ON GREEN'S FUNCTION FOR THE LAPLACE OPERATOR IN AN UNBOUNDED DOMAIN

by

DENTON ELWOOD HEWGILL

B.Sc., University of British Columbia, 1963.

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

in the Department

of

Mathematics.

We accept this thesis as conforming to the
required standard

THE UNIVERSITY OF BRITISH COLUMBIA

July, 1966.

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Mathematics

The University of British Columbia
Vancouver 8, Canada

Date Sept 6, 1966.

Abstract

Supervisor: Dr. Colin W. Clark.

This thesis investigates the Green's functions for the operator T defined by

$$\mathcal{D}(T) = H_o^1(E) \cap \{f \in L_2(E) \mid \Delta f \in L_2(E)\}$$

$$Tf = -\Delta f \qquad \text{for} \qquad f \in \mathcal{D}(T).$$

Here $H_o^1(E)$ is a standard Sobolev space, Δ is the Laplacian, and E is a domain in R_n which is taken to be "quasi-bounded". In particular we assume that E lies in the half-space $x_1 > 0$ and is bounded by the surface obtained by rotating $\varphi(x_1)$ about the x_1-axis, where φ is continuous, $\varphi(x_1) > 0$ and $\varphi^k \in L_1(0,+\infty)$ for some $k > 0$.

The Green's function $G(x,y,\lambda)$ for the operator $T + \lambda$ is obtained as the limit of the Green's functions for the well known problem on the truncated domain $E_X = E \cap [x_1 < X]$. Most of the expected properties of the function are developed including the inequality

$$0 \leq G(x,y,\lambda) < K(\rho\sqrt{-\lambda}) \qquad \rho = |x-y|$$

where K is the fundamental singularity for the domain.

The eigenvalues and eigenfunctions are constructed, and it is shown that

$$\lambda_{X,n} \to \lambda_n \qquad \text{as } X \to \infty \qquad \text{for each } n,$$

where $\lambda_{X,n}$ and λ_n are the eigenvalues for the problem on E_X and E respectively. Furthermore, it is shown that the eigenvalues $\{\lambda_n\}$ are positive with no finite limit point, and the corresponding eigenfunctions are complete.

A detailed calculation involving the inequality displayed above shows that some iterate $(G^{(k_o)})$ of $G(x,y,\lambda)$ is a Hilbert-Schmidt kernel. From this property of $G^{(k_o)}$ it follows that the series $\Sigma\, \lambda_n^{-2k_o}$ is convergent. From the convergence of this series three results are derived. The first one is an expansion formula in terms of the complete set of eigenfunctions, and the second is that some iterate of the Green's function tends to zero on the boundary. The last one is the construction of the solution $H(x,\lambda,f)$ for the boundary value problem

$$\Delta H + \lambda H = f$$
$$H(x,\lambda,f) \to 0 \qquad \text{as} \qquad x \to \partial E$$

for a sufficiently regular f on E.

The final property of the Green's function, namely, that $G(x,y,\lambda)$ tends to zero on the boundary, is proved using the fact that $G^{(k_o)}$ is zero on the boundary, and certain inequalitites estimating the iterates $G^{(k)}$. $G(x,y,\lambda)$ is also shown to be unique.

The asymptotic formula

$$N_\tau(\lambda) \sim \frac{\lambda}{4\pi} \int_E \tau(x)\, dx ,$$

a generalization of the usual asymptotic formula of Weyl for the eigenvalues, first given by C. Clark, is derived for these quasi-bounded domains. Finally, the usual asymptotic formula due to Carleman for the eigenfunctions is shown to remain valid.

Table of Contents

Acknowledgment

The author wishes to thank his advisor Dr. Colin W. Clark for suggesting the topic of this thesis and for his helpful advice during its preparation. He also wishes to thank Dr. C.A. Swanson and E. Gerlach for their careful reading of the draft form of this thesis. Furthermore, the financial assistance of the National Research Council through a scholarship is gratefully acknowledged.

1. Introduction

The eigenvalue problem for the Laplacian $(-\Delta)$, namely

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0 \qquad x \in E \\ u(x) &= 0 \qquad x \in \partial E, \end{aligned}$$

has been the subject of much discussion in the literature. Courant and Hilbert [5] extensively investigated the problem when the domain E is bounded. Titchmarsh [10], when he discussed the Schrödinger equation gave an alternate treatment of the problem and extended many of the results to the whole plane.

Rellich [9] was the first to obtain results when the domain E was unbounded but different from the whole plane. He showed that when the domain was "narrow at infinity" along a given ray, the problem had a discrete set of eigenvalues. Molčanov [8] generalized Rellich's result to include domains which were "narrow at infinity" in a wider sense, specifically, those domains which do not contain infinitely many disjoint spherical balls of equal positive radius. Glazman in his recent book [7] gave a discussion of problems closely connected to the eigenvalue problem for unbounded domains. Still more recently, Clark [4] published a paper "An embedding theorem in function spaces" which allows the treatment of an arbitrary elliptic operator instead of the Laplacian.

The present discussion will center around the existence and properties of the Green's function (GF) for the problem. The general method of attack and the arguments in many cases follow those used by Titchmarsh [10].

The first result, whose proof is an application of the Ascoli-Arzela theorem to the well known case when the domain is bounded, is the existence of the GF. It is important to note that although this construction gives many of the properties which a GF must possess, it does not show the GF tends to zero at the boundary. From this construction a basic result easily follows, namely that the GF is bounded by the fundamental singularity for the domain. The fundamental singularity (a Bessel function) is the GF for the problem on the whole space. This relation, which is not needed in Titchmarsh's development, forms a crucial link in the present argument.

In Part 3 we discuss the eigenvalues and eigenfunctions of the problem, or more precisely, those of the operator T defined by

$$\mathscr{D}(T) = H_0^1(E) \cap \{f \in L_2(E) \mid \Delta f \in L_2(E)\}$$

$$Tf = -\Delta f \quad \text{when} \quad f \in \mathscr{D}(T),$$

where E satisfies the "narrowness at infinity" condition given by Clark [4]. The main result to be proved in this section is that the eigenfunctions of T are contained in $H_0^1(E)$. From this result we conclude that

$$\lambda_{X,n} \to \lambda_n \quad \text{as} \quad X \to \infty \quad \text{for each } n,$$

where the $\lambda_{X,n}$ are the eigenvalues for the truncated domain

E_X (see Part II) and the λ_n are the eigenvalues for T defined on E. This result appears to be new but not unexpected. This section concludes with some discussion about the eigenfunctions near the boundary.

At various points in the argument restrictions are applied to the domain. Most of these restrictions are smoothness conditions for the boundary; however the most important restriction, which is introduced in Part 4, is an assumption about the rate at which the domain narrows at infinity. This condition is as follows: for example, when dimension of $E=2$, let the boundary of E be the positive x_1-axis and the set $\{x_1, \varphi(x_1)\}$ $x_1 \geq 0$, where φ is a positive continuous function; then we assume that there exists an integer k such that $\varphi^k(x_1)$ is integrable to infinity.

It does not seem possible to treat the GF directly as Titchmarsh does; instead in Part 4 we consider the iterates of the GF, which are much smoother. An important theorem in this respect is that there exists an iterate of the GF which is a Hilbert-Schmidt kernel. From the Hilbert-Schmidt property we obtain that the series $\Sigma\, \lambda_n^{-2k}$ converges for some k which is dependent on the narrowness of E. Once it is known that this series converges we can show that the iterate which is Hilbert-Schmidt actually tends to zero on the boundary. This result, as we show in Part 5, in turn implies that the GF itself tends to zero at the boundary and that it is unique.

The last section, Part 6, discusses application of the GF to asymptotic problems for the eigenvalues and eigenfunctions. The first application is a proof of the asymptotic formula

$$N_\tau(\lambda) \sim \frac{\lambda}{4\pi} \int_E \tau(x)\, dx$$

(See Theorem 6.1 for notation), which reduces to the well known formula of Weyl, i.e.

$$N(\lambda) \sim \frac{\lambda}{4\pi} \cdot \text{area } E$$

when the domain E has finite area. This asymptotic formula was first given by Clark [3] for a smaller class of domains. The second application proves that the usual asymptotic formula due to Carleman for the eigenvalues extends to unbounded E.

Throughout the work all lemmas and theorems are numbered successively, for example Theorem 2.4 means Theorem 4 Part 2. Within a section equations and definitions are referred to by a number but if we wish to refer to equation 6 of Part 4, then we would write equation (4.6). References are referred to by the author's name and a number corresponding to his paper in the bibliography at the end.

Part 2. Construction of the Green's function and some of its elementary properties.

In this section we shall give a construction of the Green's function for the Laplacian on domain which will be defined below.

Notations:

Let R_n denote Euclidean n-space, let $x=(x_1, x_2, \ldots, x_n)$ denote a typical point in R_n and let $|x-y|$ be the Euclidean norm. Let E denote a simply connected unbounded domain R_n, with boundary ∂E. For a given E define

$$E' = \{p \in E \mid d(p, \partial E) \geq b\} ,$$

where b is a fixed positive constant which determines E', and d denotes distance. Furthermore, let

$$L = \{\lambda \in \mathbb{C} \,\Big|\, |\lambda| \leq K, \ \text{ and } \ |\text{Im } \lambda| \geq \nu \ \text{ or } \ \text{Re } \lambda \leq -2\nu\},$$

where K and ν are fixed positive constants which determine L. We will sometimes write $L=L(\nu)$ to indicate the dependence of L on ν. Finally we set

$$E_X = \{x \in E \,\Big|\, |x_i| < X, \ \ i=1, 2, \ldots n\} \qquad X > 0 .$$

Consider the boundary value problem

$$\begin{aligned} \Delta\, u(x) + \lambda\, u(x) &= 0 \qquad & x \in E \\ u(x) &= 0 \qquad & x \in \partial E \end{aligned} \tag{1}$$

where Δ denotes the Laplacian. We define the operator T in $L_2(E)$ by:

$$\left.\begin{aligned} \mathscr{D}(T) &= H_0^1(E) \cap \{f \in L_2(E) \mid \Delta f \in L_2(E)\} \\ Tf &= -\Delta f \qquad \text{if} \qquad f \in \mathscr{D}(T) \end{aligned}\right\} \tag{2}$$

where $H_0^m(E)$ denotes the standard Sobolev space (see e.g. Dunford and Schwartz [6, p. 1652]) with the norm

$$\|f\|_m = \left[\int_E \sum_{|\alpha| \leq m} |D^\alpha f(x)|^2 dx \right]^{1/2} \tag{3}$$

in which we use the standard notations

$$\left.\begin{aligned} D_i &= \frac{\partial}{\partial x_i} \\ D^\alpha &= D_1^{\alpha_1} \dots D_n^{\alpha_n} \quad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \\ |\alpha| &= \Sigma\, \alpha_i \ . \end{aligned}\right\} \tag{4}$$

The classical theory for problem (1) states that if E is bounded, then T is a self-adjoint operator in the Hilbert space $L_2(E)$, and has a discrete spectrum consisting of isolated eigenvalues λ_n, $n=1, 2, 3, \dots$ etc., each eigenvalue corresponding to an eigenfunction $u_n(x)$.

Let us assume that E_X is sufficiently regular to allow us to construct a Green's function $G_X(x,y,\lambda)$ for problem (1) in the sense of Titchmarsh [10, Ch 14]. In the present section we shall need the following properties of $G_X(x,y,\lambda)$:

$G_X(x,y,\lambda)$ has a standard singularity for $x=y$. (See Lemma 2.7).

If $x \in E_X$, $\lambda \in$ some L, then $G_X(x,y,\lambda)$ tends to zero as y approaches the boundary of E_X. If we define the operator $G_{X,\lambda}$ by

$$G_{X,\lambda} f(x) = \int_{E_X} G_X(x,y,\lambda) f(y)\, dy \quad f \in L_2(E_X) \tag{5}$$

then we need two theorems:

$$G_{X,\lambda} u_n(x) = (\lambda_n - \lambda)^{-1} u_n(x) \quad \text{for } \lambda \text{ non real} \tag{6}$$

and

$$\|G_{X,\lambda} f\| \leq \nu^{-1} \|f\| \quad \text{where } \lambda \in L(\nu). \tag{7}$$

All these results are proved in Titchmarsh [10, Ch 14] except for (7), which is our first lemma.

Lemma 2.1. Suppose that either $|\mathrm{Im}\, \lambda| \geq \nu > 0$ or that $\mathrm{Re}\, \lambda \leq -2\nu$ then

$$\|G_{X,\lambda} f\| \leq \nu^{-1} \|f\| . \tag{8}$$

Proof. We know from the classical theory that $G_{X,\lambda}$ is the resolvent operator of a non-negative self-adjoint operator T in $L_2(E_X)$ i.e.

$$G_{X,\lambda} = (T-\lambda I)^{-1}.$$

Now if $\lambda=\alpha+i\beta$, $\alpha \leq -2\nu < 0$ and $|\beta| \leq \nu$, then

$$\|T-\lambda I\| \geq \|T+2\nu I\| - \|\beta I\|$$
$$\geq \|T+2\nu I\| - \nu .$$

But since $T \geq 0$, $T+2\nu I \geq 2\nu I$, so that $\|T+2\nu I\| \geq 2\nu$. Hence $\|T-\lambda I\| \geq \nu$, and therefore

$$\|G_{X,\lambda}\| = \|(T-\lambda I)^{-1}\| \leq \nu^{-1} ,$$

which is equivalent to (8). One can easily show, using the results of Titchmarsh [10, Ch 12], that (8) holds for $|\mathrm{Im}\ \lambda| \geq \nu > 0$, completing the lemma.

The following construction of the Green's function for the problem (1) will be given only for the plane. A similar construction is available for higher dimensions. The argument, up to a point, will be similar to Titchmarsh's argument for the whole space R_2; however, the presence of the boundary of E requires us to be much more precise about the nature of the convergence.

Let

$$g(x) = g(x,u) = \begin{cases} \frac{1}{2\pi}\left[\log \frac{R}{r} - \frac{1}{2}\left(1 - \frac{r^2}{R^2}\right)\right] & \text{if } r \leq R \\ 0 & \text{if } r > R, \end{cases}$$

where $r=|x-u|$. We remove the singularity from $G_X(x,y,\lambda)$ by subtracting the function $g(x)$; we set

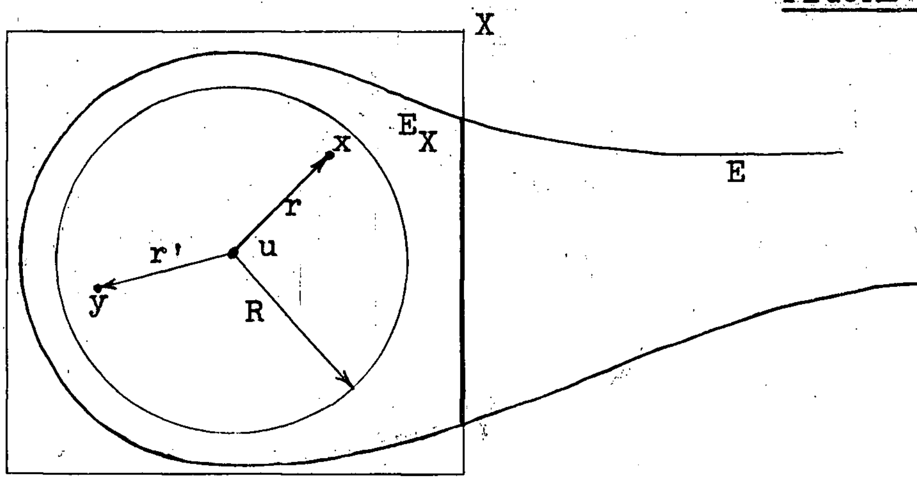
$$\Gamma_X(x,y,\lambda) = G_X(x,y,\lambda) - g(x,y) \tag{9}$$

Note: Γ_X is dependent on R, since g is.

Theorem 2.2. The set $[X|X>0]$ has a subsequence $[X_k]$, $X_k \to \infty$, such that the sequence $[G_{X_k}(x,y,\lambda)]$ converges pointwise to a function $G(x,y,\lambda)$ for all $x, y \in E$ and all λ not on the non-negative part of the real axis. Furthermore, given $E' \subset E$, $L(\nu)$ and $X > 0$, the sequence $[\Gamma_{X_k}(x,y,\lambda)]$ converges uniformly for $x, y \in E'_X$ and $\lambda \in L(\nu)$.

Proof.

FIGURE 1



Let $f(x) = G_X(x,y,\lambda)$ and $g(x)$ be as above; for $x, y, u \in E_X$ write $r = |x-u|$ and $r' = |y-u|$, as is figure 1. Let the circle $|x-u| = r<R$ be contained in E_X. By Green's formula for the region $r<R$ (cf. Titchmarsh [10, p. 34]), we have

$$G_X(u,y,\lambda) - g(u,y) = \frac{1}{\pi R^2} \int_{r \le R} G_X(x,y,\lambda)dx + \tag{10}$$
$$+ \lambda \int_{r \le R} g(x,u)\ G_X(x,y,\lambda)dx \quad .$$

Let $\Gamma_X(x,y,\lambda) = G_X(x,y,\lambda) - g(x,y)$ for this R, and

$$F(x,u) = \begin{cases} \pi^{-1} R^{-2} + \lambda g(x,y) & \text{if } r \leq R \\ 0 & \text{if } r > R. \end{cases}$$

Substituting these relations into (10), we obtain

$$\Gamma_X(u,y,\lambda) = \int_{E_X} G_X(x,y,\lambda)\, F(x,u)dx$$
$$= (G_X(\cdot,y,\lambda),\ F(\cdot,u)) \ .$$

The first problem in the proof is to show that $\Gamma_X(u,y,\lambda)$ is uniformly bounded for $u, y \in E^!_{X_o}$ and $\lambda \in L(\nu_o)$ (where X_o and ν_o are arbitrary fixed constants). We first apply the Schwarz inequality to Γ_X to obtain

$$|\Gamma_X(u,y,\lambda)| \leq \|G_X(\cdot,y,\lambda)\|\ \|F(\cdot,u)\|. \tag{11}$$

For the second norm in (11), we have

$$\|F(\cdot,u)\|^2 \leq \frac{2}{\pi^2 R^4} \int_{r\leq R} dx + 2|\lambda|^2 \int_{r\leq R} |g(x,u)|^2\, dx \ ,$$

from the definition of $F(x,u)$. Therefore

$$\|F(\cdot,u)\|^2 \leq K(u,R,|\lambda|),$$

where $K(u,R,|\lambda|)$ is bounded if u and λ are bounded, the distance between the boundary of E and u is greater than R, and R is bounded away from zero. The first term in (11), namely $\|G_X(\cdot,y,\lambda)\|$, is more difficult to estimate. One proceeds as follows: Observing that the form of $\Gamma(u,y,\lambda)$ satisfies

Lemma 2.1 (note $\Gamma_X(u,y,\lambda) = G_{X,\lambda}F(u,y))$, we have

$$\begin{aligned} \| \Gamma_X(u,\cdot,\lambda)\|^2 &\leq \nu^{-2} \|F(\cdot,u)\|^2 \\ &\leq \nu^{-2} K(u,R,|\lambda|), \end{aligned} \tag{12}$$

where either $|\mathrm{Im}\ \lambda| \geq \nu > 0$ or $\mathrm{Re}\ \lambda \leq -2\nu$ and $\nu > 0$. From the definition of Γ_X in (9) one has

$$\|G_X(u,\cdot,\lambda)\|^2 \leq 2\|\Gamma_X(u,\cdot,\lambda)\|^2 + 2\|g(u,\cdot)\|^2,$$

from which it follows that

$$\|G_X(u,\cdot,\lambda)\|^2 \leq (1+\nu^{-2})\ K(u,R,|\lambda|) \tag{13}$$

In these formulas it is important to note that K is independent of X. Combining these with the Schwarz inequality above, one obtains the final estimate

$$|\Gamma_X(u,y,\lambda)| \leq [(1+\nu^{-2})\ K(y,R,|\lambda|)\ K(u,R,|\lambda|)]^{1/2},$$

where y and u cannot be closer to the boundary of E than the distance R. We can shorten this into

$$|\Gamma_X(u,y,\lambda)| \leq K(u,y,R,|\lambda|), \tag{14}$$

where K is bounded if for some given E', X_o and L_o, $u, y \in E'_{X_o}$ and $\lambda \in L_o$.

We now apply the same argument as Titchmarsh [10, p. 35], to arrive at

$$| \Gamma_X(u',y,\lambda) - \Gamma_X(u,y,\lambda)| < \epsilon, \tag{15}$$

where $|u'-u| < \delta = \delta(\epsilon)$ independent of (X). In order to do this, one just needs to consider the representation of Γ_X, equation (10), and make appropriate estimates using the X-uniform bound (14). By symmetry a similar result holds for the y variable.

We now have the desired equicontinuity in u and y, but we must also have it in the λ variable, in order to apply the Ascoli-Arzela theorem. This is achieved as follows: by the resolvent equation known for $G_X(x,y,\lambda)$ we have

$$D_\lambda \Gamma_X = D_\lambda G_X = (G_X(u,\cdot,\lambda), \; G_X(\cdot,y,\lambda)) .$$

The right hand side is bounded, as X tends to infinity, for $u, y \in E'_{X_o}$ and $\lambda \in L_o$, by the estimate (13) and the Schwarz inequality. Since the partial derivative of Γ_X with respect to λ is uniformly bounded, Γ_X will be uniformly continuous in λ.

The above calculations and remarks show that the set of functions $[\Gamma_X(x,y,\lambda)]$ as $X \to \infty$, for $x, y \in E'_{X_o}$ and $\lambda \in L_o$ (E', X_o and L_o being fixed), is equicontinuous in each of the three variables separately. The Ascoli-Arzela theorem says that such a set is compact, i.e. there exists a function $\Gamma(x,y,\lambda)$ such that $\Gamma_{X_k}(x,y,\lambda)$ tends to $\Gamma(x,y,\lambda)$, uniformly for some subsequence $\{X_k\}$ tending to infinity, when $x, y \in E'_{X_o}$ and $\lambda \in L_o$.

Now by a simple diagonalization process, we can obtain a sequence $[\Gamma_{X_k}(x,y,\lambda)]$ which converges (pointwise) for all $x, y \in E$, and $\lambda \in \mathbb{C}$ with λ not on the non-negative real axis. Namely, let

$$X_o < X_1 < \ \ldots \ \to \infty$$
$$E'_o \subset E'_1 \subset \ \ldots \ \to E$$
$$L_o \subset L_1 \subset \ \ldots \ \to \{\lambda \mid \operatorname{Im} \lambda \neq 0 \ \text{ or } \ \lambda < 0\}.$$

We could choose for example E'_i to be $E'_n = \{p \in E \mid d(p, \partial E) \geq n^{-1}\}$ etc. Let $[X(0,n)]$ be a sequence approaching infinity such that $[\Gamma_{X(0,n)}]$ converges uniformly when x, y and λ are restricted to E'_o, X_o and L_o. Let $[X(1,n)]$ be a subsequence of $[X(0,n)]$ such that $[\Gamma_{X(1,n)}]$ converges uniformly for x, y, λ restricted to E'_1, X_1 and L_1, and so on. Then the diagonal sequence $X^* = [X(n,n)]$ is such that $[\Gamma_{X^*}]$ converges pointwise for desired values of x, y, λ. We note that this sequence also has the property that, given any E', X and $L(\nu)$, $[\Gamma_X]$ converges uniformly for $x, y \in E'_X$ and $\lambda \in L(\nu)$. This completes the proof of Theorem 2.2.

A consequence of the above theorem is that we have a Green's function for E with the representation

$$G(x,y,\lambda) = \Gamma(x,y,\lambda) - g(x,y).$$

Furthermore, Γ satisfies the integral equation

$$\Gamma(u,y,\lambda) = \frac{1}{\pi R^2} \int_{r \leq R} G(x,y,\lambda)dx + \lambda \int_{r \leq R} g(x,u)\ G(x,y,\lambda)dx \qquad (16)$$

where $r=|x-u|$ and R is so small that the circle $|x-u|\leq R$ is contained in E. We shall now examine some of the elementary properties of the Green's function.

<u>Theorem 2.3</u>. $G(x,y,\lambda)$ is continuous for $x\neq y$ and it is such that

$$G(x,y,\lambda) = \frac{1}{2\pi}\log\frac{1}{p} + O(1) \quad \text{as} \quad p \to 0$$

where $p = |x-y|$, and also

$$\frac{\partial\, G(x,y,\lambda)}{\partial p} = -\frac{1}{2\pi p} + O(1) \quad \text{as} \quad p \to 0.$$

<u>Theorem 2.4</u>. $G(x,y,\lambda)$ has continuous partial derivatives up to the second order except at $x=y$, and

$$\{\Delta + \lambda\}\, G(x,y,\lambda) = 0 \quad \text{if} \quad x \neq y\ .$$

The above two theorems are proved in the same manner as in the case of bounded E.

<u>Theorem 2.5</u>.

$$\|G(u,\cdot,\lambda)\|^2 \leq \nu^{-2} K(u,R,|\lambda|)$$

where K is bounded if $u \in E'_{X_o}$ and $\lambda \in L_o(\nu)$.

Proof. By inequality (11) we have

$$\|\Gamma_X(u,\cdot,\lambda)\|^2 \leq \nu^{-2} K(u,R,|\lambda|).$$

Then for E' and X_o fixed $(E^*=E'_{X_o})$

$$\int_{E^*} |\Gamma_X(u,y,\lambda)|^2 dy \leq \nu^{-2} K(u,R,|\lambda|) \qquad (X\geq X_o).$$

First let $X \to \infty$ through the sequence defined in Theorem 2.2.

We then have

$$\int_{E^*} |\Gamma(u,y,\lambda)|^2 dy \;\leq\; \nu^{-2}\, K(u,R,|\lambda|)$$

where the right hand side is independent of E^*. Thus, letting E^* tend to E, one has by Fatou's lemma

$$\|\Gamma(u,\cdot,\lambda)\|^2 \;\leq\; \nu^{-2}\, K(u,R,|\lambda|)\ .$$

If we combine this inequality with the definition of Γ and the result $\|g(x,\cdot)\|^2 = AR^2$, then Theorem 2.5 will follow.

We define the operator G_λ as follows

$$G_\lambda\, f(x) \;=\; \int_E G(x,y,\lambda)\, f(y)\, dy \qquad f \in L_2(E).$$

The integral exists in view of equation (13) for all $x \in E$ and all λ contained in some L.

<u>Theorem 2.6</u>. If $H(x,\lambda,f) = -G_\lambda f(x)$, where $f \in L_2(E)$ and λ is not on the non-negative real axis, then

$$\|H(\cdot,\lambda,f)\| \leq \nu^{-1}\, \|f\|\ ,$$

$$\{\Delta+\lambda\}\, H(x,\lambda,f) = f(x) \quad \text{and}$$

$$H(x,\lambda) = O(|\lambda|^{1/2}\nu^{-1}) \quad \text{uniformly for} \quad x \in E'_{X_0}\ ,$$

$$|\lambda| \geq \delta > 0 \quad \text{and} \quad \operatorname{Im}\lambda = \nu \neq 0.$$

We shall prove the first result and remark that the other two are proved by the usual methods. (See Titchmarsh [10, Ch 12]).

Proof. The first item to show is that $H_X(x,\lambda,f) \to H(x,\lambda,f)$

uniformly $x \in E'_{X_0}$, where $H_X(x,\lambda,f) = -G_{X,\lambda}f(x)$. Now

$$H_X - H = \int_E G(x,y,\lambda)\, f(y)\, dy - \int_{E_X} G_X(x,y,\lambda)\, f(y)\, dy .$$

If we set $G_X = 0$ outside E_X we have

$$H_X - H = \int_{E-E^*} [G(x,y,\lambda) - G_X(x,y,\lambda)]\, f(y)\, dy + \int_{E^*} [G(x,y,\lambda) - G_X(x,y,\lambda)]\, f(y)\, dy,$$

for any E^* equal to some E'_{X_0}. By the Schwarz inequality the second term tends to zero as $X \to \infty$ since G_X tends to G uniformly over E^*. Applying the Schwarz inequality to the first term, one has

$$|\text{first term}| \leq \int_{E-E^*} |G(x,y,\lambda) - G_X(x,y,\lambda)|^2 dy \cdot \int_{E-E^*} |f|^2 dx$$

$$\leq \{2\|G(x,\cdot,\lambda)\|^2 + 2\|G_X(x,\cdot,\lambda)\|^2\} \int_{E-E^*} |f|^2 dx.$$

This term then can be made small uniformly in X by noting $f \in L_2(E)$, $x \in E'$ and the results of equation (13) and Theorem 2.5. We now have the desired convergence for H_X.

If $\lambda \in L(\nu)$ and $E^* \subset E'_X$, we have by Lemma 2.1

$$\int_{E^*} |H_X(x,\lambda,f)|^2 dx \leq \nu^{-2} \int_{E'_X} |f|^2 dx$$

$$\leq \nu^{-2} \int_E |f|^2 dx = \nu^{-2}\, \|f\|^2 .$$

If we let $X \to \infty$ through the sequence defined in Theorem 2.2, we have by the uniform convergence for H_X, proved above,

$$\int_{E^*} |H(x,\lambda,f)|^2 dx \leq \nu^{-2} \|f\|^2 .$$

If we let E^* tend to E, we have by Fatou's lemma

$$\|H(\cdot,\lambda,f)\| \leq \nu^{-1} \|f\|. \qquad \text{Q.E.D.}$$

<u>Remark</u>: At this point in the argument we do not know the boundary behaviour of $G(x,y,\lambda)$. Although we know that for each fixed X, $G_X(x,y,\lambda)$ goes to zero on the boundary as y tends to the boundary, our convergence theorem (2.2) is not strong enough to imply the result for $G(x,y,\lambda)$.

We now turn our attention to the fundamental singularity for the domain and its relation to the Green's function. Here we consider the case of a general dimension n.

<u>Lemma 2.7</u>. (Brownell [2, p. 555, Lemma 2.1]).

There exists a real positive function $H_{n,\omega}(r)$ defined for all real, positive r and ω, and all integers $n \geq 1$, which is real-analytic in r, and such that the following holds:

$$\Delta H_{n,\omega}(r) = \omega^2 H_{n,\omega}(r)$$

$$\lim_{r\to 0+} [H'_{n,\omega}(r)\, r^{n-1}\, \sigma_n] = -1$$

$$0 < H_{n,\omega}(r) \leq M_n\, r^{-(n-2)} \exp(-\omega r/4) \qquad n \geq 3$$

$$0 < H_{2,\omega}(r) \leq M_2 \frac{1 + \log(1+(\omega r)^{-1})}{1 + \sqrt{\omega r}} \exp(-\omega r),$$

where the M_n's are constants independent of ω and r, and σ_n is the volume of the unit ball in R_n. The $H_{n,\omega}(r)$ so defined is called the fundamental singularity for $\Delta-\omega^2$.

For the details of the proof see Brownell's paper [2, p. 555].

<u>Theorem 2.8</u>. Suppose that $G(x,y,\lambda)$ is obtained as a pointwise limit of functions $G_X(x,y,\lambda)$, as in Theorem 2.2 for $n=2$. Then

$$0 \leq G(x,y,\lambda) \leq H_{n,\omega}(p) \leq K(p\omega)$$

where $p=|x-y|$, $K(p\omega)$ is the bound for $H_{n,\omega}(p)$ given in Lemma 2.7, and $\lambda=-\omega^2$ where $\omega > 0$.

Proof. Since E_X is bounded the maximum principle can be applied to prove

$$0 \leq G_X(x,y,\lambda) \leq H_{n,\omega}(p) ,$$

but for fixed $x, y \in E$, $G_X(x,y,\lambda)$ tends to $G(x,y,\lambda)$ as X goes through the sequence defined in Theorem 2.2. Thus

$$0 \leq G(x,y,\lambda) \leq H_{n,\omega}(p) \qquad \text{all } x, y \in E.$$

Note that this relation implies that $G(x,y,\lambda)$ tends to zero, for fixed x and λ, as y tends to infinity.

<u>Theorem 2.9</u>. $G(x,y,\lambda) = G(y,x,\lambda) \quad x, y \in E.$

Proof. We know that

$$G_X(x,y,\lambda) = G_X(y,x,\lambda)$$

for $x, y \in E$ and X sufficiently large. Since the left hand side tends to $G(x,y,\lambda)$ and the right hand side tends to $G(y,x,\lambda)$, as X tends to infinity, the result follows.

The final result in this section is

Theorem 2.10. $D_\lambda G(x,y,\lambda) = G^{(2)}(x,y,\lambda)$ and

$$D^n_\lambda G(x,y,\lambda) = n!\ G^{(n+1)}(x,y,\lambda)$$

for λ a negative number and $x, y \in E$.

Proof. First we must establish the "resolvent equation":

$$(\lambda-\lambda')\ (G(\cdot,x,\lambda),\ G(\cdot,y,\lambda')) = G(y,x,\lambda) - G(x,y,\lambda') \tag{16}$$

for λ and λ' negative.

The usual proof of (16) requires that $G(x,y,\lambda)$ be zero on the boundary; however we want this lemma independent of the boundary result so we proceed differently.

We have

$$(\lambda-\lambda') \int_{E_X} G_X(s,x,\lambda)\ G_X(s,y,\lambda')ds = G_X(y,x,\lambda) - G_X(x,y,\lambda') \tag{17}$$

by Green's theorem, since E_X is bounded. Let $x, y \in E^*$ where E^* is some fixed $E'_{X_0} \subset E$. The right hand side of (17) converges to the right hand side of (16) by Theorem 2.2. Thus we need to show that the difference (18) tends to zero as $X \to \infty$. (Here we extend $G_X(s,y,\lambda)$ by zero to $s \notin E_X$.)

$$
\begin{aligned}
&(G(\cdot,x,\lambda),\ G(\cdot,y,\lambda')) - (G_X(\cdot,x,\lambda),\ G_X(\cdot,y,\lambda')) \\
&= (G(\cdot,x,\lambda) - G_X(\cdot,x,\lambda),\ G(\cdot,y,\lambda')) + \qquad (18)\\
&\qquad + (G_X(\cdot,x,\lambda),\ G(\cdot,y,\lambda') - G_X(\cdot,y,\lambda')) .
\end{aligned}
$$

By the Schwarz inequality the first term of this expression is less than

$$\|G(\cdot,x,\lambda) - G_X(\cdot,x,\lambda)\|^2 \cdot \|G(\cdot,y,\lambda')\|^2 . \qquad (19)$$

The second factor of (19) is bounded since $y \in E^*$ and λ' is fixed. Now consider the first factor of (19), which equals

$$\int_{E-E^*} |G(s,x,\lambda) - G_X(s,x,\lambda)|^2 ds + \int_{E^*} |G(s,x,\lambda) - G_X(s,x,\lambda)|^2 ds . \qquad (20)$$

The latter term of (20) goes to zero for each fixed E^* as X goes to infinity, since G_X converges uniformly on E^*. The result will now be complete if we can show that the first term of (20) can be made small, independently of X, by an appropriate choice of E^*.

$$
\begin{aligned}
|\text{1st term of (20)}| &\le 2 \int_{E-E^*} |G(s,x,\lambda)|^2 ds + 2 \int_{E-E^*} |G_X(s,x,\lambda)|^2 ds \\
&\le 4 \int_{E-E^*} |K(p\omega)|^2 ds
\end{aligned}
$$

by Theorem 2.8, where $p = |s-x|$ and $\lambda = -\omega^2$, $\omega > 0$. Thus the first term of (20) can be made small by picking E^*, and the choice of E^* will be independent of X. A similar argument can be applied to the second term in (18), so that (18) approaches zero as $X \to \infty$, and equation (16) is proved.

From equation (16) we have

$$(G(\cdot,x,\lambda),\ G(\cdot,y,\lambda')) = (\lambda-\lambda')^{-1}(G(y,x,\lambda) - G(x,y,\lambda')).$$

If we let $\lambda' \to \lambda$, the theorem is proved. Note that G is symmetric in x and y by Theorem 2.9. Similar proofs will show the results for higher iterates.

Part 3. The Eigenvalues and Eigenfunctions of the Problem.

In this section we shall introduce conditions on E that will allow construction of eigenvalues and eigenfunctions for problem (2.1). It turns out that a certain condition on E called "narrowness at infinity" will be sufficient, provided E satisfies certain regularity conditions. Rellich [9] and Molčanov [8] gave "narrowness at infinity" conditions sufficient for problem (2.1) to have a discrete spectrum. Clark [4] gave a condition which we shall use to construct the eigenfunction and eigenvalues. The condition (called condition I) is as follows:

I Corresponding to each $X \geq 0$ there exist positive numbers $d(X)$ and $\delta(X)$ satisfying

a) $d(X) + \delta(X) \to 0$ as $X \to \infty$

b) $d(X) / \delta(X) \leq M < \infty$ for all X

c) for each $x \in E - E_X$ there exists a point y such that $|x-y| < d(X)$ and $E \cap \{z \mid |z-y| < \delta(X)\} = \emptyset$.

Condition I implies that E is narrow at infinity in the following sense:

The set E is said to be "narrow at infinity" if

$$\lim_{X \to \infty} \rho(E - E_X) = 0,$$

where $\rho(A)$, for A an arbitrary set in R_n, is defined by

$$\rho(A) = \sup_{x \in A} d(x, \partial A).$$

It is clear that $\rho(A)$ is the supremum of the radii of the spheres inscribable in A.

When Theorem 5 from Clark [4] is applied to the operator T (equation 2.2), with condition I on E, we can conclude:

<u>Lemma 3.1.</u> T is a self-adjoint operator in the Hilbert space $L_2(E)$; the spectrum $\sigma(T)$ is discrete and has no finite limit points; for $\lambda \notin \sigma(T)$ the resolvent operator $R_\lambda(T) = (\lambda I - T)^{-1}$ is completely continuous.

<u>Remark</u>: This result generalizes the result of Rellich [9, p. 335] to a larger class of domains.

Let $\sigma(T) = [\lambda_n']$, where $\lambda_1' \leq \lambda_2' \leq \lambda_3' \ldots$ etc. Let $\lambda_{X,n}$ and $u_{X,n}(x)$ be the eigenvalues and eigenfunctions for the problem:

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0 \qquad x \in E_X \\ u(x) &= 0 \qquad x \in \partial E_X \end{aligned} \tag{1}$$

which are known to exist since E_X is bounded. Since $E_X \subset E$ we have, by elementary variational principles (cf. e.g. Glazman [7]), $\lambda_{X,n} \geq \lambda_n'$. In view of the fact that $\lambda_{X,n}$ is a non-increasing function of X for each fixed n, we have

$$\lambda_{X,n} \to \lambda_n \geq \lambda_n' \quad \text{as} \quad X \to \infty \quad \text{for each } n. \tag{2}$$

<u>Lemma 3.2.</u> (Titchmarsh [10, p. 334] theorem 22.14).

For $p=1, 2, 3, \ldots$ let $H_p(x)$ have continuous partial derivatives up to the second order and satisfy the differential equation

$$(\Delta + \lambda_p - q)H_p = f,$$

where f and q have continuous partial derivatives of the first order. As $p \to \infty$, let H_p tend to a limit $H(x)$ uniformly over some given region, and let λ_p tend to a limit λ. Then $H(x)$ has continuous partial derivatives up to the second order, and the equation

$$(\Delta + \lambda - q)\, H(x) = f(x)$$

is satisfied.

The same result holds if we are merely given that $H_p \to H$ in mean square.

Lemma 3.3. There exists a set of functions $\{u_n(x)\}$, such that $u_{X,n}(x)$ tends to $u_n(x)$ in $L_2(E)$ for each n and some subsequence of $\{X\}$ tending to infinity (this subsequence can be picked from the subsequence given in Theorem 2.2); moreover

$$\Delta\, u_n(x) + \lambda_n\, u_n(x) = 0 \qquad x \in E;$$

u_n has continuous partial derivatives up to the second order; the u_n are orthonormal, and λ_n are positive.

Proof. Extend $u_{X,n}(x)$ by zero the $E-E_X$. Since

$$\|u_{X,n}\|_1^2 = \|u_{X,n}\|_0^2 + \int_E |\nabla u_{X,n}|^2 dx$$

we have

$$\|u_{X,n}\|_1^2 \leq 1 + \lambda_{X,n} ,$$

and if we take $X \geq X_o$, we have

$$\|u_{X,n}\|_1^2 \leq 1 + \lambda_{X_o,n} . \tag{3}$$

Now, by theorem 3 of Clark [4], the embedding map $H_o^1(E) \subset L_2(E)$ is completely continuous. Since by (3) the sequence $u_{X,n}$ is bounded in $H_o^1(E)$, it must therefore have a subsequence convergent in $L_2(E)$. Let $u_n(x)$ be the limit of this convergent subsequence. We shall remove, by a diagonalization procedure, the restriction that the sequence chosen may depend on n. For example, find a subsequence X_1 such that $[u_{X_1,1}]$ converges, then find a subsequence X_2 of X_1 such that $[u_{X_2,2}]$ converges, and so on. We then take the diagonal of this process to show the result

$$\|u_n(x) - u_{X,n}(x)\| \to 0 \quad \text{as} \quad X \to \infty \quad \text{for each} \quad n. \tag{4}$$

Since we also have $\lambda_{X,n} \to \lambda_n$ for each n, we can apply Lemma 3.2 to obtain

$$\Delta u_n(x) + \lambda_n u_n(x) = 0, \qquad x \in E.$$

Lemma 3.2 also says that u_n will have continuous partial derivatives up to the second order. Furthermore u_n are orthonormal since

$$(u_n, u_m) = \lim_{X\to\infty} (u_{X,n}, u_{X,m}) = \delta_{m,n} .$$

Since

$$\int_{E_X} |\nabla u_{X,n}|^2 dx = \lambda_{X,n} \leq \lambda_{X_o,n} \quad \text{if} \quad X \geq X_o,$$

and $D_i u_{X,n} \to D_i u_n$ inside E, Fatou's Lemma shows

$$\int_E |\nabla u_n|^2 dx \leq \lambda_{X_o,n} \tag{5}$$

If we now recall that $\lambda_{X,n} \to \lambda_n$, we can sharpen the result (5) to $\|\nabla u\|^2 \leq \lambda_n$ by a simple contradiction argument. From this it follows that all the λ_n are non-negative.

<u>Theorem 3.4</u>. Let E be such that through every point of the boundary passes a circle which lies otherwise entirely inside E (roughly, this means that the point is not the vertex of an outward-pointing angle). Also, let E be "star-shaped". Then the "eigenfunction" u_n is contained in $H_o^1(E)$.

<u>Remarks</u>:

1. The proof will be given for dim $E=2$, but holds for all dimensions.
2. The hypothesis that E is "star shaped" can be dispensed with entirely by using a partition of unity.
3. The hypothesis that the boundary of E has interior circles, etc., can be weakened.

The proofs of the above three remarks will not be given since Theorem 3.4 will be sufficient for our purpose as it is proved.

Proof of Theorem 3.4. Write $u=u_n$ for the present. We note that

$u \in H^1(E)$ by inequality (5). The proof that Titchmarsh [4, p. 99] gives to show that $u_n(x) \to 0$ as $x \to \partial E$ for the case of E bounded works for our case also. Thus, to complete the proof, we need to show that u can be approximated by $C_0^\infty(E)$ functions in the $\|\,\|_1$ norm. We shall approximate u in several steps.

Let $g(\beta)$ be a function such that $g(\beta) \in C^\infty(R_1)$; $0 \leq g(\beta) \leq 1$; $g(\beta) = 0$ for $\beta \geq 1$; and $g(\beta) = 1$ for $\beta \leq 0$, then define $g_R(x)$ as $g_R(x) = g(|x|-R)$. We want to show that $\|u-g_Ru\|_1$ can be made small for sufficiently large R. Now

$$\|u-g_Ru\|_1^2 = \int_E |u-g_Ru|^2 dx + \int_E |\nabla(u-g_Ru)|^2 dx$$

$$= \int_{|x|\geq R} |u-g_Ru|^2 dx + \int_{|x|\geq R} |\nabla(u-g_Ru)|^2 dx. \tag{6}$$

The first integral in (6) is less than $4\int_{|x|\geq R} u^2 dx$, since $0 \leq g_R(x) \leq 1$. Thus this integral can be made small for large R since $u \in L_2(E)$. Consider now the second integral in (6)

$$\int_{|x|\geq R} |\nabla(u-g_Ru)|^2 dx \leq 2\int_{|x|\geq R} |\nabla u|^2 dx + 2\int_{|x|\geq R} |\nabla(g_Ru)|^2 dx.$$

The first integral here can be made small since $|\nabla u| \in L_2(E)$. Consider the remaining integral

$$\int_{|x|\geq R} |\nabla(g_Ru)|^2 dx = \int_{R+1\geq|x|\geq R} |\nabla(g_Ru)|^2 dx$$

$$\leq 2\int_{|x|\geq R} g_R^2 |\nabla u|^2 dx + 2\int_{R\leq|x|\leq R+1} u^2 |\nabla g_R|^2 dx. \tag{7}$$

The first integral in (7) can be made small, for sufficiently large R, since $g_R \leq 1$ and $|\nabla u| \in L_2(E)$. From the definition of $g_R(x)$ we have

$$\max_{R \leq |x| \leq R+1} |\nabla g_R(x)| = \max_{R \leq |x| \leq R+1} |\nabla g(|x|-R)|$$

$$= \max_{R \leq |x| \leq R+1} |g'(|x|-R)| = \max_{0 \leq \beta \leq 1} |g'(\beta)|.$$

Hence $|\nabla g_R|$ is less than a constant which is independent of R. Thus the last integral in (7) can be made arbitrarily small, from which it follows that $\|u-g_R u\|_1$ can be made small for sufficiently large R. Thus we may assume, without loss of generality, that u has bounded support.

The next step is to show that u can be approximated, in the $\| \ \|_1$ norm, with functions whose supports are compact in E. We know from above that $u=0$ for $x \in \partial E$ or $|x| \geq R+1$. Define u to be zero for all other values of x outside E. Since E is "star-shaped", we have that if $x \in E$, then $(1-\mathcal{E})x \in E$ for $0 \leq \mathcal{E} \leq 1$. This can be achieved by translating $(0, 0)$ into E if necessary. Let $u_1(x) = u((1-\mathcal{E})x)$. By uniform continuity

$$u((1-\mathcal{E})x) \to u(x) \quad \text{as} \quad \mathcal{E} \to 0 \quad \text{if} \quad |x| \leq R+1.$$

However, since $u \in C^1$ we have, by a further application of the principle of uniform continuity,

$$D_i u_1 \to D_i u \qquad \text{as } \mathcal{E} \to 0 \qquad i=1, 2, \ldots, n.$$

This means that

$$\|u_1 - u\|_1 = \int_{E_{R+1}} |u-u_1|^2 dx + \int_{E_{R+1}} |\nabla(u-u_1)|^2 dx$$

can be made small, for sufficiently small $\mathcal{E}$, and hence we can assume, without loss of generality, that u is in $C_o^1(E)$.

To complete the proof we need to show that $u(\in C_o^1(E))$ can be approximated in the norm of $H_o^1(E)$. To do this, let $J_{\mathcal{E}}$ be the mollifier function (cf. Agmon [1, p.5]). A standard result (cf. Agmon [1] Theorem 1.5) shows that $J_{\mathcal{E}}\, u(x) \in C_o^\infty(E)$ for sufficiently small $\mathcal{E}$. A further result (cf. Agmon [1] Theorem 1.10) shows that $J_{\mathcal{E}}\, u$ tends to u, as $\mathcal{E} \to 0$, in the norm of $H_o^1(E)$ since u has compact support in E. This completes the proof that $u_n \in H_o^1(E)$.

<u>Lemma 3.5</u>. (The Parseval formula).

If $f \in L_2(E)$ and $C_n = (f, u_n)$, then

$$\|f\|^2 = \sum_{n=o}^{\infty} |C_n|^2 .$$

Lemma 3.5 can be proved by slightly modifying the proof in Titchmarsh [4, p.104].

<u>Remark</u>: If f and g are in $L_2(E)$ we can show, by applying the Parseval formula to $f \pm g$, that

$$(f,g) = \sum_{n=o}^{\infty} a_n b_n ,$$

where $a_n = (f, u_n)$ and $b_n = (g, u_n)$.

As a summary of the preceeding results we may state the following theorem:

<u>Theorem 3.6</u>. Let the domain E be "narrow at infinity" and satisfy certain regularity conditions, then the eigenfunctions constructed in Lemma 3.3 constitute a complete set of orthonormal functions in $H^1_o(E)$ satisfying the equation $\Delta\, u_n(x) + \lambda_n u_n(x) = 0$ for $x \in E$ and the boundary condition $u_n = 0$ on the boundary of E.

In future, when we refer to the eigenvalues and eigenfunctions of (2.1) we mean the eigenfunctions and eigenvalues of T constructed in Lemma 3.1 and 3.3. Note: since the eigenfunctions are complete we have shown $\lambda_n = \lambda'_n$ (cf. equation (2)).

In the next theorem we prove the important inversion property of the Green's function.

<u>Theorem 3.7</u>. If λ is not on the non-negative real axis, then

$$G_\lambda\, u_n(x) = (\lambda_n - \lambda)^{-1}\, u_n(x) \; . \tag{8}$$

Proof. We shall prove (8) by letting X go to infinity in

$$G_{X,\lambda} u_{X,n}(x) = (\lambda_{X,n} - \lambda\,)^{-1}\, u_{X,n}(x) \; . \tag{9}$$

First the right hand side of (9) converges to the right hand side of (8) by the results given earlier in this part.

Now consider the difference

$$(G(x,\cdot,\lambda),\ u_n) - (G_X(x,\cdot,\lambda),\ u_{X,n}). \tag{10}$$

We can consider $u_{X,n}$ and $G_X(x,y,\lambda)$ to be zero outside E_X and thus we are able to use E as the domain of integration in the second inner product. Expression (10) is now equal to

$$((G-G_X)(x,\cdot,\lambda),\ u_n) + (G_X(x,\cdot,\lambda),\ u_n-u_{X,n}) . \tag{11}$$

By the Schwarz inequality the second term of (11) is bounded by

$$\|G_X(x,\cdot,\lambda)\|\ \|u_n - u_{X,n}\|.$$

The first term in the above is bounded by Theorem 2.2 equation 2.13 (independent of X). Since $\|u_n-u_{X,n}\|$ goes to zero as X goes to infinity, the second term of (11) goes to zero as X goes to infinity. If we let $E^* = E'_{X_o}$ where E' and X_o are arbitrary, then the first term of (11) can be written as

$$\int_{E-E^*} [G_X - G]\, u_n\, dy + \int_{E^*} [G_X - G]\, u_n\, dy . \tag{12}$$

The second term in (12) goes to zero for fixed E^* by Theorem 2.2. By the Schwarz inequality the first term in (12) is less than

$$\left\{\int_{E-E^*} |G_X - G|^2 dy \cdot \int_{E-E^*} u_n^2\, dy\right\}^{1/2}$$

$$\leq \|(G_X-G)(x,\cdot,\lambda)\| \cdot \left\{\int_{E-E^*} u_n^2\, dy\right\}^{1/2} .$$

Now since $\|G_X(x,\cdot,\lambda)-G(x,\cdot,\lambda)\|$ is bounded independent of X (Theorem 2.2 and 2.5) and the remaining piece $\int_{E-E^*} u_n^2\,(y)dy$ can be made as small as we please (since $u_n \in L_2(E)$), we have

$$\lim_{X\to\infty} G_{X,n}\, u_{X,n}(x) = G_\lambda\, u_n(x),$$

completing the proof of Theorem 3.7.

We shall finish this part with two lemmas on the eigenfunctions.

Lemma 3.8. $u(x) \to 0$ as $x \to \infty$ where u is any eigenfunction corresponding to an eigenvalue λ.

Proof. u has the representation (cf. Titchmarsh equation 22.9.3)

$$u(y) = \frac{1}{\pi R^2} \int_{r \le R} u(x)dx + \lambda \int_{r \le R} g(x,y)\, u(x)dx \tag{13}$$

where $g(x,y) = \frac{1}{2\pi}[\log \frac{R}{r} - \frac{1}{2}(1 - \frac{r^2}{R^2})] \quad r \le R$

$= 0 \quad r > R,$

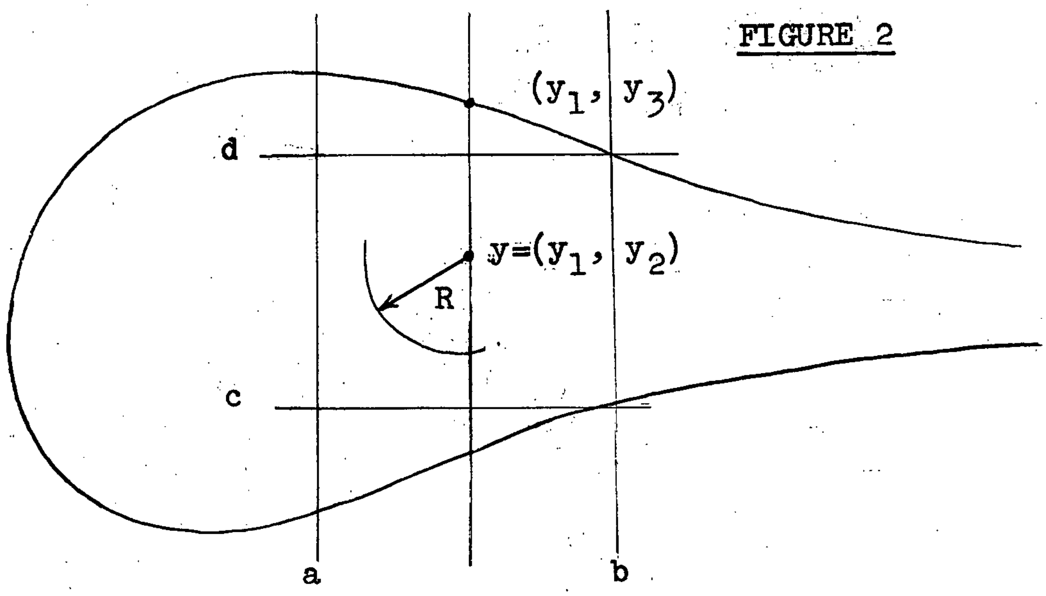
and $r = |x-y|$. If we apply the Schwarz inequality to (13), then

$$\begin{aligned} |u(y)| &\le \frac{1}{\pi R^2}[\int_{r \le R} dx \cdot \int_{r \le R} u^2dx]^{1/2} + \lambda[\int_{r\le R} u^2dx \cdot \int_{r \le R} g^2(x,y)dx]^{1/2} \\ &\le [\pi^{-1/2} R^{-1} + \lambda AR][\int_{r \le R} u^2dx]^{1/2}, \end{aligned} \tag{14}$$

since $\int_{R_2} g^2(x,y)dx = (AR)^2$ where A is a constant.

Consider the following diagram, where $r \leq R \subseteq E$:

FIGURE 2



Since $u(y_1,y_3) = 0$, we have

$$u(y_1, y_2) = - \int_{y_2}^{y_3} D_t\, u(y_1, t)dt.$$

Therefore

$$|u(y_1, y_2)|^2 \leq (y_3 - y_2) \int_{y_2}^{y_3} [D_t\, u(y_1, t)]^2 dt$$

$$\leq R \int_{0}^{y_3} [D_t\, u]^2 dt.$$

Now, integrating this inequality with respect to y_1 from a to b, we get

$$\int_a^b |u(y_1, y_2)|^2 dy_1 \leq R \int_a^b \int_0^{y_3} [D_t u(y_1, t)]^2 dt\, dy_1 .$$

If we now use the fact that $|\nabla u| \in L_2(E)$, we have

$$\int_a^b |u(y_1, y_2)|^2 dy_1 \le R\, o(1) \quad \text{as} \quad a \to \infty .$$

If this expression is now integrated with respect to y_2, one has

$$\int_c^d \int_a^b |u(y_1, y_2)|^2 dy_1\, dy_2 \le R\, o(1)(d-c) \le R^2\, o(1).$$

Thus, making the sides of the square touch the circle $r \le R$ (see figure 2), we have

$$\int_{r \le R} |u(x)|^2 dx \le R^2\, o(1) \quad \text{as} \quad y \to \infty.$$

Combining this result with (14), we get

$$|u(y)| \le [\pi^{-1/2} R^{-1} + \lambda\, A\, R]\, [R^2\, o(1)]^{1/2} \quad \text{as} \quad y \to \infty.$$

Hence it follows that $u(x) \to 0$ as $x \to \infty$.

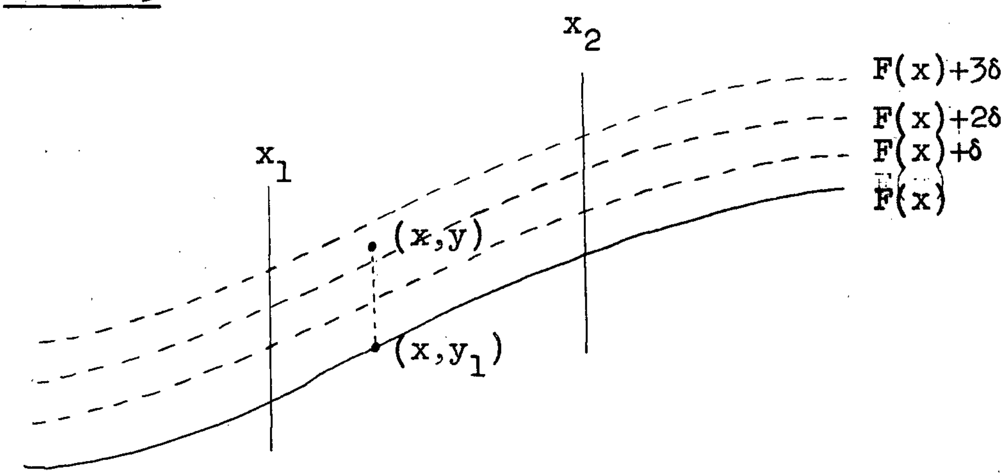
For the purpose of the next proof we assume (x,y) is a point in R_2.

Lemma 3.9. Suppose that the boundary of E is given locally by $y = F(x)$. Then for (x,y) in a region which includes a piece of the boundary $[x, F(x)]$, $|u_n(x,y)| < K\, \lambda_n$, where K is independent of (x,y) and n, and $\dim E = 2$.

Remark: The proof of this lemma is a detailed proof of Titchmarsh's remark [10, p. 108].

Proof. Consider the following diagram

FIGURE 3.



where δ is sufficiently small, i.e. we are considering a small piece of the lower boundary of E. Since $u_{X,n}$ is zero on the boundary, we have

$$u_{X,n}(x,y)^2 = [\int_{y_1}^{y} D_t\, u_{X,n}(x,y)dt\ ]^2 .$$

Thus

$$u_{X,n}(x,y)^2 \leq 3\ \delta \int_{y_1}^{y} [D_t\, u_{X,n}(x,t)\ ]^2 dt$$

by the Schwarz inequality. If we integrate this expression with respect to y from $F(x)+\delta$ to $F(x)+2\delta$, then

$$\int_{F(x)+\delta}^{F(x)+2\delta} u_{X,n}(x,y)^2 dy \leq 3\delta \int_{F(x)+\delta}^{F(x)+2\delta} dy \int_{y_1}^{y} [\ D_t\, u_{X,n}(x,t)\ ]^2 dt. \tag{15}$$

If we now interchange the order of integration in the last integral, we see that y varies over a range not exceeding 3δ, thus the last integral is less than

$$9\ \delta^2 \int_{y_1}^{y} [\ D_t\, u_{X,n}(x,t)\ ]^2 dt.$$

Next integrate (15) with respect to x over $[x_1,x_2]$; this gives

$$\int_{x_1}^{x_2} dx \int_{F(x)+\delta}^{F(x)+2\delta} [u_{X,n}]^2 dy \leq 9\ \delta^2 \int_{x_1}^{x_2} \int_{y_1}^{y} [\ D_t\ u_{X,n}(x,t)\ ]^2 dt\ dx. \quad (16)$$

Let us first consider the last term of (16). We know from the case of bounded E_X that

$$\int_{E_X} |\nabla\ u_{X,n}|^2 dx = \lambda_{X,n} \ .$$

Thus if $X \geq X_o$, then

$$\int_{E_X} |\nabla\ u_{X,n}|^2 dx \leq \lambda_{X_o,n} \ .$$

Hence

$$\limsup_{X \to \infty} \int_{E_X} |\nabla\ u_{X,n}|^2 dx \leq \lambda_{X_o,n} \ ,$$

but since X_o is arbitrary

$$\limsup_{X \to \infty} \int_{E_X} |\nabla\ u_{X,n}]^2 dx \leq \lambda_n,$$

since $\lambda_{X,n} \to \lambda_n$ as $X \to \infty$. Thus the right hand side of (16) is bounded by $9\ \delta^2\lambda_n$ as X tends to infinity, but since $u_{X,n} \to u_n$ in the mean, we have by Fatou's theorem

$$\int_{x_1}^{x_2} dx \int_{F(x)+\delta}^{F(x)+2\delta} [\, u_n]^2 dy \;\leq\; 9\, \delta^2\, \lambda_n \,. \tag{17}$$

We are now in a position to make the required estimate on u_n. By Titchmarsh [10, equation 22.9.3] we have

$$u_n(u) = \frac{1}{\pi R^2} \int_{r\leq R} u_n(s)ds + \frac{\lambda_n}{2\pi} \int_{r\leq R} [\, \log \frac{R}{r} - \frac{1}{2}(1-\frac{r^2}{R^2})\,]u_n(s)ds, \tag{18}$$

where $r \leq R$ is a circle $[s : |s-u| \leq R]$ which is contained in E. Now suppose that for x in $[x_1, x_2]$ we have $|F(x+h)-F(x)| \leq M|h|$. Thus if $|x-x_o| \leq \zeta$, then

$$F(x) + \delta \leq F(x_o) + M\zeta + \delta \,,$$

which is in turn less than $F(x_o) + 5\delta/4$ if $\zeta \leq \frac{1}{4}\delta/M$. Likewise

$$F(x) + 2\delta \geq F(x_o) - M\zeta + 2\delta \geq F(x_o) + \frac{7}{4}\delta.$$

Hence the circle, center $(x_o,\, F(x_o) + \frac{3}{2}\delta)$ and radius $\frac{1}{4}\delta \min\,(M^{-1}, 1)$, lies between the curves $y = F(x) + \delta$ and $y = F(x) + 2\delta$. Now, in expression (18) set $u = (x_o,\, F(x_o) + \frac{3}{2}\delta)$ and $R = \frac{1}{4}\delta \min\,(M^{-1}, 1)\, \lambda_o^{1/2}\, \lambda_n^{-1/2}$. Note that the circle $r \leq R$ will be in the region of interest since $\lambda_o^{1/2}\, \lambda_n^{-1/2} < 1$ for all n.

By the Schwarz inequality the first term in (18) is less than

$$\frac{1}{\pi R^2}\, [\int_{r\leq R} dx \cdot \int_{r\leq R} u_n^2(s)ds\,]^{1/2}$$

which in turn less than

$$M_1 \frac{1}{\delta\ \lambda_n^{-1/2}} [\ 9\ \delta^2\ \lambda_n\ ]^{1/2}$$

by inequality (17). Thus the first term of (18) is less than $M_2\ \lambda_n$, where M_2 is independent of δ, u and n.

Again by the Schwarz inequality the second term of (18) is less than

$$\frac{\lambda_n}{2\pi} [\ \int_{r \leq R} [\ \log \frac{R}{r} - \frac{1}{2}(1 - \frac{r^2}{R^2})\ ]^2 dx \cdot \int_{r \leq R} u_n^2(s) ds\ ]^{1/2}$$

$$\leq \frac{\lambda_n}{2\pi} [\ CR^2 \cdot 9\ \delta^2\ \lambda_n\ ]^{1/2} = C_1\ \lambda_n\ [\ \delta^2\ \lambda_n^{-1}\ \lambda_n\ \delta^2\ ]^{1/2}$$

$$= C_1\ \lambda_n\ \delta^2,$$

where C, C_1 are constants independent of u, n, and δ.

Combining all these results, we have $|u_n(x)| \leq M_2\ \lambda_n + C_1\ \lambda_n\ \delta^2$. If we now assume that $\delta \leq 1$, we have the desired result $|u_n(x,y)| \leq K\ \lambda_n$ where K is independent of (x,y) and n.

Part 4. The iterates of the Green's function and their properties.

For the sake of simplicity we assume in this part that the domain E has a boundary formed by the x_1-axis and a function $\varphi(x_1)$, where $\varphi(0) = 0$, $\varphi(x_1) > 0$ if $x_1 > 0$, and $\varphi(x_1)$ satisfies the smoothness conditions in Theorem 4.2. We also assume all conditions in part 2 and 3, so that we shall have a Green's function $G(x,y,\lambda)$ for E, eigenvalues λ_n for T, and eigenfunctions $u_n(x)$.

If we are working in a dimension $n \geq 3$ we shall consider E to be the domain in R_n formed by revolving $\varphi(x_1)$ about the x_1-axis.

<u>Definition</u>. Let the iterates of the Green's function be defined as follows:

$$G^{(1)}(x,y,\lambda) = G(x,y,\lambda) \quad \text{and}$$

$$G^{(i+1)}(x,y,\lambda) = (G(x,\cdot,\lambda),\ G(\cdot,y,\lambda)) \qquad (i \geq 1).$$

These iterates are well defined by Theorems 2.5 and 2.6.

<u>Theorem 4.1</u>. The iterate $G^{(2)}(x,y,\lambda)$ for $\dim E = 3$ is continuous on $E \times E$ and satisfies

$$0 \leq G^{(2)}(x,y,\lambda) \leq M \exp(-\omega|x-y|/8) \tag{1}$$

where $\lambda = -\omega^2$, $\omega > 0$, and M is a constant independent of ω, x and y.

Lemma. Let E be an open subset of a cylinder (of finite cross section) in R_n. Then, as $\delta \to 0$,

$$\lim_{u \to y} \int_E \frac{dx}{|x-u|^\alpha |x-y|^\beta} \begin{cases} = O(1) & \text{if } \alpha + \beta < n \\ = O(\log 1/\delta) & \text{if } \alpha + \beta = n \\ = O(\delta^{n-\alpha-\beta}) & \text{if } \alpha + \beta > n, \end{cases}$$

where $\alpha + \beta > 1$ and $\delta = |u-y|$.

Proof. If $u, y \in E_X$, then the integral we are interested in can be broken up as follows:

$$\int_{E_{X+1}} \frac{dx}{|x-u|^\alpha |x-y|^\beta} + \int_{E-E_{X+1}} \frac{dx}{|x-u|^\alpha |x-y|^\beta} . \tag{2}$$

Since E_{X+1} is bounded, we can apply Titchmarsh [10, p. 323] to the first integral in (2) to obtain

$$\lim_{u \to y} \int_{E_{X+1}} \frac{dx}{|x-u|^\alpha |x-y|^\beta} \begin{cases} = O(1) & \alpha + \beta < n \\ = O(\log 1/\delta) & \alpha + \beta = n \\ = O(\delta^{n-\alpha-\beta}) & \alpha + \beta > n . \end{cases}$$

To complete the proof we need only show that the last integral in (2) is bounded. To see this we note that, for $x \in E-E_{X+1}$, $|x-u| \geq 1$ and $|x-y| \geq 1$; $E - E_{X+1}$ is contained in a tube of finite cross section, and $\alpha + \beta > 1$.

Proof of Theorem 4.1. By the property of the fundamental singularity for E (Theorem 2.8), we have

$$G^{(2)}(x,y,\lambda) \leq \int_E K(\omega|x-z|)\, K(\omega|y-z|)dz,$$

where $K(\omega p) = M\, p^{-1} \exp(-\omega p/4)$. If we now apply the triangle inequality to the integrand of $G^{(2)}$, then

$$G^{(2)}(x,y,\lambda) \leq$$

$$M^2 \exp(-\tfrac{\omega}{8}|x-y|) \cdot \int_E \frac{dz}{|x-z|\,|y-z|} \cdot \exp(-\tfrac{\omega}{8}(|x-z| + |z-y|))$$

$$\leq M^2 \exp(-\omega|x-y|/8) \int_E \frac{dz}{|x-z|\,|y-z|}$$

$$\leq M' \exp(-\omega|x-y|/8)$$

by the Lemma, since $\alpha + \beta = 2 < n = 3$.

Remarks: For $\dim E = 2$ (proof below), we have

$$G^{(2)}(x,y,\lambda) \leq M \exp(-\omega|x-y|/2) \quad \text{if} \quad \omega \geq 1 . \tag{3}$$

However if $\dim E = 4$, and we try to use the same method of proof as in $\dim E = 3$, we get

$$G^{(2)}(x,y,\lambda) \leq M \exp(-\omega|x-y|/8) \log \frac{1}{|x-y|} . \tag{4}$$

Instead of (4) we can show

$$G^{(k-1)}(x,y,\lambda) \leq M \exp(-\omega|x-y|/2^k) \tag{5}$$

if $k \geq n$, and if $\dim E = n \geq 3$. One proves this by a repeated application of the method used to prove (1).

Proof of (3) for $\dim E = 2$. Since the Lemma does not apply to this case we proceed as follows:

We have

$$0 \leq G^{(2)}(x,y,\lambda) \leq \int_E K(\omega|x-z|)\, K(\omega|y-z|)\, dz$$

where $K(\omega r) = f(\omega r)\, e^{-\omega r}$, and

$$f(\xi) = M(1 + \log(1 + \xi^{-1})) \,/\, (1 + \xi^{1/2})$$

by Lemma 2.7.

Consider the integration above over three sets, namely $N(x,\omega^{-1}\delta)$, $N(y,\omega^{-1}\delta)$ and $E^* = E - \{N(x,\omega^{-1}\delta) \cup N(y,\omega^{-1}\delta)\}$, where $N(x,a) = \{z : |x-z| < a\}$, and $\delta < 1$ is such that $|x-y| > 2\delta$. We need the following estimates for $f(\xi)$: if $\xi \geq \delta$, then $f(\xi) \leq M(1 + \log(1 + \delta^{-1})) = M_1$, and if $\xi < 1$, then $f(\xi) \leq M_2 \log \xi^{-1}$. Now consider the integral over $N(x,\omega^{-1}\delta)$, which is bounded by

$$\exp\,(-\omega|x-y|) \int_{N(x,\omega^{-1}\delta)} f(\omega|x-z|)\, f(\omega|y-z|)\, dz\ .$$

If $z \in N(x,\omega^{-1}\delta)$, then $\omega|x-z| \leq \delta < 1$, from which it follows that $f(\omega|x-z|) \leq M_2 \log|x-z|^{-1}$, if we assume $\omega \geq 1$. Also $|y-z| \geq 2\delta - \omega^{-1}\delta$ or $\omega|y-z| \geq \delta(2\omega - 1) \geq \delta$, from which it follows that $f(\omega|y-z|) \leq M_1$. If we substitute these inequalities into the integral over the set $N(x,\omega^{-1}\delta)$, then that integral is bounded by

$$M_1\, M_2 \int_{N(x,\omega^{-1}\delta)} \log \frac{1}{|x-z|}\, dz\ ,$$

which is bounded independent of ω and x, as long as $\omega \geq 1$. By symmetry a similar result holds for $N(y,\omega^{-1}\delta)$.

Consider now the integral over E^*. If $z \in E^*$ then $\omega|x-z| \geq \delta$ and $\omega|y-z| \geq \delta$. Therefore the integral over E^* is less than

$$\exp(-\omega|x-y|/2)M_1^2 \int_{E^*} \exp(-\omega(|x-z| + |z-y|)/2)\, dz$$

by an argument similar to that above. Hence the integral over E^* is bounded independent of ω, x and y if $\omega \geq 1$, $\delta \leq 1$, and $|x-y| \geq 2\delta$. Combining these integrals, we have

$$0 \leq G^{(2)}(x,y,\lambda) \leq M_\delta \exp(-\omega|x-y|/2) \quad \text{if} \quad \omega \geq 1, \tag{6}$$

and $|x-y| \geq 2\delta$, where M_δ is independent of x, y and ω.

Since $G^{(2)}(x,y,\lambda)$ is continuous at $x=y$, the result (3) is proved.

<u>Theorem 4.2</u>. If E is a plane region such that

(a) there exists a $t_o > 0$ such that

$$\sup_{t \geq t_o} \left| \frac{\varphi'(t)}{\varphi(t)} \right| < \infty \quad \text{and}$$

(b) for every $\beta > 0$, there exists a $k_\beta > 0$ such that

$$\varphi(t) > k_\beta e^{-\beta t} \quad t \geq t_o,$$

then there exists a constant ω_o such that for $\omega \geq \omega_o$ and integers $k \geq 2$,

$$\int_E \int_E |G^{(2k)}(x,y,\lambda)|^2 \, dx \, dy \leq M_\omega \int_0^\infty |\varphi(s)|^{2k} ds ,$$

where $\lambda = -\omega^2$ and M_ω is a constant depending only on ω.

Remark: The proof of this theorem for $\dim E = 2$ is due to Clark.

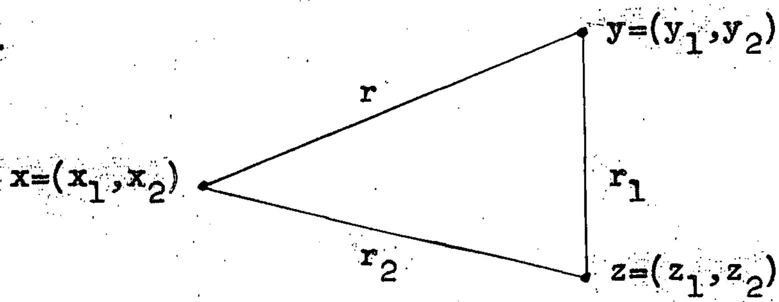
Proof. First let us show the theorem for $k = 2$; that is, consider $G^{(4)}(x,y,\lambda)$. By applying the definition of the iterates of the Green's function, and Theorem 2.9 (symmetry of $G(x,y,\lambda)$), one can show:

$$G^{(4)}(x,y,\lambda) = \int_E G^{(2)}(x,z,\lambda) \, G^{(2)}(z,y,\lambda) dx.$$

By Theorem 4.1 (Equation (3)) we have

$$G^{(4)}(x,y,\lambda) \leq M \int_E \exp(-\omega(|x-z| + |z-y|)/2) dz.$$

FIGURE 4.



By considering figure 4 and applying the triangle inequality, we can see that

$$G^{(4)}(x,y,\lambda) \leq M \, e^{-\omega r/4} \int_0^\infty \int_0^{\varphi(z_1)} \exp(-\omega|x_1-z_1|/4) dz_2 dz_1 .$$

If we carry out the integration with respect to z_2 we get

$$G^{(4)}(x,y,\lambda) \leq M\ e^{-\omega r/4} \int_0^\infty \varphi(z_1)\ \exp(-\omega|x_1-z_1|/4)dz_1 \ . \qquad (7)$$

The remaining integral must be analyzed in two parts, the first part being over $[0, t_o]$ and the second part over $[t_o, \infty)$, where t_o is the positive constant given in hypothesis (a). Let $x_1 \geq t_o$ and consider:

$$\int_0^{t_o} \varphi(z_1)\ \exp(-\omega|x_1-z_1|/4)dz_1$$

$$\leq M \int_0^{t_o} \exp(-\omega|x_1-z_1|/4)dz_1 \leq M_1\ e^{-\omega x_1/4}$$

$$\leq M_2\ \varphi(x_1) \qquad \text{using hypothesis (b).}$$

M, M_1, and M_2 are constants depending only on ω.

We now examine the second part of the integral, namely the one over $[t_o, +\infty)$.

First we note that hypothesis (a) implies that for sufficiently large K,

$$(*) \quad \max_{t \geq t_o} \varphi(t)\ \exp(-K|t-t_1|) = \varphi(t_1) \quad \text{for any} \quad t_1 > t_o \ .$$

To see this consider

$$D_t\ [\ \varphi(t)\ \exp(-K|t-t_1|)\ ] = \exp(-K|t-t_1|)[\ \varphi'(t)-K\varphi(t)\ ]<0$$

if $t > t_1$ and K is sufficiently large, and

$$= \exp(-K|t-t_1|)[\ \varphi'(t)+K\varphi(t)\ ]>0$$

if $t_o \leq t_1 < t$, where K is sufficiently large.

This proves (*).

Applying (*) to the integral over $[t_o, \infty)$, we have

$$\int_{t_o}^{\infty} \varphi(z_1) \exp(-\omega|z_1-x_1|/4)\, dz_1$$
$$= \int_{t_o}^{\infty} \varphi(z_1) \exp(-\omega|z_1-x_1|/8) \exp(-\omega|z_1-x_1|/8)\, dz_1$$
$$\leq \varphi(x_1) \int_{t_o}^{\infty} \exp(-\omega|z_1-x_1|/8) dz_1 \leq M\, \varphi(x_1)$$

which holds for sufficiently large ω.

Thus we have, combining the two estimates for the integrals,

$$G^{(4)}(x,y,\lambda) \leq M_\omega\, \varphi(x_1) \exp(-\omega r/4) \tag{8}$$

for $x_1 \geq t_o$ and sufficiently large ω.

For reasons of simplicity we introduce the following function:

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t_o) & \text{if } t < t_o \\ \varphi(t) & \text{if } t \geq t_o . \end{cases}$$

Now by inequality (7) we have

$$G^{(4)}(x,y,\lambda) \leq M_\omega^o \exp(-\omega r/4) \qquad \text{if } x_1 < t_o . \tag{9}$$

Thus we have, by combining (8) and (9),

$$G^{(4)}(x,y,\lambda) \leq M_\omega'\, \tilde{\varphi}(x_1) \exp(-\omega r/4), \tag{10}$$

where

$$M'_\omega = \max\ (M^o_\omega/\varphi(t_o),\ M_\omega)\ .$$

Consider now the integral of (10) over E with respect to x.

$$\int_E |G^{(4)}(x,y,\lambda)|^2 dx \leq M \int_E |\tilde{\varphi}\,(x_1)|^2 \exp(-\omega r/2)\,dx$$

$$\leq M \int_0^\infty |\tilde{\varphi}(x_1)|^3 \exp(-\omega|x_1-y_1|/2)\,dx_1\ .$$

If we apply the same analysis to this as we did to equation (7), we have

$$\int_E |G^{(4)}(x,y,\lambda)|^2 dx \leq M_1 |\tilde{\varphi}(y_1)|^3$$

for sufficiently large ω. If we integrate this expression again, this time with respect to y, we have

$$\int_E \int_E |G^{(4)}(x,y,\lambda)|^2 dx\ dy \leq M_1 \int_E |\tilde{\varphi}(y_1)|^3 dy$$

$$\leq M_1 \int_0^\infty |\tilde{\varphi}(y_1)|^4 dy_1 \leq M_2 \int_0^\infty |\varphi(s)|^4 ds\ .$$

Thus we have shown that there exists a positive number ω_o such that

$$\int_E \int_E |G^{(4)}(x,y,\lambda)|^2 dx\ dy \leq M_\omega \int_0^\infty |\varphi(s)|^4 ds$$

for $\omega \geq \omega_o$.

It is now a simple matter to extend the result to higher values of k. For example:

$$G^{(6)}(x,y,\lambda) = \int_E G^{(4)}(x,z,\lambda)\ G^{(2)}(z,y,\lambda)dz\ .$$

By equations (3) and (10) we have

$$\begin{aligned} G^{(6)}(x,y,\lambda) &\leq M \int_E \tilde{\varphi}(x_1)\ \exp(-\omega r_2/4)\ \exp(-\omega r_1/2)dz \\ &\leq M\ \tilde{\varphi}(x_1) \int_E \exp(-\omega(r_2+r_1)/4)dz \\ &\leq M\ \tilde{\varphi}(x_1)^2\ \exp(-\omega r/8)\ , \end{aligned}$$

and therefore

$$\begin{aligned} \int_E \int_E |G^{(6)}(x,y,\lambda)|^2 dx\ dy &\leq M \int_E \int_E |\tilde{\varphi}(x_1)|^4 \exp(-\omega r/4)dx\ dy \\ &\leq M \int_0^\infty |\varphi(s)|^6 ds\ , \end{aligned}$$

by the usual arguments. The same calculation applied to larger values of k, and the proof of Theorem 4.2 is complete for $\dim E = 2$.

<u>Remarks</u> on extension of Theorem 4.2 to dimensions larger than two. The case $\dim E = 3$:

If we apply the same procedure to $G^{(4)}(\dim E = 3)$ as we did to $G^{(4)}(\dim E = 2)$, using equation (1) instead of equation (3), we can show

$$G^{(4)}(x,y,\lambda) \leq M\ e^{-\omega r/8} \int_E \exp(-\omega|x_1-z_1|/8)dz.$$

If we now recall that in three dimensions E is the solid of revolution formed by rotating $\varphi(x_1)$ about the x_1-axis, then by a change of variables in the above equations and the usual calculations, we have

$$G^{(4)}(x,y,\lambda) \leq M\, e^{-\omega r/8} \int_0^{\infty}\int_0^{2\pi}\int_0^{\varphi(z_1)} \exp(-\omega|x_1-z_1|/8)\, r\,dr\,d\theta\, dz_1$$

$$\leq M\, e^{-\omega r/8} \int_0^{\infty}\int_0^{2\pi} \exp(-\omega|x_1-z_1|/8)\, \frac{\varphi(z_1)^2}{2}\, d\theta\, dz_1$$

$$\leq M_1\, e^{-\omega r/8}\, \tilde{\varphi}(x_1)^2 .$$

If we integrate with respect to x, we have

$$\int_E |G^{(4)}(x,y,\lambda)|^2 dx \leq M \int_E \tilde{\varphi}(x_1)^4 \exp(-\omega r/4)\,dx$$

$$\leq M_1 \int_0^{\infty} \tilde{\varphi}(x_1)^6 \exp(-\omega|x_1-y_1|/4)\,dx_1$$

$$\leq M_2\, \tilde{\varphi}(y_1)^6 .$$

If we integrate again, then

$$\int_E\int_E |G^{(4)}(x,y,\lambda)|^2 dx\, dy \leq M_\omega \int_0^{\infty} |\varphi(s)|^8 ds .$$

By similar argument we can show that

$$\int_E\int_E |G^{(2k)}(x,y,\lambda)dx\, dy \leq M_\omega \int_0^{\infty} |\varphi(s)|^{4k}\, ds,$$

where $\omega \geq \omega_o$, for some ω_o and $k \geq 2$. Similar results hold for $\dim E \geq 3$.

Henceforth we consider only domains E which satisfy the conclusion of Theorem 4.2. Thus there must exist a constant k_o such that

$$\int_E \int_E |G^{(k_o)}(x,y,\lambda)|^2 dx\, dy < \infty,$$

that is $G^{(k_o)}$ must be a Hilbert-Schmidt Kernel for some k_o. Note that, although Theorem 4.2 is stated only for domains which lie above the x_1-axis, it also holds for domains which have a similar piece below the x_1-axis, for example, one described by $-\varphi_1(x_1)$ where φ_1 satisfies the same conditions as φ.

The next step in our investigation of the boundary behaviour of G is to show that $G^{(k)}(x,y,\lambda)$ tends to zero at the boundary for some k. In order to prove this result we shall need the properties of λ_n and $u_n(x)$ developed in Part 3.

Unfortunately it does not seem possible to use the compactness of $G^{(k)}$ alone to construct the eigenvalues and eigenfunctions. Even though one can show the existence of λ_n and $u_n(x)$ such that $\Delta u_n(x) + \lambda_n u_n(x) = 0$, it does not seem possible to show that $u_n(x)$ is zero on the boundary, and thereby avoid the calculations in Part 3.

Using Theorem 3.7 (Green's function inversion) and the definition of $G^{(k)}(x,y,\lambda)$, one can easily see that

$$(G^{(k)}(x,\cdot,\lambda), u_n) = (\lambda_n - \lambda)^{-k} u_n(x) ,$$

where $\{u_n(x)\}$ and $\{\lambda_n\}$ are as constructed in Part 3. Combining this with Theorem 4.2 we can prove the following result on the eigenvalues.

Theorem 4.3. The series $\sum_{n=0}^{\infty} \lambda_n^{-2k}$ converges if $k \geq k_o$, where k_o is defined by the remark following Theorem 4.2.

Proof. For any set of orthonormal functions, for example $\{u_n(x)\}$, the Bessel inequality holds. i.e.

$$\sum_{n=0}^{\infty} |c_n|^2 \leq \|f(x)\|^2 \qquad \text{where} \quad c_n = (f, u_n).$$

By the inversion result for $G^{(k)}$, we have

$$c_n = (\lambda_n - \lambda)^{-k} u_n(y) \qquad \text{when} \quad f(x) = G^{(k)}(x,y,\lambda) \ .$$

Thus for $k = k_o$,

$$\sum_{n=0}^{N} |\lambda_n - \lambda|^{-2k} u_n^{\,2}(y) \leq \|G^{(k)}(\cdot,y,\lambda)\|^2 ,$$

where N is as large as we please.

Integrating this expression with respect to y we get

$$\sum_{n=0}^{N} |\lambda_n - \lambda|^{-2k} \leq \int_E \int_E |G^{(k)}(x,y,\lambda)|^2 dx\, dy,$$

which is finite by Theorem 4.2, if $\lambda = -\omega^2$ and ω is large enough, and is independent of N. Hence $\sum_{n=0}^{\infty} |\lambda_n - \lambda|^{-2k} < \infty$. Since $\lambda_n \to \infty$, this implies that $\sum_{n=0}^{\infty} \lambda_n^{-2k}$ converges, which completes Theorem 4.3.

Theorem 4.4. (The expansion formula). Let $k = k_o$, and let $f(x)$

have sufficient derivatives so that the functions f, $\Delta f(x)$, $\Delta(\Delta f(x))$, ..., $\Delta^{(k)} f(x)$ are all continuous, contained in $L_2(E)$, and tend to zero at the boundary of E.

Then

$$f(x) = \sum_{n=0}^{\infty} c_n u_n(x) \quad \text{where} \quad c_n = (f, u_n).$$

Proof. Let $c_n^{(i)} = (f^{(i)}, u_n)$ where

$$f^{(1)}(x) = -\Delta f(x) \quad \text{and}$$
$$f^{(i)}(x) = -\Delta f^{(i-1)}(x) \qquad i > 1.$$

By Glazman [7, Theorem 34, p. 90] we have $(\Delta u, v) = (\Delta v, u)$ where $\Delta u \in L_2(E)$ and both u and v are zero on ∂E. Thus if we set $v = u_n$ and $u = f^{(i)}$, we have

$$(f^{(i+1)}, u_n) = (\lambda_n u_n, f^{(i)}) \quad \text{or} \quad c_n^{(i+1)} = \lambda_n c_n^{(i)} = \lambda_n^{i+1} c_n.$$

Let $$g(x) = \sum_{n=0}^{\infty} c_n u_n(x) = \sum_{n=0}^{\infty} c_n^{(k)} \lambda_n^{-k} u_n(x) . \tag{11}$$

The method of proof will be to show that (11) converges and defines a function which is identical to $f(x)$.

We first must show that for some constant C we have

$$\sum_{n=0}^{\infty} (\lambda_n^{2k} + 1)^{-1} u_n^2(x) \leq C \tag{*}$$

uniformly for $x \in E'_{x_0}$. As in Theorem 4.3 we have

$$\sum_{n=0}^{\infty} |\lambda_n - \lambda|^{-2k} u_n^2(x) \leq \|G^{(k)}(x, \cdot, \lambda)\|^2 .$$

We have by Theorem 2.6

$$\|G^{(k)}(x,\cdot,\lambda)\| \leq \nu^{-1}\| G^{(k-1)}(x,\cdot,\lambda)\|$$

if $\lambda \in L(\nu)$, and likewise

$$\|G^{(k)}(x,\cdot,\lambda)\| \leq \nu^{-k+1}\| G(x,\cdot,\lambda)\| .$$

Since $\|G(x,\cdot,\lambda)\| \leq K$ for $x \in E'_{X_o}$ and $\lambda \in L_o$, we have

$$\sum_{n=0}^{\infty} |\lambda_n - \lambda|^{-2k} u_n^2(x) \leq K'$$

for all $x \in E'_{X_o}$ and $\lambda \in L_o$. Since $\lambda_n \to \infty$, (*) follows.

We now consider the tail end of the series (11). If $\lambda_{N+1} > 1$, then by the Schwarz inequality

$$\sum_{n=N+1}^{\infty} c_n^{(k)} \lambda_n^{-k} u_n(x) \leq [\sum_{n=N+1}^{\infty} (c_n^{(k)})^2 \cdot \sum_{n=N+1} a_n(\lambda^{2k}+1)^{-1} u_n^2(x)]^{1/2},$$

where $a_n = (\lambda_n^{2k}+1)\lambda_n^{-2k} \leq$ constant. Note that the series $\Sigma [c_n^{(k)}]^2$ converges, being the "Fourier" series of $f^{(k)}$ which is in $L_2(E)$ by hypothesis. In view of (*) and $f^{(k)} \in L_2(E)$ the tail end of the series can be made small by the choice of N, uniformly for $x \in E'_{X_o}$. Hence series (11) defines a continuous function inside E. Also

$$g^2(x) \leq \sum_{n=o}^{\infty} [c_n^{(k)}]^2 \cdot \sum_{n=o}^{\infty} \lambda_n^{-2k} u_n^2(x),$$

from which it follows

$$\|g(x)\|^2 \leq \sum_{n=o}^{\infty} [c_n^{(k)}]^2 \cdot \sum_{n=o}^{\infty} \lambda_n^{-2k} ,$$

which is bounded by Theorem 4.3 (note $k = k_o$).

A similar argument shows that

$$g_m(x) = \sum_{n=o}^{m} c_n^{(k)} \lambda_n^{-k} u_n(x)$$

converges, in mean square, to $g(x)$ over E. Hence, by the Schwarz inequality, for any n

$$\lim_{m \to \infty} (u_n, g-g_m) = 0 .$$

However we know that $(u_n, g_m) = c_n$ $(m \geq n)$, from which it follows that $(u_n, g) = c_n$. Thus the function $f(x) - g(x)$ is an $L_2(E)$-function, all of whose "Fourier coefficients" vanish. Hence by the Parseval Theorem $\|f - g\| = 0$. Since the integrand is continuous inside E, it must vanish everywhere inside E. This completes the proof of Theorem 4.4.

We are now in a position to prove the main result in this part, namely that $G^{(k)}(x,y,\lambda)$ tends to zero as $x \to \partial E$. The proof will depend directly on the knowledge that $G^{(k)}$ is Hilbert-Schmidt and thus the series $\Sigma\, \lambda_n^{-2k}$ convergent. The method is derived from Titchmarsh [10, p. 106].

Theorem 4.5. For fixed $x \in E$ and fixed $\lambda \in \mathbb{C}$ not on the non-negative real axis, $G^{(k)}(x,y,\lambda)$ tends to zero as y approaches the boundary of E. ($k \geq k_o + 2$ and $\dim E = 2$.)

Proof. Let $r = |x-u|$ and define

$$F(x,u) = -\frac{1}{2\pi} \log \frac{r}{R}\, g(r) ,$$

where R is such that the circle $|x-u| < R$ is inside E for fixed u, and $g(r)$ has the following properties:

$$g \in C_0^\infty(E) , \quad g(r) = 1 \quad r \leq R/2 \quad \text{and}$$

$$g(r) = 0 \quad r \geq R .$$

Since $G^{(k)}$ is not singular, we have in the Green's formula for $G^{(k)}(x,y,\lambda)$ and $F(x,u)$ as functions of x :

$$\int_{r\leq R} (G^{(k)}(x,y,\lambda)\, \Delta F(x,u) - F(x,u)\, \Delta G^{(k)}(x,y,\lambda))dx$$
$$= G^{(k)}(u,y,\lambda) + \int_{r=R} (G^{(k)} \frac{\partial F}{\partial n} - F \frac{\partial G^{(k)}}{\partial n}\, ds, \qquad (12)$$

the singularity of $F(x,u)$ at $x = u$ giving rise to the term $G^{(k)}(u,y,\lambda)$. Consider the boundary term: $F(r) = 0$ for $r = R$, and $\frac{\partial F}{\partial n} = \frac{\partial F}{\partial r} = 0$ for $r = R$, by the definition of F. Thus the boundary term of (12) vanishes. Upon substitution of

$$\Delta G^{(k)}(x,y,\lambda) = -\lambda G^{(k)}(x,y,\lambda) - G^{(k-1)}(x,y,\lambda) \qquad (k>1),$$

which follows directly from the definition of $G^{(k)}$ and Theorem 2.6, into (12), we have

$$G^{(k)}(x,y,\lambda) = \int_{r\leq R} [\Delta F(x,u) + \lambda F(x,u)]\, G^{(k)}(x,y,\lambda)\, dx$$
$$+ \int_{r\leq R} F(x,u)\, G^{(k-1)}(x,y,\lambda)\, dx. \qquad (13)$$

Recall the result $\sum_{n=0}^{\infty} a_n b_n = (f,g)$ where $a_n = (f,u_n)$ and

$b_n = (g, u_n)$, which followed directly from the Parseval Theorem. Let

$$d_n'(u) = (\Delta F(\cdot,u) + \lambda F(\cdot,u),\ u_n) \quad \text{and} \quad d_n(u) = (F(\cdot,u),\ u_n),$$

and recall that $\quad (G^{(k)}(\cdot,y,\lambda),\ u_n) = (\lambda_n - \lambda)^{-k}\ u_n(y)$.

Applying these results to (13) we get

$$G^{(k)}(u,y,\lambda) = \sum_{n=o}^{\infty} d_n'(u)\ (\lambda_n - \lambda)^{-k}\ u_n(y)$$
$$+ \sum_{n=o}^{\infty} d_n(u)\ (\lambda_n - \lambda)^{-k+1}\ u_n(y). \qquad (14)$$

We want to show that $G^{(k)}(u,y,\lambda)$ goes to zero as $y \to \partial E$. Consider the second series in (14); a similar result will apply to the first. By Lemma 3.9 we have $|u_n(y)| < K\,\lambda_n$ for y near to the boundary. First consider the tail end of the series on the right hand side of (14), namely

$$\sum_{n=N}^{\infty} d_n(u)(\lambda_n - \lambda)^{-k+1} u_n(y) \leq \Big(\sum_{n=N}^{\infty} d_n^2(u) \cdot \sum_{n=N}^{\infty} |\lambda_n - \lambda|^{-2k+2} u_n^2(y)\Big)^{1/2}$$
$$\leq \Big(\sum_{n=o}^{\infty} d_n^2(u) \cdot \sum_{n=N}^{\infty} |\lambda_n - \lambda|^{-2k+2}\ K^2\ \lambda_n^2\Big)^{1/2}$$
$$\leq K'\ \|F(\cdot,u)\|\ \Big(\sum_{n=N}^{\infty} \lambda_n^{-2k+4}\Big)^{1/2},$$

since $\lambda_n \to \infty$. Now by hypothesis $k \geq k_o + 2$ and so the series $\Sigma\ \lambda_n^{-2k+4}$ converges. Thus given any $\mathcal{E}$, $\mathcal{E} > 0$ we can find an N so large that

$$\Big|\sum_{n=N}^{\infty} d_n(u)\ (\lambda_n - \lambda)^{-k+1}\ u_n(y)\Big| < \mathcal{E}/2$$

uniformly in y, y in a neighbourhood of the boundary.

If y is sufficiently close to the boundary of E we have

$$|d_n(u)\ (\lambda_n - \lambda)^{-k}\ u_n(y)| \leq \mathcal{E}/2N$$

for $n=0, 1, \ldots, N-1$, since u and λ are fixed and $u_n(y)$ tends to zero for each n. Combining all these estimates we get

$$\left|\sum_{n=o}^{\infty} d_n(u)\ (\lambda_n - \lambda)^{-k+1}\ u_n(y)\right| \leq \left|\sum_{n=o}^{N-1} d_n(u)\ (\lambda_n - \lambda)^{-k+1}\ u_n(y)\right| + \left|\sum_{n=N}^{\infty} d_n(u)\ (\lambda_n - \lambda)^{-k+1}\ u_n(y)\right|$$

$$\leq N(\mathcal{E}/2N) + \mathcal{E}/2 = \mathcal{E}\ .$$

Thus $G^{(k)}(x,y,\lambda)$ goes to zero as y tends to the boundary of E, when x and λ are fixed.

We can prove the following theorems in a similar manner.

<u>Theorem 4.6.</u> The function $H(x,\lambda,f) = -(G(x,\cdot,\lambda),f)$ satisfies the equation $\Delta H + \lambda H = f$ and the boundary condition $H(x,\lambda,f) \to 0$ as $x \to \partial E$, as long as f, Δf, $\Delta(\Delta f)$, ..., $\Delta^{(k)} f$ are all continuous, contained in $L_2(E)$ and tend to zero at the boundary of E, where $k = k_o$. λ is a complex number not on the non-negative real axis.

Proof. $H(x,\lambda,f)$ satisfies the equation $\Delta H + \lambda H = f$ by Theorem

2.6. By the definition of $H(x,\lambda,f)$ we have

$$H(x,\lambda,f) = \sum_{n=o}^{\infty} c_n(\lambda - \lambda_n)^{-1} u_n(x) \ , \tag{15}$$

where $c_n = (f, u_n)$ by the Parseval Theorem.

As we showed in the proof of the expansion theorem (Theorem 4.4) $c_n^{(k)} = \lambda_n^{\,k} c_n$, where $\Sigma [c_n^{(k)}]^2 < \infty$. Thus, substituting this result into (15), we get

$$H(x,\lambda,f) = \sum_{n=o}^{\infty} c_n^{(k)} \lambda_n^{-k} (\lambda - \lambda_n)^{-1} u_n(x) .$$

We now apply the same analysis to this equation as we did to equation (14) in the previous theorem, to conclude that $H(x,\lambda,f)$ goes to zero on the boundary. Note: $k = k_o$ will be sufficient to carry out the calculations.

Remark: Note that Theorem 4.3 $(\Sigma \lambda_n^{-2k} < \infty)$ holds for a domain E which lies in the half-space $x_1 > 0$ and is bounded by the surface obtained by rotating a $\varphi^*(x_1)$, where φ^* satisfies the conditions of φ, about the x_1-axis. This follows from the fact that the eigenvalues of E dominate the eigenvalues of the surface of revolution (see Glazman [7, p.229]). Furthermore Theorem 4.3 now implies that the other theorems of the section, namely, Theorem 4.4, 4.5 and 4.6 hold for such a domain. Some modifications are necessary for example the singularity of the function $F(x,u)$ in Theorem 4.5.

Part 5. Boundary behaviour and uniqueness of the Green's Function.

We are now in a position to prove that the Green's function tends to zero at the boundary of E. We continue to make the assumptions of Part 4, so that some iterate $G^{(k)}(x,y,\lambda)$ satisfies Theorem 4.5.

<u>Theorem 5.1</u>. For $\lambda = -\omega^2$, $\omega > 0$ and $x \in E$, $G(x,y,\lambda)$ tends to zero as y tends to the boundary of E.

Proof. The proof is performed step by step. One proceeds as follows: Suppose for example that

$$G^{(3)}(x,y,\lambda) \to 0 \qquad \text{as} \qquad y \to \partial E .$$

Let us assume (λ_o is negative) that

$$G^{(2)}(x,y,\lambda_o) \not\to 0 \qquad \text{as} \qquad y \to \partial E .$$

Thus there exists a δ and a sequence $y_n \to z \in \partial E$ such that

$$G^{(2)}(x,y_n,\lambda_o) \geq \delta > 0 \qquad \text{for all } n \tag{1}$$

and

$$|x-y_n| \geq a > 0 .$$

Recall the estimate (similar results holding for other iterates and dimensions) for $\dim E = 2$:

$$0 \leq G^{(2)}(x,y,-\omega^2) \leq M \exp(-\omega|x-y|/2) .$$

Thus we have

$$0 \leq G^{(2)}(x,y_n,-\omega^2) \leq M \exp(-\omega|x-y_n|/2)$$

$$\leq M_a \exp(-\omega\, a/2)$$

where M_a is a constant independent of ω (see equation 4.6 of Theorem 4.1). From this it follows that there exists a $\lambda_1 (= -\omega_1^2)$ such that

$$0 \leq G^{(2)}(x,y_n,\lambda_1) \leq \delta/4 \qquad \text{for all } n. \tag{2}$$

By theorem 2.10 we have

$$D_\lambda\, G^{(2)}(x,y,\lambda) = G^{(3)}(x,y,\lambda)$$

if λ is negative, from which it follows that

$$G^{(2)}(x,y_n,\lambda_1) - G^{(2)}(x,y_n,\lambda_o) = \int_{\lambda_o}^{\lambda_1} G^{(3)}(x,y_n,s)ds. \tag{3}$$

Since the proof of Theorem 4.5 clearly shows that

$$G^{(3)}(x,y_n,\lambda) \to 0 \qquad \text{as} \qquad y_n \to z \in \partial E$$

uniformly for $\lambda_1 \leq s \leq \lambda_o < 0$, we have

$$\lim_{y_n \to z \in \partial E} \int_{\lambda_o}^{\lambda_1} G^{(3)}(x,y_n,s)ds = 0 .$$

Thus from (3) it follows that as $n \to \infty$

$$|G^{(2)}(x,y_n,\lambda_1) - G^{(2)}(x,y_n,\lambda_o)| \to 0, \tag{4}$$

which contradicts equations (1) and (2). Thus $G^{(2)}(x,y,\lambda)$ tends to zero, as $y \to \partial E$, for each x and negative λ.

To prove the final result we use the following argument: By Theorem 2.10 and the same argument used to prove equation (4) it follows that

$$|G(x,y_n,\lambda_1) - G(x,y_n,\lambda_o) - (\lambda_1 - \lambda_o)\, G^{(2)}(x,y_n,\lambda_o)| \to 0$$

as $n \to \infty$. The argument proceeds in exactly the same manner, except for the introduction of the term $(\lambda_1 - \lambda_o)\ G^{(2)}(x,y_n,\lambda_o)$ which we already know goes to zero as $n \to \infty$. For higher values of k the procedure is clear, that is, reduce the result one step at a time until one has $G(x,y,\lambda) \to 0$ as $y \to \partial E$.

Theorem 5.2. The Green's function $G(x,y,\lambda)$ is unique for E if λ is negative.

Proof. Let G_1 and G_2 be two Green's functions for the same negative λ and set

$$f(x,\lambda) = G_1(x,y,\lambda) - G_2(x,y,\lambda) \qquad (\in L_2(E)).$$

By the results of Part 2 we have

$$(\Delta + \lambda)\, f(x,\lambda) = 0 \qquad x \in E$$

if λ is not on the non-negative real axis. If λ is real and negative we can show $f(x,\lambda) \equiv 0$, by the maximum principle. The maximum principle states that if $\Delta u(x) + a(x)\, u(x) = 0$ for x in a bounded open set $\mathcal{K}$, if $a(x) \leq 0$ on $\mathcal{K}$, and if $u(x)$

attains its maximum at an interior point of $\mathcal{K}$, then $u(x) \equiv$ constant on $\mathcal{K}$. To apply this to $f(x,\lambda)$ suppose that $f(x,\lambda) \not\equiv 0$ on E, so, without loss of generality, there is a point $x_o \in E$ with $f(x_o,\lambda) = \sigma > 0$. Let $\mathcal{K} = E_X$ where X is chosen so large that (cf. Theorem 2.6) $x_o \in E_X$ and $|f(x,\lambda)| \leq \sigma/2$ for $|x| \geq X$. Then $|f(x,\lambda)| < \sigma$ for any $x \in \partial\mathcal{K}$, $f(x_o,\lambda) = \sigma$, and therefore $f(x,\lambda)$ must achieve its maximum at an interior point of $\mathcal{K}$. Consequently $f(x,\lambda) \equiv$ constant on $\mathcal{K}$, and thus $f(x,\lambda) \equiv 0$ on $\mathcal{K} = E_X$, by Theorem 5.1. Since X may be chosen arbitrarily large, $f(x,\lambda) = 0$ on E.

Part 6. Applications of the Green's Function.

In this part we shall prove two asymptotic properties, one for the eigenvalues and one for the eigenfunctions as defined in part 3. Throughout this part we shall assume that E satisfies all the conditions of the previous theorems in order that we shall have a Green's function which is zero on the boundary of E. Furthermore, we assume that the dimension of E is 2.

Let

$$F(x) = F(x,y,\lambda) = \frac{1}{4} i H_0^{(1)} (\rho\sqrt{\lambda}) - G(x,y,\lambda) \tag{1}$$

where $\rho = |x-y|$. The first term on the right hand side is the Green's function for Δ in the whole plane. $F(x)$ is continuous at $x=y$ since all Green's functions have the same type of singularity as $\rho \to 0$.

Our first task is to obtain bounds and asymptotic estimates for the function $F(x,y,-\mu)$ and its derivatives with respect to μ.

If we set $\lambda = -\mu$ where μ is real and positive, then

$$\Delta F(x) = \mu F(x).$$

Now we wish to apply the maximum principle to $F(x)$; however, since E is not bounded we need to know that $F(x) \to 0$ as $x \to \infty$. By Theorem 2.8 we have that $G(x) \to 0$, as $x \to \infty$ for fixed λ and y. Furthermore $H_0^{(1)}$ goes to zero as $\rho \to \infty$, so we have the required result that $F(x) \to 0$ as $x \to \infty$. Now if we apply the maximum principle in the usual manner (see Titchmarsh [10, p.169] we have that $F(x)$ must assume its maximum on the boundary of E.

If x is interior to E and y is on the boundary, $G(x,y,-\mu) = 0$, and so

$$F(x,y,-\mu) = \frac{1}{4}\, i\, H_0^{(1)}\,(\rho\sqrt{-\mu}) = (2\pi)^{-1}\, K_0(\rho\sqrt{\mu})$$

in the usual notations of Bessel functions. Now $K_0(t)$ is a positive, strictly decreasing function of t. Hence the right hand side lies between 0 and $(2\pi)^{-1}\, K_0(a\sqrt{\mu})$, where a is the distance from x to the nearest point on the boundary. If we now regard x as fixed and y varying, we obtain

$$0 \leq F(x,y,-\mu) \leq (2\pi)^{-1}\, K_0(a\sqrt{\mu}) \tag{2}$$

for all x and y. Since F is continuous at $x=y$ this is true for $x=y$.

If we again apply the results in Titchmarsh [10, p. 170] we can show

$$- \max\,\{(2\pi\mu)^{-1}\, K_0(a\sqrt{\mu}),\ (4\pi\mu^{1/2})^{-1}\, a\, K_1(a\sqrt{\mu})\}$$

$$\leq F_\mu(x,y) \leq 0$$

where $F_\mu = \frac{\partial}{\partial\mu} F$. The next step is to extend these results to higher derivatives.

Let $F_{k\cdot\mu} = D_\mu^k F$. From Titchmarsh [10, p.170] we know

$$F_{1\cdot\mu}(x,y,-\mu) = -\frac{\rho}{4\pi}\,\mu^{-1/2}\,K_1(\rho\sqrt{\mu}) \qquad y \in \partial E\ .$$

We wish to show that

$$F_{k\cdot\mu}(x,y,-\mu) = \frac{(-\rho)^k}{2^{k+1}\pi}\,\frac{K_k(\rho\sqrt{\mu})}{\mu^{k/2}} \qquad y \in \partial E\ . \tag{3}$$

First of all, we can show

$$D_\mu^k\,K_o(\rho\sqrt{\mu}) = \frac{(-\rho)^k}{2^k}\,\frac{K_k(\rho\sqrt{\mu})}{\mu^{k/2}}$$

by applying the equations

$$\begin{aligned} t\,K_\nu'(t) &= -\nu\,K_\nu(t) - t\,K_{\nu-1}(t) \\ &= \ \nu\,K_\nu(t) - t\,K_{\nu+1}(t)\ . \end{aligned} \tag{4}$$

Equations (4) can be found in Watson [11, p.79 Sec. 3.71]. The next step is to differentiate the last term in (1), but by Theorem 2.10

$$D_\lambda^k\,G(x,y,\lambda) = k!\ \ G^{(k+1)}(x,y,\lambda)\ ,$$

and thus for $y \in \partial E$ the derivative of the term $G(x,y,-\mu)$ will be zero since μ is real and $G^{(k)}(x,y,-\mu) = 0$ for $y \in \partial E$. This completes the calculation for equation (3).

From equation (4) it is easy to see that

$$D_t\ (t^k\,K_k\,(t)) = -t^k\,K_{k-1}(t)\ .$$

However, since K is a positive function, it follows that $t^k\,K_k(t)$ is a positive strictly decreasing function of t. Hence, if $y \in \partial E$, then

$$0 \leq F_{2\cdot\mu}(x,y,-\mu) \leq \frac{a^2}{8\pi} \frac{K_2(a\sqrt{\mu})}{\mu} , \tag{5}$$

where a is the distance between x and the nearest boundary point.

In order to apply the maximum principle to $F_{2\cdot\mu}$ we must first show that

$$\Delta F_{k\cdot\mu} = \mu F_{k\cdot\mu} + k F_{(k-1)\cdot\mu} . \tag{6}$$

Since both $H_0^{(1)}$ and G are Green's functions (see equation (1)), F satisfies the distribution equation

$$\Delta F = \mu F + \delta_y .$$

We obtain equation (6) by differentiating the distribution equation k-times with respect to μ, and observing that $D_\mu \delta_y = 0$.

As a special case of (6) we have

$$\Delta F_{2\cdot\mu} = \mu F_{2\cdot\mu} + 2 F_\mu . \tag{7}$$

We shall work with (7) in order to obtain a bound for $F_{2\cdot\mu}$, using the bounds already known for F_μ and F. Higher results can be shown by induction.

Suppose $F_{2\cdot\mu}$ has a negative minimum inside E (note $F_{2\cdot\mu} \geq 0$ on ∂E and it tends to zero for large x). Therefore

$$\Delta F_{2\cdot\mu} \geq 0 \qquad \mu F_{2\cdot\mu} < 0 \qquad \text{and} \qquad F_\mu \leq 0 ,$$

but this contradicts equation (7), so $F_{2\cdot\mu}$ is non-negative on E. Suppose next that $F_{2\cdot\mu}$ takes on a value, inside E, greater than the right hand side of equation (5), then $F_{2\cdot\mu}$ must have a positive maximum inside E.

i.e.

$$\mu F_{2\cdot\mu} + 2 F_{\mu} = \Delta F_{2\cdot\mu} \leq 0 ,$$

and hence

$$F_{2\cdot\mu} \leq -\frac{2}{\mu} F_{\mu} \leq \max \frac{2}{\mu} \left\{ \frac{K_0(a\sqrt{\mu})}{2\pi\mu} , \frac{a K_1(a\sqrt{\mu})}{4\pi\mu^{1/2}} \right\} .$$

Thus we have

$$0 \leq F_{2\cdot\mu}(x,y,-\mu) \leq \max\left\{\frac{K_0(a\sqrt{\mu})}{\pi \mu^2} , \frac{a K_1(a\sqrt{\mu})}{2\pi \mu^{3/2}} , \frac{a^2 K_2(a\sqrt{\mu})}{8 \pi \mu} \right\} .$$

By an inductive argument we have

$$|F_{k\cdot\mu}(x,y,-\mu)| \leq \max_{i=0,1,\ldots k} \left\{ \frac{k!}{i!} \frac{a^i K_i(a\sqrt{\mu})}{2^{i+1} \mu^{k-i/2}} \right\}.$$

We are now in a position to extend the asymptotic formula for the eigenvalues, given by Clark in [3], to our domain E. Let $\tau(x)$ be any function such that

$$0 \leq \tau(x) \leq A\, a(x)^{k'+\mathcal{E}} ,$$

where A and $\mathcal{E}$ are positive constants; $a(x)$ is the distance between x and the nearest point on the boundary, and k is a integer such that the integral of $a(x)^{k'}$ over E is finite and

$$\int_E \int_E |G^{(m)}(x,y,\lambda)|^2 dy\, dx < \infty \quad \text{where} \quad m = \left[\frac{k'}{2}\right]$$

is also finite. Furthermore let

$$N_\tau(\lambda) = \begin{cases} \tau_0 + \tau_1 + \dots + \tau_j & \lambda_{j-1} \le \lambda < \lambda_j \\ 0 & \lambda < \lambda_0 \end{cases}$$

where $\tau_n = (\tau, |u_n|^2)$, u_n being the usual eigenfunctions defined in Part 3.

Theorem 6.1. If $\dim E = 2$, and $N_\tau(\lambda)$ and $\tau(x)$ are as defined above, then

$$N_\tau(\lambda) \sim \frac{\lambda}{4\pi} \int_E \tau(x)\, dx .$$

Proof. If λ and λ' are not eigenvalues, then

$$G(x,y,\lambda) - G(x,y,\lambda') = \sum_{n=0}^{\infty} \frac{u_n(x)\, u_n(y)\, (\lambda - \lambda')}{(\lambda_n - \lambda)\, (\lambda_n - \lambda')}$$

by the results of the previous sections. If we let $\lambda = -\mu$ and $\lambda' = -\mu'$, where μ and μ' are positive, then

$$(\mu' - \mu) \sum_{n=0}^{\infty} \frac{u_n(x)\, u_n(y)}{(\lambda_n+\mu)(\lambda_n+\mu')} = \frac{1}{2} [K_0(\rho\sqrt{\mu}) - K_0(\rho\sqrt{\mu'})]$$

$$- F(x,y,-\mu) + F(x,y,-\mu') .$$

If we divide both sides of this expression by $\mu' - \mu$ and let μ' tend to μ, we have (if we can show the series converges uniformly with respect to μ')

$$\sum_{n=0}^{\infty} \frac{u_n(x)\, u_n(y)}{(\lambda_n + \mu)^2} = \frac{1}{2} D_\mu K_0(\rho\sqrt{\mu}) - F_\mu(x,y) . \tag{8}$$

To show the series converges uniformly, consider the tail end of the series:

$$\left| \sum_{n=N}^{\infty} \frac{u_n(x)}{(\lambda_n+\mu)} \cdot \frac{u_n(y)}{(\lambda_n+\mu')} \right| \leq$$

$$\leq \left[\sum_{n=N}^{\infty} \frac{u_n^2(x)}{(\lambda_n+\mu)^2} \cdot \sum_{n=o}^{\infty} \frac{u_n^2(y)}{(\lambda_n+\mu')^2} \right]^{1/2} .$$

Now since μ' tends to μ, $-\mu'$ can be contained in a set L_o (see Part 2), and thus

$$\sum_{n=o}^{\infty} \frac{u_n^2(y)}{(\lambda_n+\mu')^2} \leq K$$

by Lemma 3.5 and Theorem 2.5 (note y is fixed), where K is independent of y and μ'. Thus the series is uniformly convergent since the tail end can be made small (by choice of N) independent of μ'.

Since

$$D_\mu \, K_o(\rho\surd\mu) = \frac{\rho \, K_1(\rho\surd\mu)}{2\,\pi\,\mu^{1/2}}$$

and $K_1(t) \sim t^{-1}$ as $t \to 0$, we have

$$\sum_{n=o}^{\infty} \frac{u_n(x)^2}{(\lambda_n+\mu)^2} = \frac{1}{4\pi\mu} - F_\mu(x,x) \tag{9}$$

by letting $y \to x$ in expression (8), i.e. $\rho \to 0$. We prove that the series is uniformly convergent in y by a method similar to that which was used before.

The next step is to differentiate the expression (9) to build up the following set of equations:

$$\sum_{n=o}^{\infty} \frac{u_n^2(x)}{(\lambda_n+\mu)^{k+1}} = \frac{1}{4\pi k\mu^k} + \frac{(-1)^k}{k!} F_{k\cdot\mu}(x,x) . \tag{10}$$

In order to pass the derivative through the summation sign we must show that the resulting series is uniformly convergent with respect to the variable in question. Again consider the tail end of a series like those in (10); i.e.

$$\sum_{n=N}^{\infty} \frac{u_n{}^2(x)}{(\lambda_n+\mu)^{k+2}} \leq \frac{1}{(\lambda_N+\mu)^k} \sum_{n=N}^{\infty} \frac{u_n{}^2(x)}{(\lambda_n+\mu)^2}$$

where k is a free index greater than zero. Thus as usual

$$\sum_{n=N}^{\infty} \frac{u_n{}^2(x)}{(\lambda_n+\mu)^{k+2}} \leq \frac{1}{(\lambda_n+\mu)^k} \|G(x,\cdot,-\mu)\| \leq \lambda_n{}^{-1} K,$$

where K is bounded, since x is fixed and $-\mu \in L_o$. Thus the series in (10) is uniformly convergent in μ for any $k \geq 2$, and hence we can differentiate (9) with respect to μ as many times as we please.

Let $\tau(x)$ be defined as in the statement of this theorem. Multiply expression (10) by $\tau(x)$ and integrate; this gives (if we can pass the integral sign through the summation)

$$\sum_{n=o}^{\infty} \int_E \frac{\tau(x)\, u_n{}^2(x)}{(\lambda_n + \mu)^{k+1}} dx = \frac{1}{4\pi k\mu^k} \int_E \tau(x)dx + $$

$$+ \frac{(-1)^k}{k!} \int_E \tau(x)\, F_{k\cdot\mu}(x,x)dx. \tag{11}$$

We shall show directly that the integral and summation sign in (11) can be interchanged. Let

$$S(x) = \sum_{n=o}^{\infty} \frac{u_n^{\,2}(x)}{(\lambda_n+\mu)^{k'+1}}\, \tau(x)\ .$$

Consider

$$\left| \int_E [S(x) - \sum_{n=o}^{N} \frac{u_n^{\,2}(x)}{(\lambda_n+\mu)^{k'+1}}\, \tau(x)]\, dx \right|$$

$$= \int_E \sum_{n=N}^{\infty} \frac{u_n^{\,2}(x)}{(\lambda_n+\mu)^{k'+1}}\, \tau(x)\ dx$$

$$\leq (\lambda_N + \mu)^{-1} \int_E \tau(x) \sum_{n=o}^{\infty} \frac{u_n^{\,2}(x)}{(\lambda_n+\mu)^{k'+1}}\, dx$$

$$\leq K\, \lambda_N^{-1} \int_E \sum_{n=o}^{\infty} \frac{u_n^{\,2}(x)}{(\lambda_n+\mu)^{k'}}\, dx$$

since $\tau(x)$ is eventually less than one. If we let $m = [\frac{k'}{2}]$, then the tail end is less than

$$K\, \lambda_N^{-1} \int_E \int_E |G^{(m)}(x,y,\lambda)|^2 dy\ dx$$

which is bounded by the hypothesis on k'. Thus we can make the original difference as small as we please by a sufficiently large choice of N, and hence we can pass the integration sign through the summation sign in (11) for $k \geq k'$.

If we use the definition of τ_n, namely $\tau_n = (\tau, u_n^2)$, the expression on the left hand side of (11) can be expressed as

$$\sum_{n=o}^{\infty} (\lambda_n + \mu)^{-k-1}\, \tau_n\ .$$

We wish to express this series as an integral in order to apply a Tauberian theorem. Consider

$$\int_0^\infty \frac{N_\tau(\lambda)}{(\lambda+\mu)^{k+2}}\,d\lambda = \sum_{j=0}^\infty \int_{\lambda_j}^{\lambda_{j+1}} \frac{d\lambda}{(\lambda+\mu)^{k+2}}(\tau_0+\ldots+\tau_j)$$

$$= \sum_{j=0}^\infty \frac{(\tau_0+\ldots\tau_j)}{k+1}\left\{\frac{1}{(\lambda_j+\mu)^{k+1}} - \frac{1}{(\lambda_{j+1}+\mu)^{k+1}}\right\}$$

$$= \frac{1}{k+1}\sum_{j=0}^\infty \frac{\tau_j}{(\lambda_j+\mu)^{k+1}} .$$

Thus by combining this result with (11) we have

$$\int_0^\infty \frac{N_\tau(\lambda)}{(\lambda+\mu)^{k+2}}\,d\lambda = (4\pi\, k(k+1)\mu^k)^{-1}\int_E \tau(x)dx$$

$$+ \frac{(-1)^k}{(k+1)!}\int_E \tau(x)\, F_{k\cdot\mu}(x,x)dx. \tag{12}$$

Let us assume for the moment that

$$\int_E \tau(x)\, F_{k\cdot\mu}(x,x)dx = O(\mu^{-k-\mathcal{E}}) \quad (\mu \to \infty) \tag{13}$$

where $\mathcal{E}$ is some fixed positive number. This result will be proved after the application of the following Tauberian theorem to expression (12), (see Titchmarsh [10, p.364]). If

$$\int_0^\infty \frac{f(y)}{(x+y)^\alpha}\,dy \sim \frac{c}{x^\beta}$$

where $\alpha > 1$ and $0 < \beta < \alpha$, then

$$f(x) \sim \frac{c\,\Gamma(\alpha)}{\Gamma(\alpha-\beta)\Gamma(\beta)}\, x^{\alpha-\beta-1} .$$

Thus we have from equation (12) and the hypothesis, equation (13), that

$$N_\tau(\lambda) \sim \frac{\Gamma(k+2)}{\Gamma(2)\,\Gamma(k)}\;\frac{\lambda}{4\pi(k+1)k}\int_E \tau(x)\,dx\ .$$

Hence

$$N_\tau(\lambda) \sim \frac{\lambda}{4\pi}\int_E \tau(x)\,dx\ ,$$

which is what we wished to prove.

The theorem will be complete if we can show the hypothesis (13). Recall the condition on $\tau(x)$, namely

$$0 \le \tau(x) \le A\ a(x)^{k'+\mathcal{E}}$$

where A and $\mathcal{E}$ are constants and k' is such that $a(x)^{k'}$ is integrable over E, where $a(x)$ is the distance between x and the nearest point on the boundary. If we apply the estimate

$$K_\nu(t) \sim \left(\frac{\pi}{2t}\right)^{1/2} e^{-t} \qquad \text{as} \quad t \to \infty,$$

which is valid for all ν (see Watson [11, sec. 7.23), to the bound we found for $F_{2\cdot\mu}(x,y,-\mu)$, we see that for $a\sqrt{\mu} \ge 1/2$,

$$F_{2\cdot\mu}(x,y,-\mu) \le C\ \exp(-a\sqrt{\mu})\ \max\left[\frac{1}{\mu^2(a\sqrt{\mu})^{1/2}}, \frac{a}{\mu^{3/2}(a\sqrt{\mu})^{1/2}}, \frac{a^2}{\mu(a\sqrt{\mu})^{1/2}}\right]$$

$$\le C\ \mu^{-5/4}\ a^{3/2}\ \exp(-a\sqrt{\mu})$$

since the last term in the parentheses dominates the others. The general case for $F_{k\cdot\mu}(x,x)$ gives

$$|F_{k\cdot\mu}(x,x)| \le C\ \mu^{-(2k+1)/4}\ a^{k-1/2}\ e^{-a\sqrt{\mu}}$$

where C is a constant and $a\sqrt{\mu} \geq 1/2$.

Next we wish to examine $|F_{k\cdot\mu}(x,x)|$ for $a\sqrt{\mu} \leq 1/2$. To do this we use the estimates

$$K_o(t) = O(|\log t|) \qquad \text{as} \quad t \to 0$$

and

$$K_i(t) = O(t^{-i}) \qquad \text{as} \quad t \to 0 \quad (i \geq 1).$$

The first term in the inequality for $F_{k\cdot\mu}(x,x)$ gives rise to

$$\frac{K_o(a\sqrt{\mu})}{\mu^k} \leq C\,\mu^{-k}\,|\log a\sqrt{\mu}| \qquad \text{for} \quad a\sqrt{\mu} < 1/2.$$

The i^{th} term $(i > 0)$ gives

$$\frac{a^i\,K_i(a\sqrt{\mu})}{\mu^{k-i/2}} \leq C\,\frac{a^i}{a^i\,\mu^{i/2}} \cdot \frac{1}{\mu^{k-i/2}} = C\,\mu^{-k}$$

for $a\sqrt{\mu} < 1/2$. Thus the first term dominates and we have

$$|F_{k\cdot\mu}(x,x)| \leq C\,\mu^{-k}\,|\log a\sqrt{\mu}| \qquad \text{for} \quad a\sqrt{\mu} < 1/2\,.$$

We shall now attempt to estimate the integral in equation (13).

Let

$$E_m = \{x \in E \;\big|\; a(x)\sqrt{\mu} \leq 2^{m-1}\} \quad m = o, 1, 2, \ldots\ldots$$

If we use the hypothesis on $\tau(x)$ and the above estimate which applies to E_o $(a(x)\sqrt{\mu} \leq 1/2)$, then

$$\left|\int_{E_o} \tau(x)\,F_{k\cdot\mu}(x,x)\,dx\right| \leq C \int_{E_o} a(x)^{k+\mathcal{E}}\,\mu^{-k}\,|\log a(x)\sqrt{\mu}|\,dx$$

which is less than

$$C\ \mu^{-k}\left\{\int_{E_o} a(x)^{k+\mathcal{E}}\ |\log a(x)|\,dx + \frac{1}{2}\int_{E_o}|\log \mu|\,a(x)^{k+\mathcal{E}}\,dx\right\}.$$

Consider the first integral:

$$\int_{E_o} a(x)^{k+\mathcal{E}}\ |\log a(x)|\,dx \le C \int_{E_o} a(x)^{k+\mathcal{E}/2}dx$$

$$\le C \max_{x\in E_o} a(x)^{\mathcal{E}/2} \int_{E_o} a(x)^k\,dx,$$

but for $x \in E_o$ $a(x) \le \frac{\mu}{2}^{-1/2}$ and hence the maximum over E_o of $a(x)^{\mathcal{E}/2}$ is less than $\frac{\mu}{2}^{-\mathcal{E}/4}$.

Also

$$\int_{E_o} |\log \mu|\,a(x)^{k+\mathcal{E}}\,dx$$

is less than $C'\ \mu^{-\mathcal{E}/4}$. Putting these together we have

$$\left|\int_{E_o} \tau(x)\ F_{k\cdot\mu}(x,x)dx\right| \le C\,\mu^{-k-\mathcal{E}/4} \quad \text{as} \quad \mu \to \infty.$$

Consider next the integral over $E_{m+1} - E_m$ for $m=0, 1, 2, \ldots$ i.e.

$$\left|\int_{E_{m+1}-E_m} \tau(x)\ F_{k\cdot\mu}(x,x)dx\right| \le$$

$$\le C\ \mu^{-(2k+1)/4} \int_{E_{m+1}-E_m} a(x)^{k+\mathcal{E}}\ a(x)^{k-1/2}\ e^{-a(x)\sqrt{\mu}}\,dx\ .$$

We know from the definition of E_m, that if $x \in E_{m+1}-E_m$ then

$$2^{m-1} < a(x)\sqrt{\mu} \leq 2^m ,$$

and so our integral over $E_{m+1} - E_m$ is less than

$$C\ \mu^{-(2k+1)/4} \max_{x \in E_{m+1} - E_m} \{a(x)^{k+\mathcal{E}-1/2}\ e^{-a(x)\sqrt{\mu}}\} \int_E a(x)^k dx,$$

which in turn is less than

$$C\ \mu^{-(2k+1)/4} \{(\frac{2^m}{\sqrt{\mu}})^{k+\mathcal{E}-1/2} \exp(-2^{m-1})\} \int_E a(x)^k dx ,$$

since $a(x) \leq \mu^{-1/2}\, 2^m$ and $a(x)\sqrt{\mu} > 2^{m-1}$. Therefore

$$\left| \int_{E_{m+1} - E_m} \tau(x)\ F_{k\cdot\mu}(x,x) dx \right| \leq C'\ \mu^{-k-\mathcal{E}/2}\ 2^{m(k+\mathcal{E}-1/2)} \exp(-2^{m-1}),$$

but since

$$\int_{E-E_o} \tau\ F_{k\cdot\mu}\ dx = \sum_{m=o}^{\infty} \int_{E_{m+1}-E_m} \tau\ F_{k\cdot\mu}\ dx$$

we have

$$\int_{E-E_o} \tau\ F_{k\cdot\mu}\ dx \leq C'\ \mu^{-k-\mathcal{E}/2} \sum_{m=o}^{\infty} 2^{m(\mathcal{E}+k-1/2)} \exp(-2^{m-1}),$$

where the infinite series clearly converges. Thus

$$\int_{E-E_o} \tau\ F_{k\cdot\mu}\ dx \leq C\ \mu^{-k-\mathcal{E}/2} ,$$

from which it follows that

$$\int_E \tau\ F_{k\cdot\mu}\ dx \leq C'\ (\mu^{-k-\mathcal{E}/2} + \mu^{-k-\mathcal{E}/4})$$

$$\leq C\ \mu^{-k-\mathcal{E}/4} \qquad \text{as} \quad \mu \to \infty .$$

The proof of the asymptotic formula is now complete.

Remark: Theorem 6.1 is a generalization of the well known asymptotic formula for the eigenvalues (see Titchmarsh [10, p.172]). If E is a bounded set we can set $\tau(x) = 1$ throughout E. If $\tau(x) = 1$, then $\tau_n = 1$ for all n, from this it follows that $N_\tau(\lambda) = N(\lambda)$, where $N(\lambda)$ is the number of eigenvalues less than λ. Thus for $\tau(x) = 1$ Theorem 6.1 reduces to

$$N(\lambda) \sim \frac{\lambda}{4\pi} \text{ area } E.$$

<u>Theorem 6.2.</u> If

$$\psi(\lambda) = \sum_{\lambda_n \leq \lambda} u_n^2(x) ,$$

then

$$\psi(\lambda) = \frac{\lambda}{4\pi} + O(\lambda^{1/2}) \qquad \text{for each } x.$$

Proof. Using the results and notation of Theorem 6.1, we have for $\dim E = 2$

$$\sum_{n=o}^{\infty} \frac{u_n^2(x)}{(\lambda_n+\mu)^{k+1}} = \frac{1}{4\pi k\mu^k} + \frac{(-1)^k}{k!} F_{k\cdot\mu}(x,x) .$$

Furthermore

$$\int_0^{\infty} \frac{\psi(\lambda)\, d\lambda}{(\lambda+\mu)^{k+2}} = \frac{1}{k+1} \sum_{n=o}^{\infty} \frac{u_n^2(x)}{(\lambda_n+\mu)^{k+1}}$$

$$= \frac{1}{4\pi k(k+1)\mu^k} + \frac{(-1)^k}{k!} F_{k\cdot\mu}(x,x) .$$

Thus, since $F_{k,\mu}(x,x) \leq C \exp(-a(x)\sqrt{\mu})$ if $a(x)\sqrt{\mu} \geq 1$ where $a(x)$ depends on x only, we have

$$\int_0^\infty \frac{\psi(\lambda)}{(\lambda+\mu)^{k+2}} d\lambda \sim \frac{1}{4\pi k(k+1)} \cdot \frac{1}{\mu^k} \quad \text{as} \quad \mu \to \infty .$$

If we apply the Tauberian theorem used in Theorem 6.1, then

$$\psi(\lambda) \sim \frac{\lambda}{4\pi} \quad \text{as} \quad \lambda \to \infty .$$

This result can be improved to

$$\psi(\lambda) = \frac{\lambda}{4\pi} + O(\lambda^{1/2})$$

by the methods of Titchmarsh [10, p.198].

BIBLIOGRAPHY

1. SHMUEL AGMON, Lectures on elliptic boundary value problems, Princeton, New Jersey, Van Nostrand (1965).

2. F.H. BROWNELL, Spectrum of the Static Potential Schrödinger equation over E_n. Annals of Mathematics, Volume 54, No. 3. (Nov. 1951), p. 554-594.

3. COLIN W. CLARK, An asymptotic formula for the eigenvalues of the Laplacian operator in an unbounded domain, to appear in the Bull, Amer. Math. Soc.

4. COLIN W. CLARK, An embedding theorem for function spaces, to appear in the Pacific J. Math.

5. R. COURANT AND D. HILBERT, Methods of Mathematical Physics, I and II, Interscience: New York (1953, 1962).

6. DUNFORD AND SCHWARTZ, Linear Operators, I and II, Interscience: New York (1957, 1963).

7. I.M. GLAZMAN, Direct methods of qualitative spectral analysis of singular differential operators. (Russian), Fizmatgiz: Moscow (1963).

8. A.M. MOLČANOV, On the conditions for discreteness of the spectrum of second order self-adjoint differential operators (Russian), Trudy Mosk. Mat. Obshchestva 2 (1953), 169-200.

9. F. RELLICH, Das Eigenwertproblem of $\Delta u+\lambda u = 0$ in Halbröhren, in "Essays presented to R. Courant". The Courant Anniversary Volume (1948), p. 329-344.

10. E.C. TITCHMARSH, Eigenfunction Expansions Associated with Second Order Differential Equations. Part II, Oxford University Press: Oxford (1958).

11. G.N. WATSON, Theory of Bessel Functions Second Edition, Cambridge University Press (1944).