PSEUDOCONVEXITY AND THE ENVELOPE OF HOLOMORPHY
FOR FUNCTIONS OF SEVERAL COMPLEX VARIABLES

by

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ABSTRACT

We first handle some generalizations from the theory of functions of a single complex variable, including results regarding analytic continuation.

Several "theorems of continuity" are considered, along with the associated definitions of pseudoconvexity, and these are shown to be equivalent up to a special kind of transformation. By successively applying a form of analytic continuation to a function $f$, a set of pseudoconvex domains is constructed, and the union of these domains is shown to be the envelope of holomorphy of $f$. 
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INTRODUCTION

The theory of several complex variables is about seventy years old. A relatively dormant period was reached when scholars found that their generalizations from the theory of a single complex variable began to fail, notable exceptions being discovered principally by F. Hartogs about 1910. Some striking differences in the theories are listed. Except for the case $n=1$, for $f(z_1, z_2, \ldots, z_n)$ in $D \subset \mathbb{C}^n$, we have in general

(i) holomorphic functions do not produce conformal maps (rather the result is a "pseudoconformal" map, in which only the projection onto $\mathbb{C}^1$ is necessarily conformal).

(ii) holomorphic functions do not have isolated zeros (or singularities).

(iii) not every domain $D \subset \mathbb{C}^n$ is a domain of holomorphy.

In particular, we also have except for $n=1$,

(iv) the boundary of a domain of holomorphy is always connected.

(v) the exterior of a bounded set cannot be a domain of holomorphy.

About twenty years later, due to the work of H. Behnke, H. Cartan, and P. Thullen on domains, and the contribution by
S. Bergman of the invariant metric and the "kernel function", the theory branched off in several new directions and became recognized as being independent. However, another period of dormancy was reached; this time the restriction was the old-fashioned machinery mathematicians were using in an attempt to solve contemporary problems. The most recent breakthrough is the language of "fibre-bundles", which is now being applied successfully to the study of "complex spaces", which are the extensions to \( n \) complex dimensions of the familiar Riemann surfaces.

We define \( f(z) = f(z_1, z_2, \ldots, z_n) \) to be "holomorphic" in \( D \subset \mathbb{C}^n \) iff at all \( z^{(0)} \in D \), \( \frac{\partial f}{\partial z_k} \) exists, and \( f \) satisfies the Cauchy-Riemann equations

\[
\frac{\partial f}{\partial \bar{z}_k} = 0 \quad k = 1, 2, \ldots, n.
\]

Observe that \( \frac{\partial}{\partial \bar{z}_k} = \frac{1}{2} (\frac{\partial}{\partial x_k} + i \frac{\partial}{\partial y_k}) \). Let

\[ \Phi(D) = \{ f : f(z) \text{ is holomorphic in } D \} \]

Equivalently, \( f \in \Phi(D) \iff f \) can be developed into a multiple power series

\[
f(z) = \sum_{k_1, k_2, \ldots, k_n = 0} a_{k_1, k_2, \ldots, k_n} (z_1 - z_1^{(0)})^{k_1} (z_2 - z_2^{(0)})^{k_2} \ldots (z_n - z_n^{(0)})^{k_n}
\]
that converges uniformly in a neighbourhood of \( z^{(0)} \) for any choice of \( z^{(0)} \in D \).

One of the early generalizations was Cauchy's integral formula, namely that if \( f(z) \) satisfies the Cauchy-Riemann equations for \( z \in \mathbb{C} \), then

\[
f(z) = \frac{1}{2\pi i} \oint_C \frac{f(t)}{t-z} \, dt,
\]

where \( C \) is a 'contour' (simple, closed, rectifiable, oriented). The generalization yields the following:

If \( f(z) \) is holomorphic in a neighbourhood of the product domain \( D_1 \times D_2 \times \cdots \times D_n \), then

\[
f(z) = \left( \frac{1}{2\pi i} \right)^n \oint_{\partial D_1} \oint_{\partial D_2} \cdots \oint_{\partial D_n} \frac{f(t)}{(t_1-z_1)(t_2-z_2)\cdots(t_n-z_n)} \, dt_1 \, dt_2 \cdots dt_n.
\]

We state without proof the following fundamental theorem, the proof of which depends basically on the above formula.

**Theorem A:** If \( f(z) = f(z_1, z_2, \ldots, z_n) \in \mathcal{O}(P) \), where

\[
P = \{ |z_1-z_1^{(0)}| < r_1, |z_2-z_2^{(0)}| < r_2, \ldots, |z_n-z_n^{(0)}| < r_n \},
\]

then

\[
f(z) = k_1, k_2, \ldots, k_n = 0 \, a_{k_1 k_2 \ldots k_n} (z_1-z_1^{(0)})^{k_1} (z_2-z_2^{(0)})^{k_2} \cdots (z_n-z_n^{(0)})^{k_n}
\]

uniquely, for all \( z \in P \), where

\[
a_{k_1 k_2 \ldots k_n} = \frac{1}{k_1! k_2! \cdots k_n!} \left( \frac{k_1 + k_2 + \cdots + k_n}{\partial z_1^{k_1} \partial z_2^{k_2} \cdots \partial z_n^{k_n}} \right) z_1^{z_1^{(0)}} z_2^{z_2^{(0)}} \cdots z_n^{z_n^{(0)}}.
\]
Moreover, \( f(\mathbf{z}) \) converges absolutely and uniformly in \( D \).

An important and very useful tool is the maximum principle.

**Theorem B**: Let \( f(\mathbf{z}) \in \mathbb{C}(D) \), \( f(\mathbf{z}) \) not constant. Then \( |f(\mathbf{z})| \) does not assume its maximum in \( D \).

**Proof**: Assume the theorem is false. Let \( \mathbf{z}^{(0)} \in \text{int} D \) be a maximal point of \( f(\mathbf{z}) \), and let \( N(\mathbf{z}^{(0)}; r) \) be a (polycylindrical) neighbourhood of \( \mathbf{z}^{(0)} \) such that \( N \subset D \). Now the holomorphic functions

\[
f(\mathbf{z}_1, \mathbf{z}_2^{(0)}, \ldots, \mathbf{z}_n^{(0)}), f(\mathbf{z}_1^{(0)}, \mathbf{z}_2, \mathbf{z}_3^{(0)}, \ldots, \mathbf{z}_n^{(0)}), \ldots f(\mathbf{z}_1^{(0)}, \ldots, \mathbf{z}_{n-1}^{(0)}, \mathbf{z}_n^{(0)})
\]

of a single complex variable must then be constant, for they assume their maxima at the centres of the composite disks of \( N \). But this implies that \( f(\mathbf{z}) \) is constant in \( N \) with respect to all of \( \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n \), by Theorem A. We assert that \( f(\mathbf{z}) \) must have been constant, contrary to our assumption. For if there were a point \( \mathbf{z}^{(1)} \neq \mathbf{z}^{(0)} \) in \( D \), by the connectivity of \( D \) we can construct an arc from \( \mathbf{z}^{(0)} \) to \( \mathbf{z}^{(1)} \) and cover it with polycylindrical neighbourhoods \( \{N_{\lambda}\} \). By the Heine-Borel theorem there exists a finite subcovering \( \{N_j\} \) which we may choose to overlap in such a way that each neighbourhood contains the centre of the previous. From our earlier result, \( f(\mathbf{z}) \) is constant in all these neighbourhoods by construction, and
thus $f(a^{(0)}) = f(a^{(1)})$. Since our choice of $a^{(1)}$ was arbitrary, the contradiction follows.

**Corollary:** $|f(a)|$ assumes its maximum on $\partial D$, the boundary of $D$, provided the function is continuous in $\bar{D}$, the closure of $D$.

**Proof:** Weierstrass' theorem on the maximum of a continuous function.

This thesis is concerned with the "envelope of holomorphy" of a given (univalent) domain $D$, which is the smallest domain of holomorphy containing $D$. The generalization of the convex envelope from $n$ real dimensions leads us to consider non-univalent regions. This is analogous to continuing a holomorphic function of one complex variable and getting a multiple-valued function as the result. We can make the continuation single-valued by considering the associated Riemann surface. We may allow complex manifolds to be envelopes, but examples show they will often be non-univalent. Hence there are no norms, and we must find a more general definition for a pseudoconvex region (equivalent to a region of holomorphy), preferably not dependent on the concept of "plurisubharmonic function", to be defined later.

From the familiar unit disk of complex analysis, we can branch off in our generalization in one of two ways. We could take as our basic domain the polycylinder, which in fact
we do in this paper for some of the results, or we could choose our domain to be the \( n \) - ball. Modern theories have extended the polycylinder to the analytic polyhedron, and the \( n \) - ball to the pseudoconvex domain. The latter parts of this thesis are concerned with the second extension.
PSEUDOCONVEXITY AND THE ENVELOPE OF HOLOMORPHY

We begin by recalling some definitions from complex analysis, and by stating some results and definitions which are directly relevant to the topic.

The space under consideration is \( \mathbb{C}^n = \mathbb{C}^n \), consisting of \( n \)-tuples of complex variables \( z = (z_1, z_2, \ldots, z_n) \). The topology will be that generated by the euclidean norm; all topologies generated by arbitrary norms are equivalent. A "region" is an open set, and a "domain" is a connected region. We let \( \Omega(D) \) denote the convex hull, and \( E(D) \) the "envelope of holomorphy" of the domain \( D \). \( E(D) \) is the largest domain to which every function holomorphic in \( D \) may be continued. The envelope of holomorphy has become useful to theoretical physicists in the derivation of dispersion relations of quantum field theory.

We will call \( D \) a "domain of holomorphy" iff \( D = E(D) \), and some equivalent definitions are clear. \( H \) is a "region of holomorphy" iff there exists a function holomorphic in \( H \) but not in any larger region. A "Riemann domain" is a pair \( \langle X, h \rangle \), where \( X \) is a topological space and \( h : X \to \mathbb{C}^n \) is a local homeomorphism. We will restrict ourselves to univalent (schlicht) domains, with occasional reference to non-univalent domains. A sufficient condition for the existence of \( E(D) \) is that \( D \) be a Riemann domain.
E(D) has been determined only for certain special domains, the definitions of which follow.

I. The "polycylinder" is probably the most commonly encountered domain in dealing with functions of several complex variables. It is represented by
\[ \{ z : \|a_1\| < r_1, \|a_2\| < r_2, \ldots, \|a_n\| < r_n \} \]. This domain is the product of n disks
\[ \{ z_1 : \|z_1\| < r_1 \} \times \{ z_2 : \|z_2\| < r_2 \} \times \cdots \times \{ z_n : \|z_n\| < r_n \} \].

II. The "hypersphere" is well known. It can be represented as
\[ \{ z : \|z_1\|^2 + \|z_2\|^2 + \cdots + \|z_n\|^2 < r \} \].

III. A "circular" domain \( D_{\mathbb{C}^n} \) with centre \( (a_1, a_2, \ldots, a_n) \) contains with each \( z^{(0)} \in D \) the points
\[ (a_1 + (z_1^{(0)} - a_1)e^{i\theta_1}, a_2 + (z_2^{(0)} - a_2)e^{i\theta_2}, \ldots, a_n + (z_n^{(0)} - a_n)e^{i\theta_n}) \in D, \]
where \( \theta_1, \theta_2, \ldots, \theta_n \) are real and \( 0 \leq \theta_k \leq 2\pi, \ k = 1, 2, \ldots, n. \)
From the definition we observe that this domain admits the automorphisms
\[ z_k = (z_k^{(0)} - a_k)e^{i\alpha_k} + a_k, \quad 0 \leq \alpha_k \leq 2\pi, \quad 1 \leq k \leq n. \]

IV. The "Hartogs" domain is given by
\[ \{(z, w) : z \in D, \ r(z) < |w - w_0| < R(z) \} \] where
\( D_{\mathbb{C}^n}, \ w \in \mathbb{C} \), \( r(z), R(z) > 0 \). In its most general form this domain admits the automorphisms
\[ \overline{a} = a \]
\[ w^* = (w-w_0)e^{i\theta} + w_0 \quad \theta \text{ real.} \]

V. An open set \( \mathcal{K} \subset \mathbb{C}^n \) is called a "tube" if there is an open set \( \mathcal{K} \subset \mathbb{R}^n \), called the "base" of \( \mathcal{K} \), such that 
\[ \mathcal{K} = \{ \mathbf{z} : \text{Re} \mathbf{z} \in \mathcal{K} \} \]

VI. "Product" domains are well known also, and included here for completeness. They are simply a generalization of the polycylinder and are given by 
\[ \{ \mathbf{z} : a_1 \in D_1, a_2 \in D_2, \ldots, a_n \in D_n \} \] 
where \( D_1, D_2, \ldots, D_n \) are plane domains.

The plurisubharmonic functionals are valuable tools in the theory. A real-valued function \( V(a_1, a_2, \ldots, a_n) = V(\mathbf{a}) \) on a region \( D \subset \mathbb{C}^n \) will be called plurisubharmonic iff

(i) \(-\infty < V(\mathbf{a}) < \infty \) in \( D \)

(ii) \( V(\mathbf{a}) \) is upper semicontinuous in \( D \)

(iii) \( V(\mathbf{a}) \) is subharmonic in the intersection of every analytic plane \( \{ \mathbf{z} : \mathbf{z} = a(0) + \lambda \mathbf{a} \} \) with \( D \), where \( a \in \mathbb{C}^n \), \( \lambda \in \mathbb{C} \)

or

(iii)' \( V(a(0) + \lambda a) \) is subharmonic in \( \lambda \) on the open set of \( \lambda \in \mathbb{C} \) such that \( a(0) + \lambda a \in D \).

There is a close relation between the familiar convex
functions and domains and the above concepts. If we replace "straight line" by "analytic plane", "linear majorant" by "harmonic majorant", "R^n" by "f^n", then the plurisubharmonic functions have the same definition as convex functions. Many theorems hold for both types.

We define
\[ \mathcal{R}(f; \mathbf{a}) = \sup \{ \delta : \sum k_1 a_1 k_2 \ldots a_n < \delta, \ a \in N(0; \delta) \} \]
where \( f \in \mathcal{O}(D) \), and \( N(0; \delta) \) denotes the open \( \delta \)-ball about the origin. It is usual also to define a distance functional \( d_{D}^{(N)}(\mathbf{a}) \), which measures the distance of \( \mathbf{a} \) from the boundary \( \partial D \) of \( D \) with respect to the norm \( N \). A more rigorous definition is \( d_{D}^{(N)}(\mathbf{a}) = \sup r \), where the supremum is taken over all \( r \) such that \( \{ \mathbf{a}' : \| \mathbf{a}' - \mathbf{a} \|_N < r \} \subset D \). Where the norm is understood, we drop the superscript \( (N) \). Thus a necessary and sufficient condition for a Riemann domain \( D \) to be a domain of holomorphy is the existence of \( f \in \mathcal{O}(\mathbb{D}) \) such that
\[ \mathcal{R}(f; \mathbf{a}) = d_{D}(\mathbf{a}) \]
for all \( \mathbf{a} \in D \). A domain \( D \) will be called "pseudoconvex" iff \( -\log d_{D}(\mathbf{a}) \) is plurisubharmonic.

**Proposition 1:** \( d_{D}^{(N)}(\mathbf{a}) \) is continuous with respect to the topology generated by a norm \( N \) provided \( D \subset \mathbb{C}^n \).

**Proof:** Since the topology generated by the euclidean norm is equivalent to the topology generated by \( N \), we can employ the usual continuity argument. By definition, if
\[ \|\mathbf{a} - \mathbf{a}'\| < d_D(\mathbf{a}) , \text{ then } \mathbf{a} \in D . \] If \[ \|\mathbf{a} - \mathbf{a}''\| < \varepsilon , \text{ then by the triangle inequality we have } \|\mathbf{a} - \mathbf{a}'\| < \|\mathbf{a} - \mathbf{a}''\| + \varepsilon . \]

Thus for all \( \mathbf{a} \) such that \[ \|\mathbf{a} - \mathbf{a}''\| < d_D(\mathbf{a}) - \varepsilon , \text{ we have } \mathbf{a} \in D , \] and then by definition \[ d_D(\mathbf{a}) > d_D(\mathbf{a}) - \varepsilon . \] Also, by symmetry, \[ d_D(\mathbf{a}) > d_D(\mathbf{a}) - \varepsilon . \]

Therefore, \[ |d_D(\mathbf{a}) - d_D(\mathbf{a})| < \varepsilon \] if \[ \|\mathbf{a} - \mathbf{a}''\| < \varepsilon . \] If \( D = \emptyset \), then \( d_D(\mathbf{a}) = \infty \).

Another important functional is that representing the distance \( d_a(\mathbf{a}) \) of the point \( \mathbf{a} \) on \( \pi : \{ \mathbf{a}' : \mathbf{a}' = \mathbf{a} + \lambda \mathbf{a} \} \) from the boundary of \( \pi \cap D \). That is, \( d_a(\mathbf{a}) = \sup r \) where the supremum is taken over \( \{ \mathbf{a}' : \mathbf{a}' = \mathbf{a} + \lambda \mathbf{a} , \|\mathbf{a}\| = 1 , |\lambda| < r \} \subset D \). It is clear that \( d_a(\mathbf{a}) = \inf \{ d_a(\mathbf{a}) \} \) where \( \|\mathbf{a}\| = 1 \).

This brings us to a fundamental theorem for continuation of functionals.

**Proposition 2:** Let \( f \in \Phi(D) \), \( f \) a functional. Then \( f \) can be continued from \( D \) into

\[ D^* = \{ \mathbf{a} : \mathbf{a} = \mathbf{a}' + \alpha \mathbf{a} + \beta \mathbf{b} , \alpha \in \mathbb{S} , |\beta| < \rho |h(\alpha)|^{-1} \} , \text{ where } S^\perp \text{ is simply connected}, \mathbb{S} \subseteq \mathbb{D}^\perp , \text{ and } h \text{ is a nonzero functional defined and continuous on the projection of } \mathbb{S} \text{ and holomorphic in the projection of } S \text{ in the } \alpha\text{-plane and satisfying the} \]
boundary condition

\[ |h(\alpha)| \frac{d}{d \alpha, D} \langle \vec{\alpha} \rangle \geq \rho > 0 \quad \alpha \in \mathcal{S}. \]

**Proof:** The continuation of \( f \) as a function of \( \beta \) is straightforward; the difficulty is the extension to \( f \) as a function of \( \alpha \). For we can expand \( f(\vec{\alpha}(0) + \alpha \vec{a} + \beta \vec{b}) \) in powers of \( \beta \) if we fix \( \alpha \). Now from our corollary to the maximum principle we see that \( f \) converges in \( D^* \) and the expansion represents our continuation. Choose \( c \) so that \( \|c\| = 1 \), let \( \gamma \) be a complex parameter, and let \( B = \{ \vec{z} : \vec{z} = \vec{z}(0) + \alpha \vec{a} + \beta \vec{b} + \gamma_c \} \). Consider the domain \( D_1 = \{ \vec{z} : \vec{z} \in B, |\beta| < (\rho - \varepsilon) |h(\alpha)|^{-1}, \alpha \in \mathcal{S}, |\gamma| < \delta \} \).

Then given \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( D_1 \subset D \). Moreover, \( D_1 \) is relatively compact in \( B \) (we will write \( D_1 \subset B \)), and \( f|_B \) is holomorphic with respect to \( \alpha, \beta, \gamma \), and by Hartogs' theorem (Caratneodory [1]), it is bounded by some number \( M(c) \).

Expanding \( f|_{D_1} \) in a power series in \( \beta \) and \( \gamma \), we have

\[ f|_{D_1}(\vec{z}) = \sum_{J, K=0}^{\infty} \frac{1}{J! K!} \left( \frac{\partial^J f(\vec{z}(0) + \alpha \vec{a} + \beta \vec{b} + \gamma \vec{c})}{\partial \beta^J \partial \gamma^K} \right)_{\beta=0}^{\gamma=0} \beta^J \gamma^K, \]

subject to the restrictions on \( D_1 \). Applying Cauchy's inequality, a straightforward generalization of the formula for functions of a single complex variable, we deduce

\[ |h(\alpha)|^{-1} \frac{1}{J! K!} \left| \frac{\partial^J f|_{D_1}(\vec{z})}{\partial \beta^J \partial \gamma^K} \right|_{\beta=0}^{\gamma=0} \leq \frac{M(c)}{\alpha \in \mathcal{S}}. \]
The left-hand side assumes its maximum with respect to $S$ on $\partial S$ and thus the inequality is valid for $\alpha \in \bar{S}$. Hence we have uniform convergence in every compact subset of $D_2$, which is $D_1$ with the condition $\alpha \in \partial S$ replaced by $\alpha \in S$, and we have continued $f|_{D_1}(\mathbf{z})$ into $D_2$.

Since our choice of $c$ was arbitrary up to its norm, we let $c$ range through all values of $\mathbf{z}^n$, and $\epsilon \to 0$. Thus we have imposed holomorphy on $f$ in any neighbourhood of $D^*$. (A routine calculation shows our new definition of $f(\mathbf{z})$ is consistent). $f$ is single-valued in $D^*$ since $D^*$ is simply-connected, and the theorem is proved. $f$ must be holomorphic at $\mathbf{z}(\alpha) \in D^*$ since it is holomorphic in a neighbourhood of $\mathbf{z}(\alpha)$ when restricted to an arbitrary analytic plane through $\mathbf{z}(\alpha)$.

**Corollary:** $f$ a functional, $f_{\infty}(D) \Rightarrow f_{\infty}(D')$, where $D' = \{ \mathbf{z} : \|\mathbf{z}(\alpha)\| + \| \mathbf{z} - \mathbf{z}(\alpha) \| < \rho(1(\alpha))^{-1}, \alpha \in S \}$ and the conditions of the theorem hold with replacement of $d_{a,D}(\mathbf{z})$ by $d_D(\mathbf{z})$.

We say $D$ is "holomorph-convex" if $D \subset X$ as a compact subset, where $\langle X, h \rangle$ is a Riemann domain, and there exists $f_{\infty}(X)$ such that $|f(\mathbf{z})| > \| f \|_{D, \mathbf{z} \in D} \sup_{\mathbf{z} \in D} |f(\mathbf{z})|$ for all $\mathbf{z} \in X$. If $X = \bigcup_{n=1}^{\infty} D_n$, $D_n$ is holomorph-convex for all $n$, and $D_n \supseteq D_{n+1}$, we say $X$ is holomorph-convex. A holomorph-convex Riemann domain is a domain of holomorphy.
An important result of relatively recent date is the following: \( H \) is a (univalent) region of holomorphy iff it is pseudoconvex. The more difficult necessary condition was proved for \( n \) complex dimensions by K. Oka in 1954. It is this equivalence theorem which underlies and motivates this paper. Closely related and also fundamental to the theory is the Kontinuitätssatz, the name of which is not translated to prevent confusion with other "theorems of continuity." The proof of the Kontinuitätssatz is given in the appendix for completeness and as an illustration of techniques encountered. Each of the various continuity theorems have proved valuable in certain applications. We will show that they are often convenient rephrasings of the same theorem. These theorems are important to us because our definition of a pseudoconvex domain depends directly upon them.

In the following, when we speak generally of a "projection", we mean onto some fixed subspace of \( \mathbb{C}^n \). Domains \( D_1, D_2 \) will be called "equivalent", \( D_1 \sim D_2 \), if there is a one-one bicontinuous mapping between the points of \( D_1 \) and \( D_2 \) such that the corresponding points have the same projection on a space \( \mathbb{C}^m \). \( D_2 \) will be said to be "contained in \( D_1 \) in the sense of Behnke-Thullen", \( D_2 \subset D_1 \), if there is a one-one continuous map between the points of \( D_2 \) and a particular subset of \( D_1 \) which has the same projection as \( D_2 \). We will assume our domains are bounded and contain no singular points.
Property 1: Divide $\phi^n_a$ so that $(z_1, z_2, \ldots, z_n)$ becomes $(w_1, w_2, \ldots, w_{n-1}, \tilde{w})$, and let $(x, y)$ be the coordinates of $P \in \partial D$. Then to $P$ there corresponds $p > 0$ so that if $S = \{(w, \tilde{w}) : w_i = x_i \quad i = 1, 2, \ldots, n-1, 0 < |w - y| < p\}$, there exists a set $S^*$ in a neighbourhood of $P$ such that $p(S^*) = S$ (the operator $p$ indicates projection). Given $\delta > 0$ such that $\delta < p$, there exists $r > 0$ such that if $A = \{z : |w_i - x_i| < r \quad i = 1, 2, \ldots, n-1, |\tilde{w} - y| = \delta\}$, then there exists $A^*$ contained in a unique univalent subdomain of $D$ such that $p(A^*) = A$, and $S^* \cap A^* \neq \emptyset$.

Property 2: To the point $P$ there corresponds a pair of positive numbers $\delta, \delta'$ where $\delta < r$, in such a way that to all points $w'$ of the $(n-1)$-polycylinder $|w'_i - x_i| < \delta$ $(i = 1, 2, \ldots, n-1)$ there corresponds at least one point $\tilde{w}'$ on the circumference $|\tilde{w}' - y| = \delta'$ satisfying the following condition:

Let $P(x, y)$ be the point of $A^*$ in $C^n_{w', \tilde{w}}$, let $s$ be the segment joining the points $\tilde{w}'$ and $y$, and let $L$ be the segment in $\phi^n_{w', \tilde{w}}$ of the form $(w', s)$. Then if we trace a curve in $D$ from $P$ and such that the projection is always on $L$, we must necessarily come to a point of $A^*$.

In the literature, a domain has been said to satisfy the theorem of continuity if it satisfied one of the following conditions.
Continuity Condition 1 (CC1):

Let $D \subset \mathbb{C}^n$, $P \in \mathbb{D}$. Then $D$ satisfies CC1, $D \subset CC1$, if whenever all points of $\partial D$ have Property 1 they have Property 2.

If $D \subset \mathbb{C}^n$, $D \subset CC1$, and $D \subset CC1$ after any one-one pseudoconformal transformation of $\mathbb{C}^n$ into the above-mentioned neighbourhood of $P$, then we will say $D$ is "pseudoconvex". Thus for $n=1$, all domains are pseudoconvex.

Continuity Condition 2 (CC2):

Consider the following geometrical configuration.
Let $D \subset \mathbb{C}^n$, $P=(x, y) \in \mathbb{D}$, where $(w_1, w_2, w_3, \ldots, w_{n-1}) = \hat{w}$ is a new representation of $\mathbb{C}^n$. Let $S \subset \mathbb{S}$ be a hypersphere with centre $\alpha + x$ such that $\partial S \cap \{x\} \neq \emptyset$, let $H$ be another hypersphere with centre $x$, let $\alpha$ be a linearly simply-connected domain, namely that part of $H$ outside $S$, and let $\mathbb{B}=(\alpha, y) \subset \hat{w}$. Then we will say $D$ satisfies Continuity Condition 2, $D \subset CC2$, if for arbitrary $\alpha$ and $H$ and for all $P \in \mathbb{D}$, there does not exist a point set in the neighbourhood of $P$ having projection $\mathbb{B}$.

Again we have the corresponding definition of pseudo-convexity and the observation that $D \subset \mathbb{C}^n \Rightarrow D \subset CC2$.
Continuity Condition 3 (CC3):

D is a domain of \( \mathbb{C}^n \), \( n > 1 \), satisfies CC3, \( D \in \mathcal{CC}3 \), if for all domains \( D' \) such that \( D' \sim \mathcal{D}, D' \subseteq D \), there exists \( \mathcal{L}' \) with the property \( \mathcal{L}' \sim \mathcal{L}, D' \subseteq \mathcal{L}' \subseteq D \), where \( \mathcal{D} \) is a simply connected domain consisting of the pointwise union of the two domains

\[
\mathcal{L}_1 : \quad |\tilde{w}_1 - \tilde{w}_1^{(o)}| < r, \quad \rho' < |w - w_0| < \rho
\]

\[
\mathcal{L}_2 : \quad |\tilde{w}_1 - \tilde{w}_1^{(o)}| < r, \quad |w - w_0| < \rho, \quad \text{for } i = 1, 2, \ldots, n-1,
\]

and \( \mathcal{L} = \{(w, \tilde{w}) : |\tilde{w}_1 - \tilde{w}_1^{(o)}| < r, |w - w_0| < \rho, i = 1, 2, \ldots, n-1\} \).

As usual we have made a variable modification, from \((a_1, a_2, \ldots, a_n)\) to \((w, \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{n-1}) = (w, \tilde{w})\). For \( n = 1 \) we consider all domains to satisfy CC3.

By means of CC3, our definition of pseudoconvexity assumes a slightly different form. Let \( D \subseteq \mathbb{C}^n \), \( n > 1 \), and \( P \in \partial D \). Let \( \mathcal{L} \) be a polycylinder about \( P \) of radius \( \rho \), and let \( \Delta \) be the component \( D \cap \mathcal{L} \) such that \( P \in \partial \Delta \). If for all \( P \) there is a number \( \rho_0 > 0 \) such that for all \( \rho < \rho_0 \) we have \( \Delta \in \mathcal{CC}3 \) and \( \Delta \in \mathcal{CC}3 \) for all one-one pseudoconformal transformations of \( \mathcal{L} \), we say \( D \) is "pseudoconvex".

Note that CC1, CC2 are local conditions and CC3 is global.
Domains satisfying the above continuity conditions can be shown to satisfy the Kontinuitätssatz. We will show that the three definitions of pseudoconvexity are equivalent.

**Theorem:** The three continuity conditions are equivalent up to a one-one pseudoconformal transformation.

**Proposition 3:** \( CC_1 \Rightarrow CC_2 \).

**Proof:** Let \( D \subset \Phi^n_g \) and \( D \subset CC_1 \). Let \((a_1, w) = (a_1, a_2, w_1, w_2, ..., w_{n-2})\) be a new representation of variables. In the geometrical configuration of \( CC_2 \), assume the existence of a set \( B \subset \Phi^n_g, w \) contained in some neighbourhood of \( P \in D \), and such that \( p(B) = \beta \). By means of suitable translations and rotations map \( a \rightarrow (0, 0) \) and \( x \rightarrow (R, 0) \), where \( R \) is the radius of \( S \), and the images of the other point sets involved will be designated by the same symbols. In the new space, \( \Phi^n_u \), except for the point \((R, 0)\), any point of \( u_1 = R \) satisfies \( \text{Re}(u_1) < R \), and lies outside \( S \). Now if \( B \) exists as described, then clearly for \( D \subset \Phi^n_g, w \), we cannot have \( D \subset CC_1 \). Since this is a contradiction, we have \( D \subset CC_2 \).

**Lemma:** Let \( D_1, D_2 \in CC_2 \), and let \( D_0 \subset D_1 \cap D_2 \) be a (connected) component. Then \( D_0 \in CC_2 \).

**Proof:** Let \( t \) be a one-one pseudoconformal mapping of a neighbourhood of \( P \in D_0 \) into \( \Phi^n_g \). Assume there exists a point
set with the property mentioned in CC2, with respect to \( t(P) \) and \( t(D_0) \). Note that \( t(D_0), t(D_1), t(D_2) \) are only defined in a neighbourhood of \( t(P) \). Now

\[
\{t(D_1), t(D_2) \in CC2 \Rightarrow t(P) \in \text{int } t(D_0),
\]

which is a contradiction.

**Proposition 4**: \( CC2 \Rightarrow CC3 \).

**Proof**: Let \((s_1, s_2, \ldots, s_{n-1}, w)\) be a new representation of \( \Phi^*_n \).

Consider the geometrical configuration of CC3 with DeCC2.

If there exists a domain \( D' \subset D \) such that \( D' \sim \mathcal{L} \), we must show the existence of a domain \( \mathcal{L}' \) with the property \( \mathcal{L}' \sim \mathcal{L} \) and \( D' \subset \mathcal{L}' \subset \mathcal{L} \).

We first perform a translation so that

\[
(a_0, w_0) \rightarrow (0,0)
\]

Now consider the following construction. Let

\[
\mathbf{a}^* \in \{ \mathbf{a} : |a_1| < r, 1 = 1, 2, \ldots, n-1 \}, \quad w^* \in \Theta : |w| = \rho^*,
\]

where \( \rho^* \) is the average length of \( \rho \) and \( \rho' \), and let the path from \((\mathbf{a}^*, w^*)\) to \((\mathbf{a}^*, 0)\) in \( \Phi^*_n \) be \( L \). From the initial point \( Q \in D' \), construct a continuous path in \( D \) having projection entirely on \( L \). If this path intersects \( aD \), let \( P = (\mathbf{a}^*, \tilde{w}) \) be the point of intersection. If no intersection occurs, then regardless of \( w^* \), \( \mathbf{a}^* \) lies in \( \mathcal{S}_1 : \{ \mathbf{a} : |a_1| < r_1, 1 = 1, 2, \ldots, n-1 \} \).

Suppose now there is a point

\[
\mathbf{a} \in \mathcal{S}_2 : \{ \mathbf{a} : |a_1| < r, |s_j| < r_1, j = 2, 3, \ldots, n+1 \}
\]

so that given \((\mathbf{a}, b)\),
for be\(\Theta\) there corresponds \(P(a,b_1)\). Define \(\varphi \in \Phi^n_\mathcal{S}\) by
\[
\varphi = \{(z_i) : |z_i| < r, z_i = a_i, \ i = 2, 3, \ldots, n-1\},
\]
and let \((a^*, w^*) \in (\varphi, \Theta)\).
Also define the metric \(d(a, w) = \sqrt{|k/a_1|^2 + |w|^2}\), \(k > 0\). If
\((a^*, w^*)\) corresponds to \((a^*, w^*)\), we set up another correspondence between \(d(P')\) and \((a^*, w^*)\); otherwise we make no correspondence. The set \(\{d\}\) has an upper bound, say \(d^*\), since no intersection occurs between the constructed path and \(\partial D\) for all \(a \in \mathcal{S}\).

Now \(d(a, b_1) > d_1 = \sqrt{(k/r_2)^2 + (\rho''^2)}\) when \(0 < r_2 < r\)
and \(k\) is sufficiently large. Also, \(d^* > d_1\), and \(d < d_1 < d^*\)
(when \(d\) exists), for all \((a^*, w^*) \in (\varphi, \Theta)\) such that
\(r_2 < |a^*_1| < r\). Thus \(d(x, y^*) = d^*\) for some point \((x, y) \in (\varphi, \Theta)\)
corresponding to the boundary point \(P_0 = (x, y^*)\), where
\(x_1 = a_1\), \(i = 2, 3, \ldots, n-1\).

Let \(D^*, P_0^*\) be the images of \(D, P_0\) under the
transformation of \(\Phi^n_\mathcal{S}\) to \(\Phi^n_{u, v}\) given by

\[
T : u_1 = k/a_1, \quad u_2 = a_2, \quad u_3 = a_3, \ldots, u_{n-1} = a_{n-1}, \quad v = w.
\]
Let \(S\) be a hypersphere in \(\Phi^2_{u_1, v}\) about \((0, 0)\) such that
\(P_1(k/x_1, y^*) \in \mathcal{C}\), let \(h\) be a small hypersphere centered
at \(P_1\), let \(n' = \mathcal{C}(S) \cap h\) (\(C\) indicates complement), and in
\(\Phi^n_{u, v}\) let
\[
G = (u, v) : (u_1, v) \in h', \quad u_1 = x_1, \quad i = 2, 3, \ldots, n-1\}.
\]
Now there exists \( G_1 \subset \Phi_{u,v} \) such that \( G_1 \subset N(P_o^*; \delta) \) for some \( \delta > 0 \), and \( p(G_1) = \overline{G} \) (see Property 1). Then we have arrived at a contradiction since \( P_o^* \in \partial D^* \), and \( D^* \in CC2 \) with respect to the neighborhood of \( P_o^* \). Thus there corresponds no boundary point of \( D \) for any choice of \( (\xi^*, w^*) \in (\mathbb{P}^2, \emptyset) \).

Similar reasoning applied to

\[ \exists_j : \{ \xi : |a_1| < r, |a_2| < r, |a_k| < r, k = 3, 4, \ldots, n-1 \} \]\n
and eventually to \( \exists_n : \{ \xi : |a_k| < r, k = 1, 2, \ldots, n-1 \} \) gives us the desired result; namely the existence of a domain \( \mathcal{C}' \) such that \( \mathcal{C}' \sim \mathcal{C}, D' \subset \mathcal{C} \subseteq D \).

**Proposition 5:** \( CC3 \Rightarrow CC1 \).

**Proof:** This is clear.

**Corollary:** The three definitions of pseudoconvexity are equivalent.

All holomorphic functions can be continued into their pseudoconvex envelope. In particular, the tube domain is easily handled. The following statements are equivalent. For the tube \( \mathcal{T}_X \), the pseudoconvex envelope is that tube with the convex hull of \( \mathcal{T}_X \) as its base. A tube domain is pseudoconvex iff it is convex. Or, the envelope of holomorphy of \( \mathcal{T}_X \) is \( \mathcal{C}(\mathcal{T}_X) \), its convex hull. The accepted proof of this last fact is due to S. Bochner, and is based on the expansion of the analytic function in \( \mathcal{T}_X \) in multiple Legendre polynomials. It would seem that the following lemmas lead to a more direct
Lemma: \( T_T^X \) is a tube \( \iff \) \( T_T^X \) admits the automorphisms 
\[
(*) \quad z_k' = z_k + ia_k
\]
\[k = 1, 2, \ldots, n, \quad a_k \text{ real constants.}
\]

Lemma: \( E(T_T^X) \) is a tube.

Proof: Let \( G \) be an automorphism of the type \( (*) \). Now whenever \( G(T_T^X) = T_T^X \), we have \( E(T_T^X) = G(E(T_T^X)) \) [Cartan-Thullen [1], Theorem 3, Corollary 1]. But this means \( E(T_T^X) \) is a tube, by the first lemma.

Since \( \zeta(T_T^X) \) is a domain of holomorphy, [Bochner-Martin [1], p. 91], we have \( \zeta(T_T^X) \supseteq E(T_T^X) \). The converse follows from the next lemma.

Lemma: The base of \( E(T_T^X) \) is convex.

Another approach to the proof is to observe that the conformal map \( w_k = e^{\gamma_k} \) takes \( T_T^X \) into a covering surface over a Reinhardt domain in \( \phi^1_{w_k} \). And in the logarithmic sense, the Reinhardt domain of holomorphy is convex.

G. Källen and A.S. Wightman have been working on a means of applying the Kontinuîtatsssats directly to determine the envelope of holomorphy of a given domain. Bremermann ([4]) has found an iterative procedure which is the best method to date. This is the most recent of the approximation theorems; the envelope
is determined as the limit of a sequence of pseudoconvex domains.

**Proposition 6:** Let \( \{D_n\} \) be a sequence of pseudoconvex domains in \( \mathbb{C}^n \). Then \( \lim_{n \to \infty} D_n \) is pseudoconvex if it exists and is finite.

**Proof:** Assume without loss of generality that the domains \( D_n \) are pseudoconvex in the sense of CC3. Assume also that \( n \geq 2 \), and consider the sequence \( \mathcal{S}_1 \sim D_1, \mathcal{S}_2 \sim D_1 \cap D_2, ..., \mathcal{S}_n \sim D_1 \cap D_2 \cap ... \cap D_n, ... \).

Then since the intersection of pseudoconvex domains is pseudoconvex, \( \mathcal{S}_n \) is pseudoconvex for all \( n \). By CC3 and observing that \( \mathcal{S}_n \sim \mathcal{S}_{n-1} \) for all \( n \), we have \( \Delta_1 \sim \mathcal{S}_n \) is pseudoconvex. Let \( \Delta_n \sim \mathcal{S}_{\cap \cap} \) if it exists. Then \( \Delta_n \sim \Delta_{n-1} \), and we define \( \Delta_0 \) to be a domain satisfying \( \Delta_n \sim \Delta_0 \) for all \( n \), and \( \Delta_0 \) such that \( \Delta_n \sim \Delta \) for all \( n \). \( \Delta_0 \) is called the "limit" of the sequence \( \{D_n\} \), unique up to the equivalence class.

Let \( \mathcal{L} = \{ (w, \bar{w}) : |w_i - w_1^{(0)}| < r_1, |\bar{w} - \bar{w}_0| < r_2 \} \)

\( i=1,2,...,n-1 \) be a polycylinder in \( \mathbb{C}^n \), where

\( (z_1, z_2, ..., z_n) \sim (w_1, w_2, ..., w_{n-1}, \bar{w}) \), and let \( \mathcal{L'} \) be the subset of points of \( \mathcal{L} \) satisfying at least one of the conditions

1. \( r_1' < |\bar{w} - \bar{w}_0| < r_2 \)
2. \( |w_i - w_1^{(0)}| < r_1' < r_1 \) \( i=1,2,...,n-1 \).

Assume \( \Delta_0 \) exists, and suppose there exists a domain \( \mathcal{S} \) such that \( \mathcal{S} \sim \mathcal{L'} \), and \( \mathcal{S} \subseteq \Delta_0 \). By the existence of \( \Delta_0 \) and
by its first property, there corresponds a polycylinder $\mathcal{P}$ around any $P \in \Delta_0$ and a positive integer $n'_0$ such that if $P_n \in \Delta_n$ corresponds to $P \in \Delta_0$, then $P_n$ is contained in some subdomain of $\Delta_n$ which is equivalent to $\mathcal{P}$ for all $n \geq n'_0$. Let $n_o = \min\{n'_0\}$.

Assume now that $\mathcal{P}$ is relatively compact in $\Delta_0$, $\mathcal{P} \subset \Delta_0$. The above reasoning and Borel's lemma imply the existence of a domain $\mathcal{L}^*$ such that $\mathcal{L}^* \sim \mathcal{L}$, and $\mathcal{P} \subset \mathcal{L}^* \subset \Delta_0$. Thus $\Delta_0 \in \mathbb{C}^3$.

It remains to check that $\Delta_0 \in \mathbb{C}^3$ after an arbitrary one-one pseudoconformal transformation.

Suppose such a transformation $\tau$ is made on a polycylinder $\phi \subset \Phi^n$. Consider $\tau(\phi \cap \Delta_0)$. The limit of the new sequence is $\tau(\phi \cap \Delta_0)$. Since $\tau(\phi \cap \Delta_0)$ is pseudoconvex for all $n$, then $\tau(\phi \cap \Delta_0) \in \mathbb{C}^3$. Therefore $\Delta_0$ is pseudoconvex.

The above proposition has been proved by H. Behnke and K. Stein for regions of holomorphy; that is, the limit of a sequence of regions of holomorphy is also a region of holomorphy.

**Lemma**: Let $V(\overline{z})$ be the largest plurisubharmonic minorant in the domain $D$ of the function $-\log d_D(\overline{z})$, let

$$D^* = \bigcup_{\overline{z} \in D} \{\overline{z}': \|\overline{z}' - \overline{z}\|e^{-V(\overline{z})}\},$$

and assume $E(D)$ is univalent. Then if $f(\overline{z}) \in \mathcal{O}(D)$, it can be continued from
D into $D^*$.

**Proof:** We assume $D = D^n$ (otherwise the statement is trivial). Then $-\log d_D(\bar{z}) > -\infty$ for all $\bar{z} \in D$. Let $S = \{ f : f$ is plurisubharmonic in $D , |f(\bar{z})| \leq -\log d_D(\bar{z}), \bar{z} \in D \}$. Bremermann ([7]) has proved the plurisubharmonicity of the upper envelope $U(\bar{z}) = \lim \sup_{\bar{z} \to \bar{z}'} \sup_{\alpha} \{ U_{\alpha}(\bar{z}') \}$ of a family $\{ U_{\alpha} \}$ of functions plurisubharmonic in a region $D$ and bounded in every closed subregion of $D$. In our case we have, then, by our definition of $V$, $V(\bar{z}) = \sup_{\bar{z} \to \bar{z}'} \sup_{\alpha} f(\bar{z}')$. Let us indirectly define $d(\bar{z})$ by $V(\bar{z}) = -\log d(\bar{z})$. As shown in the above-mentioned article, the function $V^*(\bar{z}) = -\log d_{E(D)}(\bar{z})$ is plurisubharmonic in $D$.

Now we have $V^*(\bar{z}) \leq -\log d_D(\bar{z})$ since $d_{E(D)}(\bar{z}) \geq d_D(\bar{z})$. Therefore $V^*(\bar{z}) \leq V(\bar{z})$, and $d(\bar{z}) \leq d_{E(D)}(\bar{z})$. Thus if $D^* = \bigcup_{\bar{z} \in E(D)} \{ \bar{z} : ||\bar{z}' - \bar{z}|| < d(\bar{z}) \}$, we have $D^* \subseteq E(D)$. But $E(D)$ was assumed to be univalent. Thus any $f \in \phi(D)$ has a single-valued holomorphic continuation to $E(D)$, and therefore to $D^*$.

Our next result shows in fact that the envelope of holomorphy is the limit of an increasing sequence of domains. One way to get such a sequence is by repeated application of the lemma. The sequence is $D, D^* = D_1, (D^*)^* = D_2, D_3, D_4, \ldots$. This construction is called "Bremermann's i-process".
The proof of the next theorem illustrates that when we deal with the union of the domains of such a sequence, the limit (the envelope of holomorphy) need not even be finite.

**Proposition 7:** \( \bigcup_{n=1}^{\infty} D_n = \mathbb{E}(D) \).

**Proof:** Let \( U = \bigcup_{n=1}^{\infty} D_n \). As usual, to insure \( -\log d_U(\mathbf{z}) > -\infty \), we assume \( U \neq \emptyset \), the result being trivial otherwise. Let \( \mathbf{u}_0 \in U \).

Choose \( n_0 \) a positive integer and a neighbourhood \( N(\mathbf{u}_0; \delta) \) so that \( N \subset D_n \) for all \( n > n_0 \). Now

\[
D_{n+1} = \bigcup_{\mathbf{z} \in D_n} \{ \mathbf{z}' : ||\mathbf{z}' - \mathbf{z}|| < e^{-N_n(\mathbf{z})} \}
\]

by the lemma, where \( N_n(\mathbf{z}) \) is the largest plurisubharmonic minorant in \( D_n \) of \( -\log d_{D_n}(\mathbf{z}) \).

But this implies \( -\log d_{D_{n+1}}(\mathbf{z}) \leq N_n(\mathbf{z}) \leq -\log d_D(\mathbf{z}) \) for all \( \mathbf{z} \in N, n > n_0 \). Also, \( \lim_{n \to \infty} N_n(\mathbf{z}) = -\log d_U(\mathbf{z}) \) for all \( \mathbf{z} \in N \), by virtue of \( \lim_{n \to \infty} -\log d_{D_n}(\mathbf{z}) = -\log d_U(\mathbf{z}) \). Furthermore, \( N_{n+1}(\mathbf{z}) \leq N_n(\mathbf{z}) \) for all \( \mathbf{z} \in N \). Now we see that \( \lim_{n \to \infty} N_n(\mathbf{z}) \) is plurisubharmonic in \( N \), if we recall that the result holds for subharmonic functions in \( \Omega \), and apply (iii)' of the definition of a plurisubharmonic function. We have then that

\[
-\log d_U(\mathbf{z}) \text{ is plurisubharmonic in } N.
\]

But by the generality of our choice of \( \mathbf{u}_0 \) and \( N \) in \( U \), we have that \( -\log d_U(\mathbf{z}) \) is plurisubharmonic for all \( \mathbf{z} \in U \). Therefore \( U \) is pseudoconvex. But \( D_n \subset \mathbb{E}(D) \) for all \( n \) (see proof of lemma). Therefore \( U \subset \mathbb{E}(D) \), and since \( \mathbb{E}(D) \) is the smallest pseudoconvex region containing \( D \), \( \mathbb{E}(D) \subset U \). Thus \( U = \mathbb{E}(D) \) and the proposition is proved.
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