CHARACTERIZATION OF TRANSFORMATIONS PRESERVING
RANK TWO TENSORS OF A TENSOR PRODUCT SPACE

by

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ABSTRACT

Let $U \otimes V$ be a tensor product space over an algebraically closed field $F$; let $\dim U = m$ and $\dim V = n$; let $T$ be a linear transformation on $U \otimes V$ such that $T$ preserves rank two tensors.

We show that $T$ preserves rank one tensors and this enables us to characterize $T$ for all values of $m$ and $n$. 
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1. INTRODUCTION.

Let $U$ and $V$ be $m$-dimensional and $n$-dimensional vector spaces over an algebraically closed field $F$. The tensor product of $U$ and $V$ denoted by $U \otimes V$ is the dual space of the space of all multilinear functions mapping $U \times V$ into $F$.

An element $x \in U \otimes V$ has rank $k$ if $x = \sum_{i=1}^{k} x_i \otimes y_i$ and $x_1, \ldots, x_k$ are linearly independent and $y_1, \ldots, y_k$ are linearly independent. We define $R_k(U \otimes V)$ to be $\{ x \in U \otimes V \mid \text{rank of } x = k \}$. If $x \in R_k(U \otimes V)$ and $x = \sum_{i=1}^{k} x_i \otimes y_i$, then, by definition,

$$U(x) = \langle x_1, \ldots, x_k \rangle \quad \text{and} \quad V(x) = \langle y_1, \ldots, y_k \rangle.$$ 

$U(x)$ and $V(x)$ are well defined by Lemma 1.2 of [1].* If $x = \sum_{i=1}^{k} x_i \otimes y_i$ and $x_1, \ldots, x_k$ are linearly independent, then the rank of $x = \dim \langle y_1, \ldots, y_k \rangle$ by Lemma 1.1 of [1].

The subspaces in $R_2(U \otimes V)$ are of four types, by Theorem 2.4 of [1].

Type 1: $U(x)$ is constant as $x$ ranges of $H$.

($H$ is a subspace in $R_2(U \otimes V)$)

* Numbers in square brackets refer to the bibliography.
Type 2: $V(x)$ is constant as $x$ ranges over $H$.

Type 3: There exists $u \in U$ and $v \in V$ such that each element of the subspace has a representation of the form $x \otimes u + v \otimes y$ where $x \in U$ and $y \in V$.

Type 4: Those subspaces which are referred to as "special type 3 subspaces" in [1]. In Lemma 1.1, we show that every type 4 space has a basis of the form:

$$u \otimes x_1 + y_1 \otimes v$$
$$u \otimes x_2 + y_2 \otimes v$$
$$y_1 \otimes x_2 - y_2 \otimes x_1$$

where $\dim \langle u, y_1, y_2 \rangle = 3$ and $\dim \langle v, x_1, x_2 \rangle = 3$.

The maximum dimensions of type 1, type 2 and type 3 spaces are $m-1$, $n-1$ and the minimum of $\{ m-1, n-1 \}$ respectively; by Theorem 2.5 in [1]. These maximum dimensions can always be achieved. Except for a pair of subspaces of types 3 and 4, the intersection of two different types of subspaces is at most one-dimensional, by Theorem 2.6 in [1].

2. SOME PROPERTIES OF RANK TWO SUBSPACES

Lemma 1.1: Let $H$ be a rank 2 subspace of dimension three.

A/ If $H$ is not of types 1, 2, or 3 then $H$ has a basis of the form:
\[ X_1 = u \otimes x_1 + y_1 \otimes v \]
\[ X_2 = u \otimes x_2 + y_2 \otimes v \]
\[ X_3 = y_1 \otimes x_2 - y_2 \otimes x_1 , \]
and \( u, y_1, y_2 \) are linearly independent and \( v, x_1, x_2 \) are linearly independent.

B/ If \( H \) is a type 4 subspace and \( X_1 \in H, X_2 \in H \) then \( H = \langle X_1, X_2, X_3 \rangle \).

**Proof:**  (A) From the proof of Theorem 2.4 in [1], we can write \( H = \langle X_1, Y_1, Z_1 \rangle \) where

\[ X_1 = u \otimes x_1 + y_1 \otimes v \]
\[ Y_1 = u \otimes x_2 + y_2 \otimes v \]
\[ Z_1 = z_1 \otimes z_2 + z_3 \otimes z_4 \] and \( u, y_1, y_2 \) are linearly independent and \( v, x_1, x_2 \) are linearly independent.

Now, \( u \not\in U(Z_1) \). If \( u \in U(Z_1) \) then \( Z_1 = u \otimes z_1 + z_1 \otimes z_2 \). This implies that \( z_2 \in \langle x_1, v \rangle \cap \langle x_2, v \rangle = \langle v \rangle \), by Lemma 2.1 of [1]. Then, \( H \) is a type 3 space. This contradicts the assumption that \( H \) is not of the types 1, 2, or 3. We use, \( u \not\in U(Z_1) \) to show that \( U(Z_1) \subset \langle u, y_1, y_2 \rangle \). First, we show \( \dim(\langle z_1, z_3 \rangle \cap \langle u, y_1 \rangle) = 1 \). Suppose, \( \dim(\langle z_1, z_3 \rangle \cap \langle u, y_1 \rangle) = 0 \).

Then, for \( X_1 + Z_1 \) to be rank 2, \( \langle x_1, v \rangle = \langle z_2, z_4 \rangle \). By Lemma 2.3 of [1], \( H \) is type 2. This is a contradiction. Suppose \( \dim(\langle z_1, z_3 \rangle \cap \langle u, y_1 \rangle) = 2 \). By Lemma 2.2 of [1], \( H \) is type 1. Therefore, \( \dim(\langle z_1, z_3 \rangle \cap \langle u, y_1 \rangle = 1 \) and similarly \( \dim(\langle z_1, z_3 \rangle \cap \langle u, y_2 \rangle) = 1 \). This implies, as
u \not\in U(Z_1), \text{ that } U(Z_1) \subset \langle u, y_1, y_2 \rangle \text{ and similarly } V(Z_1) \subset \langle v, x_1, x_2 \rangle.

Now, U(Z_1) and \langle y_1, y_2 \rangle \text{ are both } 2\text{-dimensional subspace of } \langle u, y_1, y_2 \rangle \text{ and therefore they intersect in at least one dimension, say } \langle y \rangle. \text{ Let } \alpha y_1 + \beta y_2 = y.

Then, we can form

\[
X_2 = u \otimes x_1 + y_1 \otimes v \\
Y_2 = u \otimes x'_1 + y \otimes v \quad (Y_2 = \alpha x_1 + \beta Y_1) \\
Z_2 = z_1 \otimes z'_2 + y \otimes z'_3.
\]

By Lemma 2.1 of [1], \langle z'_2 \rangle = \langle x'_2 \rangle. \text{ Let } x'_2 = x,

\[
z'_2 = \lambda_2 x, \quad x'_1 = \lambda x_1, \quad y'_1 = \lambda_1 y_1, \quad z''_1 = \lambda_2 z'_1, \quad z''_3 = \lambda_2 z'_3.
\]

Let

\[
X_3 = \lambda_1 x_2 = u \otimes x_1 + y_1 \otimes v \\
Y_3 = Y_2 = u \otimes x + y \otimes v \\
Z_3 = \lambda_2 z_2 = z''_1 \otimes x + y \otimes z''_3.
\]

Now, \( z''_1 \in \langle u, y, y'_1 \rangle \). \text{ Let } \( z''_1 = \alpha u + \beta y + \gamma y'_1 \).

We can assume \( \beta = 0 \) since \( Z_3 = (\alpha u + \gamma y'_1) \otimes x + y \otimes (z''_3 + \beta x) \).

Let \( X_4 = X_3 \), \( Y_4 = Y_3 \), and \( Z_4 = Z_3 - \alpha Y_3 = \gamma y'_1 \otimes x + y \otimes z''_3 \),

where \( z''_3 = z''_2 - \alpha v \). \text{ Let } \( z = y'_1, z_4 = 1/\gamma z''_3 \) and we arrive at a basis of the form

\[
X_5 = u \otimes x'_1 + z \otimes v \\
Y_5 = u \otimes x + y \otimes v \\
Z_5 = z \otimes x + y \otimes z_4 \quad (Z_5 = \gamma^{-1} z_4)
\]

By Lemma 2.1 of [1], \( z_4 = \lambda x'_1 \) for some \( \lambda \in F \). \text{ Therefore, letting } x'_1 = w, \text{ the basis is}
To prove (A), it remains to show that \( \lambda = -1 \). Consider
\[
X + Y + Z = u \otimes (w+x) + (z+y) \otimes v + z \otimes x + y \otimes \lambda w
\]
\[
= u \otimes (w+x) + z \otimes (w+x) = z \otimes w + (z+y) \otimes v + y \otimes \lambda w
\]
\[
= (u+z) \otimes (w+x) + (z+y) \otimes v + (\lambda y-z) \otimes w.
\]

Now, \( w+x, v, w \) are independent. Therefore,
\[
dim \langle u+z, z+y, \lambda y-z \rangle = 2. \quad \text{This implies that}
\]
\[
z+y = \mu(\lambda y-z) \quad \text{for some } \mu \in F. \quad \text{Therefore, } \lambda = -1.
\]

(8) To prove B, we assume \( X_1, X_2 \) are in the basis of a type 4 subspace, \( H \), and show that \( H = \langle X_1, X_2, X_3 \rangle \).

From the proof of A, \( H = \langle X, Y, Z \rangle \). Also from A,
\[
z = y_1 = \lambda, y_1
\]
\[
x = 1/\lambda_2, z_2
\]
\[
y = a'y_1 + b'y_2
\]
\[
w = x_1 = \lambda_1 x_1.
\]

Therefore, \( Z = z \otimes x - y \otimes w \)
\[
= \lambda_1 y_1 \otimes \lambda_2^{-1} z_2 - (a'y_1 + b'y_2) \otimes \lambda_1 x_1
\]
\[
= y_1 \otimes (\lambda_1 \lambda_2^{-1} z_2 - a'y_1 x_1) - y_2 \otimes b' \lambda_1 x_1.
\]

Therefore, \( H \) has an element of the form,
\[
y_1 \otimes w' - y_2 \otimes x_1.
\]

To show \( w' = x_2 \), first consider
\[ S = \alpha(u \otimes x_2 + y_2 \otimes v) + \beta(y_1 \otimes w' - y_2 \otimes x_1) \]
\[ = \alpha u \otimes x_2 + \gamma y_1 \otimes w' + y_2 \otimes (\alpha v - \beta x_1) \]

Now, \( S \in H \). Therefore, \( S \) is rank 2. This implies, since \( u, y_1, y_2 \) are linearly independent and \( x_2, x_1, v \) are linearly independent; that, for every \( \alpha, \beta \neq 0 \), there exists \( \gamma, \gamma' \) such that \( w' = \gamma x_2 + \gamma'(\alpha v - \beta x_1) \). Obviously \( \gamma' = 0 \) and \( w' = \delta x_2 \) for some \( \delta \in F \). To show \( \delta = 1 \), let \( X_1 = y_1 \otimes \delta x_2 - y_2 \otimes x_1 \). Now let \( X_2 = y_1 \otimes x_2 - \delta^{-1} y_2 \otimes x \). Use \( X_1 + X_2 + X_3 \), as we used \( X + Y + Z \) in the proof of \( A \), to show \( -\delta^{-1} = -1 \). Therefore, \( \delta = 1 \) and \( H = \langle X_1, X_2, X_3 \rangle \).

**Corollary 1.1:** Distinct spaces of type \( 4 \) intersect in at most one dimension.

**Lemma 1.2:** Let \( V_1 = \langle \{ u' \otimes x_1 + v' \otimes y_1 \} \rangle \) \( i = 1, \ldots, m-1 \rangle \) be a type \( 1 \) subspace. Let \( V_2 = \langle \{ u'' \otimes z_1 + v'' \otimes w_1 \} \rangle \) \( i = 1, \ldots, m-1 \rangle \) be a type \( 1 \) subspace. If \( \dim \langle V_1 \cap V_2 \rangle \geq 1 \) then \( \langle u', v' \rangle = \langle u'', v'' \rangle \).

**Proof:** Assume without loss of generality that
\[ u' \otimes x_1 + v' \otimes y_1 = u'' \otimes z_1 + v'' \otimes w_1 \]. This means that
\[ u' \otimes x_1 + v' \otimes y_1 - u'' \otimes z_1 - v'' \otimes w_1 = 0 \]. If \( \dim \langle u', v', u'', v'' \rangle = 4 \) then \( x_1 = y_1 = z_1 = w_1 = 0 \) which is a contradiction. Suppose \( \dim \langle u', v', u'', v'' \rangle = 3 \). We may assume that \( v'' = \alpha u' + \beta v' + \gamma u'' \). Therefore
\[ u' \otimes (x_1 - \alpha w_1) + v' \otimes (y_1 - \beta w_1) + u'' \otimes (-z_1 - \gamma w_1) = 0 \]
This implies $x_1 = \alpha w_1, y_1 = \beta w_1$, and therefore $x_1$ and $y_1$ are dependent. This is a contradiction. Therefore, 
$\dim <u',v',u'',v''> = 2$ or, in other words, $<u',v'> = <u'',v''>$.

The next Lemma is analogous to Lemma 1.2 for type 2 subspaces.

**Lemma 1.3:** Let $V_1 = \langle \{ x_i \otimes u' + y_i \otimes v' \} \rangle \ i = 1, \ldots, n-1 > \ be \ a \ type \ 2 \ subspace. \ Let \ V_2 = \langle \{ z_i \otimes u'' + w_i \otimes v'' \} \rangle \ i = 1, \ldots, n-1 > \ be \ a \ type \ 2 \ subspace. \ If \ \dim \langle V_1 \cap V_2 \rangle \geq 1 \ then \ <u',v'> = <u'',v''>$.

**Lemma 1.4:** Suppose $n \geq 4$ and $m \geq 4$. Let $X = \langle \{ u \otimes x_i + y_i \otimes v \} \rangle \ i = 1, \ldots, \min(m-1,n-1) > \ be \ a \ type \ 3 \ subspace. \ Let \ Y = \langle \{ u' \otimes z_i + w_i \otimes v' \} \rangle \ i = 1, \ldots, \min(m-1,n-1) > \ be \ a \ type \ 3 \ subspace. \ If \ \dim \langle X \cap Y \rangle \geq 2 \ then \ <u> = <u'> and <v> = <v'>.

**Proof:** Suppose $<u> \neq <u'>$. Without loss of generality, assume

$$X_1 = u \otimes x_1 + y_1 \otimes v = u' \otimes z_1 + w_1 \otimes v',$$

$$X_2 = u \otimes x_2 + y_2 \otimes v = u' \otimes z_2 + w_2 \otimes v'. $$

Then $<u,y_1> = <u',w_1> = <u,u'>$ and $<u,y_2> = <u',w_2> = <u,u'>$. Let $y_1 = \alpha u + \beta u'$ and $y_2 = \alpha' u + \beta' u'$. It is essential that $\beta \neq 0, \beta' \neq 0$; otherwise, $X_1$ and $X_2$ are rank one. Consider,
\[ \beta' \beta^{-1} x_1 - x_2 = \beta' \beta^{-1} (u \otimes x_1 + y_1 \otimes v) - (u \otimes x_2 + y_2 \otimes v) \]
\[ = u \otimes (\beta' \beta^{-1} x_1 - x_2) + \{ \beta' \beta^{-1} (au + \beta u') - \\
\alpha' u - \beta u' \} \otimes v \]
\[ = u \otimes (\beta' \beta^{-1} x_1 - x_2) + (\beta' \beta^{-1} \alpha - \alpha') u \otimes v. \]

This is rank 1 which contradicts the assumption that \( X_1 \) and \( X_2 \) form a type 3 subspace. Therefore \( \langle u \rangle = \langle u' \rangle \). Similarly \( \langle v \rangle = \langle v' \rangle \).
CHAPTER TWO

In this chapter, we assume $T$ is a linear transformation and $T(R_2) \subseteq R_2$. We show that $T(R_1) \subseteq R_1$ for all cases except $m = n = 3$. The latter case is dealt with in the next chapter.

**Lemma 2.1:** (a) If $\dim V \geq 4$ then, for all $u \in U$, $v \in V$, $T(u \otimes v)$ has rank $\leq 2$.

(b) If $\dim U \geq 4$ then, for all $u \in U$, $v \in V$, $T(u \otimes v)$ has rank $\leq 2$.

**Proof:** Assume $\dim V \geq 4$. Let $u \otimes v$ be any rank 1 tensor. We can express $u \otimes v$ as $u \otimes (\alpha' x_1 - x_2)$ where $\dim(\langle x_1, x_2 \rangle) = 2$, and $\alpha' \neq 0$, $\alpha' \neq 1$. Extend $x_1, x_2$ to a set of four independent vectors $x_1, x_2, x_3, x_4$. Consider the following two spaces:

$S_1 = \langle u \otimes x_1 + v \otimes x_4, u \otimes (x_1 + x_2) + v \otimes x_3, u \otimes x_3 + v \otimes x_1 \rangle$ and

$S_2 = \langle u \otimes x_2 + v \otimes \alpha' x_4, u \otimes (x_1 + x_2) + v \otimes x_3, u \otimes x_3 + v \otimes x_1 \rangle$.

Any linear combination of tensors in $S_1$ is rank two.

Consider $X = \alpha(u \otimes x_1 + v \otimes x_4) + \beta(u \otimes (x_1 + x_2) = v \otimes x_3)$

$$+ \gamma (u \otimes x_3 + v \otimes x_1).$$

If $X$ is rank 1 or 0, then either $\alpha = \beta = \gamma = 0$ or

$$\langle \alpha x_1 + \beta x_1 + \beta x_2 + \gamma x_3 \rangle = \langle \alpha x_4 + \beta x_3 + \gamma x_1 \rangle.$$

The latter implies $\alpha = 0$ since $x_1, x_2, x_3, x_4$ are linearly independent and $\alpha x_4$ occurs only on the righthand side. This implies $\beta = \gamma = 0$ for similar reasons. Therefore $S_1$ and similarly
$S_2$ are rank 2 subspaces. Extend $S_1$ and $S_2$ to $(m-1)$-dimensional rank 2 subspaces. Now, $T$ maps $S_1$ and $S_2$ into $(m-1)$-dimensional subspaces. Also, $\dim(S_1 \cap S_2) \geq 2$. Therefore, $T$ maps $S_1$ and $S_2$ into subspaces of the same type. (When $\dim V=4$, $S_1$ and $S_2$ cannot be mapped into type 3 and type 4 subspaces. This is proven at the end.)

Now, $T(u \otimes (\alpha'x_1 - x_2)) = T(\alpha'(u \otimes x_1 + v \otimes x_4) - (u \otimes x_2 + v \otimes \alpha'x_4))$. By Lemmas 1.2, 1.3 and 1.4 we know $T(u \otimes (\alpha'x_1 - x_2))$ can have rank no greater than two.

It remains to show that if $\dim V=4$, $S_1$ and $S_2$ cannot be mapped into type 4 and type 3 spaces. Suppose $T(S_1)$ is a type 3 subspace and $T(S_2)$ is a type 4 subspace. Then,

$$
T(u \otimes (x_1 + x_2) + v \otimes x_3) = u' \otimes x_1 + x_2 \otimes v'
$$
$$
T(u \otimes x_3 + v \otimes x_1) = u' \otimes y_1 + y_2 \otimes v'
$$
$$
T(u \otimes x_1 + v \otimes x_4) = u' \otimes z_1 + z_2 \otimes v'
$$

By Theorem 1.1, there exists $\alpha', \beta, \gamma$ such that

$$
X = T(\alpha'(u \otimes (x_1 + x_2) + v \otimes x_3) + \beta(u \otimes x_3 + v \otimes x_1) + \gamma(u \otimes x_2 + v \otimes \alpha x_4))
$$
$$
= x_2 \otimes y_1 - y_2 \otimes x_1. \text{ Obviously, } \gamma \neq 0.
$$

Let $X' = T(\alpha'(u \otimes (x_1 + x_2) + v \otimes x_3) + \beta(u \otimes x_3 + v \otimes x_1) + \gamma(u \otimes x_2 + v \otimes \alpha x_4))$.

$$
= u' \otimes (\alpha'x_1 + \beta y_1 + \gamma z_1) + (\alpha_2 x_2 + \beta y_2 + \gamma z_2) \otimes v
$$

Consider, $X' - X = \gamma T(u \otimes (x_1 - x_2) + v \otimes (1-\alpha)x_4)$

$$
= u' \otimes (\alpha'x_1 + \beta y_1 + \gamma z_1) + (\alpha x_2 + \beta y_2 + \gamma z_2) \otimes v'
$$

$$
- x_2 \otimes y_1 - y_2 \otimes x_1.
$$
Since \( \lambda \neq 0, X' - X \) is a rank 4 tensor. Therefore, \( T \) maps a rank two tensor into a rank four tensor. This is a contradiction. By a similar proof, \( T(S_1) \) is a type 4 subspace implies \( T(S_2) \) cannot be a type 3 subspace.

By application of Lemma 2.1, we have proved the following Lemma.

**Lemma 2.2:** \( T \) maps rank 1 tensor into tensors of rank \( \leq 2 \).

**Theorem 1:** Except possibly when \( m = n = 3 \), \( T(R_1) \subset R_1 \).

**Proof:** From Lemma 2.2, \( T(R_1) \subset \{0\} \cup R_1 \cup R_2 \). Now, if \( T(x \otimes y) = 0 \) then; if \( m > 1, n > 1 \); there is a rank 1 tensor mapped into a rank 2 tensor. Therefore, it is sufficient to show that no rank 1 tensor can be mapped into a rank 2 tensor.

Suppose \( T(u_1 \otimes v_m) \) is rank 2. Extend \( u_1 \) to a basis of \( U \); say, \( (u_1, \ldots, u_m) \); and extend \( v_m \) to a basis of \( V \); say, \( (v_1, \ldots, v_m) \). Consider the space \( S = \langle S_1, \ldots, S_m \rangle \) where

\[
S_1 = u_1 \otimes v_m \\
S_2 = u_1 \otimes v_1 + u_2 \otimes v_m \\
S_3 = u_1 \otimes v_2 + u_2 \otimes v_1 \\
\vdots \\
S_m = u_1 \otimes v_{m-1} + u_2 \otimes v_{m-2}.
\]

\( T(S) \) is a rank two, \( m \)-dimensional subspace. This is established.
If every linear combination, \( \alpha S_1 + \alpha_1 S_2 + \alpha_2 S_3 + \ldots + \alpha_{m-1} S_m \), is rank two unless \( \alpha_1 = \alpha_2 = \ldots = \alpha_{m-1} = 0 \). Consider

\[
X = \alpha u_1 \otimes v_m + \alpha_1 (u_1 \otimes v_1 + u_2 \otimes v_m) + \sum_{i=2}^{m-1} \alpha_i (u_1 \otimes v_i + u_2 \otimes v_{i-1})
\]

\[
= u_1 \otimes (\alpha v_m + \sum_{i=1}^{m-1} \alpha_i v_i) + u_2 \otimes (\alpha_1 v_m + \sum_{i=2}^{m-1} \alpha_i v_{i-1})
\]

If \( X \) is not rank 2, then

\[
\langle \alpha v_m + \sum_{i=1}^{m-1} \alpha_i v_i \rangle = \langle \alpha_1 v_m + \sum_{i=2}^{m-1} \alpha_i v_{i-1} \rangle.
\]

Now, \( v_{m-1} \) does not appear on the right-hand side. This implies \( \alpha_{m-1} = 0 \). This means \( v_{m-2} \) does not appear on the right-hand side and \( \alpha_{m-2} = 0 \). By this method it is shown that \( \alpha_1 = \alpha_2 = \ldots = \alpha_{m-1} = 0 \). Therefore, no linear combination of \( S_1, \ldots, S_m \) is rank 1 except \( \alpha (u_1 \otimes v_m) \), \( \alpha \in F \). Since by assumption \( T(u_1 \otimes v_m) \) has rank 2, \( T(S) \) is a rank two \( m \)-dimensional subspace. Unless \( m = 3 \), there are no such subspaces. This is a contradiction. Similarly, consider the subspace, \( S' \), spanned by

\[
\begin{align*}
&u_1 \otimes v_m \\
u_2 \otimes v_1 + u_1 \otimes v_m \\
u_3 \otimes v_1 + u_2 \otimes v_m \\
&\quad \ldots \\
u_n \otimes v_1 + u_{n-1} \otimes v_m
\end{align*}
\]

\( T(S') \) is a \( n \)-dimensional rank 2 subspace. This is a contradiction unless \( n = 3 \). Therefore \( T \) maps no rank 1 tensor into a rank 2 tensor, with the possible exception when \( n = m = 3 \).
CHAPTER THREE

In this chapter we assume \( m = n = 3 \) and show in this case also that \( T(R_2) \subseteq R_2 \) implies \( T(R_1) \subseteq R_1 \).

**Lemma 3.1:** If \( m = n = 3 \) then \( T(R_2) \subseteq R_2 \) implies \( T(R_1) \subseteq R_1 \cup R_3 \).

**Proof:** First, we show that no rank 1 tensor is mapped into 0. Suppose \( T(u \otimes v) = 0 \). Extend \( u \) and \( v \) to bases of \( U \) and \( V \) respectively; say, \( U = \langle u, x_2, y_2 \rangle \) and \( V = \langle v, x_1, y_1 \rangle \). Choose any \( \alpha \neq 0 \) and consider the family of subspaces, \( S(\alpha) \), with the following basis:

\[
\begin{align*}
& u \otimes y_1 + x_2 \otimes \alpha x_1 \\
& y_2 \otimes y_1 + x_2 \otimes v \\
& x_2 \otimes y_1.
\end{align*}
\]

\( T(S(\alpha)) \) is a 3-dimensional, rank 2 subspace if every tensor in \( S(\alpha) \) and not in \( \langle x_2 \otimes y_1 \rangle \) has rank 2. Suppose \( \alpha' (u \otimes y_1 + x_2 \otimes \alpha x_1) + \beta' (y_2 \otimes y_1 + x_2 \otimes v) + \gamma' (x_2 \otimes y_1) = (\alpha' u + \beta' y_2 + \gamma' x_2) \otimes y_1 + x_2 \otimes (\alpha' \alpha x_1 + \beta' v) \) is not rank 2. Then, \( \alpha' = \beta' = 0 \) and we have \( \gamma' (x_2 \otimes y_1) \). But \( T(\gamma' x_2 \otimes y_1) \) is rank two as \( T(u \otimes v) = 0 \). Therefore, every linear combination of tensors in the basis of \( S(\alpha) \) is mapped into a rank 2 tensor. This implies \( T(S(\alpha)) \) is a type 4 subspace. Since all \( S(\alpha) \) intersect in 2 dimensions, \( T(S(\alpha)) \) is the same space for every \( \alpha \neq 0 \).
Choose \( \alpha \neq 1 \). Now \( T(S(1)) \subseteq T(S(\alpha)) \). This implies, for some \( a, b, c, \in F \),
\[
T(u \otimes y_1 + x_2 \otimes x_1) = T((au + by_2 + cx_2) \otimes y_1 + x_2 \otimes (a\alpha x_1 + bv))
\]
This implies,
\[
T(((a-1)u + by_2 + cx_2) \otimes y_1 + x_2 \otimes ((a\alpha - 1)x_1 + bv)) = 0
\]
Therefore,
\[
((a-1)u + by_2 + cx_2) \otimes y_1 + x_2 \otimes ((a\alpha - 1)x_1 + bv) \text{ is not rank 2.}
\]
Therefore, one of the following three cases must hold:

Case 1: \( (a\alpha - 1)x_1 + bv = 0 \). This implies \( a\alpha = 1, b = 0 \) and \( T(((a\alpha - 1)u + cx_2) \otimes y_1) = 0 \).

Case 2: \( (a - 1)u + by_2 + cx_2 = 0 \). This implies \( a = 1, b = 0, c = 0 \) and \( T(x_2 \otimes (\alpha - 1)x_1) = 0 \).

Case 3: \( \langle x_2 \rangle = \langle (a - 1)u + by_2 + cx_2 \rangle \). This implies \( a = 1, b = 0 \) and \( T(x_2 \otimes (cy_1 + (a - 1)x_1)) = 0 \).

Now, \( T(u \otimes v) = 0 \). Therefore, Case 1 must hold with \( c = 0 \) and \( T(u \otimes y_1) = 0 \). Since the extension of \( v \) to a basis \( v, x_1, y_1 \) is arbitrary, we have that \( T(u \otimes y) = 0 \) for all \( y \in V \).

Now, the problem is symmetric with respect to \( u \) and \( v \). Therefore, \( T(x \otimes v) = 0 \) for all \( x \in U \). Now choose \( y \) independent of \( v \) and \( x \) independent of \( u \) and we have a contradiction; namely, \( T(u \otimes y + x \otimes v) = 0 \).
This shows that no rank one vector is mapped into 0.

Suppose a rank one tensor, call it \( x_2 \otimes y_1 \), is
mapped into a rank $2$ tensor. Extend $x_2$ and $y_1$ to bases of $U$ and $V$. Let $U = \langle u, x_2, y_2 \rangle$ and $V = \langle v, x_1, y_1 \rangle$.

Then, by considering the spaces, $s(\alpha)$ as defined above, we arrive at cases 1, 2 or 3 as above. Therefore, we have the contradiction that a rank 1 vector is mapped into 0. Therefore $T(R_1) \subset R_1 \cup R_3$.

**Lemma 3.2:** Let $m = n = 3$. If $T(R_2) \subset R_2$ then $T(R_1) \subset R_1$ or $T(R_1) \subset R_3$.

**Proof:** Assume $T(R_1) \not\subset R_1$ and $T(R_1) \not\subset R_3$. Then, $T(R_1) \subset R_1 \cup R_3$ by Lemma 3.1. Now, we can find $x \otimes y$ and $x' \otimes y'$ such that $x, x'$ are linearly independent; $y, y'$ are linearly independent; $T(x \otimes y)$ is rank 1 and $T(x' \otimes y')$ is rank 3. If this is not the case then $T(R_1) \subset R_1$ or $T(R_1) \subset R_3$ and we are finished.

Let $T(x \otimes y) = x_1 \otimes y_1$ and $T(x' \otimes y') = x_1 \otimes z_1 + x_2 \otimes z_2 + x_3 \otimes z_3$.

Consider,

$$T(\alpha x \otimes y + x' \otimes y') = x_1 \otimes (\alpha y_1 + z_1) + x_2 \otimes z_2 + x_3 \otimes z_3.$$  

This must be rank 2, for all $\alpha \neq 0$. Therefore, there exist $\beta, \gamma \in F$ such that $\alpha y_1 + z_1 = \beta z_2 + \gamma z_3$ or $\alpha y_1 = \beta z_2 + \gamma z_3 - z_1$. Let $\alpha_1 \in F$, $\alpha_2 \in F$ and $\alpha_1 \neq \alpha_2$. There exist $\beta_1, \gamma_1, \beta_2, \gamma_2 \in F$ such that

$$\alpha_1 y_1 = \beta_1 z_2 + \gamma_1 z_3 - z_1$$
$$\alpha_2 y_1 = \beta_2 z_2 + \gamma_2 z_3 - z_1.$$  

This implies
\[\beta_1 \alpha_1^{-1} z_2 + \gamma_1 \alpha_1^{-1} z_3 - \alpha_1^{-1} z_1 = \beta_2 \alpha_2^{-1} z_2 + \gamma_2 \alpha_2^{-1} z_3 - \alpha_2^{-1} z_1.\]

Now, \(\dim\langle z_1, z_2, z_3 \rangle = 3\). Therefore, \(\alpha_1^{-1} = \alpha_2^{-1}\). This is a contradiction. Therefore, \(T(R_1) \subset R_1\) or \(T(R_1) \subset R_3\).

**Theorem 2:** Let \(m = n = 3\). \(T(R_2) \subset R_2\) implies \(T(R_1) \subset R_1\).

**Proof:** From Lemma 3.2, it is sufficient to show that \(T(R_1) \not\subset R_3\). Assume \(T(R_1) \subset R_3\). Let
\[T(u_1 \otimes v_1) = u_1 \otimes y_1 + u_2 \otimes y_2 + u_3 \otimes y_3\]
and
\[T(u_1 \otimes v_2) = u_1 \otimes z_1 + u_2 \otimes z_2 + u_3 \otimes z_3\]
where \(v_1, v_2\) are linearly independent. Let \(A : V \rightarrow V\) such that \(Ay_i = z_i\), \(i = 1, 2, 3\). \(A\) has an eigenvalue, \(\lambda\). Then
\[T(u_1 \otimes (v_2 - \lambda v_1)) = u_1 \otimes (A - \lambda I)y_1 + u_2 \otimes (A - \lambda I)y_2 + u_3 \otimes (A - \lambda I)y_3.\]

There exists \(\alpha, \beta, \gamma\) not all 0, such that \(\alpha y_1 + \beta y_2 + \gamma y_3\) is the eigenvector corresponding to \(\lambda\). Therefore,
\[(A - \lambda I)(\alpha y_1 + \beta y_2 + \gamma y_3) = 0\]
and \((A - \lambda I)y_1, (A - \lambda I)y_2, (A - \lambda I)y_3\) are dependent. This means \(T(u_1 \otimes (v_2 - \lambda v_1))\) is not rank 3 which contradicts the assumption that \(T(R_1) \subset R_3\). Therefore, \(T(R_1) \subset R_1\).
Theorem 3: If \( F \) is algebraically closed and \( T(R_2) \subset R_2 \) then \( T(R_1) \subset R_1 \).

Proof: The result follows immediately from Theorems 1 and 2.

For algebraically closed fields of characteristic 0, the structure of \( T \) is given by the following theorem, which is quoted from [2].

Theorem 4: Let \( T(R_1) \subset R_1 \). "Let \( T_1 \) be the linear transformation of \( V \otimes U \) into \( U \otimes V \) which maps \( y \otimes x \) onto \( x \otimes y \). If \( m = n \), let \( \phi \) be any non-singular linear transformation of \( U \) onto \( V \). Then if \( m \neq n \), there exist non-singular linear transformations \( A \) and \( B \) on \( U \) and \( V \), respectively, such that \( T = A \otimes B \). If \( m = n \), there exist non-singular \( A \) and \( B \) such that either \( T = A \otimes B \) or \( T = T_1(\phi A \otimes \phi^{-1} B) \)."

For algebraically closed fields of all characteristics, Theorem 4 holds; but the proof is, as yet, unpublished.
BIBLIOGRAPHY
