ON THE SPACES
OF
THE CONVEX CURVES IN THE PROJECTIVE PLANE

by

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ABSTRACT

Two topologies \((Z,L)\) and \((Z,L_1)\) for the family of the non-degenerate convex curves in the projective plane are considered, where \((Z,L)\) is the topology from the Lane's neighborhood system and \((Z,L_1)\) is the topology from the parabolic neighborhood system. It is shown that the definition of convexity in the affine plane can be extended to the projective plane so that the Blaschke selection theorem remains true for the projective convex sets. With the help of this theorem, the topological space \((Z,L)\) is compactified by adding Lane's compactifying elements. Furthermore, it is shown that \((Z,L)\) is metrizable but \((Z,L_1)\) is not metrizable. The Lane's topology \((X,L)\), as a subspace of \((Z,L)\) for the non-degenerate conics, is both metrizable and separable. A subspace \((X,\tau)\) of \((Z,L_1)\) is studied which is metrizable but not separable.
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INTRODUCTION

Lane defined a neighborhood system for the non-degenerate conics in the affine plane [1]. He showed that the space of non-degenerate conics under this neighborhood system could be compactified by adding certain degenerate conics.

In the present work, Lane's neighborhood system is extended to the family \( Z \) of non-degenerate convex curves in the projective plane. The topology is denoted by \((Z,L)\). In the first part, the enlarged space \((Z,L)\) is shown to be compact if the compactifying elements used by Lane are added to the set \( Z \). This result is obtained by showing that the Blaschke selection theorem holds for the projective convex sets. The boundedness condition which is essential for the Blaschke selection theorem is not needed for the corresponding result in the projective plane.

In the second part it is shown that if the neighborhood system for the projective curves is refined by the addition of parabolic neighborhoods, the resulting topology \((Z,L_1)\) is no longer metrizable.

In the third part, the set \( X \) of all non-degenerate conics in an affine plane is considered. The induced topology \((X,L)\) from \((Z,L)\) is exactly the topology defined by Lane for the non-degenerate conics. \((X,L)\) is both metrizable and separable. The interesting result is that the subspace \((X,T)\) of \((Z,L_1)\) is metrizable but not separable.
CHAPTER 1: The Blaschke Theorem for the Projective Convex Curves and the Compactification of the Space \((Z,L)\).

1.1 The projective plane \(P^2\) is the set of all 3-triples \((x_1,x_2,x_3)\) for which \((x_1,x_2,x_3) \neq (0,0,0)\), with the convention that \(x = (x_1,x_2,x_3)\) and \(y = (y_1,y_2,y_3)\) express the same point if and only if there exists a number \(\lambda \neq 0\) such that \(x_i = \lambda y_i\) for \(i = 1,2,3\). i.e. \(x = \lambda y\). Therefore a point in \(P^2\) is a class \([x]\) of 3-triples \(x = (x_1,x_2,x_3)\), such that \(y \in [x]\) if and only if \(x = \lambda y\) for some non zero number \(\lambda\). Each element in the class can be seen as a representative of the point.

A sequence of points \(x^i\) is said to be point-wise convergent to \(x\) if there is a representative \(x^*\) of \(x^i\), \(x^*\) of \(x\) such that \(\lim_{i \to \infty} x^*_k = x_k^*\) for \(k = 1,2,3\).

We define a metric \(\pi\), such that the convergent in \((P^2,\pi)\) is point-wise.

For \(x = (x_1,x_2,x_3)\), let \(|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}\), then for any real \(\lambda \neq 0\). \(\frac{x}{|x|} = \frac{\lambda x}{|\lambda x|}\), where the sign depends on the sign of \(\lambda\). If \(x^i \to x\) pointwise, then there are representatives \(x^*_i\) and \(x^*\) of \(x^i\) and \(x\) respectively such that \(x^*_i \to x^*\). Hence we have \(|x^*_i| \to |x^*|\). Therefore \(\frac{x^*_i}{|x^*_i|}\) is arbitrary near to \(\frac{x^*}{|x^*|}\) for large \(i\).
Moreover \( \frac{x}{|x|} \) is either \( \frac{x^*}{|x^*|} \) or \( -\frac{x^*}{|x^*|} \), and for each

\[ i, \quad \frac{x_k}{|x^1|} \text{ is either } \frac{x_k^*}{|x^1*|} \text{ or } -\frac{x_k^*}{|x^1*|} \]

Thus the minimum of \( \sum_{k=1}^{3} \left| \frac{x_k^1}{|x^1|} - \frac{x_k}{|x|} \right| \) and \( \sum_{k=1}^{3} \left| \frac{x_k}{|x^1|} + \frac{x_k}{|x|} \right| \) will tend to 0 if \( x^1 \) tends to \( x \) pointwise.

By this analysis we know that if we define the distance between \( x \) and \( y \) by

\[ \pi(x,y) = \min \left\{ \sum_{k=1}^{3} \left| \frac{x_k}{|x|} - \frac{y_k}{|y|} \right|, \sum_{k=1}^{3} \left| \frac{x_k}{|x|} + \frac{y_k}{|y|} \right| \right\} \]

then \( \pi(x^1,x) \to 0 \) if and only if \( x^1 \to x \) pointwise.

It follows from a routine computation that \( \pi \) is a metric.

An affine plane \( A^2 \) is the complement of a line \( L \) in \( P^2 \). In particular if \( L_0 : X_3 = 0 \), let the corresponding affine plane be \( A^2_0 \). In this case \( x \in A^2_0 \) implies that \( x_3 \neq 0 \). Let \( \bar{x} = \left( \frac{x_1}{x_3}, \frac{x_2}{x_3}, 1 \right) = (\bar{x}_1, \bar{x}_2, 1) \), where

\[ \bar{x}_1 = \frac{x_1}{x_3}, \quad \bar{x}_2 = \frac{x_2}{x_3} \]

then \( x \in [x] \) and \( x,y \in [x] \Rightarrow \bar{x} = \bar{y} \)

(This means that \( \bar{x} \) and \( \bar{y} \) have exactly the same coordinates).

We define \( (\bar{x}_1, \bar{x}_2) \) to be the affine coordinates of \( \bar{x} \) in \( A^2_0 \).
If \( x^1 \rightarrow x \) in \( P^2 \) and \( x^1, x \in A^2_0 \), then there exists certain representatives \( x^{1*} \) and \( x^* \) of \( x^1 \) and \( x \) such that \( x^{1*}_k \rightarrow x^*_k \). Since \( x^*_3 \neq 0 \), \( x^{1*}_3 \neq 0 \), we have

\[
\frac{x^{1*}}{x^*_3} \rightarrow \frac{x^*}{x^*_3}, \text{ i.e. } \frac{x^1}{x^*_3} \rightarrow \frac{x}{x^*_3}, \text{ pointwise.}
\]

Conversely, it is evident that \( \frac{x^1}{x^*_3} \rightarrow \frac{x}{x^*_3} \) pointwise \( \Rightarrow x^1 \rightarrow x \) in \( (P^2, \pi) \).

This shows that the relative topology \( (A^2_0, \pi) \) is equivalent to the usual topology on the affine space \( A^2_0 \).

The space \( (P^2, \pi) \) is compact. To show this first we observe that \( (P^2, \pi) \) has a countable base, that is the family of open discs with rational radii and the rational points as centers, where the rational point means the class \([x]\) which contains some 3-triples \((x_1, x_2, x_3)\) such that \( x_1, x_2, \) and \( x_3 \) are rational numbers. We know that in a space which satisfies the second countability axiom (i.e. a space which has a countable base), compactness is equivalent to sequential compactness (i.e. any sequence has a convergent subsequence). Therefore proving that \( (P^2, \pi) \) is compact is equivalent to proving that it is sequentially compact.

Let \( \{x^1\} \) be given. We can normalize \( x^1 \), in such a way that \( \sum_{k=1}^{3} x^1_k^2 = 1 \), or \( |x^1| = 1 \). Consider the
coordinate sequences \( \{ x_k^i \} \), \( k = 1,2,3 \). Since \( |x^i| = 1 \), there exists some \( k \), \( 1 \leq k \leq 3 \), such that the subsequence \( \{ x_k^i \} \) has the property that \( |x_k^i| > 0 \) for all \( x_1 \).

Without loss of generality, we may assume that \( k = 3 \).

(a) Suppose that \( |x_3^i| \geq \alpha, \alpha > 0 \), for all \( x_1 \).

This means that the sequence \( \{ x_3^i \} \) in the affine space \( A^2 \) is bounded. By the Bolzano-Weierstrass theorem, there exists a convergent subsequence \( \{ x_3^i \} \) of \( \{ x_1^i \} \) in \( A^2 \). Therefore \( \{ x_3^i \} \) is a convergent subsequence of \( \{ x_1^i \} \) in \( (P^2, \pi) \).

(b) If \( |x_3^i| \to 0 \), then at least one of the sequences \( \{ |x_1^i| \}, \{ |x_2^i| \} \) does not tend to \( 0 \), say \( \{ |x_1^i| \} \to 0 \).

Hence there is \( \beta > 0 \), such that \( |x_1^i| > \beta \) for all but a finite number of \( x_1 \). Consider the affine space \( A^2_1 \) obtained by deleting the line \( x_1 = 0 \) from \( P^2 \).

Then \( \{ x_1^i \} \) is bounded in \( A^2_1 \) except for a finite number of \( x_1 \). Therefore we have a convergent subsequence of \( \{ x_1^i \} \) in \( (P^2, \pi) \).

In any case \( \{ x_1^i \} \) has a convergent subsequence in \( (P^2, \pi) \). Hence \( (P^2, \pi) \) is compact.
1.2 In an affine plane $A^2$, a set $A$ is defined to be convex if and only if for any $x, y \in A$, $\lambda x + (1-\lambda)y \in A$ for $0 \leq \lambda \leq 1$. The set $B$ whose boundary is the hyperbola $x^2 - y^2 = 1$ such that $(0,0) \not\in B$ is not convex in $A^2$.

If we use the projective coordinates $x = \frac{x_1}{x_3}$, $y = \frac{x_2}{x_3}$, then $x^2 - y^2 = 1$ becomes $x_1^2 - x_2^2 = x_3^2$. We know that $A^2$ is the affine space by assigning $x_3 = 0$ to be the line at infinity. Now if we assign the line $x_1 = 0$ to be the line at infinity and delete this line from $P^2$, we get another affine space $A^{2*}$. In this new affine space the curve $x_1^2 - x_2^2 = x_3^2$ becomes an ellipse $x^2 + y^2 = 1$. Therefore the set $B$ is convex with respect to $A^{2*}$. By this motivation, we define a set $B$ to be convex in $P^2$ if a line $L$ exists such that $B$ is convex in the affine space $A^2$ which is obtained by deleting $L$ from $P^2$. That is, $L$ is the line at infinity with respect to $A^2$.

A point set is a convex curve in $P^2$ if it is a boundary of a convex set in $P^2$. A convex curve is called non-degenerate if it is the boundary of a convex, compact region in an affine plane such that the region contains at least one interior point under the usual Euclidean metric of the affine plane. Otherwise, it is called degenerate.
A non-degenerate convex curve $\sigma$ divides $P^2$ into three disjoint sets, one is the point set of $\sigma$ and two other separated sets. (since it is affine convex with respect to some affine space). We call the region, in which there is at least one line which does not intersect $\sigma$, the exterior of $\sigma$, denoted by $\sigma^*$. The other set, other than $\sigma$, is called the interior of $\sigma$ and is denoted by $\sigma_*$.

Let $Z$ be the set of all non-degenerate convex curves in $P^2$. For any $\sigma \in Z$ if there are $\epsilon, n$ in $Z$ such that $\epsilon \subset \sigma_*$ and $\sigma \subset n_*$, then denote $(\epsilon, n)$ the set of all curves $\sigma$ in $Z$ such that $\sigma \subset (\epsilon^* \cap n^*)_*$ and $\epsilon \subset \sigma_*$. If we define $(\epsilon, n)$ to be a neighborhood of $\sigma$, then this neighborhood system satisfies the Hausdorff neighborhood axioms [2]. Let $(Z, L)$ denote the topological space on $Z$ constructed by the neighborhood system just defined. We call it Lane's topology for $Z$.

We construct another topology for $Z$ as follows. If $A$ is a subset of $P^2$, we call
$$V_r(A) = \{x; x \in P^2 \text{ and } \pi(x, A) < r\}$$
a parallel set of $A$, where $\pi(x, A) = \inf\{\pi(x, y), y \in A\}$.

Define a mapping $D$ on $Z \times Z$ into $R$ (the real numbers) by
$$D(\sigma_1, \sigma_2) = \inf\{r; \sigma_1 \subset V_r(\sigma_2) \text{ and } \sigma_2 \subset V_r(\sigma_1)\}.$$
Then $D$ is a Hausdorff metric on $Z$ [2]. We denote $(Z, D)$ to be the metric space on $Z$ with metric $D$.

1.3 The topological spaces $(Z, D)$ and $(Z, L)$ are equivalent.

Proof: Let $\beta_D$ be the set of all open spheres in $(Z, D)$ and $\beta_L$ be the set of all open neighborhood of $(Z, L)$. To prove this theorem it is sufficient to prove that each set in $\beta_D$ is open in $(Z, L)$ and each set in $\beta_L$ is open in $(Z, D)$.

(a) Let $S(\sigma, r)$ be an open sphere in $(Z, D)$; that is to say, $r$ is a positive number, $\sigma \in Z$, and

$$S(\sigma, r) = \{\sigma'; \sigma' \in Z \text{ and } D(\sigma', \sigma) < r\}.$$ 

Denote $e' = \sigma_* \cap \text{bdry } V_\tau(\sigma)$ and $h' = \sigma_* \cap \text{bdry } V_\tau(\sigma)$, where $\text{bdry } V_\tau(\sigma)$ is the boundary of the parallel set $V_\tau(\sigma)$. Then $e'$ and $h'$ are closed in $(P^2, \tau)$. This can be seen by proving that $e' = \overline{\sigma_*} \cap \text{bdry } V_\tau(\sigma)$ and $h' = \overline{\sigma_*} \cap \text{bdry } V_\tau(\sigma)$, where $\overline{\sigma_*}, \overline{\sigma_*}$ are the closure of $\sigma_*, \sigma_*$ respectively. For any $x \in \overline{\sigma_*} \cap \text{bdry } V_\tau(\sigma)$, we have $\pi(x, \sigma) = r > 0$. Thus $x \notin \sigma$, i.e. $x \in \sigma_* \cap \text{bdry } V_\tau(\sigma) = e'$, so we have $\overline{\sigma_*} \cap \text{bdry } V_\tau(\sigma) \subset e'$. It is trivial that $e' \subset \overline{\sigma_*} \cap \text{bdry } V_\tau(\sigma)$. Hence $e' = \overline{\sigma_*} \cap \text{bdry } V_\tau(\sigma)$. Similarly, we can prove that $h' = \overline{\sigma_*} \cap \text{bdry } V_\tau(\sigma)$.
For an arbitrary $\xi$ in $S(\sigma,r)$, $\xi$ and $\xi'$ are disjoint, so are $\xi$ and $\eta'$. Let $A^2$ be an affine space in which $\xi$ is convex and let $d$ be the usual Euclidean metric for $A^2$. Then $t_1 = \inf\{d(x,y); x \in \xi, y \in \xi'\}$ is positive, for if $t_1 = 0$, then there exists $x_n \in \xi$ and $y_n \in \xi'$ such that $d(x_n, y_n) \to 0$. By the relation of the metric $d$ and the projective metric $\pi$ which was stated before, we have $d(x_n, y_n) \to 0$ implies $\pi(x_n, y_n) \to 0$. But this is impossible, since both $\xi$ and $\xi'$ are disjoint and closed in $(P^2, \pi)$. Similarly $t_2 = \inf\{d(x,z); x \in \xi, z \in \eta'\}$ is positive. Let $t = \min\{t_1, t_2\}$ and $\xi, \eta$ be the boundary curves of the parallel set $V_t(\xi)$ in the affine space $A^2$. Both $\xi$ and $\eta$ are convex curves [3] in $A^2$ and $\xi \in (\xi, \eta) \subset S(\sigma,r)$.

We have proved that for any convex curve $\xi$ in $S(\sigma,r)$, there is an open neighborhood $(\xi, \eta)_{\xi}$ of $\xi$ in $(Z, L)$ such that $(\xi, \eta)_{\xi} \subset S(\sigma,r)$. Hence we have

$$S(\sigma,r) = \bigcup_{\xi \in S(\sigma,r)} \xi \subset \bigcup_{\xi \in S(\sigma,r)} (\xi, \eta)_{\xi} \subset S(\sigma,r).$$

i.e. $S(\sigma,r) = \bigcup_{\xi \in S(\sigma,r)} (\xi, \eta)_{\xi}$ is open in $(Z, L)$. Therefore each set $\beta_{D}$ is open in $(Z, L)$.

(b) To show that each set in $\beta_{L}$ is open in $(Z, D)$, let $(\xi, \eta)$ be given and $\sigma$ be an element in it. Define

$$t = \min\{\inf[\pi(x,y); x \in \xi, y \in \xi'], \inf[\pi(x,u); x \in \xi, u \in \eta]\},$$

i.e. $t$ is the minimum of the distance between the sets
σ and ε and that of σ and h. Then t is positive (the reason is similar to (a)) and the open sphere S(σ, t) is contained in (ε, h). Now for any σ in (ε, h) there corresponds an open neighborhood S(σ, t) of σ such that S(σ, t) ⊂ (ε, h). Therefore (ε, h) is the union of all those open sets in (Z, D), hence it is open in (Z, D).

Thus we have completed the proof of the theorem.

In order to compactify the space (Z, L), let us introduce an enlarged space (Y, D), where Y is the family of all closed subsets of (P^2, π) and D is the Hausdorff metric.

Since the ground space (P^2, π) is compact, we have

1.4 (Y, D) is compact [2].

Let Z^* be the subset of Y which consists of all closed convex curves in P^2, both degenerate and non-degenerate. According to what we have defined, a degenerate closed convex curve is then a point, a segment, a line or a pair of distinct lines.

Let (Z^*, D) denote the subspace of (Y, D) with relative topology. We want to show that it is a compactification of (Z, D).

1.5 Let σ_n be a sequence of convex curves and 1 be a fixed line which intersects σ_n at P_n^1 and P_n^2. If L_n^1, L_n^2, are supporting lines of σ_n at P_n^1, P_n^2 respectively, and L_n^1 ∩ L_n^2 = Q_n, L_n^1 \rightarrow L_1^1, L_n^2 \rightarrow L_2^2, and
If the interior of triangle $\triangle QP_1P_2$, which is denoted by $\Delta$, is the limit of the interior triangles $\Delta_n : \triangle Q_nP_1^np_2^n$, then if an interior point $y$ of $\Delta$ is a limit of points $(\sigma_n)_* \cap \Delta_n$ and $Q$ is a limit point of $(\sigma_n)_* \cap \Delta_n$, every interior point of $\Delta$ is within all except a finite number of $(\sigma_n)_*$.

Proof: Let $x$ be an arbitrary interior point of $\Delta$.
Since $L_n^1 \to L^1$ and $L_n^2 \to L^2$, $x$ is in $\Delta_n$ for sufficiently large $n$. There exists a sequence of points $V_n, y_n$ of $\sigma_n$ such that $V_n \to Q$, $y_n \to y$ since $Q, y$ are limit points of $\sigma_n$. All but a finite number of $y_n$ are in $\Delta$, since $y$ is an interior point of $\Delta$.
Suppose that $x$ is exterior to infinitely many of $\sigma_n$.
Let $M_n$ be the point of intersection of the line $Q_n x$ and $\sigma_n$, if $x$ is exterior to $\sigma_n$. $M_n$ is in $\Delta_n$ since the triangle $y_nP_1^np_2^n$ is in $(\sigma_n)_*$. Draw a support line $h_n$ to $\sigma_n$ at $M_n$.

For such sequence $h_n$, we have a convergent subsequence which has limit say $h$. The corresponding subsequence of $\{M_n\}$ tends to a point on $Qx$, say $M$. Then $M \notin Q$, since
12.

$x \notin Q$. $h_n$ supports the point $V_n$ and $V_n \to Q$. This implies that $h$, the limit of a subsequence of $h_n$, supports $Q$. On the other hand, $h$ supports the line $P^1P^2$, since the segment $P^1P^2 \to P^1P^2$. Then $h$ must be in the exterior of the triangle $QP^1P^2$. This will force $h$ to be a line through $Q$ and then $M = Q$. This is a contradiction. Hence $x$ should be interior to all but a finite number of $\sigma_n$. The proof is now completed.

This theorem shows that the interior of $\sigma_n$ will tend to fill out the interior of the triangle $QP^1P^2$.

1.6 Let \{\sigma_n\} be a sequence of convex curves in $Z^*$, then there exists a convergent subsequence of \{\sigma_n\} in $(Z^*,D)$.

Proof: If $l_n$ is a support line of $\sigma_n$ at point $P_n$, then because of the compactness of the projective space, a sequence of $\sigma_n$ can be selected so that $l_n \to l$ and $P_n \to P$.

(1) If for each line $m$, $m \neq l$, $P \in m$, $P$ is the only point for which each of its neighborhoods contains points of infinitely many $\sigma_n$, then either $\sigma_n$ converges to $l$ or to a segment of $l$ in $(Z^*,D)$.

To prove this, first we observe that the limit points of any subsequence of \{\sigma_n\} are all on the line $l$. By a limit point $x$ of a sequence \{\sigma_n\} we mean a point $x$ any neighborhood of which contains points of infinitely many $\sigma_n$. 

If there exists some point $x$ on $l$ which is not a limit point of any subsequence of $\{\sigma_n\}$, then any line through $x$, except $l$, will not intersect $\sigma_n$ except a finite number of $\sigma_n$. Assign one of these lines to be the line at infinity, then all but a finite number $\{\sigma_n\}$ are bounded by a certain finite region in this affine space since no point on this line is a limit point of any subsequence of $\{\sigma_n\}$. By the Blaschke Selection theorem, there is a convergent subsequence $\{\sigma_{n_1}\}$ of $\{\sigma_n\}$ (in $(Z^*,D)$) and the limit of $\{\sigma_{n_1}\}$ is a segment of $l$. (Actually we are applying the Blaschke theorem to the sequence $\{\tilde{\sigma}_n\}$ of convex regions, where $\tilde{\sigma}_n$ is the convex region with the boundary $\sigma_n$. We get the convergent subsequence $\{\tilde{\sigma}_{n_1}\}$ which tends to a convex set, say $\tilde{\sigma}$. Therefore $\{\sigma_{n_1}\}$ will tend to the boundary of $\tilde{\sigma}$ which is a convex curve). If every point on $l$ is a limit point of some subsequence of $\{\sigma_n\}$, then $l$ is the limit of $\{\sigma_n\}$.

(2) If a line $m$ exists which contains a limit point $Q$ of some subsequence of $\{\sigma_n\}$ other than $P$, then a sequence of $\{\sigma_n\}$ exists each of which contains a point $Q_n$ and $Q_n \rightarrow Q$. Let the supporting lines of $\sigma_n$ at $Q_n$ tend to $p$ and $p \cap l = V$. 
(a) If \( p \neq m \). Then \( p \) and \( l \) separate \( \mathbb{P}^2 \) into two regions. We can select a subsequence of \( \{\sigma_n\} \) such that all of them are in the same region, say the union of \( I, I' \) and \( I'' \), where \( I \) is the triangle \( PQV \), \( I' \) is the region bounded by \( l, m, p \) with two vertices \( P \) and \( Q \), \( I'' \) is the region bounded by \( p, l, V \), with a vertex \( V \).

If \( V \) is either not a limit point of points of \( (\sigma_n)^* \) interior to \( I \) or not a limit point of points of \( (\sigma_n)^* \) interior to \( I'' \), then there exists a line \( L \) which does not contain the limit points of \( \{\sigma_n\} \). Assign \( L \) to be the line at infinity, we get an affine space. All but a finite number of \( \sigma_n \) are bounded in a certain finite region of this affine space. Apply the Blaschke theorem, we get a convergent subsequence and the limit is an affine convex curve.

If both of the above statements are false, then infinitely many of \( \sigma_n \) will be in \( I, I' \) and \( I'' \). Now we have the situation that both \( p \) and \( l \) are the limit of supporting lines and \( V, P, Q \) are the limit points of \( \{\sigma_n\} \). Apply the theorem 1.5 to the triangle \( PQV \), we see that \( \{\sigma_n\} \) fills out the triangle \( PQV \).
Let $m$ rotate around $P$, the triangle $PQV$ will be extended and covers $I'$. Therefore $\{\sigma_n\}$ fills out the region $I \cup I'$. Apply the similar argument to the region $I''$. We conclude that $\{\sigma_n\}$ fill out the whole region $I \cup I' \cup I''$ and that the limit points of $\{a_n\}$ are all on $pU1$.

We want to show that every point on $pU1$ is a limit point of $\{\sigma_n\}$. Suppose that a point $x_0$ on $pU1$ is not a limit point of $\{\sigma_n\}$, then there exists a neighborhood $N(x_0)$ of $x_0$ such that $N(x_0)$ contains points of only a finite number of $\sigma_n$. This implies that any interior point $x_1$ of $N(x_0)$ such that $x_1 \neq x_0$ and $x_1 \in N(x_0) \cap (I \cup I' \cup I'')$ is not interior to all but a finite number of $\sigma_n$. This contradicts to the result that $\{\sigma_n\}$ fill out the whole region $I \cup I' \cup I''$. Hence every point on $pU1$ is a limit point of $\{\sigma_n\}$.

Now we can say that $\{\sigma_n\}$ converges to $pU1$ (in Hausdorff sense), for if not, then there is a real number $r > 0$ and a sequence of points $x_{n_1}$ of $\sigma_{n_1}$ such that $x_{n_1} \notin (pU1)r$ (a parallel set of the set $pU1$). Since $P^2$ is compact, there exists a convergent subsequence of $\{x_{n_1}\}$ which tends to $y_o$. Then $y_o \notin pU1$ and $y_o$ is a limit point of $\{\sigma_n\}$. This contradicts to the fact that all limit points of $\{\sigma_n\}$ are on $pU1$. Hence $\{\sigma_n\}$ converges to $pU1$ (in Hausdorff sense).
(b) If $p = m$, we consider the following situations:

(a) Suppose a limit point $R$ of $\{\sigma_n\}$ exists with $R \notin m$, $R \notin 1$. We can construct a limit of supporting lines $k$ which contains $R$. $k$ can contain at most one of $P, Q$. Suppose $P \in k$, then we are in the last situation of (a) if we consider the limit lines $k, m$ and $R \in k, Q \in m$ with $P, Q, R$ the limit points of $\{\sigma_n\}$. If $P \notin k$, then $1, m, k$ are not concurrent and all of them are the limits of support lines of $\sigma_n$. Therefore a subsequence of $\{\sigma_n\}$ can be selected such that each curve of the subsequence is in the same triangle which is one of the projective triangles formed by $1, m, k$. Then the Blaschke theorem can be applied to this case.

(β) Suppose that no limit point of $\{\sigma_n\}$ exists which is neither on $1$ nor on $m$. If $P$ is the only limit point of $\{\sigma_n\}$ on $1$, then this reduces to (1) by considering the line $m$ and $P \in m$. If there is a limit point $S$ on $1$ and $S \notin P$, then we can construct a limit line $j$ with $S \in j$. Suppose $j = 1$, this reduces to the last situation of (a) by considering the limit lines $1$ and $m$ with limit points $P, Q, S$, where $P \in 1$, $S \in 1$, $Q \in m$. If $j \neq 1$, then we have three non-concurrent limit lines $1, m, j$. The Blaschke theorem can be applied to this case, the reason is similar to that in (a).

The proof is now completed.
Blaschke showed that if a sequence of convex sets $A_n$ are bounded by a finite square, then there exists a convergent subsequence of $\{A_n\}$, and the limit is a convex set. Now the theorem 1.6 shows that the Blaschke theorem holds for the projective convex sets and the condition of boundedness can be removed.

1.7 Let us turn back to our problem, that is to show that $(Z^*,D)$ is a compactification of $(Z,D)$.

Since $(Z^*,D)$ is a subspace of $(Y,D)$ and $(Y,D)$ is a metric space, $(Z^*,D)$ is a metric space too. Thus it is a $T_1$-space (A space $X$ is a $T_1$-space if for any $x,y$ in $X$, there exists neighborhoods $N_x$ and $N_y$ of $x$ and $y$ respectively, such that $y \notin N_x$ and $x \notin N_y$). By theorem 1.6, $(Z^*,D)$ is sequentially compact. A $T_1$ and sequentially compact space is compact. Hence $(Z^*,D)$ is compact. Since each element in $Z^*$ is a limit of $Z$, we can say that $(Z^*,D)$ is a compactification of $(Z,D)$. We know that $(Z,D)$ and $(Z,L)$ are topologically equivalent. Hence we can embed $(Z,L)$ into $(Z^*,D)$ and then $(Z^*,D)$ is a compactification of $(Z,L)$.
CHAPTER 2: Some Topological Properties of $(Z,L_1)$.

2.1 Let $L$ be an arbitrary fixed line in the projective plane. In addition to the neighborhoods that we defined for the elements of $Z$ in the space $(Z,L)$, we admit the following sets as neighborhoods: Define a curve $\sigma$ to be differentiable around a point $P$ if there is a neighborhood $N$ of $P$ such that $\sigma$ is differentiable in $N$. If $\sigma$ is in $Z$ which is tangent to $L$ at a point $P$ and is differentiable around $P$, then take $\epsilon, \eta$ both in $Z$ such that both of them are tangent to $L$ at $P$ and are differentiable around $P$ and that $\epsilon \subset \sigma^*, \sigma \subset \eta^*$. Here "\subset" is in the wide sense, that is $\epsilon$ is contained in $\sigma^*$ except the point $P$. Let $(\epsilon, \eta)$ denote the set of all closed convex curves $\xi$ in $Z$ such that $\xi \subset (\epsilon^* \cap \eta^*)$ in wide sense, $\epsilon \subset \xi^*$ and $\xi$ is tangent to $L$ at $P$ and is differentiable around $P$. Define such a $(\epsilon, \eta)$ to be a parabolic neighborhood of $\sigma$.

Adding these new neighborhoods to $(Z,L)$, we get the new topology, denoted by $(Z,L_1)$. Then $(Z,L_1)$ is a refinement of $(Z,L)$.

2.2 $(Z,L_1)$ is a $T_1$-space.

Proof: Since $(Z,L)$ is metrizable, it is a $T_1$-space. Given any two elements $\sigma_1, \sigma_2$ in $Z$, we can find a neighborhood $N(\sigma_1)$ of $\sigma_1$ in $(Z,L)$ which does not contain $\sigma_2$ and a
neighborhood \( N(\sigma_2) \) of \( \sigma_2 \) in \((Z,L)\) which does not contain \( \sigma_1 \). Since \((Z,L_1)\) is a refinement of \((Z,L)\), \( N(\sigma_1), N(\sigma_2) \) are also neighborhoods of \( \sigma_1, \sigma_2 \) in \((Z,L_1)\) respectively. Hence \((Z,L_1)\) is a \( T_1 \)-space.

2.3 \((Z,L_1)\) is regular.

Proof: Given \( \sigma \in Z \) and a neighborhood \( N(\sigma) \) of \( \sigma \) in \((Z,L_1)\). We want to show that \( N(\sigma) \) contains a closed neighborhood \( W \) of \( \sigma \). For \( N(\sigma) \), there is an open neighborhood \((\epsilon,\eta)\) of \( \sigma \) such that \((\epsilon,\eta) \subseteq N(\sigma) \). If \((\epsilon,\eta)\) is open in \((Z,L)\), then there exists a closed neighborhood \( W \) of \( \sigma \) such that \( W \subseteq (\epsilon,\eta) \), since \((Z,L)\) is regular. If \((\epsilon,\eta)\) is not open in \((Z,L)\), i.e. \( \epsilon,\eta \) and \( \sigma \) are all tangent to \( L \) at a point \( P \) and are differentiable around \( P \). Between \( \sigma \) and \( \eta \) we want to construct a convex curve \( \beta \) such that \( \beta \) is tangent to \( L \) and is differentiable around \( P \).

Since \( \sigma,\eta \) are convex and \( \sigma \subseteq \eta \) and both have a common tangent at \( P \), both can be realized in an affine plane with the common tangent \( y = 0 \) with point of contact \((0,0)\).

There exists a neighborhood \((-a,a)\) of \( 0 \) so that \( \eta \) can be represented by \( y = y_1(x), x \in (-a,a) \) with \(-\infty < \dot{y}_1(x) < \infty \) and \( \sigma \) by \( y = y_2(x), -\infty < \dot{y}_2(x) < \infty \), \( x \in (-a,a) \) since \( \eta \) and \( \sigma \) are differentiable around \((0,0)\). We have \( y_1(x) < y_2(x) \quad x \in (-a,a) \).
Since the convexity of a curve $\xi$ with equation $y(x)$ is equivalent to the fact that $y(x)$ is increasing, provided that $y(x)$ is differentiable on its domain of definition, we have both $y_1(x), y_2(x)$ are increasing in $(-a,a)$. Therefore

$$y(x) = \lambda y_1(x) + (1-\lambda)y_2(x) \quad \text{for} \quad x \in (-a,a) \quad \text{and}$$

$$0 < \lambda < 1$$

is increasing and $y(x) \leq \eta^*$ for $x \in (-a,a)$ except $(0,0)$. Let $\sigma_\rho^*$ be a parallel set of $\sigma$ with $\rho > 0$. For sufficiently small $\rho$ part of $\sigma_\rho^*$ will be outside of $\eta$.

Now $\sigma_\rho^* \to \sigma$ as $\rho \to 0$. Therefore for sufficiently small $\rho$, $\sigma_\rho^*$ intersects $y(x), x \in (-a,a)$, at two distinct points $x_1$ and $x_2$ for which $(x,y(x)) \leq \sigma_\rho$ for $-a \leq x_1 \leq x \leq x_2 \leq a$. Let $\beta$ be the curve obtained by replacing the part of $\sigma_\rho^*$ in between $(x,y(x_1))$ and $(x_2,y(x_2))$ by $y(x), x_1 \leq x \leq x_2$ for sufficiently small $\rho, \rho > 0$. Then $\beta \subset \eta^*$ and $\beta \subset \sigma^*$.

\[ \text{Figure III} \]

$y(x), x_1 \leq x \leq x_2$ is tangent to $L$ at $P$ and is differentiable around $P$, so is $\beta$. $\beta$ is convex, since $\beta$ (\( \beta \) and its interior) is the intersection of the convex region $\bar{\sigma}_\rho^*$ and the convex region bounded by $y$ and its end tangents. Furthermore, since

$\eta^*$
Similarly we can construct a convex curve $\alpha$ in between $\epsilon$ and $\sigma$ such that $\alpha$ is tangent to $L$ at $P$ and is differentiable around $P$. Thus we have $\sigma \epsilon (\alpha, \beta) \subset (\epsilon, h) \subset N(\sigma)$. Where $(\alpha, \beta)$ is the closure of the set $(\alpha, \beta)$.

Hence $(Z, L_1)$ is regular.

2.4 $(Z, L_1)$ satisfies the first axiom of countability.

Proof: Given $\sigma$. Let $\epsilon_r, h_r$ be the boundary curves of the parallel set $V_r(\sigma)$, then $\epsilon_r$ and $h_r$ are both convex and $(\epsilon_r, h_r)$ is a neighborhood of $\sigma$. Let $G = \{(\epsilon_r, h_r); \text{for all rational } r\}$, then $G$ is a countable family of neighborhoods of $\sigma$. If $\sigma$ is not tangent to $L$ or is tangent to $L$ at $P$ but is not differentiable around $P$, then $G$ forms a countable base for the neighborhood system of $\sigma$. For if $W$ is a neighborhood of $\sigma$, then there exists $\epsilon$ and $h$ both in $Z$ such that $(\epsilon, h)$ is a neighborhood of $\sigma$ and $(\epsilon, h) \subset W$. If $a$ is the distance between $\epsilon$ and $\sigma$, $b$ is the distance between $\sigma$ and $h$, then there exists a rational number $r$ such that $r < \min(a, b)$. Therefore $(\epsilon_r, h_r) \subset (\epsilon, h) \subset W$. Hence $G$ forms a local base at $\sigma$.

If $\sigma$ is tangent to $L$ at a point $P$ and is differentiable around $P$, then $G$ does not form a local base at $\sigma$.

So we add the following neighborhoods to $G$.
Draw a line through $P$ and cut $\sigma$ at $x$. Given a fixed number $\lambda > 0$, let $\sigma_\lambda$ denote the locus of the point $q$ on segment $P_x$ such that $P_q = \lambda P_x$. Then $\sigma_\lambda$ is a convex curve. To show this it is sufficient to show that for any two points $q$ and $q'$ on $\sigma_\lambda$, the segment $qq'$ is in $\sigma_\lambda$ ($\sigma_\lambda \cap$ interior of $\sigma_\lambda$).

There exists $x' \in \sigma$ such that $P_q' = \lambda P_x$. For $0 < \mu < 1$

$$z = \mu q + (1-\mu)q' = \mu [(1-\lambda)P + \lambda x] + (1-\mu)[(1-\lambda)P + \lambda x']$$

$$= (1-\lambda)P + \lambda [\mu x + (1-\mu)x']$$

$$= (1-\lambda)P + \lambda y$$

where $y = \mu x + (1-\mu)x' \in \sigma$, since $\sigma$ is convex. Let $y'$ be the point where the line $Py$ intersects $\sigma$. Then $Py' > Py$. Therefore $Pz = \lambda Py < \lambda Py'$, hence $z \in \sigma_\lambda$, and $\sigma_\lambda$ is convex. Since $P_q = \lambda P_x$ and $\sigma$ is tangent to $L$ at $P$ and is differentiable around $P$, so is $\sigma_\lambda$.

Let $\mathcal{C}$ be the family of all $\sigma_\lambda$ for positive rational $\lambda$. Denote $\mathcal{C} = \{(\sigma_\lambda, \sigma_{\lambda'}) \mid \sigma_\lambda, \sigma_{\lambda'} \in \mathcal{C} \text{ and } \lambda < 1, \lambda' > 1 \}$. Then $G_U \mathcal{C}$ forms a countable local base for $\sigma$.

For if a neighborhood $W$ of $\sigma$ is given then there exists a neighborhood $(\epsilon, \eta)$ of $\sigma$ such that $(\epsilon, \eta) \subset W$. We may
suppose that both $\epsilon$ and $\eta$ are tangent to $L$ at a point $P$ and are differentiable around $P$ (for otherwise we have already shown that there is $(\epsilon_r, h_r) \in G$ such that $\sigma \in (\epsilon_r, h_r) \subset (\epsilon, h) \subset W$). Since $\sigma$ can be approached by $\{\sigma_\lambda\} \lambda < 1$ and $\{\sigma_\lambda\} \lambda' > l$ as $\lambda \to 1$, $\lambda' \to l$, now $(\epsilon, \eta)$ is a neighborhood of $\sigma$, hence there exists at least one rational $\lambda < 1$ and one rational $\lambda' > 1$ such that both $\sigma_\lambda$ and $\sigma_{\lambda'}$ are in $(\epsilon, h)$. Thus we have $\sigma \in (\sigma_\lambda, \sigma_{\lambda'}) \subset (\epsilon, h) \subset W$.

Hence $(Z, L_1)$ satisfies the first axiom of countability.

2.5 Although $(Z, L_1)$ is $T_1$ and regular, it is non-metrizable. For if $(Z, L_1)$ is metrizable, then the separated sets $(A$ and $B$ are separated if $(A \cap B) \cap (A \cap B) = \emptyset$) have disjoint neighborhoods $(A$ neighborhood of a set $A$ is the open set which contains $A$).

Let $A = (\epsilon, \eta)$ where $\epsilon$ and $\eta$ both tangent to $L$ at a point $P$ and are differentiable around $P$.

Let $B$ be the set of all convex curves in $Z$ which lie entirely in the interior of $\epsilon$ or partly in the interior of $\epsilon$ partly on the curve $\epsilon - \{P\}$, where $\epsilon - \{P\}$ means the open curve $\epsilon$ without the point $P$. Then
\[ \overline{A} = (\epsilon, \eta) = (\epsilon, \eta) \cup \{\epsilon, \eta\} \]

\( \overline{B} \) consists of all convex curves of the set in \( \epsilon \cup \epsilon \). It is evident that \( A \cap \overline{B} = \emptyset \). Observe that \( \overline{A} \) contains convex curves which go through \( P \), but the curves of \( B \) do not. Therefore \( \overline{A} \cap B = \emptyset \). This shows that \( A \) and \( B \) are separated. But now every neighborhood of \( B \) consists of some curves, such as \( \delta \) shown in the figure, of the set \( A \). That is \( A \) and \( B \) can never have the disjoint neighborhoods. This is a contradiction. Hence \( (Z, L_1) \) is not metrizable.

2.6 \( (Z, L_1) \) is non-compact even if we include the degenerate closed convex curves. Because, if \( \{\sigma_n\} \) is a sequence of curves in \( Z \) such that \( \sigma_n \rightharpoonup \sigma \) pointwise, but none of \( \{\sigma_n\} \) is tangent to \( L \), where \( \sigma \) is tangent to \( L \) at a point \( P \) and is differentiable around \( P \). Then \( \{\sigma_n\} \) converges to \( \sigma \) in the space \( (Z^*, D) \) but not in \( (Z^*, L_1) \) and has no convergent subsequence either, where \( (Z^*, L_1) \) is the topological space obtained by adding the degenerate closed convex curves to \( (Z, L_1) \) and the neighborhood system for the degenerate convex curves is inherited from that of \( (Z^*, D) \).
Similar to 2.4, we can prove that $(Z^*, L_1)$ satisfies the first axiom of countability. Then its compactness is equivalent to sequential compactness. By the above example, we conclude that $(Z^*, L_1)$ is not compact.
CHAPTER 3: The topological space of non-degenerate conics in affine space.

3.1 Now let us consider the subset $X$ of $Z$ which consists of all the non-degenerate conics

$$\sigma: \sum_{i,k=1}^{3} a_{ik} x_i x_k = 0, \quad a_{ik} = a_{ki}, \quad |a_{ik}| = 0$$

where $|a_{ik}|$ is the determinant of the coefficient matrix of $\sigma$. If we take the fixed line $L$ in Chapter II to be the line at infinity, then the equation of $\sigma$ in this affine space $A^2$ has the form:

$$\sigma: b_{11}x^2 + 2b_{12}xy + b_{22}y^2 + 2b_{23}x + 2b_{33}y + b_{33} = 0$$

$$|b_{ik}| \neq 0.$$ We still use $X$ to denote the set of all non-degenerate conics in $A^2$.

Let us denote $(X, \tau)$ to be the topological space on $X$ which is the restriction of the subspace $X$ of $(Z, L_1)$ to the affine space $A^2$.

By the heredity of the topological space (the property that every subspace of a space shares with the space), $(X, \tau)$ is $T_1$, regular and satisfies the first axiom of countability.

Each conic in $X$ has a unique equation with respect to a fixed affine coordinates $(x,y)$. Therefore there is an one-one correspondence between $X$ and a subspace of $E^5$ (Euclidean 5-space).
Define $\nu : X \rightarrow \mathbb{R}^5$ by

$$\nu(\sigma) = (b_{11}, b_{12}, b_{22}, b_{13}, b_{23}, b_{33})$$

where $\sigma : b_{11}x^2 + b_{12}xy + b_{22}y^2 + b_{13}x + b_{23}y + b_{33} = 0$

Let $B = \nu(X)$, then $\nu$ is one-one on $X$ to $B$. If $\nu$ were a homeomorphism, then our space $(X, \tau)$ would be metrizable since $B$, a subspace of the metric space $\mathbb{R}^5$, is a metric space. But $\nu$ is not a homeomorphism. Because if we let

$$\sigma : y^2 - x = 0$$

and let $\sigma_n$ be the parabola which is obtained by rotating $\sigma$ counterclockwise through an angle $\frac{2\pi}{n}$ around the origin.

Then $\sigma_n$ has the equation

$$x^2 \sin^2 \frac{2\pi}{n} - 2xy \sin \frac{2\pi}{n} \cos \frac{2\pi}{n} + y^2 \cos^2 \frac{2\pi}{n} - x \cos \frac{2\pi}{n} - y \sin \frac{2\pi}{n} = 0.$$

Now $\nu(\sigma_n) = (\sin^2 \frac{2\pi}{n}, -2\sin \frac{2\pi}{n} \cos \frac{2\pi}{n}, \cos^2 \frac{2\pi}{n}, -\cos \frac{2\pi}{n}, -\sin \frac{2\pi}{n}, 0)$ and $\nu(\sigma_n) \rightarrow (0, 0, 1, -1, 0, 0)$ as $n \rightarrow \infty$,

but $\sigma_n \not\rightarrow \sigma$ in $(X, \tau)$. Because if we take a neighborhood $(\epsilon, n)$ of $\sigma$, where $\epsilon : y^2 = x - 1$, $\epsilon$ intersects $\sigma_n$ for each $n \geq 2$. This shows that $\sigma_n \not\rightarrow (\epsilon, n)$ for all $n \geq 2$. Hence $\sigma_n$ is not convergent to $\sigma$ in the space $(X, \tau)$. Therefore $\nu$ is not a homeomorphism.

Now let us try the other way to check the metrizability of $(X, \tau)$. We say that a family of sets is discrete if for every point $x$ of the space there is a neighborhood $N(x)$
of $x$ such that $N(x)$ has a non-void intersection with at
the most one set of the family. A countable union of discrete
families of sets is called $\sigma$-discrete.

By Bing's metrization theorem [4], a topological space
is metrizable if and only if it is $T_1$, regular and it has
a $\sigma$-discrete base. We show that $(X, \tau)$ has a $\sigma$-discrete
base. Therefore it is metrizable (We know already that
$(X, \tau)$ is $T_1$ and regular).

3.2 $(X, \tau)$ has a $\sigma$-discrete base.

Proof: Let $G_0$ be the family of all non-degenerate ellipses
$\sigma$ such that $\sigma$ is either the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where } a \text{ and } b \text{ are rational numbers}$$

or the curve obtained by rotating the curve (1) around its
center through an angle $\theta$, $\theta$ rational. $G_0$ is countable.

Let $\mu$ be a translation which maps the origin into the
point $P = (x, y)$, with both $x$ and $y$ rationals. We call
such a point $P$ a rational point.

Applying $\mu$ to $G_0$ we get another countable family $G_P$.
Each ellipse in $G_P$ has $P$ as its center. Let $G = \bigcup G_P$,
where $P$ runs through all rational points. $G$ is still
countable.

Similarly we can construct a countable family $\mathcal{H}$ of
hyberbolas. Let $\mathcal{H} = G \cup \mathcal{H}$. Form the ordered pairs from the
members of $\mathcal{H}$ such that $(\epsilon, \eta)$ is an open set in $(X, \tau)$. 
Denote \( \mathcal{F} \) to be the family of all such open sets, then \( \mathcal{F} \) is countable.

Let \( \alpha \) be the parabola with the equation

\[
 y^2 = 4cx , \quad \text{where } c \text{ is rational} .
\]

Let \( P \) be a rational point \((h,0)\) with \( h > 0 \) and denote \( \alpha_P \) to be the parabola

\[
 y^2 = 4c(x-h) .
\]

Then \((\alpha, \alpha_P)\) is an open set in \((X,\tau)\). Rotating the coordinate axes through an angle \( \theta, \ 0 \leq \theta < 2\pi \), we get the image \((\alpha, \alpha_P)_\theta\) of \((\alpha, \alpha_P)\). \((\alpha, \alpha_P)_\theta\) is still open in \((X,\tau)\). Denote \( \mathcal{T}_{P,c} = \bigcup_{0<\theta<2\pi} (\alpha, \alpha_P)_\theta \). Apply \( \mu \) to \( \mathcal{T}_{P,c} \) we get the new family \( \mathcal{V}^{\mu}_{P,c} \). Let

\[
 \mathcal{E} = \bigcup_{\mu} \bigcup_{P,c} \mathcal{V}^{\mu}_{P,c}
\]

where \( \mu \) runs through all translations which map the origin into a rational points, \( c \) takes all rational values and \( P \) runs through the rational points of the form \((h,0), \ h > 0 \). \( \mathcal{E} \) is then a countable union of the subfamilies \( \mathcal{V}^{\mu}_{P,c} \).

Since \( \mathcal{F} \) is countable, we can arrange the members of \( \mathcal{F} \) in a sequence \( \{V_n\}_{n=1}^\infty \). Then \( \mathcal{F} = \{V_n ; n=1,2,\ldots\} \).

Consider \( \{V_n\} \) as a family consisting of a single set \( V_n \) and set

\[
 \mathcal{G} = \bigcup_{n=1}^\infty \{V_n\} \bigcup_{\mu} \bigcup_{P,c} \mathcal{V}^{\mu}_{P,c} .
\]
\( \mathcal{E} \) is a countable unions of the families \( \{V_n\} \) and \( \mathcal{L}_{P,c} \). Now we are going to show that \( \mathcal{E} \) is a \( \sigma \)-discrete base for \( (X, \tau) \).

(a) \( \mathcal{E} \) is a base for \( (X, \tau) \)

To show this, we have to prove that for any \( \sigma \in X \) and any open set \( G \) to which \( \sigma \) belongs, there exists a set \( V \) in some subfamilies of \( \mathcal{E} \) such that \( \sigma \in V \subseteq G \).

First we see that there exists an open set \( (e, h) \) such that \( \sigma \in (e, h) \subseteq G \). We want to show that there is \( (e', h') \in \mathcal{E} \) such that \( \sigma \in (e', h') \subseteq (e, h) \subseteq G \).

1) Suppose that \( \sigma \) is an ellipse.

Since the set of all rational numbers is dense in \( \mathbb{R} \) (reals) and the set of all rational points is dense in \( \mathbb{R}^2 \), we can find two ellipses \( e', h' \) with

\[
e' \subseteq e^* \cap \sigma^* , \quad h' \subseteq \sigma^* \cap h^*, \quad \sigma \in (e', h') \subseteq (e, h)
\]

such that the centers of \( e' \) and \( h' \) are rational points and the major and minor diameters of \( e' \) and \( h' \) are rational. Furthermore we define \( e', h' \) so that the major axes of \( e' \) and \( h' \) make rational angles with \( x \)-axis. In other words, there exists \( e', h' \) both in \( \mathcal{E} \) such that \( \sigma \in (e', h') \subseteq (e, h) \subseteq G \).

i.e. there is some \( n \), such that

\[
\sigma \in V_n \subseteq (e, h) \subseteq G.
\]
ii) If $\sigma$ is a hyperbola, then similar to case i) there exists some $V_m$ so that $\sigma \in V_m \subset (\epsilon,\eta) \subset G$.

iii) If $\sigma$ is a parabola and if the axis of $\sigma$ forms an angle $\theta$ with the x-axis in the positive sense, then there is a rational point $P = (h,0)$, $h > 0$, a translation $\mu$ and a rational number $c$ so that

$$(a,a_p)^\Theta \subset \mathcal{C}_{P,c}$$

$(a,a_p)^\Theta$ is the image of $(a,a_p)^\Theta$ by the translation $\mu$.

Take $V = V_n$ in case i), $V = V_m$ in case ii), $V = (a,a_p)^\Theta$ in case iii), our assertion (a) is proved.

(b) $\mathcal{E}_\eta$ is $\sigma$-discrete.

Since each family $\{V_n\}$ consists of only a single set $V_n$, it is evidently a discrete family for each $n$. Therefore we have only to show that each of the families $\mathcal{C}_{P,c}^\mu$ is discrete.

Suppose $\sigma \in X$ and the angle between the axis of $\sigma$ and x-axis is $\theta$. For a fixed family $\mathcal{C}_{P,c}^\mu$, i.e. $P,c$ and $\mu$ are fixed, if $\sigma$ happens to be a point of a set $A$ in $\mathcal{C}_{P,c}^\mu$ (here a point means a non-degenerate conic), then $A$ is a neighborhood of $\sigma$ and $A \cap B = \emptyset$ for any $B \in \mathcal{C}_{P,c}^\mu$ and $B \neq A$. If $\sigma$ is not a point of any member of $\mathcal{C}_{P,c}^\mu$, then there is some $\mu',P',c'$ and a set $A'$ such that $\sigma \in A' \in \mathcal{C}_{P',c'}^\mu$, because $\mathcal{E}_\eta$ is
a base. Since $A'$ is open, it is a neighborhood of $\sigma$ and we observe that $A'$ has a non-empty intersection with at the most one member of $\mathcal{T}_{P,c}^\mu$ (If $A \in \mathcal{T}_{P,c}^\mu$ and $A \cap A' \neq \emptyset$, then the axes of the parabolas in $A$ and $A'$ are parallel. Now given a direction $\theta$, there exists only one set $B$ in $\mathcal{T}_{P,c}^\mu$ and one $B'$ in $\mathcal{T}_{P',c'}^\mu$ so that the axes of the parabolas in $B$ and $B'$ have the same direction $\theta$). This shows that $\mathcal{E}$ is $\sigma$-discrete. Hence $\mathcal{E}$ is $\sigma$-discrete. This completes the proof of 3.2.

3.3 We know that a metric space is separable if and only if it has a countable base. Although $(X, \tau)$ has a $\sigma$-discrete base, it has no countable base. Hence it is not separable. Here we show that $(X, \tau)$ does not have a countable base.

Let $G$ be a base for $(X, \tau)$. For the parabola

$$\sigma: y^2 = 4ax, \quad a > 0,$$

we choose a special neighborhood $(\epsilon, h)$ of $\sigma$ by taking $\epsilon$ and $h$ of the form $y^2 = 4b(x-c)$.

Since $G$ is a base, there is an open set $G$ in $G$ such that $\sigma \in G \subseteq (\epsilon, h)$. We know that $(\epsilon, h)$ contains only the parabolas with the axes parallel to those of $\epsilon$ and $h$. Hence we can assume that $G = (\epsilon', h')$ for some parabolas $\epsilon', h'$ of the form
\[ y^2 = 4a_1(x-a_2) \]

Now rotate the coordinate axes by an angle \( \theta \) around the origin and denote \( \sigma_\theta, \epsilon_\theta, \eta_\theta \) to be the image of \( \sigma, \epsilon, \eta \), respectively. \((\epsilon_\theta, \eta_\theta)\) is a neighborhood of \( \sigma_\theta \)

By the same argument, there exists \((\epsilon^\theta, \eta^\theta) \in G\) such that \( \sigma_\theta \in (\epsilon^\theta, \eta^\theta) \subset (\epsilon_\theta, \eta_\theta) \) and the axes of \( \epsilon^\theta, \eta^\theta, \epsilon_\theta, \eta_\theta \) are parallel.

Let \( \mathcal{A} \) be the family of all those \((\epsilon^\theta, \eta^\theta), 0 \leq \theta < 2\pi\). Then \( \mathcal{A} \) is a subfamily of \( G \) and is not countable. Furthermore, each member of \( \mathcal{A} \) cannot be written as the union of other members in \( \mathcal{A} \). (Actually any two members of \( \mathcal{A} \) are disjoint). This shows that \( \mathcal{A} \) cannot be reduced to a countable family which is still a base of \((X,T)\).

Since \( \mathcal{A} \) is a subfamily of \( G \) and is not countable, \( G \) is not countable. \( G \) is an arbitrary base, therefore any base of \((X,T)\) is not countable.

By the proof we see that \((X,T)\) is not separable because of the parabolas. The subspace which consists of all ellipses and hyperbolas is both metrizable and separable, because this subspace has a countable base (The family \( \mathcal{F} \) in the proof of 3.2 is a countable base for this subspace).

Another method of obtaining a metrizable and separable space is discussed in the next section.
3.4 Define a relation $R$ between the elements of $X$ as follows: Let $x, y$ be elements in $X$, if $x$ and $y$ are parabolas and have the same foci or $x$ and $y$ not both parabolas but have the same centers $q$, then $xRy$ if and only if $x$ coincides with $y$ by rotating $x$ around $q$. It is evident that $R$ is an equivalence relation. Therefore we can take the quotient space $X/R$ with the quotient topology, say $\tau'$. The elements in $X/R$ are the residue classes. Select an element from each residue class. In particular we can choose a representative element of each class to be the one which has the equation of the form

$$
(y-b)^2 = 4c(x-a) \quad \text{with } c > 0
$$

or

$$
\frac{(x-a)^2}{a^2} + \frac{(y-b)^2}{b^2} = 1 \quad \text{with } a \geq b
$$

or

$$
\frac{(x-a)^2}{a^2} - \frac{(y-b)^2}{b^2} = 1
$$

Then the selection is unique.

Let $V$ be the family of all such representative elements. Define the neighborhoods of the elements of $V$ as we did in Chapter I for $(Z,L)$. Denote $(V,S)$ to be the topology constructed by this neighborhood system. If $\sigma$ is a residue class in $X/R$ and $\sigma$ is its representative element in $V$, then define $f: X/R \to V$.
by \( f(\sigma) = \sigma \). We have \( f \) one to one of \( X/R \) onto \( V \). We shall show that \( f \) is a homeomorphism.

3.5 \( f \) is continuous.

**Proof:** Let \( P \) by the projection of \( X \) onto \( X/R \)

\[
\begin{array}{ccc}
X & \xrightarrow{P} & X/R & \xrightarrow{f} & V \\
\end{array}
\]

Since \((X/R, \tau')\) is a quotient space with quotient topology, \( f \) is continuous if and only if \( f \circ P \) is continuous.

Let \( G \) be an open set in \( V \), then \( G \) can be expressed as a union of members of the base, say \( G = \bigcup_a V_a \), where

\[ V_a = (\epsilon_\alpha, \eta_\alpha) \]. Now

\[
(f \circ P)^{-1}(G) = P^{-1}(f^{-1}(G)) = P^{-1}(\bigcup_a f^{-1}(V_a)) = \bigcup_a P^{-1}(f^{-1}(V_a))
\]

Observe that \( P^{-1}(f^{-1}(V_\alpha)) = \bigcup_{0 < \theta \leq 2\pi} V_\alpha^\theta \), where \( V_\alpha^\theta = (\epsilon_\alpha^\theta, \eta_\alpha^\theta) \)

and \( \epsilon_\alpha^\theta \) is obtained by rotating \( \epsilon_\alpha \) around its focus (if it is a parabola) or around its center by an angle \( \theta \). \( \eta_\alpha^\theta \) has the similar meaning. Since each \( (\epsilon_\alpha^\theta, \eta_\alpha^\theta) \) is open in \((X, \tau)\) so is their union. i.e. \( P^{-1}(f^{-1}(V_\alpha)) \) is open in \((X, \tau)\).

Therefore \((f \circ P)^{-1}(G)\) is open in \((X, \tau)\). That is \( f \circ P \) is continuous and hence \( f \) is also continuous.

3.6 \( f \) is an open mapping.

**Proof:** Let \( B \) be an open set in \((X/R, \tau')\), then \( P^{-1}(B) \) is open in \((X, \tau)\) (Since \( P \) is open), thus we can express
\[ p^{-1}(B) = \bigcup_{\beta} W_{\beta} \]

where \( W_{\beta} \) is an element in the base of \((X, \tau)\). i.e., we can write \( W_{\beta} = (\epsilon, h) \). Then \( f \cdot p(W_{\beta}) = (\epsilon', h') \), where \( \epsilon', h' \) are in \( V \) and \( \epsilon \in \epsilon' \), \( h \in h' \). Now

\[
f(B) = f(P \cdot p^{-1}(B)) = f \cdot p(P^{-1}(B)) = f \cdot p(\bigcup_{\beta} W_{\beta})
= \bigcup_{\beta} (f \cdot p(W_{\beta})).
\]

\( f(B) \) is a union of open sets in \((V, S)\), hence it is open.

This shows that \( f \) is an open mapping.

Hence \( f \) is a homeomorphism. Therefore we can identify the topological space \((X/\mathcal{R}, \tau')\) with \((V, S)\).

Let us compare the spaces \((V, S)\) and \((X, \tau)\). We know that they have the same definition of neighborhood system, the only difference between \((V, S)\) and \((X, \tau)\) is that each of the conics in \( X \) which has its axes not parallel to the coordinate axes disappears in \( V \). Therefore if \( \sigma \in V \), then \( \sigma \in X \). Let \((\epsilon, h)\) be a neighborhood of \( \sigma \) in \( X \), if \( \epsilon \) and \( h \) both in \( V \), then deleting all the conics in \((\epsilon, h)\) which are not the elements of \( V \). The remaining set which we still denote \((\epsilon, h)\) is a neighborhood of \( \sigma \) in \((V, S)\). But if \( \epsilon \) or \( h \) is not in \( V \), we can construct \( \epsilon', h' \) such that both \( \epsilon' \) and \( h' \) are in \( V \) and \( \sigma \in (\epsilon', h') \subset (\epsilon, h) \).

Take out all the elements in \((\epsilon', h')\) which are not in \( V \), the remaining set \((\epsilon', h')\) is a neighborhood of \( \sigma \) in \((V, S)\).
All these statements show that if $\sigma \in V$ and $(e, h)$ is a neighborhood of $\sigma$ in $(X, \tau)$, then there exists a neighborhood $(e', h')$ of $\sigma$ in $(V, S)$ such that $(e', h') \subset (e, h)$.

In 3.1 we mentioned that $(X, \tau)$ is $T_1$ and regular. By the help of the above statement, we see that $(V, S)$ is also $T_1$ and regular.

Now we return to the construction of $e'$ and $h'$, if $e$ or $h$ is not in $V$. Suppose that $h$ is not in $V$. Set $\delta = \inf \{d(x, y); x \in e, y \in h\}$ $d$ is the usual Euclidean metric.

(a) If $\delta > 0$, then let $h'$ be the conic which is obtained by translating $\sigma$ along its real axis by the distance $\delta$ (if $\sigma$ is a parabola or a hyperbola) or by magnifying $\sigma$ such that its major and minor diameters increase $\delta$ in length (in case that $\sigma$ is an ellipse) In each case we have $h' \subset h' \cap \sigma$ and $h' \in V$.

(b) If $\delta = 0$, this only happens when both $\sigma$ and $h$ are hyperbolas such that they have at least one common asymptote.

Let $a_1, a_2$ denote the vertices of $\sigma$ and $b_1, b_2$ be the points of intersection of $h$ and the real axis of $\sigma$. Let $c_1, c_2$ be the points in the interior of the segment $a_1b_1, a_2b_2$ respectively. Denote $h'$ to be
the hyperbola with vertices $c_1$ and $c_2$ and the asymptotes the same as those of $\sigma(c_1,c_2$ are equal distant from $a_1,a_2$ respectively). Then $h' \subset \sigma \cap h_*$ and $h' \in \mathcal{V}$.

Similarly we can construct $\varepsilon'$, such that $\varepsilon' \subset \varepsilon \cap \sigma_*$ and $\varepsilon' \in \mathcal{V}$. Therefore we get $(\varepsilon',h') \subset (\varepsilon,h)$ and $(\varepsilon',h')$ is a neighborhood of $\sigma$ in $(\mathcal{V},S)$.

3.7 $(\mathcal{V},S)$ has a countable base.

Proof: Let $\mathcal{G}_1$ be the family of all ellipses in $\mathcal{V}$ with centers at rational points and rational major and minor axes. Let $\mathcal{G}_2$ be the corresponding family for hyperbolas. Denote $\mathcal{G}_3$ to be the family of all parabolas in $\mathcal{V}$ of which the vertices and foci are points with rational coordinates. As $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are countable. $\mathfrak{S} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ is also countable.

If $\mathcal{U}$ is the family of all open neighborhoods $(\varepsilon,h)$ with $\varepsilon, h \in \mathfrak{S}$, then $\mathcal{U}$ is countable. We show that $\mathcal{U}$ is a base for $(\mathcal{V},S)$.

Given $\sigma \in \mathcal{V}$ and a neighborhood $G$ of $\sigma$, there exists a neighborhood $(\varepsilon,h)$ of $\sigma$ such that $(\varepsilon,h) \subset G$.

If $(\varepsilon,h) \in \mathcal{U}$, the result is clear. If $(\varepsilon,h) \notin \mathcal{U}$, because the rational points are dense in the $E^2$ plane, $(\varepsilon',h')$ exists in $\mathcal{U}$ so that $\sigma \in (\varepsilon',h') \subset (\varepsilon,h)$. Hence $\mathcal{U}$ is a base for $(\mathcal{V},S)$.
(V,S) is $T_1$, regular and has a countable base. By Urysohn's metrization theorem, the space (V,S) is metrizable and separable. That is $(X/R,\tau')$ is both metrizable and separable, since it is topologically equivalent to (V,S).

3.8 The topology defined by Lane for the non-degenerate conics is another topology which is both metrizable and separable.

If $\sigma$ is a non-degenerate ellipse or a hyperbola, Dr. Lane defines [1] a neighborhood of $\sigma$ to be the region which lies outside a non-degenerate conic $\epsilon$ and inside a non-degenerate conic $\eta$, where $\epsilon \subset \sigma^*$ and $\sigma \subset \eta^*$. i.e. the neighborhood of $\sigma$ is $(\epsilon,\eta)$.

If $\sigma$ is a non-degenerate parabola, he defines [5] a neighborhood of $\sigma$ to be the region which lies outside an ellipse (non-degenerate) $\epsilon$ and inside a branch of non-degenerate hyperbola $\eta$, where $\epsilon \subset \sigma^*$ and $\eta \subset \sigma^*$. All of these definitions give a topology for $X$. We call it Lane's topology and denote it by (X,L).

Recall the space (Z,L) in which a neighborhood of a convex curve $\sigma$ is defined to be the set $(\epsilon,\eta)$ of all curves $\xi \subset (\epsilon \cap \eta)$ and $\epsilon \subset \xi^*$, where $\epsilon,\eta$ are two fixed curves in Z with $\epsilon \subset \sigma^*$, $\sigma \subset \eta^*$. Let a line L be assigned to be a line at infinity, and $A^2$ be the affine space with L as line at infinity. If $\sigma, \epsilon, \eta$ are non-degenerate conics, we consider the following situations:
1. If $\sigma$ is an ellipse or a hyperbola in $A^2$, then the set $(e,h) \cap X$ is just a neighborhood of $\sigma$ in $X$ which Lane defined.

2. If $\sigma$ is a parabola in $A^2$, then $e$ must be an ellipse and $h$ a hyperbola in $A^2$, since $e \subset \sigma^*$ and $\sigma \subset h^*$.

Therefore the set $(e,h) \cap X$ is just what Lane defined a neighborhood of the parabola $\sigma$ in $X$.

Now consider $X$ as a subset of $Z$ (in fact, $X$ is a subset of $Z$ in the affine space $A^2$). Then the relative topology for $X$ is constructed by the family of $(e,h) \cap X$ for all neighborhoods $(e,h)$ in $(Z,L)$. Being a subspace of the metrizable space $(Z,L)$, this relative topology for $X$ is metrizable. As we analyzed, $(e,h) \cap X$ is the neighborhood system what Lane defined for the elements of $X$. Hence the above relative topology for $X$ is just the Lane's topology for $X$, i.e. $(X,L)$. Hence $(X,L)$ is metrizable.

Furthermore, it follows from the proof of the metrization of $(X,\tau)$ that $(X,L)$ has a countable base. Thus $(X,L)$ is both metrizable and separable.

3.9. The following diagram illustrates the relations between the spaces studied in this thesis.
Here the "vertical line" means that the lower space is a subspace of the upper space. The "diagonal line" means that the upper space is a refinement of the lower space. The "horizontal line" means identity or topologically equivalent.

"cpt." means compactification. "W" means "in wide sense". For example (Z,L) means that (X,L) is the space obtained by confining a subspace of (Z,L) to the affine space.
BIBLIOGRAPHY


