ON COVERING SYSTEMS

by

Donald James Mallory

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ABSTRACT

This paper is concerned with the relationships between certain covering systems useful for set differentiation and with their application to density theorems and approximate continuity.

The covering systems considered are the Vitali systems (which we call V-systems), the systems introduced by Sion (which we call S-systems), and a modification of the tile systems (which we call T-systems).

It is easily checked from the definitions that V-systems are S-systems, and under slight restrictions, T-systems. We show also that under certain conditions S-systems are T-systems, and that in general the converses do not hold.

Density theorems and the relationships between approximate continuity and measurability of functions are discussed for these systems.

In particular, we prove that for T-systems measurable functions are approximately continuous and hence a density theorem holds.

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1. INTRODUCTION

Differentiation of two set functions μ, ν at a point x involves taking the limit of the ratio $\frac{\mu A}{\nu A}$ as A "converges" to x in some sense which requires that A belong to some family of sets $N(x)$ (e.g. spheres with center x). For the development of a reasonable theory of differentiation certain restrictions must be placed on the families $N(x)$. The best known restriction is that they form a Vitali system. However, other systems have been considered.

In this paper we study the relationships between three systems: Vitali systems, which we call V-systems; the systems having property (V) introduced by Sion in [4][1], which we call S-systems; and a modification of the tile systems studied in Hahn and Rosenthal [2], which we call T-systems. The main difference between tile systems and T-systems is that sets in the first are assumed to be measurable whereas in the latter they need not be.

The main results in section 3 state that V-systems are S-systems; under certain conditions, V-systems are T-systems; under more stringent conditions, S-systems are T-systems. The converses in general do not hold.

In Section 4, we prove that for T-systems, measurable functions are approximately continuous and apply this result to obtain a density theorem. This generalizes similar results for tile systems and parallels similar results for S-systems.

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[1]Numbers in brackets refer to the bibliography at the end of the paper.

2. NOTATION AND TERMINOLOGY

The following notation and terminology will be used throughout this paper.

.1 ω is the set of all integers greater than zero.

.2 $(A \sim B) = \{x : x \in A,\ x \notin B\}$

.3 $\sigma F = \bigcup_{\alpha \in F} \alpha$

.4 μ is a (outer) measure on X iff μ is a function defined on all subsets of X, and

(i) $\mu(0) = 0$

(ii) $0 \leq \mu A \leq \sum_{n \in \omega} \mu Bn$

if $A \subset \bigcup_{n \in \omega} Bn \subset X$

.5 For μ a measure on X a set A is μ-measurable iff for every $B \subset X$

$$\mu B = \mu(B \cap A) + \mu(B \sim A)$$

.6 $f^{-1} V = \{x : f(x) \in V\}$

.7 For μ a measure on X, a function f on X to a topological space Y is μ-measurable iff, for every open $V \subset Y$, $f^{-1} V$ is a μ-measurable set.

.8 For X a topological space and $x \in X$, nbhdx denotes the family of all open sets α such that $x \in \alpha$.

3. COVERING SYSTEMS

In this section, X is a topological space and μ is a measure on X. In the next set of definitions we introduce the three types of covering systems that will be compared: a V-system is one having the Vitali property, an S-system is the one introduced by Sion in [4], and a T-system is a modification of the tile system discussed by Hahn and Rosenthal [2].

3.1 <u>Definitions</u>.

.1 N is a covering system iff N is a function on X such that, for every $x \in X$, $N(x) \subset$ nbhdx and for every $U \in$ nbhdx there exists $\alpha \in N(x)$ with $\alpha \subset U$.

.2 For every $A \subset X$, $\overline{N}(A)$ is the collection of all families F such that

(i) $F \subset \bigcup_{x \in A} N(x)$

(ii) for every $x \in A$ and $U \in$ nbhdx there exists $\alpha \in F$ with $x \in \alpha \subset U$

.3 N is a V-system for μ iff N is a covering system and for every $A \subset X$ and $F \in \overline{N}(A)$ there exists a countable disjoint family F' of μ-measurable sets such that $F' \subset F$ and $\mu(A \sim \sigma F') = 0$.

.4 N is an S-system for μ with factor λ iff N is a covering system, $1 \leq \lambda < \infty$, and for every $A \subset X$ and $F \in \overline{N}(A)$ there exists a countable family $F' \subset F$ such that $\mu(A \sim \sigma F') = 0$ and for every $B \subset \sigma F'$

$$\sum_{a \in F'} \mu(a \cap B) \leq \lambda \cdot \mu B$$

.5 N is a T-system for μ iff for every $A \subset X$, $F \in \overline{N}(A)$, and $\epsilon > 0$ there exists a countable family $F' \subset F$ such that $\mu(A \sim \sigma F') = 0$ and

$$\sum_{a \in F'} \mu a \leq \mu A + \epsilon$$

3.2 Conditions on the Measure.

We shall say that μ satisfies condition 3.2 iff μ is a measure on X such that

(i) $\mu X < \infty$

(ii) open sets are μ-measurable

(iii) for every $A \subset X$ and $\epsilon > 0$ there exists an open set B such that $A \subset B$ and

$$\mu B \leq \mu A + \epsilon$$

We now have the following results.

3.3 Theorem.

Let N be a V-system for μ. Then N is an S-system for μ with factor 1.

Proof. Given $A \subset X$, $F \in \overline{N}(A)$, there exists a countable, disjoint family F' of μ-measurable sets such that $F' \subset F$ and $\mu(A \sim \sigma F') = 0$. Then for any $B \subset \sigma F'$

$$\sum_{a \in F'} \mu(a \cap B) = \mu B$$

so that N is an S-system with factor 1.

3.4 Remark.

The converse does not hold, even if μ satisfies condition 3.2, as is shown by the following example.

Let X be the interval $(0, 1)$ with the usual topology. Let the rationals in $(0, 1)$ be ordered $r_1, r_2, \ldots$ and assign to $\{r_i\}$ a measure $\mu\{r_i\} = \frac{1}{2^i}$ and to $(0, 1) \sim \{r_1, r_2 \ldots\}$ measure zero.

For $x \neq 1/2$, let $N(x)$ be the family of all open intervals containing $\{x\}$ which have irrational end points and do not contain $\{1/2\}$. Let $N(1/2)$ consist of all open intervals in X which contain $\{1/2\}$ and have rational end points. Then

(1) N is an S-system for μ with factor 2.

Proof. Let $A \subset X$, $F \in \overline{N}(A)$. The rationals in $B = A \sim \{1/2\}$ are ordered $r_{i1}, r_{i2} \ldots$ with $i_j < i_{j+1}$. We define the sequence α by recursion as follows: let $r_{i1} \in \alpha_1 \in F$, $\alpha_1 \in N(r_{i1})$. For any $n \in \omega$, having defined $\alpha_1, \alpha_2 \ldots, \alpha_n$ let k be the first integer such that $r_{ik} \in B \sim \bigcup_{i=1}^{n} \alpha_i$. Since the α_i are intervals with irrational end-points and r_{ik} is rational, let $\alpha_{n+1} \in F$, $\alpha_{n+1} \in N(r_{ik})$ and $\alpha_{n+1} \cap \bigcup_{i=1}^{n} \alpha_i = 0$. Thus, α is a sequence of disjoint intervals covering B. Let $1/2 \in \alpha_0 \in F$ and $F' = \bigcup_{i=0}^{\infty} \{\alpha_i\}$. Then F' covers the rationals in A so that $\mu(A \sim \sigma F') = 0$ and for any $B \subset \sigma F'$,

$$\sum_{\beta \in F'} \mu(B \cap \beta) = \sum_{i=1}^{\infty} \mu(B \cap \alpha_i) + \mu(B \cap \alpha_0) \leq 2\mu B.$$

(2) N is not a V-system for μ.

Proof. Let $A = X$, $F = \bigcup_{x \in X} N(x)$, and let $F' \subset F$ be any countable subfamily such that $\mu(A \sim \sigma F') = 0$, i.e. F' covers the rationals.

Then there exists $\alpha \in F'$ such that $1/2 \in \alpha$, and α has rational end-points. Let x be such an end-point. $\mu\{x\} > 0$, therefore there must be an open interval $\alpha_1 \in F'$ such that $x \in \alpha_1$. But then $\alpha \cap \alpha_1 \neq 0$, and N is not a V-system for μ.

3.5 Theorem.

Let N be a V-system for μ and let μ satisfy part (iii) of condition 3.2. Then N is a T-system for μ.

Proof. Let $A \subset X$, $F \in \overline{N}(A)$, and $\epsilon > 0$. Then since μ satisfies part (iii) of condition 3.2, there exists an open set B such that $A \subset B$ and

$$\mu(B) < \mu A + \epsilon$$

Let $F' = \{\alpha : \alpha \in F \text{ and } \alpha \subset B\}$

Then $F' \in \overline{N}(A)$ and there exists a disjoint countable family $G \subset F'$ such that

$$\mu(A \sim \sigma G) = 0$$

Since the elements of G are μ-measurable

$$\sum_{\alpha \in G} \mu\alpha = \mu\sigma G \leq \mu B < \mu A + \epsilon .$$

Therefore N is a T-system.

3.6 Lemma.

Let $A \subset X$, $1 \leq \lambda < \infty$. If F is a countable family of μ-measurable sets such that $\mu(A \sim \sigma F) = 0$, and for any $B \subset \sigma F$

$$\sum_{\alpha \in F} \mu(\alpha \cap B) \leq \lambda\mu B,$$

then for any $k > 1$ there exists a subfamily $G \subset F$ such that

$$\mu\sigma G \geq \frac{\mu A}{k\lambda} \text{ and } \sum_{\alpha \in G} \mu\alpha < \frac{k}{k-1}\mu\sigma G.$$

Proof. Let the elements of F be ordered, i.e.

$F = \{a_1, a_2 \ldots\}$. Let $G = \{a_{i1}, a_{i2} \ldots\} \subset F$ where the i_n are defined by recursion as follows: $i_1 = 1$ and for $n \in \omega$, i_{n+1} is the smallest $j \in \omega$, if any, such that $j > i_n$ and

$$\mu(\bigcup_{m=1}^{n} a_{im} \cap a_j) < \frac{\mu a_j}{k}$$

Then $\sigma G = a_1 \cup \bigcup_{n \in \omega} (a_{i_{n+1}} \sim \bigcup_{j=1}^{n} a_{ij})$ and

$$\mu\sigma G > \mu a_{i1} + \sum_{n \in \omega} \frac{k-1}{k} \mu a_{i_{j+1}} > \frac{k-1}{k} \sum_{n \in \omega} \mu a_{i_n}.$$

Therefore $\sum_{a \in G} \mu a < \frac{k}{k-1} \mu\sigma G.$

Also since $\sigma G \subset \sigma F$,

$$\sum_{a \in G} \mu a + \sum_{a \in F \sim G} \mu(a \cap \sigma G) \leq \lambda\mu\sigma G$$

And since $\mu\sigma G \leq \sum_{a \in G} \mu a$,

$$\sum_{a \in F \sim G} \mu(a \cap \sigma G) \leq (\lambda - 1)\mu\sigma G.$$

But for $a \in F \sim G$, $\mu(a \cap \sigma G) \geq \frac{\mu a}{k}$.

Therefore $\sum_{a \in F \sim G} \mu a \leq \sum_{a \in F \sim G} k\mu(a \cap \sigma G) \leq k(\lambda - 1)\mu\sigma G.$

Thus
$$\mu A \leq \mu\sigma F \leq \sum_{a \in F \sim G} \mu a + \mu\sigma G \leq [k(\lambda - 1) + 1]\mu\sigma G$$
$$\leq k\lambda\mu\sigma G,$$

and $$\mu\sigma G \geq \frac{\mu A}{k\lambda}$$

3.7 Theorem.

Let μ satisfy condition 3.2. If N is an S-system for μ with factor λ then N is a T-system for μ.

Proof. Let N be an S-system for μ with factor λ. Let $C \subset X, \mu C > 0$, $F \in \overline{N}(C)$. For any $\epsilon > 0$ choose $k > 3$, $k > \epsilon$ such that $\frac{\mu C}{k-1} < \frac{\epsilon}{10}$, and define A_n, δ_n, A_n', F_n, and G_n

by recursion so that:

$A_1 = C$ and for any $n \in \omega$;

$\delta_n =$ minimum of $\frac{\epsilon}{10.2^n}, \frac{\mu A_n}{2k\lambda}$;

A'_n is an open set, $A_n \subset A'_n$, $\mu A'_n < \mu A_n + \delta_n$;

$F_n \subset F$, F_n is countable, $\sigma F_n \subset A'_n$, $\mu(A_n \sim \sigma F_n) = 0$,

and for $B \subset \sigma F_n$, $\sum_{\alpha \in F_n} \mu(\alpha \cap B) \leq \lambda \mu B$ (F_n exists since N is an S-system);

$G_n \subset F_n$, $\mu \sigma G_n \leq \frac{\mu A_n}{k\lambda}$, and $\sum_{\alpha \in G_n} \mu\alpha \leq \frac{k}{k-1} \mu\sigma G_n$,

(G_n exists by the previous lemma).

$A_{n+1} = A_n \sim \sigma G_n$;

$A'_{n+1} \subset A'_n$.

Then
$$\mu A_{n+1} \leq \mu(A'_n \sim \sigma G_n) = \mu A'_n - \mu\sigma G_n$$
$$\leq \left(1 + \frac{1}{2k\lambda}\right) \mu A_n - \frac{\mu A_n}{k\lambda} = \left(1 - \frac{1}{2k\lambda}\right) \mu A_n .$$

By induction, $\mu A_{n+1} \leq \left(1 - \frac{1}{2k\lambda}\right)^n \mu A_1$.

Let N be so large that $\left(1 - \frac{1}{2k\lambda}\right)^N \mu A_1 < \frac{\epsilon}{3k\lambda}$.

Then $\mu A_{N+1} < \frac{\epsilon}{3k\lambda}$ and

$$\mu\sigma F_{N+1} \leq \mu A'_{N+1} < \left(1 + \frac{\epsilon}{2k\lambda}\right) \frac{\epsilon}{3k\lambda} < \frac{\epsilon}{k\lambda}$$

(since $\epsilon < k$). Therefore

$$\sum_{\alpha \in F_{N+1}} \mu\alpha \leq \lambda\mu\sigma F_{N+1} < \frac{\epsilon}{k} .$$

And since for each $n \in \omega$ $\mu A_{n+1} \leq \mu A'_n - \mu\sigma G_n$,

$$\mu A'_n \geq \mu A_{n+1} + \mu\sigma G_n .$$

Also $\mu A_n \geq \mu A'_n - \delta_n$ for each $n \in \omega$. Therefore

$$\mu C \geq \mu A_1' - \delta_1 \geq \mu A_2 + \mu\sigma G_1 - \delta_1$$

$$\geq \mu A_3 + \mu\sigma G_2 + \mu\sigma G_1 - (\delta_1 + \delta_\sigma) .$$

By induction, $\mu C \geq \sum_{i=1}^{n} \mu\sigma G_i + \mu A_{n+1} - \sum_{i=1}^{n} \delta_i ,$

and as $n \to \infty$, $\mu A_{n+1} \leq (1 - \frac{1}{2k\lambda})^n \; A_1 \to 0$, we have

$$\mu C \geq \sum_{i \in \omega} \mu\sigma G_i - \sum_{i \in \omega} \delta_i \geq \sum_{i \in \omega} \mu\sigma G_i - \frac{\epsilon}{10} .$$

Let $H = \bigcup_{i \in \omega} G_i \cup F_{N+1}$. Then H is countable, $H \subset F$, and $\mu(A \sim \sigma H) = 0$. Furthermore,

$$\sum_{\alpha \in H} \mu\alpha = \sum_{i \in \omega} \sum_{\alpha \in G_i} \mu\alpha + \sum_{\alpha \in F_{n-1}} \mu\alpha \leq \sum_{i \in \omega} \frac{k}{(k-1)}$$

$$\mu\sigma G_i + \frac{\epsilon}{k} \leq \frac{k}{k-1} (\mu C + \frac{\epsilon}{10}) + \frac{\epsilon}{k}$$

$$= \mu C + \frac{\mu C}{k-1} + \frac{k\epsilon}{10(k-1)} + \frac{\epsilon}{k}$$

$$< \mu C + \frac{\epsilon}{10} + \frac{\epsilon}{5} + \frac{\epsilon}{3} < \mu C + \epsilon .$$

Since C was any set in X, and F any element of $\overline{N}(C)$, N is a T-system for μ.

3.8 <u>Remark</u>.

The converse does not always hold, as is shown by the following example.

The construction in this example was used by Banach [1] to show that the family of all open rectangles is not a Vitali system for Lebesgue measure on the plane. It will be used here to show that this class of sets does not form an S-system for Lebesgue measure on the plane, although it is known to be a T-system [2].

Let Q be the open unit square $(0, 1) \times (0, 1)$ and μ be Lebesgue measure on Q.

We first observe that if for $\nu \in \omega$, $C_\nu > 0$, $m \in \omega$, and $a = (a_1, a_2) \in Q$ we let

$$W_m^\nu(a) = \left\{(x, y) : 0 \leq x - a_1 \leq \frac{1}{m},\ 0 \leq y - a_2 \leq \frac{1}{m},\ (x - a_1)(y - a_2) \leq \frac{e^{-C_\nu}}{m^2}\right\}$$

$$F_\nu = \left\{\alpha : \alpha = W_m^\nu(a) \subset Q \text{ for some } a \in Q,\ m \geq \nu\right\}$$

then we can easily check (see [1]) that

$$W_m^\nu(a) = \frac{e^{-C_\nu}(C_\nu + 1)}{m^2}$$

so that F_ν satisfies the hypotheses of the Vitali Covering Theorem ([2], page 265) in the plane. Hence there exists a countable disjoint subfamily $F'_\nu \subset F_\nu$ such that

$$\mu(Q \sim \sigma F'_\nu) = 0 .$$

For each $x \in Q$, let $N(x)$ be the family of open rectangles α such that $x \in \alpha \subset Q$. It is known that N is a T-system for μ ([2], page 284). We shall show that it is not an S-system for μ for any factor λ.

Suppose N is an S-system for μ with factor λ. For each $\nu \in \omega$ let $C_\nu > 0$ and such that

$$\sum_{\nu \in \omega} \frac{4}{C_\nu + 1} < \frac{1}{n} \leq \frac{1}{\lambda} \quad \text{where} \quad n \in \omega$$

Taking $W_m^\nu(a)$, F_ν, F'_ν as above let

$$P = \bigcap_{\nu \in \omega} \sigma F'_\nu .$$

Then $P \subset Q$ and

$$\mu(Q \sim P) = \mu(Q \sim \bigcap_{\nu \in \omega} \sigma F'_\nu) = \mu \bigcup_{\nu \in \omega} (Q \sim \sigma F'_\nu) \leq \sum_{\nu \in \omega} \mu(Q \sim \sigma F'_\nu) = 0$$

so that $\mu P = 1$.

Given $x \in P$, $\nu \in \omega$, there exists exactly one $\alpha \in F'_\nu$ with $x \in \alpha$. Let $p_\nu(x) \in Q$ and $m_\nu(x) \in \omega$, $m_\nu(x) \geq \nu$ be such that $\alpha = W^\nu_{m_\nu(x)}(p_\nu(x))$. Let $B_\nu(x)$ be the open rectangle with center x and $p_\nu(x)$ as a vertex, and let

$$G = \{\alpha : \alpha = B_\nu(x) \text{ for some } x \in P \text{ and } \nu \in \omega\}.$$

Then $G \in \overline{N}(P)$ so that there exists a countable $G' \subset G$ such that $\mu(P \sim \sigma G') = 0$ and for any $B \subset \sigma G'$,

$$\sum_{\alpha \in G'} \mu(\alpha \wedge B) \leq \lambda \mu B.$$

Recalling that $n \in \omega$ and $n \geq \lambda$, we see that for a fixed $\nu \in \omega$ no more than n elements $B_\nu(x)$ all having the same vertex $p_\nu(x)$ can be in G', for if there were, the intersection D of $n+1$ of them would be open, $\mu D > 0$ and

$$\sum_{\alpha \in G'} \mu(\alpha \wedge D) \geq (n+1)\mu D > \lambda \mu D.$$

Also one can check that

$$\mu B_\nu(x) \leq \frac{4\, e^{-C\nu}}{(m_\nu(x))^2} = \frac{4}{C_\nu + 1} \; W^\nu_{m_\nu(x)}(p_\nu(x)).$$

Let $\qquad H_\nu = \{\alpha : \alpha = B_\nu(x) \in G' \text{ for some } x \in P\}$.

Then $\qquad \displaystyle\sum_{\alpha \in H} \mu\alpha \leq \frac{4n}{C_\nu + 1} \sum_{p_\nu(x) \in Q} \mu W^\nu_{m_\nu(x)}(p_\nu(x)) = \frac{4n}{C_\nu + 1},$

and $\qquad \displaystyle\sum_{\alpha \in G'} \mu\alpha = \sum_{\nu \in \omega} \sum_{\alpha \in H_\nu} \mu\alpha \leq \sum_{\nu \in \omega} \frac{4n}{C_\nu + 1}$

$$= n \sum_{\nu \in \omega} \frac{4}{C_\nu + 1} < 1.$$

Therefore $\mu(P \sim \sigma G') > 0$, contradicting the choice of G'. Thus N cannot be an S-system for any λ.

4. APPROXIMATE CONTINUITY AND DENSITY

Throughout this section we assume X and Y are topological spaces, f is a function on X to Y, N is a covering system for X and μ is a measure on X.

In the following definitions the concepts depend on μ and N. We omit these variables for the sake of simplicity.

4.1 Definitions.

$\underset{t \to x}{\text{Limit}}\, f(t) = y$ iff for every $\epsilon > 0$ and neighbourhood V of y there exists a neighbourhood U of x such that for every $W \in N(x)$ with $W \subset U$ we have $\mu(W \sim f^{-1}V) \leq \epsilon \mu W$.

f is approximately continuous at x iff

$$\underset{t \to x}{\text{limit}}\, f(t) = f(x).$$

4.2 Conditions on the Measure.

We shall say that μ satisfies condition 4.2 iff

(i) $\mu X < \infty$

(ii) for every $A \subset X$ there exists a μ-measurable A' such that $A \subset A'$ and

$\mu A = \mu A'$.

(iii) for every $A \subset X$ and $\epsilon > 0$ there exists an open set $A' \supset A$ for which

$\mu A' \leq \mu A + \epsilon$.

It has been shown that if f is a μ-measurable function and the range of f has a countable base then f is approximately continuous μ-almost everywhere provided either

(i) N is a T-system for μ and μ satisfies condition 3.2 (see [2], page 288) or

(ii) N is an S-system for μ and μ satisfies only parts (i) and (iii) of condition 4.2 (see [4]).

The proof under assumption (i) given in [2] makes use of density theorems. We follow the direct method used in [4] and get the same conclusion provided

(iii) N is a T-system for μ and μ satisfies condition 4.2.

For the proof of this theorem we need the following two results.

4.3 <u>Theorem</u>.

If μ satisfies conditions (i) and (iii) of 4.2, f is μ-measurable, $\epsilon > 0$, and Y has a countable base, then there exists $C \subset X$ such that $\mu(X \sim C) < \epsilon$ and f is continuous on C.

Proof. See [4], theorem 3.5.

4.4 <u>Lemma</u>.

If μ satisfies part (ii) of condition 4.2, F is countable, $A \subset X$, $\epsilon > 0$, $\mu(A \sim \sigma F) = 0$ and $\sum_{W \in F} \mu W \leq \mu A + \epsilon$, then for every $B \subset X$

$$\sum_{W \in F} \mu(B \cap W) \leq \mu B + \epsilon .$$

Proof. Let $B \subset X$ and let $B' \subset X$ be a μ-measurable set such that $B \subset B'$ and $\mu B' = \mu B$. Then for A, ϵ, and F as defined above

$$\mu A \leq \mu \sigma F = \mu(\sigma F \cap B') + \mu(\sigma F \sim B') \leq$$

$$\sum_{W \in F} \mu(W \cap B') + \sum_{W \in F} \mu(W \sim B') = \sum_{W \in F} \mu W \leq \mu A + \epsilon .$$

If $\sum_{W \in F} \mu(W \cap B') > \mu B' + \epsilon$ then

$$\sum_{W \in F} \mu W > \mu B' + \epsilon + \sum_{W \in F} \mu(W \sim B') \geq \mu B' + \epsilon$$

$$+ \mu(\sigma F \sim B') \geq \mu(\sigma F \cap B') + \mu(\sigma F \sim B') + \epsilon$$

$$= \mu \sigma F + \epsilon \geq \mu A + \epsilon$$

which contradicts the assumptions.

Thus $$\sum_{W \in F} \mu(B \cap W) \leq \sum_{W \in F} \mu(B' \cap W) \leq \mu B' + \epsilon$$

$$= \mu B + \epsilon .$$

4.5 Theorem.

If μ satisfies condition 4.2, N is a T-system for μ, Y has a countable base, and f is a μ-measurable function, then f is approximately continuous for μ-almost all x.

Proof. For each $n \in \omega$, let

$An = \{x : x \in X$ and there exists a neighbourhood V of f(x) for which there are arbitrarily small $W \in N(x)$ with $\mu(W \sim f^{-1}V) > \frac{1}{n} \mu W\}$.

Then $B = \{x : x \in X$ and f is not approximately continuous at $x\} = \bigcup_{n \in \omega} An$

and $\mu B = 0$ iff $An = 0$ for all $n \in \omega$.

Using 4.3, let $C \subset X$, $\mu(X \sim C) < \epsilon$ and f be continuous on C. Given $n \in \omega$, to show that $\mu An = 0$, let $A' = An \cap C$ and for each $x \in A'$, let V_x be a neighbourhood of f(x) such that there are arbitrarily small $W \in N(x)$ with $\mu(W \sim f^{-1}V_x) > \frac{1}{n} \mu W$. Let $U_x \in N(x)$ be such that $C \cap U_x \subset f^{-1}V_x$ and let

$F = \{W :$ for some $x \in A'$, W N(x), $W \subset U_x$ and $\mu(W \sim f^{-1}V_x) > \frac{1}{n} \mu W\}$.

Then $F \in \overline{N}(A')$.

Since N is a T-system there exists a countable $F_1 \subset F$ such that $\mu(A' \sim \sigma F_1) = 0$ and $\sum_{W \in F_1} \mu W < \mu A' + \epsilon$. Let $D = \sigma F_1 \sim C$. Then $\mu D < \epsilon$ and for every $W \in F_1$ there is an $x \in A'$ with $W \subset U_x$ so that

$$D \cap W = W \sim C = W \sim (C \cap U_x) \supset W \sim f^{-1}Vx.$$

Therefore $\mu(D \cap W) \geq \mu(W \sim f^{-1}Vx) > \frac{1}{n}\mu W$

and by lemma 4.4

$$2\epsilon \geq \mu D + \epsilon \geq \sum_{W \in F_1} \mu(D \cap W) > \frac{1}{n} \sum_{W \in F_1} \mu W$$

$$\geq \frac{1}{n} \mu \sigma F_1 \geq \frac{1}{n} \mu A'.$$

Therefore $\mu An \leq \mu(An \cap C) + \mu(An \sim C) \leq 2\epsilon n + \epsilon$.

Since ϵ is arbitrary $\mu An = 0$.

The following theorem has been proved by Sion for N an S-system for μ, in [4], and proved for N a T-system and μ satisfying condition 3.2 in [2].

4.6 Theorem.

If N is a T-system for μ, μ satisfies condition 4.2, and A is a μ-measurable set, then for μ-almost all $x \in X \sim A$,

$$\underset{W \in N(x)}{\text{limit}} \frac{\mu(A \cap W)}{\mu W} = 0.$$

Proof. Let f be the characteristic function of A. By 4.5 for μ-almost all $x \in X$, f is approximately continuous at x. Let

$$B = \{x : x \in X \sim A \text{ and } f \text{ is approximately continuous at } x\}$$

Let $\epsilon > 0$ and V a neighbourhood of 0 which excludes 1. Then for every $x \in B$ there exists $U \in N(x)$ such that for every $W \in N(x)$ with $W \subset U$,

$$\mu(W \cap A) = \mu(W \sim f^{-1}V) \leq \epsilon \cdot \mu W.$$

Since N is a T-system,

$$\mu\{x : x \in X \text{ and } \mu W = 0 \text{ for some } W \in N(x)\} = 0.$$

Therefore for μ-almost all $x \in B$, and therefore for μ-almost all $x \in X \sim A$, $\dfrac{\mu(W \cap A)}{\mu W} \leq \epsilon$.

Since $\epsilon > 0$ is arbitrary, for μ-almost all $x \in X \sim A$,

$$\frac{\mu(A \cap W)}{\mu W} = 0.$$

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