CHARACTERIZATION OF TRANSFORMATIONS PRESERVING RANK TWO TENSORS OF A TENSOR PRODUCT SPACE

by

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ABSTRACT

Let $U \otimes V$ be a tensor product space over an algebraically closed field F ; let $\dim U = m$ and $\dim V = n$; let T be a linear transformation on $U \otimes V$ such that T preserves rank two tensors.

We show that T preserves rank one tensors and this enables us to characterize T for all values of m and n .

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CHAPTER ONE

1. INTRODUCTION.

Let U and V be m-dimensional and n-dimensional vector spaces over an algebraically closed field F. The tensor product of U and V denoted by $U \otimes V$ is the dual space of the space of all multilinear functions mapping $U \times V$ into F.

An element $x \in U \otimes V$ has rank k if $x = \sum_{i=1}^{k} x_i \otimes y_i$ and $x_1, \ldots, x_k$ are linearly independent and $y_1, \ldots, y_k$ are linearly independent. We define $R_k(U \otimes V)$ to be $\{ x \in U \otimes V \mid \text{rank of } x = k \}$. If $x \in R_k(U \otimes V)$ and $x = \sum_{i=1}^{k} x_i \otimes y_i$ then, by definition,

$U(x) = \langle x_1, \ldots, x_k \rangle$ and $V(x) = \langle y_1, \ldots, y_k \rangle$.

$U(x)$ and $V(x)$ are well defined by Lemma 1.2 of [1].* If $x = \sum_{i=1}^{k} x_i \otimes y_i$ and $x_1, \ldots x_k$ are linearly independent, then the rank of $x = \dim \langle y_1, \ldots, y_k \rangle$ by Lemma 1.1 of [1].

The subspaces in $R_2(U \otimes V)$ are of four types, by Theorem 2.4 of [1].

Type 1: $U(x)$ is constant as x ranges of H.
(H is a subspace in $R_2(U \otimes V)$)

* Numbers in square brackets refer to the bibliography.

Type 2: $V(x)$ is constant as x ranges over H .

Type 3: There exists $u \in U$ and $v \in V$ such that each element of the subspace has a representation of the form $x \otimes u + v \otimes y$ where $x \in U$ and $y \in V$.

Type 4: Those subspaces which are referred to as "special type 3 subspaces" in [1]. In Lemma 1.1, we show that every type 4 space has a basis of the form:

$$u \otimes x_1 + y_1 \otimes v$$
$$u \otimes x_2 + y_2 \otimes v$$
$$y_1 \otimes x_2 - y_2 \otimes x_1 \quad \text{where } \dim \langle u, y_1, y_2 \rangle = 3$$

and $\dim \langle v, x_1, x_2 \rangle = 3$.

The maximum dimensions of type 1, type 2 and type 3 spaces are $m-1$, $n-1$ and the minimum of $\{m-1, n-1\}$ respectively; by Theorem 2.5 in [1]. These maximum dimensions can always be achieved. Except for a pair of subspaces of types 3 and 4, the intersection of two different types of subspaces is at most one-dimensional, by Theorem 2.6 in [1].

2. SOME PROPERTIES OF RANK TWO SUBSPACES

Lemma 1.1: Let H be a rank 2 subspace of dimension three.

A/ If H is not of types 1, 2, or 3 then H has a basis of the form:

$$X_1 = u \otimes x_1 + y_1 \otimes v$$
$$X_2 = u \otimes x_2 + y_2 \otimes v$$
$$X_3 = y_1 \otimes x_2 - y_2 \otimes x_1 ,$$

and u, y_1, y_2 are linearly independent and v, x_1, x_2 are linearly independent.

B/ If H is a type 4 subspace and $X_1 \in H$, $X_2 \in H$ then $H = \langle X_1, X_2, X_3 \rangle$.

Proof: (A) From the proof of Theorem 2.4 in [1], we can write $H = \langle X_1, Y_1, Z_1 \rangle$ where

$$X_1 = u \otimes x_1 + y_1 \otimes v$$
$$Y_1 = u \otimes x_2 + y_2 \otimes v$$
$$Z_1 = z_1 \otimes z_2 + z_3 \otimes z_4$$ and u, y_1, y_2 are linearly independent and v, x_1, x_2 are linearly independent.

Now, $u \notin U(Z_1)$. If $u \in U(Z_1)$ then $Z_1 = u \otimes z_1' + z_2' \otimes z_3'$. This implies that $z_3' \in \langle x_1, v \rangle \cap \langle x_2, v \rangle = \langle v \rangle$, by Lemma 2.1 of [1]. Then, H is a type 3 space. This contradicts the assumption that H is not of the types 1, 2, or 3. We use, $u \notin U(Z_1)$ to show that $U(Z_1) \subset \langle u, y_1, y_2 \rangle$. First, we show $\dim(\langle z_1, z_3 \rangle \cap \langle u, y_1 \rangle) = 1$. Suppose, $\dim(\langle z_1, z_3 \rangle \cap \langle u, y_1 \rangle) = 0$. Then, for $X_1 + Z_1$ to be rank 2, $\langle x_1, v \rangle = \langle z_2, z_4 \rangle$. By Lemma 2.3 of [1], H is type 2. This is a contradiction. Suppose $\dim(\langle z_1, z_3 \rangle \cap \langle u, y_1 \rangle) = 2$. By Lemma 2.2 of [1], H is type 1. Therefore, $\dim(\langle z_1, z_3 \rangle \cap \langle u, y_1 \rangle = 1$ and similarly $\dim(\langle z_1, z_3 \rangle \cap \langle u, y_2 \rangle) = 1$. This implies, as

$u \notin U(Z_1)$, that $U(Z_1) \subset \langle u, y_1, y_2 \rangle$ and similarly $V(Z_1) \subset \langle v, x_1, x_2 \rangle$.

Now, $U(Z_1)$ and $\langle y_1, y_2 \rangle$ are both 2-dimensional subspace of $\langle u, y_1, y_2 \rangle$ and therefore they intersect in at least one dimension, say $\langle y \rangle$. Let $\alpha' y_1 + \beta' y_2 = y$. Then, we can form

$$X_2 = u \otimes x_1 + y_1 \otimes v$$
$$Y_2 = u \otimes x_2' + y \otimes v \quad (Y_2 = \alpha' X_1 + \beta' Y_1)$$
$$Z_2 = z_1' \otimes z_2' + y \otimes z_3' .$$

By Lemma 2.1 of [1], $\langle z_2' \rangle = \langle x_2' \rangle$. Let $x_2' = x$, $z_2' = \lambda_2 x$, $x_1' = \lambda x_1$, $y_1' = \lambda_1 y_1$, $z_1'' = \lambda_2^2 z_1'$, $z_3'' = \lambda_2 z_3'$. Let

$$X_3 = \lambda_1 X_2 = u \otimes x_1' + y_1' \otimes v$$
$$Y_3 = Y_2 = u \otimes x + y \otimes v$$
$$Z_3 = \lambda_2 Z_2 = z_1'' \otimes x + y \otimes z_3'' .$$

Now, $z_1'' \in \langle u, y, y_1' \rangle$. Let $z_1'' = \alpha u + \beta y + \gamma y_1'$.
We can assume $\beta = 0$ since $Z_3 = (\alpha u + \gamma y_1') \otimes x + y \otimes (z_3'' + \beta x)$.
Let $X_4 = X_3$, $Y_4 = Y_3$, and $Z_4 = Z_3 - \alpha Y_3 = \gamma y_1' \otimes x + y \otimes z_3'''$, where $z_3''' = z_3'' - \alpha v$. Let $z = y_1'$, $z_4 = 1/\gamma z_3'''$ and we arrive at a basis of the form

$$X_5 = u \otimes x_1' + z \otimes v$$
$$Y_5 = u \otimes x + y \otimes v$$
$$Z_5 = z \otimes x + y \otimes z_4 \quad (Z_5 = \gamma^{-1} Z_4)$$

By Lemma 2.1 of [1], $z_4 = \lambda x_1'$ for some $\lambda \in F$. Therefore, letting $x_1' = w$, the basis is

$$X = u \otimes w + z \otimes v$$

$$Y = u \otimes x + y \otimes v$$

$$Z = z \otimes x + y \otimes \lambda w$$

To prove (A), it remains to show that $\lambda = -1$. Consider

$$X + Y + Z = u \otimes (w+x) + (z+y) \otimes v + z \otimes x + y \otimes \lambda w$$
$$= u \otimes (w+x) + z \otimes (w+x) - z \otimes w + (z+y) \otimes v + y \otimes \lambda w$$
$$= (u+z) \otimes (w+x) + (z+y) \otimes v + (\lambda y - z) \otimes w .$$

Now, $w+x, v, w$ are independent. Therefore, $\dim \langle u+z, z+y, \lambda y - z \rangle = 2$. This implies that $z+y = \mu(\lambda y - z)$ for some $\mu \in F$. Therefore, $\lambda = -1$.

(B) To prove B, we assume X_1, X_2 are in the basis of a type 4 subspace, H, and show that $H = \langle X_1, X_2, X_3 \rangle$. From the proof of A, $H = \langle X, Y, Z \rangle$. Also from A,

$$z = y_1' = \lambda_1 y_1$$
$$x = 1/\lambda_2 \, z_2'$$
$$y = \alpha' y_1 + \beta' y_2$$
$$w = x_1' = \lambda_1 x_1 .$$

Therefore, $Z = z \otimes x - y \otimes w$

$$= \lambda_1 y_1 \otimes \lambda_2^{-1} z_2' - (\alpha' y_1 + \beta' y_2) \otimes \lambda_1 x_1$$
$$= y_1 \otimes (\lambda_1 \lambda_2^{-1} z_2' - \alpha' \lambda_1 x_1) - y_2 \otimes \beta' \lambda_1 x_1 .$$

Therefore, H has an element of the form,

$$y_1 \otimes w' - y_2 \otimes x_1 .$$

To show $w' = x_2$, first consider

$$S = \bar{\alpha}(u \otimes x_2 + y_2 \otimes v) + \bar{\beta}(y_1 \otimes w' - y_2 \otimes x_1)$$
$$= \bar{\alpha}u \otimes x_2 + \bar{\beta}y_1 \otimes w' + y_2 \otimes (\bar{\alpha}v - \bar{\beta}x_1)$$

Now, $S \in H$. Therefore, S is rank 2. This implies, since u, y_1, y_2 are linearly independent and x_2, x_1, v are linearly independent; that, for every $\bar{\alpha}, \bar{\beta} \neq 0$, there exists γ, γ' such that $w' = \gamma x_2 + \gamma'(\bar{\alpha}v - \bar{\beta}x_1)$. Obviously $\gamma' = 0$ and $w' = \delta x_2$ for some $\delta \in F$. To show $\delta = 1$, let $X_3' = y_1 \otimes \delta x_2 - y_2 \otimes x_1$. Now let $X_3'' = y_1 \otimes x_2 - \delta^{-1} y_2 \otimes x$. Use $X_1 + X_2 + X_3''$, as we used $X + Y + Z$ in the proof of A, to show $-\delta^{-1} = -1$. Therefore, $\delta = 1$ and $H = \langle X_1, X_2, X_3 \rangle$.

<u>Corollary 1.1</u>: Distinct spaces of type 4 intersect in at most one dimension.

<u>Lemma 1.2</u>: Let $V_1 = \langle \{u' \otimes x_i + v' \otimes y_i\} \ i = 1, \ldots, m-1 \rangle$ be a type 1 subspace. Let $V_2 = \langle \{u'' \otimes z_i + v'' \otimes w_i\} \ i = 1, \ldots, m-1 \rangle$ be a type 1 subspace. If $\dim \langle V_1 \cap V_2 \rangle \geq 1$ then $\langle u', v' \rangle = \langle u'', v'' \rangle$.

<u>Proof</u>: Assume without loss of generality that $u' \otimes x_1 + v' \otimes y_1 = u'' \otimes z_1 + v'' \otimes w_1$. This means that $u' \otimes x_1 + v' \otimes y_1 - u'' \otimes z_1 - v'' \otimes w_1 = 0$. If $\dim \langle u', v', u'', v'' \rangle = 4$ then $x_1 = y_1 = z_1 = w_1 = 0$ which is a contradiction.. Suppose $\dim \langle u', v', u'', v'' \rangle = 3$. We may assume that $v'' = \alpha u' + \beta v' + \gamma u''$. Therefore

$$u' \otimes (x_1 - \alpha w_1) + v' \otimes (y_1 - \beta w_1) + u'' \otimes (-z_1 - \gamma w_1) = 0$$

This implies $x_1 = \alpha w_1, y_1 = \beta w_1$, and therefore x_1 and y_1 are dependent. This is a contradiction. Therefore, $\dim\langle u', v', u'', v''\rangle = 2$ or, in other words, $\langle u', v'\rangle = \langle u'', v''\rangle$.

The next Lemma is analogous to Lemma 1.2 for type 2 subspaces.

Lemma 1.3: Let $V_1 = \langle\{x_i \otimes u' + y_i \otimes v'\}\ i = 1,\ldots,n-1\rangle$ be a type 2 subspace. Let $V_2 = \langle\{z_i \otimes u'' + w_i \otimes v''\}\ i = 1,\ldots,n-1\rangle$ be a type 2 subspace. If $\dim\langle v_1 \cap v_2\rangle \geq 1$ then $\langle u', v'\rangle = \langle u'', v''\rangle$.

Lemma 1.4: Suppose $n \geq 4$ and $m \geq 4$. Let $X = \langle\{u \otimes x_i + y_i \otimes v\}\ i = 1,\ldots,\min(m-1,n-1)\rangle$ be a type 3 subspace. Let $Y = \langle\{u' \otimes z_i + w_i \otimes v'\}\ i = 1,\ldots,\min(m-1,n-1)\rangle$ be a type 3 subspace. If $\dim\langle X \cap Y\rangle \geq 2$ then $\langle u\rangle = \langle u'\rangle$ and $\langle v\rangle = \langle v'\rangle$.

Proof: Suppose $\langle u\rangle \neq \langle u'\rangle$. Without loss of generality, assume

$$X_1 = u \otimes x_1 + y_1 \otimes v = u' \otimes z_1 + w_1 \otimes v'$$
$$X_2 = u \otimes x_2 + y_2 \otimes v = u' \otimes z_2 + w_2 \otimes v'$$

Then $\langle u, y_1\rangle = \langle u', w_1\rangle = \langle u, u'\rangle$ and $\langle u, y_2\rangle = \langle u', w_2\rangle = \langle u, u'\rangle$. Let $y_1 = \alpha u + \beta u'$ and $y_2 = \alpha' u + \beta' u'$.
It is essential that $\beta \neq 0$, $\beta' \neq 0$; otherwise, X_1 and X_2 are rank one. Consider,

$$\begin{aligned}\beta'\beta^{-1}X_1 - X_2 &= \beta'\beta^{-1}(u\otimes x_1 + y_1\otimes v) - (u\otimes x_2 + y_2\otimes v)\\ &= u\otimes(\beta'\beta^{-1}x_1 - x_2) + \{\beta'\beta^{-1}(\alpha u + \beta u') -\\ &\quad \alpha' u - \beta u'\}\otimes v\\ &= u\otimes(\beta'\beta^{-1}x_1 - x_2) + (\beta'\beta^{-1}\alpha - \alpha')\, u\otimes v\ .\end{aligned}$$

This is rank 1 which contradicts the assumption that X_1 and X_2 form a type 3 subspace. Therefore $\langle u\rangle = \langle u'\rangle$. Similarly $\langle v\rangle = \langle v'\rangle$.

CHAPTER TWO

In this chapter, we assume T is a linear transformation and $T(R_2) \subset R_2$. We show that $T(R_1) \subset R_1$ for all cases except $m = n = 3$. The latter case is dealt with in the next chapter.

<u>Lemma 2.1</u>: (a) If $\dim V \geq 4$ then, for all $u \in U$, $v \in V$, $T(u \otimes v)$ has rank ≤ 2.

(b) If $\dim U \geq 4$ then, for all $u \in U$, $v \in V$, $T(u \otimes v)$ has rank ≤ 2.

<u>Proof</u>: Assume $\dim V \geq 4$. Let $u \otimes v$ be any rank 1 tensor. We can express $u \otimes v$ as $u \otimes (\alpha' x_1 - x_2)$ where $\dim(\langle x_1, x_2 \rangle) = 2$, and $\alpha' \neq 0$, $\alpha' \neq 1$. Extend x_1, x_2 to a set of four independent vectors x_1, x_2, x_3, x_4. Consider the following two spaces:

$S_1 = \langle u \otimes x_1 + v \otimes x_4,\ u \otimes (x_1 + x_2) + v \otimes x_3,\ u \otimes x_3 + v \otimes x_1 \rangle$ and

$S_2 = \langle u \otimes x_2 + v \otimes \alpha' x_4,\ u \otimes (x_1 + x_2) + v \otimes x_3,\ u \otimes x_3 + v \otimes x_1 \rangle$.

Any linear combination of tensors in S_1 is rank two.

Consider $X = \alpha(u \otimes x_1 + v \otimes x_4) + \beta(u \otimes (x_1 + x_2) = v \otimes x_3)$

$+ \gamma(u \otimes x_3 + v \otimes x_1)$.

$= u \otimes (\alpha x_1 + \beta x_1 + \beta x_2 + \gamma x_3) + v \otimes (\alpha x_4 + \beta x_3 + \gamma x_1)$

If X is rank 1 or 0, then either $\alpha = \beta = \gamma = 0$ or $\langle \alpha x_1 + \beta x_1 + \beta x_2 + \gamma x_3 \rangle = \langle \alpha x_4 + \beta x_3 + \gamma x_1 \rangle$. The latter implies $\alpha = 0$ since x_1, x_2, x_3, x_4 are linearly independent and αx_4 occurs only on the righthand side. This implies $\beta = \gamma = 0$ for similar reasons. Therefore S_1 and similarly

S_2 are rank 2 subspaces. Extend S_1 and S_2 to $(m-1)$-dimensional rank 2 subspaces. Now, T maps S_1 and S_2 into $(m-1)$-dimensional subspaces. Also, $\dim(S_1 \cap S_2) \geq 2$. Therefore, T maps S_1 and S_2 into subspaces of the same type. (When $\dim V=4$, S_1 and S_2 cannot be mapped into type 3 and type 4 subspaces. This is proven at the end.) Now, $T(u\otimes(\alpha' x_1 - x_2)) = T(\alpha'(u\otimes x_1 + v\otimes x_4) - (u\otimes x_2 + v\otimes\alpha' x_4))$. By Lemmas 1.2, 1.3 and 1.4 we know $T(u\otimes(\alpha' x_1 - x_2))$ can have rank no greater than two.

It remains to show that if $\dim V=4$, S_1 and S_2 cannot be mapped into type 4 and type 3 spaces. Suppose $T(S_1)$ is a type 3 subspace and $T(S_2)$ is a type 4 subspace. Then,

$$T(u\otimes(x_1+x_2) + v\otimes x_3) = u'\otimes x_1' + x_2'\otimes v'$$
$$T(u\otimes x_3 + v\otimes x_1) = u'\otimes y_1' + y_2'\otimes v'$$
$$T(u\otimes x_1 + v\otimes x_4) = u'\otimes z_1' + z_2'\otimes v'$$

By Theorem 1.1, there exists α', β, γ such that

$$X = T(\alpha'(u\otimes(x_1+x_2) + v\otimes x_3) + \beta(u\otimes x_3+v\otimes x_1) + \gamma(u\otimes x_2 + v\otimes \alpha x_4))$$
$$= x_2'\otimes y_1' - y_2'\otimes x_1' . \quad \text{Obviously, } \gamma \neq 0 .$$

Let $X' = T(\alpha'(u\otimes(x_1+x_2) + v\otimes x_3) + \beta(u\otimes x_3 + v\otimes x_1) + \gamma(u\otimes x_1 + v\otimes x_4))$.

$$= u'\otimes(\alpha' x_1' + \beta y_1' + \gamma z_1') + (\alpha x_2' + \beta y_2' + \gamma z_2')\otimes v'$$

Consider, $X' - X = \gamma T(u\otimes(x_1 - x_2) + v\otimes(1-\alpha)x_4)$

$$= u'\otimes(\alpha' x_1' + \beta y_1' + \gamma z_1') + (\alpha x_2' + \beta y_2' + \gamma z_2')\otimes v' - x_2'\otimes y_1' + y_2'\otimes x_1' .$$

Since $\gamma \neq 0$, $X' - X$ is a rank 4 tensor. Therefore, T maps a rank two tensor into a rank four tensor. This is a contradiction. By a similar proof, $T(S_1)$ is a type 4 subspace implies $T(S_2)$ cannot be a type 3 subspace.

By application of Lemma 2.1, we have proved the following Lemma.

Lemma 2.2: T maps rank 1 tensor into tensors of rank ≤ 2.

Theorem 1: Except possibly when $m = n = 3$, $T(R_1) \subset R_1$.

Proof: From Lemma 2.2, $T(R_1) \subset \{0\} \cup R_1 \cup R_2$. Now, if $T(x \otimes y) = 0$ then; if $m > 1$, $n > 1$; there is a rank 1 tensor mapped into a rank 2 tensor. Therefore, it is sufficient to show that no rank 1 tensor can be mapped into a rank 2 tensor.

Suppose $T(u_1 \otimes v_m)$ is rank 2. Extend u_1 to a basis of U; say, $(u_1, \ldots, u_m)$; and extend v_m to a basis of V; say, $(v_1, \ldots, v_m)$. Consider the space $S = \langle S_1, \ldots, S_m \rangle$ where

$$S_1 = u_1 \otimes v_m$$
$$S_2 = u_1 \otimes v_1 + u_2 \otimes v_m$$
$$S_3 = u_1 \otimes v_2 + u_2 \otimes v_1$$
$$\vdots$$
$$S_m = u_1 \otimes v_{m-1} + u_2 \otimes v_{m-2} .$$

$T(S)$ is a rank two, m-dimensional subspace. This is established if every linear combination, $\alpha S_1 + \alpha_1 S_2 + \alpha_3 S_3 + \ldots + \alpha_{m-1} S_m$, is rank two unless $\alpha_1 = \alpha_2 = \ldots = \alpha_{m-1} = 0$. Consider

$$X = \alpha u_1 \otimes v_m + \alpha_1 (u_1 \otimes v_1 + u_2 \otimes v_m) + \sum_{i=2}^{m-1} \alpha_i \,(u_1 \otimes v_i + u_2 \otimes v_{i-1})$$

$$= u_1 \otimes (\alpha v_m + \sum_{i=1}^{m-1} \alpha_i v_i) + u_2 \otimes (\alpha_1 v_m + \sum_{i=2}^{m-1} \alpha_i v_{i-1})$$

If X is not rank 2, then

$$\langle \alpha v_m + \sum_{i=1}^{m-1} \alpha_i v_i \rangle = \langle \alpha_1 v_m + \sum_{i=2}^{m-1} \alpha_i v_{i-1} \rangle .$$

Now, v_{m-1} does not appear on the righthand side. This implies $\alpha_{m-1} = 0$. This means v_{m-2} does not appear on the righthand side and $\alpha_{m-2} = 0$. By this method it is shown that $\alpha_1 = \alpha_2 = \ldots = \alpha_{m-1} = 0$. Therefore, no linear combination of $S_1, \ldots, S_m$ is rank 1 except $\alpha(u_1 \otimes v_m)$, $\alpha \in F$. Since by assumption $T(u_1 \otimes v_m)$ has rank 2, $T(S)$ is a rank two m-dimensional subspace. Unless $m = 3$, there are no such subspaces. This is a contradiction. Similarly, consider the subspace, S', spanned by

$$\begin{array}{l} u_1 \otimes v_m \\ u_2 \otimes v_1 + u_1 \otimes v_m \\ u_3 \otimes v_1 + u_2 \otimes v_m \\ \cdot \\ \cdot \\ \cdot \\ u_n \otimes v_1 + u_{n-1} \otimes v_m \end{array}$$

$T(S')$ is a n-dimensional rank 2 subspace. This is a contradiction unless $n = 3$. Therefore T maps no rank 1 tensor into a rank 2 tensor, with the possible exception when $n = m = 3$.

CHAPTER THREE

In this chapter we assume $m = n = 3$ and show in this case also that $T(R_2) \subset R_2$ implies $T(R_1) \subset R_1$.

Lemma 3.1: If $m = n = 3$ then $T(R_2) \subset R_2$ implies $T(R_1) \subset R_1 \cup R_3$.

Proof: First, we show that no rank 1 tensor is mapped into 0 . Suppose $T(u \otimes v) = 0$. Extend u and v to bases of U and V respectively; say, $U = \langle u, x_2, y_2 \rangle$ and $V = \langle v, x_1, y_1 \rangle$. Choose any $\alpha \neq 0$ and consider the family of subspaces, $S(\alpha)$, with the following basis:

$$u \otimes y_1 + x_2 \otimes \alpha x_1$$
$$y_2 \otimes y_1 + x_2 \otimes v$$
$$x_2 \otimes y_1 .$$

$T(S(\alpha))$ is a 3-dimensional, rank 2 subspace if every tensor in $S(\alpha)$ and not in $\langle x_2 \otimes y_1 \rangle$ has rank 2. Suppose $\alpha'(u \otimes y_1 + x_2 \otimes \alpha x_1) + \beta'(y_2 \otimes y_1 + x_2 \otimes v) + \gamma'(x_2 \otimes y_1)$ $= (\alpha' u + \beta' y_2 + \gamma' x_2) \otimes y_1 + x_2 \otimes (\alpha' \alpha x_1 + \beta' v)$ is not rank 2. Then, $\alpha' = \beta' = 0$ and we have $\gamma'(x_2 \otimes y_1)$. But $T(\gamma' x_2 \otimes y_1)$ is rank two as $T(u \otimes v) = 0$. Therefore, every linear combination of tensors in the basis of $S(\alpha)$ is mapped into a rank 2 tensor. This implies $T(S(\alpha))$ is a type 4 subspace. Since all $S(\alpha)$ intersect in 2 dimensions, $T(S(\alpha))$ is the same space for every $\alpha \neq 0$.

Choose $\alpha \neq 1$. Now $T(S(1)) \subset T(S(\alpha))$. This implies, for some $a, b, c, \in F$,

$$T(u \otimes y_1 + x_2 \otimes x_1) = T((au + by_2 + cx_2) \otimes y_1 + x_2 \otimes (a\alpha x_1 + bv))$$

This implies,

$$T(\{(a-1)u + by_2 + cx_2\} \otimes y_1 + x_2 \otimes \{(a\alpha - 1)x_1 + bv\}) = 0$$

Therefore,

$\{(a-1)u + by_2 + cx_2\} \otimes y_1 + x_2 \otimes \{(a\alpha-1)x_1 + bv\}$ is not rank 2. Therefore, one of the following three cases must hold:

Case 1: $(a\alpha-1)x_1 + bv = 0$. This implies $a\alpha = 1$, $b = 0$ and $T(\{(\alpha^{-1}-1)u + cx_2\} \otimes y_1) = 0$

Case 2: $(a-1)u + by_2 + cx_2 = 0$. This implies $a = 1$, $b = 0$, $c = 0$ and $T(x_2 \otimes (\alpha-1)x_1) = 0$.

Case 3: $\langle x_2 \rangle = \langle (a-1)u + by_2 + cx_2 \rangle$. This implies $a = 1$, $b = 0$ and $T(x_2 \otimes (cy_1 + (\alpha-1)x_1)) = 0$.

Now, $T(u \otimes v) = 0$. Therefore, Case 1 must hold with $c = 0$ and $T(u \otimes y_1) = 0$. Since the extension of v to a basis v, x_1, y_1 is arbitrary, we have that $T(u \otimes y) = 0$ for all $y \in V$.

Now, the problem is symmetric with respect to u and v. Therefore, $T(x \otimes v) = 0$ for all $x \in U$. Now choose y independent of v and x independent of u and we have a contradiction; namely, $T(u \otimes y + x \otimes v) = 0$. This shows that no rank one vector is mapped into 0.

Suppose a rank one tensor, call it $x_2 \otimes y_1$, is mapped into a rank 2 tensor. Extend x_2 and y_1 to bases of U and V. Let $U = \langle u, x_2, y_2 \rangle$ and $V = \langle v, x_1, y_1 \rangle$. Then, by considering the spaces, $s(\alpha)$ as defined above, we arrive at cases 1,2 or 3 as above. Therefore, we have the contradiction that a rank 1 vector is mapped into 0. Therefore $T(R_1) \subset R_1 \cup R_3$.

Lemma 3.2: Let $m = n = 3$. If $T(R_2) \subset R_2$ then $T(R_1) \subset R_1$ or $T(R_1) \subset R_3$.

Proof: Assume $T(R_1) \not\subset R_1$ and $T(R_1) \not\subset R_3$. Then, $T(R_1) \subset R_1 \cup R_3$ by Lemma 3.1. Now, we can find $x \otimes y$ and $x' \otimes y'$ such that x, x' are linearly independent; y, y' are linearly independent; $T(x \otimes y)$ is rank 1 and $T(x' \otimes y')$ is rank 3. If this is not the case then $T(R_1) \subset R_1$ or $T(R_1) \subset R_3$ and we are finished.

Let $T(x \otimes y) = x_1 \otimes y_1$ and

$$T(x' \otimes y') = x_1 \otimes z_1 + x_2 \otimes z_2 + x_3 \otimes z_3 .$$

Consider,

$$T(\alpha x \otimes y + x' \otimes y') = x_1 \otimes (\alpha y_1 + z_1) + x_2 \otimes z_2 + x_3 \otimes z_3 .$$

This must be rank 2, for all $\alpha \neq 0$. Therefore, there exist $\beta, \gamma \in F$ such that $\alpha y_1 + z_1 = \beta z_2 + \gamma z_3$ or $\alpha y_1 = \beta z_2 + \gamma z_3 - z_1$. Let $\alpha_1 \in F$, $\alpha_2 \in F$ and $\alpha_1 \neq \alpha_2$. There exist $\beta_1, \gamma_1, \beta_2, \gamma_2 \in F$ such that

$$\alpha_1 y_1 = \beta_1 z_2 + \gamma_1 z_3 - z_1$$
$$\alpha_2 y_1 = \beta_2 z_2 + \gamma_2 z_3 - z_1$$

This implies

$\beta_1\alpha_1^{-1}z_2 + \gamma_1\alpha_1^{-1} z_3 - \alpha_1^{-1} z_1 = \beta_2\alpha_2^{-1}z_2 + \gamma_2\alpha_2^{-1}z_3 - \alpha_2^{-1}z_1$. Now, $\dim(\langle z_1,z_2,z_3\rangle) = 3$. Therefore, $\alpha_1^{-1} = \alpha_2^{-1}$. This is a contradiction. Therefore, $T(R_1) \subset R_1$ or $T(R_1) \subset R_3$.

<u>Theorem 2</u>: Let $m = n = 3$. $T(R_2) \subset R_2$ implies $T(R_1) \subset R_1$.

<u>Proof</u>: From Lemma 3.2, it is sufficient to show that $T(R_1) \not\subset R_3$. Assume $T(R_1) \subset R_3$. Let $T(u_1 \otimes v_1) = u_1 \otimes y_1 + u_2 \otimes y_2 + u_3 \otimes y_3$ and $T(u_1 \otimes v_2) = u_1 \otimes z_1 + u_2 \otimes z_2 + u_3 \otimes z_3$ where v_1, v_2 are linearly independent. Let $A : V \to V$ such that $Ay_i = z_i$, $i = 1,2,3$. A has an eigenvalue, λ . Then

$$T(u_1 \otimes (v_2 - \lambda v_1)) = u_1 \otimes (A - \lambda I)y_1 + u_2 \otimes (A - \lambda I)y_2 + u_3 \quad (A - \lambda I)y_3 .$$

There exists α,β,γ not all 0 , such that $\alpha y_1 + \beta y_2 + \gamma y_3$ is the eigenvector corresponding to λ. Therefore, $(A - \lambda I)(\alpha y_1 + \beta y_2 + \gamma y_3) = 0$ and $(A - \lambda I)y_1$, $(A - \lambda I)y_2$, $(A - \lambda I)y_3$ are dependent. This means $T(u_1 \otimes (v_2 - \lambda v_1))$ is not rank 3 which contradicts the assumption that $T(R_1) \subset R_3$. Therefore, $T(R_1) \subset R_1$.

CHAPTER FOUR

Theorem 3: If F is algebraically closed and $T(R_2) \subset R_2$ then $T(R_1) \subset R_1$.

Proof: The result follows immediately from Theorems 1 and 2.

For algebraically closed fields of characteristic 0, the structure of T is given by the following theorem, which is quoted from [2].

Theorem 4: Let $T(R_1) \subset R_1$. "Let T_1 be the linear transformation of $V \otimes U$ into $U \otimes V$ which maps $y \otimes x$ onto $x \otimes y$. If $m = n$, let φ be any non-singular linear transformation of U onto V. Then if $m \neq n$, there exist non-singular linear transformations A and B on U and V, respectively, such that $T = A \otimes B$. If $m = n$, there exist non-singular A and B such that either $T = A \otimes B$ or $T = T_1(\varphi A \otimes \varphi^{-1} B)$."

For algebraically closed fields of all characteristics, Theorem 4 holds; but the proof is, as yet, unpublished.

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