \([0, 1)\) respectively.

To show that \( f \) is continuous at \( a \), we have to show that for each sequence \( \{\beta_n\} \) converging to \( a \), \( \lim_{n \to \infty} f(\beta_n) = f(a) \) (= 1). (By Prop. (5.1) and theorem (5.34) [19].)

Without loss of generality, we may assume that \( \beta_n < \beta_{n+1} \) for each \( n \in \mathbb{N} \). Since \( f \) is an isotone mapping \( \{f(\beta_n)\} \) is a monotonic, nondecreasing sequence. It is clear that \( \{f(\beta_n)\} \) is bounded above by 1. Thus \( \lim_{n \to \infty} f(\beta_n) \) exists. Let
\[
b = \lim_{n \to \infty} f(\beta_n).
\]
Then \( b \leq 1 \). If \( b < 1 \), then by (5), we know that \( f^{-1} \) is continuous at \( b \), so that \( \lim_{n \to \infty} f(\beta_n) = b \) which implies \( \lim_{n \to \infty} \beta_n = f^{-1}(b) \). But \( f^{-1}(b) \neq \alpha \) and \( \lim_{n \to \infty} \beta_n = \alpha \).

This is impossible. Hence \( b = 1 \), or \( \lim_{n \to \infty} f(\beta_n) = 1 = f(a) \).

Consequently, \( f \) is continuous at \( a \). Now, we want to show that \( f^{-1} \) is continuous at 1. We know that the space \([0, 1]\)
satisfies the second axiom of countability, hence again it is enough to show that for any sequence \( \{b_n\} \) which converges to 1, \( \lim_{n \to \infty} f^{-1}(b_n) = f^{-1}(1) = \alpha \). We may also assume that \( \{b_n\} \) is a monotonic non-decreasing sequence converging to 1.

Since \( f^{-1} \) is an isotone mapping \( \{f^{-1}(b_n)\} \) is a monotonic non-decreasing sequence. It is also clear that \( \{f^{-1}(b_n)\} \) is bounded above by 1. Thus \( \lim_{n \to \infty} f^{-1}(b) \) exists. Let \( \beta = \lim_{n \to \infty} f^{-1}(b_n) \). Then \( \beta \leq \alpha \). If \( \beta < \alpha \), then by (5) as \( f^{-1} \) is continuous at \( \beta \), \( \lim_{n \to \infty} f^{-1}(b_n) = \beta \) implies \( \lim_{n \to \infty} b_n = f(\beta) \). So that we have \( f(\beta) < 1 \), and \( \lim_{n \to \infty} b_n = 1 \).

This is impossible. Hence \( \beta = \alpha \), or \( \lim_{n \to \infty} f^{-1}(b_n) = \alpha = f^{-1}(1) \).

That is, \( f^{-1} \) is continuous at 1. Therefore \( f \) is an isotone-homeomorphism of \([1,\alpha]\) onto \([0,1]\). By transfinite induction, the first part of the proposition is proved.

Let \( \alpha \in L \), \( \alpha \neq 1 \). Then \([1,\alpha+1]\) is isotone-homeomorphic to \([0,1]\), so that \((1,\alpha+1)\) an open neighborhood of \( \alpha \), is isotone-homeomorphic to \((0,1)\). Q.E.D.

(5.5) Proposition: \( L \) is countably compact but is not paracompact hence it is not a compact space.

Proof: Let \( A \) be any countably infinite subset of \( L \). Then, \( A \) will be contained in the union of \( \{I_\alpha : \alpha \in \Delta \subset W\} \), where \( \Delta \) has at most countably many elements. Let \( \alpha_0 \) be the least upper bound of \( \Delta \). Then \([1,\alpha_0]\) is homeomorphic to \([0,1]\).

Hence \([1,\alpha_0]\) is compact. Now \( A \subset [1,\alpha_0] \), \( A \) must have a limit point in \([1,\alpha_0] \subset L \). That is, \( L \) is countably compact.
Now, we will see that $L$ is not paracompact. In view of Prop. (5.4), every point $\alpha \in L$, $\alpha \neq 1$ has an open neighborhood homeomorphic to $E^1$. Moreover, $[1,2)$ is an open set of $L$ which is a half open interval in $E^1$. Thus, $L$ is locally metrizable (see [9] p. 80). But $L$ is not metrizable. By Theorem 2-68 ([9], p. 81) $L$ is not paracompact. Consequently, it is not compact. Q.E.D.

(5.6) Proposition: Of any two disjoint closed sets in $L$, one is bounded.

Proof: If $F_1$ and $F_2$ are two cofinal closed sets, we can choose an increasing sequence $\{\alpha_n : n \in \mathbb{N}\}$ where $\alpha_n \in F_1$ if $n$ is odd, and $\alpha_n \in F_2$ if $n$ is even. Then, since $F_1$ and $F_2$ are closed $\sup_{n \in \mathbb{N}} \alpha_n = \lim_{n \to \infty} \alpha_n \in F_1 \cap F_2$. This is a contradiction. Q.E.D.

From Prop. (5.4), we know that each point $\alpha \neq 1$ of the long line has an open neighborhood which is homeomorphic to an open interval, while each open neighborhood of $1$ is homeomorphic to the half open interval $[0,1)$, say. The long line then can be considered as a 1-dimensional manifold with a boundary point $1$. Hence we can define the differentiable functions on $L$.

(5.7) Proposition: Every function $f \in C(L)$ is a constant on a tail $L - L(\alpha)$, where $\alpha$ depends on $f$, and $L(\alpha) = \{\sigma \in L : \sigma < \alpha\}$.

Proof: It is clear that every tail $L - L(\alpha)$ is countably compact, (in fact, it is homeomorphic to $L$ itself). Thus,
each image set \( f[L - L(\alpha)] \) is a countably compact subset of \( E^1 \) and hence compact, (as \( E^1 \) has a countable base), so that the intersection \( \cap_{\alpha \in L} f[L - L(\alpha)] \) of the nested family is non-empty. Choose a number \( r \) belonging to this intersection. Then the closed set \( f^{-1}(r) \) is cofinal in \( L \). Now, for each \( n \in \mathbb{N} \) the closed set \( A_n = \{ x \in L : |f(x) - r| \geq \frac{1}{n} \} \) is disjoint from \( f^{-1}(r) \). Hence, by Prop. (5.6), \( A_n \) is bounded. That is, there is \( \alpha_n \in L \) such that \( \alpha_n \leq \alpha_n \) for all \( \alpha_n \in A_n \). Let \( \alpha_0 \in L \), and \( \alpha_0 \geq \sup_{n \in \mathbb{N}} \alpha_n \). We have \( f[L - L(\alpha_0)] = \{ r \} \). Q.E.D.

Let \( L^* \) be the union space of \( L \) and the point \( \Omega \), the first uncountable ordinal. Then, it is clear that \( L^* \) is a compact 1-dimensional manifold. For each \( f \in C^m(L) \), we extend \( f \) to a function \( f^* \) on \( L^* \) by defining that \( f^*(\Omega) \) is the final constant value of \( f \). Evidently, \( f^* \in C^m(L^*) \), and \( f^* \) is the unique differentiable extension of \( f \). On the other hand, for each \( g \in C^m(L^*) \), the restriction of \( g \) to \( L \) belongs to \( C^m(L) \). It follows that \( C^m(L) \) is isomorphic with \( C^m(L^*) \), under the mapping \( f \rightarrow f^* \).

Since \( L^* \) is compact, we already have a complete description of the maximal ideals in \( C^m(L^*) \): every ideal is fixed, and the maximal ideals assume the form \( M_\sigma = \{ f^* \in C^m(L^*) : f^*(\sigma) = 0 \} \), where \( \sigma \in L^* \). (By Theorem (3.3).) By virtue of the isomorphism of \( C^m(L^*) \) with \( C^m(L) \), the maximal ideals in \( C^m(L) \) are in one-one correspondence with those of \( C^m(L^*) \). Moreover, the fixed maximal ideals in \( C^m(L) \) correspond to the ideals \( M_\sigma \) in \( C^m(L^*) \) for each \( \sigma \in L \), leaving just one free
maximal ideal in $C^m(L)$, namely, $M_0 = \{ f \in C^m(L) : f^* \in M_\Omega \}$, the one that corresponds to $M_\Omega$. Though $M_0$ is free, it is not hyper-real, for $C^m(L)/M_0 \ncong C^m(L^*)/M_\Omega$. Hence $L$ is not $m$-realcompact.

§ 6. Homomorphisms, $C^m$-mappings and $C^m$-diffeomorphisms.

In this section we will describe the relation between any $C^m$-mappings (see (6.1)) from $X \subset E^{n_1}$ into $Y \subset E^{n_2}$ and homomorphisms from $C^m(Y)$ to $C^m(X)$. We shall find that, in a sense, every homomorphism from one ring of $C^m$-differentiable functions into another is induced by a $C^m$-mapping (see (6.20)). We will also show that any two $m$-realcompact spaces $X$ and $Y$ are $C^m$-diffeomorphic (see (6.1)) iff $C^m(X)$ and $C^m(Y)$ are isomorphic (see (6.19)).

(6.1) Definition: Let $X \subset E^{n_1}$ and $Y \subset E^{n_2}$. A mapping $\tau : X \to Y$ is said to be a $C^m$-mapping at a point $p$, if each component of $\tau(x) = (\tau_1(x_1, \ldots, x_{n_1}), \ldots, \tau_{n_2}(x_1, \ldots, x_{n_1}))$ is $C^m$ at $p$. If $\tau$ is $C^m$ at each point of $X$, then $\tau$ is said to be a $C^m$-mapping on $X$. And $\tau$ is $C^m$-diffeomorphism if $\tau$ is a $C^m$-mapping, one-one, onto and its inverse mapping $\tau^{-1}$ is also a $C^m$-mapping. We will say then $X$ and $Y$ are $C^m$-diffeomorphic. Note that by def. (6.1), it is clear that $X$ and $Y$ are $C^m$-diffeomorphic implies $n_1 = n_2$.

(6.2) Definition: An $f \in C^m(X)$ is said to be a local $i$-th projection at a point $a$ if there exists a neighborhood $U$ of $a$ such that $f|U = \mathcal{I}$, where $\mathcal{I}$ will always denote the $i$-th projection of the space $E^n$. 
Lemma: Let $X$ be any subset of $E^n$. For each $a \in X$ and $r > 0$, there are $h_i, (1 \leq i \leq n)$, $h_i \in C^{**}(X)$ such that $h_i(x) = x_i$ for all $x \in cl_x B_r(a)$. Then we call $h_i, (1 \leq i \leq n)$, the bounded local projections at $a$.

Proof: Choose $r' > r$. As shown in Theorem (2.2), there exists $g \in C^{**}(E^n)$ such that $g(x) = 1$ if $x \in cl_{E^n} B_r(a)$, $0$ if $x \in E^n - B_{r'}, (a)$, $0 < g(x) < 1$, elsewhere.

Let $i_i(x) = x_i$, $1 \leq i \leq n$, the projections of $E^n$. Set $h_i(x) = i_i(x) \cdot g(x)$. Then, it is clear that $h_i \in C^{**}(E^n) \subset C^{**}(X)$ and satisfies the required condition. Q.E.D.

Let $C^m_o$ be a subset of $C^m(Y)$ (or $C^{m*}(Y)$), and $\tau$ be a mapping from $X \subset E^{n_1}$ to $Y \subset E^{n_2}$. Then we will see what $C^m_o$ should be in order that $g \cdot \tau \in C^m(X)$ (or $C^{m*}(X)$) for all $g \in C_o^m$ implies $\tau$ is a $C^m$-mapping from $X$ into $Y$.

Theorem: Let $\tau$ be a mapping from $X$ to $Y$ and $C^m_o$ be a subset of $C^m(Y)$. Then

1. $\tau$ is a $C^m$-mapping implies $g \cdot \tau \in C^m(X)$ for all $g \in C_o^m$,

2. If $g \cdot \tau \in C^m(X)$ for each $g \in C_o^m$, and $C^m_o$ includes all projections of $X$, then $\tau$ is a $C^m$-mapping on $X$.

Proof: (1) It is clear that $g \cdot \tau \in C^m(X)$ for each $g \in C_o^m$.

(2) Since $g \cdot \tau \in C^m(X)$ for each $g \in C_o^m$ which includes all projections on $X$, we have, in particular, $i_i \cdot \tau(x) = \tau_i(x) \in C^m(X)$ for $1 \leq i \leq n_2$. Hence, by Def. (6.1) $\tau$ is a $C^m$-mapping.
(6.5) Theorem: Let $\tau$ be a mapping from $X$ to $Y$ and $C^m_0$ be a subset of $C^{m*}(Y)$. Then

1. $\tau$ is $C^m$-mapping implies $g \circ \tau \in C^{m*}(X)$ for all $g \in C^m_0$.

2. If $g \circ \tau \in C^m(X)$ for each $g \in C^m_0$, and $C^m_0$ includes all local projections defined as in Lemma (6.3), then $\tau$ is a $C^m$-mapping on $X$.

Proof: (1) It is clear that $g \circ \tau \in C^m(X)$ for each $g \in C^m_0$. Moreover, $g \in C^{m*}(Y)$ and $\tau[X] \subset Y$, hence $g \circ \tau \in C^{m*}(X)$.

(2) The proof is similar to that of Theorem (6.4) (2) with the projections replaced by the bounded local projections.

(6.6) Definition: Let $\varphi$ be given mapping from a set $A$ into a set $B$. For each mapping $g$ from $B$ into $D$, the composition $g \circ \varphi$ carries $A$ into $D$. Thus, $\varphi$ induces a mapping $\varphi': D^B \to D^A$, explicitly $\varphi'(g) = g \circ \varphi$, and $\varphi'$ is said to be an induced mapping of $\varphi$.

There is a duality between the properties one-one and onto (provided that $D$ has more than one element): $\varphi'$ is one-one iff $\varphi$ is onto, and $\varphi'$ is onto iff $\varphi$ is one-one.

We are concerned with a $C^m$-mapping $\tau$ of $X$ into $Y$, where the role of $D$ is taken by $E^1$. The appropriate subset of $E^{1Y}$ will be either $C^m(Y)$ or $C^{m*}(Y)$. Evidently, the induced mapping $\tau'$, defined by $\tau'(g) = g \circ \tau \in C^m(X)$ for each $g \in C^m(Y)$ [resp. $C^{m*}(Y)$] is a homomorphism from $C^m(Y)$ into $C^m(X)$ [resp. $C^{m*}(Y)$ into $C^{m*}(X)$]. Indeed, for any $g,g' \in C^m(Y)$ and any $x \in X$.

$$(g+g')(\tau(x)) = g \circ \tau(x) + g' \circ (\tau(x)) = (g \circ \tau)(x) + (g' \circ \tau)(x) = (g \circ \tau + g' \circ \tau)(x)$$
so that \( \tau'(g+g') = (g+g') \circ \tau = g \circ \tau + g' \circ \tau = \tau'(g) + \tau'(g) \).

Similarly, \( \tau'(g \cdot g') = (g \circ \tau) \cdot (g' \circ \tau) = (\tau'(g)) \cdot (\tau'(g)) \).

(6.7) Proposition: The homomorphism \( \tau' \) carries the constant functions onto the constant functions of \( C^m(\tau[X]) \) identically.

Proof: For any \( x \in X \) and \( r \in \mathcal{R} \), we have \( \tau'(r)(X) = r(\tau(x)) = r \). Hence \( \tau'(r) = r \) on \( \tau[X] \). Q.E.D.

(6.8) Proposition: The homomorphism \( \tau' \) determines the mapping \( \tau \) uniquely.

Proof: If \( \sigma \) is also determined by \( \tau' \), then \( \sigma' = \tau' \).
Thus for each \( x \in X \), \( g(\sigma(x)) = g(\tau(x)) \) for all \( g \in C^m(Y) \).
By \( m \)-complete regularity of \( X \), \( \sigma(x) = \tau(x) \). Hence \( \sigma = \tau \).
Q.E.D.

We now examine the duality relation between \( \tau \) and \( \tau' \).

(6.9) Definition: A subset \( A \) of \( X \) is \( C^m \) [resp. \( C^{m*} \)]-embedded in \( X \) if for each \( f \in C^m(A) \) [resp. \( C^{m*}(A) \)], there is \( g \in C^m(X) \) [resp. \( C^{m*}(X) \)] such that \( g | A = f \).

(6.10) Theorem: Let \( \tau \) be a \( C^m \)-mapping from \( X \subset E^{n_1} \) into \( Y \subset E^{n_2} \), and \( \tau' \) be the induced homomorphism \( g \rightarrow g \circ \tau \) from \( C^m(Y) \) into \( C^m(X) \) [resp. \( C^{m*}(Y) \) into \( C^{m*}(X) \)].

(1) \( \tau' \) is an isomorphism (into) iff \( \tau[X] \) is dense in \( Y \).

(2) \( \tau' \) is onto iff \( \tau \) is a \( C^m \)-diffeomorphism whose image is \( C^m \)-embedded [resp. \( C^{m*} \)-embedded].

Proof: (1) \( \tau' \) is an isomorphism iff \( \tau'(g) = 0 \) implies \( g = 0 \). But the latter means \( (\tau'(g))(x) = 0 \) for all \( x \in X \).
implies $g = \Theta$ on $Y$. That is, $\tau'$ is an isomorphism iff $g = \Theta$ on $\tau[X]$ implies $g = \Theta$ on $Y$. By $m$-complete regularity of $Y$, we have that $\tau'$ is an isomorphism iff $\tau[X]$ is dense in $Y$.

(2) Necessity: By hypothesis, for each $f \in C^m(X)$, there exists $g \in C^m(Y)$ such that $\tau(g) = f$. We shall show that $\tau$ is one-one. If $\tau(x_1) = \tau(x_2)$, then $g(\tau(x_1)) = g(\tau(x_2))$ for all $g \in C^m(Y)$. That is, $(\tau'(g))(x_1) = (\tau'(g))(x_2)$ for all $g \in C^m(Y)$. Since $\tau'$ is onto, $f(x_1) = f(x_2)$ for all $f \in C^m(X)$. By $m$-complete regularity of $X$, $x_1 = x_2$. Hence, $\tau$ is one-one. Therefore, the inverse mapping $\tau^{-1}$ of $\tau$ is well defined as a mapping from $\tau[X]$ to $X$. We know that $f \circ \tau^{-1}$ is the function $g \mid \tau[X] \in C^m(\tau[X])$ for all $f \in C^m(X)$, and $C^m(X)$ includes all projections, it follows from Theorem (6.4)(2) that $\tau^{-1}$ is a $C^m$-mapping. Hence $\tau$ is a $C^m$-diffeomorphism from $X$ onto $\tau[X]$. Moreover, for any $h \in C^m(\tau[X])$, $h \circ \tau \in C^m(X)$, in other words, there exists $f \in C^m(X)$ such that $h \circ \tau = f$ or $h = f \circ \tau^{-1}$. The latter is $g \mid \tau[X]$ for some $g \in C^m(Y)$ such that $\tau'(g) = f$. Hence $\tau[X]$ is $C^m$-embedded in $Y$.

Sufficiency: By hypothesis, $\tau^{-1}$ is a $C^m$-mapping from $\tau[X]$ onto $X$. Consider any $f \in C^m(X)$, the function $f \circ \tau^{-1} \in C^m(\tau[X])$, by hypothesis, has an extension $g \in C^m(Y)$ such that $g \mid \tau[X] = f \circ \tau^{-1}$. That is $f = g \circ \tau = \tau'(g)$. Hence $\tau'$ is onto.

The proof for $C^{m*}$ is exactly the same, since we know, by Lemma (6.3) that $C^{m*}(X)$ contains bounded local projections.
Corollary: If \( \tau \) is a \( C^m \)-diffeomorphism from \( X \) onto \( Y \), then \( \tau' \) is an isomorphism of \( C^m(Y) \) onto \( C^m(X) \).

Proof: Since \( \tau[X] = Y \), by the theorem, \( \tau \) is both isomorphism and onto.

Corollary: If \( \tau \) is a \( C^m \)-diffeomorphism of a compact space \( X \subset E^{n1} \) to \( Y \subset E^{n2} \), then the induced mapping \( \tau' \) is onto.

Proof: By hypothesis, \( \tau[X] \) is compact in \( Y \), so it is compact in \( E^{n2} \), hence it is closed in \( E^{n2} \). By Whitney's Analytic Extension Theorem [30], for each \( g \in C^m(\tau[X]) \) there exists \( f \in C^m(E^{n2}) \subset C^m(Y) \) such that \( f | \tau[X] = g \). Hence \( \tau[X] \) is \( C^m \)-embedded in \( Y \). Moreover, \( \tau \) is a \( C^m \)-diffeomorphism hence \( \tau' \) is onto. (By Theorem (6.10)).

Next, we examine the inverse problem of determining when a given homomorphism of \( C^m(Y) \) into \( C^m(X) \) is induced by some \( C^m \)-mapping from \( X \) into \( Y \). Here, we shall first consider the homomorphisms from \( C^m(Y) \) into \( \mathcal{R} \), that is, the case in which \( X \) consists of just one point.

Proposition: Any nonzero homomorphism \( \varphi \) from \( C^m(Y) \) [or \( C^m^*(Y) \)] into \( \mathcal{R} \) is onto \( \mathcal{R} \). In fact \( \varphi(r) = r \) for all \( r \in \mathcal{R} \).

Proof: We know that \( \varphi(g) = \varphi(g \cdot u) = \varphi(g) \cdot \varphi(u) \) for all \( g \in C^m(Y) \), and \( \varphi \) is not identically zero, where \( u \) is the unity of \( C^m(Y) \). Hence \( \varphi(u) \) is the unity in \( \mathcal{R} \). That is, \( \varphi(u) = 1 \). The mapping from \( \mathcal{R} \) to \( \mathcal{R} \), defined be \( r \rightarrow \varphi(r) \) is a nonzero homomorphism. By (0.22) in [7].
it is the identity. In other words, \( \phi(r) = r \). Hence \( \phi \) is onto.

(6.14) Proposition: The correspondence between the homomorphisms of \( C^m(Y) \) [or \( C^{m*}(Y) \)] onto \( \mathcal{R} \), and the real maximal ideals is one-one.

Proof: The kernel of a homomorphism, \( \phi \), of \( C^m(Y) \) onto \( \mathcal{R} \) is a maximal in \( C^m(Y) \) and a real ideal (as \( C^m(Y)/\ker \phi \sim \mathcal{R} \)). On the other hand, each real maximal ideal is the kernel of such a homomorphism. Indeed, for each real maximal ideal \( M \), let \( \varphi \) be the isomorphism from \( C^m(Y)/M \) onto \( \mathcal{R} \), and define \( \phi : C^m(Y) \to \mathcal{R} \) by \( \phi(f) = \varphi(M(f)) \in \mathcal{R} \), for any \( f \in C^m(Y) \). Clearly, \( \phi \) is onto, and \( \ker \phi = \{ f \in C^m(Y) : \phi(f) = 0 \} = \{ f \in C^m(Y) : \varphi(M(f)) = 0 \} = \{ f \in C^m(Y) : f \in M \} \).

Now, for any \( f, g \in C^m(Y) \), \( \phi(f+g) = \varphi(M(f+g)) = \varphi(M(f) + M(g)) = \varphi(M(f)) + \varphi(M(g)) = \phi(f) + \phi(g) \), and \( \phi(f \cdot g) = \varphi(M(f \cdot g)) = \varphi(M(f) \cdot M(g)) = \varphi(M(f)) \cdot \varphi(M(g)) = \phi(f) \cdot \phi(g) \). Moreover, by (0.23) [7], distinct homomorphism onto \( \mathcal{R} \) have distinct kernels. Hence it is one-one.

The proof for \( C^{m*} \) is similar.

(6.15) Proposition: \( Y \) is \( m \)-realcompact iff to each homomorphism \( \phi \) from \( C^m(Y) \) onto \( \mathcal{R} \) - i.e. each nonzero homomorphism into \( \mathcal{R} \) - there corresponds a unique point \( y \) of \( Y \) such that \( \phi(g) = g(y) \) for all \( g \in C^m(Y) \).

Proof: Necessity: Let \( \phi \) be any homomorphism from \( C^m(Y) \) onto \( \mathcal{R} \). Then, by Prop. (6.14), \( \ker \phi \) is a real maximal ideal in \( C^m(Y) \). By \( m \)-realcompactness of \( Y \), \( \ker \phi = M_y \) for unique \( y \in Y \). Hence \( \phi(g) = \phi(M_y(g)) = g(y) \).
Sufficiency: The hypothesis says that the kernel of each homomorphism of \( C^m(Y) \) onto \( R \) is a fixed ideal. In view of Prop. (6.14), each real maximal ideal of \( C^m(Y) \) is then fixed. Hence \( Y \) is \( m \)-realcompact. Q.E.D.

Our first result about homomorphisms from \( C^m(Y) \) into \( C^m(X) \) for \( X \) is a generalization of Prop. (6.15).

(6.16) Theorem: Let \( \varphi \) be a homomorphism from \( C^m(Y) \) into \( C^m(X) \) such that \( \varphi(u) = u \). If \( Y \) is \( m \)-realcompact, then there exists a unique \( C^m \)-mapping \( \tau \) of \( X \) into \( Y \) such that \( \tau' = \varphi \).

Notice that the condition \( \varphi(u) = u \) is necessary.

Proof of the Theorem: For each point \( x \in X \), the mapping \( \sigma : g \to (\varphi(g))(x) \) is a homomorphism from \( C^m(Y) \) in \( R \).

Since \( (\varphi(u))(x) = u(x) = 1 \neq 0 \), \( \sigma \) is not zero homomorphism.

By Prop. (6.15), there is a point \( y \in Y \) such that \( \varphi(g)(x) = g(y) \) for all \( g \in C^m(Y) \). We set \( y = \tau(x) \). Then that the mapping \( \tau \) from \( X \) into \( Y \), thus defined, satisfies \( \varphi(g) = g \cdot \tau \) for each \( g \in C^m(Y) \) is clear. Since \( \varphi(g) \in C^m(X) \) for each \( g \in C^m(Y) \), Theorem (6.4) shows that \( \tau \) is a \( C^m \)-mapping. In view of Prop. (6.8), \( \tau \) is a unique \( C^m \)-mapping for which \( \tau' = \varphi \). Q.E.D.

(6.17) Corollary: An \( m \)-realcompact space \( Y \) contains a \( C^m \)-image of \( X \) iff \( C^m(X) \) contains a homomorphic image of \( C^m(Y) \) that includes the constant functions on \( X \).

Proof: Necessity: Let \( \tau \) be a \( C^m \)-mapping from \( X \) into \( Y \). Then \( \tau' \), the induced mapping of \( \tau \), is a homomorphism from
C^m(Y) into C^m(Y). By Prop. (6.7) C^m(X) \supset \tau'(C^m(Y)) which induces the constant functions on X.

Sufficiency: \tau'(C^m(Y)) contains the constant functions on X, there is g \in C^m(Y) such that \tau'(g) = u. Then, \tau'(u) = \tau'(u) \circ u = \tau'(u) \circ \tau'(g) = \tau'(u \circ g) = \tau'(g) = u. Thus, the result follows immediately from Theorem (6.16). Q.E.D.

(6.18) Corollary: An m-realcompact space Y contains a dense C^m image of X iff C^m(X) contains an isomorphic image of C^m(Y) that includes the constant functions on X.

(6.19) The Main Theorem: Two m-realcompact spaces X and Y are C^m-diffeomorphic iff C^m(X) and C^m(Y) are isomorphic.

Proof: The necessity follows from (6.11).

Sufficiency: Let \phi be an isomorphism of C^m(Y) onto C^m(X). Then \phi^{-1} is an isomorphism of C^m(X) onto C^m(Y).

By Theorem (6.16) there exist unique C^m-mappings \tau and \tau_1 from X into Y and from Y into X, respectively, such that \phi(g) = g \circ \tau, and \phi^{-1}(f) = f \circ \tau_1, for each g \in C^m(Y) and f \in C^m(X). Then, g(y) = \phi^{-1}(g \circ \tau)(y) = (g \circ \tau) \circ \tau_1(y) = g \circ (\tau \circ \tau_1)(y) = g \circ (\tau \circ \tau_1)(y) for all y \in Y. That is, \tau \circ \tau_1 is the identity mapping onto itself. Similarly, \tau_1 \circ \tau is the identity mapping of X onto itself. Thus \tau and \tau_1 are the inverse mappings of each other. Hence X and Y are C^m-diffeomorphic. Q.E.D.

Remark: S. B. Myers [16], L. E. Pursell [20] and M. Nakai [17] have dealt with C^m-differentiable n-manifolds. This Theorem is applicable to any closed subset of E^n.
In spite of the remark made in Theorem (6.16), every homomorphism is induced, in essence, by a $C^m$-mapping.

(6.20) Theorem: Let $\varphi$ be a homomorphism from $C^m(Y)$ into $C^m(X)$, $Y$ be $m$-realcompact. Then the set $E = \{x \in X : \varphi(u)(x) = 1\}$ is open-and-closed in $X$. Moreover, there exists a unique $C^m$-mapping $\tau$ from $E$ into $Y$, such that for all $g \in C^m(Y)$ $\varphi(g)(x) = g(\tau(x))$ for all $x \in E$, and $\varphi(g)(x) = 0$ for all $x \in X - E$.

Proof: As with any homomorphism, the element $e = \varphi(u)$ is an idempotent in $C^m(X)$. Hence it is the characteristic function of the set $E = e^{-1}(1)$. Since $e$ is continuous (as a matter of fact it is $C^m$), $E$ is open-and-closed in $X$.

Moreover, again, as with any homomorphism, $e$ is the unity element of the image ring $\varphi[C^m(Y)]$. It follows that $
abla(g)[X - E] = \varphi(g)\varphi(u)[X - E] = \varphi(g)[X - E] \cdot \varphi(u)[X - E] = [0]$ for all $g \in C^m(Y)$.

If $E \neq \emptyset$, consider the homomorphism $\sigma$ from $C^m(Y)$ into $C^m(E)$, defined by $\sigma(g) = \varphi(g) | E$. It is clear that $\sigma(u) = u \in C^m(E)$ for $\sigma(u) = e | E = u$. By Theorem (6.16), there exists a unique $C^m$-mapping $\tau$ of $E$ into $Y$ such that $\tau | E = \sigma$. We have then for any $g \in C^m(Y)$, $\varphi(g)(x) = (\sigma(g)(x)) = g \circ \tau(x)$ for all $x \in E$ and $\varphi(g)(x) = 0$ for all $x \in X - E$.

Q.E.D.

(6.21) Theorem: Let $Y$ be a compact subspace of $E^{n^2}$, and $\varphi$ be a homomorphism from $C^m(Y) = C^m_*(Y)$ into $C^m_*(X)$. Then the set $E = \{x \in X : \varphi(u)(x) = 1\}$ is open-and-closed in $X$. Moreover, there exists a unique $C^m$-mapping $\tau$ from $E$ into $Y$, 
such that for any $g \in C^m(Y)$,

$$\varphi(g)(x) = g(\tau(x)) \quad \text{for all } x \in E,$$

and

$$\varphi(g)(x) = 0 \quad \text{for all } x \in X - E.$$

The proof is similar to Theorem (6.20).

(6.22) Corollary: Let $\varphi$ be a homomorphism from $C^m(Y)$ into a ring of $C^m$ functions. If $Y$ is m-realcompact, then there exists a unique closed subset $F$ of $Y$ such that the kernel of $\varphi$ is the $z$-ideal of all $C^m$ functions that vanish on $F$.

Proof: Let $\varphi$ be a homomorphism of $C^m(Y)$ onto $\mathcal{R}$ which is a subring of $C^m(X)$, and $E = \{x \in X : \varphi(u)(x) = 1\}$. By Theorem (6.20), there exists a $C^m$-mapping $\tau$ from $E$ into $Y$ such that $\varphi(g)(x) = g(\tau(x))$ for all $x \in E$, and $\varphi(g)(x) = 0$ for all $x \in X - E$, for any $g \in C^m(Y)$. Let $F = cl_Y \tau[E]$, and $I = \{g \in C^m(Y) : Z(g) \supseteq F\}$. Then $I$ is a $z$-ideal (see (1.15)). On the other hand, $\ker \varphi = \{g \in C^m(Y) : \varphi(g) = 0\}$. It is then clear that $\ker \varphi \ni I$. For any $g_0 \in \ker \varphi$,

$$\varphi(g_0)(x) = 0 \in \mathcal{R} \subseteq C^m(X)$$

so that $\varphi(g_0)[X] = \{0\}$. That is,

$$\varphi(g_0)(x) = g_0(\tau(x)) = 0 \text{ for all } x \in E,$$

or $g_0(\tau[E]) = \{0\}$.

Thus $Z(g_0) \supseteq \tau[E]$ so that $Z(g_0) = cl_Y Z(g_0) \supseteq cl_Y \tau[E] = F$.

Hence $g_0 \in I$. Consequently $\ker \varphi = I$. The uniqueness of $F$ follows from the choice of $F$.

Q.E.D.

(6.23) Proposition: An m-realcompact space $Y$ contains a $C^m$-embedded $C^m$ image of $X$ iff $C^m(X)$ is a homomorphic image of $C^m(Y)$.

Proof: The necessity is clear. For sufficiency, let $\varphi$ be a homomorphism of $C^m(Y)$ onto $C^m(X)$, and $k \in C^m(Y)$ such that
\( \varphi(k) = u \). Then \( \varphi(u) = \varphi(k)\varphi(u) = \varphi(k\cdot u) = \varphi(k) = u \). By Theorem (6.16), we have a unique \( C^m \)-mapping of \( X \) into \( Y \) such that \( \tau' = \varphi \). Hence \( \tau' \) is onto. By Theorem (6.10)(2), then \( \tau[X] \) is \( C^m \)-embedded.

(6.24) Proposition: A compact space \( Y \) contains a \( C^{m*} \)-embedded \( C^m \) image of \( X \) iff \( C^{m*}(X) \) is a homomorphic image of \( C^{m*}(Y) \).

Proof is similar to that of Prop. (6.23).

§7 The Embedding Theorems.

K. D. Magill, in 1965, [14], established the algebraic conditions relating \( C(Y) \) and \( C(X) \) which are both necessary and sufficient for embedding \( X \) in \( Y \). We will generalize his results to the rings of \( C^m \)-differentiable functions.

(7.1) Definition: Let \( \mathcal{C} \) be a collection of real ideals in a ring. Then \( \cap \mathcal{C} \) is said to be a \( \delta \)-real ideal.

(7.2) Definition: Let \( \mathcal{B} \) be a subring of a ring \( A \). Then we say that \( \mathcal{B} \) is \( \delta \)-dense in \( A \) if for every pair \( I \) and \( I' \) of \( \delta \)-real ideals of \( A \) with \( I - I' \not= \emptyset \), \( I - I' \) contains an element of \( \mathcal{B} \).

(7.3) Definition: A homomorphism from a ring \( A' \) into a ring \( A \) is a \( \delta \)-homomorphism if it is nontrivial and the image of \( A' \) is \( \delta \)-dense in \( A \).

For example, a homomorphism onto is a \( \delta \)-homomorphism.

(7.4) Definition: A set of elements of a ring is said to be a subreal, if it is contained in a real ideal of the ring.
(7.5) Definition: A $\delta$-homomorphism is a $\delta F$-homomorphism, if the image of every real ideal containing the kernel is subreal.

(7.6) Definition: A $\delta$-homomorphism from a ring $A'$ into a ring $A$ is a $\delta G$-homomorphism if for every real ideal $M$ of $A'$ whose image is subreal, there exists an element $a \notin M$ such that the image of every real ideal not containing $a$ is subreal.

Let $\varphi$ be a homomorphism of a ring of functions $A'$ into another ring of functions $A$. We will say that $\varphi$ has property (7-1) to mean that for every $g \in A$ and $x \notin Z(g)$, there is an $f \in A'$ such that $x \notin Z(\varphi(f))$, and $Z(g) \subset Z(\varphi(f))$.

(7.7) Theorem: Let $Y$ be an $m$-realcompact space and $X \subset E^{n_1}$, and $\varphi$ a homomorphism from $C^m(Y)$ into $C^m(X)$. Then $\varphi$ has the property (7-1) iff there is a homeomorphism $\tau$ from $X$ into $Y$ such that $\varphi(f) = f \circ \tau$ for all $f \in C^m(Y)$ and $\tau$ is a $C^m$-mapping. In addition, if $\varphi(C^m(Y))$ contains all projections of $X$, then $\tau$ is a $C^m$-diffeomorphism into $Y$.

Proof: Sufficiency: Suppose $\tau$ is a $C^m$-diffeomorphism from $X$ into $Y$ such that $\varphi(f) = f \circ \tau$ for all $f \in C^m(Y)$ and that $x \notin Z(g)$ where $g \in C^m(X)$. Then $\tau(x) \notin c_{\tau Y} \tau[Z(g)]$. For if $\tau(x) \in c_{\tau Y} \tau[Z(g)]$, then for each open neighborhood $U$ of $\tau(x)$, $U \cap \tau[Z(g)] \neq \emptyset$. However, we know that $x \notin Z(g)$ so that there exists an open neighborhood $V$ of $x$ such that $V \cap Z(g) = \emptyset$. Thus $\tau[V] \cap \tau[Z(g)] = \emptyset$ (as $\tau$ is one-one), and $\tau[V]$ is open in $\tau[X]$. Hence, there exists an open set $G$ in $Y$ such that $G \cap \tau[X] = \tau[V]$. We know then that
\[ t(x) \in \tau[V] \subseteq G, \quad \text{and} \quad G \cap \tau[Z(g)] = G \cap \tau[Z(g)] \cap \tau[X] = (G \cap \tau[X]) \cap \tau[Z(g)] = \tau[V] \cap \tau[Z(g)] = \emptyset, \quad \text{which is a contradiction.} \]

Now, since \( \tau(x) \notin c_\ell[Y\tau[Z(g)]] \), by \( m \)-complete regularity of \( Y \), there is an \( f \in C^m(Y) \) such that \( Z(f) \supset c_\ell[Y\tau[Z(g)]] \) and \( \tau(x) \notin Z(f) \). Hence \( x \notin \tau^{-1}(Z(f)) \) and \( \tau^{-1}[Z(f) \cap \tau[X]] \supseteq Z(g) \). But, \( \tau^{-1}[Z(f) \cap \tau[X]] = \{x \in X : f \circ \tau(x) = 0\} = Z(f \circ \tau) = Z(\varphi(f)) \). This proves the sufficiency.

**Necessity:** Let \( \varphi \) be a homomorphism of \( C^m(Y) \) into \( C^m(X) \) satisfying (7-1). We define a mapping \( \tau \) from \( X \) onto \( Y \) as follows. Let \( x \in X \) be given. Then the mapping \( \varphi_x : C^m(Y) \rightarrow \mathbb{R} \) defined by \( \varphi_x(f) = \varphi(f)(x) \) is a homomorphism. Moreover, \( x \notin Z(\varphi_x) \) so by (7-1), there exists \( f_0 \in C^m(Y) \) such that \( x \notin Z(\varphi(f_0)) \). Hence \( \varphi_x(f_0) = \varphi(f_0)(x) \neq 0 \).

That is, \( \phi_x \) is a non-zero homomorphism. By Prop. (6.15) since \( Y \) is \( m \)-realcompact, there exists a unique \( y \in Y \) such that \( \varphi_x(f) = f(y) \) for all \( f \in C^m(Y) \). We then define \( \tau(x) = y \).

Then we have \( \varphi(f) = f \circ \tau \) for all \( f \in C^m(Y) \). It remains for us to show that \( \tau \) is a \( C^m \)-mapping, \( \tau^{-1} \) exists and is continuous.

Since \( f \circ \tau \in C^m(X) \) for all \( f \in C^m(Y) \), by Theorem (6.4) \( \tau \) is a \( C^m \)-mapping. Now, if \( x \neq x' \) in \( X \), there is a \( g \in C^m(X) \) such that \( g(x) = 0 \) and \( g(x') = 1 \). By (7-1), there is an \( f \in C^m(Y) \) such that \( x' \notin Z(\varphi(f)) \) and \( x \in Z(\varphi(f)) \).

Hence \( f(\tau(x')) = [\varphi(f)](x') \neq 0 \), while \( f(\tau(x)) = (\varphi(f))(x) = 0 \).

That is, \( \tau(x) \neq \tau(x') \) so that \( \tau \) is one-one. Therefore \( \tau^{-1} \) is well-defined. Now, \( \tau^{-1} \) is a mapping from \( \tau[X] \) onto \( X \).

In order to show that \( \tau^{-1} \) is continuous, it is enough to show that \( \tau \) is open. Let \( U \) be any open set in \( X \), and \( x \in U \).
be arbitrary. Then \( x \not\in Z(g) \) for some \( g \in C^m(X) \). By (7-1), there is an \( f \in C^m(Y) \) such that \( x \not\in Z(\psi(f)) \) and \( Z(g) \subset Z(\psi(f)) \). It follows that \( \tau(x) \in \tau[X] \cap [Y - Z(f)] \subset \tau[X - Z(g)] \). That is, \( \tau(x) \) is an interior point of \( \tau[X] \) (as \( Y - Z(f) \) is open in \( Y \)). Thus \( \tau[U] \) is a set of interior points. Hence it is open so that \( \gamma^{-1} \) is continuous.

Now, we assume that \( \psi(C^m(Y)) \) contains all projections of \( X \). Since \( \gamma^{-1} \) is a mapping from \( \tau[X] \) onto \( X \), and for every \( g \in \psi(C^m(Y)) \), \( g = f \circ \tau \) for some \( f \in C^m(Y) \) so that \( g \circ \gamma^{-1} = (f \circ \tau) \circ \gamma^{-1} = f \in C^m(Y) \subset C^m(\tau[X]) \). Since \( \psi(C^m(Y)) \) contains all local projections of \( X \), by Prop. (6.4), \( \gamma^{-1} \) is a \( C^m \)-mapping on \( \tau[X] \).

Q.E.D.

Let \( F \) be a closed subset of \( X \). Then \( M_F = \{ f \in C^m(X) : Z(f) \supset F \} \) is an ideal in \( C^m(X) \). Moreover, \( M_F = \cap \{ M_x : x \in F \} \). Hence it is a \( \delta \)-real ideal. If \( X \) is \( m \)-realcompact, then the converse is also true. For, let \( M_1 \) be any \( \delta \)-real ideal of \( C^m(X) \), then \( M_1 = \cap \{ M_{\alpha} : M_{\alpha} \) is real and \( \alpha \in \Delta \} \). However, for each \( \alpha \), there exists \( x \in X \) such that \( M_x = M_{\alpha} \). Thus, \( M_1 = \cap \{ M_x : x \in E \} \), where \( E = \{ x \in X : x \) corresponds to \( \alpha \) for some \( \alpha \in \Delta \} \). Let \( F = cl_X E \). We will show that \( M_E = \cap \{ M_x : x \in E \} = \cap \{ M_x : x \in F \} = M_F \). Indeed, each \( f \in M_F \), \( f \in M_E \) is clear. Hence \( M_E \supset M_F \). Take any \( g \in M_E \). We have \( Z(g) \supset E \). Thus \( cl_X Z(g) = Z(g) \supset cl_X E = F \). That is, \( g \in M_F \). Therefore, \( M_E = M_F \). Since \( E \) is uniquely determined by \( \Delta \) and \( F \) is uniquely determined by \( E \), \( F \) is unique. Consequently, \( X \) is \( m \)-realcompact iff every \( \delta \)-real ideal of \( C^m(X) \) is of the form \( M_F \) for some closed subset \( F \) of \( X \).
Theorem: Let $Y$ be an arbitrary topological space and $X$ be an $m$-realcompact space. Then the following statements concerning a homomorphism $\varphi$ from $C^m(Y)$ into $C^m(X)$ are equivalent.

1. $\varphi$ satisfies property (7-1).
2. $\varphi$ is a $\delta$-homomorphism.
3. The image of $C^m(Y)$ separates points and closed sets and is contained in no $\delta$-real ideal of $C^m(X)$.

Proof: (1) implies (2). Let $M_F^r$ and $M_F^i$ be two $\delta$-real ideals of $C^m(X)$ with $M_F^r - M_F^i \neq \emptyset$. Then $F' \neq F$ and there is an element $x \in F' - F$ and a $g \in C^m(X)$ such that $Z(g) \supset F$ but $x \notin Z(g)$. According to (1) there exists an $f \in C^m(Y)$ such that $x \notin Z(\varphi(f))$ and $Z(g) \subseteq Z(\varphi(f))$. Hence $\varphi(f) \in M_F^r - M_F^i$. That is, $\varphi$ is a $\delta$-homomorphism.

(2) implies (1). For any two $\delta$-real ideals $M_F^r$, $M_F^i$, in $C^m(X)$ such that $M_F^r - M_F^i \neq \emptyset$, there exists an $f \in C^m(Y)$ such that $\varphi(f) \in M_F^r - M_F^i$. In particular, for any zero-set $Z(g)$ such that $x \notin Z(g)$, we have $M_{Z(g)}^r - M_X^i \neq \emptyset$. By hypothesis, there exists an $f \in C^m(Y)$ such that $\varphi(f) \in M_{Z(g)}^r - M_X^i$. Thus, $Z(\varphi(f)) \supset Z(g)$ but $x \notin Z(\varphi(f))$.

(2) implies (3). Let $M_F^r$ be an arbitrary $\delta$-real ideal in $C^m(X)$. If $F = X$, then $M_F^r = (0)$ and since $\varphi$ is nonzero, the image of $C^m(Y)$ can not be contained in $M_F^r = (0)$. On the other hand, if $F \neq X$, choose $x \notin F$. Then $M_X^r - M_F^i \neq \emptyset$ and hence must contain an element of $\varphi(C^m(Y))$. In either case, $\varphi(C^m(Y))$ is not contained in $M_F^r$ for any closed subset $F$ of $X$. We will now show that $\varphi(C^m(Y))$ separates points and closed subsets. If $F$ is any
closed subset and \( x \notin F \), then \( M_p - M_x \neq \phi \). By hypothesis there exists \( f \in C^m(Y) \) such that \( \varphi(f) \in M_p - M_x \). Hence, \( \varphi(f)(x) \neq 0 \) and \( Z(\varphi(f)) \supset F \). That is, \( (\varphi(f))(x) \notin (\varphi(f))(F) \).

(3) implies (2). Suppose (3) and \( \varphi \) is not a \( \delta \)-homomorphism. That is, \( \varphi(C^m(Y)) \) is not \( \delta \)-dense in \( C^m(X) \). Hence, for some, \( M_p, M_{p'} \), we have \( M_p - M_{p'}, \neq \phi \) but \( (M_p - M_{p'}) \cap \varphi(C^m(Y)) = \phi \).

However, \( M_p, \subset M_x \) for each \( x \in F' \). Thus \( M_p - M_{p'}, \supset M_p - M_x \) for each \( x \in F' \) and \( M_p - M_{p'}, = M_p - \cup \{ M_x : x \in F' \} = \phi \). Hence, there exists at least one \( x_o \in F' \) such that \( M_p - M_{x_o} \neq \phi \). But \( (M_p - M_{x_o}) \cap \varphi(C^m(Y)) \subset (M_p - M_{x_o}) \cap \varphi(C^m(Y)) = \phi \). That is, there does not exist any \( f \in C^m(Y) \) such that \( \varphi(f) \in M_p - M_{x_o} \). In other words, there is not any \( f \in C^m(Y) \) such that \( \varphi(f)(x) \notin \text{cl}_X \varphi(f)(F) \). This is a contradiction.

Q.E.D.

(7.9) Theorem: Let \( X \) and \( Y \) be \( m \)-realcompact. A homomorphism from \( C^m(Y) \) into \( C^m(X) \) is a \( \delta \)-homomorphism iff there is a homeomorphism \( \tau \) from \( X \) into \( Y \) such that \( \varphi(f) = f \circ \tau \) for all \( f \in C^m(Y) \) and \( \tau \) is a \( C^m \)-mapping. Moreover, if \( \varphi(C^m(Y)) \) contains all projections of \( X \), then \( \tau^{-1} \) is a \( C^m \)-mapping, that is \( \tau \) is a \( C^m \)-diffeomorphism.

Proof follows from Theorems (7.7) and (7.8).

(7.10) Lemma: Let \( X \) and \( Y \) be any subsets of \( E^m_1 \) and \( E^m_2 \), respectively, and \( \tau \) be a \( C^m \)-mapping from \( X \) into \( Y \). Define a homomorphism \( \varphi \) from \( C^m(Y) \) into \( C^m(X) \) by \( \varphi(f) = f \circ \tau \).

Then for fixed ideals \( M_y \) and \( M_x \) of \( C^m(Y) \) and \( C^m(X) \) respectively, \( \varphi[M_y] \subset M_x \) iff \( \tau(x) = y \).

Proof: Sufficiency: Suppose \( \tau(x) = y \). Then for \( f \in M_y \),
\[ f(y) = 0. \] On the other hand, \( \varphi(f(x)) = f(\tau(x)) = f(y) = 0. \) Thus \( \varphi(f) \in M_x. \) Hence \( \varphi(M_y) \subseteq M_x. \)

**Necessity:** Suppose \( \varphi(M_y) \subseteq M_x. \) Let \( f \in \mathcal{C}^m(Y), \) \( f(y) = r. \) Then \( f - r \in M_y \) and \( \varphi(f) - r = \varphi(f-r) \in M_x. \) Thus \( \varphi(f)(x) = r. \) Hence \( \varphi(f)(x) = f \cdot \tau(x) = f(y) \) for all \( f \in \mathcal{C}^m(Y) \) which implies \( \tau(x) = y. \) (By m-complete regularity).

(7.11) **Theorem:** Let \( X \subseteq E^1 \) and \( Y \subseteq E^2 \) be \( m \)-realcompact. A homomorphism \( \varphi \) from \( \mathcal{C}^m(Y) \) into \( \mathcal{C}^m(X) \) is a \( \delta \Phi \)-homomorphism iff there is a homeomorphism \( \tau \) from \( X \) into \( Y \) such that \( \varphi(f) = f \cdot \tau \) for all \( f \in \mathcal{C}^m(Y), \) \( \tau[X] \) is a closed subset of \( Y, \) and \( \tau \) is a \( \mathcal{C}^m \)-mapping. Moreover, if \( \varphi(\mathcal{C}^m(Y)) \) contains all projections of \( X \), then \( \tau^{-1} \) is also a \( \mathcal{C}^m \)-mapping.

**Proof:** **Sufficiency:** Let \( M \) be a real ideal of \( \mathcal{C}^m(Y) \) which contains the kernel of \( \varphi, \) ker \( \varphi. \) Then \( M = M_y \) for some \( y \in Y. \) Moreover, since \( \tau[X] \) is closed, \( y \in \tau[X] \). For otherwise, there would be an \( f \in \mathcal{C}^m(Y) \) such that \( Z(f) \subseteq \tau[X] \) and \( y \notin Z(f). \) This would imply \( f \in \ker \varphi - M_x \) which is a contradiction. Hence \( \tau(x) = y \) for some \( x \in X, \) and by Lemma (7.10), \( \varphi(M_y) \subseteq M_x, \) that is, the image \( M \) is subreal in \( \mathcal{C}^m(X). \)

**Necessity:** Suppose \( \varphi \) is a \( \delta \Phi \)-homomorphism. By Theorem (7.9), there is a homeomorphism \( \tau \) from \( X \) into \( Y \) such that \( \varphi(f) = f \cdot \tau \) for all \( f \in \mathcal{C}^m(X), \) and \( \tau \) is a \( \mathcal{C}^m \)-mapping. We will show that \( \tau[X] \) is a closed subset of \( Y. \) Choose \( y \in ct_Y(\tau[X]) = F. \) Then, \( \ker \varphi = M_F \subseteq M_y. \) Hence, there is a real ideal \( M_x \) of \( \mathcal{C}^m(X) \) such that \( \varphi(M_x) \subseteq M_x. \) By Lemma (7.10), we have \( y = \tau(x). \) Thus \( y \in \tau[X]. \) That
is, $\tau[X] = c^y \tau[X]$. 

The last part follows from that of Theorem (7.7). Q.E.D.

(7.12) Theorem: Let $X \subset E^{n_1}$ and $Y \subset E^{n_2}$ be $m$-realcompact. A homomorphism $\varphi$ from $C^m(Y)$ into $C^m(X)$ is a $G\delta$-homomorphism iff there exists a homeomorphism $\tau$ from $X$ into $Y$ such that $\varphi(f) = f \circ \tau$ and $\tau$ is a $C^m$-mapping for all $f \in C^m(Y)$ and $\tau[X]$ is an open subset of $Y$. Moreover, if $\varphi(C^m(Y))$ contains all projections of $X$, then $\tau^{-1}$ is a $C^m$-mapping.

Proof: Sufficiency: Let $M_y$ be a real ideal of $C^m(Y)$ whose image is subreal in $C^m(X)$. Then, for some $x \in X$, $\varphi[M_y] \subset M_x$. From Lemma (7.10) again, $\tau(x) = y$. Then $y \notin Y - \tau[X]$ which is a closed subset of $Y$ and there is an $f \in C^m(Y)$ such that $Z(f) \supset Y - \tau[X]$ and $y \notin Z(f)$. Thus, $f \notin M_y$ and it follows that the image of real ideal not containing $f$ is subreal.

Necessity: Suppose $\varphi$ is a $G\delta$-homomorphism. Then it is $G\delta$-homomorphism so that, by Theorem (7.9), there exists a homeomorphism $\tau$ from $X$ into $Y$ which is a $C^m$-mapping such that $\varphi(f) = f \circ \tau$. Let $y \in \tau[X]$. Then, $y = \tau(x)$ for some $x \in X$. By Lemma (7.10), $\varphi(M_y) \subset M_x$. Hence, there exists an $f \notin M_y$ and the image of every real ideal not containing $f$ is subreal. From this it follows that $y \in Y - Z(f) \subset \tau[X]$. Indeed, for each $y' \in Y - Z(f)$, $f \notin M_y$, $\varphi(M_y) \subset M_x$, for some $x'$. By Lemma (7.10), $y' = \tau(x') \in \tau[X]$. Hence $\tau[X]$ is open.

The last part follows from that of Theorem (7.7). Q.E.D.
§ 8 A Representation Theorem for Transformations of Rings of $C^m$-differentiable Functions.

Let $X$ and $Y$ be completely regular Hausdorff spaces, $T$ be a transformation of $C(Y)$ into $C(X)$ which is much more general than a homomorphism. We know that $T$ can be represented by means of a continuous mapping from a subset of $E^1 \times X$ and a continuous mapping $\tau$ from a subset of $X$ into $\mathcal{V}Y$ if $T$ satisfies certain conditions. (Theorem 1, [12]). We will observe that if $T$ is a transformation of $C^m(Y)$ into $C^m(X)$, where $Y$ is $m$-realcompact, then $T$ will have a representation if $T$ satisfies certain conditions (see Theorem (8.5)). We start with some generalizations of classical results.

(8.1) Lemma: Let $f(x) = f(x_1, \ldots, x_n)$ be defined in an open subset $G$ of $E^n$, and the first partial derivatives in $x_i$ and $x_j$ exist and be continuous. Then, if $f_{x_i x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ exists and is continuous in $G$, then $f_{x_j x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ exists and is continuous in $G$. (Here we may assume that $i < j$).

Proof: Let $\Delta$ be an abbreviation for the mixed second difference. Then $\Delta = f(x_1, \ldots, x_i+\theta, \ldots, x_j+k, \ldots, x_n) - f(x_1, \ldots, x_i+h, \ldots, x_n)$

$= f(x_1, \ldots, x_i, \ldots, x_j+k, \ldots, x_n) + f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)$

(8-1)

We write $F(x_i) = f(x_1, \ldots, x_j+k, \ldots, x_n) - f(x_1, \ldots, x_n)$.

Then (8-1) becomes $\Delta = F(x_i+h) - F(x_i)$. By the Mean Value theorem, we have $\Delta = hF'(x_i + \theta h)$, where $0 < \theta < 1$. That is, $\Delta = h[f_{x_i}(x_1, \ldots, x_i+\theta h, \ldots, x_j+k, \ldots, x_n) - f_{x_i}(x_1, \ldots, x_i+\theta h, \ldots, x_n)]$. 
Again, by the Mean Value Theorem, we have

\[ \Delta = hkf_{x_1 x_j} (x_1, \ldots, x_j + \Theta h, \ldots, x_j + \Theta' k, \ldots, x_n), \quad 0 < \Theta' < 1. \]

\[ = f(x_1, \ldots, x_1 + h, \ldots, x_j + k, \ldots, x_n) - f(x_1, \ldots, x_1 + h, \ldots, x_n) \]
\[ - f(x_1, \ldots, x_1, \ldots, x_j + k, \ldots, x_n) + f(x_1, \ldots, x_n) \quad \text{(by (8-1))} \]
\[ (8-2). \]

Dividing (8-2) by \( k \), we have

\[ hf_{x_1 x_j} (x_1, \ldots, x_1 + \Theta h, \ldots, x_j + \Theta' k, \ldots, x_n) = \frac{1}{k} [f(x_1, \ldots, x_1 + h, \ldots, x_j + k, \ldots, x_n) - f(x_1, \ldots, x_1 + h, \ldots, x_n)] - \frac{1}{k} [f(x_1, \ldots, x_1, \ldots, x_j + k, \ldots, x_n) - f(x_1, \ldots, x_n)]. \]
\[ (8-3). \]

Consider \((x_1, \ldots, x_n)\) as a certain fixed point, and take the limit of (8-3) as \( k \to 0 \). Since \( f_{x_j} \) and \( f_{x_1 x_j} \) exist we have

\[ f_{x_j} (x_1, \ldots, x_1 + h, \ldots, x_n) - f_{x_j} (x_1, \ldots, x_n) \]
\[ = hf_{x_1 x_j} (x_1, \ldots, x_1 + \Theta h, \ldots, x_j, \ldots, x_n) \quad (8-4). \]

Next, divide (8-4) by \( h \), and take the limit of it as \( h \to 0 \).

Since \( f_{x_1 x_j} \) is continuous, we have

\[ \lim_{h \to 0} \frac{1}{h} [f_{x_j} (x_1, \ldots, x_1 + h, \ldots, x_n) - f_{x_j} (x_1, \ldots, x_n)] = f_{x_1 x_j} (x). \]

Or \( f_{x_j x_1} = f_{x_1 x_j} \). We know that since \( f_{x_1 x_j} \) is continuous in \( G \) so is \( f_{x_j x_1} \).

(8.2) Corollary: Let \( G \) be a differentiable \( n \)-manifold.

Then, \( f_{x_1 x_j} = \frac{\partial^2 f}{\partial x_j \partial x_1} \) exists and is continuous on each coordinate neighborhood system implies that \( f_{x_j x_1} = \frac{\partial^2 f}{\partial x_1 \partial x_j} \) exists and is
continuous on each coordinate neighborhood system.

Proof: Note that the notations $f(x_1, \ldots, x_n)$, $f_{x_i x_j}$, etc. refer only to the local coordinate neighborhoods. The proof is the same as that of Lemma (8.1).

(8.3) Lemma: Let $f(x)$ be defined in an open subset $G$ of $\mathbb{R}^n$, and $f_{x_1^{k_1} \ldots x_n^{k_n}} = \frac{\partial^{k_1 + \ldots + k_n} f}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}$ exist and be continuous in $G$. Then each partial derivative of $f$ in one of the permutations of $x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_n, \ldots, x_n$, $(m = \sum_{i=1}^{n} k_i$ letters) exists, is continuous, and equal to $f_{x_1^{k_1} \ldots x_n^{k_n}}$.

Proof: We will first prove this in the case of $f_{x_i x_j x_k}$. Let $F(x) = f_{x_i}(x)$. Then, by assumption, $F_{x_j x_k}$ exists and is continuous. By Lemma (8.1), $F_{x_k x_j} = F_{x_j x_k}$ so it is continuous. Thus $f_{x_i x_k x_j}$ exists, is continuous and $f_{x_i x_k x_j} = f_{x_i x_j x_k}$.

Since $f_{x_i x_j x_k}$ is continuous, $f_{x_i x_j}$ is differentiable in $x_k$. By Lemma (8.1) $f_{x_j x_i} = f_{x_i x_j}$ and is differentiable in $x_k$.

Thus, $f_{x_j x_i x_k} = f_{x_i x_j x_k}$ is continuous. The proof for the other permutation is similar.
Now, assume that it is true for all \( m \leq p - 1 \). We will show that it is true for \( m = p \). Without loss of generality, let us assume that \( f_{k_1'} \ldots x_n' \) exists and is continuous, where \( \prod_{i=1}^{\Omega} k_i = p \). We know that every permutation is a product of transpositions \((i,j) \to (j,i)\). Moreover, the above argument applies to each transposition. Hence, each partial derivative of \( f \) is one of the permutations of \( x_1', \ldots, x_n' \).

\[ f_{k_1'} \ldots x_n' \]

exists and is continuous, and equal to

\[ f_{k_1'} \ldots x_n' \]. Therefore the Lemma is true for all \( n \).

Notice that this result is true in a differentiable \( n \)-manifold in each local coordinate neighborhood.

(8.4) Lemma: Let \( f(t,x) = f(t,x_1,\ldots,x_n) \) be defined in an open subset \( X \subset E^{n+1} \), where \( X = E^1 \times X_n \), \( X_n \) is an open subset of \( E^n \). Let \( g(x) \) be a real-valued function defined on the open subset \( X_n \), \( \frac{\partial f}{\partial t} \), denoted by \( f_0 \), be non-zero for all points of \( X \), and \( f_{k_0'} \ldots k_1' \ldots x_n' = \frac{\partial^m f}{\partial x_n' \ldots \partial x_1' \partial t^0} \)

exist and be continuous for each choice of \( k_0, \ldots, k_n' \) such that

\[ 1 \leq m' = \prod_{i=0}^{\Omega} k_i' \leq \prod_{i=1}^{\Omega} k_i = m \]. Then \( \frac{\partial^m f(g(x),x)}{\partial x_1' \ldots \partial x_n'} \) exists and
is continuous iff \( \frac{\partial^m g(x)}{\partial x_1 \ldots \partial x_n} \) exists and is continuous.

Proof: The sufficiency follows immediately from the Chain rule for partial derivatives and the hypothesis.

Necessity: By assumption and Lemma (8.3), without loss of generality, we may consider the first partial derivative of \( f(g(x),x) \) with respect to \( x_1 \). Then we have:

\[
\frac{\partial f(g(x),x)}{\partial x_1} \cdot h = f(g(x_1+h,\ldots,x_n), x_1+h,\ldots,x_n) - f(g(x_1,\ldots,x_n), x_1,\ldots,x_n),
\]

\( x_1,\ldots,x_n \) + \( \epsilon h \), where \( \epsilon \to 0 \) as \( h \to 0 \). However, the right hand side can be written as \( [f(g(x_1+h,\ldots,x_n), x_1+h,\ldots,x_n) - f(g(x_1,\ldots,x_n), x_1,\ldots,x_n)] + [f(g(x_1+h,\ldots,x_n), x_1,\ldots,x_n) - f(g(x_1,\ldots,x_n), x_1,\ldots,x_n)] + \epsilon h = f_1(g(x_1+h,\ldots,x_n), x_1+\theta h,\ldots,x_n) + \epsilon h \), where \( 0 < \theta, \theta' < 1 \), by the Mean Value Theorem. Thus

\[
\frac{\partial f(g(x),x)}{\partial x_1} = f_1(g(x_1+h,\ldots,x_n), x_1+\theta h,\ldots,x_n) + f_0(g(x_1+\theta' h,\ldots,x_n), x_1,\ldots,x_n).
\]

Let \( h \to 0 \). We have \( \frac{\partial f(g(x),x)}{\partial x_1} = f_0(g(x),x) \frac{\partial g(x)}{\partial x_1} + f_1(g(x),x) \).

By hypothesis, we have \( \frac{\partial g(x)}{\partial x_1} = \frac{\partial f(g(x),x)}{\partial x_1} - f_1(g(x),x) \) / \( f_0(g(x),x) \).

Hence \( \frac{\partial g(x)}{\partial x_1} \) exists and is continuous. We know that

\[
\frac{\partial f(g(x),x)}{\partial x_1}, f_1(g(x),x), \text{ and } f_0(g(x),x) \text{ have continuous 1st}
\]
partial derivatives because \( \partial g(x)/\partial x_1 \) is continuous. Hence by differentiating (8-5), we can show that \( g(x) \) has continuous 2nd partial derivatives. Proceeding by induction, we conclude that \( g(x) \) has continuous \( m \)-th partial derivatives as we require. Q.E.D.

If \( X_n \), in the lemma, is replaced by a differentiable \( n \)-manifold, the result is also true in each local coordinate neighborhood.

Now, we are in the position to show the representation theorem.

(8.5) Theorem: Let \( X \) be a differentiable \( n \)-manifold and \( Y \) an \( m \)-realcompact space, \( T \) be a transformation of \( C^m(Y) \) into \( C^m(X) \) such that

(a) For \( x \in X \), if \( f, g \in C^m(Y) \) such that \( (Tf)(x) = (Tg)(x) \) then \( (T(f+h))(x) = (T(g+h))(x) \) and \( (T(f\circ h))(x) = (T(g\circ h))(x) \) for all \( h \in C^m(Y) \).

(b) For each \( x \in X \), the mapping \( \zeta_x : t \rightarrow (Tt)(x) \) from \( E^1 \) into \( E^1 \) is \( C^m \) in \( t \) onto \( (T[C^m(Y)])(x) \).

(c) For each \( x \in X \), the mapping \( t \rightarrow (Tt)^k(x) \) is \( C^{m-k} \) in \( t \), where \( 0 \leq k \leq m \), and since \( Tt \in C^m(X) \), \( (Tt)^k \) is a \( k \)-th partial derivative of \( Tt \).

(d) \( \zeta_x'(t) \neq 0 \) for all \( t \in E^1 \) and \( x \in X \), where \( \zeta_x \) is defined in (b).

Set \( E = \{ x \in X : (T[C^m(Y)])(x) \) contains more than one point} \). Then there is a \( C^m \)-mapping \( \tau \) of \( E \) into \( Y \), and a continuous mapping \( \omega \) from \( E^1 \times E \) into \( E^1 \) such that \( \omega(t,x_0) \) is a homeomorphism onto \( (T[C^m(Y)])(x_0) \) for each \( x_0 \in E \), \( \omega(t,x) \) is
\( C^m \) in \( t \) and \( x \) separately, and satisfying

\[(1) \quad (Tf)(x) = \omega(f \circ \tau(x),x) \quad \text{if} \quad x \in E \]
\[= (T\Theta)(x) \quad \text{if} \quad x \in X - E \quad \text{for each} \quad f \in C^m(Y) \]

Moreover, \( E \) is an open subset of \( X \), and \( \tau[E] \) is dense in \( Y \) iff \( T \) is one-one.

In addition, if \( T \) maps \( C^m(Y) \) onto \( C^m(X) \), then \( \tau \) is a \( C^m \)-diffeomorphism of \( X \) onto a \( C^m \)-embedded subset of \( Y \).

Proof: Let \( \omega \) be the mapping from \( E^1 \times E \) into \( E^1 \) defined by \( \omega(t,x) = (Tt)(x) \). First of all, we will show that (a) implies that \( \omega(t,x_0) \) is one-one for each \( x_0 \in E \), where \( t \) is variable, indeed, if \( \omega(r,x_0) = \omega(s,x_0) \) for some \( r \neq s \), then \( (Tr)(x_0) = (Tg)(x_0) \). We would have \( (T(r-s))(x_0) = (T\Theta)(x_0), \) and \( (T(r-s \cdot \frac{1}{r-s}))(x_0) = (Tu)(x_0), (T(\Theta \cdot \frac{1}{r-s}))(x_0) = (T\Theta)x_0 \). Thus, for each \( h \in C^m(Y) \), \( (Th \cdot u)(x_0) = (Th \cdot \Theta)u_0 ) \).

That is, \( (Th)(x_0) = (T\Theta)(x_0) \). This shows that \( x_0 \notin E \), which is a contradiction. Thus \( \omega(t,x_0) \) is one-one, so that its inverse, denoted by \( \alpha(t,x_0) \), exists. By hypothesis (b) and the Brouwer Invariance of Domain Theorem, \( \alpha(t,x_0) \) is continuous. Since \( \omega(r,x) = (Tr)(x) \) where \( x \) is variable, \( \omega(r,x) \) is \( C^m \) in \( x \) for each \( r \in E^1 \). By hypothesis (b) \( \omega(t,x_0) = (Tt)(x_0) \) is \( C^m \) in \( t \). By Lemma (P.675 [12]) both \( \omega \) and \( \alpha \) are jointly continuous.

For each \( x \in E \), we define a mapping \( \phi_x \) of \( C^m(Y) \) into \( R \) by \( \phi_x(f) = \alpha((Tf)(x),x) \). We will show that \( \phi_x \) is a homomorphism. Given any \( f,g \) in \( C^m(Y) \), let \( \phi_x(g) = s \), so that \( (Tf)(x) = \omega(r,x) = (Tr)(x) \) and \( (Tg)(x) = \omega(s,x) = \)}
(Ts)(x). Then, by (a), \((T(f+g))(x) = (T(r+g))(x) = (T(r+s))(x) = \varphi_x(f) + \varphi_x(g)\). Also \((T(f\cdot g))(x) = (T(r\cdot g))(x) = r\cdot s\) hence \(\varphi_x(f\cdot g) = \varphi_x(f)\cdot \varphi_x(g)\). Moreover, \(\varphi_x\) is a non-zero homomorphism, for \(\alpha(t,x)\) is one-one mapping, where \(x\) is fixed. By Prop. (6.13) \(\varphi_x\) is onto. From (6.15) and the \(m\)-realcompactness of \(Y\), there is a unique point \(y \in Y\), say, such that \(\varphi_x(f) = f(y)\) for all \(f \in C^m(Y)\). Thus, we can define a mapping \(\tau\) of \(E\) into \(Y\) as follows: For each \(x \in E\), \(\tau(x)\) is the unique point \(y\) of \(Y\) such that \(\varphi_x(f) = f(y) = f(\tau(x))\) for all \(f \in C^m(Y)\). That is, \(f(\tau(x)) = \alpha((Tf)(x),x)\). It follows from the definition of \(\alpha\) that \((Tf)(x) = \omega(f \cdot \tau(x),x)\) for each \(x \in E\). We now show that \((Tf)(x) = (T\varnothing)(x)\) for \(x \in X - E\). We know that for each \(x \in X - E\), \((T[C^m(Y)])(x)\) is only one point. That is, \((Tf)(x) = (Tg)(x)\) for any \(f,g \in C^m(Y)\). In particular, if we let \(g = \varnothing\), we have \((Tf)(x) = (T\varnothing)(x)\) for all \(f \in C^m(Y)\). Thus, the equation (1) holds. From the definition of \(E\), and the discussion in the first paragraph of the proof, we conclude that the set \(\{x \in X : (Tu)(x) = (T\varnothing)(x)\}\) which coincides with \(X - E\) is closed so that \(E\) is open in \(X\). Hence \(E^1 \times E\) is a differentiable \((n+1)\)-manifold.

Now, we are ready to show that \(\tau\) is \(C^m\). From (c) and the definition of \(\omega\), we have \((T_t)^k(x_o) = \omega_k x_2 \ldots (n+1)^{k_{n+1}}(t,x_o)\) (see the notation in Lemma (8.4)), is \(C^{m-k}\) in \(t\) for each
\( x_0 \in X \), where \( \sum_{i=2}^{n+1} k_i = k \). That is, its \((m-k)\)-th derivative exists and is continuous in \( t \). On the other hand, for every fixed \( r \in \mathbb{R}^1 \), \((T_r)^{(k)}(x) = w_{\frac{k}{2} \ldots (n+1)}(r, x)\) is differentiable in \( x \). By Lemma (P.675 [12]) (the first paragraph of the proof) \( w_{\frac{k}{2} \ldots (n+1)}(t, x)\) is jointly continuous in \( t \) and \( x \). Thus, by Lemma (8.1), all mixed partial derivatives exist and are continuous. From (d) we have \( \omega_1(t, x) \neq 0 \), for \((t, x) \in \mathbb{R}^1 \times E^1 \). By Lemma (8.3), and the fact that \((Tf)(x) = w(f \circ \tau(x), x)\) is \( C^m \), we have \( f \circ \tau \in C^m(E) \). Since \( f \) is arbitrary in \( C^m(Y) \), by Theorem (6.4) \( \tau \) is a \( C^m \)-mapping. On the other hand, by Lemma (8.3), and the continuity of \( w_{\frac{k}{2} \ldots (n+1)}(t, x) \) in \( t \) and \( x \), we have for each \( g \in C^m(X) \), \( w(g(x), x)\) is \( C^m \) on \( E \) - this fact will be needed later.

In virtue of \( m \)-complete regularity, we know that \( \tau[E] \) is not dense in \( Y \) if only if there exist \( f \) and \( g \in C^m(Y) \) that coincide on \( \tau[E] \) but not on \( Y \). However, the latter is equivalent to the condition that there exist \( f \) and \( g \in C^m(Y) \) such that \( f \neq g \) but \( w(f \circ \tau(x), x) = w(g \circ \tau(x), x) \) for all \( x \in E \), that is, \((Tf)(x) = (Tg)(x)\) for all \( x \in E \). Hence \( T \) is not one-one (as we know that \((Tf)(x) = (Tg)(x) \) for all \( x \in X - E \)). Therefore \( \tau[E] \) is dense in \( Y \) iff \( T \) is one-one.

In addition, we assume now \( T \) is onto. Then \( T[C^m(Y)] = C^m(X) \) which includes all constant functions. Thus, for each \( x \in X \), \((T[C^m(Y)])(x) = C^m(X)(x) \supset \{r(x) = r : r \in \mathbb{R}^1\} = E^1 \).
On the other hand, for each \( f \in \mathcal{C}^m(X) \), \( f(x) = r \) for some \( r \in E \). Hence \( (T[\mathcal{C}^m(Y)])(x) = E \) so that \( x \in E \).

Consequently, \( X = E \). We will show that \( \tau \) is one-one. If \( \tau(x) = \tau(x_0) = y_0 \), then \( f(\tau(x)) = f(\tau(x_0)) \). Suppose \( x \neq x_0 \).

Since \( T \) is onto, there exist \( f \) and \( g \in \mathcal{C}^m(Y) \) such that \((Tf)(x) = (Tg)(x)\) but \( (Tf)(x_0) \neq (Tg)(x_0) \). Then \( \omega(f \circ \tau(x), x) = \omega(g \circ \tau(x), x) \) and as \( \omega(t, x) \) is one-one for fixed \( x \), \( f(\tau(x)) = g \circ \tau(x) \) so that \( f(\tau(x_0)) = g \circ \tau(x_0) \).

That is, \( f(y_0) = g(y_0) \). On the other hand, \( \omega(f(y_0), x_0) = (Tf)(x_0) \neq (Tg)(x_0) = \omega(g(y_0), x_0) \) and \( \omega(t, y_0) \) is one-one, so we have \( f(y_0) \neq g(y_0) \) which is a contradiction. Hence \( x = x_0 \), i.e. \( \tau \) is one-one. Therefore \( \tau^{-1} \) is well-defined on \( \tau[X] \).

Now, for each \( g \in \mathcal{C}^m(X) \), the mapping \( f \) defined by \( f(x) = \omega(g(x), x) \) for all \( x \in E \) is \( \mathcal{C}^m(X) \) (as shown above).

Thus, \( g(x) = \alpha(f(x), x) \) for all \( x \in X \). Since \( T \) is onto, there is an \( f_0 \in \mathcal{C}^m(Y) \) such that \( T(f_0) = f \). We then have \( g(x) = \alpha((Tf_0)(x), x) = f_0 \circ \tau(x) \) for all \( x \in X \). In other words, \( \mathcal{C}^m(X) \subset \{ h \circ \tau : h \in \mathcal{C}^m(Y) \} \). But, we know that \( h \circ \tau \in \mathcal{C}^m(X) \) for all \( h \in \mathcal{C}^m(Y) \). Hence \( \mathcal{C}^m(X) = \{ h \circ \tau : h \in \mathcal{C}^m(Y) \} \). Now, for each \( f \in \mathcal{C}^m(X) \), \( f = h \circ \tau \) for some \( h \in \mathcal{C}^m(Y) \) hence \( f \circ \tau^{-1} = h | \tau[X] \in \mathcal{C}^m(\tau[X]) \). By Theorem (6.4), \( \tau^{-1} \) is \( \mathcal{C}^m \) on \( \tau[X] \).

Finally, we will show that \( \tau[X] \) is \( \mathcal{C}^m \)-embedded in \( Y \).

For each \( g \in \mathcal{C}^m(\tau[X]) \), the function \( f \), defined by \( f(x) = \omega(g \circ \tau(x), x) \) for each \( x \in \tau[X] \), is in \( \mathcal{C}^m(X) \). Thus, there exists \( g_0 \in \mathcal{C}^m(Y) \) such that \( Tg_0 = f \), or \( (Tg_0)(x) = f(x) \)

\( \omega(g_0 \circ \tau(x), x) = \omega(g \circ \tau(x), x) \) for each \( x \in X \). Hence \( g \circ \tau(x) = g_0 \circ \tau(x) \) for each \( x \in X \), or \( g_0 | \tau[X] = g \), that is \( g_0 \) is an extension of \( g \). Q.E.D.
(8.6) Theorem: Let \( X \) and \( Y \) be any two differentiable \( n_1, n_2 \)-manifolds, or subspaces of \( E^{n_1} \) and \( E^{n_2} \), respectively, \( \tau \) be a \( C^m \)-mapping from \( X \) into \( Y \), and \( \omega \) a \( C^m \)-mapping from \( E^1 \times X \) into \( E^1 \) such that \( \omega(t,x_0) \) is a homeomorphism into \( E^1 \) for each \( x_0 \in X \). Then the transformation \( T \) defined for each \( f \in C^m(Y) \) by \((Tf)(x) = \omega(f \circ \tau(x),x)\) is into \( C^m(X) \), and \( T \) satisfies conditions (a) and (b) of Theorem (8.5), \( T \) is one-one iff \( \tau[X] \) is dense in \( Y \).

Moreover, if \( \tau \) is a \( C^m \)-diffeomorphism of \( X \) onto a \( C^m \)-embedded subset of \( Y \), then \( T \) maps onto \( C^m(X) \).

Proof: That \( Tf \) is in \( C^m(X) \) follows immediately from the Chain rule, definitions of \( \omega(t,x) \) and \( \tau \), and the fact that \( f \in C^m(Y) \). Next, we will show \( T \) satisfies conditions (a) and (b) of Theorem (8.5). If \( f, g \in C^m(Y) \) are such that \((Tf)(x) = (Tg)(x)\) for any \( x \in X \), then \( \omega(f \circ \tau(x),x) = \omega(g \circ \tau(x),x) \). But \( \omega(t,x_0) \) is one-one for fixed \( x_0 \in X \), so we have \( f \circ \tau(x) = g \circ \tau(x) \). Now, for each \( h \in C^m(Y) \), \((T(f+h))(x) = \omega((f+h) \tau(x),x) = \omega(f \circ \tau(x) + h \circ \tau(x),x) = \omega(g \circ \tau(x) + h \circ \tau(x),x) = \omega((g+h) \tau(x),x) = (T(g+h))(x) \). Similarly \((T(f \circ h))(x) = (T(g \circ h))(x) \). Hence \( T \) satisfies (a). That \( T \) satisfies (b) is a consequence of definition of \( T \).

The proof that \( T \) is one-one iff \( \tau[X] \) is dense in \( Y \) is similar to that \( T \) is one-one iff \( \tau[E] \) is dense in \( Y \) in Theorem (8.5). By hypothesis of \( \omega \), for \( x_0 \in X \), let \( \alpha(t,x_0) \) be the inverse of \( \omega(t,x_0) \). Now, for each \( x \in X \), we have \((Tf)(x) = \omega(f \circ \tau(x),x) \). Thus, \( f \circ \tau(x) = \alpha((Tf)(x),x) \) on \( X \). Hence \( \alpha((Tf)(x),x) \) is \( C^m \) on \( X \), for each \( f \in C^m(Y) \).
Finally, assume that \( \tau \) is a \( C^m \)-diffeomorphism of \( X \) onto a \( C^m \)-embedded subset of \( Y \). Given \( F \in C^m(X) \), define \( g \) by \( g(y) = g \circ \tau(x) = \alpha(F \circ \tau^{-1}(y), \tau^{-1}(y)) \) for all \( x = \tau^{-1}(y) \in X \). As shown in the last paragraph, \( g \in C^m(\tau[X]) \). We know that \( \tau[X] \) is a \( C^m \)-embedded subset of \( Y \), so there exists \( g_o \in C^m(Y) \) such that \( g_o | \tau[X] = g \). We know that \( (Tg_o)(x) = \omega(g \circ \tau(x), \tau^{-1} \circ \tau(x)) \). By definitions of \( g \) and \( \alpha \), we have \( F \circ \tau^{-1}(y) = \omega(g(y), x) \) for all \( x \in X \) and \( y = \tau(x) \in \tau[X] \). Hence \( (Tg_o)(x) = F \circ \tau^{-1}(\tau(x)) = F(x) \) for each \( x \in X \). That is, \( Tg_o = f \) or \( T \) maps \( C^m(Y) \) onto \( C^m(X) \). Q.E.D.

Note that no extra condition is needed for \( X \) or \( Y \).

§9 The Rings of \( C^m \)-differentiable Functions on Spaces which are not \( m \)-realcompact, and some Algebraic Properties of \( C \) not applicable in \( C^m \).

We have shown that if \( X \) and \( Y \) are two \( m \)-realcompact spaces, then \( C^m(X) \) and \( C^m(Y) \) are isomorphic iff \( X \) and \( Y \) are \( C^m \)-diffeomorphic. We shall make some observations about other cases in this section.

We will first of all construct a subring of \( C^m(X) \), where \( X \subset \mathbb{R}^n \). Let \( \mathcal{L}(x_1, \ldots, x_n) = x_1 \) for \( 1 \leq i \leq n \), (as defined in (6.2)), \( S_1 = \{ r : r \text{ is the constant function defined on } X \text{ with value } r \in \mathbb{R} \} \), and \( S_2 = \{ \mathcal{L}: 1 \leq i \leq n \} \). Let \( S = S_1 \cup S_2 \) and \( \mathcal{O} \) be the ring generated by \( S \). Then, it is clear that \( \mathcal{O} \) is a commutative ring. Next, we embed \( \mathcal{O} \) in the quotient ring, \( \mathbb{R}(X) \), which consists of all quotients \( f/g, f, g \in \mathcal{O} \) and \( Z(g) = \{ x \in X : g(x) = 0 \} = \emptyset \). Evidently
\( \mathcal{R}(x) \) is a commutative ring of rational functions on \( X \) with unity \( u \), the constant function of value 1, zero element \( 0 \), the constant function of value 0.

In the following lemmas and theorems \( A_1 \) and \( A_2 \) are rings of functions with the following properties: \( \mathcal{R}(X) \subseteq A_1 \subseteq C^m(X) \), \( \mathcal{R}(Y) \subseteq A_2 \subseteq C^m(Y) \) and if \( f \in A_1 \) (or \( A_2 \)) with \( Z(f) = \emptyset \), then \( f^{-1} \in A_1 \) (or \( A_2 \)).

(9.1) Lemma: There is a function \( f \in M_a = \{ f \in A_1 : f(a) = 0 \} \), for \( a \in X \) such that \( Z(f) = \{ a \} \), and \( f \) belongs to no other free or fixed maximal ideal of \( A_1 \).

Proof: Let \( a = (a_1, \ldots, a_n) \in X \subseteq \mathbb{R}^n \), and \( f(x) = \sum_{i=1}^{n} (x_i - a_i)^2 \).

Then, it is clear that \( f \in M_a \) and belongs to no other fixed maximal. Suppose that \( M \) is a free maximal ideal in \( A_1 \) such that \( f \in M \). Since \( M \) is free, there is \( g \in M \) such that \( g(a) \neq 0 \). Let \( h = f^2 + g^2 \). We have \( Z(h) = \emptyset \), so that its inverse \( h^{-1} \in A_1 \). Since \( M \) is an ideal \( u = hh^{-1} \in M \).

That is, \( M \) is the whole ring which is impossible. Hence \( f \notin M \), or \( f \) belongs to \( M_a \) only. Q.E.D.

Let \( Y \) be a subset of \( \mathbb{R}^n \) where \( n' \) may be equal to \( n \). Let \( \mathcal{R}(Y) \) be the ring defined in the same manner as \( \mathcal{R}(X) \) with domain \( Y \), the ring of rational functions on \( Y \).

(9.2) Lemma: If \( \varphi \) is an isomorphism from \( A_1 \) onto \( A_2 \), then for any \( M_a \subseteq A_1 \), \( \varphi(M_a) \) is a fixed maximal ideal in \( A_2 \).

Proof: Since \( \varphi \) is an isomorphism onto, \( \varphi(M_a) \) is a maximal ideal in \( A_2 \). By Lemma (9.1) there is \( f_\varphi(x) = \sum_{i=1}^{n} (x_i - a_i)^2 \).
such that $Z(f_o) = \{a\}$ belongs to $M_a$ only. Consider $Z(\varphi(f_o))$. If $Z(\varphi(f_o)) = \emptyset$, then $\varphi(f_o)$ is a unit in $A_2$ so that $\varphi(M_a)$ is the whole ring $A_2$ which is impossible. Hence $Z(\varphi(f_o)) \neq \emptyset$. On the other hand, if $Z(\varphi(f_o))$ contains more than one point, say $b, b' \in Y$, then $\varphi(f_o) \in M_b$ and $\varphi(f_o) \in M_{b'}$, so that $f_o$ would belong to at least two maximal ideals which again is impossible for $f_o$ belongs to only one maximal ideal. Thus $Z(\varphi(f_o)) = \{b\}$, say. Hence $\varphi(M_a) = M_b$.

(9.3) Lemma: Let $B_1$ and $B_2$ be subrings of $C(X)$ and $C(Y)$ respectively, which contain all constant functions, $\varphi$ be an isomorphism from $B_2$ into $B_1$, and $X$ be connected. Then $\varphi$ is the identity on the constant functions.

Proof: It is clear that $\varphi(u) = u$ and $\varphi(\Theta) = \Theta$. By property of isomorphism, we have $\varphi(n) = n$ for all positive integer $n$. Now, $\varphi(u-u) = \varphi(\Theta) = \Theta$ or $\varphi(u) + \varphi(-u) = \Theta$. Thus $\varphi(-u) = -\varphi(u) = -u$, so that $\varphi(-n) = -n$ for all positive integer $n$. Moreover, $\varphi(q) = \frac{q}{p} = \varphi(p \cdot \frac{q}{p}) = p \varphi\left(\frac{q}{p}\right)$ (by property of isomorphism).

That is, $\varphi\left(\frac{q}{p}\right) = \frac{q}{p}$ for any positive integer $p$ and any integer $q$.

We will show that for constant function $k$, $\varphi(k)$ is also a constant function. Indeed, if $k$ is an irrational number, $k-r \neq 0$ for all rational numbers $r$. Thus $\frac{1}{k-r}$ exists for all rational numbers $r$. Moreover, $\varphi(k-r) \cdot \varphi\left(\frac{1}{k-r}\right) = \varphi(k-r) \cdot \frac{1}{k-r} = \varphi(u) = u$, we have $\varphi\left(\frac{1}{k-r}\right) = \frac{1}{\varphi(k-r)} = \frac{1}{\varphi(k)-r}$, for all rational numbers $r$. Now, if $\varphi(k)$ is not
a constant function on \( X \), then since \( X \) is connected and \( \varphi(k) \) is continuous, we must have \( \varphi(k)(X) \) is a connected subset in \( E \). Thus \( \varphi(k)(X) \) contains some rational numbers say \( r_0 \). Then \( \varphi\left(\frac{1}{k-r_0}\right) = \frac{1}{\varphi(k)-r_0} \) is undefined at \( x \in X \) with \( \varphi(k)(x) = r_0 \). This is a contradiction. Consequently \( \varphi(k) \) must be constant, so that \( \varphi \), restricted to the subring of constant functions on \( Y \), is a non-zero homomorphism of \( R \) into itself. By Theorem (0.22)[7], \( \varphi \) is the identity.

Q.E.D.

(9.4) Theorem: Let \( X \) and \( Y \) be two arbitrary subsets of \( E^n \), and \( \varphi \) be an isomorphism from \( A_2 \) onto \( A_1 \) leaving all constant functions unchanged. Then, \( \varphi \) induces a mapping \( \tau : X \to Y \) defined by \( \varphi(g) = g \circ \tau \) and \( \tau \) is a \( C^m \)-diffeomorphism.

Proof: Define \( \tau \) to be a mapping from \( X \) to \( Y \) as follows:
\[
\tau(x) = \mathfrak{N}[\varphi^{-1}(M_x)] .
\] Since \( \varphi \) is an isomorphism, and it onto, its inverse mapping \( \varphi^{-1} \) is an isomorphism of \( A_1 \) onto \( A_2 \).

By Lemma (9.2), \( \varphi^{-1}(M_x) \) is a fixed maximal ideal in \( A_2 \). Thus, \( \tau \) is a single valued mapping. Evidently, \( M_{\tau}(x) = \varphi^{-1}(M_x) \). Let \( x \) and \( x' \) be in \( X \) and \( x \neq x' \). Then, by Lemma (9.2), again \( \varphi^{-1}(M_x) = M_y \) and \( \varphi^{-1}(M_{x'}) = M_{y'} \) for some \( y, y' \in Y \). If \( y = y' \), then \( \varphi^{-1}(M_x) = M_y = \varphi^{-1}(M_{x'}) \) which implies \( M_x = M_{x'} \). This is impossible for \( x \neq x' \).

Thus \( y \neq y' \). Hence \( \tau \) is one-one. Let \( y_0 \) be arbitrary in \( Y \). Then \( M_{y_0} \) is a fixed maximal ideal in \( A_2 \) and
\[
\varphi(M_{y_0}) = M_{x_0} \text{ for some } x_0 \in X .
\] Thus \( y_0 = \mathfrak{N}[\varphi^{-1}(M_{x_0})] = \tau(x_0) \).
This shows that \( \tau \) is onto. Now, for each \( g \in A_2 \) and each \( x \in X \), let \( \phi(g)(x) = r \). Then \( \phi(g) - r \in M_x \), so \( g - \phi^{-1}(r) = \phi^{-1}(\phi(g) - r) \in M_\tau(x) \). Hence \( \phi(g) = g \cdot \tau \). Similarly, \( \phi^{-1}(f) = f \cdot \tau^{-1} \), where \( \tau^{-1} : Y \to X \) is defined by \( \tau^{-1}(y) = \pi_2(\phi(M_y)) \). We know that for all \( f \in A_2 \), \( f \cdot \tau \in A_1 \). In particular, if \( f = \mathcal{I} \) (as defined above), where \( 1 \leq i \leq n \), then \( \tau_1 \) is \( C^m \) for each \( 1 \leq i \leq n \). Similarly, each component of \( \tau^{-1} \) is \( C^m \). That is, \( \tau \) is a \( C^m \)-diffeomorphism. Q.E.D.

(9.5) Corollary: Let \( X \) and \( Y \) be two connected subsets of \( \mathbb{E}^n \), and \( \phi \) be an isomorphism of \( A_2 \) onto \( A_1 \). Then \( \phi \) induces a \( C^m \)-diffeomorphism, \( \tau \), from \( X \) onto \( Y \) such that \( \phi(g) = g \cdot \tau \) for each \( g \in A_2 \).

Proof: We know that both \( A_1 \) and \( A_2 \) satisfy the condition in Lemma (9.3). Hence \( \phi \) is the identity on the constant functions between \( A_2 \) and \( A_1 \). Then the conditions in Theorem (9.4) are satisfied. The result follows immediately from Theorem (9.4). Q.E.D.

(9.6) Theorem: Let \( X \) and \( Y \) be two subsets of \( \mathbb{E}^n \), and \( \tau \) be a \( C^m \)-diffeomorphism of \( X \) onto \( Y \). Then the induced mapping \( \tau' \) defined by \( \tau'(g) = g \cdot \tau \) is an isomorphism of \( A_2 \) onto \( A \) which is the identity on the constant functions.

Proof: For each \( g \in A_2 \), \( g \cdot \tau \in A_1 \), and for each \( f \in A_1 \), \( f \cdot \tau^{-1} \in A_2 \) are clear. Now, for any \( f \in A_2 \), \( \tau'(f \cdot \tau^{-1}) = (f \cdot \tau^{-1}) \tau = f \). Thus, \( \tau' \) is onto. Now, if \( \tau'(g) = \mathcal{O} \), then \( (g \cdot \tau)(x) = 0 \) for each \( x \in X \) or \( g \cdot \tau[X] = g[Y] = \{0\} \).
as $\tau$ is onto. That is, $g = \theta$. This shows that $\tau$ is one-one. Finally, for each constant function $r$, $\tau'(r)(x) = r(\tau(x)) = r$, for each $x$. That is $\tau'(r) = r$. Q.E.D.

Remark: If $A_1 = \mathcal{R}(X)$ and $A_2 = \mathcal{R}(Y)$, then $\tau$ and $\tau^{-1}$ are not only $C^m$, each of their components is a rational function. We will name this mapping as rational-homeomorphism. We also know that there is a non-linear rational-homeomorphism. That is, let $X = Y = \mathbb{E}^n - (0,\ldots,0)$ and $\tau(x) = (\tau_1(x),\ldots,\tau_n(x))$ be defined as $\tau_i(x) = \frac{x_1}{x_1^2 + \ldots + x_n^2}$ for $1 \leq i \leq n$. Then its inverse is known to be $\tau^{-1}(y) = (\phi_1(y),\ldots,\phi_n(y))$ with $\phi_j(y) = \frac{y_j}{y_1^2 + \ldots + y_n^2}$, $1 \leq j \leq n$.

Next, we will see some algebraic properties of the rings of continuous functions which are inapplicable in the rings of $C^m$-differentiable functions, where $1 \leq m \leq \infty$.

(1) The first one is that the rings of continuous functions are lattice-ordered. But the rings of $C^m$-differentiable functions are not. For instance, let $X = \mathbb{E}^1$. Consider $C^m(X)$. Then we know that $i(x) = x$, $i \in C^m(X)$ but $|i| \notin C^m(X)$. Thus neither $f \wedge 0$ nor $f \vee 0$, in general, is in $C^m(X)$.

(2) We know that, in the ring of continuous functions, for a $z$-ideal $I$, $I(f) \geq 0$ if and only iff $f$ is non-negative on some zero-set of $I$ (see [7](5.4)(a)). Also, $I$ is a $z$-ideal implies that $I$ is convex. Hence, by Theorem (5.2)[7], $I(f) \geq 0$ if there is $g \in C(X)$ such that $g \geq 0$ and $g = f$ (mod $I$). In the ring of differentiable function such a $g$ need not exist. For example: Consider $X = \mathbb{E}^1$ and $C^1(X)$.
Let $I = \{ f \in C^1(X) : Z(f) \supset [0,1] \}$. Then since $I$ is a $z$-ideal, it is convex. However it is not absolutely convex, for $C^1(X)$ is not lattice-ordered. Now, let $f_0(x) = x - x^2$. It is clear that $f_0 \geq 0$ on a zero-set of $I$. But, if $g \in C^1(X)$ so that $g \geq 0$ and $g$ agrees with $f_0$ on $[0,1]$, then the derivative of $g$ does not exist at 0 or 1. Indeed, 
\[
\lim_{\Delta x \to 0^+} \frac{g(\Delta x) - g(0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x - \Delta x^2}{\Delta x} = 1 , \text{ but}
\]
\[
\lim_{\Delta x \to 0^-} \frac{g(\Delta x) - g(0)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{g(\Delta x)}{\Delta x} \leq 0 , \text{ as } g(x) \geq 0 \text{ for all } x .
\]
Thus, $g'(x)$ does not exist at $x = 0$. Similarly,
\[
\lim_{\Delta x \to 0^+} \frac{g(\Delta x + 1) - g(1)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{g(\Delta x + 1)}{\Delta x} \geq 0 , \text{ but}
\]
\[
\lim_{\Delta x \to 0^-} \frac{g(\Delta x + 1) - g(1)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{1 + \Delta x - (1+\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{\Delta x - \Delta x^2 - 2\Delta x}{\Delta x} = -1 .
\]
That is, $g$ does not exist at $x = 1$. Hence, there is not any $g \in C^1(X)$ with $g \geq 0$ and $f_0 \equiv g \ (\text{mod } I)$ though we know that $f_0$ is nonnegative on a zero-set of $I$.

(3) If $I$ and $J$ are $z$-ideals in $C(X)$, then $IJ = I \cap J$. This is not true in $C^m(X)$. Let $X = E^1$, consider in $C^1(E^1)$, $I = J = M_0 = \{ f \in C^1(E^1) : f(0) = 0 \}$. Then $I \cap J = M_0$. But $IJ \neq I \cap J$. Indeed, $i \in M_0 = I \cap J$ where $i(x) = x$. If $i = gh$ for some $g \in I$ and $h \in J$, then $g(0) = 0 = h(0)$, and $g,h$ are $C^1$-differentiable. Accordingly, we have $g(x) = g(0) + ax + \varepsilon x$ , $h(x) = h(0) + a'x + \varepsilon'x$ , where $a$ and $a'$ are constant and $\varepsilon, \varepsilon' \to 0$ as $x \to 0$. Thus, $g(x)h(x) = (a+\varepsilon)(a'+\varepsilon)x^2$. Now, $i'(x) = 1$
for all $x \in E^1$. However, $(g \cdot h)'(x)|_{x=0} = \lim_{x \to 0} \frac{g(x)h(x) - g(0)h(0)}{x} = \lim_{x \to 0} \frac{(a+\epsilon)(a'+\epsilon')x^2}{x} = 0$. Hence $1 = 0$ which is impossible. That is, $i \in (I \cap J) - IJ$. This example also shows that the following is inapplicable to $C^m(X)$ or $C^{m*}(X)$. If $P$ and $Q$ are prime ideals in $C$ (or $C^*$), then $PQ = P \cap Q$. For $C^{m*}(X)$, we take $X = (-n,n)$ and use the same argument as above.
PART II

THE RINGS OF Lc- (OR L-) FUNCTIONS

We now study the same sort of properties when \( C^m(X) \) is replaced by the ring of functions satisfying a Lipschitz condition on each compact subset of a metric space.

§10. Rings, Ideals and Some Properties of Lipschitzian or Lc-Functions

(10.1) Definition: A real-valued function \( f \) defined on a metric space \( (X, d) \) is said to be a function satisfying a Lipschitz condition on each compact subset of \( (X, d) \), if for each compact \( A \subseteq X \), there is a positive number \( K_A \) such that for any two points \( x, x' \in A \), \(|f(x) - f(x')| \leq K_A d(x, x')\). For the sake of brevity, we will call such a function Lc-function. (see uniform Lipschitz condition on each compact subset P.354[6])

(10.2) Definition: A real-valued function \( f \) defined on a metric space \( (X, d) \) is said to be a Lipschitzian function (or to satisfy a Lipschitz condition) of constant \( K \), if there exists a positive number \( K \) such that for any two points \( x, x' \in X \), \(|f(x) - f(x')| \leq K d(x, x')\). Note that if the metric space is compact, then an Lc-function is, clearly, an L-function.

Remark: (1) The Lipschitz condition implies uniform continuity. Indeed, if \( f \) satisfies a Lipschitz condition of constant \( K \) on \( (X, d) \), then for any given positive number \( \varepsilon \), choose \( \delta = \varepsilon / K \), we have for any \( x, x' \in X \), \(|f(x) - f(x')| \leq K d(x, x') \leq \varepsilon \).
Given \( K \epsilon /K = \epsilon \) whenever \( d(x, x') < \delta = \epsilon /K \).

(2) However, a continuous function may not be a Lipschitzian function. For example, \( f(x) = x^{1/3} \) is continuous on \( A = \{ x : -1 \leq x \leq 1 \} \). But

\[
|f(x) - f(x')| = |x^{1/3} - x'^{1/3}|
= |x - x'|/|x^{2/3} + x'^{1/3} + x'^{2/3}|
\]

is not bounded. Hence \( f \) is not a Lipschitzian function.

(3) It is also easy to see that an \( L_c \)-function is a continuous function. But, a continuous function need not be an \( L_c \)-function. Consider \( X = \mathbb{E}^1 \) and \( f(x) = x^{1/3} \). Then, as it was shown in Remark (2), for any compact subset containing the origin, there does not exist any constant satisfying the required condition.

(10.3) Definition: Let \((X, d)\) be a metric space. \( L_c(X) = \{ f : f \) is a real-valued \( L_c \)-function on \( X \} \). \( L(X) = \{ f : f \) is a real-valued, bounded function on \( X \), and is a Lipschitzian function of some constant \( K \} \). \( L \)-function means Lipschitzian function.

(10.4) Theorem: The family \( L_c(X) \) and \( L(X) \) are commutative rings with unity under pointwise addition, subtraction and multiplication. In such case, the unity is \( u \), the function with constant value \( 1 \), and \( 0 \) the constant function with value \( 0 \), is the zero element of them.

Remark: The condition that \( f \) is bounded on \( X \) can not be omitted in \( L(X) \). For, let \((X, d')\) be an unbounded metric space. Then, \( f_p(x) = d(p, x) \) is a Lipschitzian function with
constant 1. For \( |f_p(x) - f_p(x')| = |d(p,x) - d(p,x')| \leq d(x,x') \). However, let \( g(x) = f_p(x)\ast f_p(x) = (d(p,x))^2 \) is not Lipschitzian of any constant. Indeed, \( |g(x) - g(x')| = |(d(p,x))^2 - (d(p,x'))^2| = [d(p,x) + d(p,x')]\ast d(p,x) - d(p,x') \). Here we know that \( |d(p,x) - d(p,x')| \) is unbounded. That is, \( g \) is not Lipschitzian.

The reason that we discuss this particular ring \( L(X) \) is that in \( \S 13 \), we shall discuss that \( L(X) \) is a Banach algebra, and \( L_c \)-realcompactness.

Proof of the Proposition: First consider \( L(X) \). Let \( f \) and \( g \) be arbitrary from \( L(X) \) of constants \( K_1 \) and \( K_2 \), respectively, and \( |f| \leq K_1 \), \( |g| \leq K_2 \). Then, for any \( x,x' \in X \).

\[
|f(x) + g(x) - (f(x) + g(x'))| = |(f(x) - f(x')) + (g(x) - g(x'))| \leq |f(x) - f(x')| + |g(x) - g(x')| \leq K_1 d(x,x') + K_2 d(x,x') \leq (K_1 + K_2)d(x,x').
\]

Take \( K = K_1 + K_2 \). Moreover \( f + g \) is bounded by \( K' + K'' \). Hence \( f + g \in L(X) \). Now,

\[
|f \ast g(x) - f \ast g(x')| = |f(x)g(x) - f(x')g(x')| \leq |f(x)||g(x) - g(x')| + |g(x')||f(x) - f(x')| \leq K'K_2d(x,x') + K''K_1d(x,x') = (K_2K_1 + K''K_1)d(x,x').
\]

Also \( |f \ast g| = |f| \ast |g| \leq K'K'' \). Hence \( f \ast g \in L(X) \). Evidently, \( u \) and \( \Theta \in L(X) \).

For \( L_c(X) \), take any \( f,g \in L_c(X) \) and any compact subset \( A \) of \( X \). Then, by the first part we know that \( f|A,g|A \in L(A) \). Hence, \( (f + g)|A, (fg)|A \in L(A) \). Since \( A \) is an arbitrary subset of \( X \), by definition of \( L_c(X) \), \( f + g, fg \in L_c(X) \).

It is obvious that \( L(X) \subset L_c(X) \). Hence \( u, \Theta \in L_c(X) \). Q.E.D.

Remark: If \( f \in L(X) \), then \( |f| \in L(X) \). For \( |f(x)| - |f(x')| \leq |f(x) - f(x')| \leq K \cdot d(x,x') \) for all \( x,x' \in X \), and \( |f| \) is
bounded. Hence, if \( f, g \in L(X) \), then \( |f - g| \in L(X) \) so that 
\[
    f \vee g = \frac{f + g + |f - g|}{2} \quad \text{and} \quad f \wedge g = \frac{f + g - |f - g|}{2}
\]
e \( L(X) \). That is, \( L(X) \) is a lattice-ordered ring (p.7 [7]). Similarly, we can show that \( Lc(X) \) is a lattice-ordered ring.

(10.5) Proposition: If \( f \) is a Lipschitzian function on \((X, d)\) with constant \( K \), then \( |f| \wedge n \in L(X) \) with the same constant.

Proof: We know that \( |f| \wedge n \) is bounded by \( n \). Both \( |f| \) and \( n \) are Lipschitzian. Hence \( |f| \wedge n = \frac{|f| + n - |f| - n}{2} \) is Lipschitzian.

(10.6) Lemma: Let \( f \in L(X) \) and \( r \leq |f| \) for some positive number \( r \). Then \( f^{-1} (= 1/f) \in L(X) \).

Proof: We know that \( |f(x) - f(x')| < Kd(x, x') \) for some positive number \( K \). We also have 
\[
    \left| \frac{1}{f}(x) - \frac{1}{f}(x') \right| = \left| \frac{f(x) - f(x')}{{f(x)f(x')}} \right| \leq \frac{|f(x) - f(x')|}{r^2},
\]
Moreover \( 1/f \leq 1/r \). Hence \( 1/f \in L(X) \).

(10.7) Lemma: Let \( f \in L^*_c(X) = \{ f \in L_c(X) : f \text{ is bounded} \} \), and \( r \leq |f| \) for some positive number \( r \). Then \( f^{-1} (= 1/f) \in L^*_c(X) \).

Proof: Let \( A \) be any compact subset of \( X \). Then, as shown in Lemma (10.6), we know \( 1/f \mid A \in L(A) \). Hence \( 1/f \in L^*_c(X) \).

(10.8) Proposition: For any \( p \in X \), define \( f_p(X) = d(p, x) \). Then \( f_p \in L(X) \) iff \((X, d)\) is bounded. We shall call such a function \( f_p \) a distant function.
Proof: We have already shown that \( f_p \) is a Lipschitzian function in the Remark of Theorem (10.4). Thus \( f_p \in L(X) \) iff \( f_p \) is bounded. Hence the result follows immediately from the boundedness of a metric space. Q.E.D.

(10.9) Definition: The set \( Z(f) = \{ x \in X : f(x) = 0 \} \) is said to be the zero-set of \( f \). Let \( Z(X) = \{ Z(f) : f \in L_c(X) \} = \{ Z(f) : f \in L(X) \} \) (see (11.8)).

(10.10) Definition: If \( f \in L_c(X), (L^*_c(X) \text{ or } L(X)) \) has a multiplicative inverse in \( L_c(X), (L^*_c(X) \text{ or } L(X)) \) is said to be a unit in \( L_c(X), (L^*_c(X) \text{ or } L(X)) \).

Remarks: (1) It is clear that \( f \in L_c(X) \) is a unit in \( L_c(X) \) iff \( Z(f) = \emptyset \) and \( f \in L^*_c(X), (L(X)) \) is a unit in \( L^*_c(X) (L(X)) \) iff \( |f| \geq r \) for some \( r \in \mathbb{R}^+ \), \( r > 0 \).

(2) Consider an unbounded metric space \( (X,d) \), and \( f(x) = \frac{1}{1 + d(p,x)} \). Then \( |f| \leq 1 \), and \( |f(x) - f(x')| = \frac{|d(p,x) - d(p,x')|}{(1 + d(p,x))(1 + d(p,x'))} \leq d(x,x') \). Hence \( f \in L(X) \) and \( Z(f) = \emptyset \) but \( |f| \neq r \) for all positive number \( r \). Consequently \( f^{-1} \notin L(X) \) so that \( f \) is not a unit.

(10.11) Definition: A nonempty subfamily \( \mathcal{F} \) of \( Z(X) \) is said to be a z-filter on \( (X,d) \), if it satisfies the conditions (1), (ii) and (iii) of (1.7).

(10.12) Proposition: If \( I \) is a proper ideal in \( L_c(X) \), then the family \( Z[I] = \{ Z(f) : f \in I \} \) is a z-filter on \( X \).

Proof is similar to (1.8).
Remark: If $L_c(X)$ is replaced by $L(X)$ or $L_c^*(X)$, then the result need not be true. Let $(X,d)$ be unbounded, and $f(x) = \frac{1}{d(x,p) + 1}$ for a fixed point $p \in X$. Then, we know that $f \in L(X)$ (and $L_c^*(X)$) but $f^{-1} \notin L(X)$ (nor $L_c^*(X)$).

Set $I = (f)$, the ideal generated by $f$. We know then that $\phi = Z(f) \in Z[I]$. That is, $Z[I]$ is not a $z$-filter.

(10.13) Proposition: If $\mathfrak{F}$ is a $z$-filter on $X$, then the families $Z_l^{-1}[\mathfrak{F}] = \{f \in L_c(X) : Z(f) \in \mathfrak{F}\}$, $Z_l^{-1}[\mathfrak{F}] = \{f \in L_c^*(X) : Z(f) \in \mathfrak{F}\}$ and $Z_l^{-1}[\mathfrak{F}] = \{f \in L(X) : Z(f) \in \mathfrak{F}\}$ are ideals in $L_c(X)$, $L_c^*(X)$ and $L(X)$, respectively.

Proof is similar to (1.9).

Notice that we will use $Z_l^{-1}$ to denote the appropriate notation of $Z_l^{-1}$, $Z_l^{-1}$ or $Z_l^{-1}$.

Remarks: (1) $Z[Z_l^{-1}[\mathfrak{F}]] = \mathfrak{F}$. This shows that every $z$-filter is of the form $Z[I]$ for some ideal in $L_c(X)$, $L_c^*(X)$ or $L(X)$.

(2) It is clear that $Z_l^{-1}[Z[I]] \supset I$. The inclusion may be proper. For example, let $(X,d)$ be a (bounded) metric space, and $f_\circ(x) = (f_p(x))^2 = [d(p,x)]^2$. Then $f_\circ \in L_c(X)$ ($L_c^*(X)$ or $L(X)$). Let $I = (f_\circ)$. This consists of all functions $f$ in $L(X)$ such that $f = f_\circ \circ g$ for some $g \in L_c(X)$ ($L_c^*(X)$ or $L(X)$). In particular, every function in $I$ vanishes at $p$. Hence every zero-set in $Z[I]$ contains the point $p$. As a matter of fact, since $Z[I]$ is a $z$-filter that includes the set $\{p\} = Z(f_\circ)$, it must be the family of all zero-sets containing $p$. 
The ideal \( M_0 = Z^{-1}[Z[I]] \) evidently consists of all functions in \( L_c(X) \) (\( L^*_c(X) \) or \( L(X) \)) which vanish at \( p \). Hence \( M_0 \supset I \). However, \( M_0 \not= I \). For instance, \( f_p(x) \) vanishes at \( p \) and \( f_p \in L_c(X) \) (\( L^*_c(X) \) or \( L(X) \)). Thus \( f_p \in M_0 \).

Suppose \( f_p \in I \). Then, \( f_p = f^o_g \) for some \( g \in L_c(X) \) (\( L^*_c(X) \) or \( L(X) \)). But, then \( g(x) = [d(p,x)]^{-1} \) is discontinuous at the point \( p \). Thus \( g \notin L_c(X) \) (\( L^*_c(X) \) or \( L(X) \)). This is a contradiction.

Note that \( Z[M_0] = Z[I] \), in spite of the fact that \( M_0 \not= I \).

It is also obvious that \( M_0 \) is a fixed maximal ideal.

(10.14) Definition: A z-ultrafilter on \( X \) is a maximal z-filter on \( X \).

(10.15) Proposition: If \( M \) is a maximal ideal in \( L_c(X) \), then \( Z[M] \) is a z-ultrafilter on \( X \).

Proof is similar to (1.11).

(10.16) Proposition: If \( \mathcal{A} \) is a z-ultrafilter on \( X \), then \( Z^{-1}[\mathcal{A}] \) is a maximal ideal in \( L_c(X) \).

Proof is similar to (1.12).

(10.17) Definition: An ideal \( I \) in \( L_c(X) \) is said to be a z-ideal if \( Z(f) \in Z[I] \) implies \( f \in I \). That is, \( I = Z^{-1}[Z[I]] \).

Example: Every fixed maximal ideal is a z-ideal.

Note that if \( I \) is an ideal in \( L^*_c(X) \) (or \( L(X) \)) such that there is an \( f \in I \) with \( Z(f) = \emptyset \) and \( f^{-1} \notin L^*_c(X) \) (or \( L(X) \)),
then $I$ is clearly not a $z$-ideal.

(10.18) Proposition: If $M$ is a maximal ideal in $L_c(X)$ ($L^*(X)$, $L(X)$), and $Z(f)$ meets every member of $Z[M]$, then $f \in M$.

Proof is similar to (1.13).

(10.19) Proposition: If $\mathcal{F}$ is a $z$-ultrafilter on $X$, and a zero-set $Z$ meets every member of $\mathcal{F}$, then $Z \in \mathcal{F}$.

Proof is similar to (1.14).

(10.20) Definition: An ideal $I$ in $L_c(X)$ ($L^*(X)$ or $L(X)$) is said to be a prime ideal if $f \cdot g \in I$ implies $f \in I$ or $g \in I$.

(10.21) Theorem: For any $z$-ideal $I$ in $L_c(X)$ ($L^*(X)$ or $L(X)$), the following are equivalent.

1. $I$ is prime.
2. $I$ contains a prime ideal.
3. For all $g \cdot h \in L_c(X)$ ($L^*(X)$ or $L(X)$), $g \cdot h = \Theta$, then $g \in I$ or $h \in I$.
4. For every $f \in L_c(X)$ ($L^*(X)$ or $L(X)$), there is a zero-set in $Z[I]$ on which $f$ does not change sign.

Proof: That (1) implies (2) is clear. If $I$ contains a prime ideal $P$, and $g \cdot h = \Theta$, then $g \cdot h \in P$. Hence $g \in P \subseteq I$ or $h \in P \subseteq I$. This shows that (2) implies (3). To see that (3) implies (4), for every $f \in L_c(X)$, consider $(f \vee \Theta) \cdot (f \wedge \Theta) = \Theta \in I$. By (3) $f \vee \Theta \in I$ or $f \wedge \Theta \in I$.

Either case implies (4). Finally, we will show that (4) implies (1). Given $g \cdot h \in I$, consider the function $|g| - |h|$. By
hypothesis, there is a zero-set $Z$ of $Z[I]$ on which $|g| - |h|$ is nonpositive, say. Then every zero of $h$ on $Z$ is zero of $g$. Hence $Z(g) \supset Z \cap Z(g) = Z \cap Z(g \cdot h) \in Z[I]$ so that $Z[g] \in Z[I]$. Since $I$ is a z-ideal, $g \in I$. Thus $I$ is prime.

The proof for $L^*(X)$ or $L(X)$ is exactly the same with $L^*(X)$ or $L(X)$ taking the place of $L(X)$.

§11 L-Complete Regularity and L-Normality.

(11.1) Definition: A metric space $(X,d)$ is said to be L-completely regular if, for every closed subset $F$ of $X$, and $x \in X - F$, there is a function $f \in L(X)$ such that $f(x) = 1$ and $f(F) = \{0\}$.

(11.2) Définition: A metric space $(X,d)$ is said to be L-normal if, for any two disjoint closed subsets $F$, $F'$, there is an $f \in L(X)$ such that $f(F) = \{0\}$ and $f(F') = \{1\}$.

Note that a metric space is a Hausdorff space so that L-normality implies L-complete regularity.

(11.3) Theorem: A metric space $(X,d)$ is L-completely regular iff the family $Z(X)$ of all zero-sets of $L(X)$ is a base for the closed subsets of $X$.

Proof is similar to (2.4).

(11.4) Lemma: Let $A$ be a nonempty subset of a metric space $(X,d)$ and $f \in L(A)$. Then there is a $g \in L(X)$ such that $g|A = f$.

Proof: Suppose $|f| \leq n$, and $K$ is the Lipschitzian constant of $f$. For each $x \in X$, define $f_0(x) = \sup_{x' \in A} \{f(x') - Kd(x',x)\}$. 

Then for each \( x_0 \in A \), \( f_0(x_0) = \sup_{x' \in A} \{ f(x') - Kd(x',x_0) \} \).

We know that \( K > 0 \) and \( d(x',x) \geq 0 \). If there exists \( x_1 \in A \) such that \( f(x_1) - Kd(x_1,x_0) > f(x_0) - Kd(x_0,x_0) = f(x_0) \), then \( f(x_1) - f(x_0) > Kd(x_1,x_0) \) which contradicts the fact that \( |f(x) - f(x')| \leq Kd(x,x') \) for all \( x,x' \in A \). Thus, \( f(x_0) \geq f(x') - Kd(x',x_0) \) for all \( x' \in A \). Hence

\[
\sup_{x' \in A} \{ f(x') - Kd(x,x') \} = f(x_0) \quad \text{for} \quad x_0 \in A.
\]

That is,

\[
f_0|A = f.
\]

We will show that \( f_0 \) is Lipschitzian on \( X \) of constant \( K \). For arbitrary \( x,x' \in X \), and \( x_0 \in A \),

\[
f(x_0) - Kd(x_0,x') + Kd(x,x') \geq f(x_0) - Kd(x_0,x) \geq f(x_0) - Kd(x,x') - Kd(x',x).
\]

Hence \( f_0(x') = Kd(x',x) = \sup_{x_0 \in A} \{ f(x_0) - Kd(x_0,x') \} + Kd(x,x') \)

\[
\geq \sup_{x_0 \in A} \{ f(x_0) - Kd(x_0,x) \} = f_0(x).
\]

On the other hand,

\[
f_0(x') - Kd(x',x) = \sup_{x_0 \in A} \{ f(x_0) - Kd(x_0,x') \} - Kd(x',x) = \sup_{x_0 \in A} \{ f(x_0) - Kd(x_0,x') \} - Kd(x',x) \leq \sup_{x_0 \in A} \{ f(x_0) - Kd(x,x_0) \}
\]

\[
= f_0(x).
\]

Hence \( |f_0(x) - f_0(x')| \leq Kd(x,x') \). Finally, let \( g = f_0 \wedge n \). By Prop. (10.5) \( g \in L(X) \). Hence \( g \) is required.

Q.E.D.

(11.5) Theorem: (1) Every metric space \((X,d)\) is \(L\)-completely regular.

(2) Every compact metric space \((X,d)\) is \(L\)-normal.

Proof: (1) Let \( F \) be a closed subset of \( X \) and \( p \in X - F \). Then \( d(F,p) \neq 0 \). Let \( f \) be a function defined as follows:
$f[F] = \{1\},$ and $f(p) = 0$. Let $K = (d(F,p))^{-1}$. We will show that $f$ is Lipschitzian on $F \cup \{p\}$ with constant $K$.

If $x,x' \in F$, then $|f(x) - f(x')| = 0 \leq K \cdot d(x,x')$. If $x \in F$ and $x' = p$, then $|f(x) - f(x')| = 1$, and $K \cdot d(x,x') = \frac{d(x,x')}{d(F,p)} \geq 1$. Hence $|f(x) - f(x')| \leq K \cdot d(x,x')$ for $x,x' \in F \cup \{p\}$. It is clear that $f$ is bounded by 1. That is, $f \in L(F \cup \{p\})$. By Lemma (11.4) there is an $f_0 \in L(X)$ such that $f_0|F \cup \{p\} = f$. Hence $f_0[F] = \{1\}$ and $f_0(p) = 0$.

For (2), we only have to replace $\{p\}$ by a closed set $F'$ disjoint from $F$.

Remark: The compactness cannot be omitted in (2). For instance: Let $X = \mathbb{E}^2$, $F = \{(x,y) \in \mathbb{E}^2 : xy = 1\}$ and $F' = \{(x,y) \in \mathbb{E}^2 : xy = -1\}$. Then $F$ and $F'$ are two disjoint closed sets in $\mathbb{E}^2$. However, it is clear that there is not any $f \in L(\mathbb{E}^2)$ such that $f[F] = \{1\}$ and $f[F'] = \{0\}$.

(11.6) Lemma: Let $B_{r_i}(x_0) = \{x \in X : d(x,x_0) < r_i\}$ $i = 1,2$, where $0 \leq r_1 < r_2$. Then, there is an $f \in L(X)$ such that:

$$f(x) = \begin{cases} 0 & \text{for } x \in \text{cl}_X B_{r_1}(x_0) \\ 0 < f(x) < 1 & \text{for } x \in B_{r_2}(x_0) - \text{cl}_X B_{r_1}(x_0) \\ f(x) = 1 & \text{for } x \in X - B_{r_1}(x_0) \end{cases}$$

Proof: Let $\varphi(t) = \begin{cases} 0 & \text{for } t \leq r_1 \\ \frac{t-r_1}{r_2-r_1} & \text{for } r_1 \leq t \leq r_2 \\ 1 & \text{for } t \geq r_2 \end{cases}$, and $f(x) = \varphi(d(x,x_0))$. \[82.\]
Then, it is obvious that $f$ satisfies (11-1). Moreover, for any $x, x' \in X$, 
\[ |f(x) - f(x')| = |\varphi(d(x, x_0)) - \varphi(d(x', x))| \leq \frac{1}{r_2 - r_1} |d(x, x_0) - d(x', x_0)| \leq \frac{1}{r_2 - r_1} d(x, x'). \]

Moreover, $f$ is bounded by 1. Hence $f \in L(X)$. Q.E.D.

(11.7) Proposition: For every neighborhood $U$ of a point $x_0$ of $(X,d)$, there is a zero-set which is a neighborhood of $x_0$ contained in $U$.

Proof: Since $U$ is a neighborhood of $x_0$, by regularity, there is an $r_1 > 0$ such that $c_{x,x_0} B_{r_1}(x_0) \subset U$. Choose $r_2 > r_1$, by Lemma (11.6), there is an $f \in L(X)$ satisfying (11-1).

Hence $Z(f) = c_{x,x_0} B_{r_1}(x_0) \subset U$. Q.E.D.

(11.8) Proposition: Every closed subset $A$ of $(X,d)$ is an $L$-zero-set.

Proof: Let $f(x) = d(A, x)$ for each $x \in X$. Then, that $Z(f) = A$ is clear. We will show that $f$ satisfies a Lipschitz condition with constant 1. That is, $|f(x) - f(y)| \leq d(x, y)$ for all $x, y \in X$. For any $x, y \in X$, $d(A, x) = \inf \{d(a, x) : a \in A\} \leq \inf \{d(a, y) + d(y, x) : a \in A\} = \inf \{d(a, y) : a \in A\} + d(y, x) = d(A, y) + d(x, y)$. Thus $d(A, x) - d(A, y) \leq d(x, y)$. Similarly, $d(A, y) - d(A, x) \leq d(x, y)$. Hence $|f(x) - f(y)| \leq d(x, y)$ for all $x, y \in X$. Let $g = f^{\wedge 1}$.

By Prop. (10.5), $g \in L(X)$. Moreover, $Z(g) = Z(f) = A$. Q.E.D.
§12 Fixed, Free Ideals and Compact Spaces.

The fixed ideals and free ideals are defined the same as in (3.1). Moreover, the three Remarks of (3.1) are verified with $L^c(X), (L^c(X) \text{ or } L(X))$ in place of $C^m(X)$. We will also use $I(f)$ to denote the residue class of $f \text{ mod } I$.

(12.1) Theorem: (1) The fixed maximal ideals in $L^c(X), (L^c(X) \text{ or } L(X))$ are precisely the sets $M_p = \{ f \in L^c(X), (L^c(X) \text{ or } L(X)) : f(p) = 0 \}$. 

(2) The ideals $M_p$ are distinct for distinct $p$.

(3) For each $p$, $L^c(X)/M_p, (L^c(X)/M_p \text{ or } L(X)/M_p)$ is isomorphic with the real field $\mathbb{R}$. In fact, the mapping $M_p(f) = f(p)$ is the unique isomorphism of $L^c(X)/M_p, (L^c(X)/M_p \text{ or } L(X)/M_p)$ onto $\mathbb{R}$, where $p \in X$.

Proof is similar to (3.3).

(12.2) Proposition: If $(X,d)$ is compact, then every ideal $I$ in $L^c(X) = L(X)$ is fixed.

Proof is similar to (3.5).

(12.3) Proposition: If $X$ is compact, then the correspondence $p \rightarrow M_p$ is one-one from $X$ onto the set of all maximal ideals in $L^c(X)$ ($L(X)$).

Proof is similar to (3.6).

(12.4) Lemma: A zero-set $Z \in Z(X)$, $Z \neq \emptyset$ is compact iff it belongs to no free $z$-filter. (See (3.7)).

Proof is similar to (3.9).
(12.5) Proposition: Let $\mathcal{A}$ be a $z$-ultrafilter on $(X,d)$ and each of its members be noncompact. Then $\mathcal{A}$ is free.

Proof: Suppose $p \in \bigcap \mathcal{A}$. Then, consider $f_p(x) = d(p,x)$. We know that $Z(f_p) = \{p\}$ which is compact and meets each member of $\mathcal{A}$ at $p$. Hence $Z(f_p) \in \mathcal{A}$ (by (10.19)). This is a contradiction.

Remark: As we have shown in the Remark of (3.10), $\mathcal{A}$ must be a $z$-ultrafilter. We consider $X = E^1$ and $f_0(x) = \sin x$. By the Mean Value Theorem, $\sin x - \sin x' = (\cos x_1)(x-x')$ where $x_1$ is between $x$ and $x'$. Hence we have $|f(x) - f(x')| = |\cos x_1| \cdot |x-x'| \leq d(x,x')$. That is, $f_0 \in L_c(X), (L_c^*(X) \text{ or } L(X))$. Let $I = (f_0)$. Then $Z[I]$ is a $z$-filter but not a $z$-ultrafilter. We know that each member of $Z[I]$ is unbounded so that it is noncompact. But $\bigcap Z[I] = Z(f_0)$.

(12.6) Theorem: For a metric space $(X,d)$, the following are equivalent.

1. $X$ is compact.
2. Every ideal in $L_c(X)$ is fixed, i.e. every $z$-filter is fixed.
3. Every maximal ideal in $L_c(X)$ is fixed, i.e. every $z$-ultrafilter is fixed.

Proof is similar to (3.11).
§13 The Banach Space $L(X)$ and $L_c$-realcompactness.

As we have shown in §4, every residue field of $L_c(X)$, $L_c^*(X)$ or $L(X)$ modulo a maximal ideal contains a canonical copy of the real field $\mathbb{R}$. We shall show later that for every maximal ideal $M$ of $L_c^*(X)$ (or $L(X)$), $L_c^*(X)/M$ (or $L(X)/M$) is isomorphic with $\mathbb{R}$.

(13.1) Definition: For $f \in L(X)$, $\|f\|_\infty = \sup \{|f(x)| : x \in X\}$, $\|f\|_d = \sup \{|f(x) - f(x')| : x, x' \in X, x \neq x'\}$, and $\|f\| = \|f\|_\infty + \|f\|_d$.

(13.2) Proposition: $L(X)$ is a Banach algebra under the above-defined norm.

Proof: (i) That $\|f\| = 0$ iff $f = 0$ is clear.

(ii) For $f, g \in L(X)$, $\|f + g\| = \|f + g\|_\infty + \|f + g\|_d \leq \|f\|_\infty + \|g\|_\infty + \|f\|_d + \|g\|_d = \|f\| + \|g\|$.

(iii) $\|\lambda f\| = |\lambda| \cdot \|f\|$ for each $\lambda \in \mathbb{R}$ is clear.

(iv) $\|f \cdot g\| = \|f \cdot g\|_\infty + \|f \cdot g\|_d \leq \|f\|_\infty \cdot \|g\|_\infty + \|f\|_d \cdot \|g\|_d$. But, $\|f \cdot g\|_d = \sup \{|f(x)g(x) - f(x')g(x')| : x \neq x', x, x' \in X\}$, and $|f(x)g(x) - f(x')g(x')| \leq |f(x')| \cdot |g(x) - g(x')| + |g(x')| \cdot |f(x) - f(x')|$. We have $\|f \cdot g\|_d \leq \|f\|_\infty \cdot \|g\|_d + \|g\|_\infty \cdot \|f\|_d$; hence $\|f \cdot g\| \leq \|f\|_\infty \cdot \|g\|_d + \|f\|_\infty \cdot \|g\|_d + \|g\|_\infty \cdot \|f\|_d \leq \|f\| \cdot \|g\|$.

(v) Let $\{f_n\}$ be an arbitrary Cauchy sequence under the above-defined norm. Then, it is clear that $\{f_n\}$ is a Cauchy sequence under the norm $\|\cdot\|_\infty$. We know that $C^*(X)$ is a Banach algebra, hence there is $f \in C^*(X)$ such that $\{f_n\}$ converges to $f$ under $\|\cdot\|_\infty$. We will show that $f \in L(X)$. We know that
every Cauchy sequence is uniformly bounded. Indeed, for each 
\( \varepsilon > 0 \), there exists a positive integer \( N(\varepsilon) \) such that 
\[ \|f_n - f_m\| < \varepsilon \text{ for all } n, m \geq N(\varepsilon) . \]
Thus \( \|f_{N(\varepsilon)} - f_m\| < \varepsilon \).

Let \( K_1 = \sup \{\|f_{N(\varepsilon)} - f_j\| : j < N(\varepsilon)\} \), and \( K = \max (\varepsilon, K_1) \).

Then, \( \|f_{N(\varepsilon)} - f_m\| < K \text{ for all } m \). Hence \( \|f_m\| - \|f_{N(\varepsilon)}\| < K \) (by property (ii)), or \( \|f_m\| < \|f_{N(\varepsilon)}\| + K \) for all \( m \).

Let \( K_0 \) be such that \( \|f_n\| \leq K_0 \) for all \( n \). Now

\[ \frac{|f(x) - f(x')|}{d(x, x')} \leq \frac{|f(x) - f_n(x)|}{d(x, x')} + \frac{|f_n(x) - f_n(x')|}{d(x, x')} + \frac{|f_n(x') - f(x')|}{d(x, x')} \]
is true for all \( x, x' \in X \), and \( x \neq x' \), and all \( n \). Suppose \( f \notin L(X) \). That is, for any positive number \( K \), there exist \( x, x' \in X \), \( x \neq x' \) such that

\[ \frac{|f(x) - f(x')|}{d(x, x')} > K . \]
In particular, for \( K = K_0 + 1 \),

\[ \frac{|f(x_o) - f(x_o')|}{d(x_o, x_o')} > K_0 + 1 \text{ for some } x_o, x_o' \in X , x_o \neq x_o' . \]

However, on the other hand, for these two points \( x_o, x_o' \), \( d(x_o, x_o') > 0 \), there is a positive integer \( N \) such that

\[ \|f - f_n\|_\infty < \frac{d(x_o, x_o')}{2} \text{ for all } n \geq N . \]
Thus

\[ \frac{|f(x_o) - f_n(x_o)|}{d(x_o, x_o')} + \frac{|f_n(x_o) - f_n(x_o')|}{d(x_o, x_o')} + \frac{|f_n(x_o') - f(x_o')|}{d(x_o, x_o')} \leq 1 + K_0 . \]
This is a contradiction. Hence \( f \in L(X) \).

Finally we have to show that \( \|f_n - f\| \to 0 \) as \( n \to \infty \).
Since \(|f_n - f| = |f_n - f|_\infty + |f_n - f|_d\) and \(|f_n - f|_d \to 0\) as \(n \to \infty\), we only have to show \(|f_n - f|_d \to 0\) as \(n \to \infty\). Since \([f_n]\) is a Cauchy sequence under the norm \(\|\cdot\| = \|\cdot\|_\infty + \|\cdot\|_d\), given any \(\varepsilon > 0\), there is a positive integer \(N\) such that 
\[ |f_m - f_n| < \varepsilon \quad \text{whenever } m, n \geq N. \]
This implies that
\[ |f_m - f_n|_d < \varepsilon \quad \text{whenever } m, n \geq N. \]
That is
\[ \left| \frac{(f_m - f_n)(x) - (f_m - f_n)(x')}{{d(x,x')}} \right| < \varepsilon \quad (13-1), \quad \text{for } m, n \geq N \text{ and all } x \neq x' \in X. \]
Rewriting (13-1), we have
\[ \left| \frac{f_m(x) - f_m(x')}{{d(x,x')}} - \frac{f_n(x) - f_n(x')}{{d(x,x')}} \right| < \varepsilon, \quad \text{for } m, n \geq N \text{ and all } x \neq x' \in X. \]
Now, we hold \(m\) fixed, for any fixed pair of \(x \neq x' \in X\), we have \(\frac{f_m(x) - f_m(x')}{d(x,x')}\) is fixed. Let it be \(r\), and \(r_n = \frac{f_n(x) - f_n(x')}{d(x,x')}\). We know, by the uniform convergence of \([f_n]\) to \(f\) under the sup norm, \([r_n] \to \frac{f(x) - f(x')}{d(x,x')}\) as \(n \to \infty\). Moreover \(|r - r_n| < \varepsilon\) for \(n \geq N\).
Thus, we have
\[ \left| r - \frac{f(x) - f(x')}{d(x,x')} \right| \leq \varepsilon. \]
Or
\[ \left| \frac{f_m(x) - f_m(x')}{d(x,x')} - \frac{f(x) - f(x')}{d(x,x')} \right| \leq \varepsilon, \quad \text{for } m \geq N \]
and all \(x \neq x' \in X\). That is, \(|f_m - f|_d \leq \varepsilon, \quad \text{for } m \geq N\).
The proof is then complete.
We are now back to show that every maximal ideal in \( L(X) \) or \( L^*_c(X) \) is real. We shall discuss in the case of \( L^*_c(X) \). The proof for \( L(X) \) is exactly the same except for the notation.

In order to show that every \( L^*_c(X)/M \) is isomorphic to the real field, \( \mathbb{R} \), by the theory of totally ordered fields (see P. 209 [28]), we first classify the elements of \( L^*_c(X)/M \) as positive, negative or zero in such a way that \( f + g \) and \( fg \) are positive and \(-f\) is negative when \( f \) and \( g \) are positive. (See [27]).

(13.3) Definition: \( f \) is said to be positive if \( f = |f| \) (mod \( M \)) and \( f \not\equiv 0 \) (mod \( M \)). \( f \) is said to be negative if \(-f\) is positive. And \( f \) is said to be zero if \( f \in M \).

Let \( \mathcal{P} \) and \( \mathcal{N} \) denote the classes of positive and negative elements of \( L^*_c(X) \) (or \( L(X) \), respectively). In order to justify this classification, we shall show the following lemmas.

(13.4) Lemma: For each \( f \in L^*_c(X) \) (or \( L(X) \)), one and only one of the three relations \( f \in \mathcal{P} \), \( f \in M \) or \( f \in \mathcal{N} \) holds.

Proof: We know that \( (-f + |f|)(f + |f|) = \theta \in M \). Since \( \pm f + |f| \in L^*_c(X) \) and \( L^*_c(X)/M \) is a field, at least one of the relation \(-f + |f| = 0 \) and \( f + |f| = 0 \) (mod \( M \)) is valid. If both hold, then \( 2f = (f + |f|) - (-f + |f|) = 0 \) (mod \( M \)). Thus \( f = 0 \) (mod \( M \)). Q.E.D.

(13.5) Lemma: For any two elements \( f, g \in L^*_c(X) \) (or \( L(X) \)), if \( 0 \leq f \leq g \) and \( g \in M \), then \( f \in M \).

Proof: If \( f \not\in M \), then \( f \not\equiv 0 \) (mod \( M \)) so that there exists an
element $h \in L^*_c(X)$ such that, $f \cdot h = 1 \pmod{M}$. By (13.4), and the fact that $f \cdot h = 1$, at least one of the relations $-h = |h| \pmod{M}$, and $h = |h| \pmod{M}$ holds. If the first should hold we would have $1 + f|h| = 1 - f \cdot h = 0 \pmod{M}$. However, by hypothesis, $f \geq 0$, $1 + f|h| \geq 1$ so that $1/1 + f|h| \in L^*_c(X)$ (by (10.7)). Thus $u = \frac{(1 + f|h|)}{1 + f|h|} \in M$. This is a contradiction. If the second should hold, we would have $1 - f|h| = 1 - f \cdot h = 0 \pmod{M}$ or $g|h| + (1 - f|h|) = 0 \pmod{M}$. But $f \leq g$, so we have $g|h| + (1 - f|h|) = 1 + (g - f)|h| \geq 1$. Again, we have a contradiction. Hence $f = 0 \pmod{M}$. Q.E.D.

(13.6) Lemma: For any $f, g \in L^*_c(X)$ (or $L(X)$), if $f \in \overline{P}$, and $f \equiv g \pmod{M}$, then $g \in \overline{P}$.

Proof: Since $f \in \overline{P}$, $f = |f|$ and $f \not\equiv 0 \pmod{M}$. By hypothesis $g \equiv f \not\equiv 0 \pmod{M}$. Thus, we have only either $-g = |g| \pmod{M}$ or $g = |g| \pmod{M}$. Suppose $-g = |g| \pmod{M}$ holds, we would combine it with $f = |f| \pmod{M}$ to obtain $|f| + |g| = 0 \pmod{M}$. However, $0 \leq |g| \leq |f| + |g|$, by (13.5) $|g| = 0 \pmod{M}$ which is a contradiction. Consequently $g = |g| \pmod{M}$. That is, $g \in \overline{P}$, as we showed above $g \not\equiv 0 \pmod{M}$. Q.E.D.

(13.7) Lemma: For any $f, g \in L^*_c(X)$ (or $L(X)$), if $f \in \overline{P}$ and $g \in \overline{P}$, then $f + g, f \cdot g \in \overline{P}$.

Proof: By Definition (13.3) $f = |f|$, $f \not\equiv 0 \pmod{M}$ and $g = |g|$, $g \not\equiv 0 \pmod{M}$. We then have that $f \cdot g = |f||g| = |f\cdot g|$ and $f \cdot g \not\equiv 0 \pmod{M}$ as $L^*_c(X)/M$ is a field. That is,
Suppose \((f + g) = |f + g|\). Then \(|f| + |g| + |f + g| = f + g - (f + g) = 0\) (mod M). We have \(0 \leq |f| \leq |f| + |g| + |f + g|\), \(0 \leq |g| \leq |f| + |g| + |f + g|\) which imply \(|f| = 0\) and \(|g| = 0\) (mod M). Thus, \(f = 0, g = 0\) (mod M). This is a contradiction. Hence \(f + g \neq |f + g|\) (mod M). If \(f + g = 0\) (mod M), then \(|f| + |g| = f + g\) (mod M). Again, from \(0 \leq |f| \leq |f| + |g|\) and \(0 \leq |g| \leq |f| + |g|\). We have \(|f| = 0\) and \(|g| = 0\) (mod M) which is a contradiction. Therefore \(f + g = |f + g|\) and \(|f + g| \neq 0\) (mod M). That is, \(f + g \notin \mathcal{F}\).

Combining these four Lemmas and the definition of a totally ordered field (see P. 209, [28]), we have:

(13.8) Proposition: \(L^*_c(X)/M\), (or \(L(X)/M\)) is a totally ordered field.

(13.9) Lemma: Let \(M\) be a maximal ideal in \(L^*_c(X)\), (or \(L(X)\)) and \(L^*_c(X)\), (or \(L(X)\)) be normed by the sup norm \(\|\cdot\|_\infty\). Then \(M\) is closed in \(L^*_c(X)\) (or \(L(X)\) respectively) under \(\|\cdot\|_\infty\).

Proof: In view of (2M.1) [7], \(clM\) is either a proper ideal of \(L^*_c(X)\) or \(L^*_c(X)\) itself. Suppose \(clM = L^*_c(X)\). Then \(u \in clM\) and for any neighborhood of \(u, N_\varepsilon(u), N_\varepsilon(u) \cap M \neq \phi\). In particular, take \(\varepsilon = 1/2\). Then \(N_{1/2}(u) \cap M \neq \phi\). That is, there is an \(f \in M\) such that \(\|u - f\|_\infty < 1/2\). Thus \(\sup_{x \in X} |u(x) - f(x)| < 1/2\), so that \(|u(x) - f(x)| < 1/2\) for each \(x \in X\) or \(u(x) - |f(x)| < 1/2\) for each \(x \in X\). Hence \(1/2 < |f(x)|\) for each \(x \in X\). By Lemma (10.7) \(1/f \in L^*_c(X)\).

In other words, \(M\) has a unit so that \(M = L^*_c(X)\). This is
impossible. Hence $\mathfrak{c}_M$ is a proper ideal containing $M$. We must have $M = \mathfrak{c}_M$.

Q.E.D.

(13.10) Proposition: For each maximal ideal $M$ in $\mathbb{L}_c^*(X)$, (or $L(X)$), $\mathbb{L}_c^*(X)/M$ is an archimedean ordered field. That is, $\mathbb{L}_c^*(X)/M$, (or $L(X)/M$) $\cong \mathbb{R}$.

Proof: It is enough to show that for any $f \in \mathbb{F}$ such that $f$ is not a constant function, there is a positive integer $n$ such that $f - 1/n \in \mathbb{F}$. Suppose that there does not exist such a positive integer. Then we would have $f - 1/n \notin \mathbb{F}$ for all $n$ (as $f$ is not a constant function). That is, $(f - 1/n) + |f - 1/n| \in M$ for all $n$. Consider now the sequence $\{g_n = (f - 1/n) + |f - 1/n| : n \in \mathbb{N}\}$ which has $f + |f|$ as the limit under the norm $\|\cdot\|_\infty$. Indeed, $f + |f| - [(f - 1/n) + |f - 1/n|] = |f| + 1/n - |f - 1/n| \geq 0$. On the other hand, $|f - 1/n| \geq |f| - 1/n$, hence we have $0 \leq f + |f| - [(f - 1/n) + |f - 1/n|] \leq 2/n$. By (13.9) Lemma, $f + |f| \in \mathfrak{c}_M = M$. This show that $-f \equiv |f| \pmod{M}$. This is a contradiction.

Q.E.D.

We have proved that every maximal ideal in $\mathbb{L}_c^*(X)$ (or $L(X)$) is real. In view of Theorem (12.6), we have that every real maximal ideal in $L(X)$ is fixed if and only if $(X,d)$ is compact.

(13.11) Definition: A metric space $(X,d)$ is said to be \textit{Lc-realcompact} if every real maximal ideal is fixed.

Now, we will give an example to show that there is an Lc-realcompact space which is not compact. However, the
existence of non-Lc-realcompact spaces remains as an open question.

(13.12) Lemma: An ideal in $L_c(X)$ is free iff for every compact subset $A$ of $X$ there exists an $f \in I$ having no zero in $A$.
Proof is similar to (4.3).

(13.13) Proposition: Let $X$ be any closed subspace of $E^n$. Then $X$ is Lc-realcompact.
Proof: Suppose that $L_c(X)/M \simeq R$ for a free maximal ideal $M$.
Let $g(x) = \frac{1}{\|x\|^2 + 1}$. Then $g \in L_c(X)$ and is a unit is clear.
Hence $g \notin M$ and $M(g) \not= 0$. For any $r \in R$, $r > 0$, $g < r$
for all but a compact subset of $E^n$, say $A$. Then $B = A \cap X$
is compact in $X$ as $X$ is closed. Let $A' = cl_X(X - B)$
which is closed in $X$ so is closed in $E^n$. Thus there is an
$f \in L(E^n) \subset L_c(E^n)$ such that $Z(f) = A'$. We will show that
$Z(f) \subset Z(M)$. It is enough to show that $Z(f) \supset Z$ for some
$Z \in Z(M)$. However, we know that $B$ is compact in $X$. By
Lemma (13.12), there is $f_1 \in M$ such that $Z(f_1) \cap B = \emptyset$.
That is, $Z(f_1) \subset X - B \subset cl_X(X - B) = Z(f)$. Hence $Z(f) \in
Z(M)$. $g \leq f$ on the zero-set $Z(f)$ and $f - g \geq 0$. Hence
$M(f - g) \geq 0$. Or $M(r) - M(g) \geq 0$. That is, $r \geq M(g)$.
Since $r$ is any positive number, $M(g)$ is infinitely small.
This is a contradiction. Therefore $M$ must be fixed. Q.E.D.
§14  Lc, L-Mappings and Lc, L-Homeomorphisms.

(14.1) Definition: A mapping \( \tau \) from \((X,d_1)\) to \((Y,d_2)\) is said to be an Lc-[resp. L-] mapping if, for each compact subset \( A \) of \((X,d_1)\), there is a positive number \( K_A \) such that
\[
d_2(\tau(x), \tau(x')) \leq K_A d(x,x') \quad \text{for all } x,x' \in A, \quad \text{[if there is a positive number } K \text{ such that } d_2(\tau(x), \tau(x')) \leq K d(x,x') \quad \text{for all } x,x' \in X].
\]

(14.2) Definition: A mapping \( \tau \) from \((X,d_1)\) to \((Y,d_2)\) is said to be an Lc (L-\( \phi \)-) homeomorphism, if \( \tau \) is one-one, onto, and both \( \tau \) and its inverse are Lc-[resp. L-] mapping.

(14.3) Lemma: Let \((X,d_1)\) and \((Y,d_2)\) be two compact metric spaces, and \( \tau \) be a mapping from \( X \) into \( Y \) such that \( f \circ \tau \in L(X) \) for each \( f \in L(Y) \). Then \( \tau \) is an L-mapping.

Proof: Let \( \varphi \) be a mapping from \( L(Y) \) into \( L(X) \) defined by \( \varphi(f) = f \circ \tau \). Then \( \varphi \) is clearly a homomorphism. By Theorem (2.5.17) [21], \( \varphi \) is continuous under the norm \( \| \cdot \| = \| \cdot \|_\infty + \| \cdot \|_d \) defined in (13.1), so that \( \varphi \) is bounded ([13] Theorem 7A).

Now, for \( q \in Y \) let \( f_q(Y) = d(q,y) \) for each \( y \in Y \) and \( D = \{ f_q : q \in Y \} \). Then, \( \| f_q \| = \| f_q \|_\infty + \| f_q \|_d = \text{Diam}(Y) + 1 \).

For \( \| f_q \|_d = \sup \left\{ \frac{d_2(q,y) - d_2(q,y')}{d_2(y,y')} : y,y' \in Y, y \neq y' \right\} \)
\[ \leq \sup \left\{ \frac{d_2(y,y')}{d_2(y,y')} = 1 \right\}. \] Hence the set \( D \) is bounded by \( \text{Diam}(Y) + 1 \), so that \( \varphi(D) \) is a bounded subset in \( L(X) \), say by \( K \). In other words \( \| f_q(\tau(x)) - f_q(\tau(x')) \| < K \) for all \( q \in Y \),
so that $\| f \circ \tau(x) - f \circ \tau(x') \|_{d_2} < K$. Hence

$$\left| \frac{d_2(q,\tau(x)) - d_2(q,\tau(x'))}{d_1(x,x')} \right| < K$$

for all $x,x' \in X$, $x \neq x'$. In particular, if we take $q = \tau(x)$, then

$$\frac{d_2(\tau(x),\tau(x'))}{d_1(x,x')} < K$$

for all $x,x' \in X$, $x \neq x'$. If $x = x'$, then we have $d_2(\tau(x),\tau(x')) = d_1(x,x') = 0$. Thus

$$d_2(\tau(x),\tau(x')) \leq K d_1(x,x')$$

for all $x,x' \in X$.

Q.E.D.

(14.4) Proposition: Let $(X,d_1),(Y,d_2)$ be metric spaces and $\tau$ be a mapping from $(X,d_1)$ to $(Y,d_2)$.

1. If $\tau$ is an $L_\tau$-mapping, then $f \circ \tau \in L_c(X)$ for all $f \in L_c(Y)$.

2. If $f \circ \tau \in L_c(X)$ for all $f \in L_c(Y)$, then $\tau$ is an $L_\tau$-mapping of $(X,d_1)$ into $(Y,d_2)$.

Proof: (1) Take any compact subset $A$ of $X$. Then we have two constants $K_1$ and $K_2$ such that $d_2(\tau(x),\tau(x')) \leq K_1 d(x,x')$ and $|f(y) - f(y')| \leq K_2 d_2(y,y')$ for $x,x' \in A$ and $y,y' \in \tau[A]$, where $\tau[A]$ is a compact subset of $Y$ as $\tau$ is continuous and $A$ is compact. Hence, we have

$$|f \circ \tau(x) - f \circ \tau(x')| \leq K_2 d_2(\tau(x),\tau(x')) \leq K_2 K_1 d_1(x,x')$$

for any $x,x' \in A$. Therefore, $f \circ \tau \in L_c(X)$.

(2) Consider any compact subset $A \neq \emptyset$ of $X$. We will show that $\tau$ is an $L$-mapping on $A$. We know from Theorem (3.8) [7] that $\tau$ is continuous. Hence $\tau[A]$ is compact. Let $\phi$ be a mapping from $L_c(Y)$ to $L_c(X)$ defined by $\phi(f) = f \circ \tau$ for all $f \in L_c(Y)$. Then, it is obvious that $\phi$ is a
homomorphism of \( L_c(Y) \) into \( L_c(X) \). We restrict \( \phi \) to 
\[ L_c(Y)|\tau[A] = \{ f|\tau[A] : f \in L_c(Y) \} \], then \( \phi \) is into \( L_c(X)|A = \{ g|A : g \in L_c(X) \} \). Since \( A \) is compact, \( L_c(X)|A \subset L(A) \).
Also for each \( f_0 \in L(A) \), by (11.4), there is an \( f \in L(X) \subset L_c(X) \) such that \( f|A = f_0 \). Thus \( L(A) \subset L_c(X)|A \) so that \( L(A) = L_c(X)|A \). Similarly, we have \( L_c(Y)|\tau[A] = L(\tau[A]) \).
By Lemma (14.3), \( \tau \) is an \( L \)-mapping on \( A \). Since \( A \) is arbitrary, \( \tau \) is then an \( L_c \)-mapping. Q.E.D.

We will investigate the relation between arbitrary \( L_c \)-mappings from \((X,d_1)\) into \((Y,d_2)\) and the ring homomorphisms of \( L_c(Y) \) to \( L_c(X) \). We shall see that every homomorphism from a ring \( L_c(X) \) into another \( L_c(Y) \), in the same sense, is induced by an \( L_c \)-mapping (see (6.1) and (14.13)).

Let \( \tau : X \rightarrow Y \) be an \( L_c \)-mapping. Then, the induced mapping \( \tau' \), defined by \( \tau'(g) = g \circ \tau \in L_c(X) \) for each \( g \in L_c(Y) \) is a homomorphism from \( L_c(Y) \) into \( L_c(X) \). It carries the constant functions onto constant functions identically. Moreover, it also determines the mapping \( \tau \) uniquely. The proofs are similar to (6.7) and (6.8) respectively.

(14.5) Theorem: Let \( \tau \) be an \( L_c \)-mapping from \((X,d_1)\) to \((Y,d_2)\) and \( \tau' \) be the induced homomorphism \( g \rightarrow g \circ \tau \) from \( L_c(Y) \) into \( L_c(X) \).

(1) \( \tau' \) is an isomorphism (into) iff \( \tau[X] \) is dense in \( Y \).

(2) \( \tau' \) is onto iff \( \tau \) is an \( L_c \)-homeomorphism of \( X \) onto an \( L_c \)-embedded subset of \( Y \), i.e. for each \( f \in L_c(\tau[X]) \), there is an \( f_0 \in L_c(Y) \) with \( f_0|\tau[X] = f \).

Proof is similar to (6.10).
We now examine the inverse problem of determining when a given homomorphism of \( L_c(Y) \) into \( L_c(X) \) is induced by some \( L_c \)-mapping from \( X \) into \( Y \). We will first consider the homomorphisms from \( L_c(Y) \) into \( \mathbb{R} \), that is, the case in which \( X \) is a single point.

(14.6) Proposition: Every nonzero homomorphism \( \varphi \) from \( L_c(Y) \) into \( \mathbb{R} \) is onto \( \mathbb{R} \). In fact, \( \varphi(r) = r \) for all \( r \in \mathbb{R} \). Proof is similar to (6.13).

(14.7) Proposition: The correspondence between the homomorphisms of \( L_c(Y) \) onto \( \mathbb{R} \), and the real maximal ideals is one-one. Proof is similar to (6.14).

(14.8) Proposition: \( Y \) is \( L_c \)-realcompact iff to each homomorphism \( \varphi \) from \( L_c(Y) \) onto \( \mathbb{R} \) - i.e. each nonzero homomorphism into \( \mathbb{R} \) - there corresponds a unique point \( y \) of \( Y \) such that \( \varphi(g) = g(y) \) for all \( g \in L_c(Y) \). Proof is similar to (6.15).

(14.9) Theorem: Let \( \varphi \) be a homomorphism from \( L_c(Y) \) into \( L_c(X) \) such that \( \varphi(u) = u \). If \( Y \) is \( L_c \)-realcompact, then there exists a unique \( L_c \)-mapping \( \tau \) of \( X \) into \( Y \) such that \( \tau' = \varphi \). Proof is similar to (6.16).

(14.10) Corollary: An \( L_c \)-realcompact metric space \( (Y,d_2) \) contains an image of an \( L_c \)-mapping of \( (X,d_1) \) iff \( L_c(X) \) contains a homomorphic image of \( L_c(Y) \) that includes the constant functions on \( X \).
Proof is similar to (6.17).

(14.11) Corollary: An Lc-realcompact space, \((Y,d_2)\) contains a dense image of an Lc-mapping of \((X,d_1)\) iff \(L_c(X)\) contains an isomorphic image of \(L_c(Y)\) that includes the constant functions on \(X\).

Proof follows immediately from (14.5) and (14.9).

(14.12) The Main Theorem: Two Lc-realcompact metric spaces \((X,d_1)\) and \((Y,d_2)\) are Lc-homeomorphic iff \(L_c(Y)\) and \(L_c(X)\) are isomorphic.

Proof is similar to (6.19).

Note that, in particular, that if we let \((X,d_1)\) and \((Y,d_1)\) be compact metric spaces, then they are Lc-homeomorphic iff \(L_c(X) = L(X)\) and \(L_c(Y) = L(Y)\) are isomorphic.

D.R. Sherbert has a similar result. (See Theorem 5.1 [26].)

In spite of the Theorem (14.9), every homomorphism is induced, in essence, by an Lc-mapping.

(14.13) Theorem: For any metric space \((X,d_1)\) and an Lc-realcompact metric space \((Y,d_2)\), let \(\varphi\) be a homomorphism from \(L_c(Y)\) into \(L_c(X)\). Then the set \(E = \{x \in X : \varphi(u)(x) = 1\}\) is open-and-closed in \(X\). Moreover, there exists a unique Lc-mapping \(\tau\) from \(E\) into \(Y\), such that for all \(g \in L_c(Y)\), \(\varphi(g)(x) = g(\tau(x))\) for all \(x \in E\), and \(\varphi(g)(x) = 0\) for all \(x \in X - E\).

Proof is similar to (6.20).
99.

(14.14) Theorem: Let \((Y,d_2)\) be a compact space and \(\varphi\) be a homomorphism from \(L^*_c(Y) = L(Y)\) into \(L^*_c(X)\). Then the set \(E = \{p \in X : \varphi(u)(p) = 1\}\) is open-and-closed in \(X\). Moreover, there exists a unique \(L^*_c\)-mapping from \(E\) into \(Y\), such that for any \(g \in L(Y)\)
\[
\varphi(g)(x) = g(\varphi(x)) \quad \text{for all } x \in E, \quad \text{and}
\]
\[
\varphi(g)(x) = 0 \quad \text{for all } x \in X - E.
\]

Proof is similar to (6.20).

(14.15) Corollary: Let \(\varphi\) be a homomorphism from \(L_c(Y)\) into a ring of \(L^*_c\)-functions. If \(Y\) is \(L^*_c\)-realcompact, then there exists a unique closed set \(F\) in \(Y\) such that the kernel of \(\varphi\) is the \(z\)-ideal of all \(L^*_c\)-functions that vanish on \(F\).

Proof is similar to (6.22).

(14.16) Proposition: An \(L^*_c\)-realcompact space \((Y,d_2)\) contains an \(L^*_c\)-embedded image of an \(L^*_c\)-mapping of \((X,d_1)\) iff \(L^*_c(X)\) is a homomorphic image of \(L_c(Y)\).

Proof is similar to (6.23).

(14.17) Proposition: A compact space \((Y,d_2)\) contains an \(L^*_c\)-embedded image of an \(L^*_c\)-mapping of \((X,d_1)\) iff \(L^*_c(X)\) is a homomorphic image of \(L^*_c(Y) = L(Y)\).

Proof is similar to (6.23).

§15 Embedding Theorems

The definitions (7.1) to (7.6) are applicable in this section.
(15.1) Theorem: Let \((Y,d_2)\) be an \(\mathcal{L}_c\)-realcompact metric space, \((X,d_1)\) be any metric space, and \(\varphi\) a homomorphism from \(\mathcal{L}_c(Y)\) into \(\mathcal{L}_c(X)\). Then, \(\varphi\) has the property (7-1) iff there is a homeomorphism \(\tau\) from \(X\) into \(Y\) such that \(\varphi(f) = f \circ \tau\) for all \(f \in \mathcal{L}_c(Y)\) and \(\tau\) is an \(\mathcal{L}_c\)-mapping. In addition, if \(\varphi(\mathcal{L}_c(Y)) = \mathcal{L}_c(X)\), then \(\tau\) is an \(\mathcal{L}_c\)-homeomorphism.

Proof: The proof of the first part is the same as Theorem (7.7). Now, let \(\varphi^o\) be a mapping from \(\varphi(\mathcal{L}_c(Y)) = \mathcal{L}_c(X)\) into \(\mathcal{L}_c(Y)\), defined by \(\varphi^o(f) = f \circ \tau^{-1}\). Since each \(f \in \varphi(\mathcal{L}_c(Y))\) has the form \(f = f_o \circ \tau\) for some \(f_o \in \mathcal{L}_c(Y)\), we have \(\varphi^o(f) = (f_o \circ \tau) \circ \tau^{-1} = f_o\). We then show that \(\tau^{-1}\) is an \(\mathcal{L}_c\)-mapping, by Prop. (14.4).

Q.E.D.

(15.2) Theorem: Let \((Y,d_2)\) be a metric space, \((X,d_1)\) be an \(\mathcal{L}_c\)-realcompact metric space. Then the following statements concerning a homomorphism \(\varphi\) from \(\mathcal{L}_c(Y)\) into \(\mathcal{L}_c(X)\) are equivalent.

(1) \(\varphi\) has the property (7-1).

(2) \(\varphi\) is a \(\delta\)-homomorphism.

(3) The image of \(\mathcal{L}_c(Y)\) separates points and closed sets and it is contained in no \(\delta\)-real ideal of \(\mathcal{L}_c(X)\).

Proof: The proof is similar to Theorem (7.8).

(15.3) Theorem: Let \((X,d_1)\) and \((Y,d_2)\) be two \(\mathcal{L}_c\)-realcompact metric spaces. Then a homomorphism, \(\varphi\), from \(\mathcal{L}_c(Y)\) into \(\mathcal{L}_c(X)\) is a \(\delta\)-homomorphism iff there is a homeomorphism \(\tau\) from \(X\) into \(Y\) such that \(\varphi(f) = f \circ \tau\) for all \(f \in \mathcal{L}_c(X)\), and \(\tau\) is an \(\mathcal{L}_c\)-mapping. In addition, if \(\varphi(\mathcal{L}_c(Y)) = \mathcal{L}_c(X)\) itself,
then $\tau$ is an Lc-homeomorphism.

Proof: The first part is similar to Theorem (7.9). The second part is the same as that in Theorem (15.1).

(15.4) Lemma: Let $(X,d_1)$ and $(Y,d_2)$ be any metric spaces, and $\tau$ be an Lc-mapping from $X$ into $Y$. Define a homomorphism $\varphi$ from $L_c(Y)$ into $L_c(X)$ by $\varphi(f) = f \circ \tau$. Then for any ideals $M_X$ and $M_Y$ of $L_c(X)$ and $L_c(Y)$, respectively, $\varphi[M_Y] \subset M_X$ iff $\tau(x) = y$.

Proof is similar to Lemma (7.10).

(15.5) Theorem: Let $(X,d_1)$ and $(Y,d_2)$ be Lc-realcompact metric spaces. Then a homomorphism, $\varphi$, from $L_c(Y)$ into $L_c(X)$ is a $\delta F$-homomorphism iff there exists a homeomorphism, $\tau$, from $X$ into $Y$ such that $\varphi(f) = f \circ \tau$ for all $f \in L_c(Y)$, $\tau[X]$ is a closed subset of $Y$, and $\tau$ is an Lc-mapping. In addition, if $\varphi[L_c(Y)] = L_c(X)$, then $\tau$ is an Lc-homeomorphism.

Proof is similar to (7.11) and the last part of Theorem (15.1).

(15.6) Theorem: Let $(X,d_1)$ and $(Y,d_2)$ be two Lc-realcompact metric spaces. Then a homomorphism, $\varphi$, from $L_c(Y)$ into $L_c(X)$ is a $\delta G$-homomorphism iff there exists a homeomorphism, $\tau$, from $X$ into $Y$ such that $\varphi(f) = f \circ \tau$ for all $f \in L_c(Y)$, $\tau[X]$ is an open subset of $Y$, and $\tau$ is an Lc-mapping. In addition, if $\varphi$ is onto, then $\tau$ is a homeomorphism.

Proof is similar to (7.12) and the last part of Theorem (15.1).
16 The Rings of Lc-Functions Defined on the Metric Spaces other than Lc-Realcompact Spaces, and Rings of Lipschitzian Functions on Compact Subsets of $E^n$.

(16.1) Lemma: Suppose $p \in X$. Then there exists $f \in L_c(X)$ such that $Z(f) = \{p\}$ and such that $f$ belongs to no maximal ideal other than $M_p$.

Proof: Consider $f(x) = d(p,x)$ for all $x \in X$. We know that $f$ belongs to $M_p$ and not any other fixed maximal ideal. Suppose $f$ belongs to a free maximal ideal $M$. Then by the property of free ideals, we have $g \in M$ such that $g(p) \neq 0$.

Let $h = f^2 + g^2$. We have $Z(h) = \emptyset$. Thus its inverse $1/h$ exists. Take any compact subset $A$ of $X$. For this $A$, there exist two positive numbers $K_A$ and $m$ such that $|h(x) - h(x')| < K_A \cdot d(x,x')$ and $|h(x)| > m$ for all $x,x' \in A$.

Hence $|1/h(x) - 1/h(x')| = \frac{|h(x) - h(x')|}{|h(x) \cdot h(x')|} \leq \frac{1}{m^2} |h(x) - h(x')| \leq \frac{K_A}{m^2} d(x,x')$. Therefore $1/h \in L_c(X)$. Since $M$ is an ideal, we would have $u = h \cdot 1/h \in M$. That is, $M = L_c(X)$.

This is impossible. Q.E.D.

(16.2) Lemma: If $\varphi$ is an isomorphism from $L_c(Y)$ onto $L_c(X)$, then for $M_q \in L_c(Y)$, and $\varphi(M_q) = M_p$ for some $p \in X$, that is, $\varphi(M_q)$ is a fixed maximal ideal in $L_c(X)$.

Proof: Since $\varphi$ is an isomorphism onto, $\varphi(M_q)$ is a maximal ideal. By Lemma (16.1) $f_q(y) = d(q,y)$, $f_q \in M_q$ only. Consider $Z(\varphi(f_q))$. If $Z(\varphi(f_q)) = \emptyset$, then as shown in the last lemma $\varphi(f_q)$ is a unit so that $\varphi(M_q)$ is the whole ring $L_c(X)$ which is impossible. Hence $Z(\varphi(f_q)) \neq \emptyset$. On the other hand,
if \( Z(\varphi(f_q)) \) contains more than one point, say \( p_1 \) and \( p_2 \), then \( \varphi(f_q) \in M_{p_1} \) and \( M_{p_2} \) so that \( f_q \) would belong to at least two maximal ideals which again is impossible for \( f_q \) belongs to only one maximal ideal. Hence \( Z(\varphi(f_q)) = \{p\} \), say. Therefore \( \varphi(M_q) = M_p \). Q.E.D.

(16.3) Theorem: Let \((X, d_1)\) and \((Y, d_2)\) be any two metric spaces, and \( \varphi \) be an isomorphism from \( L_c(Y) \) onto \( L_c(X) \) leaving all constant functions unchanged. Then \( \varphi \) induces a mapping \( \tau: X \to Y \) defined by \( \varphi(g) = g \circ \tau \) and \( \tau \) is an Lc-homeomorphism. Conversely, if \( \tau \) is an Lc-homeomorphism of \( X \) onto \( Y \), then the induced mapping \( \varphi \), defined by \( \varphi(g) = g \circ \tau \), is an isomorphism from \( L_c(Y) \) onto \( L_c(X) \).

Proof is similar to (9.4) and the converse is obvious.

(16.4) Corollary: Let \((X, d_1)\) and \((Y, d_2)\) be two connected metric spaces and \( \varphi \) be an isomorphism of \( L_c(Y) \) onto \( L_c(X) \). Then \( \varphi \) induces a Lc-homeomorphism, \( \tau \), from \( X \) on \( Y \) such that \( \varphi(g) = g \circ \tau \) for each \( g \in L_c(Y) \).

Proof is similar to (9.5).

Now we discuss the similar properties as we have done in §9 for \( C^m(X) \). Hereafter, we consider \( X \) and \( Y \) as compact subsets of \( E^n \).

(16.5) Lemma: The projection functions \( \iota(x_1, \ldots, x_n) = x_i \) for \( 1 \leq i \leq n \) belong to \( L(X) \).

Proof: We know that \( |\iota(x) - \iota(x')| = |x_i - x_i'| \leq d(x, x') \) for \( 1 \leq i \leq n \), and \( \iota(X) \) is bounded. Hence \( \iota \in L(X) \) for
Proposition: Let \( L' \) be a subfamily of \( L(X) \), and \( \tau \) be a mapping from \( X \) into \( Y \).

1. If \( \tau \) is an \( L \)-mapping, then \( f \circ \tau \in L(X) \) for all \( f \in L(Y) \).

2. If \( L' \) contains all projections and \( f \circ \tau \in L(X) \) for each \( f \in \mathcal{L} \), then \( \tau \) is an \( L \)-mapping from \( X \) into \( Y \).

Proof: (1) is obvious.

(2) For any \( x, x' \in X \), and any \( f \in L' \), \( |f \circ \tau(x) - f \circ \tau(x')| \leq K_{f \circ \tau} \cdot d(x, x') \), where \( K_{f \circ \tau} = \|f \circ \tau\|_d \) . In particular, if \( f = \varphi_i \) for \( 1 \leq i \leq n \), then \( |(\tau(x))_i - (\tau(x'))_i| \leq K_{\varphi_i \circ \tau} \cdot d(x, x') \), where \( (\tau(x))_i \) denotes the \( i \)-th coordinate of \( \tau(x) \), for \( 1 \leq i \leq n \). Hence

\[
d(\tau(x), \tau(x')) = \left( \sum_{i=1}^{n} |(\tau(x))_i - (\tau(x'))_i|^2 \right)^{1/2} \leq \left( \sum_{i=1}^{n} K_{\varphi_i \circ \tau}^2 \right)^{1/2} d(x, x').
\]

Let \( K = \left( \sum_{i=1}^{n} K_{\varphi_i \circ \tau}^2 \right)^{1/2} \). Then \( K \) is independent of \( x \) and \( x' \) as each \( K_{\varphi_i \circ \tau} \) is. That is, \( d(\tau(x), \tau(x')) \leq K \cdot d(x, x') \) for all \( x \) and \( x' \in X \). Hence \( \tau \) is an \( L \)-mapping. Q.E.D.

A subring \( B(X) \) of \( L(X) \) is said to have the property (16-1) if \( B(X) \) is a subring of \( L(X) \), \( B(X) \supset \mathcal{R}(X) \) and \( f \in B(X) \) with \( Z(f) = \emptyset \) implies \( f^{-1}(= 1/f) \in B(X) \), where \( \mathcal{R}(X) \) is defined as in §9.

We know that such a \( B(X) \) exists. For instance, let \( \mathcal{B}_0(X) = \{f \in L(X) : f \in C^3(X)\} \). Then, it is obvious that \( \mathcal{R}(X) \subset \mathcal{B}_0(X) \subset L(X) \) and \( f \in \mathcal{B}_0(X) \) with \( Z(f) = \emptyset \) implies \( f^{-1} \in \mathcal{B}_0(X) \).
(16.7) Lemma: There is a function \( f \in M_x = \{ f \in B(X) : f(x) = 0 \} \) such that \( Z(f) = \{ x \} \) and \( f \) belongs to no other free or fixed maximal ideal.

Proof is similar to (9.1).

(16.8) Lemma: If \( \varphi \) is an isomorphism from \( B(Y) \) onto \( B(X) \), then for \( M_q \subset B(Y) \), \( \varphi(M_q) \) is a fixed maximal ideal in \( B(X) \), where \( B(X) \) and \( B(Y) \) have property (16-1).

Proof is similar to (9.2).

(16.9) Theorem: Let \( X \) and \( Y \) be two compact subsets of \( \mathbb{E}^n \), and \( \varphi \) be an isomorphism from \( B(Y) \) onto \( B(X) \) leaving the constant functions unchanged, where \( B(X) \) and \( B(Y) \) have the property (16-1). Then \( \varphi \) induces an \( L \)-homeomorphism, \( \tau \), from \( X \) onto \( Y \), defined by \( \varphi(g) = g \circ \tau \).

Proof is similar to (9.4).

(16.10) Corollary: Let \( X \) and \( Y \) be connected compact subsets of \( \mathbb{E}^n \), and \( \varphi \) be an isomorphism from \( B(Y) \) onto \( B(X) \), where \( B(X) \) and \( B(Y) \) have the property (16-1). Then \( \varphi \) induced an \( L \)-homeomorphism, \( \tau \), from \( X \) onto \( Y \), defined by \( \varphi(g) = g \circ \tau \).

Proof is similar to (9.5).
PART III

THE RINGS OF ANALYTIC FUNCTIONS

§17 Rings of Analytic Function on any Subset of the Complex Plane

All the functions considered in this section are complex single-valued. \( \mathbb{C} \) will denote the complex plane.

(17.1) Definition: Let \( G \) be an open subset of \( \mathbb{C} \). A function \( f \) on \( G \) is said to be an analytic function on \( G \), if for each \( p \in G \), there is a power series \( \sum_{n=0}^{\infty} a_n(z-p)^n \) which converges on \( D = \{ z : |z-p| < R \} \) and \( f(z) = \sum_{n=0}^{\infty} a_n(z-p)^n \) for \( z \in D \), where \( R > 0 \) and \( D \subset G \), and \( a_n \) is a complex number for each \( n = 0,1,2,\ldots \).

(17.2) Definition: Let \( X \) be an arbitrary subset of \( \mathbb{C} \). A function \( f \) on \( X \) is said to be an analytic function on \( X \), if for each \( p \in X \), there is a power series \( \sum_{n=0}^{\infty} a_n(z-p)^n \) which converges for \( |z-p| < R \), and \( f(z) = \sum_{n=0}^{\infty} a_n(z-p)^n \) for all \( z \in X \) and \( |z-p| < R \), where \( R > 0 \).

(17.3) Definition: Let \( X \) and \( Y \) be two arbitrary subspaces of \( \mathbb{C} \). A mapping \( \tau \) from \( X \) to \( Y \) is said to be an analytic mapping if \( \tau \) is an analytic function on \( X \) and valued in \( Y \). \( \tau \) is said to be a conformal mapping if \( \tau \) is one-one, onto. (See §2 Ch. II [2].)

(17.4) Definition: Let \( X \) be an arbitrary subset of \( \mathbb{C} \), and \( \mathcal{O}(x) = \{ f : f \text{ is an analytic function on } X \} \).
Let $f$ and $g$ be any two elements from $\mathcal{O}(X)$, and $f + g$, and $f \cdot g$ be the pointwise sum, difference, and product of $f$ and $g$, respectively. Then, for each $p \in X$ we have power series $\sum_{n=0}^{\infty} a_n(z - p)^n$ and $\sum_{n=0}^{\infty} \beta_n(z - p)^n$ converging on $|z - p| < R_1$, and $|z - p| < R_2$, respectively, and $f(z) = \sum_{n=0}^{\infty} \alpha_n(z - p)^n$, $g(z) = \sum_{n=0}^{\infty} \beta_n(z - p)^n$ for $z \in X$ and $|z - p| < R_1$, $|z - p| < R_2$, respectively, where $R_1 > 0$ and $R_2 > 0$.

Thus $\sum_{n=0}^{\infty} (\alpha_n + \beta_n)(z - p)^n = \sum_{n=0}^{\infty} \alpha_n(z - p)^n + \sum_{n=0}^{\infty} \beta_n(z - p)^n$, and $\sum_{n=0}^{\infty} (\sum_{k=0}^{\infty} \alpha_k \beta_{n-k})(z - p)^n = (\sum_{n=0}^{\infty} \alpha_n(z - p)^n)(\sum_{n=0}^{\infty} \beta_n(z - p)^n)$ converges for $|z - p| < R$ and $(f + g)(z) = f(z) + g(z) = \sum_{n=0}^{\infty} (\alpha_n + \beta_n)(z - p)^n$ and $f \cdot g(z) = f(z) \cdot g(z) = \sum_{n=0}^{\infty} (\sum_{k=0}^{\infty} \alpha_k \beta_{n-k})(z - p)^n$ for $z \in X$ and $|z - p| < R$, where $R = \min\{R_1, R_2\}$.

Hence $f + g$ and $f \cdot g$ are analytic at $p$. Since $p$ is arbitrary, $f + g$ and $fg \in \mathcal{O}(X)$. Suppose $f \in \mathcal{O}(X)$ such that $Z(f) = \{z \in X : f(z) = 0\} = \emptyset$. For $p \in X$, there exists a power series $\sum_{n=0}^{\infty} \alpha_n(z - p)^n$ which converges on $|z - p| < R$, $R > 0$ and $f(z) = \sum_{n=0}^{\infty} \alpha_n(z - p)^n$ for $z \in X$ and $|z - p| < R$.

It is clear that $\alpha_0 \neq 0$. Then, that $\frac{1}{\sum_{n=0}^{\infty} \alpha_n(z - p)^n}$ also has a power series which converges at least on $|z - p| < R$, (See P. 145 [1]), and $\frac{1}{f(z)} = (\sum_{n=0}^{\infty} \alpha_n(z - p)^n)^{-1}$ for $z \in X$.

$|z - p| < R$ is clear. Hence $1/f \in \mathcal{O}(X)$. Moreover, $\Theta$, and
u, the constant function with values 0, and 1, respectively, are two entire functions on the complex plane, so that \( \Theta, u \in \mathcal{O}(X) \). The operations of addition and multiplication thus defined are associative and commutative, and the distributive laws hold. Hence we have the following theorem:

(17.5) Theorem: The family \( \mathcal{O}(X) \), defined in (17.4), forms a commutative ring with unity.

(17.6) Lemma: For \( p \in X \), there is an \( f \in M_p = \{ f \in \mathcal{O}(X) : f(p) = 0 \} \) such that \( Z(f) = \{ p \} \), and \( f \) belongs to no maximal ideal other than \( M_p \).

Proof: Let \( f(z) = z - p \). Then that \( f \in M_p \) and \( f \) belongs to no other fixed maximal ideal is clear. Now, suppose that \( M \) is a free maximal ideal such that \( f \in M \). Since \( M \) is free, there is \( g \in M \in \mathcal{O}(X) \) such that \( g(p) \neq 0 \). We know that \( g \) is analytic at \( p \) so there is a power series

\[
\sum_{n=0}^{\infty} a_n (z - p)^n
\]

which converges for \( |z - p| < R \), and

\[
g(z) = \sum_{n=0}^{\infty} a_n (z - p)^n \quad \text{for } z \in X \text{ and } |z - p| < R,
\]

where \( R > 0 \). It is clear that \( a_0 \neq 0 \). Let \( k \) be the first positive integer such that \( a_k \neq 0 \). Then \( g(z) = a_0 + a_k (z - p)^k + \ldots \), for \( z \in X \) and \( |z - p| < R \). Let \( h(z) = \frac{g(z) - a_0}{(z - p)^k} \). Then for each \( p' \in X \), \( p' \neq p \), there is a power series

\[
\sum_{n=0}^{\infty} \beta_n (z - p')^n
\]

which converges for \( |z - p'| < R_1 \), \( R_1 > 0 \), and \( g(z) = \sum_{n=0}^{\infty} \beta_n (z - p')^n \) for \( z \in X \) and \( |z - p'| < R_1 \).
Thus, \[ g(z) = \frac{\sum \beta_n(z-p')^n - a_0}{(z-p)^k} = \frac{\sum \beta'_n(z-p')^n}{[(p'-p)+(z-p')]^k} \]

for \( z \in X \) and \( |z - p'| < R_1 \) where \( \beta'_0 = \beta_0 - a_0 \) and \( \beta'_1 = \beta_1 \), \( i = 1, 2, \ldots \), will be a power series, say \( \sum \gamma_n(z-p')^n \) which converges for \( |z - p'| < R_1 \), and evidently \( h(z) = \sum \gamma_n(z-p')^n \)

for \( z \in X \) and \( |z - p'| < R_1 \). Suppose \( p' = p \). Then the power series \( \sum a_{k+n}(z-p)^n \) converges for \( |z - p| < R \), and \( h(z) = \sum a_{k+n}(z-p)^n \) for \( z \in X \) and \( |z - p| < R \).

Hence \( h \in \mathcal{O}(X) \). Now, \( g(z) = a_0 + (z-p)^kh(z) \), so that \( a_0 = g(z) - (z-p)^{k-1}f_h(z) = g(z) - (z-p)^{k-1}f(z)h(z) \).

Since \( f, g \in M \) which is an ideal, \( a_0 \in M \). However, it is impossible, as \( a_0 \neq 0 \) is a unit. Hence the assertion is proved.

(17.7) Lemma: If \( \varphi \) is an isomorphism from \( \mathcal{O}(X) \) onto \( \mathcal{O}(Y) \), then \( \varphi(M_p) \) is a fixed maximal ideal.

Proof: That \( \varphi(M_p) \) is a maximal ideal is clear. From Lemma (17.6), there is an \( f_o \in M_p \) such that \( Z(f_o) = \{p\} \), and \( f_o \) belongs to no other maximal ideal. Consider \( Z(\varphi(f_o)) \). If \( Z(\varphi(f_o)) = \emptyset \), then \( \varphi(f_o) \) is a unit so that \( \varphi(M_p) \) is the whole ring, \( \mathcal{O}(X) \). This is impossible. Hence \( Z(\varphi(f_o)) \neq \emptyset \). But if \( Z(\varphi(f_o)) \) contains more than one point, say \( q_1 \) and \( q_2 \), then \( \varphi(f_o) \in M_{q_1} \) and \( M_{q_2} \) so that \( f_o \) would belong to at least two maximal ideals which is again impossible, for \( f_o \).
belongs to only one maximal ideal. Hence \( Z(\varphi(f_0)) = \{q\} \), \( q \in Y \). Now, since \( \varphi(M_p) \) is a maximal ideal, \( \varphi(f_0) \in M_q \), and \( \varphi(f_0) \) belongs to only one maximal \( \varphi(M_p) = M_q \). Hence \( \varphi(M_p) \) is a fixed maximal ideal. Q.E.D.

(17.8) Theorem: Let \( X \) and \( Y \) be two subsets of \( \mathcal{C} \), and \( \varphi \) be an isomorphism from \( \mathcal{O}(Y) \) onto \( \mathcal{O}(X) \) such that it is the identity on the constant functions. Then \( \varphi \) induces a mapping \( \tau : X \to Y \), defined by \( \varphi(g) = g \circ \tau \), and \( \tau \) is a conformal mapping of \( X \) onto \( Y \).

Proof: Define \( \tau \) to be a mapping from \( X \) to \( Y \) as follows:
\[
\tau(p) = \cap Z[\varphi^{-1}(M_p)] .
\]
Since \( \varphi \) is an isomorphism, and onto, its inverse mapping \( \varphi^{-1} \) is an isomorphism of \( \mathcal{O}(X) \) onto \( \mathcal{O}(Y) \). By Lemma (17.7), \( \varphi^{-1}(M_p) \) is a fixed maximal ideal in \( \mathcal{O}(Y) \). Thus \( \tau \) is a single-valued mapping. Evidently,
\[
M_\tau(p) = \varphi^{-1}(M_p) .
\]
Let \( p \) and \( p' \) be in \( X \) and \( p \neq p' \).

Then, by Lemma (17.7), \( \varphi^{-1}(M_p) = M_q \), and \( \varphi^{-1}(M_{p'}) = M_q' \), for some \( q, q' \in Y \). If \( q = q' \), then \( \varphi^{-1}(M_p) = M_q = \varphi^{-1}(M_{p'}) \) or \( M_p = M_{p'} \). This is impossible for \( p \neq p' \).

Thus, \( q \neq q' \) and so \( \tau(p) = q \neq q' = \tau(p') \). Hence \( \tau \) is one-one. Let \( q_o \) be arbitrary in \( Y \). Then \( M_{q_o} \) is a maximal ideal in \( \mathcal{O}(X) \), and \( \varphi(M_{q_o}) = M_{p_o} \) for some \( p_o \in X \). Thus
\[
q_o = \cap Z[\varphi^{-1}(M_{p_o})] = \tau(p_o) .
\]
This shows that \( \tau \) is onto. Now, for each \( g \in \mathcal{O}(Y) \), and \( p \in X \), let \( \varphi(g)(p) = \alpha \), and \( \alpha \) be the corresponding constant function on \( X \). Then, \( \varphi(g) - \alpha \in M_p \), \( g - \varphi^{-1}(\alpha) = \varphi^{-1}(\varphi(g) - \alpha) \in M_{\tau(p)}, g(\tau(p)) = \varphi^{-1}(\alpha)(\tau(p)) = \alpha(\tau(p)) = \alpha = \varphi(g)(p) \). Hence \( \varphi(g) = g \circ \tau \). Similarly,
\[ \varphi^{-1}(f) = f \circ \tau^{-1}, \text{ where, } \tau^{-1} : Y \to X \text{ is defined by } \tau^{-1}(q) = \cap \varphi(M_q) \]. If we choose \( g(w) = w \) on \( Y \), and \( f(z) = z \) on \( X \), then \( \tau(p) = g \circ \tau(p) \) and \( \tau^{-1}(q) = f \circ \tau^{-1}(q) \) are analytic. Hence \( \tau \) is a conformal mapping. 

Q.E.D.

(17.9) Corollary: Let \( X \) and \( Y \) be two connected subsets of \( C \), and \( \varphi \) be an isomorphism from \( \mathcal{U}(Y) \) onto \( \mathcal{U}(X) \) such that it is the identity on the real constant functions. Then \( \varphi \) induces a mapping \( \tau : X \to Y \). Either \( \tau \) or its conjugate mapping \( \overline{\tau} \) is a conformal mapping according as \( \varphi(g) = g \circ \tau \) or \( \varphi(g) = g \circ \overline{\tau} \).

Proof: By Theorem (17.8), the mapping \( \tau \) defined by \( \tau(p) = \cap \varphi^{-1}(M_p) \), is one-one and onto. Let \( \alpha \) be the constant function of value \( \alpha \). We assume that \( \varphi(r) = r \) if \( r \) is real and \( (\varphi(i))^2 = \varphi(-1) = -1 \), \( \varphi(i) = \pm i \) as \( X \) is connected. Hence, \( \varphi(\alpha) = \alpha \) or \( \overline{\alpha} \). For each \( g \in \mathcal{U}(X) \) and \( p \in X \), let \( \varphi(g)(p) = \alpha_0 \). Then \( \varphi(g) - \alpha_0 \in M_p \), \( g - \varphi^{-1}(\alpha_0) = \varphi^{-1}(\varphi(g) - \alpha_0) \in M_\tau(p) \), or \( g(\tau(p)) = \varphi^{-1}(\alpha_0)(\tau(p)) = \alpha_0 \) or \( \overline{\alpha_0} \) according as \( \varphi(\alpha_0) = \alpha_0 \) or \( \overline{\alpha_0} \). Thus \( \varphi(g) = g \circ \tau \) or \( \overline{g \circ \tau} \) according as \( \varphi(\alpha) = \alpha \) or \( \overline{\alpha} \). Similarly \( \varphi^{-1}(f) = f \circ \tau^{-1} \) or \( \varphi^{-1}(f) = \overline{f \circ \tau^{-1}} \) according as \( \varphi(\alpha) = \alpha \) or \( \overline{\alpha} \), where \( \tau^{-1} \) is defined by \( \tau^{-1}(q) = \cap \varphi(M_q) \). In particular, we choose \( g(w) = w \) on \( Y \), and \( f(z) = z \) on \( X \), then \( \tau(p) = g \circ \tau(p) \) and \( \tau^{-1}(q) = f \circ \tau^{-1}(q) \); \( \overline{\tau(p)} = g \circ \overline{\tau(p)} \) and \( \overline{\tau^{-1}(q)} = f \circ \overline{\tau^{-1}(q)} \) are analytic. To show that \( \tau \) and \( \overline{\tau} \) are one-one is similar to what we have shown in (17.8). Hence \( \tau \) or \( \overline{\tau} \) is a conformal mapping. 

Q.E.D.
Remarks: (1) In Theorem (17.8), the condition that \( \varphi \) is the identity on the constant functions cannot be omitted. We know that an isomorphism of \( \mathcal{A}(Y) \) onto \( \mathcal{A}(X) \) always leaves the constant functions with rational values unchanged. Without this restriction on \( \varphi \), the result of the Theorem will not be true. For example, consider \( X = \{p\}, Y = \{q\} \). Then \( \mathcal{A}(X) = \{a : a \in \mathbb{C}\} \) and \( \mathcal{A}(Y) = \{a' : a' \in \mathbb{C}\} \). In order to establish the required result, let us first prove the following Lemma:

(17.10) Lemma: There exists a non-zero automorphism of \( \mathbb{C} \) onto itself which is different from the mappings \( \varphi_1 : z \to z \), the identity, and \( \varphi_2 : z \to \bar{z} \), the conjugate mapping.

Proof: Let \( \mathbb{Q} \) and \( \mathbb{R} \) be the rational and real fields, respectively. Consider \( \mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subset \ldots \subset \mathbb{R} = \mathbb{C} \), where \( \mathbb{Q}(\sqrt{2}) \) means the extension field of \( \mathbb{Q} \) by the adjunction of \( \sqrt{2} \), and so on. Let \( \varphi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2}) \) be defined by \( \varphi(\sqrt{2}) = -\sqrt{2} \) and \( \varphi(r) = r \) if \( r \in \mathbb{Q} \). Then \( \varphi \) is an isomorphism of \( \mathbb{Q}(\sqrt{2}) \) onto itself. \( \varphi \) can be extended to an isomorphism, \( \varphi' \) of \( \mathbb{Q}(\sqrt{2}, \sqrt{3}) \) onto itself (say \( a + b\sqrt{2} + c\sqrt{3} \to a - b\sqrt{2} + c\sqrt{3} \), or \( a - b\sqrt{2} - c\sqrt{3} \), where \( a, b \) and \( c \in \mathbb{Q} \) ) which is not the identity on the real numbers. And so on. We have a linearly ordered set of fields and a linearly ordered set of isomorphisms. By Zorn's Lemma [10], there exists a maximal field, that is \( \mathbb{C} \), and a maximal isomorphism \( \Phi \) from \( \mathbb{C} \) onto itself. \( \Phi \) is then not the identity on the real field. Hence \( \Phi \) is neither \( \varphi_1 \) nor \( \varphi_2 \) . Q.E.D.

Now, define \( \varphi : \mathcal{A}(X) \to \mathcal{A}(Y) \) in the obvious way. Then \( \varphi(a') = a' \) for some \( a' \in \mathcal{A}(Y) \). On the other hand,
\(a' \circ \tau = a',\) where \(\tau: X \to Y\) is defined by \(\tau(p) = q.\) Hence \(\varphi(a') \neq a' \circ \tau.\)

However, L. Bers showed in 1948 that if \(X\) and \(Y\) are two domains in \(\mathbb{C}\) and, if \(\mathcal{O}(X)\) and \(\mathcal{O}(Y)\) are isomorphic, then there exists either a conformal or an anti-conformal mapping which maps \(X\) onto \(Y\) (Theorem 1, [3]).

(2) In Corollary (17.9), neither of the conditions "\(\varphi\) leaves all real constant functions unchanged" and "\(X\) is connected" can be omitted. The Remark (1) above shows that the result is not valid if \(X\) is connected and \(\varphi\) does not leave all the real constant functions unchanged. Now, consider \(X = \{p_1, p_2\}\) and \(Y = \{q_1, q_2\}\) where \(p_1 \neq p_2\), and \(q_1 \neq q_2.\)

Then \(\mathcal{O}(X) = \{(\alpha, \beta) : \alpha, \beta \in \mathbb{C}\}\) and \(\mathcal{O}(Y) = \{(\alpha', \beta') : \alpha', \beta' \in \mathbb{C}\}\).

Let \(\varphi: \mathcal{O}(Y) \to \mathcal{O}(X)\) be defined as follows: \(\varphi((\alpha, \beta)) = (\alpha, \overline{\beta}).\) That \(\varphi\) is an isomorphism and leaves all the real constant functions unchanged is clear. Then \(\tau(p_1) = \cap(\mathbb{Z}[\varphi^{-1}(M_{p_1})]) = q_1.\) Choose \(g = (\alpha, \beta) \in \mathcal{O}(Y),\) where \(\beta\) is a complex number. Then \(\varphi(g) = (\alpha, \overline{\beta}) \neq g\) where \(g \circ \tau = (\alpha, \beta)\). Hence \(\varphi(g) \neq g \circ \tau.\)

L. Bers has shown that let \(X\) and \(Y\) be two domains possessing boundary points. Then, every isomorphism of \(\mathcal{O}(Y)\) onto \(\mathcal{O}(X)\) induces a conformal or an anti-conformal mapping of \(X\) onto \(Y\) (See Theorem 2 [3]).

(17.11) Theorem: Let \(X\) and \(Y\) be two subsets of \(\mathbb{C},\) and \(\tau\) be a conformal mapping of \(X\) onto \(Y.\) Then the induced mapping \(\tau'\) defined by \(\tau'(g) = g \circ \tau\) is an isomorphism of \(\mathcal{O}(Y)\) onto \(\mathcal{O}(X)\) which leaves the constant functions unchanged.
Proof: Let \( \tau'(g) = g \circ \tau \) for each \( g \in \mathcal{O}(Y) \). For each \( p \in X \), \( \tau(p) \in Y \). Since \( g \in \mathcal{O}(Y) \) and \( \tau(p) \in Y \), there exists a power series \( \sum_{n=0}^{\infty} \beta_n(w - \tau(p))^n \) which converges for

\[
|w - \tau(p)| < R_1 \quad \text{where} \quad R_1 > 0 , \quad \text{and} \quad g(w) = \sum_{n=0}^{\infty} \beta_n(w - \tau(p))^n \]

for \( |w - \tau(p)| < R_1 \) and \( w \in Y \). Since \( \tau \) is analytic on \( X \), there exists a power series \( \sum_{n=0}^{\infty} \gamma_n(z - p)^n \) which converges for \( |z - p| < R_2 \) where \( R_2 > 0 \), and \( \tau(z) = \sum_{n=0}^{\infty} \gamma_n(z - p)^n \)
for \( |z - p| < R_2 \) and \( z \in X \). Therefore \( \sum_{m=0}^{\infty} \beta_m(\sum_{n=0}^{\infty} \gamma_n(z - p)^n - \tau(p))^m \) is convergent and equal to \( g(\tau(z)) \) for each \( z \in X \) such that \( |z - p| < R_2 \), \( \tau(z) \in Y \), and \( |\tau(z) - \tau(p)| < R_1 \).

Moreover, there exists \( R > 0 \), such that \( \sum_{m=0}^{\infty} \beta_m(\sum_{n=0}^{\infty} \gamma_n(z - p)^n - \tau(p))^m \) converges on \( |z - p| < R \), and \( g(\tau(z)) = \sum_{m=0}^{\infty} \beta_m(\sum_{n=0}^{\infty} \gamma_n(z - p)^n - \tau(p))^m \) for \( |z - p| < R \) and \( z \in X \). By the Weierstrass' double-series theorem [11], \( \sum_{m=0}^{\infty} \beta_m(\sum_{n=0}^{\infty} \gamma_n(z - p)^n - \tau(p))^m \) will be a power series in terms of \( (z - p) \), say, \( \sum_{n=0}^{\infty} \alpha_n(z - p)^n \) which converges for each \( z \), \( |z - p| < R \), and \( g(\tau(z)) = \sum_{n=0}^{\infty} \alpha_n(z - p)^n \) for \( |z - p| < R \)
and \( z \in X \). We know that \( p \) is arbitrary in \( X \). Hence \( g \circ \tau \in \mathcal{O}(X) \). Similarly, for each \( f \in \mathcal{O}(X) \), \( f \circ \tau^{-1} \in \mathcal{O}(Y) \). Also \( \tau'(f \circ \tau^{-1}) = (f \circ \tau^{-1})' \tau = f \). Hence \( \tau' \) is onto. Now, if \( \tau'(g) = \Theta \), then \( g \circ \tau(z) = 0 \) for each \( z \in X \) or \( g \circ \tau[X] = g[Y] = \{0\} \) as \( \tau \) is onto. That is, \( g = \Theta \). This shows that \( \tau \)
is one-one. Finally, for any constant function $a$, $\tau'(a)(z) = a(\tau(z)) = a$ for all $z \in X$. That is, $\tau'(a) = a$. Q.E.D.

Let $Z(J(X)) = \{Z(f) : f \in \mathcal{O}(X)\}$, where $Z(f) = \{z \in X : f(z) = 0\}$ as defined before. We know that not every closed set is a zero-set of some analytic function. Hence $Z(J(X))$ is not a base for the closed subset of $X$ in its relative topology if $X$ is not a discrete subset of $\mathbb{C}$. However, we have the following result.

(17.12) Theorem: Let $X$ be a subset of $\mathbb{C}$. $X$ is compact iff every maximal ideal in $\mathcal{O}(X)$ is fixed.

Proof: The necessity is clear.

Sufficiency: Suppose that for each maximal ideal $M$ in $\mathcal{O}(X)$, $M$ is fixed, and $X$ is not compact. Then there exists a sequence of points, say $\{Z_n : n \in \mathbb{N}\} \subset X$ which converges to a point $Z_0 \notin X$. Case I. If $Z_0 = \infty$, then $\{Z_n : n \in \mathbb{N}\}$ has no finite limit point. By the Weierstrass factor Theorem [11], there exists an entire function, $f_n$, which has zeros exactly at $\{Z_k : k \geq n\}$ for each $n$. Clearly $f_n \in \mathcal{O}(X)$. Case II. If $Z_0 \neq \infty$, then consider the conformal mapping $f_0(Z) = \frac{1}{Z-Z_0}$ which is analytic except at the point $Z_0$. Then $f_0$ maps $\{Z_n : n \in \mathbb{N}\}$ into a sequence $\{W_n : n \in \mathbb{N}\}$ in the $w$-plane which has no finite limit. By the Weierstrass factor Theorem [11], there exists an entire function $g$ which has zeros exactly at $\{W_n : n \in \mathbb{N}\}$. Let $h_0 = (g \cdot f_0)|X$. Then $h_0 \in \mathcal{O}(X)$ which has zero exactly at $\{Z_n : n \in \mathbb{N}\}$. Similarly, we can find an $f_n \in \mathcal{O}(X)$ such that
$f_n$ has zeros exactly at $\{Z_k : k \geq n\}$ for each $n$. In either case we have $\{f_n : n \in \mathbb{N}\}$ with $f_n$ has zeros exactly at $\{Z_k : k > n\}$. Let $M_0$ be a maximal ideal which contains $\{f_n : n \in \mathbb{N}\}$. Then $\cap Z[M] \subseteq \bigcap_{n=1}^{\infty} Z(f_n) = \emptyset$. Thus, $M_0$ is free. This is impossible. Hence $X$ must be compact. Q.E.D.
BIBLIOGRAPHY


