CHARACTERIZATION OF RANK TWO SUBSPACES OF A TENSOR PRODUCT SPACE

by

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ABSTRACT

Let U,V be two vector spaces of dimensions n and m, respectively, over an algebraically closed field F; let $U \otimes V$ be their tensor product; and let $R_k(U \otimes V)$ be the set of all rank k tensors in $U \otimes V$; that is $R_k(U \otimes V)$

$$= \left\{ Z = \sum_{i=1}^{k} x_i \otimes y_i \;\middle|\; x_i;\ i = 1,\ldots,k \text{ and } y_i;\ i = 1,\ldots,k \right.$$

are each linearly independent in U and V respectively$\}$. We first obtain conditions on two vectors X and Y that they be members of a subspace H contained in $R_k(U \otimes V)$.

In chapter 2, we restrict our consideration to the rank 2 case, and derive a characterization of subspaces contained in $R_2(U \otimes V)$. We show that any such subspace must be one of three types, and we find the maximum dimension of each type. We also find the dimension of the intersection of two subspaces of different types.

Finally, we show that any maximal subspace has a dimension which depends only on its type.

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CHAPTER ONE

1. INTRODUCTION

Let F be an algebraically closed field; let U be an n-dimensional vector space over F; let V be an m-dimensional vector space over F; and let $U \otimes V$ be the tensor product space of U and V. An element X, of $U \otimes V$ is called pure if X can be represented as $X = x \otimes y$ for some $x \in U$ and $y \in V$, and, any element in $U \otimes V$ can be represented as a sum of pure tensors. The definition of each of the above terms can be found in any standard algebra text such as [1][1].

For any element $Z \in U \otimes V$, Z is said to be of rank k if $Z = \sum_{i=1}^{k} x_i \otimes y_i$, where $x_1, \ldots, x_k$ are linearly independent and $y_1, \ldots, y_k$ are linearly independent. This definition, taken from [2], page 1215, is used throughout this thesis, and is the basis for the ensuing material.

We define $R_k(U \otimes V)$ to be $\{Z \in U \otimes V \mid \text{rank }(Z)=k\} \cup \{0\}$; and, for any $Z \in R_k(U \otimes V)$, $Z = \sum_{i=1}^{k} x_i \otimes y_i$, we define

$U(Z) = \langle x_1, \ldots, x_k \rangle$ and $V(Z) = \langle y_1, \ldots, y_k \rangle$. In lemma 1.2

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1 Numbers in the square bracket refer to the bibliography at the end.

we show that $U(Z)$ and $V(Z)$ are well defined. In chapter one, using some elementary theorems of algebra, we obtain some properties of the elements of $R_k(U \otimes V)$.

2. RANK k

First we note that the maximum rank of any element of $U \otimes V$ is at most the minimum of n and m. The following theorem is a useful criterion for finding the rank of an element X, of $U \otimes V$.

Theorem 1.1: Let $X = \sum_{i=1}^{s} x_i \otimes y_i \in U \otimes V$. If $x_1, \ldots, x_s$ are linearly independent, then $\dim \langle y_1, \ldots, y_s \rangle = \text{rank}(X)$.

Proof: Suppose $\dim \langle y_1, \ldots, y_s \rangle = t \leq s$. Without loss of generality, we can assume $y_1, \ldots, y_t$ form a basis of $\langle y_1, \ldots, y_s \rangle$. Then each y_i, $t < i \leq s$, is a linear combination of $y_1, \ldots, y_t$. Thus for $i = t+1, \ldots, s$, $y_i = \sum_{j=1}^{t} \alpha_{ij} y_j$. Upon substituting these in X, we have

$$
\begin{aligned}
X &= \sum_{j=1}^{t} x_j \otimes y_j + \sum_{i=t+1}^{s} x_i \otimes \sum_{j=1}^{t} \alpha_{ij} y_j \\
&= \sum_{j=1}^{t} x_j \otimes y_j + \sum_{i=t+1}^{s} \sum_{j=1}^{t} \alpha_{ij} x_i \otimes y_j .
\end{aligned}
$$

Interchanging the summation signs we get

$$X = \sum_{j=1}^{t} \left(x_j + \sum_{i=t+1}^{s} \alpha_{ij} x_i\right) \otimes y_j .$$

Therefore $\text{rank}(X) \leq t = \dim \langle y_1, \ldots, y_s \rangle$.

Suppose $$\sum_{j=1}^{t} \alpha_j(x_j + \sum_{i=t+1}^{s} \alpha_{ij}x_i) = 0 ,$$

then $$\sum_{j=1}^{t} \alpha_j x_j + \sum_{j=1}^{t} \alpha_j \sum_{i=t+1}^{s} \alpha_{ij}x_i = 0.$$

Again, by interchanging the summation signs we have

$$\sum_{j=1}^{t} \alpha_j x_j + \sum_{i=t+1}^{s} (\sum_{j=1}^{t} \alpha_j\alpha_{ij})x_i = 0 .$$

Let $\sum_{j=1}^{t} \alpha_j\alpha_{ij} = \beta_i$, $i = t+1, \ldots, s$, then the summation becomes

$$\sum_{j=1}^{t} \alpha_j x_j + \sum_{i=t+1}^{s} \beta_i x_i = 0 .$$

Since $x_1,\ldots,x_s$ are linearly independent, each α_j and β_i must be zero. Thus $\{x_j + \sum_{i=t+1}^{s} \alpha_{ij}x_i , j = 1,\ldots,t\}$ is a linearly independent set. By assumption $y_1,\ldots,y_t$ are linearly independent, and thus

$$\text{rank } (X) = t = \dim \langle y_1,\ldots,y_s\rangle.$$

We next show that the subspaces, $U(X)$ and $V(X)$ are independent of the representation of X.

<u>Lemma 1.2</u>: For any $X \in R_k(U\otimes V)$, $X \neq 0$, $U(X)$ and $V(X)$ are independent of the representation of X as a sum of k pure tensors.

<u>Proof</u>: Let X have two representations as $X = \sum_{i=1}^{k} x_i \otimes y_i$ and $X = \sum_{i=1}^{k} x_i' \otimes y_i'$; and let $U(X) = \langle x_1,\ldots,x_k\rangle$ and $U'(X) = \langle x_1',\ldots,x_k'\rangle$.

Suppose $U(X) \neq U'(X)$. If $U(X) \cap U'(X) = 0$, then $x_1,\dots,x_k$, $x_1',\dots,x_k'$ are linearly independent and so

$$0 = \sum_{i=1}^{k} x_i \otimes y_i - \sum_{i=1}^{k} x_i' \otimes y_i' .$$ Theorem 1.1 implies

$\dim \langle y_1,\dots,y_k, y_1',\dots,y_k' \rangle = 0$, and thus y_i and y_i' are all zero for $i = 1,\dots,k$. Hence $X = 0$, which is a contradiction.

If $U(X) \cap U'(X) \neq 0$, let $z_1,\dots,z_r$ be a basis of $U(X) \cap U'(X)$. We can extend $z_1,\dots,z_r$ to a basis of $U(X)$ by $v_{r+1},\dots,v_k$ and to a basis of $U'(X)$ by $u_{r+1},\dots,u_k$. Each x_i can be represented as

$$x_i = \sum_{j=1}^{r} \gamma_{ij} z_j + \sum_{j=r+1}^{k} \gamma_{ij} v_j \; ; \quad \text{for } i = 1,\dots,k$$

and

$$x_i' = \sum_{j=1}^{r} \eta_{ij} z_j + \sum_{j=r+1}^{k} \eta_{ij} u_j \; ; \text{ for } i = 1,\dots,k .$$

It follows that

$$0 = \sum_{i=1}^{k}\Big(\sum_{j=1}^{r} \gamma_{ij} z_j + \sum_{j=r+1}^{k} \gamma_{ij} v_j \Big) \otimes y_i - \sum_{i=1}^{k}\Big(\sum_{j=1}^{r} \eta_{ij} z_j + \sum_{j=r+1}^{k} \eta_{ij} u_j \Big) \otimes y_i'$$

$$0 = \sum_{i=1}^{k} \sum_{j=1}^{r} \gamma_{ij} z_j \otimes y_i + \sum_{i=1}^{k} \sum_{j=r+1}^{k} \gamma_{ij} v_j \otimes y_i - \sum_{i=1}^{k} \sum_{j=1}^{r} \eta_{ij} z_j \otimes y_i' - \sum_{i=1}^{k} \sum_{j=r+1}^{k} \eta_{ij} u_j \otimes y_i'$$

$$0 = \sum_{j=1}^{r} z_j \otimes \Big(\sum_{i=1}^{k} \gamma_{ij} y_i - \sum_{i=1}^{k} \eta_{ij} y_i' \Big) + \sum_{j=r+1}^{k} v_j \otimes \sum_{i=1}^{k} \gamma_{ij} y_i - \sum_{j=r+1}^{k} u_j \otimes \sum_{i=1}^{k} \eta_{ij} y_i' .$$

Since $z_1,\dots,z_r$, $v_{r+1},\dots,v_k$, $u_{r+1},\dots,u_k$ are linearly independent,

$$\left(\sum_{i=1}^{k} \gamma_{ij} y_i - \sum_{i=1}^{k} \eta_{ij} y_i'\right) \; ; \quad j = 1,\ldots,r$$

$$\sum_{i=1}^{k} \gamma_{ij} y_i \qquad ; \quad j = r+1,\ldots,k$$

$$\sum_{i=1}^{k} \eta_{ij} y_i' \qquad ; \quad j = r+1,\ldots,k$$

are all zero. But since X has rank k, $y_1,\ldots,y_k$ are linearly independent and $y_1',\ldots,y_k'$ are linearly independent. Therefore $\gamma_{ij} = 0$; $i = 1,\ldots,k$; $j = r+1,\ldots,k$ and $\eta_{ij} = 0$; $i = 1,\ldots,k$; $j = r+1,\ldots,k$. Hence $x_i = \sum_{j=1}^{r} \gamma_{ij} z_i$; for $i = 1,\ldots,k$ and $x_i' = \sum_{j=1}^{r} \eta_{ij} z_j$; for $i = 1,\ldots,k$. This implies that $U(X) \subseteq U(X) \cap U'(X)$ and $U'(X) \subseteq U(X) \subseteq U(X) \cap U'(X)$, and hence $U(X) = U'(X)$. An identical argument holds for $V(X)$ and $V'(X)$.

The next theorem gives a first step in characterizing the subspaces.

<u>Theorem 1.3</u>: Let H be a subspace of $U \otimes V$ contained in $R_k(U \otimes V)$ and let X,Y be two elements in H. Then $\dim(U(X) \cap U(Y)) + \dim(V(X) \cap (V(Y)) \geq k$

<u>Proof</u>: Suppose $\dim(U(X) \cap U(Y)) = 0$, and $X = \sum_{i=1}^{k} u_i \otimes v_i$, $Y = \sum_{i=1}^{k} x_i \otimes y_i$. Then $u_1,\ldots,u_k,x_1,\ldots,x_k$ are linearly independent.

For all α and $\beta \in F$, $\alpha X + \beta Y \in H$, i.e. $\alpha X + \beta Y$ has rank k. Therefore

$X + Y = \sum_{i=1}^{k} u_i \otimes v_i + \sum_{i=1}^{k} x_i \otimes y_i \in R_k(U \otimes V)$. By Theorem 1.1, $\dim \langle v_1,\ldots,v_k,y_1,\ldots,y_k\rangle = k$. But $\dim \langle y_1,\ldots,y_k\rangle = k$ and $\dim \langle v_1,\ldots,v_k\rangle = k$ since both X and $Y \in R_k(U \otimes V)$. Therefore $\dim(V(X) \cap V(Y)) = k$.

Suppose $\dim (U(X) \cap U(Y)) = r > 0$ and let $z_1,\ldots,z_r$ be a basis of $U(X) \cap U(Y)$. We can extend $z_1,\ldots,z_r$ to a basis of $U(X)$ by $a_{r+1},\ldots,a_k$, and to a basis of $U(Y)$ by $b_{r+1},\ldots,b_k$. Then we can represent each u_i and x_i as $u_i = \sum_{j=1}^{r} \alpha_{ij} z_j + \sum_{j=r+1}^{k} \alpha_{ij} a_j$; for $i = 1,\ldots,k$, and $x_i = \sum_{j=1}^{r} \beta_{ij} z_j + \sum_{j=r+1}^{k} \beta_{ij} b_j$; for $i = 1,\ldots,k$.

Thus $X + Y =$

$$\sum_{i=1}^{k} \Big(\sum_{j=1}^{r} \alpha_{ij} z_j + \sum_{j=r+1}^{k} \alpha_{ij} a_j \Big) \otimes v_i + \sum_{i=1}^{k} \Big(\sum_{j=1}^{r} \beta_{ij} z_j + \sum_{j=r+1}^{k} \beta_{ij} b_j \Big) \otimes y_i$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{r} \alpha_{ij} z_j \otimes v_i + \sum_{i=1}^{k} \sum_{j=r+1}^{k} \alpha_{ij} a_j \otimes v_i + \sum_{i=1}^{k} \sum_{j=1}^{r} \beta_{ij} z_j \otimes y_i$$

$$+ \sum_{i=1}^{k} \sum_{j=r+1}^{k} \beta_{ij} b_j \otimes y_i$$

$$= \sum_{j=1}^{r} z_j \otimes \sum_{i=1}^{k} \alpha_{ij} v_i + \sum_{j=r+1}^{k} a_j \otimes \sum_{i=1}^{k} \alpha_{ij} v_i + \sum_{j=1}^{r} z_j \otimes \sum_{i=1}^{k} \beta_{ij} y_i + \sum_{j=r+1}^{k} b_j \otimes \sum_{i=1}^{k} \beta_{ij} y_i \, .$$

Combining terms we have $X + Y =$

$$\sum_{j=1}^{r} z_j \otimes \Big(\sum_{i=1}^{k} \alpha_{ij} v_i + \sum_{i=1}^{k} \beta_{ij} y_i \Big) + \sum_{j=r+1}^{k} a_j \otimes \sum_{i=1}^{k} \alpha_{ij} v_i + \sum_{j=r+1}^{k} b_j \otimes \sum_{i=1}^{k} \beta_{ij} y_i \, .$$

Now X and Y are in $R_k(U \otimes V)$; by lemma 1.2, we have that

$< \sum_{i=1}^{k} \alpha_{ij} v_i; \quad j = 1,\ldots,k> = V(X)$ and $< \sum_{i=1}^{k} \beta_{ij} y_i; \ j=1,\ldots,k> = V(Y)$.

For convenience in notation let $s_j = \sum_{i=1}^{k} \alpha_{ij} v_i$ and $t_j = \sum_{i=1}^{k} \beta_{ij} y_i$;

for $j = 1,\ldots,k$. Then we have

$$X+Y = \sum_{j=1}^{n} z_j \otimes (s_j + t_j) + \sum_{j=r+1}^{k} a_j \otimes s_j + \sum_{j=r+1}^{k} b_j \otimes t_j .$$

Let $A = V(X) + V(Y)$ and let $k+t$ be the dimension of A. Then

$$\begin{aligned} \dim (V(X) \cap V(Y)) &= \dim V(X) + \dim V(Y) - \dim A \\ &= k + k - (k+t) = k - t . \end{aligned}$$

Since we have assumed $\dim (U(X) \cap U(Y)) = r$, in order to prove the theorem we must show $\dim (V(X) \cap V(Y)) = k-t \geq k-r$ or equivalently that $t \leq r$. To this end consider the space $B = <s_1+t_1,\ldots,s_r+t_r, s_{r+1},\ldots,s_k,t_{r+1},\ldots t_k>$. B is a k-dimensional space, since $X + Y \in R_k(U \otimes V)$. Let $C = B + <s_1,\ldots,s_r>$. Then $\dim C \leq \dim B + r = k+r$. But now each s_j and each t_j ; $j = 1,\ldots,k$ is in C and therefore $A \leq C$. Thus $k+t \leq k+r$ or $t \leq r$ and $\dim (V(X) \cap V(Y)) = k-t \geq k-r$. Therefore $\dim (U(X) \cap U(Y)) + \dim (V(X) \cap V(Y)) \geq k$.

CHAPTER TWO

1. SUBSPACES OF $R_2(U\otimes V)$.

In this chapter we restrict our discussion to the case where k=2. We consider only those subspaces which are contained in $R_2(U\otimes V)$. We begin with three lemmas which will enable us to characterize the three types of subspaces.

Lemma 2.1: Let X and Y be independent elements of H, where H is a subspace of $U\otimes V$ contained in $R_2(U\otimes V)$. Suppose $U(X) \neq U(Y)$ and $V(X) \neq V(Y)$. Let $\langle u\rangle = U(X) \cap U(Y)$ and $\langle v\rangle = V(X) \cap V(Y)$. If we have representations $X = u\otimes x_1+y_1\otimes v'$ and $Y = u\otimes x_2+y_2\otimes v''$ then v' and $v'' \in \langle v\rangle$.

Proof: Since $U(X) \neq U(Y)$ and $V(X) \neq V(Y)$, it follows from theorem 1.3 that both $U(X) \cap U(Y)$ and $V(X) \cap V(Y)$ have dimension equal to one. Since X and Y $\in$ H, $\alpha X + \beta Y \in H$ for all $\alpha,\beta \in F$,

$$\alpha X+\beta Y = u\otimes(\alpha x_1+\beta x_2) + y_1\otimes\alpha v' + y_2\otimes\beta v''.$$

If $\langle v'\rangle = \langle v''\rangle$, then v' and v'' are both in $V(X) \cap V(Y)$, i.e., v' and $v'' \in \langle v\rangle$. On the other hand, suppose $\langle v'\rangle \neq \langle v''\rangle$. If $\alpha x_1+\beta x_2 \in \langle v',v''\rangle$ for all α,β then $\langle x_1,x_2\rangle = \langle v',v''\rangle$. But then, since $\langle x_1\rangle \neq \langle v'\rangle$ and $\langle x_2\rangle \neq \langle v''\rangle$ we have $\langle x_1,x_2\rangle = \langle x_1,v'\rangle = V(X) = \langle v',v''\rangle = \langle x_2,v''\rangle = V(Y)$, contradicting the hypothesis $V(X) \neq V(Y)$. If $\alpha x_1+\beta x_2 \notin \langle v',v''\rangle$ for some α,β, then for such α and β, $\alpha x_1+\beta x_2$, v',v'' are linearly independent. Also since $U(X) \neq U(Y)$, u,y_1,y_2 are linearly independent. This implies that $\text{rank}(\alpha X+\beta Y)=3$ for this choice of α and β. Therefore v', v'' must be in $\langle v\rangle$.

Next we prove a lemma which characterizes another type of subspace. But first we note that if $X \in R_2(U \otimes V)$, $X \neq 0$ and if $\{u,v\}$ is any basis of $U(X)$, then X has a representation of the form $u \otimes y_1 + v \otimes y_2$.

<u>Lemma 2.2</u>: Suppose $U(X) = U(Y)$ for some pair of independent elements of H, where H is a subspace in $R_2(U \otimes V)$. Then for any $Z \in H$, $U(Z) = U(X)$.

<u>Proof</u>: Let $X = u \otimes x_1 + v \otimes x_2$, $Y = u \otimes y_1 + v \otimes y_2$. First, we show $V(X) \neq V(Y)$. Suppose $V(X) = V(Y)$. If $x_1 = \gamma y_1$, then $X - \gamma Y = v \otimes (x_2 - \gamma y_2)$ is rank one, and if $x_2 = \gamma' y_2$, then $X - \gamma' Y = u \otimes (x_1 - \gamma' y_1)$ is rank one. Therefore $\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle$ implies $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle$. Let $A: \langle x_1, y_1 \rangle \to \langle x_2, y_2 \rangle$ be a linear transformation where $Ax_1 = x_2$ and $Ay_1 = y_2$. Let $x = \alpha x_1 + \beta y_1$ be an eigenvector of A with eigenvalue λ. Then

$$\begin{aligned} \alpha X + \beta Y &= u \otimes (\alpha x_1 + \beta y_1) + v \otimes (\alpha x_2 + \beta y_2) \\ &= u \otimes x + v \otimes Ax = u \otimes x + v \otimes \lambda x \\ &= (u + \lambda v) \otimes x \quad \text{is rank one.} \end{aligned}$$

Therefore $V(X) \neq V(Y)$.

Now let $Z = z_1 \otimes z_2 + z_3 \otimes z_4$. We must show that $\dim(U(Z) \cap U(X)) = 2$. First, we assume $\dim(U(Z) \cap U(X)) = 0$. Then since $U(X) = U(Y)$, $\dim(U(Z) \cap U(Y)) = 0$. By theorem 1.3, $\dim(V(Z) \cap V(X)) = 2$ and $\dim(V(Z) \cap V(Y)) = 2$. This implies $V(X) = V(Y)$, a contradiction.

Next, suppose $U(Z) \cap U(X) = \langle u \rangle$. Then by theorem 1.3, $\dim(V(Z) \cap V(X)) = 1$ and $\dim(V(Z) \cap V(Y)) = 1$. Therefore X, Y, and Z have representations:

$$X = u \otimes x_1 + v \otimes x_2$$

$$Y = u \otimes y_1 + v \otimes y_2$$

$$Z = u \otimes z_1 + w \otimes z_2$$

where u,v,w are linearly independent. Note, since x_2 and y_2 are linearly independent, that z_2 cannot be a multiple of both x_2 and y_2. Therefore, if z_2 and y_2 are linearly independent, $\langle y_1, y_2\rangle = \langle z_1, z_2\rangle$. Since $V(X) \neq V(Y)$, $\langle z_1, z_2\rangle \neq \langle x_1, x_2\rangle$. This proves that $z_2 = \varphi x_2$ (lemma 2.1) and $z_1 \neq \lambda x_1$.
Thus our representations become:

$$X = u \otimes x_1 + v \otimes x_2$$

$$Y = u \otimes y_1 + v \otimes y_2$$

$$Z = u \otimes z_1 + w' \otimes x_2$$

Consider $W = X + Y + Z$.

$$W = u \otimes (x_1 + y_1 + z_1) + v \otimes (x_2 + y_2) + w' \otimes x_2$$

where x_1, z_1, x_2 are linearly independent and $\langle y_1, y_2\rangle = \langle z_1, x_2\rangle = \langle y_2, x_2\rangle$, since $y_2 \neq \alpha x_2$. It follows that x_1, y_2, and x_2 are linearly independent, and hence $x_1 + y_1 + z_1$, $x_2 + y_2$ and x_2 are linearly independent, since y_1 and z_1 are in $\langle x_2, y_2\rangle$. But this makes rank $W = 3$, a contradiction. Similarly if z_2 and x_2 are linearly independent then $\langle z_1, z_2\rangle = \langle x_1, x_2\rangle$, $z_2 = \theta' y_2$, and $z_1 \neq \lambda' y_1$. Using the same arguments we can derive a rank three tensor from $X + Y + Z$. Therefore $\dim (U(Z) \cap U(X)) \neq 1$, and $U(Z) = U(X) = U(Y)$.

In lemma 2.2 we used the condition $U(X) = U(Y)$. A similar result is true for $V(X) = V(Y)$. We state it without proof in

Lemma 2.3: Suppose $V(X) = V(Y)$ for some pair of independent elements of H, where H is a subspace in $R_2(U\otimes V)$. Then for any $Z \in H$, $V(Z) = V(X)$.

We are now in a position to give a characterization of the subspaces in $R_2(U\otimes V)$.

Theorem 2.4: The subspaces H, in $R_2(U\otimes V)$ are of three types:

1) $U(X)$ is constant as X ranges over H.
2) $V(X)$ is constant as X ranges over H.
3) If $\dim H \neq 3$, then there exists $x \in U$ and $y \in V$ such that for all $X \in H$, X has a representation of the form $x\otimes v + u\otimes y$, where $v \in V$ and $u \in U$. This is the general type 3 case.

If $\dim H = 3$, then there are spaces of a type distinct from the above three types. The following is an example of such a space:

$$H = \langle X,Y,Z\rangle \text{ where } \quad X = u\otimes x_1 + x_2\otimes v,$$
$$Y = u\otimes y_1 + y_2\otimes v,$$
$$\text{and} \quad Z = x_2\otimes y_1 - y_2\otimes x_1.$$

We will call this the special type 3 case.

Proof: Let X and Y be two linearly independent elements of H. Then for all $\alpha,\beta \in F$, $\alpha X + \beta Y \in H$. By theorem 13., $\dim(U(X) \cap U(Y)) + \dim(V(X) \cap V(Y)) \geq 2$. If $\dim(V(X) \cap V(Y)) = 0$, then $U(X) = U(Y)$ and by lemma 2.2 we have case 1. If $\dim(V(X) \cap V(Y)) = 2$, then $V(X) = V(Y)$ and by lemma 2.3 we have case 2. If $\dim(V(X) \cap V(Y)) = 1$, then $\dim(U(X) \cap U(Y)) \geq 1$. If $\dim(U(X) \cap U(Y)) = 2$, then $U(X) = U(Y)$ and by lemma 2.2 we again have case 1. If $\dim(U(X) \cap U(Y)) = 1$, then we have case 3. If $\dim H = 2$, then by lemma 2.1 we have the general case 3. If $\dim H = 3$, $\dim(V(X) \cap V(Y)) = 1$ and $\dim(U(X) \cap U(Y)) = 1$, then $H = \langle X,Y,Z\rangle$. By lemma 2.1, we know that X and Y have representations

$$X = u\otimes x_1 + x_2\otimes v$$

$$Y = u\otimes y_1 + y_2\otimes v$$

and Z is such that $\dim(U(Z) \cap U(X)) = 1$, $\dim(U(Z) \cap U(Y)) = 1$. Also $\dim(V(Z) \cap V(X)) = 1$ and $\dim(V(Z) \cap V(Y)) = 1$. If $U(Z) \cap U(X) = U(Z) \cap U(Y) = \langle u\rangle$ then by lemma 2.1, $V(Z) \cap V(X) = V(Z) \cap V(Y) = \langle v\rangle$ and we have the general case 3.

If $U(Z) \cap U(X) \neq U(Z) \cap U(Y)$ then $U(Z) \subset \langle u,x_2,y_2\rangle$ and $u \notin U(Z)$. Also $V(Z) \cap V(X) \neq V(Z) \cap V(Y)$ gives $V(Z) \subset \langle v,x_1,y_1\rangle$ and $v \notin V(Z)$. This case can arise. In fact, let $Z = x_2\otimes y_1 - y_2\otimes x_1$. We show that $\alpha X + \beta Y + \gamma Z$ has rank 2 for all choices of α, β and γ. We can assume $\beta \neq 0$.

$$\alpha X+\beta Y+\gamma Z = u\otimes(\alpha x_1+\beta y_1)+(\alpha x_2+\beta y_2)\otimes v + \gamma x_2\otimes y_1 - \gamma y_2\otimes x_1$$

$$= \beta u\otimes(\tfrac{\alpha}{\beta}x_1 + y_1)+(\tfrac{\alpha}{\beta}x_2 + y_2)\otimes\beta v + \gamma x_2\otimes y_1 - \gamma y_2\otimes x_1$$

$$+ \gamma x_2\otimes \tfrac{\alpha}{\beta}x_1 - \gamma x_2\otimes \tfrac{\alpha}{\beta}x_1$$

$$= (\beta u + \gamma x_2)\otimes(\tfrac{\alpha}{\beta}x_1 + y_1) + (\tfrac{\alpha}{\beta}x_2 + y_2)\otimes(\beta v - \gamma x_1)$$

which is rank two. Suppose $\alpha X + \beta Y + \gamma Z = 0$, then either $\beta u + \gamma x_2 = 0$ and $\frac{\alpha}{\beta}x_2 + y_2 = 0$ or $\frac{\alpha}{\beta}x_1 + y_1 = 0$ and $\beta v - \gamma x_1 = 0$. In either case $\gamma = 0$ implies $\alpha = \beta = 0$. Therefore X, Y, and Z are linearly independent, and $H = \langle X,Y,Z\rangle$ has dimension 3.

(It can be shown that spaces of this exceptional type must have a basis of the form given, but we do not include the details here since we do not make use of this fact.)

Now, if $\dim H \geq 4$, let X, Y, Z, W be linearly independent in H where H is not type 1 or 2. By lemma 2.1 there exist representations such that

$$X = u \otimes x_1 + x_2 \otimes v \;, \qquad Y = u \otimes y_1 + y_2 \otimes v \quad \text{and}$$
$$W = u' \otimes w_1 + w_2 \otimes v' , \qquad Z = u' \otimes z_1 + z_2 \otimes v' .$$

If $\langle u \rangle = \langle u' \rangle$, then $\langle v \rangle = \langle v' \rangle$ and we have the general case 3. Hence suppose $\langle u \rangle \neq \langle u' \rangle$. Then $\langle u', w_2 \rangle \subset \langle u, x_2 y_2 \rangle$, and $\langle u', z_2 \rangle \subset \langle u_1 x_2 y_2 \rangle$. Therefore $\langle w_2 z_2 \rangle \subset \langle u, x_2, y_2 \rangle$ since $w_2 \neq \theta z_2$. Therefore $u' \in \langle u, x_2, y_2 \rangle$ and there exists a and b $\in F$ such that $u' \in \langle u, ax_2 + by_2 \rangle$. Similarly $u \in \langle u', w_2, z_2 \rangle$ and there exist c and d $\in F$ such that $u \in \langle u', cw_2 + dz_2 \rangle$.

Consider $\quad U(aX+bY) = \langle u, ax_2 + by_2 \rangle$ and

$\qquad\qquad U(cW+dZ) = \langle u, cw_2 + dz_2 \rangle$.

Both spaces equal $\langle u, u' \rangle$ and we have two linearly independent vectors in H which have the same U space. By lemma 2.2 we have case 1, a contradiction. Therefore $\langle u \rangle = \langle u' \rangle$ and $\langle v' \rangle = \langle v \rangle$. Thus we have a characterization of every subspace contained in $R_2(U \otimes V)$.

2. DIMENSIONS AND INTERSECTIONS

First, we recall that n and m are the dimensions of U and V respectively.

Theorem 2.5: The maximum dimension of type 1, type 2 and general type 3 subspaces is m-1, n-1 and minimum (m-1,n-1) respectively.

Proof: Type 1

$$\begin{aligned}\text{Let } X_1 &= x_1 \otimes u_1 + x_2 \otimes v_1 \\ X_2 &= x_1 \otimes u_2 + x_2 \otimes v_2 \\ &\vdots \\ X_k &= x_1 \otimes u_k + x_2 \otimes v_k\end{aligned}$$

be an independent set of elements of H. If k=m, then since $u_1,\ldots,u_m$ are linearly independent, they form a basis of V; and $v_1,\ldots,v_m$ form another basis of V. Let A: $V \to V$ be a linear transformation such that $Au_i = v_i$ for $i = 1,\ldots,m$.

Let $u = \sum_{i=1}^{m} \alpha_i u_i$ be an eigenvector of A with eigenvalue λ.

Consider $\sum_{i=1}^{m} \alpha_i X_i = x_1 \otimes \sum_{i=1}^{m} \alpha_i u_i + x_2 \otimes \sum_{i=1}^{m} \alpha_i v_i = x_1 \otimes u + x_2 \otimes Au$

$= x_1 \otimes u + x_2 \otimes \lambda u = (x_1 + \lambda x_2) \otimes u$ which is rank one. Therefore $k \leq m-1$. Similarly $k \leq n-1$ for type 2 subspaces.

Type 3.

As shown in theorem 2.4, there exist 3 dimensional subspaces which are maximal. These are considered as a special case of type 3 subspaces. In the general case, let $X_1,\ldots,X_t$

be an independent set of vectors in a subspace H, where H is the general type 3 subspace. Assume $m \geq n > 3$. By lemma 2.1 and theorem 2.4, there exist representations of $X_1, \ldots, X_t$ such that

$$X_1 = x_1 \otimes v_1 + u_1 \otimes y_1$$
$$X_2 = x_1 \otimes v_2 + u_2 \otimes y_1$$
$$\vdots$$
$$X_t = x_1 \otimes v_t + u_t \otimes y_1$$

Here $u_1, \ldots, u_t$ are linearly independent, for, if some $u_r = \sum_{i \neq r} \alpha_i u_i$, then

$$X_r - \sum_{i \neq r} \alpha_i X_i = x_1 \otimes (v_r - \sum_{i \neq r} \alpha_i v_i) + (u_r - \sum_{i \neq r} \alpha_i u_i) \otimes y_1$$
$$= x_1 \otimes (v_r - \sum_{i \neq r} \alpha_i v_i) \text{ is rank one.}$$

Similarly, $v_1, \ldots, v_t$ are linearly independent. Next we show that $x_1 \notin \langle u_1, \ldots, u_t \rangle$.

If $x_1 = \sum_{i=1}^{t} \alpha_i u_i$, then $\sum_{i=1}^{t} \alpha_i X_i = x_1 \otimes \sum_{i=1}^{t} \alpha_i v_i + \sum_{i=1}^{t} \alpha_i u_i \otimes y_1$

$$= x_1 \otimes \left(\sum_{i=1}^{t} \alpha_i v_i + y_1 \right) \text{ which is rank one.}$$

Using a similar argument we can also show that $y_1 \notin \langle v_1, \ldots, v_t \rangle$. It follows that the maximum dimension of a type 3 subspace is less than or equal to $\min(m-1, n-1)$. If $m \geq n=3$, then the special case 3 subspace gives a maximum subspace of dim=3.

These maximal dimensions can be achieved. For example, for type 1 subspaces, let x_1, x_2 be independent in U and let $e_1, \ldots, e_m$ be a basis of V. Then set

$$\begin{aligned} X_1 &= x_1 \otimes e_1 + x_2 \otimes e_2 \\ X_2 &= x_1 \otimes e_2 + x_2 \otimes e_3 \\ &\;\vdots \\ X_{m-1} &= x_1 \otimes e_{m-1} + x_2 \otimes e_m. \end{aligned}$$

If $\sum_{i=1}^{m-1} \alpha_i X_i = 0$, then $0 = \sum_{i=1}^{m-1} \alpha_i (x_1 \otimes e_i) + \sum_{i=1}^{m-1} \alpha_i (x_2 \otimes e_{i+1})$;

and each $\alpha_i = 0$; $i = 1, \ldots, m-1$ as $x_i \otimes e_j$, $i = 1,2$; $j = 1, \ldots, m$ are linearly independent in $U \otimes V$. Moreover, $\sum_{i=1}^{m-1} \alpha_i X_i = x_1 \otimes \sum_{i=1}^{m-1} \alpha_i e_i + x_2 \otimes \sum_{i=1}^{m-1} \alpha_i e_{i+1}$. If not all $\alpha_i = 0$, then $\sum_{i=1}^{m-1} \alpha_i e_i \neq 0$ and $\sum_{i=1}^{m-1} \alpha_i e_{i+1} \neq 0$. Choose j to be the least integer such that $\alpha_j \neq 0$. Then $\sum_{i=1}^{m-1} \alpha_i e_i = \alpha_j e_j + \sum_{i>j} \alpha_i e_i$ and $\sum_{i=1}^{m-1} \alpha_i e_{i+1} = \alpha_j e_{j+1} + \sum_{i>j} \alpha_i e_{i+1}$ which implies that $\sum_{i=1}^{m-1} \alpha_i e_i$ and $\sum_{i=1}^{m-1} \alpha_i e_{i+1}$ are linearly independent. Therefore $\sum_{i=1}^{m-1} \alpha_i X_i$ is always a rank two tensor. In the type 2 case an example of the n-1 dimensional subspace is readily given in a similar fashion.

For the special type 3 case, we have already shown the

3 dimensional maximal subspace. In the general type 3 case, we can assume $m \geq n \geq 4$. Let $e_1, \ldots, e_m$ be a basis of V, and let $f_1, \ldots, f_n$ be a basis of U.

Set

$$X_1 = f_n \otimes e_1 + f_1 \otimes e_m$$
$$X_2 = f_n \otimes e_2 + f_2 \otimes e_m$$
$$\vdots$$
$$X_{n-1} = f_n \otimes e_{n-1} + f_{n-1} \otimes e_m.$$

If $\sum_{i=1}^{n-1} \alpha_i X_i = 0$, then $0 = \sum_{i=1}^{n-1} \alpha_i (f_n \otimes e_i) + \sum_{i=1}^{n-1} \alpha_i (f_i \otimes e_m)$;

and each $\alpha_i = 0$ as $f_j \otimes e_i$; $i = 1, \ldots, m$; $j = 1, \ldots, n$ are linearly independent in $U \otimes V$. If not all $\alpha_i = 0$ then

$$\sum_{i=1}^{n-1} \alpha_i X_i = f_n \otimes \sum_{i=1}^{n-1} \alpha_i e_i + \sum_{i=1}^{n-1} \alpha_i f_i \otimes e_m$$

is rank two because $f_n \notin \langle f_1, \ldots, f_{n-1} \rangle$ and $e_m \notin \langle e_1, \ldots, e_{n-1} \rangle$.

The next theorem is concerned with the intersection of two different types of subspaces.

Theorem 2.6: The intersection of two different types of subspaces is at most one dimensional.

Proof: Type 1 and type 2 intersection.

Suppose X_1 and X_2 are in the intersection and linearly independent. Since X_1 and X_2 are in a type 1 subspace, $U(X_1) = U(X_2)$. Since X_1 and X_2 are in a type 2 subspace, $V(X_1) = V(X_2)$. In the proof of lemma 2.2 we showed that if $U(X_1) = U(X_2)$ then $V(X_1) \neq V(X_2)$ for any pair of linearly independent elements of H. Therefore $X_1 = \lambda X_2$, and the intersection is one dimensional.

Type 1 and type 3 intersection.

Suppose X_1 and X_2 are in the intersection and $X_1 \neq \alpha X_2$. Since X_1 and X_2 are both in a type 3 subspace, there exists representations such that $X_1 = x \otimes y_1 + x_1 \otimes y$ and $X_2 = x \otimes y_2 + x_2 \otimes y$. Since the intersection is a subspace, $\alpha X_1 + \beta X_2 = x \otimes (\alpha y_1 + \beta y_2) + (\alpha x_1 + \beta x_2) \otimes y$ is in the intersection for all α and $\beta \in F$. Since both elements are in a type 1 subspace, $U(X)$ is fixed for all elements in the intersection, i.e., $\langle x, x_1 \rangle = \langle x, x_2 \rangle = \langle x, \alpha x_1 + \beta x_2 \rangle$. If dim $\langle x_1, x_2 \rangle \neq 1$, then $\langle x_1, x_2 \rangle = \langle x, x_1 \rangle$, and there exist α and β such that $\alpha x_1 + \beta x_2 = x$. This choice of α and β makes $\alpha X_1 + \beta X_2 =$ $x \otimes (\alpha y_1 + \beta y_2 + y)$, a pure tensor. Therefore $\langle x_1 \rangle = \langle x_2 \rangle$ or $x_1 = \theta x_2$; but then $X_1 - \theta X_2 = x \otimes (y_1 - \theta y_2)$ is rank one unless $y_1 = \theta y_2$. But this implies $X_1 = \theta X_2$, a contradiction. Therefore, the intersection is one dimensional.

Type 2 and type 3 intersection.

We can use an argument similar to that given for the type 1 and type 3 intersection. In this case the $V(X)$ space is constant.

This completes the proof of the theorem. The remaining intersections to consider are those of subspaces of the same type. The special type 3 subspace can have a two dimensional intersection with a general type 3 subspace but it can have only a one dimensional intersection with another special type 3 subspace.

Theorem 2.7: The intersection of two different subspaces of the same type can be at most m-2, n-2 or min(m-2,n-2) for type 1, type 2 and general type 3 subspaces, respectively. These maximums can all be achieved.

Proof: Suppose $X_1,\ldots,X_k$ are a set of linearly independent vectors in the intersection of two subspaces, H_1 and H_2, of type 1 such that $H_1 \neq H_2$. We know that $k \leq m-1$. We shall show that $k \leq m-2$. If $k = m-1$, there exist $X_1,\ldots,X_{m-1}$ which are linearly independent. Since H_1 and H_2 are both of type 1, $\dim H_1 = \dim H_2 = m-1$, by Theorem 2.5. Thus $X_1,\ldots,X_{m-1}$ forms a basis of both H_1 and H_2; and $H_1 = H_2$, a contradiction. Thus $k \leq m-2$. Now to show that k may equal m-2, let u,v be linearly independent in U, and let $e_1,\ldots,e_m$ be a basis of V.

Set
$$X_1 = u\otimes e_1 + v\otimes e_2$$
$$X_2 = u\otimes e_2 + v\otimes e_3$$
$$\vdots$$
$$X_{m-1} = u\otimes e_{m-1} + v\otimes e_m$$
and $X'_{m-1} = u\otimes e_{m-1} + v\otimes(e_m+e_1)$.

Let $H_1 = \langle X_1,\ldots,X_{m-1}\rangle$ and $H_2 = \langle X_1,\ldots,X_{m-2},X'_{m-1}\rangle$. As shown in the examples following theorem 2.5, both H_1 and H_2 are m-1 dimensional subspaces. To show $H_1 \neq H_2$ we need only show $X'_{m-1} \neq \sum_{i=1}^{m-1} \alpha_i X_i$. Suppose $X'_{m-1} = \sum_{i=1}^{m-1} \alpha_i X_i$, then

$u \otimes e_{m-1} + v \otimes (e_m + e_1) = u \otimes \sum_{i=1}^{m-1} \alpha_i e_i + v \otimes \sum_{i=1}^{m-1} \alpha_i e_{i+1}$. This implies that $\sum_{i=1}^{m-1} \alpha_i e_i = e_{m-1}$ and $\sum_{i=1}^{m-1} \alpha_i e_{i+1} = e_m + e_1$. This is impossible, however, since the first equality gives $\alpha_{m-1} = 1$ and $\alpha_i = 0$, $i < m-1$; and the second becomes $e_m = e_m + e_1$. Therefore $H_1 \neq H_2$ and $\dim(H_1 \cap H_2) = \dim \langle X_1, \ldots, X_{m-2} \rangle = m-2$, by construction.

The proof for type 2 subspaces is exactly the same using U instead of V. This gives an intersection of dimension at most n-2 for type-2 subspaces.

For the general type 3 subspaces we can assume that $3 < n \leq m$. Suppose $X_1, \ldots, X_k$ are linearly independent in the intersection of H_1 and H_2, and $H_1 \neq H_2$, then, by an argument similar to that for the type 1 case, $k \leq n-2$. To show k may equal n-2, let $f_1, \ldots, f_n$ be a basis of U and $e_1, \ldots, e_m$ be a basis of V. Set

$$X_1 = f_n \otimes e_1 + f_1 \otimes e_m$$

$$X_2 = f_n \otimes e_2 + f_2 \otimes e_m$$

$$\vdots$$

$$X_{n-1} = f_n \otimes e_{n-1} + f_{n-1} \otimes e_m$$

and

$$X'_{n-1} = f_n \otimes e_{n-1} + (f_n + f_{n-1}) \otimes e_m.$$

Let $H_1 = \langle X_1, \ldots, X_{n-1} \rangle$ and $H_2 = \langle X_1, \ldots, X_{n-2}, X'_{n-1} \rangle$. Since $f_i \otimes e_j$; $i = 1, \ldots, n$; $j = 1, \ldots, m$ are linearly independent in

$U \otimes V$, the subspaces H_1 and H_2 are both n-1 dimensional. To show $H_1 \neq H_2$ we need only show $X'_{n-1} \neq \sum_{i=1}^{n-1} \alpha_i X_i$. Suppose

$$X'_{n-1} = \sum_{i=1}^{n-1} \alpha_i X_i, \text{ then } f_n \otimes e_{n-1} + (f_n + f_{n-1}) \otimes e_m =$$

$$f_n \otimes \sum_{i=1}^{n-1} \alpha_i e_i + \sum_{i=1}^{n-1} \alpha_i f_i \otimes e_m.$$ Since $f_n \neq \sum_{i=1}^{n-1} \alpha_i f_i$ and

$e_m \neq \sum_{i=1}^{n-1} \alpha_i e_i$ each side of the equation is a rank two tensor.

Therefore $\sum_{i=1}^{n-1} \alpha_i e_i = e_{n-1}$ and $\sum_{i=1}^{n-1} \alpha_i f_i = f_n + f_{n-1}$. But this is impossible, since $f_n \notin \langle f_1, \ldots, f_{n-1} \rangle$. Thus $H_1 \neq H_2$; and by construction, $H_1 \cap H_2 = \langle X_1, \ldots, X_{n-2} \rangle$ has dimension n-2. This concludes the proof of the theorem.

For the special type 3 subspaces, the intersection of two special type 3 subspaces is at most one dimensional. If $X = u \otimes x_1 + x_2 \otimes v$ and $Y = u \otimes y_1 + y_2 \otimes v$ are in two special type 3 subspaces, then $Z = x_2 \otimes y_1 - y_2 \otimes x_1$ or $-Z$ must be the third vector in both subspaces; and the spaces coincide. Therefore two distinct subspaces can have at most one dimension in common. The intersection of a general type 3 subspace and a special type 3 subspace can be at most two dimensional. Again, suppose $X = u \otimes x_1 + x_2 \otimes v$ and $Y = u \otimes y_1 + y_2 \otimes v$ are in both spaces. Every vector in the general type 3 subspace must now have a representation of the form $u \otimes x + y \otimes v$ but any other vector independent of X and Y in the special type 3 subspace must have a representation of the form $\gamma(x_2 \otimes y_1 - y_2 \otimes x_1)$.

3. DIMENSIONS OF MAXIMAL SUBSPACES.

In this section we prove that any maximal subspace in $R_2(U \otimes V)$ has a dimension which is dependent only on its type. In particular, all maximal subspaces of type 1 have dimension m-1, and all maximal subspaces of type 2 have dimension n-1. In the type 3 case, all maximal subspaces have dimension 3 or min(m-1, n-1).

<u>Theorem 2.8</u>: All maximal subspaces of type 3, have dimension 3 or min(m-1, n-1).

<u>Proof</u>: Since we are dealing with rank two elements both m and n must be greater than or equal to two. Suppose min(m-1,n-1)=1, then either U or V is two dimensional. Suppose U is two dimensional and {u,v} is a basis of U. Also, witʰout loss of generality, we can assume dim V $\geq$ 2. Let X and Y be in a type 3 subspace, then X and Y have representations such that:

$$X = u \otimes x_1 + x_2 \otimes y$$
$$Y = u \otimes y_1 + y_2 \otimes y.$$

We have $U = \langle u,v \rangle = U(X) = U(Y)$, and therefore either $x_2 = \theta y_2$ or $\langle x_2 y_2 \rangle = U$. If $\langle x_2, y_2 \rangle = U$, then there exist $a,b \in F$ such that $ax_2 + by_2 = u$ and $aX + bY = u \otimes (ax_1 + by_1) + (ax_2 + by_2) \otimes y = u \otimes (ax_1 + by_1 + y)$ is rank one. If $x_2 = \theta y_2$, then $X - \theta Y = u \otimes (x_1 - \theta y_1)$ is rank one or $x_1 = \theta y_1$. Therefore $X = \theta Y$ and all subspaces of type 3 are one dimensional. If min(m-1,n-1) = 2, then the dimension of U or V is 3. Suppose U is 3 dimensional and dim V $\geq$ 3. Let $\langle u,v,w \rangle = U$ and suppose X and Y are in a type 3 subspace H_1 in $R_2(U \otimes V)$. Then X and Y have representations such that

$$X = u \otimes x_1 + v \otimes y$$
$$Y = u \otimes x_2 + w \otimes y$$

and X and Y are linearly independent. This subspace is not maximal since, by theorem 2.4, we can add $Z = v \otimes x_2 - w \otimes x_1$ to get a larger subspace.

If $\min(m-1, n-1) = 3$, then we can assume $m \geq n \geq 4$. Again, as shown in theorem 2.4, there exist maximal subspaces of dimension 3. Suppose H is a general type 3 subspace, $\dim H \geq 3$ and H is maximal. Suppose $X_1, \ldots, X_k$ is a basis of H and $k < n-1 \leq m-1$. By theorem 2.4, there exist representations of $X_1, \ldots, X_k$ such that:

$$X_1 = u \otimes x_1 + y_1 \otimes v$$
$$X_2 = u \otimes x_2 + y_2 \otimes v$$
$$\vdots$$
$$X_k = u \otimes x_k + y_k \otimes v.$$

We know that $v, x_1, \ldots, x_k$ are linearly independent, and that $u, y_1, \ldots, y_k$ are linearly independent. Since $k < n-1$, there exists $y_{k+1} \in U$ such that $y_{k+1} \notin \langle u, y_1, \ldots, y_k \rangle$; and there exists $x_{k+1} \in V$ such that $x_{k+1} \notin \langle v, x_1, \ldots, x_k \rangle$. Consider $X_{k+1} = u \otimes x_{k+1} + y_{k+1} \otimes v$. Now $k+1 < n$ and thus $k+2 \leq n$, i.e., $\langle u, y_1, \ldots, y_{k+1} \rangle \subseteq U$.

Suppose $\sum_{i=1}^{k+1} \alpha_i X_i = 0$, then $0 = u \otimes \sum_{i=1}^{k+1} \alpha_i x_i + \sum_{i=1}^{k+1} \alpha_i y_i \otimes v$.

Since $v \neq \sum_{i=1}^{k+1} \alpha_i x_i$ and $u \neq \sum_{i=1}^{k+1} \alpha_i y_i$, we must have $\alpha_i = 0$; $i = 1,\ldots,k+1$. This implies that H is not maximal, a contradiction. Therefore, $k = n-1$.

<u>Theorem 2.9</u>: All maximal subspaces of type 1 have dimension m-1.

<u>Proof</u>: Suppose H is a maximal type 1 subspace with basis vectors $X_1,\ldots,X_k$ where $k < m-1$. By theorem 2.4, $X_1,\ldots,X_k$ have representations such that

$$
\begin{aligned}
X_1 &= u \otimes x_1 + v \otimes y_1 \\
X_2 &= u \otimes x_2 + v \otimes y_2 \\
&\vdots \\
X_k &= u \otimes x_k + v \otimes y_k .
\end{aligned}
$$

We know that $x_1,\ldots,x_k$ are linearly independent, and that $y_1,\ldots,y_k$ are linearly independent. We now show that $\langle x_1,\ldots,x_k\rangle \neq \langle y_1,\ldots,y_k\rangle$. If they were equal, then there would exist a linear transformation A: $\langle x_1,\ldots,x_k\rangle \to \langle y_1,\ldots,y_k\rangle$ such that $Ax_i = y_i$; $i = 1,\ldots,k$; and as in Theorem 2.5, we could construct a rank one tensor in H. Next consider the space $\langle x_1,\ldots,x_k,y_1,\ldots,y_k\rangle$. If this space is not equal to V then we can choose $x_{k+1} \notin \langle x_1,\ldots,x_k,y_1,\ldots,y_k\rangle$, and y_{k+1} such that $y_{k+1} \in \langle x_1,\ldots,x_k\rangle$ and $y_{k+1} \notin \langle y_1,\ldots,y_k\rangle$. Set $X_{k+1} = u \otimes x_{k+1} + v \otimes y_{k+1}$. ~~Then~~

Then $\sum_{i=1}^{k+1} \alpha_i X_i = u\otimes\sum_{i=1}^{k+1} \alpha_i x_i + v\otimes\sum_{i=1}^{k+1} \alpha_i y_i = 0$ implies $\alpha_i = 0$;

$i = 1,\ldots,k+1$, since $u\otimes x_i$ and $v\otimes y_i$; $i = 1,\ldots,k+1$, are linearly independent in $U\otimes V$.

$\sum_{i=1}^{k+1} \alpha_i X_i$ is always a rank two element, for if $\sum_{i=1}^{k+1} \alpha_i X_i$ is rank one, then since $\langle u\rangle \neq \langle v\rangle$, $\sum_{i=1}^{k+1} \alpha_i x_i$ must equal $\sum_{i=1}^{k+1} \alpha_i y_i$. If

$\alpha_{i+1} = 0$, then $\langle X_1,\ldots,X_k\rangle$ contains a rank one vector, contrary to hypothesis. Therefore we can assume $\alpha_{i+1} \neq 0$. Obviously $\alpha_{k+1}x_{k+1} + \sum_{i=1}^{k+1} \alpha_i x_i$ and $\sum_{i=1}^{k+1} \alpha_i y_i$ are linearly independent by the

choice of x_{k+1}. This implies H is not maximal. Therefore, we can assume $\langle x_1,\ldots,x_k,y_1,\ldots,y_k\rangle = V$.

Now we can renumber the X_i and the corresponding x_i and y_i such that $x_1,\ldots,x_k,y_1,\ldots,y_s$ is a basis of V (where $k+s = m$). After this has been done we can relabel $y_1,\ldots,y_s$ by $x_{k+1},\ldots,x_m$ and the representation becomes:

$$\begin{aligned} X_1 &= u\otimes x_1 + v\otimes x_{k+1} \\ &\vdots \\ X_s &= u\otimes x_s + v\otimes x_m \\ X_{s+1} &= u\otimes x_{s+1} + v\otimes y_{s+1} \\ &\vdots \\ X_k &= u\otimes x_k + v\otimes y_k \quad . \end{aligned}$$

We can also assume that $\langle y_{s+1}, \ldots,y_k\rangle \subset \langle x_1,\ldots,x_k\rangle$. For, if not, we can replace X_i; $i = s+1,\dots,k$ by X_i' as follows.

Suppose $y_i = y_i' + \sum_{i=k+1}^{m} \alpha_i x_i$ where $y_i' \in \langle x_1,\dots,x_k\rangle$;

let $X_i' = X_i - \sum_{j=1}^{s} \alpha_{k+j} X_j$ and we get

$$X_i' = u\otimes(x_i - \sum_{j=1}^{s}\alpha_{k+j}x_j) + v\otimes y_i' \quad \text{for } i = s+1,\dots,k.$$

Clearly $\langle x_1,\dots,x_s,x_{s+1}',\dots,x_k',x_{k+1},\dots,x_m\rangle = V$ and $\langle y_{s+1}',\dots,y_k'\rangle \subset \langle x_1,\dots,x_s,x_{s+1}',\dots,x_k'\rangle$. We can now drop the primes.

Suppose some $x_j \in \{x_1,\dots,x_s\}$ is not in $\langle y_{s+1},\dots,y_k\rangle$, then we can set $X_{k+1} = u\otimes x_{k+i} + v\otimes x_j$, where $j \neq i$.

$$\sum_{i=1}^{k+1} \alpha_i X_i = u\otimes(\sum_{i=1}^{k}\alpha_i x_i + \alpha_{k+1}x_{k+i}) + v\otimes(\sum_{i=1}^{s}\alpha_i x_{k+i} + \sum_{i=s+1}^{k}\alpha_i y_i + \alpha_{k+1}x_j).$$

If $\alpha_{k+1} = 0$, then we are back to $X_1,\dots,X_k$, and thus have a rank two vector. Therefore we can assume $\alpha_{k+1} \neq 0$, in which case $\alpha_{k+1}x_j \neq 0$ implies $\alpha_j = \alpha_{k+1}$, since x_j appears in $\sum_{i=1}^{k}\alpha_i x_i$ as $\alpha_j x_j$. This implies, however, that x_{k+j} has coefficient $\alpha_{k+1} \neq 0$ in $\sum_{i=1}^{s}\alpha_i x_{k+i}$ and x_{k+j} does not appear in $\sum_{i=1}^{k}\alpha_i x_i$.

Therefore $\sum_{i=1}^{k+1} \alpha_i X_i$ is rank two, and $\sum_{i=1}^{k+1} \alpha_i X_i = 0$ implies $\alpha_i = 0$; $i = 1,\dots,k+1$. This implies H is not maximal. Hence we can assume $\langle x_1,\dots,x_s\rangle \subseteq \langle y_{s+1},\dots,y_k\rangle$. By renumbering $X_{s+1},\dots,X_k$, we can assume $\langle y_{s+1},\dots,y_{2s}\rangle = \langle x_1,\dots,x_s\rangle$ and

$x_i = \sum_{j=1}^{s} \alpha_{ij} y_j$ for $i = 1,\ldots,s$. Let $X'_{s+i} = \sum_{j=1}^{s} \alpha_{ij} X_{s+j}$ for $i = 1,\ldots,s$. Then $X'_{s+i} = u \otimes \sum_{j=1}^{s} \alpha_{ij} x_{s+j} + v \otimes x_i = u \otimes x'_{s+i} + v \otimes x_i$ for $i = 1,\ldots,s$. Again, since (α_{ij}) is non-singular, $\langle x_1,\ldots,x_s,x'_{s+1},\ldots,x'_{2s},x_{2s+1},\ldots,x_k,x_{k+1},\ldots,x_m\rangle = V$. Dropping the primes, the representation now becomes:

$$\begin{aligned}
X_1 &= u\otimes x_1 + v\otimes x_{k+1}\\
&\vdots\\
X_s &= u\otimes x_s + v\otimes x_m\\
X_{s+1} &= u\otimes x_{s+1} + v\otimes x_1\\
&\vdots\\
X_{2s} &= u\otimes x_{2s} + v\otimes x_s\\
X_{2s+1} &= u\otimes x_{2s+1} + v\otimes y_{2s+1}\\
&\vdots\\
X_k &= u\otimes x_k + v\otimes y_k\ .
\end{aligned}$$

Now we can repeat the arguments. First, we adjust $X_{2s+1},\ldots,X_k$ to make sure $\langle y_{2s+1},\ldots,y_k\rangle \subset \langle x_{s+1},\ldots,x_k\rangle$. Then we check $x_{s+1},\ldots,x_{2s}$ to see if $\langle x_{s+1},\ldots,x_{2s}\rangle \subseteq \langle y_{2s+1},\ldots,y_k\rangle$. If not, we can set $X_{k+1} = u\otimes x_{k+i} + v\otimes x_{s+j}$, where $j \neq i$, and again show that H is not maximal. If $\langle x_{s+1},\ldots,x_{2s}\rangle \subseteq \langle y_{2s+1},\ldots,y_k\rangle$ then by renumbering $X_{2s+1},\ldots,X_k$, we can assume $\langle y_{2s+1},\ldots,y_{3s}\rangle = \langle x_{s+1},\ldots,x_{2s}\rangle$. By altering $X_{2s+1},\ldots,X_{3s}$, we can obtain the representation:

$$X_1 = u\otimes x_1 + v\otimes x_{k+1}$$

$$\vdots$$

$$X_s = u\otimes x_s + v\otimes x_m$$

$$X_{s+1} = u\otimes x_{s+1} + v\otimes x_1$$

$$\vdots$$

$$X_{2s} = u\otimes x_{2s} + v\otimes x_s$$

$$X'_{2s+1} = u\otimes x'_{2s+1} + v\otimes x_{s+1}$$

$$\vdots$$

$$X'_{3s} = u\otimes x'_{3s} + v\otimes x_{2s}$$

$$X_{3s+1} = u\otimes x_{3s+1} + v\otimes y_{3s+1}$$

$$\vdots$$

$$X_k = u\otimes x_k + v\otimes y_k .$$

We note that at each step if there is not a sufficient number of y's remaining, then some x_j is not in the space spanned by the remaining y's, and we can construct X_{k+1}. After a finite number of steps, each time considering the next sx_j's, we can find the required x_j. The representation becomes:

$$
\begin{aligned}
X_1 &= u\otimes x_1 + v\otimes x_{k+1}\\
&\vdots\\
X_s &= u\otimes x_s + v\otimes x_m\\
X_{s+1} &= u\otimes x_{s+1} + v\otimes x_1\\
&\vdots\\
X_{k-s} &= u\otimes x_{k-s} + v\otimes x_{k-2s}\\
&\vdots\\
X_{k-s+1} &= u\otimes x_{k-s+1} + v\otimes x_{k-2s+1}\\
&\vdots\\
X_k &= u\otimes x_k + v\otimes x_{k-s}\ .
\end{aligned}
$$

Choose $X_{k+1} = u\otimes x_{k+1} + v\otimes x_{k-s+2}$. Suppose

$$\sum_{i=1}^{k+1} \alpha_i X_i = u\otimes\Big(\sum_{i=1}^{k}\alpha_i x_i + \alpha_{k+1}x_{k+1}\Big) + v\otimes\Big(\sum_{i=1}^{s}\alpha_i x_{k+i} + \sum_{i=1}^{k-s}\alpha_{s+i}x_i + a_{k+1}x_{k-s+2}\Big),$$ has rank 1.

We can assume $\alpha_{k+1} \neq 0$. By comparing the coefficients of x_{k-s+2} in the two terms we conclude that $\alpha_{k-s+2} \neq 0$. Similarly, it follows that $\alpha_{k-2s+2},\ldots \alpha_2$ are all non-zero. This gives x_{k+2} a non-zero coefficient in the second term, and x_{k+2} does not appear in $\sum_{i=1}^{k}\alpha_i x_i$. Hence at each stage we have a rank two element and $X_1,\ldots,X_{k+1}$ are linearly independent. This implies H is not maximal. We conclude that $k = m-1$.

A similar argument gives the corresponding theorem for type two spaces:

Theorem 2.10: Every maximal subspace of type two has dimension n-1.

This concludes the characterization of the subspaces contained in $R_2(U \otimes V)$.

Further research could be done in extending the work of chapter two to the general rank k case. Another direction for research would lie in characterizing linear transformations of $U \otimes V$ which send rank two elements into rank two elements. In such an investigation, maximal subspaces would play an important role in the determination of the types of transformations allowed.

BIBLIOGRAPHY

[1] Paul R. Halmos, Finite-Dimensional Vector Spaces, 2nd edition, The University Series in Undergraduate Mathematics, D. Van Nostrand Co., Inc., Princeton, New Jersey, 1958.

[2] Marvin Marcus and B.N. Moyls, Transformations on Tensor Product Spaces, Pacific Journal of Mathematics, Vol. 9, No. 4, (1959), 1215-1221.