LIMITS OF INVERSE SYSTEMS OF MEASURES

by

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ABSTRACT

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In this paper we are concerned with the problem of finding 'limits' of inverse (or projective) systems of measure spaces (for a definition of these see e.g. Choksi: Inverse Limits of Measure Spaces, Proc. London Math. Soc. 8, 1958).

Our basic limit measure, \( \hat{\mu} \), is placed on the Cartesian product of the spaces instead of on the inverse limit set, \( L \). As a result we obtain an existence theorem for this measure with fewer conditions on the system than are usually needed.

We also investigate the existence of a limit measure on \( L \) by restricting our measure \( \hat{\mu} \) to \( L \). This enables us to generalize known results and to explain some of the difficulties encountered by the standard inverse limit measure. In particular we show that the product topology may be too fine to allow the limit measure to have good topological properties (e.g. to be Radon).

Another topology which is related to the product structure is introduced and we show that limit measures which are Radon w.r.t. this topology can be obtained for a wide class of inverse systems of measure spaces.
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INTRODUCTION

An inverse (or projective) system of spaces consists of a collection of spaces $X_i$, $i \in I$, where $I$ is directed (by $<$), and functions $p_{ij} : X_j \to X_i$, defined for all $i, j \in I$ with $i < j$, such that for $i < j < k$,

$$p_{ik} = p_{ij} \circ p_{jk}.$$  

The system is called an inverse system of measures if, in addition, there exists for every $i \in I$ a measure $\mu_i$ defined on a $\sigma$-field (or $\sigma$-ring) $\mathcal{A}_i$ of subsets of $X_i$, such that whenever $i < j$ and $A \in \mathcal{A}_i$, we have

$$p_{ij}^{-1}[A] \in \mathcal{A}_j$$

and

$$\mu_j(p_{ij}^{-1}[A]) = \mu_i(A).$$

Such systems are used in many areas of mathematics, for example in problems connected with stochastic processes, martingales, disintegration of measures, etc. One of the first (implicit) uses was made by Kolmogoroff [8], to obtain probabilities on infinite Cartesian product spaces. The concept was later studied explicitly by Bochner, who called such systems stochastic families (see [2]). Since then, inverse systems of measure spaces have been the subject of a number of investigations (see e.g. Choksi [4], Metevier [10], Meyer [11], Raoul [12], Scheffer [13]).

The fundamental problem in all of these investigations
is that of finding a "limit" for an inverse system of measure spaces. All previous workers in this field have concentrated on getting an appropriate 'limit' measure on the inverse limit set $L$ (see definition 1.2, Ch.III).

Such an approach presents some serious difficulties, e.g. $L$ may be empty. In this paper, we avoid dependence on $L$, and hence many of these difficulties, by constructing a 'limit' measure $\tilde{\mu}$ on the Cartesian product $\tilde{X}$ of the $X_i$'s. As a result, we are able to get existence theorems with considerably fewer conditions on the systems. (In a forthcoming paper [14], C.L. Scheffer also gets away from dependence on $L$ by working on an abstract representation space. His methods, however, seem to be very different from ours.)

Since $L \subset \tilde{X}$, we investigate the more standard inverse limit measure from the point of view of restricting $\tilde{\mu}$ to $L$. This enables us not only to extend known results but also to give a better indication of the reasons for some of the difficulties connected with the standard inverse limit measures. In particular, we show (see example 6 in the appendix) that the product topology may be too fine for any continuous limit measure to be Radon with respect to it. This leads us to the introduction of another topology which seems to be more appropriate for problems in this area.

Chapters I and II present respectively some of the results in set theory and measure theory which we require in the later chapters. In Chapter III we introduce and discuss the concept
of inverse systems of measure spaces, and prove a fundamental existence theorem for the limit measure $\tilde{\mu}$. In Chapter IV we examine the topological properties of $\tilde{\mu}$ (and related measures) on the product space $\tilde{X}$. Chapter V is devoted to an investigation of the existence and properties of inverse limit measures on $L$ obtained by restricting $\tilde{\mu}$ to $L$. The appendix consists of examples which illustrate points made in the text.

The proofs for each chapter are collected in a separate section at the end of the chapter.
CHAPTER I
SET THEORETIC PRELIMINARIES

In this chapter we give some set theoretic definitions and notation, and collect results which will be needed for future reference. We discuss families of sets closed under basic set operations, product spaces and families with a finite intersection property. These families will play an important role in our measure theory.

Our notation is fairly standard except for the part which is used for families closed under set operations; that part is modified so as to indicate the cardinality of the subfamilies involved.

1. Set Operations
1.1 Definitions and Notation

For any relation $\mathcal{R}$

.1 $\mathcal{R} [A] = \{y : \text{for some } x \in A, \ (x, y) \in \mathcal{R}\}$.

.2 $\emptyset$ denotes the empty set.

.3 $\mathbb{N}$ denotes the set of natural numbers $0, 1, 2, \ldots$, and hence the first infinite ordinal.

.4 $\aleph_\alpha$ is the cardinal corresponding to the ordinal $\alpha$ under the standard ordering of infinite cardinals.

.5 Card A denotes the cardinality of the set $A$.

.6 $A ^\sim B = \{x : x \in A, \ x \notin B\}$.

.7 $A \Delta B = (A ^\sim B) \cup (B ^\sim A)$.

For any family $\mathcal{K}$ of sets,

.8 $\bigsqcup \mathcal{K} = \bigcup_{\alpha \in \mathcal{K}} \alpha$. 
\[ \bigcap \mathcal{K} = \cap_{\alpha \in \mathcal{K}} \alpha, \]

\[ \sigma_{\alpha}(\mathcal{K}) = \{ A : A = \bigcup \mathcal{K}', \mathcal{K}' \subset \mathcal{K}, \text{Card} \mathcal{K}' < \mathcal{K}_{\alpha} \} \]
(Note that the notation \( \mathcal{K}_{\sigma} \) is commonly used for the family we call \( \sigma_{1}(\mathcal{K}) \).)

\[ \sigma(\mathcal{K}) = \{ A : A = \bigcup \mathcal{K}', \mathcal{K}' \subset \mathcal{K} \} = \bigcup \{ \sigma_{\alpha}(\mathcal{K}) : \alpha \text{ an ordinal} \} \]

\[ \delta_{\alpha}(\mathcal{K}) = \{ A : A = \bigcap \mathcal{K}', \mathcal{K}' \subset \mathcal{K}, \text{Card} \mathcal{K}' < \mathcal{K}_{\alpha} \} \]
(Note that the notation \( \mathcal{K}_{\delta} \) is commonly used for the family we call \( \delta_{1}(\mathcal{K}) \).)

\[ \delta(\mathcal{K}) = \{ A : A = \bigcap \mathcal{K}', \mathcal{K}' \subset \mathcal{K} \} = \bigcup \{ \delta_{\alpha}(\mathcal{K}) : \alpha \text{ an ordinal} \}. \]

When it is clear from the context that all the elements of \( \mathcal{K} \) are being considered as subsets of a space \( X \), we let:

\[ \mathcal{C}(\mathcal{K}) = \{ A : A \subset X \text{ and } X \sim A \in \mathcal{K} \}, \]

\[ \mathcal{F}_{\mathcal{K}} = \{ A : A \subset X \text{ and } A \cap H \in \mathcal{K} \text{ for all } H \in \mathcal{K} \}, \]

\[ \mathcal{E}_{\mathcal{K}} = \mathcal{C}(\mathcal{F}_{\mathcal{K}}). \]

Note that for any family \( \mathcal{K} \), \( \mathcal{K} \) is the smallest space \( X \) with the property that all the elements of \( \mathcal{K} \) are subsets of \( X \). Thus, unless the space \( X \) is specified in advance we shall use \( \mathcal{K} \) for \( X \).

The classes \( \mathcal{F}_{\mathcal{K}} \), \( \mathcal{E}_{\mathcal{K}} \) will play roles similar to those played by closed sets and open sets respectively in topological spaces (see sec. 3).

We now collect a few elementary lemmas for future use.
1.2 **Lemma.** For any family of sets $\mathcal{U}$ and ordinal $\alpha$,

1. $\sigma_\alpha(\sigma_\alpha(\mathcal{U})) = \sigma_\alpha(\mathcal{U}),$
2. $\sigma(\sigma(\mathcal{U})) = \sigma(\mathcal{U}),$
3. $\delta_\alpha(\delta_\alpha(\mathcal{U})) = \delta_\alpha(\mathcal{U}),$
4. $\delta(\delta(\mathcal{U})) = \delta(\mathcal{U}).$

1.3 **Lemma.** For any family of sets $\mathcal{U}$ and ordinal $\alpha$,

1. $\sigma_0(\delta_\alpha(\mathcal{U})) \supseteq \delta_\alpha(\sigma_0(\mathcal{U})),
2. $\sigma_0(\delta(\mathcal{U})) \supseteq \delta(\sigma_0(\mathcal{U})),
3. $\delta_0(\sigma_\alpha(\mathcal{U})) \subseteq \sigma_\alpha(\delta_0(\mathcal{U})),
4. $\delta_0(\sigma(\mathcal{U})) \subseteq \sigma(\delta_0(\mathcal{U})).$

1.4 **Remark.** The inclusions in lemma 1.3 may be strict as can be seen by taking for $\mathcal{U}$ the set of compact intervals of the real line. Then $\delta_1(\mathcal{U}) = \mathcal{U}$ hence $\sigma_0(\delta_1(\mathcal{U})) = \sigma_0(\mathcal{U}).$ However $\delta_1(\sigma_0(\mathcal{U}))$ contains other sets, for example the Cantor set.

1.5 **Lemma.** For any family of sets $\mathcal{U}$ and ordinal $\alpha$,

1. $\delta_\alpha(\sigma_0(\mathcal{U})) = \sigma_0(\delta_\alpha(\sigma_0(\mathcal{U}))),$
2. $\sigma(\sigma_0(\mathcal{U})) = \sigma_0(\sigma(\sigma_0(\mathcal{U}))),$
3. $\sigma_\alpha(\delta_0(\mathcal{U})) = \delta_0(\sigma_\alpha(\delta_0(\mathcal{U}))),$
4. $\sigma(\delta_0(\mathcal{U})) = \delta_0(\sigma(\delta_0(\mathcal{U}))).$

1.6 **Lemma.** Let $\mathcal{U}$ be a family of sets such that for every $A, B \in \mathcal{U}, A \cap B \in \mathcal{U}$ and there exists a finite disjoint family $\mathcal{J} \subset \mathcal{U}$ with $A \sim B = \bigcup \mathcal{J}$. Then
1. for each $D \in \sigma_0(\mathcal{K})$ there exists a finite disjoint family $\mathcal{J} \subseteq \mathcal{K}$ such that $D = \bigcup \mathcal{J}$.

2. $\sigma_0(\mathcal{K})$ is a ring.

1.7 Lemma. If $\mathcal{K}$ is a family of subsets of a space $X$ and for some ordinal $\alpha$, $\mathcal{K} = \delta_\alpha(\mathcal{K})$, then

1. $\mathcal{C}(\mathcal{K}) = \sigma_\alpha(\mathcal{C}(\mathcal{K}))$,

2. $\mathcal{J}_\mathcal{K} = \delta_\alpha(\mathcal{J}_\mathcal{K})$,

3. $\mathcal{S}_\mathcal{K} = \sigma_\alpha(\mathcal{S}_\mathcal{K})$.

1.8 Lemma. If $\mathcal{K}$ is a family of subsets of a space $X$ and for some ordinal $\alpha$, $\mathcal{K} = \sigma_\alpha(\mathcal{K})$, then

1. $\mathcal{C}(\mathcal{K}) = \delta_\alpha(\mathcal{C}(\mathcal{K}))$,

2. $\mathcal{J}_\mathcal{K} = \sigma_\alpha(\mathcal{J}_\mathcal{K})$,

3. $\mathcal{S}_\mathcal{K} = \delta_\alpha(\mathcal{S}_\mathcal{K})$.

1.9 Lemma. If $\mathcal{K}$ is a family of subsets of a space $X$ and $\alpha$ is an ordinal then

1. $\mathcal{J}_\mathcal{K} \subseteq \mathcal{J}_{\sigma_\alpha(\mathcal{K})}$,

2. $\mathcal{J}_\mathcal{K} \subseteq \mathcal{J}_{\delta_\alpha(\mathcal{K})}$,

3. $\mathcal{S}_\mathcal{K} \subseteq \mathcal{S}_{\sigma_\alpha(\mathcal{K})}$,

4. $\mathcal{S}_\mathcal{K} \subseteq \mathcal{S}_{\delta_\alpha(\mathcal{K})}$. 
2. Cartesian Products

2.1 Definitions.

.1 \((X, I)\) is a system of spaces iff to every \(i \in I\) there corresponds a space \(X_i\), i.e. \(X\) is a set valued function on \(I\).

Given such a system \((X, I)\):

.2 \(\Pi X_i = \{x : x \text{ is a function on } I \text{ such that for every } i \in I \ x_i \in X_i\}\).

.3 For every \(i \in I\), \(\pi_i\) denotes the canonical projection onto \(X_i\), i.e. \(\pi_i\) is the function on \(\Pi X_i\) such that for every \(x \in \Pi X_i\), \(\pi_i(x) = x_i\).

.4 For every \(a \in \Pi X_i\),

\[ J_a = \{i \in I : \pi_i(a) \neq X_i\}. \]

.5 \(\mathcal{K}\) is a system of families of sets w.r.t. \((X, I)\) iff \(\mathcal{K}\) is a function on \(I\) such that, for every \(i \in I\), \(\mathcal{K}_i\) is a family of subsets of \(X_i\).

For such a system \(\mathcal{K}\):

.6 \(\text{Cyl}(\mathcal{K}) = \{a : \text{for some } i \in I \text{ and some } A \in \mathcal{K}_i, \ a = \pi_i^{-1}[A]\}\).

.7 \(\text{Rect}(\mathcal{K}) = \delta_0(\text{Cyl}(\mathcal{K}))\).

Note that \(\text{Rect}(\mathcal{K})\) agrees with the usual definition of rectangles from families iff for every \(i \in I\), \(\mathcal{K}_i\) is closed under finite intersections.

We now collect some elementary lemmas for future reference.
2.2 **Lemma.** Let \((X,I)\) be a system of spaces and for each \(i \in I\), let \(\emptyset \neq A_i \subset X_i\). If \(\alpha = \prod A_i\) then

1. \(\pi_i[\alpha] = A_i\) for every \(i \in I\) and
2. \(\alpha = \bigcap \{\pi_i^{-1}[\pi_i[\alpha]] : i \in I\}\)

2.3 **Lemma.** Let \(\mathcal{K}\) be a system of families of sets w.r.t. \((X,I)\), and let \(\emptyset \neq \alpha \in \text{Rect}(\mathcal{K})\). Then

1. \(J_\alpha\) is finite.
2. \(\alpha = \bigcap \{\pi_j^{-1}[\pi_j[\alpha]] : j \in J_\alpha\}\).
3. If \(\beta \in \text{Rect}(\mathcal{K})\) and \(\alpha \cap \beta = \emptyset\) then, for some \(j \in J_\alpha \cap J_\beta\),
   \(\pi_j[\alpha] \cap \pi_j[\beta] = \emptyset\).

2.4 **Lemma.** Let \(\mathcal{K}\) be a system of families of sets w.r.t. \((X,I)\). Let \(\emptyset \neq \alpha \in \text{Rect}(\mathcal{K})\) and \(T = \bigcup \{J_\alpha : \alpha \in \mathcal{A}\}\). Then for any \(x \in \bigcup \mathcal{A}\),

\[\bigcap_{j \in T} \pi_j^{-1}[\{\pi_j(x)\}] \subset \alpha\]

for some \(\alpha \in \mathcal{A}\).

2.5 **Lemma.** Let \(\mathcal{K}\) be a system of families of sets w.r.t. \((X,I)\) such that for every \(i \in I\), \(\mathcal{K}_i = \mathcal{A}(\mathcal{K}_i)\). Then

1. for every \(\alpha, \beta \in \text{Rect}(\mathcal{K})\) there exists a finite disjoint family \(J \in \text{Rect}(\mathcal{K})\) such that \(\alpha \cap \beta = \bigcup J\).
2. \(\sigma_0(\text{Rect}(\mathcal{K}))\) is a ring.
3. for every \(\alpha \in \sigma_0(\text{Rect}(\mathcal{K}))\) there exists a finite disjoint family \(J \in \text{Rect}(\mathcal{K})\) with \(\alpha = \bigcup J\).
3. \( \aleph \)-compact Classes.

Here we define certain families of sets which resemble the family of closed compact sets in topological spaces, and study their properties.

They will later be used in extending finitely additive set functions to measures. These families (those we call \( \aleph_0 \)-compact) were first used for such purposes by Marczewski [9] and have been used by many workers since (Choksi [4], Metevier [10], Meyer [11] etc.).

We will also consider families which will act as closed sets and open sets. These are obtained in the same way as closed and open sets in forming \( k \)-spaces (see Kelley [7]).

3.1 Definitions.

For any family of sets \( \mathcal{C} \) and infinite cardinal \( \aleph \):

1. \( \mathcal{C} \) is \( \aleph \)-compact iff \( \emptyset \in \mathcal{C} \) and for every \( \mathcal{C}' \subset \mathcal{C} \) with \( \text{Card } \mathcal{C}' = \aleph \) and \( \bigcap \mathcal{C}' = \emptyset \) there exists a finite subfamily \( \mathcal{J} \subset \mathcal{C}' \) such that \( \bigcap \mathcal{J} = \emptyset \).

2. An \( \aleph \)-covering of a set \( A \) by elements of \( \mathcal{C} \) is a subfamily \( \mathcal{C}' \subset \mathcal{C} \) such that \( \text{Card } \mathcal{C}' = \aleph \) and \( A \subset \bigcup \mathcal{C}' \).

3.2 Remark. The family of closed compact sets in a topological space is \( \aleph \)-compact for every \( \aleph \). Furthermore a family which is \( \aleph \)-compact for every \( \aleph \) may be considered as a family of closed compact sets in an appropriate topology (see lemma 3.7 below).

We now consider families resulting from various operations on \( \aleph \)-compact sets and whether these families
are themselves $\aleph$-compact.

3.3 Lemma. For any ordinal $\alpha$, if $C$ is $\aleph_{\alpha}$-compact then $\delta_{\alpha+1}(C)$ is $\aleph_{\alpha}$-compact.

3.4 Lemma. Let $C$ be $\aleph$-compact. Then:

1. every subfamily of $C$ is $\aleph$-compact.
2. for every $A \in C$, $\{A \cap C : C \in C\}$ is $\aleph$-compact,
3. $\sigma_0(C)$ is $\aleph$-compact.

Note that $\sigma_1(C)$ is not in general $\aleph$-compact as this is not true for compact sets in the real line.

3.5 Lemma. Let $(X,I)$ be a system of spaces, $C$ be a system of families of sets w.r.t. $(X,I)$ such that for each $i \in I$, $C_i$ is $\aleph$-compact, and let

$$\mathcal{S} = \{A : A = \prod_{i \in I} C_i \text{ for some function } C \text{ on } I$$
$$\text{ with } C_i \in C_i \text{ for every } i \in I\},$$

$$\mathcal{F} = \{A : A = \prod_{i \in I} F_i \text{ for some function } F \text{ on } I$$
$$\text{ with } F_i \in \mathcal{F}_{C_i} \text{ for each } i \in I\}.$$

Then

1. $\text{Rect}(C)$ is $\aleph$-compact,
2. $\mathcal{S}$ is $\aleph$-compact,
3. $\mathcal{F} = \mathcal{F}_{\mathcal{S}}$.

We next check some covering properties which parallel the topological case.
3.6 *Lemma*. Let $C$ be an $\aleph$-compact family of subsets of a space $X$. Then

1. every $\aleph$-covering of $X$ by elements of $\overline{C}(C)$ can be reduced to a finite covering.
2. if $C \in C$, then every $\aleph$-covering of $C$ by elements of $\mathcal{C}$ can be reduced to a finite sub-covering.

The next lemma shows that if we have a family which is $\aleph$-compact for every $\aleph$, we may consider it as a family of closed compacts.

3.7 *Lemma*. Let $C$ be a family of subsets of a space $X$, $C$ be $\aleph$-compact for every $\aleph$, and $C = \delta(C) = \sigma_0(C)$. Then $\mathcal{C}$ is a topology in which every element of $C$ is a closed compact set.

4. *Proofs*

**Proof of 1.2** Immediate from the definitions and the fact that $\text{Card} A < \aleph$ and $\text{Card} B < \aleph$ implies $\text{Card} A \times B < \aleph$ whenever $\aleph$ is an infinite cardinal.

**Proof of 1.3.1** Let $A, B \in \delta_\alpha(\aleph)$. Then for some $I, J$, with Card $I < \aleph_\alpha$, Card $J < \aleph_\alpha$, we can write

$$A = \bigcap_{i \in I} A_i, \quad B = \bigcap_{j \in J} B_j,$$

where $A_i \in \mathcal{K}$ for every $i \in I$ and $B_j \in \mathcal{K}$ for every $j \in J$. Then

$$A \cap B = \bigcap_{i \in I} A_i \cap \bigcap_{j \in J} B_j = \bigcap_{i \in I, j \in J} (A_i \cap B_j) = \bigcap_{(i, j) \in I \times J} (A_i \cap B_j).$$
Since Card I×J<κα,

A ∩ B ∈ δα(σ₀(κ)).

Similarly if Card Iₖ < κα and

Aₖ = ∩ₖ A(i,k) ∈ δα(κ)

for k=0, 1...n, and we let S = I₀ × ... × Iₙ, then

\[ \bigcup_{k=0}^{n} Aₖ = \bigcap_{k=0}^{n} A(i,k) = \bigcup_{j∈S} A(j,k) ∈ \deltaα(σ₀(κ)), \]

since Card S < κα.

Proof of 1.3.2, 1.3.3, 1.3.4 Similar to above.

Proof of 1.5.1 Clearly

δα(σ₀(κ)) < σ₀(δα(σ₀(κ))).

By lemma 1.3.1, for any family A,

σ₀(δα(A)) < δα(σ₀(A)),

so that by letting A = σ₀(κ),

σ₀(δα(σ₀(κ))) < δα(σ₀(σ₀(κ))) = δα(σ₀(κ)).

Proof of 1.5.2, 1.5.3, 1.5.4 Similar to above.

Proof of 1.6.1 Let D ∈ σ₀(κ). Then for some new

D = ∪ₙ Dₘₙ,

where Dₘ ∈ κ for m=0, 1...n.

If n=0, 1.6.1 is clearly true. Suppose it holds for

n=k-1 and let n=k. Then

\[ \bigcup_{m=0}^{k-1} Dₘ = \bigcup S \]
for some finite disjoint family $\mathcal{B} \subset \mathcal{U}$. Thus

$$D = D_k \cup \bigcup \mathcal{B} = D_k \cup \bigcup \{B \sim D_k : B \in \mathcal{B}\},$$

and these sets are disjoint.

Since $\mathcal{B} \subset \mathcal{U}$ and $D_k \in \mathcal{U}$, for every $B \in \mathcal{B}$,

$$B \sim D_k = \bigcup \mathcal{B}_{B}$$

where $\mathcal{B}_{B}$ is a finite disjoint subfamily of $\mathcal{U}$. Thus

$$D = D_k \cup \bigcup \{L : L \in \mathcal{B}_{B} \text{ for some } B \in \mathcal{B}\},$$

and these sets form a finite disjoint subfamily of $\mathcal{U}$.

Proof of 1.6.2 $\sigma_0(\mathcal{U})$ is clearly closed under finite unions and intersections.

Let $A, B \in \sigma_0(\mathcal{U})$. Then there exist finite disjoint families $\mathcal{B}, \mathcal{J} \subset \mathcal{U}$ such that

$$A = \bigcup \mathcal{B}, \quad B = \bigcup \mathcal{J}.$$

Then

$$A \sim B = \bigcup \mathcal{B} \sim \bigcup \mathcal{J} = \bigcup_{D \in \mathcal{B}} (D \sim \bigcup \mathcal{J}) = \bigcup_{D \in \mathcal{B}, F \in \mathcal{J}} (D \sim F).$$

For $D, F \in \mathcal{U}$, $D \sim F \in \sigma_0(\mathcal{U})$, so that for every $D \in \mathcal{B}$,

$$\bigcap_{F \in \mathcal{J}} (D \sim F) \in \sigma_0(\mathcal{U})$$

and thus

$$A \sim B = \bigcup_{D \in \mathcal{B}, F \in \mathcal{J}} \left( \bigcap_{F \in \mathcal{J}} (D \sim F) \right) \in \sigma_0(\mathcal{U}).$$

Hence $\sigma_0(\mathcal{U})$ is a ring.
Proof of 1.7.1 Immediate from De Morgan's Rules.

Proof of 1.7.2 Let $F \in \mathcal{J}_\kappa$. Then for some $I$ with $\text{Card } I < \kappa$,

$$F = \bigcap_{i \in I} F_i$$

where $F_i \in \mathcal{J}_\kappa$ for every $i \in I$.

If $H \in \mathcal{J}_\kappa$, then

$$F \cap H = \bigcap_{i \in I} (F_i \cap H) \in \mathcal{J}_\kappa = \mathcal{J}_\kappa,$$  

hence $F \in \mathcal{J}_\kappa$.

Proof of 1.7.3 Immediate from 1.7.2 and 1.7.1.

Proof of 1.8.1 Similar to 1.7.1.

Proof of 1.8.2 Similar to 1.7.2.

Proof of 1.8.3 Immediate from 1.8.2 and 1.8.3.

Proof of 1.9.1 Let $A \in \mathcal{J}_\kappa$ and $F \in \mathcal{J}_\kappa$. Then there exists $\kappa' < \kappa$ with $\text{Card } \kappa' \leq \kappa$ such that $A = \bigcup \kappa'$. Hence

$$F \cap A = \bigcup_{H \in \kappa'} (F \cap H) = \bigcup_{H \in \kappa'} F \cap H.$$  

Since $F \cap H \in \mathcal{J}_\kappa$ for every $H \in \kappa'$,

$$F \cap A = \bigcup_{H \in \kappa'} F \cap H \in \mathcal{J}_\kappa.$$  

Proof of 1.9.2 Similar to the proof of 1.9.1.

Proof of 1.9.3, 1.9.4 Since $\mathcal{J}_\kappa = \mathcal{J}_\kappa$, 1.9.3 and 1.9.4 follow from 1.9.1 and 1.9.2 respectively.
Proof of 2.2 Immediate from the definition.

Proof of 2.3.1 Let \( \alpha = \bigcap_{m=0}^{n} H_m \) where \( H_m \in \text{Cyl}(\mathcal{V}) \) for \( m \leq n \), and let \( J = \bigcup_{m=0}^{n} J_m \). Then \( J \) is finite since \( J_m \) is finite for each \( m \leq n \) (by definition of \( \text{Cyl}(\mathcal{V}) \)).

Clearly \( J_\alpha \subset J \), hence \( J_\alpha \) is finite.

Proof of 2.3.2 With \( H_m \), \( m=0,1,...,n \) as above, for every \( i \in \mathcal{I} \) let

\[
B_i = \bigcap_{m=0}^{n} \pi_i^{-1}[H_m].
\]

Then \( \alpha = \prod B_i \) and the result follows from 2.2.2.

Proof of 2.3.3 Suppose no such \( j \) exists. Then for every \( i \in \mathcal{I} \) choose \( x_i \in \pi_i[\alpha] \cap \pi_i[\beta] \). Then

\[
\{x\} \subset \bigcap_{i \in \mathcal{I}} \pi_i^{-1}([x_i]) \subset \bigcap_{j \in J} \pi_j^{-1}([\pi_j[\alpha]])
\]

hence \( x \in \alpha \). Similarly \( x \in \beta \), so that \( \alpha \cap \beta \neq \emptyset \).

Proof of 2.4 Let \( x \in \bigcup \alpha \). Then there exists \( \alpha \in \mathcal{A} \) with \( x \in \alpha \).

Then

\[
\bigcap_{j \in \mathcal{T}} \pi_j^{-1}[\pi_j(x)] \subset \bigcap_{j \in J_\alpha} \pi_j^{-1}[\pi_j(x)] \subset \alpha.
\]

Proof of 2.5.1 For \( \alpha, \beta \in \text{Rect}(\mathcal{V}) \), let \( n \in \omega \) and

\[
J_\beta = \{J_0, J_1, \ldots, J_n\}.
\]

Let

\[
B_0 = \pi^{-1}_J x_{J_0} \sim \pi_{J_0}[\beta],
\]
and for $0 \leq m \leq n$ let

$$B_m = \prod_{j=0}^{m-1} [\pi_{j,m}^{-1} \circ \pi_{j,m}] \cap \pi_{j,m}^{-1} [\pi_{j,m}^{-1} [\mathcal{B}]].$$ 

Then

$$\alpha \sim \mathcal{B} = \bigcup_{m=0}^{n} (\alpha \cap B_m)$$

and these sets are disjoint since $B_\perp \cap B_k = \emptyset$, $k \neq k$.

**Proof of 2.5.2, 2.5.3** By 2.5.1, $\mathcal{U}$ satisfies the hypotheses of lemma 1.6, hence 2.5.2 and 2.5.3 hold.

**Proof of 3.3** Let $\mathcal{A} \in \mathfrak{A}_{\alpha+1}(\mathfrak{C})$ with $\text{Card } \mathcal{A} \leq \mathfrak{K}_\alpha$. For each $A \in \mathcal{A}$, there exists $\mathcal{B}_A \subseteq \mathfrak{C}$ such that $\text{Card } \mathcal{B}_A \leq \mathfrak{K}_\alpha$ and

$$A = \prod \mathcal{B}_A.$$ 

Let

$$\mathcal{C}' = \bigcup_{A \in \mathcal{A}} \mathcal{B}_A.$$ 

Then $\mathcal{C}' \subseteq \mathfrak{C}$,

$$\text{Card } \mathcal{C}' \leq \mathfrak{K}_\alpha \cdot \mathfrak{K}_\alpha = \mathfrak{K}_\alpha,$$

and $\prod \mathcal{C}' = \prod \mathcal{A}$. Thus if $\prod \mathcal{A} = \emptyset$, then $\prod \mathcal{C}' = \emptyset$ and there exists a finite family $\mathcal{F} \subseteq \mathfrak{C}$ such that $\prod \mathcal{F} = \emptyset$. For every $F \in \mathcal{F}$ there exists $A_F \in \mathcal{A}$ with $A_F \subseteq F$ so that $\bigcap_{F \in \mathcal{F}} A_F \subseteq \prod \mathcal{F} = \emptyset$.

**Proof of 3.4.1** Immediate from the definitions.

**Proof of 3.4.2** Immediate from the definitions.

**Proof of 3.4.3** (A similar fact is proven in the same way by Meyer [11].) Let $\mathcal{A} \in \mathfrak{A}_0(\mathfrak{C})$ be such that $\text{Card } \mathcal{A} \leq \mathfrak{K}_\alpha$ and for any finite family $\mathcal{A}' \subseteq \mathcal{A}$, $\prod \mathcal{A}' \neq \emptyset$. For each $A \in \mathcal{A}$ let

$$A = \bigcup \mathcal{B}_A$$

where $\mathcal{B}_A$ is a finite family in $\mathcal{C}$. Let $\mathfrak{U}$ be an ultrafilter such that $\mathcal{A} \subseteq \mathfrak{U}$. Then for every $A \in \mathcal{A}$ there exists
$B_A \in \mathcal{B}_A$ such that $B_A \in \mathcal{U}$. Then
$$\{B_A : A \in \mathcal{A}\} \subseteq \mathcal{C},$$
$$\text{Card} \{B_A : A \in \mathcal{A}\} \leq \kappa_A,$$
and for any finite family $\mathcal{J} \subseteq \{B_A : A \in \mathcal{A}\}$, $\mathcal{J} \neq \emptyset$.

Hence
$$\emptyset \neq \prod \{B_A : A \in \mathcal{A}\} = \prod \mathcal{A}.$$  

We prove lemma 3.5 in the order 3.5.2, 3.5.1, 3.5.3.

**Proof of 3.5.2** Let $\mathcal{A} \subseteq \mathcal{S}$, Card $\mathcal{A} \leq \kappa$ and $\prod \mathcal{A}' \neq \emptyset$ for any finite subfamily $\mathcal{A}' \subseteq \mathcal{A}$. For every $i \in I$, let
$$B_i = \bigcap_{A \in \mathcal{A}} \pi_i[A].$$

Then $B_i \neq \emptyset$ for any $i \in I$, since otherwise for some $A_0, A_1, \ldots, A_n \in \mathcal{A},$
$$\bigcap_{m=0}^{n} \pi_i[A_m] = \emptyset,$$
so that $\bigcap_{m=0}^{n} A_m = \emptyset$. Thus
$$\emptyset \neq \prod_{i \in I} B_i = \prod \mathcal{A}.$$

**Proof of 3.5.1** For each $i \in I$ let
$$C_i' = C_i \cup \{x_i\}.$$  

Then $C_i'$ is $\kappa$-compact and
$$\text{Rect}(C) \subseteq \{ \prod_{i \in I} C_i : C_i \subseteq C_i' \text{ for every } i \in I \},$$
which is $\kappa$-compact by 3.5.2. Hence by 3.4.1 Rect$(C)$ is $\kappa$-compact.
Proof of 3.5.3 Let $E \in \mathcal{E}$, $D \in \mathcal{D}$ and $E = \prod_{i \in I} C_i$ where $C_i \in C_i$ for every $i \in I$, and $D = \prod_{i \in I} F_i$ where $F_i \in \mathcal{F}_i$ for every $i \in I$. Then

$$E \cap D = \prod_{i \in I} (C_i \cap F_i),$$

and since $C_i \cap F_i \in C_i$ for every $i \in I$, $E \cap D \in \mathcal{E}$, hence $D \in \mathcal{J}_\mathcal{E}$.

Proof of 3.6.1 Let $\mathcal{Q}$ be an $\aleph$-covering of $X$ by elements of $\mathcal{G}(C)$. Then

$$\{X \sim A : A \in \mathcal{Q}\} \subseteq C,$$

$$\text{Card}\{X \sim A : A \in \mathcal{Q}\} \leq \aleph,$$

and

$$\bigcap \{X \sim A : A \in \mathcal{Q}\} = \emptyset.$$

Thus there exists a finite family $\mathcal{J} \subseteq \{X \sim A : A \in \mathcal{Q}\}$ with

$$\bigcap \mathcal{J} = \emptyset,$$

hence the complements of the elements of $\mathcal{J}$ form a finite subfamily of $\mathcal{Q}$ and cover $X$.

Proof of 3.6.2 Let $\mathcal{Q}$ be an $\aleph$-covering of $C$ by elements of $\mathcal{G}_C$. Then

$$\{C \cap (X \sim A) : A \in \mathcal{Q}\} \subseteq C,$$

$$\text{Card}\{C \cap (X \sim A) : A \in \mathcal{Q}\} \leq \aleph,$$

and

$$\bigcap \{C \cap (X \sim A) : A \in \mathcal{Q}\} = \emptyset.$$

Hence there exists a finite subfamily $\mathcal{J}$ of $\{C \cap (X \sim A) : A \in \mathcal{Q}\}$ such that $\bigcap \mathcal{J} = \emptyset$, so that if for each $F \in \mathcal{J}$ we choose $A_F \in \mathcal{Q}$ such that

$$F = C \cap (X \sim A_F),$$

then $C \subseteq \bigcup_{F \in \mathcal{J}} A_F$. 


Proof of 3.7  By lemmas 1.7.3 and 1.8.3, $\mathcal{C}$ is closed under arbitrary unions and finite intersections. Since $\emptyset \in \mathcal{C}$ we have $\emptyset \in \mathcal{I}$, hence $X \in \mathcal{C}$ and thus $\mathcal{C}$ is a topology.

For every $C \in \mathcal{C}$, we have $C \in \mathcal{I}$, hence $X \sim C \in \mathcal{C}$. By lemma 3.6.2 every covering of $C$ by elements of $\mathcal{C}$ can be reduced to a finite covering. Thus for every $C \in \mathcal{C}$, $C$ is closed and compact.
CHAPTER II
MEASURE THEORETIC PRELIMINARIES

In this chapter we develop the measure theoretic results which we will use in our study of inverse limits of measures. These consist of extension theorems for finitely additive and finitely subadditive set functions, and theorems concerning construction of Radon and similar measures.

We will use Carathéodory measures throughout and we will rely heavily on the standard Carathéodory extension theorem (theorem 1.3).

1. Carathéodory Measures.

1.1 Definitions.

1. $\mu$ is a Carathéodory measure on $X$ iff $\mu$ is a function on the family of subsets of $X$ such that $\mu(\emptyset) = 0$ and

\[ 0 \leq \mu(A) \leq \sum_{n} \mu(B_n) \leq \infty \]

whenever $A \subset \bigcup_{n} B_n$.

2. $A$ is $\mu$-measurable iff $\mu$ is a Carathéodory measure on a space $X$, $A \subset X$, and for every $T \subset X$,

\[ \mu(T) = \mu(T \cap A) + \mu(T \setminus A). \]

3. $\mathcal{M}_\mu = \{ A : A \text{ is } \mu\text{-measurable} \}$.

4. A Carathéodory measure $\mu$ on a space $X$ is carried by $A \subset X$ iff $\mu(X \setminus A) = 0$.

5. A Carathéodory measure $\mu$ on a space $X$ is pseudo-carried by $A \subset X$ iff $\mu(B) = 0$ whenever $B \in \mathcal{M}_\mu$ and $B \subset X \setminus A$. 
.6 If \( \mu \) is a Carathéodory measure on \( X \), the restriction of \( \mu \) to \( A \subset X \), \( \mu | A \), is the measure \( \nu \) on \( X \) defined by \( \nu(B) = \mu(B \cap A) \) for all \( B \subset X \).

.7 \( \mu \) is an outer measure on \( X \) iff \( \mu \) is a Carathéodory measure on \( X \) and for every \( A \subset X \) there exists \( B \subset X \) such that \( B \in \mathcal{M}_\mu \), \( A \subset B \) and \( \mu(A) = \mu(B) \).

.8 \( \mu \) is the Carathéodory measure on \( X \) generated by \( g \) and \( \mathcal{K} \) iff \( \mathcal{K} \) is a family of subsets of \( X \), \( g(H) \geq 0 \) for every \( H \in \mathcal{K} \), and for every \( A \subset X \),

\[
\mu(A) = \inf \{ \sum_{H \in \mathcal{K}'} g(H) : \mathcal{K}' \subset \mathcal{K}, \text{Card}(\mathcal{K}') \leq \aleph_0, \text{ and } A \subset \bigcup \mathcal{K}' \}
\]

.9 \( \mu \) is a semifinite outer measure on \( X \) iff \( \mu \) is an outer measure on \( X \) and for every \( A \subset X \)

\[
\mu(A) = \sup \{ \mu(B) : B \subset A, \mu(B) < \infty \}.
\]

1.2 Remark. If \( \mathcal{K} \) is closed under countable unions, \( g \) is countable subadditive on \( \mathcal{K} \) and \( \mu \) is the Carathéodory measure generated by \( g \) and \( \mathcal{K} \), then for every \( A \subset X \),

\[
\mu(A) = \inf \{ g(H) : H \in \mathcal{K}, A \subset H \}.
\]

We now consider what properties a measure generated as in definition 1.1.8 must have. These are contained in the following well known theorem which is essentially due to Carathéodory.

1.3 Theorem. For any non-negative set function \( g \) and family \( \mathcal{K} \) of subsets of a space \( X \), the Carathéodory measure \( \mu \),
generated by $g$ and $\mathcal{H}$ has the following properties.

1. $\mu$ is a Carathéodory measure,

2. if $A \subseteq A'$ for some $A' \in \sigma_1(\mathcal{H})$ (in particular if $\mu(A) < \infty$), then for every $\varepsilon > 0$ there exists $B \in \sigma_1(\mathcal{H})$ such that $A \subseteq B$ and

   $$\mu(B) \leq \mu(A) + \varepsilon$$

   and hence there exists $B' \in \delta_1(\sigma_1(\mathcal{H}))$ such that $A \subseteq B'$ and $\mu(A) = \mu'(B)$.

3. if $\mathcal{H}$ is a ring and $g$ is finitely additive on $\mathcal{H}$ then $\mathcal{H} \subseteq m_\mu$. (Hence in view of 1.3.2, $\mu$ is an outer measure.)

4. if $g$ is countably subadditive on $\mathcal{H}$, then $\mu(A) = g(A)$ for every $A \in \mathcal{H}$.

1.4 **Remark.** Let $\mu$ be a Carathéodory measure (or a measure on a $\sigma$-field $\mathcal{A}$) and let $\nu$ be the outer measure generated by $\mu$ and $m_\mu$ (or $\mathcal{A}$). Then by theorem 1.3, $\nu$ agrees with $\mu$ on $m_\mu$ (or $\mathcal{A}$) and $m_\mu \subseteq m_\nu$ ($\mathcal{A} \subseteq m_\nu$). For this reason we shall concentrate on outer measures throughout.

The following lemmas will be useful for extending set functions to rings.

1.5 **Lemma.** Let $\mathcal{A}$ be a class of sets such that $\mathcal{A} = \delta_0(\mathcal{A})$ and whenever $\alpha, \beta \in \mathcal{A}$ there exists a finite disjoint family $\mathcal{J} \subseteq \mathcal{A}$ with $\alpha \sim \beta = \bigcup \mathcal{J}$. If $h$ is a non-negative and
finitely additive set function on $\mathcal{A}$, then $h$ can be uniquely extended to a non-negative and finitely additive set function $h^*$ on the ring $\mathcal{B}$ generated by $\mathcal{A}$.

1.6 **Corollary.** Let $(X,I)$ be a system of spaces and let $\mathcal{K}$ be a system of families of sets w.r.t. $(X,I)$ such that for every $i \in I$, $\mathcal{K}_i$ is a $\sigma$-field. If $g$ is a non-negative finitely additive set function on $\text{Rect}(\mathcal{K})$, then $g$ can be uniquely extended to a non-negative finitely additive set function $g^*$ on $\sigma_0(\text{Rect}(\mathcal{K}))$.

2. **Approximation and Generation of Measures.**

In this section we are concerned with set functions which can be approximated from below by elements of $\aleph$-compact families.

We apply theorem 1.3 to such set functions to obtain an extension theorem similar to that of Marczewski [9], in which we include the non-$\sigma$-finite case.

We then turn to the construction of Radon-like measures, generalizing the definition of Radon measures to the non-topological case by using the families $\mathcal{J}_C$ and $\mathcal{S}_C$ as closed and open families respectively, in analogy with the construction of k-spaces (see e.g. Kelley [7]).

The particular forms of our theorems on construction of Radon and Radon-like measures are motivated by the fact that we will apply them to product spaces. There the strong conditions on the outer measures are quite naturally satisfied and we can thus dispense with the usual topological requirement (regularity).
2.1 **Definitions.** Let \( g \) be a non-negative set function on a family \( \mathcal{V} \) of subsets of a space \( X \).

1. \( C \) is an inner family for \( g \) on \( \mathcal{V} \) iff \( C \subseteq \mathcal{V} \) and when \( H \in \mathcal{V} \),
   \[
   g(H) = \sup\{g(C) : C \subseteq H, \ C \in \mathcal{C}\}.
   \]

2. \( C \) is an inner family for \( \mu \) iff \( \mu \) is an outer measure on \( X \) and \( C \) is an inner family for \( \mu \) on \( \mathcal{M}_\mu \).

3. \( \mathcal{J} \) is an outer family for \( g \) on \( \mathcal{V} \) iff \( \mathcal{J} \subseteq \mathcal{V} \) and for every \( H \in \mathcal{V} \),
   \[
   g(H) = \inf\{g(G) : G \in \mathcal{J}, \ H \subseteq G\}.
   \]

4. \( \mathcal{J} \) is an outer family for \( \mu \) iff \( \mu \) is an outer measure on \( X \) and \( \mathcal{J} \) is an outer family for \( \mu \) on \( \mathcal{M}_\mu \) (and hence also on the family of all subsets of \( X \)).

5. \( \mu \) is Radon-like w.r.t. \((\mathcal{N}, \mathcal{C}, \mathcal{J})\) iff \( \mu \) is an outer measure on \( X \), \( \mathcal{N} \) is an infinite cardinal, and the following conditions are satisfied.

   5.1 \( \mathcal{C} = \sigma_0(\mathcal{C}) = \delta_1(\mathcal{C}) \),
   
   5.2 \( \mathcal{C} \) is an \( \mathcal{N} \)-compact family of subsets of \( X \),
   
   5.3 \( \mathcal{C}(\mathcal{C}) \subseteq \mathcal{J} \subseteq \mathcal{J}(\mathcal{C}) \),
   
   5.4 \( \mathcal{J} = \delta_0(\mathcal{J}) = \sigma_1(\mathcal{J}) \).
   
   5.5 \( \mathcal{C} \) is an inner family for \( \mu \) on \( \mathcal{J} \),
   
   5.6 \( \mathcal{J} \) is an outer family for \( \mu \),
   
   5.7 \( \mu(C) < \infty \) for every \( C \in \mathcal{C} \).

If \( \mathcal{C} \) is \( \mathcal{N} \)-compact for every cardinal \( \mathcal{N} \), we shall say \( \mu \) is Radon-like w.r.t. \((\mathcal{C}, \mathcal{J})\). If in fact \( \mathcal{J} \) is a topology and \( \mathcal{C} \) is the family of closed compact sets in the topology \( \mathcal{J} \) then the above definition reduces to
that of a Radon outer measure and we shall say \( \mu \) is Radon w.r.t. \( \mathcal{J} \), or simply that \( \mu \) is Radon if the topology is clear from the context.

We first note conditions under which the families \( \mathcal{I}_c \), \( \mathcal{J}_c \) consist of measurable sets.

2.2 **Lemma.** Let \( \mu \) be an outer measure on a space \( X \) and let \( C \) be an inner family for \( \mu \). Then \( \mathcal{I}_c \subseteq m_\mu \) (hence \( \mathcal{J}_c \subseteq m_\mu \)).

The next two theorems show the role which \( \mathcal{N} \)-compact inner families play in the extension of set functions. The first theorem shows that with such a family, finite subadditivity guarantees countable subadditivity; the second theorem uses this and the fact that finite additivity yields measurability to show that the Carathéodory extension is a measure with suitable approximation properties. This will allow us to establish the existence and some approximation properties of inverse limit measures.

2.3 **Theorem.** Let \( X \) be a space and \( \mathcal{K} \) a ring of subsets of \( X \), and let \( g \) be a non-negative finitely subadditive set function on \( \mathcal{K} \). If there exists an \( \mathcal{N}_0 \)-compact family \( C \subseteq \mathcal{K} \) which is an inner family for \( g \) on \( \mathcal{K} \), then \( g \) is countably subadditive on \( \mathcal{K} \).

2.4 **Theorem.** Let \( g \) be a non-negative finitely additive set function on a ring \( \mathcal{K} \) of subsets of a space \( X \). If there exists an \( \mathcal{N}_0 \)-compact subfamily \( C \subseteq \mathcal{K} \) which is an inner family for \( g \) on \( \mathcal{K} \), then the outer measure \( \mu \) on \( X \)
generated by \( g \) and \( \mathcal{K} \) has the following properties:

1. \( \mu \) is an extension of \( g \), i.e. for all \( H \in \mathcal{K} \), \( \mu(H) = g(H) \),

2. \( \mathcal{K} \subseteq \mathcal{M}_\mu \),

3. for every \( A \in \mathcal{M}_\mu \) with \( \mu(A) < \infty \),

\[
\mu(A) = \sup \{ \mu(C) : C \in \delta_1(C), C \subseteq A \}.
\]

In our work on product spaces, the class \( \mathcal{C} \), which we use to prove the existence of a limit measure, is not in general a collection of closed compact sets in the product topology. The question then arises as to whether we can in some way adjust our measure to form a Radon measure w.r.t. the product topology. To show that under certain conditions we can do this, we prove the following theorems. They do not follow the usual existence proof for Radon measures, which uses regularity, but omit any topological conditions by using a property which arises naturally in our product space measures.

2.5 Theorem. Let \( Y \) be a topological space with topology \( \mathcal{J} \). Let

\[
\mathcal{X} = \{ A : A \text{ is closed and compact} \},
\]

and let \( \mathcal{B} \) be a base for \( \mathcal{J} \) such that

\[
\mathcal{B} = \sigma_0(\mathcal{B}) = \delta_0(\mathcal{B}).
\]

Let \( \nu \) be an outer measure on \( Y \) such that \( \nu \) is finitely additive on \( \mathcal{X} \), finite on \( \mathcal{X} \cup \mathcal{B} \), and such that

for every \( B \in \mathcal{B} \)

\[
\nu(B) = \sup \{ \nu(F) : F \subseteq B, F \text{ closed}, F \in \mathcal{M}_\nu \}
\]

If we let

\[
h(G) = \sup \{ \nu(K) : K \in \mathcal{K}, K \subseteq G \}
\]
for every $G \in \mathcal{J}$, and
\[
\mu(A) = \inf \{ h(G) : G \in \mathcal{J}, A \subset G \}
\]
for every $A \subset Y$, then:

1. $\mu$ is a Radon outer measure,
2. $\mu(G) = h(G)$ for every $G \in \mathcal{J}$.
3. $\mu(K) \geq \nu(K)$ for every $K \in \mathcal{K}$.

The following theorem is a generalization of the above one to the non-topological case. The family $\mathcal{J}$ will act as a base.

2.6 Theorem. Let $X$ be a space and $\mathcal{C}$ an $\mathcal{K}_\alpha$-compact family of subsets of $X$ such that $\emptyset \in \mathcal{C}$ and
\[
\mathcal{C} = \mathcal{J}_1(\mathcal{C}) = \mathcal{C}_0(\mathcal{C}).
\]
Let $\nu$ be an outer measure on $X$ which is finitely additive on $\mathcal{C}$ and finite on $C \cup \mathcal{J}$, where
\[
\mathcal{J} = \mathcal{C}_0(\mathcal{J}) = \mathcal{J}_0(\mathcal{J}) \subset \mathcal{C},
\]
\[
\mathcal{J}(\mathcal{C}) \subset \mathcal{C}_{\alpha+1}(\mathcal{J})
\]
and for every $D \in \mathcal{J}$,
\[
\nu(D) = \sup \{ \nu(F) : F \subset D, F \in (\mathcal{J} \cap \mathcal{M}) \}.
\]
If we now let
\[
h(G) = \sup \{ \nu(C) : C \in \mathcal{C}, C \subset G \}
\]
for every $G \in \mathcal{C}_{\alpha+1}(\mathcal{J})$, and
\[
\mu(A) = \inf \{ h(G) : G \in \mathcal{C}_{\alpha+1}(\mathcal{J}), A \subset G \},
\]
for every $A \subset X$, then
.1 μ is an outer measure on X which is Radon-like
w.r.t. \((\mathfrak{N}_\alpha, \mathfrak{C}, \sigma_{\alpha+1})\),
.2 \(\mu(G) = h(G)\) for every \(G \in \sigma_{\alpha+1}\),
.3 \(\nu(C) \leq \mu(C)\) for every \(C \in \mathfrak{C}\).

Since many approximation theorems require that the
measure be semifinite, we conclude this section by indicating
a way of amending a measure to make it semifinite.

2.7 Theorem. Let \(\nu\) be an outer measure on a space X. If
for every \(A \subseteq X\) we let
\[
\nu'(A) = \sup\{\nu(B) : B \subseteq A \text{ and } \nu(B) < \infty\}
\]
then:
.1 \(\nu'\) is a semifinite outer measure,
.2 \(\mathcal{M}_{\nu'} = \mathcal{M}_\nu\).

3. Proofs.

Proof of 1.3 See Sion [15] for .1, .2, .3.
Since we always have \(\mu(H) \leq g(H)\) for \(H \in \mathfrak{H}\); 1.3.4 follows
immediately.

Proof of 1.5 By lemma 1.6 Ch. I, every \(B \in \mathfrak{B}\) can be written
as \(\bigcup \mathcal{J}\) where \(\mathcal{J}\) is a finite disjoint subfamily of \(\mathcal{A}\). Let
\[
h^*(B) = \sum_{F \in \mathcal{J}} h(F).
\]
If also \(B = \bigcup \mathcal{K}\) where \(\mathcal{K}\) is another finite disjoint subfamily of
\(\mathcal{A}\), then
\[
\sum_{F \in \mathcal{J}} h(F) = \sum_{F \in \mathcal{J}} \left( \sum_{G \in \mathcal{K}} h(F \cap G) \right) = \sum_{G \in \mathcal{K}} h(G).
\]
so that \( h^* \) is well defined and clearly unique.

If \( A, B \in \mathcal{B} \), \( A \cap B = \emptyset \) and \( A = \bigcup \mathcal{K}, B = \bigcup \mathcal{J} \) where \( \mathcal{K}, \mathcal{J} \) are finite disjoint subfamilies of \( \mathcal{A} \), then \( \mathcal{J} \cup \mathcal{K} \) is finite and disjoint, so that

\[
\begin{align*}
\h^*(A \cup B) &= \sum_{F \in \mathcal{J} \cup \mathcal{K}} h(F) \\
&= \sum_{F \in \mathcal{J}} h(F) + \sum_{H \in \mathcal{K}} h(H) \\
&= h^*(B) + h^*(A)
\end{align*}
\]

Hence \( h^* \) is clearly finitely additive on \( \mathcal{B} \).

**Proof of 1.6** By lemma 2.5 Ch.I, \( \sigma_0(\text{Rect}(\mathcal{K})) \) satisfies the hypotheses of lemma 1.5.

**Proof of 2.2** To show that a set \( F \in \mathcal{I}_\mu \) is \( \mu \)-measurable it is sufficient to show that for every \( T \subset X \) with \( \mu(T) < \infty \),

\[
\mu(T) = \mu(T \cap F) + \mu(T \sim F).
\]

Let \( B \in \mathcal{I}_\mu, T \subset B \) and \( \mu(T) = \mu(B) \). Given \( \varepsilon > 0 \), since \( \mathcal{C} \) is an inner family for \( \mu \), there exists \( C \in \mathcal{C} \) such that \( C \subset B \) and \( \mu(B) < \mu(C) + \varepsilon \). Then \( F \cap C \subset \mathcal{I}_\mu \) so that

\[
\mu(C) = \mu(C \cap F) + \mu(C \sim F)
\]

and therefore

\[
\begin{align*}
\mu(T) &\leq \mu(T \cap F) + \mu(T \sim F) \\
&\leq \mu(B \cap F) + \mu(B \sim F) \\
&\leq \mu(C \cap F) + \mu(C \sim F) + 2\varepsilon \\
&\leq \mu(C) + 2\varepsilon \leq \mu(B) + 2\varepsilon \\
&= \mu(T) + 2\varepsilon.
\end{align*}
\]

Letting \( \varepsilon \to 0 \) we get

\[
\mu(T) = \mu(T \cap F) + \mu(T \sim F)
\]

hence \( F \) is measurable.
Proof of 2.3 Let $H_n \in \mathcal{A}$ for $n \in \omega$ and $A = \bigcup_{n \in \omega} H_n$. To see that $g(A) \leq \sum_{n \in \omega} g(H_n)$, let $t < g(A)$, $\varepsilon > 0$, and choose by recursion for every $n \in \omega$,

$C_n \in \mathcal{C}$ and $B_n \in \mathcal{A}$ so that:

1. $C_0 \subseteq A$ and $t < g(C_0)$,
2. $C_{n+1} \subseteq B_n = C_n \Delta H_n$,
3. $g(C_{n+1}) > \begin{cases} 
   g(B_n) - \frac{\varepsilon}{2^n+1} & \text{if } g(B_n) < \infty, \\
   t & \text{if } g(B_n) = \infty.
\end{cases}$

Since $\bigcap_{n \in \omega} C_n = \emptyset$ and $\mathcal{C}$ is $\aleph_0$-compact, there exists $m \in \omega$ such that $C_n = \emptyset$ for $n \geq m$. If $k = \max\{n : g(C_n) > t\}$, then, for $n \geq k$,

$g(B_n) \leq g(C_{n+1}) + \frac{\varepsilon}{2^n+1}$

and since

$g(C_n) \leq g(H_n) + g(B_n),$

we have:

$t < g(C_k) \leq g(H_k) + g(B_k) \leq g(H_k) + g(C_{k+1}) + \frac{\varepsilon}{2^{k+1}}.$

Hence by induction

$t < \sum_{i=k}^{m} \left( g(H_i) + g(C_i) + \varepsilon \right) \leq \sum_{i \in \omega} g(H_i) + \varepsilon.$

Thus $g(A) \leq \sum_{i \in \omega} g(H_i)$. 

Proof of 2.4 By lemma 2.2, $g$ is countably subadditive on $\mathcal{A}$, so that $g(H) = \mu(H)$ for every $H \in \mathcal{A}$. Since $g$ is additive on $\mathcal{A}$, $\mathcal{A} \subseteq \mathcal{M}_\mu$ (theorem 1.3 (3)).

To show 2.4.3 let $A \in \mathcal{M}_\mu$ with $\mu(A) < \infty$. Then for $\varepsilon > 0$
there exists a sequence $H_0, H_1, \ldots$ in $\mathcal{H}$ such that $A \subseteq \bigcup_{n \in \omega} H_n$ and

$$\sum_{n \in \omega} g(H_n) = \sum_{n \in \omega} \mu(H_n) < \mu(A) + \varepsilon.$$ 

There exists $k \in \omega$, such that if $H = \bigcup_{n \in \omega} H_n$, then $\mu(A) < \mu(H) + \varepsilon$. Clearly also $\mu(H \sim A) < \varepsilon$, so that there exists a sequence $G_0, G_1, \ldots$ in $\mathcal{H}$ such that

$H \sim A \subseteq \bigcup_{n \in \omega} G_n$

and

$$\sum_{n \in \omega} g(G_n) = \sum_{n \in \omega} \mu(G_n) < \varepsilon.$$ 

Let $C_0 \in \mathcal{C}$ be such that $C_0 \subseteq H \sim G_0$ and

$$\mu(C_0) > \mu(H \sim G_0) - \frac{\varepsilon}{2}.$$ 

Then choose by recursion for $n \in \omega$

$$C_n \subseteq C_{n-1} \sim G_n$$

such that

$$\mu(C_n) > \mu(C_{n-1} \sim G_n) + \frac{\varepsilon}{2^{n+1}}.$$ 

Thus

$$\mu(C_n) > \mu(C_0) - \sum_{m=1}^{n} \mu(G_m) - \sum_{m=1}^{n} \frac{\varepsilon}{2^{m+1}}$$

$$> \mu(H) - \sum_{m=0}^{n} \mu(G_m) - \sum_{m=0}^{n} \frac{\varepsilon}{2^{m+1}} > \mu(H) - 2\varepsilon.$$ 

Then we have

$$\mu( \bigcap_{n \in \omega} C_n) \geq \mu(H) - 2\varepsilon \geq \mu(A) - 3\varepsilon,$$

and

$$\bigcap_{n \in \omega} C_n \subseteq H \sim \bigcup_{n \in \omega} G_n \quad A,$$

from which we see

$$\mu(A) = \sup \{ \mu(C) : C \subseteq A, C \in \delta_1(\mathcal{C}) \}.$$
Proof of 2.5, 2.6 Since 2.5 is clearly a special case of 2.6, we proceed to prove only 2.6.

Let $C \in C$ and $C \subseteq \bigcup_{m=0}^{n} D_m$ where $D_0, D_1, \ldots, D_n \in \mathcal{B}$. Then for $\varepsilon > 0$ there exists $F \in \mathcal{F}_C$, $F \in M$ such that $F \subseteq D_0$ and $\nu(D_0 \sim F) < \varepsilon$. Hence $C \cap F \in C$ and

$$\nu(C \cap D_0) \leq \nu(C \cap F) + \varepsilon$$

so that

$$\nu(C) \leq \nu(C \sim D_0) \leq \nu(C \cap F) + \varepsilon \leq \nu(C \sim D_0) + h(D_0) + \varepsilon.$$ 

Replacing $C$ by $C \sim \bigcup_{k=0}^{m} D_k$ (which is in $C$), we get for any $m \leq n$

$$\nu(C \sim \bigcup_{k=0}^{m} D_k) \leq \nu(C \sim \bigcup_{k=0}^{m+1} D_k) + h(D_{m+1}) + \varepsilon,$$

and by induction,

$$\nu(C) \leq \sum_{m=0}^{n} h(D_m) + n \varepsilon,$$

which shows that

$$\nu(C) \leq \sum_{m=0}^{n} h(D_m).$$

Now we show that $h$ is countably subadditive on $\sigma_{\alpha+1}(\mathcal{A})$.

Let $G \subseteq \bigcup_{n \in \omega} G_n$ where $G, G_0, G_1, \ldots$ are elements of $\sigma_{\alpha+1}(\mathcal{A})$. For any $t < h(G)$, there exists $C \subseteq C$ with $C \subseteq G$ and $\nu(C) > t$. Let for each $n \in \omega$, $G_n = \bigcup_{n \in \omega} \mathcal{F}_n$ where $\mathcal{F}_n \subseteq \mathcal{F}$ and Card $\mathcal{F}_n \leq \aleph_\alpha$. Then

$$C \subseteq \bigcup_{n \in \omega} \mathcal{F}_n$$

and Card $\bigcup_{n \in \omega} \mathcal{F}_n \leq \aleph_\alpha$, so that by lemma 3.6.2 of Ch. I, there exists a finite family $\mathcal{F}' \subseteq \bigcup_{n \in \omega} \mathcal{F}_n$ such that $C \subseteq \bigcup \mathcal{F}'$. Let for each $n \in \omega$,
\[ D_n = \bigcup \{ D : D \leq G, \ D \neq G \text{ for } m < n \} \]

and let \( k \) be such that \( D_m = \emptyset \) for \( m > k \). Then \( D_n \in \mathscr{A} \) for all \( n \geq k \) and \( G \subseteq \bigcup_{m=0}^{n} D_m \), so that

\[ t < \nu(G) \leq \sum_{m=0}^{k} h(D_m) \leq \sum_{n=0}^{\infty} h(G_m) \]

since \( D_m \subseteq G_m \) and \( h \) is monotone. Thus \( h \) is countably sub-additive, hence \( \mu \) is a Carathéodory measure and \( \mu(G) = h(G) \) for all \( G \in \sigma_{\alpha+1}(\mathscr{A}) \) (theorem 1.3).

We next check that \( \sigma_{\alpha+1}(\mathscr{A}) \subseteq \mathcal{M}_{\mu} \). Let \( G \in \sigma_{\alpha+1}(\mathscr{A}) \) and \( T \subseteq X \) with \( \mu(T) < \infty \), and let \( \varepsilon > 0 \). Then there exists \( U \in \sigma_{\alpha+1}(\mathscr{A}) \) with \( T \subseteq U \) and \( \mu(U) < \mu(T) + \varepsilon \). Then choose \( C_1 \subseteq C \), \( C_1 \subseteq U \cap G \) such that

\[ \nu(C_1) + \varepsilon > h(U \cap G) = \mu(U \cap G), \]

and \( C_2 \subseteq C \), \( C_2 \subseteq U \setminus C_1 \in \sigma_{\alpha+1}(\mathscr{A}) \) such that

\[ \nu(C_2) + \varepsilon > h(U \setminus C_1) = \mu(U \setminus C_1). \]

Then

\[ \nu(C_1 \cup C_2) \leq \mu(U) = \mu(T) + \varepsilon \leq \mu(T \cap G) + \mu(T \setminus G) + \varepsilon \leq \mu(U \cap G) + \mu(U \setminus C_1) + \varepsilon \leq \nu(C_1) + \nu(C_2) + 3 \varepsilon \]

Since \( \varepsilon > 0 \) is arbitrary,

\[ \mu(T) = \mu(T \cap G) + \mu(T \setminus G). \]

Hence \( \sigma_{\alpha+1}(\mathscr{A}) \subseteq \mathcal{M}_{\mu} \), and so \( C \subseteq \mathcal{M}_{\mu} \), (and in fact \( \mathcal{A}_C \subseteq \mathcal{M}_{\mu} \)).

We next show \( \mu(C) < \infty \) for every \( C \in \mathcal{C} \). Since \( \emptyset \in \mathcal{C} \), we have \( X \in \sigma_{\alpha+1}(\mathcal{A}) \), hence there exists an \( \mathcal{N}_\alpha \)-covering of the
space $X$ by elements of $\mathcal{B}$. Hence every $C \in \mathcal{C}$ can be so covered, and thus every $C \in \mathcal{C}$ has a finite covering by elements of $\mathcal{B}$.

Since elements of $\mathcal{B}$ have finite $\mu$-measure ($h(G) \leq \nu(G)$ for every $G \in \sigma_{\alpha+1}(\mathcal{B})$), $\mu(C) < \infty$.

Clearly $\mu(C) \geq \nu(C)$ for every $C \in \mathcal{C}$, hence

$$
\mu(G) = \sup \{ \mu(C) : C \subset G, C \in \mathcal{C} \}
$$

for every $G \in \sigma_{\alpha+1}(\mathcal{B})$. Thus $\mu$ is Radon-like w.r.t. $(\mathcal{B}, \mathcal{C}, \sigma_{\alpha+1}(\mathcal{B}))$.

**Proof of 2.7** We first note that for any $A \subset X$, if $\nu(A) < \infty$ then $\nu'(A) = \nu(A)$.

To see that $\nu'$ is a Carathéodory measure, let $T \subseteq \bigcup_{n \in \omega} B_n$. Then for any $t < \nu'(T)$ there exists $A \subseteq T$ with $t < \nu(A) < \infty$, and hence

$$
t < \nu(A) \leq \sum_{n \in \omega} \nu(A \cap B_n) = \sum_{n \in \omega} \nu'(B_n).$$

Thus

$$
\nu'(T) \leq \sum_{n \in \omega} \nu'(B_n).
$$

To see that $\mathcal{M}_\nu \subseteq \mathcal{M}_{\nu'}$, let $A \in \mathcal{M}_\nu$, $T \subseteq X$ and $\nu(T) < \infty$. Then

$$
\nu(T) = \nu'(T) = \nu'(T \cap A) + \nu'(T \setminus A) = \nu(T \cap A) + \nu(T \setminus A).
$$

To see that $\mathcal{M}_\nu \subseteq \mathcal{M}_{\nu'}$, let $A \in \mathcal{M}_\nu$, $T \subseteq X$ and $\nu'(T) < \infty$. Then there exists $B \subset T$ with $\nu'(T) = \nu(B)$ and, since $\nu$ is an outer measure, $C \in \mathcal{M}_\nu$, with $B \subset C$ and $\nu(B) = \nu(C)$ so that

$$
\nu'(T) \geq \nu(T \cap C) \geq \nu(B) = \nu'(T).
$$

If we let $D = T \cap C$, then $D \subset T$,

$$
\nu'(T) = \nu(D) = \nu'(D)
$$

and $\nu'(T \setminus D) = 0$. Thus
\[ \nu'(T) = \nu(D) = \nu(D \cap A) + \nu(D \sim A) \]
\[ = \nu'(D \cap A) + \nu'(D \sim A) \]
\[ = \nu'(T \cap A) + \nu'(T \sim A). \]

hence \( A \in \mathcal{m}_{\nu'} \).

From the above construction we see also that \( \nu' \) is an outer measure, for

\[ T \subseteq C \cup (T \sim D), \]
\[ C \subseteq (T \sim D) \in \mathcal{m}_{\nu'} \]

and

\[ \nu'(T) = \nu'(C) = \nu'(C \cup (T \sim D)). \]
CHAPTER III
INVERSE SYSTEMS OF MEASURES

In this chapter we develop our notions of inverse systems of measure spaces (these are also frequently called projective systems of measure spaces) and give sufficient conditions for the existence of our basic limit measure.

Some of the definitions we shall use differ from those used by previous workers in this area. For this reason we begin by comparing our definitions with others and discussing the reasons for the changes we make.


We begin with the standard definition (see e.g. Bourbaki [3]) of an inverse (projective) system of spaces, in which, however, we include the requirement that the functions be onto.

1.1 Definition. \((X,p,I)\) is an inverse system of spaces iff

1. \(I\) is a directed set (by \(<\)),

2. \((X,I)\) is a system of spaces (i.e. for every \(i \in I\), \(X_i\) is a space),

3. for every \(i,j \in I\) with \(i < j\), \(p_{ij}\) is a function on \(X_j\) onto \(X_i\), \(p_{ii}\) is the identity function, and

\[ p_{ik} = p_{ij} \circ p_{jk} \]

whenever \(i < j < k\).

Given an inverse system of spaces \((X,p,I)\), a major problem is to determine whether one can in some
way close the system, i.e. find a "limit" set $\tilde{X}$, and "limit" functions $\tilde{p}_i$, for every $i \in I$, such that $\tilde{p}_i$ maps $\tilde{X}$ onto $X_i$ and has the property

(a) $p_i = p_{ij} \circ p_j$

whenever $i < j$.

It is not hard to see that if any such set $\tilde{X}$ and functions $\tilde{p}_i$ exist, then $\tilde{X}$ can be mapped into a subset of the set defined in 1.2 below.

1.2 Definition. The inverse limit set of an inverse system of spaces $(x,p,I)$ is

$$\{ x \in \prod_{i \in I} X_i : \pi_i(x) = p_{ij}(\pi_j(x)) \text{ whenever } i < j \}.$$  

(The notation $\operatorname{Lim}(x,p,I)$ is often used to denote the inverse limit set.)

For this reason, the inverse limit set $L$ and the projections $\pi_i$ restricted to $L$ are the only candidates usually considered for $\tilde{X}$ and $\tilde{p}_i$. The problem is then reduced to determining whether $L$ is indeed large enough so that $\pi_i[L] = X_i$. This condition is usually referred to as simple maximality and is not satisfied by all inverse systems of spaces (see e.g. Bourbaki [3]).

In many important situations (stochastic processes, product spaces, etc.) the spaces $X_i$ also carry measures $\mu_i$ which are compatible with the functions $p_{ij}$. This leads to the concept of an inverse system of measures.
It is customary (see e.g., Choksi [4], Metevier [10]) to define such a system by adding to the above definition of an inverse system of spaces.

For every \( i \in I \), \( \mu_i \) is a measure on a \( \sigma \)-field (or \( \sigma \)-ring) \( \mathcal{A}_i \) of subsets of \( X_i \), and to extend 1.1.3 to include:

whenever \( i < j \), \( p_{ij} \) is a measurable function, and

\[
\mu_j(p_{ij}^{-1}[A]) = \mu_i(A)
\]

for every \( A \in \mathcal{A}_i \).

The problem considered then is that of finding, in addition to a "limit" set \( \tilde{X} \) and "limit" functions \( \tilde{p}_i \), a "limit" measure \( \tilde{\mu} \) on the \( \sigma \)-field generated by the ring

\[
\{ p_{ij}^{-1}[A] : i \in I \text{ and } A \in \mathcal{A}_i \}
\]

so that

\[
\tilde{\mu}(p_{ij}^{-1}[A]) = \mu_i(A)
\]

for every \( i \in I \) and \( A \in \mathcal{A}_i \).

One of the most useful properties of such a measure \( \tilde{\mu} \) is the following. Suppose that for each \( i \in I \), \( f_i \) is an \( \mathcal{A}_i \)-measurable function on \( X_i \) to the reals and

\( f_j = f_i \circ p_{ij} \) for \( i < j \). Then for every \( A \in \mathcal{A}_i \), and \( j \) such that \( i < j \),

\[
A \int f_i d\mu_i = \int_{p_{ij}^{-1}[A]} f_j d\mu_j ,
\]

and if we let \( \tilde{f} = f_i \circ \tilde{p}_i \), then \( \tilde{f} \) is independent of \( i \) and
In practice, since, as we pointed out above, we can replace \( \tilde{X} \) by \( L \), all previous workers in this field have concentrated on having such a measure \( \tilde{\mu} \) carried by \( L \). As a result, the known existence theorems require strong conditions on the functions \( p_{ij} \) and on the relationship between \( L \) and countable subsystems of \((X,p,I)\).

In this paper, we try to avoid making the existence of a \( \tilde{\mu} \) dependent on a "limit" set \( \tilde{X} \) with functions \( \tilde{p}_i \) satisfying condition (a) above. We consider instead the Cartesian product \( \prod_{i \in I} X_i \) and projections \( \pi_i \), and concentrate at first on finding a measure \( \tilde{\mu} \) having the property suggested by (b) above.

Suppose again that for each \( i \in I \), \( f_i \) is an \( \sigma_i \)-measurable function on \( X_i \) to the reals, and for \( i < j \),

\[
f_j = f_i \circ p_{ij},
\]

so that

\[
\int_{A} f_i \, d\mu_i = \int_{\pi_{ij}[A]} f_j \, d\mu_j
\]

for all \( A \in \mathcal{A}_i \). If we now define \( \tilde{f}^{(i)} = f_i \circ \pi_i \), \( \tilde{f}^{(i)} \) is not independent of \( i \). However, if we can find a measure \( \tilde{\mu} \) carried by \( \prod_{i \in I} X_i \) so that for every \( j \in I \), \( A \in \mathcal{A}_j \), and \( k \) such that \( j < k \), the symmetric difference

\[
\pi_j^{-1}[A] \Delta (p_{jk} \circ \pi_k)^{-1}[A]
\]

has \( \tilde{\mu} \) measure zero, then we have

\[
(b) \int_{A} f_i \, d\mu_i = \int_{\tilde{A}} \tilde{f} \, d\tilde{\mu}
\]
whenever \( j \in I \) and \( A \in \mathcal{A}_j \), independently of \( i \), as in (b) above. We concentrate therefore on finding such a \( \tilde{\mu} \).

We note further that it is sufficient that the symmetric difference
\[
\mu_k^{-1}[A] \Delta (p_{ij} \circ p_{jk})^{-1}[A]
\]
have \( \mu_k \) measure zero for every \( A \in \mathcal{A}_1 \) whenever \( i < j < k \), in order that
\[
\int_{A} f_{i} d\mu_{i} = \int_{p_{ij}^{-1}[A]} f_{j} d\mu_{j}
\]
whenever \( i < j \). Hence in our definition of inverse system of measures we shall replace the requirement that
\[ p_{ik} = p_{ij} \circ p_{jk} \]
by such a condition. Thus an inverse system of measures in our sense need not be an inverse system of spaces.

1.3 Definition. \((X,p,\mu,I)\) is an inverse system of outer measures (i.s.o.m.) iff

1. \( I \) is a directed set (by \( < \)),

2. for \( i \in I \), \( X_i \) is a space and \( \mu_i \) is an outer measure on \( X_i \).

3. for \( i < j \), \( p_{ij} \) is a measurable function on \( X_j \) onto \( X_i \), i.e. \( p_{ij} : X_j \to X_i \) and \( p_{ij}^{-1}[A] \in \mathcal{M}_j \)
for every \( A \in \mathcal{M}_j \); \( p_{ii} \) is the identity function;
\[
\mu_k(p_{ik}^{-1}[A] \Delta (p_{ij} \circ p_{jk})^{-1}[A]) = 0
\]
and
\[ \mu_i(A) = \mu_j(p_{ij}^{-1}[A]) \]

for every \( A \in \mathcal{M}_i \) whenever \( i < j < k \).

To simplify our notation we let

\[ \mathcal{M}_i = \mathcal{M}_j. \]

We introduce next our concept of a "limit" measure for \((X, p, \mu, I)\).

1.4 Definition. \( \nu \) is a \( \pi \)-limit outer measure for \((X, p, \mu, I)\) on \( D \) iff \((X, p, \mu, I)\) is an i.s.o.m., \( D \subseteq \prod_{i \in I} X_i \), \( \nu \) is an outer measure on \( \prod_{i \in I} X_i \) which is carried by \( D \), and the following conditions are satisfied:

1. \( \pi_i^{-1}[A] \in \mathcal{M}_i \) for every \( i \in I \) and \( A \in \mathcal{M}_i \),

2. \( \nu(\pi_i^{-1}[A]) = \mu_i(A) \),

and

\[ \nu(\pi_i^{-1}[A] \Delta (p_{ij} \circ \pi_j)^{-1}[A]) = 0 \]

whenever \( i \in I \), \( A \in \mathcal{M}_i \) and \( i < j \).

2. Generation of a \( \pi \)-limit Outer Measure.

In this section we first define for an i.s.o.m. \((X, p, \mu, I)\) a set function \( g \) on the family \( \text{Rect}(\mathcal{M}) \) in a manner suggested by condition 1.4.2 of the definition of a \( \pi \)-limit outer measure. Any \( \pi \)-limit outer measure will have to agree with \( g \) on \( \text{Rect}(\mathcal{M}) \). We then generate an outer measure \( \tilde{\nu} \) on \( \prod_{i \in I} X_i \) by the standard Caratheodory process and check under what conditions \( \tilde{\nu} \) is indeed an extension of \( g \) and a \( \pi \)-limit outer measure. We begin by exhibiting in the following lemma a condition equivalent
to 1.4.2.

2.1 **Lemma.** Let \((X, p, \mu, I)\) be an i.s.o.m. and let \(\nu\) be an outer measure on \(\prod X_i\). Then \(\nu\) satisfies condition 1.4.2 iff for every \(\alpha \in \text{Rect}(\mathcal{M})\) and \(j \in \mathcal{I}\) with \(i < j\) for every \(i \in J_\alpha\), we have:

\[\nu(\alpha) = \mu_j(\bigcap_{i \in J_\alpha} p_{ij}^{-1}[\pi_i(\alpha)]).\]

We now use condition (*) to define a set function \(g\) on \(\text{Rect}(\mathcal{M})\) with which we generate a candidate for a \(\pi\)-limit outer measure.

2.2 **Definitions.** Let \((X, p, \mu, I)\) be an i.s.o.m.

1. For \(\alpha \in \text{Rect}(\mathcal{M})\),

\[g(\alpha) = \mu_j(\bigcap_{i \in J_\alpha} p_{ij}^{-1}[\pi_i(\alpha)])\]

where \(j\) is any element of \(I\) such that \(j > i\) for every \(i \in J_\alpha\). (Note that \(J_\alpha\) is finite and that in view of condition 1.3.3 of the definition of i.s.o.m., \(g\) is independent of the choice of \(j\).)

2. \(\tilde{\mu}\) is the outer measure on \(\prod X_i\) generated by \(g\) and \(\text{Rect}(\mathcal{M})\).

With no further conditions we have the following lemma.

2.3 **Lemma.** For any i.s.o.m. \((X, p, \mu, I)\):

1. \(g(\pi_i^{-1}[A]) = \mu_i(A)\) for every \(i \in \mathcal{I}\) and \(A \in \mathcal{M}_i\),

2. \(g(\pi_i^{-1}[A] \cap (p_{ij} \circ \pi_j)^{-1}[X_i \sim A]) = 0\) whenever \(i < j\) and \(A \in \mathcal{M}_j\).
3 If \( J \) is a finite directed subset of \( I \), then \( g(a) = 0 \) whenever
\[ a \vDash \{ x \in \prod_{i \in I} X_i : p_i(x_j) = x_i \text{ for } i, j \in J, i < j \} = \emptyset. \]

4 \( g \) is finitely additive on \( \text{Rect}(\mathcal{M}) \).

5 \( \tilde{\mu} \) is an outer measure on \( \tilde{X} \).

6 \( \text{Rect}(\mathcal{M}) \subseteq \tilde{\mu} \).

2.4 Remarks. From the definitions involved and the above lemmas we have the following facts:

1 If \( \tilde{\mu} \) agrees with \( g \) on \( \text{Rect}(\mathcal{M}) \), then \( \tilde{\mu} \) is a \( \pi \)-limit outer measure (note that for \( i \in I \) and \( A \in \mathcal{M}_i \),
\[ \pi_i^{-1}[A] \Delta (p_{ij} \circ \pi_j)^{-1}[A] = \]
\[ (\pi_i^{-1}[A] \cap (p_{ij} \circ \pi_j)^{-1}[X_i \sim A]) \cup (\pi_i^{-1}[X_i \sim A] \cap (p_{ij} \circ \pi_j)^{-1}[A]). \]

2 If \( \tilde{\mu} \) does not agree with \( g \) on \( \text{Rect}(\mathcal{M}) \), then no \( \pi \)-limit outer measure can exist, since \( g \) could not be countably subadditive on \( \text{Rect}(\mathcal{M}) \) and hence no outer measure could agree with \( g \).

3 If \( \nu \) is a \( \pi \)-limit outer measure then for any
\( A \subseteq \prod_{i \in I} X_i \), \( \nu(A) \leq \tilde{\mu}(A) \), for otherwise there would exist
\( \{ F \in \mathcal{F} : F \subseteq A \} \)
a countable subfamily \( \mathcal{J} \) of \( \text{Rect}(\mathcal{M}) \) which covers \( A \) and
\[ \sum_{F \in \mathcal{J}} \nu(F) < \nu(A), \] which is impossible.

We next state a condition which is sufficient to ensure that \( \tilde{\mu} \) does in fact agree with \( g \) on \( \text{Rect}(\mathcal{M}) \).
2.5 **Definition.** An i.s.o.m. \((X, p, \mu, I)\) is inner regular w.r.t. \(C\) iff \(C\) is a system of families of sets w.r.t. \((X, I)\) such that:

1. for every \(i \in I\) there \(C_i\) is an \(\mathcal{K}_0\)-compact family of sets which is an inner family for \(\mu_i\),
2. for every \(j \in I\) and \(C \in C_j\), and every \(i < j\), \(\mu_i\) is \(\sigma\)-finite on \(p_{ij}[C]\).

When we use the above condition we shall assume, without loss of generality, that \(\mu_j(C) < \infty\) for every \(j \in I\) and \(C \in C_j\), for if we let

\[C'_j = \{C \in C_j : \mu_j(C) < \infty\},\]

then \(C'_j\) satisfies 2.4.1, since 2.4.2 shows that \(\mu_j\) is \(\sigma\)-finite on every \(C \in C_j\), and thus there exists a sequence \(A \in \mathcal{M}_j\) such that \(A_n \rightarrow A_{n+1}\) and \(\mu_j(A_n) < \infty\) for every \(n \in \omega\), and \(C = \bigcup A_n\), hence

\[
\mu_j(C) = \sup\{\mu_j(A_n) : n \in \omega\}
= \sup\{\mu_j(C') : C' \in C_j, C' \subseteq A_n\text{ for some }n \in \omega\}
\leq \sup\{\mu_j(C') : C' \in C_j, C' \subseteq C, \mu_j(C') < \infty\}
\leq \sup\{\mu_j(C') : C' \in C_j, C' \subseteq C\}
= \mu_j(C),
\]

and thus for every \(A \in \mathcal{M}_j\),

\[
\mu_j(A) = \sup\{\mu_j(C) : C \in C_j, C \subseteq A\}.
\]

We can now state the main theorem of this section.

2.6 **Theorem.** If \((X, p, \mu, I)\) is an i.s.o.m. which is inner regular w.r.t. \(C\) for some \(C\), then \(\widetilde{\mu}\) is a \(\pi\)-limit outer measure for \((X, p, \mu, I)\) on \(\prod_{i \in I} X_i\).
Example 1 in the Appendix shows that without these conditions a \( \pi \)-limit outer measure may fail to exist.

3. Proofs.

Proof of 2.1. Let \( A_1 = \pi_1[\alpha] \), \( J=J_\alpha \) so that
\[
\alpha = \bigcap_{i \in J} \pi_i^{-1}[A_i].
\]
Choose \( j \in I \) so that \( j > i \) for all \( i \in J \) and let
\[
B = \bigcap_{i \in J} p_i^{-1}[A_i].
\]
Then
\[
\nu(\pi_j^{-1}[B]) = \mu_j(B),
\]
\[
\pi_j^{-1}[B] \subset \alpha \cup \bigcup_{i \in J} ((p_{ij} \circ \pi_j)^{-1}[A_i] \sim \pi_i^{-1}[A_i]),
\]
\[
\alpha \subset \pi_j^{-1}[B] \cup \bigcup_{i \in J} \pi_i^{-1}[A_i] \sim ((p_{ij} \circ \pi_j)^{-1}[A_i]).
\]
Since for each \( i \in J \),
\[
\nu(\pi_i^{-1}[A_i] \Delta (p_{ij} \circ \pi_j)^{-1}[A_i]) = 0,
\]
we conclude
\[
\nu(\alpha) = \nu(\pi_j^{-1}[B]) = \mu_j(B).
\]
Conversely suppose for each \( \alpha \in \text{Rect}(\mathcal{V}) \), \( \nu(\alpha) = \mu_j(B) \)
where \( j \) and \( B \) are as above. Then for any \( i \in I \) and \( A \in \mathcal{V}_1 \),
letting \( \alpha = \pi_i^{-1}[A] \) and \( j=i \), we have \( B=A \) and therefore
\[
\nu(\pi_i^{-1}[A]) = \mu_i(A).
\]
Moreover, for any \( j>i \), letting \( A_j = A \),
\[
A_j = X_j \sim p_i^{-1}[A],
\]
and
\[
\alpha = \pi_i^{-1}[A_i] \cap \pi_j^{-1}[A_j]
= \pi_i^{-1}[A] \sim (p_{ij} \circ \pi_j)^{-1}[A],
\]
we get
\[ B = A_j \cap p_{ij}^{-1}[A_i] = \emptyset \]

and therefore
\[ \nu(\pi_i[A] \sim (p_{ij} \circ \pi_j)^{-1}[A]) = 0. \]

Similarly, letting \( A_i = X_i \sim A \) and \( A_j = p_{ij}^{-1}[A] \) we get
\[ \nu((p_{ij} \circ \pi_j)^{-1}[A] \sim \pi_i[A]) = 0. \]

Thus
\[ \nu(\pi_i^{-1}[A] \Delta (p_{ik} \circ \pi_k)^{-1}[A]) = 0 \]

whenever \( k > i \) and \( A \in \mathcal{M}_1 \).

**Proof of 2.3.1, 2.3.2, 2.3.3.** Similar to proof of 2.1.

To show that \( g \) is finitely additive we will use the following lemma.

**Lemma A.** Let \((X, p, \mu, I)\) be an i.s.o.m., and let \( \alpha, \beta \) be disjoint elements of \( \text{Rect}(\mathcal{M}) \). Then for any \( k \) such that \( k > i \) for all \( i \in J_\alpha \cup J_\beta \),
\[ A = \bigcap_{i \in J_\alpha} p_{ik}^{-1}[\pi_i[\alpha]] \cap \bigcap_{i \in J_\beta} p_{ik}^{-1}[\pi_i[\beta]] = \emptyset. \]

**Proof.** By lemma 2.3, Ch.I, there exists \( j \in J_\alpha \cup J_\beta \) such that \( \pi_j[\alpha] \cap \pi_j[\beta] = \emptyset \). If, however, there exists \( x \in A \), then
\[ p_{jk}(x) \in \pi_j[\alpha] \cap \pi_j[\beta] = \emptyset. \]

hence no such \( x \) exists.

**Proof of 2.3.4** Let \( \alpha \subseteq \text{Rect}(\mathcal{M}) \) and let \( \mathcal{B} \subseteq \text{Rect}(\mathcal{M}) \) be a finite disjoint family such that \( \alpha = \bigcup \mathcal{B} \). Let \( K = \bigcup_{\beta \in \mathcal{B}} J_\beta \) and let \( j \in I \) be such that \( j > i \) for every \( i \in K \). Then \( J_{\alpha} \subseteq K \) so that \( j > i \) for every \( i \in J_{\alpha} \). Also let
\[ A = \bigcap_{i \in J_{\alpha}} p_{ij}^{-1}[\pi_i[\alpha]] \]
and for every $\beta \in \mathcal{B}$,

$$A_\beta = \cap_{j \in \mathbf{J}_\beta} p_{i,j}^{-1}(\pi_1(\beta)).$$

Then $\{A_\beta : \beta \in \mathcal{B}\}$ is a finite disjoint subfamily of $\mathcal{M}_j$ (from lemma A). Since for every $i \in \mathbf{J}_\alpha$ and $\beta \in \mathcal{B}$, $\pi_1(\beta) \subseteq \pi_1(\alpha)$ and $J_\alpha \subseteq J_\beta$, then $A_\beta \subseteq A$, hence $\cup_{\beta \in \mathcal{B}} A_\beta \subseteq A$.

On the other hand, for every $x \in A$, choosing $y \in \Pi X_1$ so that $y_1 = p_{i,j}(x)$ for every $i \in I$, we see that $y \in A$ and thus $y \in \mathcal{B}$ for some $\beta \in \mathcal{B}$. Then

$$\{x\} \subseteq \cap_{j \in \mathbf{J}_\beta} p_{i,j}^{-1}(\pi_1(y)) \subseteq A_\beta,$$

hence $x \in A_\beta$ and

$$A = \cup_{\beta \in \mathcal{B}} A_\beta.$$

Thus

$$g(\alpha) = \mu_j(A) = \sum_{\beta \in \mathcal{B}} \mu_j(A_\beta) = \sum_{\beta \in \mathcal{B}} g(\beta).$$

**Proof of 2.3.5, 2.3.6.** We first note that by lemma 1.6, Ch.II, we can extend $g$ to a finitely additive function $g^*$ on the ring $\sigma_0(\text{Rect}(\mathcal{M}))$. Now, the Carathéodory measure $\tilde{\mu}$ generated by $g$ and $\text{Rect}(\mathcal{M})$ is the same as that generated by $g^*$ and $\sigma_0(\text{Rect}(\mathcal{M}))$, hence by theorem 1.3, Ch.II, $\tilde{\mu}$ is an outer measure and $\sigma_0(\text{Rect}(\mathcal{M})) \not\subseteq \mathcal{M}$. 

**Proof of 2.6.** We first check some approximation properties of $g$ when $(X, p, \mu, I)$ is inner regular w.r.t. some system of families of sets $\mathcal{C}$. 
Lemma B. If \((X, p, \mu, I)\) is an i.s.o.m. which is inner regular w.r.t. \(\mathcal{C}\), then for any \(a \in \text{Rect}(\mathcal{M})\), \(t < g(a)\) and \(i \in I\), there exists \(C \in \mathcal{C}_i\) such that \(C \subset \pi_i^{-1}[a]\) and
\[
g(a \cap \pi_i^{-1}[C]) > t.
\]

Proof. Let \(k \in I\) with \(k > j\) for all \(j \in J_0 \cup \{i\}\), and let
\[
A = \bigcap_{j \in J_0} p_{jk}^{-1}[\pi_j[a]].
\]
Then \(\mu_k(A) = g(a)\) and there exists \(C' \in C_k\), \(C' \subset A\) with \(\mu_k(C') > t\). Now choose \(B \in \mathcal{M}_1\), \(B \subset \pi_1^{-1}[a]\) such that \(p_{ik}C' \subset B\) and \(\mu_1\) is \(\sigma\)-finite on \(B\). Then there exist \(C_0, C_1, \ldots \in \mathcal{C}_i\) such that for such \(n \in \omega\), \(C_n \subset C_{n+1}\), \(C_n \subset B\) and
\[
\mu_k(B \cup C_n) = 0
\]
hence
\[
\mu_k(C' \cup \bigcup_{n \in \omega} p_{ik}^{-1}[C_n]) = 0
\]
so that for some \(m \in \omega\)
\[
\mu_k(C' \cap p_{ik}^{-1}[C_m]) > t.
\]
Letting \(C = C_m\), we see that \(C \subset \pi_1^{-1}[a]\) and that
\[
g(a \cap \pi_1^{-1}[C]) = \mu_k\left( \bigcap_{j \in J_0} p_{jk}^{-1}[\pi_j[a]] \cap p_{ik}^{-1}[C]\right)
\]
\[
\geq \mu_k(C' \cap p_{ik}^{-1}[C]) > t.
\]

Lemma C. If \((X, p, \mu, I)\) is an i.s.o.m. which is inner regular w.r.t. \(\mathcal{C}\), then \(\text{Rect}(\mathcal{C})\) is an inner family for \(g\) on \(\text{Rect}(\mathcal{M})\).

Proof. Let \(a \in \text{Rect}(\mathcal{M})\) and \(J_0 = \{i_0, i_1, \ldots, i_n\}\). Given \(t < g(a)\), choose \(C_0 \in \mathcal{C}_{i_0}\), \(C_0 \subset \pi_{i_0}^{-1}[a]\) such that
\[
g(a \cap \pi_{i_0}^{-1}[C_0]) > t,
\]
and by recursion on $m$, $C_m \in C_{i_m}$, $C_m \subset \pi_{i_m}^{-1} [\alpha]$ such that

$$g(\alpha \cap \prod_{l=0}^m \pi_{i_l}^{-1} [C_l]) > t.$$ 

If

$$C = \bigcap_{l=0}^n \pi_{i_l}^{-1} [C_l],$$

then $C \in \text{Rect}(\alpha)$, $C \prec \alpha$ and $g(C) > t$.

**Lemma D.** Let $g^*$ be the extension of $g$ to a finitely additive function on $\sigma_0(\text{Rect}(\mathcal{M}))$. If $(X, \mathcal{P}, \mu, I)$ is inner regular w.r.t. $\mathcal{C}$, then $\sigma_0(\text{Rect}(\mathcal{C}))$ is an inner family for $g^*$ on $\sigma_0(\text{Rect}(\mathcal{M}))$.

**Proof.** Let $\alpha \in \sigma_0(\text{Rect}(\mathcal{M}))$. Then there exists a finite disjoint family $\mathcal{B} \subset \text{Rect}(\mathcal{M})$ such that $\alpha = \bigcup \mathcal{B}$. If $t < g^*(\alpha)$, choose for each $B \in \mathcal{B}$, $C_B \in \text{Rect}(\mathcal{C})$ so that $C_B \subset B$ and $t < \sum_{B \in \mathcal{B}} g(C_B)$. Then

$$g^*(\bigcup_{B \in \mathcal{B}} C_B) = \sum_{B \in \mathcal{B}} g^*(C_B) = \sum_{B \in \mathcal{B}} g(C_B) > t.$$ 

Theorem 2.6 now follows from theorem 2.4, Ch.II, since $\sigma_0(\text{Rect}(\mathcal{C}))$ is $\aleph_0$-compact (by lemmas 3.5.1 and 3.4.3, Ch.I).
CHAPTER IV
APPROXIMATION PROPERTIES OF LIMIT MEASURES
ON PRODUCT SPACES

In this chapter we discuss problems related to the following: given an i.s.o.m. \((X,p,\mu,I)\) in which for each \(i \in I\), \(X_i\) is a topological space and \(\mu_i\) is Radóñ, under what circumstances can we get a \(\pi\)-limit outer measure which is Radóñ for the product topology? This leads us to seek general approximation properties of \(\tilde{\mu}\) and to amend \(\tilde{\mu}\) when \(\tilde{\mu}\) is not satisfactory.

For the sake of motivation the topological case is treated first. We show that a system of bounded Radóñ measures always has a \(\pi\)-limit outer measure which is Radóñ w.r.t an appropriate topology. This topology is obtained from a family of sets \(\mathcal{C}\) whose elements are closed but not in general compact in the product topology. \(\mathcal{C}\) is, however, \(\aleph\)-compact for every cardinal \(\aleph\). We then turn to the problem of finding \(\pi\)-limit measures which are Radóñ w.r.t the product topology, and give some conditions under which such measures exist.

In the second section we generalize the results to the non-topological case (the results of the first section are in fact corollaries of results in the second section) in which the inner families are \(\aleph\)-compact instead of being the closed compact sets of some topology.
In both the topological and non-topological cases we obtain measures which are Radón or come very close to being Radón under conditions considerably weaker than those needed by previous workers who restricted their attention to the inverse limit set.

1. **The Topological Situation.**

1.1 **General Assumptions and Notation.**

Throughout this section we suppose

1. \((X,p,\mu,I)\) is an i.s.o.m.,
2. for every \(i \in I\),
3. \(X_i\) is a topological space.
4. \(\mathcal{J}_i\) is the family of open subsets of \(X_i\),
5. \(\mathcal{J}_i\) is the family of closed subsets of \(X_i\),
6. \(\mathcal{K}_i\) is the family of closed compact subsets of \(X_i\),
7. \(\mathcal{K}_i\) is an inner family for \(\mu_i\),
8. \(\mu_i(K) < \infty\) for every \(K \in \mathcal{K}_i\),
9. \(\mu_i\) is \(\sigma\)-finite on \(\pi_j[K]\) for every \(K \in \mathcal{K}_j\), whenever \(i < j\).
10. \(\tilde{X} = \prod_{i \in I} X_i\).
11. \(\tilde{\mathcal{J}}\) is the product topology on \(\tilde{X}\).
12. \(\tilde{\mathcal{J}}\) is the family of closed sets in the product topology on \(\tilde{X}\).
13. \(\tilde{\mathcal{K}}\) is the family of closed compact sets in the product topology on \(\tilde{X}\).
.7 \( \tilde{\mathcal{C}} = \delta_1(\sigma_0(\text{Rect}(\mathcal{M}))) \).

.8 \( \tilde{\mu} \) is the outer measure on \( \tilde{X} \) introduced in definition 2.2.2 Ch.III.

.9 \( g \) is the set function on \( \text{Rect}(\mathcal{M}) \) introduced in def.2.2.1 Ch.III.

1.2 Remarks. The following are immediate consequences of our assumptions.

.1 \((X,p,\mu,\lambda)\) is inner regular w.r.t. \( \mathcal{K} \) (see def.2.5 Ch.III), hence \( \tilde{\mu} \) is a \( \bar{\pi} \)-limit outer measure (theorem 2.6 Ch.II).

.2 The family \( \mathcal{C} \) consists of closed sets in the product topology \( (\mathcal{C} \subseteq \mathcal{I}) \) which are not in general compact, but \( \mathcal{C} \) is an \( \mathbb{N} \)-compact family for every cardinal \( \mathbb{N} \) (lemmas 3.3, 3.4, 3.5 Ch.I).

.3 The outer measures \( \mu_1 \) are not necessarily Rad\( \mathcal{D} \)n, but if they are bounded, then they are Rad\( \mathcal{D} \)n. We note also that our assumption 1.1.2.5 requires that for every \( A \in \mathcal{M}_1 \),

\[
\mu_1(A) = \sup\{ \mu_1(K) : K \in \mathcal{K}_1, K \subseteq A \}.
\]

This forces \( \mu_1 \) to be semifinite, since we require closed compact sets to have finite measure and for an outer measure \( \nu \) it is only necessary to check that

\[
\nu(A) = \sup\{ \nu(B) : B \subseteq A, \nu(B) < \infty \}
\]

for all \( A \in \mathcal{M}_\nu \).

We now check approximation properties of \( \tilde{\mu} \) under our general assumptions (1.1).
1.3 **Theorem.** If $A \in \mathcal{M}_\mu$ and $\tilde{\mu}(A) < \infty$, then

$$\tilde{\mu}(A) = \sup \{ \tilde{\mu}(C) : C \subseteq \mathcal{E}, C \subseteq A \}.$$ 

Clearly the condition that $\tilde{\mu}(A)$ be finite can be replaced by:

(a) $\tilde{\mu}(A) = \sup\{ \tilde{\mu}(B) : B \subseteq A, \tilde{\mu}(B) < \infty \}$.

Thus the theorem fails to hold essentially only in the pathological case in which all subsets of $A$ have either infinite measure or measure zero. The following propositions show that under our assumptions such cases are limited and that they do not occur in the sets in which we are primarily interested.

1.4 **Proposition.** For any $A \in \mathcal{M}_\mu$, at least one of $A, \tilde{\pi} \sim A$ satisfies condition (a) above.

1.5 **Proposition.** Every $A \in \sigma_0(\text{Rect}(\mathcal{M}))$ satisfies condition (a) above.

In view of proposition 1.5 we can adjust our measure so as to eliminate the pathological cases.

1.6 **Theorem.** Let, for every $A \equiv$, 

$$\tilde{\mu}'(A) = \sup \{ \tilde{\mu}(B) : B \subseteq A \text{ and } \tilde{\mu}(B) < \infty \};$$ 

then:

1. $\tilde{\mu}'$ is a semifinite $\pi$-limit outer measure,
2. $\tilde{\mu}' = \tilde{\mu}$,
3. for every $A \in \mathcal{M}_\mu'$,

$$\tilde{\mu}'(A) = \sup \{ \tilde{\mu}'(C) : C \subseteq \mathcal{E}, C \subseteq A \}.$$
Now we consider conditions under which a Radon \( \pi \)-limit outer measure exists. The following theorem shows that under the sole additional condition that the measures \( \mu_i \) are bounded (hence Radon) \( \sim \) can always be regularized to yield a \( \pi \)-limit outer measure which is Radon w.r.t an appropriate topology on \( \tilde{X} \). (Example 6 seems to indicate that this topology rather than the product topology is a natural one to consider in connection with inverse limits).

1.7 **Theorem.** If for each \( i \in I \), \( \mu_i \) is bounded, and if we let \( \mathfrak{J} \) be the topology having \( \mathfrak{J}^c \) as a base then there exists a \( \pi \)-limit outer measure which is Radon w.r.t the topology \( \mathfrak{J} \).

1.8 **Remarks.** The topology \( \mathfrak{J} \), generated by \( \mathfrak{J}^c \) is not in general the product topology \( \mathfrak{Z} \) although it contains the product topology if the spaces are compact.

The usual base for the product topology is, however, contained in Rect(\( \mathfrak{M} \)) whenever the measures \( \mu_i \) are Radon, hence the elements of the base are measurable for any \( \pi \)-limit outer measure.

We note also that if \( \mu_i(X_i) = \infty \), then for any \( K \in X_i \)

\[
\mu_i(X_i \sim K) = \infty
\]

so that \( \mu_i \) is not Radon w.r.t. \( c(\mathcal{X}_i) \).

We next seek conditions under which we can find a \( \pi \)-limit outer measure which is Radon w.r.t. the product topology \( \mathfrak{Z} \).
First we define a candidate for such a measure by regularizing $\tilde{\mu}$ with respect to $\tilde{\mathcal{I}}$.

1.9 **Definition.** $\tilde{\mu}^*$ is the set function on the subsets of $\tilde{X}$ defined by

$$\tilde{\mu}^*(A) = \inf\{h(G) \mid A \subseteq G, G \in \mathcal{E}\}$$

for every $A \subseteq \tilde{X}$, where

$$h(G) = \sup\{\tilde{\mu}(K) \mid K \subseteq G, K \in \tilde{\mathcal{K}}\}$$

for every open set $G$.

1.10 **Theorem.** If for each $i \in I$, $\mu_i$ is Radon, then:

1. $\tilde{\mu}^*$ is Radon w.r.t. the topology $\tilde{\mathcal{I}}$.

2. If $\tilde{\mu}^*$ is a $\pi$-limit outer measure then for every $E \in \mathcal{E}_0(\text{Rect}(\mathcal{\mathcal{K}}))$ with $\tilde{\mu}(E) < \infty$,

$$\tilde{\mu}^*(E) = \tilde{\mu}(E) = \sup\{\tilde{\mu}(K) \mid K \in \tilde{\mathcal{K}}, K \subseteq E\}.$$

3. If $\gamma$ is any $\pi$-limit outer measure which is Radon w.r.t. $\tilde{\mathcal{I}}$, then $\gamma = \tilde{\mu}^*$.

The problem now is to find conditions on our system which will guarantee that $\tilde{\mu}^*$ is a $\pi$-limit outer measure. In order to obtain the result in theorem 1.10.2 we find that we have to impose restrictions on the index set $I$ and/or the functions $p_{ij}$. The $\sigma$-finiteness of the measures $\mu_i$ ensures that for every $\alpha \in \text{Rect}(\mathcal{M})$,

$$g(\alpha) = \inf\{g(B) \mid B \in \text{Rect}(\mathcal{\mathcal{K}}), \alpha \subseteq B\},$$

so that $\tilde{\mu}^*(\alpha) \leq g(\alpha)$.

Our main theorem is then the following.
1.11 **Theorem.** If for each \( i \in I \), \( \mu_i \) is a \( \sigma \)-finite Radon outer measure, then \( \hat{\mu}^* \) is a \( \pi \)-limit outer measure Radon w.r.t. \( \tilde{\mathcal{H}} \) whenever any one of the following conditions holds:

1. \( \text{Card } I \leq \aleph_0 \),

2. there exists a cofinal set \( I_0 \subset I \) with \( \text{Card } I_0 \leq \aleph_0 \) and \( p_{ij} [A] \in \mathcal{K}_i \) for every \( A \in \mathcal{K}_j \) whenever \( i < j \).

3. \( p_{ij} [A] \in \mathcal{K}_i \) for every \( A \in \mathcal{K}_j \) and \( p_{ij}^{-1} [B] \in \mathcal{K}_j \) for every \( B \in \mathcal{K}_i \) whenever \( i < j \).

1.12 **Remark.** For any \( A \in \text{Rect}(\mathcal{M}) \), if

\[
\hat{\mu}^*(A) = \sup \{ \hat{\mu}^*(K) : K \in \mathcal{K}, K \subseteq A \}
\]

then

\[
\hat{\mu}^*(A) = \sup \{ \hat{\mu}^*(K) : K \in \mathcal{K}', K \subseteq A \}
\]

where

\[
\mathcal{K}' = \{ \prod_{i \in I} K_i : K_i \in \mathcal{K}_i \text{ for every } i \in I \},
\]

hence the important family of compact sets for our inverse systems is

\[
\tilde{\mathcal{K}}_0 = \delta_1(\sigma_0(\mathcal{K}')).
\]

2. **The Non-Topological Case.**

2.1 **General Assumptions and Notation.**

Throughout this section we assume:

1. \((X,p,\mu,I)\) is an i.s.o.m.

2. \( \alpha \) is an ordinal and for each \( i \in I \), \( C_i \) is an \( \mathcal{K}_\alpha \)-compact family of subsets of \( X_i \) such that:

2.1 \( C_i = \sigma_0(C_i) = \delta_1(C_i) \),
2.2 \( C_1 \) is an inner family for \( \mu_1 \).

2.3 \( \mu_1(C) < \infty \) for every \( C \in C_1 \).

2.4 \( \mu_1 \) is \( \sigma \)-finite on \( p_{ij}[C] \) for every \( C \in C_j \) whenever \( i < j \).

3 \( \tilde{X} = \prod_{i \in I} X_i \).

4 \( \tilde{\mathcal{C}} = \delta_1(\sigma_0(\text{Rect}(C))) \).

5 \( \tilde{\mathcal{C}}_0 = \delta_1(\sigma_0(\{ \prod_{i \in I} C_i : C_i \in C_i \text{ for every } i \in I \})) \).

6 \( \tilde{\mu} \) is the measure introduced in definition 2.2 Ch.III.

2.2 Remarks.

1 \( (X, p, \mu, I) \) is inner regular w.r.t. \( \mathcal{C} \) (def.2.5 Ch.III), hence \( \tilde{\mu} \) is a \( \pi \)-limit outer measure.

2 The definition of \( \tilde{\mathcal{C}} \) is entirely analogous to the definition of the same symbol in section 1. Since, however, our space \( \tilde{X} \) is not now endowed with a product topology, we shall not attempt to obtain results in terms of an analogue of the family \( \tilde{\mathcal{K}} \) in section 1 (although such an analogue could be constructed). Instead we shall state our results in terms of \( \tilde{\mathcal{C}}_0 \), which is analogous to the family \( \tilde{\mathcal{K}}_0 \) discussed in remark 1.12.

3 The outer measures \( \mu_1 \) are not necessarily Radon-like w.r.t. \( (\mathcal{K}_1, C_1, \mathcal{A}_1) \) for some \( \mathcal{A}_1 \), though, similarly to section 1, they are (with \( \mathcal{A}_1 = C(C_1) \)) whenever the \( \mu_1 \) are bounded. We note also that in any case we retain from our general assumptions (2.1) the fact that for every \( A \in \mathcal{A}_1 \)

\[ \mu_1(A) = \sup \{ \mu_1(C) : C \in C_1, C \subset A \} \]
We now show that \( \tilde{\mu} \) has significant approximation properties under our general assumptions.

2.3 **Theorem.** If \( A \in \mathcal{M}_\sim \) and \( \tilde{\mu}(A) < \infty \) then
\[
(b) \quad \tilde{\mu}(A) = \sup \{ \tilde{\mu}(C) : C \in \mathcal{C}, C \subseteq A \}.
\]

As in section 1 we can extend the above theorem to all elements of \( \mathcal{M}_\sim \) except those whose subsets all have infinite measure or measure zero, and we can again show that such cases are limited.

2.4 **Proposition.** For any \( A \in \mathcal{M}_\sim \), at least one of \( A, \tilde{X} \sim A \) satisfies condition (b) above.

2.5 **Proposition.** Every \( A \in \text{Rect}(\mathcal{M}) \) satisfies condition (b) above.

In view of proposition 2.5 we can adjust our measure so as to eliminate the pathological cases and still have a \( \pi \)-limit outer measure with inner approximation by \( \mathcal{C} \).

2.6 **Theorem.** Let for every \( A \sim X \),
\[
\tilde{\mu}'(A) = \sup \{ \tilde{\mu}(B) : B \subseteq A \text{ and } \tilde{\mu}(B) < \infty \},
\]
then:
\begin{enumerate}
\item \( \tilde{\mu}' \) is a semifinite \( \pi \)-limit outer measure,
\item \( \mathcal{M}_\sim = \mathcal{M}_{\sim}' \),
\item for every \( A \in \mathcal{M}_\sim' \),
\[
\tilde{\mu}'(A) = \sup \{ \tilde{\mu}'(C) : C \in \mathcal{C}, C \subseteq A \}.
\]
\end{enumerate}
We now turn to the problem of finding when there exist \( \tau \)-limit outer measures which are Radon-like w.r.t. \( (\mathcal{N}, \mathcal{X}, \mathcal{J}) \) for appropriate families \( \mathcal{N}, \mathcal{X} \). We shall seek such results when \( \mathcal{N} \) is either \( \mathcal{C} \) or \( \mathcal{C}_0 \).

As previously stated (remark 2.2) the measures \( \mu_i \) are Radon-like w.r.t. \( (\mathcal{N}, \mathcal{C}_1, \mathcal{J}_1) \) if they are bounded. We therefore consider first systems with only this condition added.

2.7 Theorem. If for each \( i \in I \), \( \mu_i \) is bounded, then \( \widetilde{\mu} \) is Radon-like w.r.t. \( (\mathcal{N}, \mathcal{C}, \mathcal{J}(\mathcal{C})) \).

It is not as a rule possible to extend theorem 2.7 to the case in which the measures \( \mu_i \) are infinite - see remark 1.8.

We next try to obtain results similar to theorem 2.7 for the family \( \mathcal{C}_0 \). We are able to do this for some infinite measures but we need other conditions on the system. The factors which govern our choice of conditions are essentially the same as in the topological case (see discussion preceding theorem 1.11) but in addition we have to restrict the cardinality of \( I \) to allow for the fact that \( \mathcal{C}_0 \) is only \( \mathcal{N}_\alpha \)-compact.

2.8 Theorem. For each \( i \in I \), let \( \mu_i \) be a \( \sigma \)-finite outer measure which is Radon-like w.r.t. \( (\mathcal{N}, \mathcal{C}_1, \mathcal{J}_1) \) for some \( \mathcal{J}_1 \) with \( \mathcal{C}(\mathcal{C}_1) \subset \mathcal{J}_1 \subset \mathcal{C}_1 \). Then, whenever any one of the following conditions holds, there exists a \( \tau \)-limit outer measure which is Radon-like w.r.t. \( (\mathcal{N}, \mathcal{C}_0, \sigma_{\alpha+1}(\text{Rect}(\mathcal{J}))) \).
.1 Card I \leq \aleph_0.

.2 Card I \leq \aleph_\alpha, and there exists a cofinal set I_0 \subseteq I with Card I_0 \leq \aleph_0 and, \pi_{i,j}[A] \in C_i for every \alpha, \beta \in I_0 whenever i < j.

.3 Card I \leq \aleph_\alpha, and \pi_{i,j}[A] \in C_i for every \alpha, \beta \in I_0 and \pi_{i,j}^{-1}[B] \in C_j for every \beta \in I_0 whenever i < j.

3. Proofs.

Since theorems 1.3, 1.4, 1.5, 1.6 are special cases of 2.3, 2.4, 2.5, 2.6, respectively we shall prove only the latter.

Proof of 2.3(1.3). Let g* be the finitely additive extension of g to \sigma_0(\text{Rect}(_0)). Then \sigma_0(\text{Rect}(_0)) is an inner family for g* on \sigma_0(\text{Rect}(_0)) (from lemma D in the proof of theorem 2.6 Ch.III), and is \aleph_0-compact by lemmas 3.5.1 and 3.4.3 in Ch.I. The result then follows from theorem 2.4 in Ch.II, by choosing \sigma_0(\text{Rect}(_0)) for the family 'C' in theorem 2.4 Ch.II.

Proof of 2.4(1.4). If \tilde{\mu}(\tilde{X}) < \infty, the result follows from theorem 2.3. Otherwise, for any \alpha \in I, there exists a sequence C_0, C_1, \ldots \in C_1 such that

\mu_1(X_1) = \sup\{\mu_1(C_n) : n \in \mathbb{N}\} = \infty.

Clearly \mu_1 is \sigma-finite on \bigcup C_n, so that \tilde{\mu} is infinite but \sigma-finite on \pi_1^{-1}[\bigcup C_n](\in \mathcal{P}_\mu). Thus, \tilde{\mu} is infinite and \sigma-finite on \tilde{\alpha} \cap \pi_1^{-1}[\bigcup C_n] or on \tilde{\alpha} \sim A \cap \pi_1^{-1}[\bigcup C_n], and hence the result follows easily from theorem 2.3.
Proof of 2.5(1.5). By lemma C in the proof of theorem 2.6 Ch.III, for every \( A \in \text{Rect}(\mathcal{M}) \)

\[ g(A) = \sup \{ g(C) : C \subseteq A, C \in \text{Rect}(C) \} , \]

and since \( \text{Rect}(C) \subseteq \text{Rect}(\mathcal{M}) \) and \( \tilde{\mu}(A) = g(A) \) for every \( A \in \text{Rect}(\mathcal{M}) \), then

\[ \tilde{\mu}(A) = \sup \{ \tilde{\mu}(C) : C \in \text{Rect}(C), C \subseteq A \} \]

which implies condition (b) since \( \text{Rect}(C) \subseteq \mathcal{C} \).

Proof of 2.6(1.6). By theorem 2.7 Ch.II, 2.6.1 and 2.6.2 hold. To see 2.6.3 let \( A \in \mathcal{M}_\gamma \), then

\[ \tilde{\mu}'(A) = \sup \{ \tilde{\mu}(B) : B \subseteq A, B \in \mathcal{M}_\mu, \tilde{\mu}(B) < \infty \} \]

\[ = \sup \{ \sup \{ \tilde{\mu}(C) : C \subseteq B, C \in \mathcal{C} \} : B \in \mathcal{M}_\mu, \tilde{\mu}(B) < \infty \} \]

\[ = \sup \{ \tilde{\mu}'(C) : C \subseteq A, C \in \mathcal{C} \} \]

since for every \( C \in \mathcal{C} \),

\[ \mu(C) = \mu'(C) < \infty . \]

We next prove theorem 2.7, which we will use to prove theorem 1.7.

Proof of 2.7. By lemma 3.3, Ch.I, \( \mathcal{C} \) is \( \mathcal{M}_\alpha \)-compact. By theorem 2.3 \( \mathcal{C} \) is an inner family for \( \tilde{\mu} \) so that \( \mathcal{C} \subseteq \mathcal{M}_\mu \) (lemma 2.2, Ch.II), hence

\[ \tilde{\mu}(G) = \sup \{ \tilde{\mu}(C) : C \subseteq G, C \in \mathcal{C} \} \]

for every \( G \in \mathcal{C} \). Clearly also \( \mathcal{C} \) is an outer family for \( \tilde{\mu} \) since for every \( A \in \mathcal{C} \), there exists \( B \in \mathcal{M}_\mu \), \( A \subseteq B \) with

\[ \tilde{\mu}(A) = \tilde{\mu}(B) \]

and, since \( \tilde{\mu}(\mathcal{X}) < \infty \), for every \( \varepsilon > 0 \) there exists \( C \in \mathcal{C} \), \( \mathcal{C} \subseteq \mathcal{X} \sim B \) with

\[ \mu(\mathcal{X} \sim B \sim C) < \varepsilon \]
so that \( B \subset X \sim C \in \mathcal{C} \) and \( \mu(X \sim C) < \mu(B) + \varepsilon \).

**Proof of 1.7.** From theorem 2.7 we see that \( \widetilde{\mu} \) satisfies the requirements imposed on \( \nu \) in the hypotheses of theorem 2.5, Ch.II (by using \( \mathcal{B} \) for the family of base elements \( \mathcal{B} \), and noting that \( \mathcal{C} \) consists of sets which are closed and compact in the topology \( \mathcal{J} \), and \( \mathcal{C} \subset \mathcal{M} \)). If we let, as in theorem 2.5, Ch.II,

\[ \mathcal{K} = \{ A : A \text{ is closed and compact in } \mathcal{J} \} \]

and

\[ h(G) = \sup \{ \widetilde{\mu}(K) : K \in \mathcal{K}, K \subset G \}, \]

for \( G \in \mathcal{J} \), then \( h(U) = \widetilde{\mu}(U) \nu \) for every \( U \in \mathcal{B} \) (from theorem 1.3 since \( \mathcal{B} \subset \mathcal{M} \)).

If we now let

\[ \gamma(A) = \inf \{ h(G) : G \in \mathcal{J}, A \subset G \}, \]

then, from theorem 2.5, Ch.II, \( \gamma \) is Radon w.r.t. \( \mathcal{J} \) and for every \( G \in \mathcal{J} \),

\[ \gamma(G) = h(G) = \widetilde{\mu}(G). \]

Clearly also for every \( C \in \mathcal{C} \), \( \gamma(C) \geq \widetilde{\mu}(C) \). For any \( \varepsilon > 0 \) and \( A \in \text{Rect}(\mathcal{M}) \), there exists by theorem 1.3 and the fact that \( \mathcal{B} \) is an outer family, \( C \in \mathcal{C} \) and \( \mathcal{B} \subset \mathcal{B} \) such that \( C \subset A \subset G \) and

\[ \mu(C) + \varepsilon \geq \mu(A) \geq \mu(G) - \varepsilon, \]

so that

\[ \mu(G) = \gamma(G) \geq \gamma(A) \geq \gamma(C) \geq \mu(G) - 2 \varepsilon. \]

Thus \( \gamma(A) = \mu(A) \), and since \( \mathcal{B} \subset \mathcal{M}_\gamma \), \( \mathcal{C} \subset \mathcal{M}_\gamma \), we have \( A \in \mathcal{M}_\gamma \) and \( \gamma \) is a \( \pi \)-limit outer measure.

To prove theorem 1.10 we will need the following lemmas.
Lemma A. Let for each \( i \in I \), \( \mu_i \) be Radon w.r.t. \( \mathcal{J}_i \), and let \( K_1, K_2 \in \tilde{X} \) with \( K_1 \cap K_2 = \emptyset \). Then
\[
\tilde{\mu}(K_1 \cup K_2) = \tilde{\mu}(K_1) + \tilde{\mu}(K_2).
\]

Proof. Let \( \tilde{X} \sim K_1 = \bigcup \mathcal{B} \) where \( \mathcal{B} \subset \text{Rect}(\mathcal{J}) \). Then \( K_2 \subset \bigcup \mathcal{B} \) hence there exist \( B_0, B_1, \ldots, B_n \in \mathcal{B} \) such that \( K_2 \subset \bigcup_{m=0}^{n} B_m \) and \( K_1 \subset \tilde{X} \sim \bigcup_{m=0}^{n} B_m \). Since \( \bigcup_{m=0}^{n} B_m \in \tau_{\tilde{\mu}} \),
\[
\tilde{\mu}(K_1 \cup K_2) = \tilde{\mu}(K_1) + \tilde{\mu}(K_2).
\]

Lemma B. Let \( \nu \) be a Radon measure on a topological space \( Y \) with topology \( \mathcal{K} \). Then \( \nu \) is completely determined by its values on any base \( \mathcal{B} \) with \( \sigma_{0}(\mathcal{B}) = \mathcal{B} \).

Proof. For every closed compact set \( C \),
\[
\nu(C) = \inf \{ \nu(G) : G \text{ open, } C \subset G \},
\]
\[
= \inf \{ \nu(B) : B \in \mathcal{B}, C \subset B \}
\]
since if \( C \subset G \in \mathcal{K} \), there exists \( B \in \mathcal{B} \) with \( C \subset B \subset G \). Thus the value on every closed compact set is determined by the values on \( \mathcal{B} \). But the value on every open set is determined by the values on the closed compact sets, and since the value of \( \nu \) on every set \( A \subset Y \) is determined by the value on the open sets, \( \nu \) is completely determined.

Proof of 1.10. Let
\[
\mathcal{S} = \{ A : A \in \sigma_{0}(\text{Rect}(\mathcal{J})) \text{ and } \tilde{\mu}(A) < \infty \}.
\]
Then in view of theorem 1.3, we see that the hypotheses on \( \nu, \mathcal{B}, \mathcal{K} \) in theorem 2.5 of Ch.II, are satisfied if we replace \( \nu \) \( \tilde{\mu} \), \( \mathcal{B} \) by \( \mathcal{S} \) and \( \mathcal{K} \) by \( \tilde{\mathcal{K}} \). (Note that singletons in \( X_1 \) have finite \( \mu_1 \) measure, hence \( \mathcal{S} \) is a base for \( \mathcal{J} \)). Thus \( \tilde{\mu}^* \) is Radon w.r.t. \( \mathcal{J} \).
To see 1.10.2, 1.10.3 let \( \mathcal{Y} \) be any \( \mathcal{M} \)-limit outer measure which is Radon w.r.t. \( \mathcal{M} \). Then for any \( G \in \mathcal{M} \),
\[
\mathcal{Y}(G) = \sup \{ \mathcal{Y}(K) : K \subseteq G \}
\]
\[
\widehat{\mu}^*(G) = \sup \{ \widehat{\mu}(K) : K \subseteq G \} \leq \widehat{\mu}(G)
\]
and, since \( \mathcal{Y} \leq \widehat{\mu} \) (remark 2.4.3, Ch.III), we get
\[
\mathcal{Y}(G) \leq \widehat{\mu}^*(G) \leq \widehat{\mu}(G).
\]
But for every \( A \in \mathcal{\sigma}_0(\text{Rect}(\mathcal{M})) \) we must have \( \mathcal{Y}(A) = \widehat{\mu}(A) \), hence for every \( E \in \mathcal{M} \),
\[
\mathcal{Y}(E) = \widehat{\mu}^*(E) = \widehat{\mu}(E) = \sup \{ \widehat{\mu}(K) : K \subseteq E \}.
\]
1.10.3 follows immediately by lemma B.

We will use theorem 2.8 in the proof of theorem 1.11, hence we prove theorem 2.8 first. The following lemmas will be used in the proof of theorem 2.8.

**Lemma C.** If, for each \( i \in I \), \( \mu_i \) is \( \sigma \)-finite and Radon-like w.r.t. \((\mathcal{N}_i, C_i, \mathcal{L}_i)\) for some \( \mathcal{L}_i \) such that \( C(C_1) \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_1 \),

then for every \( \varepsilon > 0 \) and \( \alpha \in \text{Rect}(\mathcal{M}) \) there exists \( G \in \text{Rect}(\mathcal{L}) \) such that \( \alpha < G \) and \( \widehat{\mu}(G \sim \alpha) < \varepsilon \).

**Proof.** Let \( \varepsilon > 0 \) and \( \alpha \in \text{Rect}(\mathcal{M}) \). Let \( n \) be the number of elements of \( J_\alpha \), and for each \( j \in J_\alpha \) choose \( G_j \in \mathcal{L}_j \) such that
\[
\pi_j[\alpha] \subseteq G_j \quad \text{and} \quad \widehat{\mu}(G_j \sim \pi_j[\alpha]) < \frac{\varepsilon}{n}.
\]
Then
\[
G = \bigcap_{j \in J_\alpha} \pi_j^{-1}[G_j] \in \text{Rect}(\mathcal{L}),
\]
and \( \alpha \in G \). Then
\[
G \sim \alpha \subseteq \bigcup_{j \in J_\alpha} \pi_j^{-1}[G_j \sim \pi_j[\alpha]],
\]
hence
\[
\tilde{\mu}(G \sim \alpha) \leq \sum_{j \in J_\alpha} \tilde{\mu}(\pi_j^{-1}[G_j \sim \pi_j[\alpha]]) < \frac{n}{n} = \varepsilon.
\]

**Lemma D.** Let \( i < j < k \), \( A \subseteq X_k \) and \( \alpha = p_{ij} \circ p_{jk}[A] \in \mathcal{M}_i \), then
\[
\mu_k(A \sim p_{ik}^{-1}[\alpha]) = 0.
\]

**Proof.** \( A \sim p_{ik}^{-1}[\alpha] \subseteq (p_{ij} \circ p_{ik})^{-1}[\alpha] \sim p_{ik}^{-1}[\alpha] \),
and, by definition 1.4, Ch.III
\[
\mu_k((p_{ij} \circ p_{jk})^{-1}[\alpha] \sim p_{ik}^{-1}[\alpha]) = 0.
\]

**Proof of 2.8.** Suppose first that \( \text{Card } I \leq \aleph_\alpha \) and let
\[
\mathscr{B} = \{ A : A \in C_0(\text{Rect}(\mathscr{A})) \text{ and } \tilde{\mu}(A) < \infty \}.
\]
We shall check that the hypotheses of theorem 2.6, Ch.II are satisfied with "\( \nu \)" replaced by "\( \tilde{\mu} \)", "\( \mathcal{C}_0 \)" and \( \mathcal{B} \) as above.
(a) From lemmas 3.5.2, 3.4.3 and 3.3 Ch.I, \( \mathcal{C}_0 \) is \( \aleph_\alpha \)-compact and clearly \( \mathcal{C}_0 = \mathcal{C}_1(\mathcal{C}_0) = \sigma_0(\mathcal{C}_0) \).
(b) To see that \( \mathcal{B} \subseteq \mathcal{C}_0 \), let
\[
\mathcal{J} = \{ \Pi C_i : C_i \subseteq C_i \text{ for every } i \in I \}.
\]
Then for any \( i \in I \) and \( G \in \mathcal{J}_i \), by lemma 3.5.3, Ch.I
\[
\pi_i^{-1}[X_i \sim G] \in \mathcal{J}_\mathcal{J},
\]
hence \( \text{Cyl}\mathcal{B} \subseteq \mathcal{J}_\mathcal{J} \). By lemmas 1.9.3, 1.9.4 of Ch.I, \( \text{Cyl}\mathcal{B} = \mathcal{C}_0 \).
Then by lemmas 1.8.3 and 1.7.3 of Ch.I,
\[
\sigma_0(\text{Rect}(\mathcal{B})) = \mathcal{C}_0,
\]
hence $\mathcal{B} \subset \mathcal{H}_0$. Clearly also $\mathcal{B} = \sigma_0(\mathcal{B}) = \delta_0(\mathcal{B})$.

(c) To see that $\sigma(\mathcal{H}_0) = \sigma_{\alpha+1}(\mathcal{B})$, let for each $i \in I$, $C_i \in \mathcal{C}_i$ and let $C = \bigcap\limits_{i \in I} C_i$. Then for each $i \in I$,

$$X \sim \pi_i^{-1}[C_i] = \pi_i^{-1}[X_i \sim C_i] \in \sigma_i(\mathcal{B}),$$

since $\mu_i$ is $\sigma$-finite and $X_i \sim C_i \in \mathcal{C}_i$. Since $\text{Card } I \leq \aleph_\alpha$, $C = \bigcap\limits_{i \in I} \pi_i^{-1}[C_i]$ and

$$\sigma_{\alpha+1}(\sigma_1(\mathcal{B})) = \sigma_{\alpha+1}(\mathcal{B}),$$

we have

$$\tilde{X} \sim \bigcap\limits_{C \in \sigma_{\alpha+1}(\mathcal{B})} C.$$

Thus since $\sigma_{\alpha+1}(\mathcal{B})$ is closed under finite intersections (from lemma 1.8.3, Ch.I, and the fact that $\mathcal{B} = \delta_0(\mathcal{B})$), we have

$$\sigma(\mathcal{H}_0) \subset \sigma_{\alpha+1}(\mathcal{B}).$$

(d) To check that for every $D \in \mathcal{B}$,

$$\widetilde{\mu}(D) = \sup\{\mu(F) : F \in \mathcal{H}_0 \land \pi_{\mathcal{J}}^{-1}[C] \in \mathcal{H}_0, \, F \subset D\},$$

note that, for every $i \in I$ and $C \in \mathcal{C}_i$, $\pi_i^{-1}[C] \in \mathcal{H}_0$, and that by lemmas 1.7, 1.8 of Ch.I,

$$\mathcal{H}_0 = \delta_1(\mathcal{H}_0) = \sigma_0(\mathcal{H}_0).$$

Hence

$$\mathcal{C} = \delta_1(\sigma_0(\text{Rect}(\mathcal{C}))) \subset \mathcal{H}_0 \subset \pi_{\mu}^{-1},$$

so that by theorem 2.3, for every $D \in \mathcal{B}$,

$$\widetilde{\mu}(D) = \sup\{\mu(F) : F \in \mathcal{H}_0, \, F \in \pi_{\mu}^{-1}, \, F \subset D\}. $$
(e) To check that \( \tilde{\mu} \) is finitely additive on \( \mathcal{C}_0 \), let \( C_1, C_2 \in \mathcal{C}_0 \) be such that \( C_1 \cap C_2 = \emptyset \). Then \( \tilde{X} \sim C_1 \in \sigma_{\alpha+1}(\mathcal{B}) \), so that there exists \( \mathcal{B}' \subset \mathcal{B} \) with Card \( \mathcal{B}' \leq \aleph_\alpha \) and \( \tilde{X} \sim C_1 = \bigcup \mathcal{B}' \). Then \( C_2 \subset \bigcup \mathcal{B}' \), and since \( \mathcal{B}' \subset \mathcal{C}_0 \), there exist \( D_0, D_1, \ldots, D_n \in \mathcal{B}' \) such that \( C_2 \subset \bigcup_{m=0}^{n} D_m \). Since \( \bigcup_{m=0}^{n} D_m \in \mathcal{C}_0 \) and \( C_1 \cap \bigcup_{m=0}^{n} D_m = \emptyset \), we have

\[
\tilde{\mu}(C_1 \cup C_2) = \tilde{\mu}(C_1) + \tilde{\mu}(C_2).
\]

Thus in view of (a), (b), (c), (d), (e) above, and theorem 2.6, Ch.II, if we let

\[
h(G) = \sup\{\tilde{\mu}(C) : C \in \mathcal{C}_0, C \subseteq G\}
\]

for every \( G \in \sigma_{\alpha+1}(\mathcal{B}) \), and

\[
\psi(A) = \inf\{h(G) : A \subseteq G, G \in \sigma_{\alpha+1}(\mathcal{B})\},
\]

for every \( A \subseteq \tilde{X} \), then \( \psi \) is an outer measure on \( \tilde{X} \), and is Radon-like w.r.t. \( (\aleph_\alpha, \mathcal{C}_0, \sigma_{\alpha+1}(\mathcal{B})) \), \( \psi(G) = h(G) \leq \tilde{\mu}(G) \) for every \( G \in \sigma_{\alpha+1}(\mathcal{B}) \), and \( \psi(C) \geq \tilde{\mu}(C) \) for every \( C \in \mathcal{C}_0 \).

We show now that \( \psi \) is a \( \pi \)-limit outer measure if the following condition is satisfied for every \( A \in \text{Rect}(\mathcal{M}) \),

\[
(*) \quad \tilde{\mu}(A) = \sup\{\tilde{\mu}(C) : C \in \mathcal{C}_0, C \subseteq A\}.
\]

Clearly if (*) holds for all \( A \in \text{Rect}(\mathcal{M}) \), it also holds for all \( A \in \sigma_1(\text{Rect}(\mathcal{M})) \). Hence if \( A \in \sigma_0(\text{Rect}(\mathcal{M})) \), \( A \subseteq G \) where \( G \in \sigma_{\alpha+1}(\mathcal{B}) \), and (*) holds, then \( h(G) = \tilde{\mu}(A) \) and therefore \( \psi(A) = \tilde{\mu}(A) \).

Since \( \mathcal{B} \subset \sigma_0(\text{Rect}(\mathcal{M})) \), we have \( h(D) = \tilde{\mu}(D) \) for all \( D \in \mathcal{B} \). Thus for any \( A \in \text{Rect}(\mathcal{M}) \), by lemma C,

\[
\psi(A) = \inf\{h(D) : D \in \mathcal{B}, A \subseteq D\} \leq \tilde{\mu}(A),
\]

so that

\[
\psi(A) = \tilde{\mu}(A) = \sup\{\tilde{\mu}(C) : C \in \mathcal{C}_0, C \subseteq A\}.
\]
\[ \{ \sup \nu(C) : C \in \mathcal{G}, C \subseteq A \} = \nu(A), \]
and since \( \nu \) is \( \sigma \)-finite, there exists \( C \in \mathcal{G}(\mathcal{G}) \) with \( C \subseteq A \) and
\[ \nu(A \sim C) = 0, \]
so that \( A \in \mathcal{M}_\nu \). Thus \( \nu \) is a \( \mathcal{M} \)-limit outer measure if (*) holds.

(1) Suppose 2.8.1 holds. Let \( \alpha \in \text{Rect}(\mathcal{M}) \) and let \( I = \{ i_0, i_1, \ldots \} \).
Then, for \( t < \mu(\alpha) \), choose by recursion \( C_n \in \mathcal{G}_n \) such that
\[ C_n \subseteq \pi_n^{-1}(\alpha) \text{ and } \mu(\alpha \cap \bigcap_{m=0}^{n-1} [C_m]) > t \]
(by lemma A, Ch. III) and let

\[ C = \bigcap_{n \in \omega} \pi_n^{-1}(C_n). \]

Then \( C \in \mathcal{G}_0 \), \( C \subseteq \alpha \) and \( \mu(C) > t \). Hence (*) holds.

(b) Suppose that 2.8.2 holds. Let \( \alpha \in \text{Rect}(\mathcal{M}) \) and \( t < \mu(\alpha) \).
Choose a cofinal subset \( \{ i_0, i_1, \ldots \} \) of \( I \) with \( j < i_0 \) for every
\( j \in J_{\alpha} \) and \( i_n < i_{n+1} \) for every \( n \). For each \( n \), let
\[ A_n = \bigcap_{j \in J_{\alpha}} \pi_{j}^{-1}(\alpha) \]
so that
\[ \mu_i(A_n) = \mu(\alpha) > t, \]
and
\[ \mu_i(A_{n+1} \Delta \pi_{i}^{-1}(\alpha) [A_n]) = 0. \]

By recursion, choose \( C_n \in \mathcal{G}_n \) so that \( C_0 = A_0 \),
\[ C_{n+1} \subseteq A_{n+1} \cap \bigcap_{m=0}^{n} \pi_{i}^{-1}(C_m) \].
and $\mu_{j}^{-1}(C_{n}) > t$. For any $j \in I$, let $n$ be the smallest integer with $j < n$ and set $K_{j} = p_{j}^{-1}[C_{n}]$, and $K = \Pi_{j \in I} K_{j}$. Then $K_{j} \in C_{j}$ and $K_{j} \subseteq \alpha_{j}$ so that $K \subseteq C_{0}$ and $K \subseteq \alpha$. To check that $\mu(K) > t$, we first note that for any finite $J < I$ we have

$$\mu(\cap_{j \in J} \alpha_{j}^{-1}[K_{j}]) > t.$$  

Indeed, for each $j \in J$, let $n_{j}$ be the smallest integer with $j < n_{j}$ and $k = \max n_{j}$. Then, for any $j \in J$,

$$C_{k} \sim p_{j}^{-1}[K_{j}] = C_{k} \sim p_{j}^{-1}[p_{j}^{-1}[C_{n_{j}}]] \subseteq C_{k} \sim p_{j}^{-1}[p_{j}^{-1} \circ p_{j}^{-1} \circ K_{j}]$$

so that by lemma D,

$$\mu_{k}(C_{k} \sim p_{j}^{-1}[K_{j}]) = 0$$

and therefore

$$\mu(\cap_{j \in J} \alpha_{j}^{-1}[K_{j}]) = \mu(\cap_{j \in J} \alpha_{j}^{-1}[K_{j}]) > t.$$  

Let $\mathcal{K} = \{H_{0}, H_{1}, \ldots\}$ Rect$(\mathcal{M})$ be a cover of $K$. By lemma C, given $\varepsilon > 0$, for each new let $G_{n} \in $ Rect$(\mathcal{M})$ be such that $H_{n} \subseteq G_{n}$ and $\mu(G_{n}) \leq \mu(H_{n}) + \frac{\varepsilon}{2^{n}}$. Since $G_{0}, G_{1}, \ldots \in C_{0}$, there exists new such that $K \subseteq \bigcup_{l=0}^{m} G_{l}$. Let $J = \bigcup_{l=0}^{m} J_{l}$. Then $J$ is finite, and for any $x \in \cap_{j \in J} \alpha_{j}^{-1}[K_{j}]$ there exists $y \in K$ with

$$y_{j} = x_{j}$$

for all $j \in J$. From lemma 2.4, Ch.I, we have

$$x \in \cap_{j \in J} \alpha_{j}^{-1}[\{\alpha_{j}(y)\}] \subseteq G_{l},$$

for some $l = 0, 1, \ldots, m$,

so that

$$\cap_{j \in J} \alpha_{j}^{-1}[K_{j}] \subseteq \bigcup_{l=0}^{m} G_{l}.$$
Thus
\[ \sum_{l \in \omega} \mu(G_l) \geq \mu(\bigcup_{l=0}^{m} G_l) > t, \]
hence,
\[ \sum_{n \in \omega} \mu(H_n) > t - \epsilon. \]

Since \( \epsilon \) is arbitrary and for every \( H \in \text{Rect}(\mathcal{F}) \), \( g(H) = \mu(H) \), we have \( \mu(K) > t \) so that (*) holds.

(c) Finally, suppose that 2.8.3 holds. Let \( a \in \text{Rect}(\mathcal{F}) \) and
\[ k > j \quad \text{for every } j \in J_\alpha. \]
For \( t \in \mu(a) \) choose \( C \in C_k \) with
\[ C \cap \bigcap_{j \in J_\alpha} p^{-1}_{jk}[\pi_j[a]], \]
and such that \( \mu(C) > t \). For any \( i \in I \) let
\[ C_i = \begin{cases} p_{ik}[C] & \text{if } i < k \\ p_{ij}[p^{-1}_{kj}[C]] & \text{for some } j \in I, j > k. \end{cases} \]

As in 2.8.2 we see that if \( K = \bigcap C_i \) then \( K \in \mathcal{C}_\alpha \), \( K \in \alpha \) and \( \mu(K) > t \), so that again (*) holds.

**Proof of 1.11.** From the definition of \( \mathcal{C}_0 \) in section 2, it follows that in the topological case, in which the \( C_i \) are closed compact sets, \( \mathcal{C}_0 \) consists of closed compact sets.
Furthermore, if in the hypotheses of theorem 2.8, the families \( \mathcal{J}_i \) are topologies, then by taking \( \alpha \) to be a sufficiently large ordinal,
\[ \sigma_{\alpha+1}(\text{Rect}(\mathcal{J})) = \mathcal{Y}, \]
where \( \mathcal{Y} \) is the product topology.

Thus theorem 2.8 shows that under the hypotheses of
Theorem 1.11: There exists a $\sigma$-limit outer measure $\nu$ such that $\mathcal{Y}$ is an outer family for $\nu$ on $\tilde{X}$, and for every $G \in \mathcal{Y}$,

$$\nu(G) = \sup\{\nu(K) : K \in \tilde{X}, K \subseteq G\}.$$  

Furthermore

$$\{G : G \in \text{Rect}(\mathcal{Y}) \text{ and } \nu(G) < \infty\}$$

is a base for $\mathcal{Y}$, so that since every $K \in \tilde{X}$ is contained in the union of a finite number of elements of this base, $\nu(K) < \infty$ for every $K \in \tilde{X}$. Hence $\nu$ is Radon w.r.t. $\mathcal{Y}$. Then by theorem 1.10.3, $\nu = \widetilde{\mu}^*$.  

CHAPTER V

LIMIT MEASURES ON THE INVERSE LIMIT SET

In this chapter we try to answer the following questions. When does a π-limit outer measure exist on the inverse limit set, L, and what 'regularity' conditions can it possess (e.g. when is it Radon)?

Our approach is from the point of view of restricting ~μ to L.

1. Definitions and Notation.

In this section we collect definitions and notation used in the sequel.

1.1 Basic Assumptions. Throughout this section we assume (X, p, μ, I) is an i.s.o.m. and

\[ P_{ik} = P_{ij} \circ P_{jk} \]

whenever i<j<k, so that (X,p,I) is actually an inverse system of spaces. We also assume that the inverse limit set, \( L_\pi \), is such that for every i\( \in \)I, \( \pi_i[L_\pi] = X_i \) (simple maximality).

1.2 Definitions (Subsystems). For any directed subset J of I,

1. (X, p, μ, J) will denote the subsystem obtained by restricting X and μ to J and p to

[\{(i,j) : i<j and i,j\in J\}]

Clearly (X,p,μ,J) is also an i.s.o.m.
.2 \( \tilde{X}_J = \prod_{i \in J} X_i \).
(In case \( J = I \) we may write \( \tilde{X} \) for \( \tilde{X}_I \).)

.3 \( L_J = \{ x \in \tilde{X}_J : \pi_i(x) = p_{ij}(\pi_j(x)) \} \) whenever \( i < j \) and \( i, j \in J \).
Thus \( L_J \) is the inverse limit set of \( (X, p, \mu, J) \).

.4 \( r_J \) is the function on \( \tilde{X} \) to \( \tilde{X}_J \) such that for every \( x \in \tilde{X} \), \( r_J(x) = x \mid J \).

.5 \( \nu \) is an inverse limit outer measure for \( (X, p, \mu, J) \)
iff \( \nu \) is an outer measure on \( \tilde{X}_J \) such that:

.5.1 \( \nu \) is carried by \( L_J \), i.e. \( \nu(\tilde{X}_J \setminus L_J) = 0 \),

.5.2 for every \( i \in J \) and \( A \in \mathcal{M}_i \), \( \pi_i^{-1}[A] \in \mathcal{M}_j \) and
\( \nu(\pi_i^{-1}[A]) = \mu_i(A) \).
(This is equivalent to the standard definition of inverse limit measure.)

In the next two definitions we introduce properties of the system and of measures which we will use in the theorems to follow.

1.3 Definition. \( (X, p, \mu, I) \) satisfies sequential maximality iff, for every countable directed subset \( J \) of \( I \), the range of \( r_J \mid L_I \) is all of \( L_J \), i.e.: for every sequence \( i_0, i_1, \ldots \) in \( I \) with \( i_n < i_{n+1} \) and sequence \( y \) with \( y_n \in X_{i_n} \) and \( p_{i_n i_{n+1}}(y_{n+1}) = y_n \) for every \( n \), there exists
such that \( x_n = y_n \) for every \( n \in \omega \).

1.4 Definition. An outer measure \( \varphi \) on a space \( S \) is almost separable iff there exists a countable family \( B \subseteq \mathcal{M}_\varphi \), and a set \( T \subseteq S \) such that \( \varphi(T) = 0 \) and for every \( x, y \in S \sim T \) with \( x \neq y \) there exists \( B \in \mathcal{B} \) with \( x \in B \) and \( y \notin B \).

In many cases we will wish to work with semifinite measures although the measure produced by our extension process may not be semifinite. The following definition indicates how we obtain such a measure from the given one.

1.5 Definition. If \( \nu \) is an outer measure on a space \( S \), \( \nu' \) is the semifinite outer measure on \( S \) derived from \( \nu \) by taking

\[
\nu'(A) = \sup\{\nu(B) : B \subseteq A \text{ and } \nu(B) < \infty\},
\]

for every \( A \subseteq S \).

1.6 Remark. It follows from theorem 2.7 Ch.II that \( \nu' \) is indeed a semifinite outer measure. If \( \nu \) is itself a semifinite outer measure then \( \nu = \nu' \).

2. Existence of an Inverse Limit Measure.

In this section we consider the problem of the existence of an inverse limit outer measure. We begin by indicating the relation between such a measure and the measure \( \tilde{\mu} \) introduced in 2.2.1 Ch.III.
2.1 **Lemma.** An inverse limit outer measure exists iff \( \tilde{\mu} \mid L_I \) is an inverse limit outer measure.

2.2 **Lemma.** If for each \( i \in I \), \( \mu_i \) is a semifinite outer measure, then \( \tilde{\mu} \mid L_I \) is an inverse limit outer measure (i.e. such a measure exists) iff \( \tilde{\mu} \) is a \( \pi \)-limit outer measure such that the semifinite outer measure \( \tilde{\mu}' \) derived from \( \tilde{\mu} \) is pseudo-carried by \( L_I \).

For an example of the pathological sets we avoid by considering \( \tilde{\mu}' \) see example 3 in the appendix.

In view of the above lemmas we devote the rest of this section to determining conditions under which \( \tilde{\mu} \) or \( \tilde{\mu}' \) is carried or pseudo-carried by \( L_I \). We have two types of conditions under which this occurs and we discuss them separately. First we consider "separability" conditions.

2.3 **Lemma.** Suppose that for every \( i \in I \), \( \mu_i \) is almost separable.

.1 If \( I \) is countable, then \( \tilde{\mu} \) is carried by \( L_I \).

.2 If sequential maximality is satisfied and for each \( i \in I \), \( \mu_i \) is semifinite, then \( \tilde{\mu}' \) is pseudo-carried by \( L_I \).

2.4 **Remark.** 2.3.1 remains true even if \( (X,p,\mu,I) \) is an i.s.o.m. for which \( (X,p,I) \) is not necessarily an inverse system of spaces. In effect the rest of the hypotheses force \( L_I \) to be large enough to carry \( \tilde{\mu} \).
By combining lemma 2.3 with the fundamental existence theorem 2.6 of Ch.III we obtain the following theorem.

2.5 **Theorem.** If \((X,p,\mu,\mathcal{I})\) is inner regular w.r.t.\(\mathcal{C}\) for some \(\mathcal{C}\) (definition 2.5 Ch.III) and if for each \(i\in\mathcal{I}\), \(\mu_i\) is almost separable, then an inverse limit outer measure exists whenever one of the following conditions holds.

1. \(\mathcal{I}\) is countable,
2. \((X,p,\mu,\mathcal{I})\) satisfies sequential maximality.

2.6 **Remarks.** Previously known existence theorems require further conditions on the images and inverse images of the functions \(p_{ij}\) than are used in theorem 2.5 (see e.g. Choksi [4], Metevier [10]).

In view of lemma 2.3.1 we can conclude that in 2.5.1 \(\tilde{\mu}\) is a \(\mathcal{I}\)-limit outer measure which is carried by \(L_\mathcal{I}\) and not just pseudo-carried. The following theorem shows that this is not the case for any non-trivial system when \(\text{Card } \mathcal{I} > \aleph_0\).

2.7 **Theorem.** If \(\mathcal{I}\) is countable, and \(X_i\) contains at least two points for uncountably many \(i\in\mathcal{I}\), then for every \(A\in\text{Rect}(\mathcal{M})\),

\[
\tilde{\mu}(A) = \tilde{\mu}(A \sim L_\mathcal{I}),
\]

hence \(\tilde{\mu}\) is not carried by \(L_\mathcal{I}\) whenever \(\tilde{\mu} \neq 0\).
From the above theorem we see that in many significant cases where an inverse limit outer measure does exist, $L_\mathcal{I}$ is not $\tilde{\mu}$-measurable. This may explain many of the difficulties encountered by inverse limit measures. For example, even when $\tilde{\mu}$ is Radó, its restriction to $L_\mathcal{I}$ may not be.

We now examine another type of conditions under which an inverse limit outer measure exists. Here we establish a "topological" relationship between $L_\mathcal{I}$ and inner families $C_i$ for the measures $\mu_i$. Conditions similar to ours have been used by previous workers (e.g. Bochner [2], Choksi [4], Metevier [10]) who worked only with $L_\mathcal{I}$ (not considering its relation with $\tilde{X}$).

The following theorem is a basic existence theorem from this point of view.

2.8 Theorem. Suppose that the assumptions of Ch.II section 2 are satisfied (see 2.1 Ch.IV). If for every sequence $i \in \mathcal{I}$ in $\mathcal{I}$ with $i_n < i_{n+1}$ for all $n \in \omega$, the family

$$\{\cap_{i_n}^{-1}[C] \cap L_\mathcal{I} : C \in C_i \text{ for some } n \in \omega\}$$

is $\aleph_0$-compact, then $\tilde{\mu}'$ is pseudo-carried by $L_\mathcal{I}$, hence $\tilde{\mu}' | L_\mathcal{I}$ is an inverse limit outer measure.

2.9 Remarks. The hypotheses of theorem 2.8 are obviously satisfied if the spaces $C_i$ are compact Hausdorff, the measures $\mu_i$ Radon, and the functions $p_{ij}$ continuous. In this case $L_\mathcal{I}$ is compact so that

$$\{\cap_{i_n}^{-1}[C] \cap L_\mathcal{I} : C \in C_i \text{ for some } i \in \mathcal{I}\}$$
consists of sets which are compact in the product topology, and thus are certainly \( \aleph_0 \)-compact.

Since in other cases it may be difficult to check the hypotheses of theorem 2.8 directly, we give in the following theorem a condition on countable subsystems which, if sequential maximality holds, will ensure the existence of an inverse limit outer measure. We should note that the following theorem is essentially that of Metevier [10], though we include the semifinite case.

2.10 Theorem. Suppose that the assumptions of Ch.IV section 2 are satisfied (see 2.1 Ch.IV), and that sequential maximality is satisfied. Then \( \tilde{\mu}' \) is pseudo-carried by \( L_\mathbb{I} \), hence \( \tilde{\mu}' \upharpoonright L_\mathbb{I} \) is an inverse limit outer measure, whenever the following conditions hold: if \( i \) is a sequence in \( \mathbb{I} \) with \( i_n < i_{n+1} \) for every \( n \in \mathbb{N} \), and

\[
\mathcal{K}_m = \{ p_{i \downarrow m}^{-1}[C] : C \in C_{i_n} \text{ for some } n \in \mathbb{N} \text{ with } m \leq n \},
\]

1 then \( \mathcal{K}_m \) is \( \aleph_0 \)-compact for every \( m \in \mathbb{N} \), and

2 \( \{ p_{i \downarrow m}^{-1}[x] \cap K : K \in \mathcal{K}_m \} \) is \( \aleph_0 \)-compact for every \( i, m \in \mathbb{N} \) with \( l < m \), and \( x \in X_i \).

We conclude this section by indicating how one can transfer an inverse limit outer measure for a system to one for a subsystem, and vice-versa.
2.11 Theorem. Suppose \( \nu \) is an inverse limit outer measure. Then for any directed subset \( J \) of \( I \), the set function \( \gamma \) generated by the family
\[
\mathcal{A} = \{ A \subset X_J : r_J^{-1}[A] \in m_J \}
\]
and the set function \( h \) on \( \mathcal{A} \), defined by
\[
h(A) = \nu(r_J^{-1}[A]) \text{ for all } A \in \mathcal{A},
\]
is an inverse limit outer measure for \((X,p,\mu,J)\).

2.12 Theorem. Let \( J \) be a cofinal subset of \( I \). Then:
1. \( r_J|L_I \) is one-to-one and onto \( L_J \).
2. If \( \nu \) is an inverse limit outer measure for \((X,p,\mu,J)\), the set function \( \gamma \) defined by
\[
\gamma(A) = \nu(r_J^{-1}[A \cap L_I])
\]
for every \( A \subset X_I \), is an inverse limit outer measure for \((X,p,\mu,I)\).

From the above theorems we see that an i.s.o.m. has an inverse limit outer measure if it can be imbedded in a system which does have one, and that theorems 2.5, 2.8 and 2.10 can be somewhat extended by requiring that their hypotheses be satisfied only for a cofinal subsystem.


We now turn to the problem of finding conditions under which we can find approximating families for inverse limit outer measures.

The following lemmas show that it may be sufficient to find such families for cofinal subsystems.
3.1 Lemma. Let \( J \) be a cofinal subset of \( I \).

1. If \( C \) is an \( \mathcal{K} \)-compact family of subsets of \( L_J \) then
\[
\{L_I \cap r_J^{-1}[C] : C \in C\}
\]
is an \( \mathcal{K} \)-compact family of subsets of \( L_I \).

2. If \( \nu \) is an inverse limit outer measure for \((X, p, \mu, J)\) and if \( C, \mathcal{G} \) are respectively inner and outer families for \( \nu \) then
\[
\{L_I \cap r_J^{-1}[C] : C \in C, C \subset L_J\}
\]
and
\[
\{L_I \cap r_J^{-1}[G] : G \in \mathcal{G} \cup \{X \sim L_I\}\}
\]
are respectively inner and outer families for the inverse limit outer measure \( \mathcal{Y} \) defined by
\[
\mathcal{Y}(A) = \nu(r_J[A \cap L_I])
\]
for every \( A \subset \widetilde{X}_I \).

3.2 Lemma. Let \( J \) be a cofinal subset of \( I \). If \( \nu \) is an inverse limit outer measure for \((X, p, \mu, J)\) which is Radon-like w.r.t. \((\mathcal{K}, C, \mathcal{G})\) then the outer measure \( \mathcal{Y} \) defined by
\[
\mathcal{Y}(A) = \nu(r_J[A \cap L_I])
\]
for all \( A \subset \widetilde{X}_I \),
is an inverse limit outer measure for \((X, p, \mu, I)\) which is Radon-like w.r.t. \((\mathcal{K}, C', \mathcal{G'})\) where
\[
C' = \{r_J^{-1}[C] : C \in C\},
\]
\[
G' = \{r_J^{-1}[G] : G \in \mathcal{G}\}.
\]
3.3 **Remark.** Note that even if all the $X_i$ are topological spaces and $\nu$ is Radón w.r.t. the product topology of $\tilde{X}_j$, we cannot conclude that $\nu$ is Radón w.r.t. the product topology on $\tilde{X}_i$ since the images of compact sets under $r_j^{-1}$ may not be compact. In this case we could use some other topology as indicated in 3.8 below and in example 6 in the appendix.

We now turn our attention to determining when an inverse limit outer measure is Radón or Radón-like. We begin with the case in which we have topological information about $L_I$.

3.4 **Theorem.** Let the assumptions 1.1 of Ch.IV hold, and suppose that for each $i \in I$, $\mu_i$ is Radon. Suppose also that $L_I$ is a closed set in the product topology. Then the set function $\hat{\mu}$ (see definition 1.9 Ch.IV) is a Radón outer measure supported by $L_I$.

It is clear that $L_I$ is closed whenever the spaces $X_i$ are Hausdorff and the functions $p_{ij}$ continuous. Thus, we can combine 3.4 with 1.11 Ch.IV to obtain the following theorem.

3.5 **Theorem.** Suppose that for each $i \in I$, $X_i$ is a Hausdorff space, $\mu_i$ is a $\sigma$-finite Radón outer measure, and $p_{ij}$ is continuous whenever $i < j$. Then $\hat{\mu}$ is an inverse limit outer measure which is Radón w.r.t. the product topology whenever any one of the following conditions hold:
There exists a countable cofinal set $I_0 \subseteq I$.

$p_{ij}^{-1}[A]$ is compact whenever $i < j$ and $A$ is a compact subset of $X_i$.

3.6 Theorem. Suppose that $I$ is countable and that for each $i \in I$,

1. $X_i$ is a topological space,
2. $\mu_i$ is a $\sigma$-finite, almost separable Radon outer measure.

Then $\widehat{\mu}^*$ is a Radon inverse limit outer measure w.r.t. the product topology.

Notice that the above theorem requires no conditions on the functions $p_{ij}$ beyond those necessary for an i.s.o.m.
(We could in fact dispense with the fact that $(X,p,I)$ forms an inverse system of spaces).

We may now use lemma 3.2 to extend theorem 3.6 to systems with a countable cofinal subset; possibly losing, however, the fact that the measure is actually Radon.

3.7 Theorem. Suppose there exists a countable cofinal subset $J$ of $I$ such that for each $i \in J$,

1. $X_i$ is a topological space,
2. $\mu_i$ is a $\sigma$-finite, almost separable Radon outer measure.

Then the outer measure $\nu$ defined by

$$\nu(A) = \widehat{\mu}^*(r_j[A \cap L_i])$$

is an inverse limit outer measure which is Radon-like w.r.t. $C$, & where
3.8 **Remark.** In theorem 3.7, we could use a topology w.r.t. which \( \nu \) is Radon. One way of doing this is to use the topology \( \mathcal{C}_0 \). For another approach which may be successful see example 6 in the appendix. There the product topology is too fine to allow any but trivial inverse limit Radon measures.

The following theorems are non-topological analogues of theorems 3.6 and 3.7.

3.9 **Theorem.** Suppose that \( I \) is countable and that for each \( i \in I \), \( \mu_i \) is a \( \sigma \)-finite almost separable outer measure on \( X_i \) which is Radon-like w.r.t. \((\mathcal{N}_\alpha, \mathcal{C}_i, \mathcal{G}_i)\) for some families \( \mathcal{C}_i, \mathcal{G}_i \) (see definition 2.1.5 Ch.II). Then there exists an inverse limit outer measure which is Radon-like w.r.t. \((\mathcal{N}_\alpha, \mathcal{G}_1(\sigma_0(\text{Rect}(\mathcal{C}))), \sigma_{\alpha+1}(\text{Rect}(\mathcal{G})))\).

3.10 **Theorem.** Suppose that there exists a countable cofinal subset \( J \) of \( I \) such that for each \( i \in J \), \( \mu_i \) is a \( \sigma \)-finite almost separable outer measure on \( X_i \) which is Radon-like w.r.t. \((\mathcal{N}_\alpha, \mathcal{C}_i, \mathcal{G}_i)\). Then there exists an inverse limit outer measure which is Radon-like w.r.t. 
\[
(\mathcal{N}_\alpha, \mathcal{C}', \mathcal{G}')
\]
where
\[
\mathcal{C}' = \{r_J^{-1}[C] \cap L_I : C \subseteq L_J, C \in \mathcal{C}_i(\sigma_0(\text{Rect}(\mathcal{C}_i) \cap J))\}
\]
\[
\mathcal{G}' = \{r_J^{-1}[G] \cap L_I : G \overset{\sim}{\subseteq} \bigvee_{j \in J} G \in \sigma_{\alpha+1}(\text{Rect}(\mathcal{G}_j) \cap J)\} \cup \{\overset{\sim}{X} \cap L_I\}.
\]
In the description of \( C' \) and \( \mathcal{G}' \) above, \( C \mid J \) means the restriction of the system of families of sets \( C \) to the set \( J \), and \( \mathcal{G}' \mid J \) is similar. The operations involving these families take place entirely in \( \tilde{X}_J \).

4. Proofs.

Proof of 2.1. Clearly we need only to show that if \( \tilde{\mu} \mid L_\| \) is not an inverse limit outer measure then no such measure exists. To do this we first establish the following lemma.

**Lemma A.** Let \( \alpha \in \text{Rect}(\mathcal{M}) \), and \( k \in I \) be such that \( k > j \) for all \( j \in J_\alpha \). If we let

\[
B = \bigcap_{j \in J_\alpha} p_{jk}^{-1}[\pi_j[\alpha]],
\]

then,

1. \( B \in \mathcal{M}_k \),
2. \( \pi_k^{-1}[B] \cap L_\| = \alpha \cap L_\| \),
3. \( g(\alpha) = \mu_k(B) \).

**Proof.** Immediate from the definitions.

Suppose that \( \tilde{\mu} \mid L_\| \) is not an inverse limit outer measure. We know from lemma 2.3.6 Ch. III that \( \text{Rect}(\mathcal{M}) \in \mathcal{M}_\| \), hence for every \( i \in I \) and \( A \in \mathcal{M}_i \),

\[
\pi_i^{-1}[A] \in \mathcal{M}_\| \mid L_\|
\]

Thus it must be that for some \( j \in I \) and \( B \in \mathcal{M}_j \)

\[
\tilde{\mu} \mid L_\|(\pi_j^{-1}[B]) \neq g(\pi_j^{-1}[B]) = \mu_j(B).
\]

Since

\[
\tilde{\mu}(\pi_j^{-1}[B]) \neq g(\pi_j^{-1}[B])
\]
we must have
\[ \tilde{\mu} \mid L_I(\pi^{-1}_j[B]) = \tilde{\mu}(\pi^{-1}_j[B] \cap L_I) < \mu_j(B). \]

Then by the definition of \( \tilde{\mu} \), there exists a countable family \( \mathcal{B} \subset \operatorname{Rect}(\mathcal{M}) \) such that
\[ \pi^{-1}_j[B] \cap L_I \subset \bigcup \mathcal{B} \]
and
\[ \sum_{D \in \mathcal{B}} g(D) < g(\pi^{-1}_j[B]). \]

Let \( \mathcal{B} = \{D_0, D_1, \ldots\} \) and for each new let \( i_n \in I \) and \( B_n \in \mathcal{M}_n \) be such that
\[ D_n \cap L_I = \pi^{-1}_{i_n}[B_n] \cap L_I \]
and \( g(D_n) = \mu_{i_n}(B_n) \) (this is possible by lemma A).

Then
\[ \pi^{-1}_j[B] \cap L_I \subset \bigcup_{n \in \omega} \pi^{-1}_{i_n}[B_n] \cap L_I \]
and
\[ \sum_{n \in \omega} \mu_{i_n}(B_n) < \mu_k[B]. \]

Hence there cannot exist an outer measure \( \nu \) carried by \( L_I \) for which
\[ \nu(\pi^{-1}_i[B]) = \mu_i(B) \]
and
\[ \nu(\pi^{-1}_{i_n}[B_n]) = \mu_{i_n}(B_n) \]
for every new, i.e. there cannot exist an inverse limit outer measure.
Proof of 2.2  Suppose first that $\tilde{\mu}$ is a $\pi$-limit outer measure and that $\tilde{\mu}'$ is pseudo-carried by $L_\pi$.

Let $A \in \mathcal{M}_\pi$. Then

$$
\tilde{\mu}(\pi_1^{-1}[A]) = \mu_1(A)
= \sup \{ \mu_1(B) : B \subseteq A, B \in \mathcal{M}_\pi, \mu_1(B) < \infty \}
= \sup \{ \tilde{\mu}(\pi_1^{-1}[B]) : B \subseteq A, B \in \mathcal{M}_\pi, \mu_1(B) < \infty \}
$$

Thus

$$
\mu_1(A) = \tilde{\mu}'(\pi_1^{-1}[A]) = \tilde{\mu}'(\pi_1^{-1}[A] \cap L_\pi)
= \tilde{\mu}(\pi_1^{-1}[A] \cap L_\pi) = \tilde{\mu} | L_\pi(\pi_1^{-1}[A]),
$$

and thus $\tilde{\mu} | L_\pi$ is an inverse limit outer measure.

Now suppose that $\tilde{\mu} | L_\pi$ is an inverse limit outer measure. Then since an inverse limit outer measure is also a $\pi$-limit outer measure it follows from remark 2.4.2 Ch.II that $\tilde{\mu}$ is a $\pi$-limit outer measure.

Suppose also that there exists $A \in \mathcal{M}_\mu$ ($=\mathcal{M}_\tilde{\mu}$) such that $A \subseteq X \sim L_\pi$ and $0 < \tilde{\mu}'(A)$. Then from the definition of $\tilde{\mu}'$ there exists a set $B \subseteq A, B \in \mathcal{M}_\mu$ (since $A \in \mathcal{M}_\tilde{\mu}$) with $0 < \tilde{\mu}(B) < \infty$. Then by definition of $\tilde{\mu}$ there exists for $0 < \varepsilon < \frac{\tilde{\mu}(B)}{2}$ a finite family $\mathcal{F} \in \text{Rect}(\mathcal{M})$ with

$$
\tilde{\mu}(\cup \mathcal{F} \sim B) < \varepsilon
$$

and

$$
\varepsilon < \tilde{\mu}(B) - \varepsilon < \tilde{\mu}(\cup \mathcal{F}).
$$

Furthermore by lemma 2.5.3 Ch.I we can choose $\mathcal{F}$ to be a disjoint family. Then

$$
\sum_{D \in \mathcal{F}} \tilde{\mu}(D \cap L_\pi) \leq \sum_{D \in \mathcal{F}} \tilde{\mu}(D \sim B) < \varepsilon.
$$
But, since \( \tilde{\mu} |_{L_{I}} \) is an inverse limit outer measure, lemma A shows that 
\[ \sum_{D \in \mathcal{D}} \tilde{\mu}(D \cap L_{I}) = \tilde{\mu}(D) \] 
for every \( D \in \mathcal{D} \), hence
\[ \sum_{D \in \mathcal{D}} \tilde{\mu}(D \cap L_{I}) = \mu(\bigcup \mathcal{D}) > \varepsilon. \]

Hence no such sets \( A, B \) exist, so \( \tilde{\mu}' \) is pseudo-carried by \( L_{I} \).

**Proof of 2.3** We first establish the following lemma.

**Lemma B.** Let for each \( i \in I \), \( \mu_{i} \) be almost separable, and let \( I_{0} \) be a countable directed subset of \( I \). Then
\[ \tilde{\mu}(X \sim r_{0}^{-1}[L_{I_{0}}]) = 0. \]

**Proof.** For each \( i \in I_{0} \) let \( T_{i} \subset X_{i} \) and \( B_{i} \subset \mathcal{B}_{i} \) be such that 
\( \mu_{i}(T_{i}) = 0 \), \( B_{i} \) is countable, and for every \( x, y \in X_{i} \sim T_{i} \) with \( x \neq y \), there exists \( B \in \mathcal{B}_{i} \) such that \( x \in B, y \notin B \). For each 
\( i, j \in I_{0} \) with \( i < j \) and \( B \in \mathcal{B}_{i} \) let
\[ B_{i,j} = r_{i}^{-1}[B] \cap r_{j}^{-1}[X_{i} \sim B]. \]

Then,
\[ \tilde{\mu}(B_{i,j}) = g(r_{i}^{-1}[B] \cap r_{j}^{-1}[X_{i} \sim B]) = g(\emptyset) = 0 \]
for every such \( i, j \). Since
\[ X \sim r_{0}^{-1}[L_{I_{0}}] \subset \bigcup \{B_{i,j} : i, j \in I_{0}, i < j \text{ and } B \in \mathcal{B}_{i}\} \cup \bigcup_{i \in I_{0}} \pi_{i}^{-1}[T_{i}] \],
we have
\[ \tilde{\mu}(X \sim r_{0}^{-1}[L_{I_{0}}]) = 0. \]

Lemma 2.3.1 follows immediately from lemma B.

To prove 2.3.2 we will use the following lemma.
Lemma C. Let $\mu_1$ be almost separable for every $i \in I$ and let sequential maximality hold. Then for every $\alpha \in \text{Rect}(\mathcal{M}),$

$$\widetilde{\mu}(\alpha) = \widetilde{\mu}(\alpha \cap L_I).$$

Proof. For $\varepsilon > 0$ let $\mathcal{N} \subseteq \text{Rect}(\mathcal{M})$ be a countable cover of $\alpha \cap L_I$ such that

$$\sum_{H \in \mathcal{N}} g(H) \leq \widetilde{\mu}(\alpha \cap L_I) + \varepsilon,$$

and let $T = \bigcup_{H \in \mathcal{N}} H \cup \alpha$ and let $K \subseteq I$ be a countable directed set with $T \subseteq K$. By sequential maximality, for each $x \in \alpha \cap r^{-1}_J [L_K]$ there exists $x' \in L_I$ such that for every $k \in K, x_k' = x_k$. Then $\pi_j(x') \in \pi_j[\alpha]$ for every $j \in J$, hence

$$x' \in \alpha \cap L_I \subseteq \bigcup_{\alpha} K,$$

so by lemma 2.4 Ch. I,

$$\bigcap_{k \in K} r^{-1}_k [\{x_k'\}] \subseteq \bigcup_{\alpha} K,$$

hence $x \in \bigcup_{\alpha} K$ and

$$\alpha \cap r^{-1}_K [L_K] \subseteq \bigcup_{\alpha} K.$$

Thus, using lemma B,

$$\widetilde{\mu}(\alpha) \leq \widetilde{\mu}(\bigcup_{\alpha} K) + \widetilde{\mu}(\tilde{x} \sim r^{-1}_K [L_K])$$

$$\leq \widetilde{\mu}(\alpha \cap L_I) + \varepsilon + 0.$$ 

Since $\varepsilon$ is arbitrary, $\widetilde{\mu}(\alpha) = \widetilde{\mu}(\alpha \cap L_I)$.

Turning now to the proof of 2.3.2, suppose that $A \in \mathcal{U}$, $(= \mathcal{M})$ is such that $A \subseteq \tilde{x} \sim L_I$ and $\widetilde{\mu}'(A) > 0$. Then as in the proof of lemma 2.2 there exists $B \subseteq A$, $B \in \mathcal{M}$ with $0 < \widetilde{\mu}(B) < \omega$, and, for $0 < \varepsilon < \frac{\widetilde{\mu}(B)}{2}$ a finite disjoint $B \subseteq \text{Rect}(\mathcal{M})$ with

$$\widetilde{\mu}(\bigcup_{B} \sim B) < \varepsilon.$$
and
\[ \tilde{\mu}(B) - \varepsilon < \tilde{\mu}(\bigcup D). \]
Thus we have again
\[ \sum_{D \in B} \tilde{\mu}(D \cap L_T) < \sum_{D \in B} \tilde{\mu}(D \sim B) < \varepsilon, \]
and from lemma C,
\[ \sum_{D \in B} \tilde{\mu}(D \cap L_T) = \sum_{D \in B} \tilde{\mu}(D) > \tilde{\mu}(B) - \varepsilon > \varepsilon, \]
which is a contradiction. Hence no such A exists and \( \tilde{\mu}' \) is pseudo-carried by \( L_T \).

Proof of 2.5. By theorem 2.6 Ch.III, \( \tilde{\mu} \) is a \( \pi \)-limit outer measure, and by lemma 2.3, \( \tilde{\mu}' \) is pseudo-carried by \( L_T \) under conditions 2.5.1 or 2.5.2. Then lemma 2.2 shows that \( \tilde{\mu} \mid L_T \) is an inverse limit outer measure.

Proof of 2.7. Let \( A \in \text{Rect}(\mathcal{M}) \), \( \mathcal{K} \in \text{Rect}(\mathcal{M}) \) be a countable cover of \( A \sim L_T \), and let \( T = \bigcup_{H \in \mathcal{K}} J_H \cup J_A \) and \( x \in A \cap L_T \). Then, since \( I \) is uncountable let \( \{i \in I : i \sim T \} \) and \( y \in x_i \) be such that \( y \neq x_i \). If we define \( x' \in X \) by letting \( x'_j = x_j \) for \( j \neq i \) and \( x_i' = y \), then \( x' \in A \sim L_T \). Hence \( x' \in \bigcup \mathcal{K} \). Thus
\[ x \in \bigcap_{j \in T} \pi^{-1}_j [x_j] = \bigcap_{j \in T} \pi^{-1}_j [x'_j] \subset \bigcup \mathcal{K} \]
(from lemma 2.4 Ch.I). Hence \( A \cap L_T \subset \bigcup \mathcal{K} \) so that \( A \subset \bigcup \mathcal{K} \) and therefore from the definition of \( \tilde{\mu} \),
\[ \tilde{\mu}(A \sim L_T) = \tilde{\mu}(A). \]
Proof of 2.8 By theorem 1.6 Ch.IV, $\tilde{\mu}'$ is a $\pi$-limit outer measure, and for every $A \in \mathcal{M}_\mu$,

$$\tilde{\mu}'(A) = \sup\{\tilde{\mu}(C) : C \subset A, C \in \mathcal{C}\},$$

where $\mathcal{C} = \delta_1(\sigma_0(\text{Rect}(c)))$.

Suppose $B \in \mathcal{M}_\mu$, $\tilde{\mu}'(B) > 0$ and $B \subset \hat{X} \sim L_I$. Then there exists for $t < \tilde{\mu}'(B)$ a sequence $C_0, C_1, \ldots$ in $\sigma_0(\text{Rect}(c))$ such that $C_{n+1} \subset C_n$ for each new (since $\text{Rect}(c) = \delta_0(\text{Rect}(c))$), $\cap_{n \in \omega} C_n \subset B$, and $\tilde{\mu}(\cap_{n \in \omega} C_n) > t$. For each new there exists by lemma 2.5.3 Ch.I, a finite disjoint family $\mathcal{B}_n \subset \text{Rect}(\mathcal{C})$ such that $C_n = \mathcal{B}_n$. Furthermore we can choose the families $\mathcal{B}$, so that if $m < n$, every $B \in \mathcal{B}_n$ is a subset of some element of $\mathcal{B}_m$.

Let $i_0$ be such that $i_0 > j$ for every $j \in U \mathcal{J}_B$ and choose by recursion $i_{n+1} \in I$ so that $i_{n+1} > i_n$ and $i_{n+1} > j$ for all $j \in U \mathcal{J}_B$. For each new let

$$D_n = \bigcup_{B \in \mathcal{B}_n} \bigcap_{j \in \mathcal{J}_B} p_{i_k}^{-1}[\pi_{i_k}[B]].$$

Then for each $m, n \in \omega$ with $m < n$

$$D_n \subset p_{i_k}^{-1}[D_m]$$

and

$$\mu_i(D_m) = \tilde{\mu}(C_m) > t$$

(this follows from lemma A above and lemma A in the proofs of Ch.III, since we have

$$\tilde{\mu}(C_m) = \sum_{B \in \mathcal{B}_m} \tilde{\mu}(B) = \sum_{B \in \mathcal{B}_m} g(B) = \mu_i(D_m)).$$
Let $0 < \varepsilon < \frac{t}{2}$ and for each new choose $K_n \in D_n$, $K_n \in C_n$ such that
\[ \mu_n(D_n \sim K_n) < \frac{\varepsilon}{2^{n+1}}. \]

For each new let
\[ E_n = \bigcap_{m=0}^{n} p^{-1}_{m+n} [K_m]. \]

Then
\[ E_n \supseteq D_n \sim \bigcup_{m=0}^{n} p^{-1}_{m+n} [D_m \sim K_m] \]

hence,
\[ \mu_n(E_n) \geq t - \sum_{m=0}^{n} \frac{\varepsilon}{2^{n+1}} > \frac{t}{2}. \]

Thus, from simple maximality
\[ \pi^{-1}_i [E_n] \cap L_I \neq \emptyset. \]

Also from lemma A it is clear that
\[ \pi^{-1}_i [E_n] \cap L_I = \bigcap_{m=0}^{n} \pi^{-1}_m [K_m] \cap L_I \]

so that
\[ \bigcap_{m=0}^{n} \pi^{-1}_m [K_m] \cap L_I \neq \emptyset. \]

for any new, hence
\[ \bigcap_{m \in \text{new}} \pi^{-1}_m [K_m] \cap L_I \neq \emptyset. \]

But, from lemma A, for every new
\[ \pi^{-1}_m [K_m] \cap L_I \subseteq C_m \cap L_I \]

hence
\[ \emptyset \neq \bigcap_{m \in \text{new}} C_m \cap L_I \subseteq B \cap L_I \]

contradicting the fact that $B \subseteq X \sim L_I$. Hence no such $B$ exists and $\tilde{\mu}'$ is pseudo-carried by $L_I$. 
Proof of 2.10 We shall check that the hypothesis of theorem 2.8 is satisfied. Let \( i \) be a sequence in \( I \) with \( i_n < i_{n+1} \) for every \( n \in w \),
\[
\mathcal{F} = \{ \pi_1^{-1}[C] : C \subseteq \bigcap_{i_n} \} \text{ and } \mathcal{F} \text{ be a sequence in } \mathcal{F} \text{ with } F_n \neq \emptyset \text{ for every } n \in w. \text{ We have to show that } \bigcap_{m \in w} F_m \neq \emptyset.
\]

For each \( m \in w \), let \( j(m) \) be the smallest integer \( k \) with
\[
F_m = \pi_1^{-1}[C_k] \cap L_{I_n}
\]
for some \( C_k \subseteq \bigcap_{i_n} \), and let \( C_m \subseteq C_{j(m)} \) be such that
\[
F_m = \pi_1^{-1}[C_m] \cap L_{I_n}.
\]
Let
\[
K_m = \bigcap \{ \pi_i^{-1}[C_n] : n \in w \text{ and } j(n) > m \}.
\]
Since, for each \( m \in w \), the family
\[
\mathcal{K}_m = \{ \pi_i^{-1}[C] : C \subseteq \bigcap_{i_n} \text{ and } n \in w, n \geq m \}
\]
is \( \Sigma_0 ^1 \)-compact and, for each \( n \in w \)
\[
\bigcap_{l=0}^{n} p_{i_l}^{-1} j(l)[C_1] = \pi_i \left[ \bigcap_{l=0}^{n} \pi_i^{-1} j(l)[C_1] \cap L_{I_n} \right] \neq \emptyset,
\]
we see that \( K_0 \neq \emptyset \). Similar considerations show that for any \( n \in w \), \( K_n \neq \emptyset \). Let \( x_0 \in K_0 \). Then
\[
p_{i_0}^{-1} \left[ \{ x_0 \} \right] \cap K_1 \neq \emptyset
\]
otherwise, by condition 2.10.1, there would exist \( m \) with
\[
p_{i_0}^{-1} \left[ \{ x_0 \} \right] \cap \bigcap_{l=0}^{m} p_{i_l}^{-1} j(l)[C_1] = \emptyset.
\]
hence,
\[ x_0 \notin \bigcap_{1=0}^{m} p_{i_0i_1}(C_i), \]
so that \( x_0 \notin K_0 \), contradicting the choice of \( x_0 \). Thus there exists \( x_1 \in K_1 \) with \( p_{i_0i_1}(x_1) = x_0 \) and by similar arguments we can choose, by recursion, for each \( n \in \mathbb{N} \), \( x_n \in K_n \) such that
\[ p_{i_mi_n}(x_n) = x_m \]
whenever \( m \leq n \). Let (by sequential maximality) \( y \in L_I \) be such that \( y_i = x_n \) for every \( n \in \mathbb{N} \). Clearly for every \( n \in \mathbb{N} \)
\[ y \in \bigcap_{n \in \mathbb{N}} \pi_n^{-1}[C_n] \cap L_I \]
thus
\[ y \in \bigcap_{n \in \mathbb{N}} \pi_n^{-1}[C_n] \cap L_I \]
so that
\[ \{ \pi_n^{-1}[C] \cap L_I : C \in C_n, n \in \mathbb{N} \} \]
is \( \kappa_0 \)-compact.

Thus the conditions of theorem 2.8 are satisfied, hence
\( \widehat{\mu} | L_I \) is an inverse limit outer measure.

Proof of 2.11 It is clear that \( h \) is countably additive on \( \mathcal{A} \) and that \( \mathcal{A} \) is a ring, hence \( \mathcal{Y} \) is an outer measure on \( \tilde{X}_J \), and \( \mathcal{A} \subseteq \mathcal{M}_\mathcal{Y} \). \( \mathcal{Y} \) is supported by \( L_J \) since
\[ \pi_j^{-1}(X_J - L_J) \subseteq \tilde{X}_J - L_I, \]
and since (using \( \tilde{\pi} \) for projection in \( \tilde{X}_J \))
\[ \mathcal{Y}(\tilde{\pi}_j^{-1}[B]) = h(\tilde{\pi}_j^{-1}[B]) = \nu(\pi_j^{-1}[B]) = \mu_j(B) \]
if \( j \in J \) and \( B \in \mathcal{M}_j \), \( \mathcal{Y} \) is an inverse limit outer measure for
Proof of 2.12.1 Immediate from the definitions.

Proof of 2.12.2 Let \( \bar{\pi}_j \) denote projection onto the \( j \)th coordinate from \( \bar{X}_j \). Then for every \( j \in J, i < j \) and \( B \in \mathcal{M}_i \),

\[
\nu(\bar{\pi}_i^{-1}[B]) = \nu(\bar{\pi}_j^{-1}[p_{ij}^{-1}[B]]) = \nu(\bar{\pi}_j^{-1}[p_{ij}^{-1}[B]]),
\]

hence

\[
\nu(\bar{\pi}_i^{-1}[B]) = \mu_j(p_{ij}^{-1}[B]) = \mu_i(B).
\]

Since \( r_j \) is 1:1 on \( L_1 \) and \( \nu(\bar{X} \sim L_1) = 0 \), \( \nu \) is an outer measure and with \( i, j, B \) as above, \( \pi_i^{-1}[B] \in \mathcal{M}_i \) since \( \nu(\pi_i^{-1}[B] \sim L_1) = 0 \) and

\[
r_j[\pi_i^{-1}[B] \cap L_1] = \pi_j[p_{ij}^{-1}[B]] \cap L_j
\]

which is in \( \mathcal{M}_j \). Hence \( \nu \) is an inverse limit outer measure.

Proof of 3.1 Immediate from the definitions and theorem 2.12.1.

Proof of 3.2 Immediate from the definitions and lemma 3.1.

Proof of 3.4 Rect(\( \mathcal{J} \)) forms a base for the product topology on \( \bar{X} \). Hence there exists \( \mathcal{B} \subset \text{Rect}(\mathcal{J}) \) such that

\[
\bar{X} \sim L_1 = \bigcup \mathcal{B}.
\]

Since for every \( i \in I, \mathcal{J}_i \subset \mathcal{M}_i \), we have

\[
\text{Rect}(\mathcal{J}) \subset \text{Rect}(\mathcal{M}) .
\]

It then follows from lemma 2.3.3 Ch.III and simple maximality that for each \( B \in \mathcal{B}, g(B) = 0 \). Let \( C \subset \bar{X} \sim L_1 \) be closed and compact. Then there exists a finite subfamily \( \mathcal{B}' \subset \mathcal{B} \) such
that \( C \subseteq \bigcup B' \). Thus
\[
\tilde{\mu}(C) = \sum_{B \in B'} g(B) = 0,
\]
and so
\[
h(\tilde{x} \sim L_I) = \sup\{\tilde{\mu}(C) : C \subseteq \tilde{x} \sim L_I, C \text{ closed and compact}\} = 0.
\]
Then
\[
\tilde{\mu}^*(\tilde{x} \sim L_I) = \inf\{h(G) : \tilde{x} \sim L_I, G, G \text{ open}\} = 0,
\]
so that \( \tilde{\mu}^* \) is supported by \( L_I \).

By lemma 1.10 Ch.IV, \( \tilde{\mu}^* \) is Radon w.r.t. the product topology.

Proof of 3.5 By theorem 1.11 Ch.IV, \( \tilde{\mu}^* \) is a Radon \( \pi \)-limit outer measure (w.r.t. the product topology). By theorem 3.4 \( \tilde{\mu}^* \) is supported by \( L_I \). Hence \( \tilde{\mu}^* \) is an inverse limit outer measure which is Radon w.r.t. the product topology.

Proof of 3.6 By theorem 1.11.1 Ch.IV, \( \tilde{\mu}^* \) is a \( \pi \)-limit outer measure which is Radon w.r.t. the product topology. By lemma 2.3 above \( \tilde{\mu}(\tilde{x} \sim L_I) = 0 \). Hence from remark 2.4.3 Ch.III, \( \tilde{\mu}^*(\tilde{x} \sim L_I) = 0 \). Thus \( \tilde{\mu}^* \) is also an inverse limit outer measure.

Proof of 3.7 Immediate from theorem 3.6 and lemma 3.2.

Proof of 3.9 By theorem 2.8.1 Ch.IV, there exists a \( \pi \)-limit outer measure which is Radon-like w.r.t.

\((K, \delta_1(\sigma_0(\text{Rect}(C))), \sigma_{\alpha+1}(\text{Rect}(\mathcal{S})))\). Let \( \nu \) be such a measure.
Then for every $A \sim \widetilde{X}$, $\nu(A) \leq \tilde{\mu}(A)$ (remark 2.4.3 Ch.III).

By lemma 2.3.1, $\tilde{\mu}(\widetilde{X} \sim L_T) = 0$, thus $\nu(\widetilde{X} \sim L_T) = 0$ and $\nu$ is an inverse limit outer measure.

**Proof of 3.10** Immediate from theorem 3.9 and lemma 3.2.
APPENDIX (EXAMPLES)

This appendix consists of examples of inverse systems of outer measures which illustrate various points such as the non-existence of \( \pi \)-limit outer measures or inverse limit outer measures, or their relationship to the topologies involved. Reference is made to these examples at appropriate places in the text and a brief summary of the roles of each example follows.

1. For this system, no \( \pi \)-limit outer measure exists.
2. A \( \pi \)-limit outer measure exists but no inverse limit outer measure can exist.
3. In this system the \( \pi \)-limit outer measure \( \tilde{\mu} \) is not semifinite even though each \( \mu_i \) is semifinite.
4. Here the \( \pi \)-limit outer measure \( \tilde{\mu} \) is not Radon w.r.t. the product topology, but the regularized outer measure \( \tilde{\mu}^* \) (see definition 1.9 Ch.IV) is a \( \pi \)-limit outer measure and is Radon w.r.t. the product topology.
5. A \( \pi \)-limit outer measure exists but no such measure can be Radon w.r.t. the product topology, although the functions \( p_{ij} \) are continuous.
6. In this example both a \( \pi \)-limit outer measure and an inverse limit outer measure exist, but neither can be Radon w.r.t. the product topology, even though the system has an "upper bound" (i.e. there exists \( j \in I \) such that \( i < j \) for every \( i \in I \)) and the functions \( p_{ij} \) are continuous except at one point in each space \( X_j \).
Here, however, if the topology of complements of closed compacts is used, then \( \widetilde{\mu} \) and \( \widetilde{\mu} \mid L \) are Radón and this topology relativized to the inverse limit set \( L \) is essentially the one which would naturally be expected.

The i.s.o.m. in example 1 has no \( \pi \)-limit outer measure, hence no inverse limit measure. Essentially the same example appears in Halmos [5], where it is used to show that indirect product measures may not exist.

1. Example. Let \( \mu \) be Lebesgue outer measure, \( \mu^* \) Lebesgue inner measure, and let \( A_0, A_1, \ldots \) be a sequence of pairwise disjoint subsets of \( [0,1] \) such that for each \( n \in \omega \), \( \mu(A_n) = 1 \), \( \mu^*(A_n) = 0 \). Then let \( B_m = \bigcup_{n=m}^{\infty} A_n \) for each \( m \in \omega \) (hence for \( m = 1 \), \( \mu(B_m) = 1 \), \( \mu^*(B_m) = 0 \)).

Define the i.s.o.m. \( (X, p, \mu, I) \) as follows: let

\[ I = \omega \text{(with the usual ordering)}, \]

and for every \( j \in I \), let

\[ X_j = [0,1], \]

\[ \mu_j = \mu \mid B_j, \]

\[ p_{ij} \text{ be the identity mapping whenever } i \leq j. \]

Then for every \( i \in I \), \( \mathcal{M}_i \) contains the Borel sets in \( [0,1] \), hence \( \mu_i \) is separable for every \( i \in I \). By lemma 2.3, Ch.V, the measure \( \widetilde{\mu} \) is carried by the inverse limit set \( L \), i.e.

\[ \widetilde{\mu}(\prod_{i \in I} X_i \sim L) = 0. \]
Furthermore
\[ g(\pi_1^{-1}[X_i \sim B_i]) = \mu_1(X_i \sim B_i) = 0, \]
and
\[ L = \{ x \in \Pi X_i : x_i = x_j \text{ for every } i, j \in I \} = \bigcup_{i \in I} \pi_1^{-1}[X_i \sim B_i], \]
since
\[ \bigcup_{i \in I} ([0,1] \sim B_i) = [0,1]. \]
Thus,
\[ \widetilde{\mu}(L) \leq \sum_{i \in I} \mu(\pi_1^{-1}[X_i \sim B_i]) \leq \sum_{i \in I} g(\pi_1^{-1}[X_i \sim B_i]) = 0, \]
so that \( \widetilde{\mu} = 0. \) It then follows (see remark 2.4.2 Ch. III) that no \( \pi \)-limit outer measure could exist for \((X,p,\mu,I)\) hence also no inverse limit outer measure.

The next example shows that an i.s.o.m. may have a \( \pi \)-limit outer measure without having an inverse limit outer measure although the system is an inverse system of spaces and its inverse limit set satisfies sequential maximality.

2. Example. Let \( S \) be an uncountable space and let \( B_0, B_1, \ldots \)
be a sequence of subsets of \( S \) such that for each \( n \in \omega \),
\[ B_{n+1} \subset B_n, \quad B_n \text{ is uncountable and } \bigcap_{n \in \omega} B_n = \emptyset. \]
Define the i.s.o.m. \((X,p,\mu,I)\) as follows: let
\[ I = \omega \text{ (with the usual ordering)} \]
and for every \( j \in I \), let
\[ X_j = S. \]
\( \mu_j \) be the outer measure on \( X_j \) defined by
\[
\mu_j(A) = 1 \quad \text{if Card } A \cap B_j > \aleph_0
\]
\[
\mu_j(A) = 0 \quad \text{otherwise},
\]
\( p_{ij} \) be the identity mapping whenever \( i < j \).

For each \( j \in I \),
\[
\{ A \subseteq X_j : \text{Card}(B_j \sim A) \leq \aleph_0 \}
\]
forms an \( \aleph_0 \)-compact class which is an inner family for \( \mu_j \).
Hence (theorem 2.6, Ch.III) \( \tilde{\mu} \) is a \( \pi \)-limit outer measure.

However for each \( i \in I \),
\[
\mathcal{g}(\pi_i^{-1}[X_i \sim B_i]) = 0
\]
and since \( S = \bigcup \{ S \sim B_i \} \)
\[
L = \{ y \in \prod_{i \in I} X_i : y_i = y_j \text{ for all } i, j \in I \} \subseteq \bigcup_{i \in I} \pi_i^{-1}[X_i \sim B_i],
\]

hence
\[
\tilde{\mu}(L) = \sum_{i \in I} \tilde{\mu}(\pi_i^{-1}[X_i \sim B_i]) = \sum_{i \in I} \mathcal{g}(\pi_i^{-1}[X_i \sim B_i]) = 0.
\]

Thus \( \tilde{\mu} | L \) is not an inverse limit outer measure and so by lemma 2.1, Ch.V, no inverse limit outer measure exists.

We now give an example of an i.s.o.m. in which all the measures are semifinite (though not \( \sigma \)-finite), and for which there exists a "pathological" set \( A \subseteq \prod_{i \in I} X_i \) such that \( A \) has infinite \( \tilde{\mu} \)-measure and all of its subsets have infinite \( \tilde{\mu} \)-measure zero, so that \( \tilde{\mu} \) is not semifinite.
3. Example. Define the i.s.o.m. \((X,p,\mu,I)\) as follows: let \(I = [0,1]\) (with the usual ordering)

and for each \(j \in I\), let

\[ X_j = [0,1], \]

\[ \mu_j \text{ be counting measure on } X_j \]

\[ p_{ij} \text{ be the identity function whenever } i \leq j. \]

Let

\[ A = \{ x \in \Pi x_i : \text{ for some } a \in (0,1), \ x_a = 0 \text{ and } x_i = a \text{ for every } i \in I \text{ with } i \neq a \}. \]

To see that \(\hat{\mu}(A) = \infty\), let \(\mathcal{B} \subseteq \text{Rect}(\mathcal{M})\) be a countable cover of \(A\), \(T = \bigcup_{B \in \mathcal{B}} J_B\) and

\[ D = \{ x \in A : x_a = 0 \text{ for some } a \notin T \}. \]

Then \(\pi_1(D) = (0,1) \sim T\) and, for every \(a \in \pi_1(D)\), there exists \(x \in D\) with \(x_i = a\) for every \(i \in T\), and \(B \in \mathcal{B}\) with \(x \in B\), so that

\(a \in \pi_1(B)\) for every \(j \in J_B\), hence

\[ a \in \bigcap_{j \in J_B} p_{ij}^{-1}[\pi_j(B)] \leq \sum_{B \in \mathcal{B}} g(B), \]

From this we see that

\[ \pi_1(D) \subseteq \bigcup_{B \in \mathcal{B}} \bigcap_{j \in J_B} p_{ij}^{-1}[\pi_j(B)]. \]

Then since \(\pi_1(D)\) is uncountable,

\[ \infty = \mu_1(\pi_1[D]) = \mu_1\left( \bigcup_{B \in \mathcal{B}} \bigcap_{j \in J_B} p_{ij}^{-1}[\pi_j(B)] \right) \leq \sum_{B \in \mathcal{B}} g(B), \]

and so \(\hat{\mu}(A) = \infty\).

We next wish to show that every subset of \(A\) which has finite \(\hat{\mu}\)-measure has in fact \(\hat{\mu}\)-measure zero. First we note that if \(x \in A\) and
where \( i \in I \) is such that \( x_i = 0 \), then \( C(x) \in \operatorname{Rect}(\mathcal{M}) \) and 
\[ g(C(x)) = 0, \] 
so that \( \hat{\mu}(\{x\}) = 0 \). Now suppose that \( E \subset A \) 
and \( \hat{\mu}(E) < \infty \). Then there exists a countable family 
\( \mathcal{K} \subset \operatorname{Rect}(\mathcal{M}) \) such that \( E \subset \bigcup \mathcal{K} \) and 
\[ \sum_{H \in \mathcal{K}} g(H) < \infty. \] 
Furthermore we may assume that \( \pi_1(H) \subset (0,1) \) since \( \pi_1(E) \subset (0,1) \).

Let \( H \in \mathcal{K} \) and 
\[ D_H = \bigcap_{j \in J_H} \pi_j^{-1}([\mathcal{M}][H]). \]
Then 
\[ g(H) = \mu_1(D_H) < \infty, \]
hence \( D_H \) contains only a finite number of points. Now, 
for each \( j \in (0,1) \), there is a unique \( x \in A \) with \( x_j = 0 \), so if 
\[ E' = \{ x \in E : x_j = 0 \text{ for some } j \in \bigcup_{H \in \mathcal{K}} J_H \} \]
then \( E' \) is countable and, for every \( x \in E \sim E' \), \( H \in \mathcal{K} \), and 
\( j \in J_H \), we have \( x_j = x_1 \). Thus if \( x \in (E \sim E') \cap H \), then 
\( x_1 \in D_H \). Since \( D_H \) is finite and for every \( t \in D_H \) there is 
a unique \( x \in A \) with \( x_1 = t \), we conclude that \( (E \sim E') \cap H \) is 
finitely, hence 
\[ E = E' \cup \bigcup_{H \in \mathcal{K}} ((E \sim E') \cap H) \]
is countable and therefore \( \hat{\mu}(E) = 0 \).

From the existence of such a set \( \mathcal{A} \), it follows that 
\( \hat{\mu} \) is not semifinite.
We next exhibit a case in which \( \widetilde{\mu} \) is a \( \pi \)-limit outer measure but is not Radon w.r.t. the product topology. The measure \( \hat{\mu}^* \) (definition 1.9, Ch.IV) is however, a \( \pi \)-limit outer measure. Note also that no inverse limit outer measure exists.

4. **Example.** For each \( j \in (0, 1) \) let \( B_j = (j, 1) \). Let \( (X, p, \mu, I) \) be the i.s.o.m. defined as follows: let

\[
I = (0, 1), \text{ ordered by } \leq,
\]

and for each \( j \in I \) let

\[
X_j = (0, 1) \text{ with topology } \mathcal{J}_j = \{X_j^0, j], (j, 1), \emptyset\}
\]

\( \mu_j \) be the outer measure on \( X_j \) defined by

\[
\mu_j(A) = 1 \quad \text{if } A \cap B_j \neq \emptyset
\]

\[
\mu_j(A) = 0 \quad \text{otherwise},
\]

\( p_{ij} \) be the identity function whenever \( i \leq j \).

Then for each \( j \in I \)

\[
\mathcal{M}_j = \{A \subseteq X_j : A \subseteq (0, j] \text{ or } (j, 1] \subseteq A\}.
\]

From this it is easily checked that \( \mu_j \) is Radon, hence by theorem 2.6, Ch.III, \( \widetilde{\mu} \) is a \( \pi \)-limit outer measure.

Consider the set \( B = \Pi_{i \in I} B_i \). \( B \) is a product of closed compact sets hence closed and compact. Suppose that \( \mathcal{A} \subseteq \text{Rect}(\mathcal{M}) \) is a countable family of sets such that \( B \subseteq \bigcup \mathcal{A} \).

Then for any point \( x \in B \) there exists \( A \in \mathcal{A} \) with \( x \in A \), hence for every \( j \in J_A \),

\[
\pi_j(x) \in \pi_j[A],
\]

and an examination of \( \mathcal{M}_j \) shows that \( B_j \subseteq \pi_j[A] \), so that in fact \( B \subseteq A \). Let

\[
k = \max\{j : j \in J_A\}.
\]
Then
\[ g(A) = \mu_k(\bigcap_{j \in J_A} p_{jk}^{-1}[\pi_j[A]]) \geq \mu_k(\bigcap_{j \in J_A} p_{jk}^{-1}[B_j]) \]
\[ = \mu_k[B_k] = 1. \]

Thus \( \widetilde{\mu}(B) = 1. \)

Now consider
\[ B' = \prod_{i \in I} X_i \sim B. \]

Let \( \mathcal{B} \subset \text{Rect}(\mathcal{M}) \) be a countable family of sets with \( B' \subset \bigcup_{D \in \mathcal{B}} D \). Then \( T \) is countable, hence there exists \( i \in I \) with \( i \notin T \). Let \( x \in B' \) be such that \( \pi_i(x) \in X_i \sim B_i \), and, for every \( j \neq i \), \( \pi_j(x) \in B_j \). Then for some \( D \in \mathcal{B}, x \in D \), so that for every \( j \in J_D \), \( \pi_j(x) \in \pi_j[D] \).

By a previous argument it follows that \( B_j \subset \pi_j[D] \) for every \( j \in J_D \), hence \( B \subset D \) so that
\[ \widetilde{\mu}(D) = g(D) = 1, \]
and hence \( \widetilde{\mu}(B') = 1. \) From this and the fact that \( \widetilde{\mu}(B) = 1 \), it follows that \( B \notin \mathcal{M}_{\widetilde{\mu}} \), hence \( \widetilde{\mu} \) is not Radón w.r.t. the product topology.

It is easily seen that for every \( \alpha \in \text{Rect}(\mathcal{M}) \), \( \widetilde{\mu}(\alpha) = 0 \) or \( 1 \), and \( \widetilde{\mu}(\alpha) = 1 \) iff \( B \sim \alpha \). From this it follows that
\[ \widetilde{\mu}(\alpha) = \sup(\widetilde{\mu}(K) : K \text{ closed and compact, } K \subset \alpha), \]
for every \( \alpha \in \text{Rect}(\mathcal{M}) \), and that every closed compact subset of \( B' \) has \( \widetilde{\mu} \)-measure zero. Since \( B' \) is open, the regularized measure \( \widetilde{\mu}^* \) (definition 1.9, Ch.IV) must be such
that \( \hat{\mu}^*(B^i) = 0 \). For every \( \beta \in \text{Rect}(\mathcal{M}) \), we have either \( \beta \subset B^i \) or \( B^i \subset \beta \), so that it is easily checked that \( \hat{\mu}^* \)

is a \( \pi \)-limit outer measure. By theorem 1.10, Ch.IV, \( \hat{\mu}^* \) is Radon w.r.t. the product topology.

We note also that the inverse limit set, \( L \), is a subset of

\[
U \bigcup_{\mathbf{n} \in \mathbb{W}} \pi_{-1}^{-1} \left[ \pi_{-1}^{-1} \beta_{-1} \right]
\]

so that

\[
\hat{\mu}(L) = \sum_{\mathbf{n} \in \mathbb{W}} g \left( \pi_{-1}^{-1} \beta_{-1} \right) = 0,
\]

hence no inverse limit outer measure exists.

The following simple example shows that there may be no \( \pi \)-limit outer measure or inverse limit outer measure which is Radon w.r.t. the product topology even when the functions \( p_{ij} \) are continuous and the system is otherwise unexceptional.

5. Example. Let \( \mu \) be a Radon outer measure on the real line \( \mathbb{R} \), such that \( \mu(\mathbb{R}) = 1 \), \( \mu \) is Radon and \( \mu \) does not have compact support (e.g. a Gaussian measure). Also let \( J \) be an uncountable set.

Define the i.s.o.m. \( (X, p, \mu, I) \) as follows: let

\( I = \) finite subsets of \( J \), ordered by inclusion, and for each \( k \in I \), let

\[
X_k = \Pi R, \text{ with the usual topology;}
\]

\[
\mu_k \text{ be the product measure on } X_k,
\]
$p_{ik}$ be projection of $X_k$ onto $X_i$ whenever $i<k$.

Then by theorem 2.6, Ch.III, $\mu$ is a $\pi$-limit outer measure. Let $C$ be a compact subset of $\prod_{i \in I} X_i$. Then for each $j \in J$, $\pi_j[C]$ is compact. Let $\varepsilon > 0$ be such that

$$\text{Card}\{j \in J : \mu_j(\pi_j[C]) < 1 - \varepsilon\} > \aleph_0,$$

and let $j_0, j_1, ...$ be a sequence in $\{j \in J : \mu_j(\pi_j[C]) < 1 - \varepsilon\}$. For each $n \in \omega$, let

$$i_n = \{j_m : m \leq n\}$$

and let

$$K_n = \bigcap_{m=0}^{n} \pi_j^{-1}[\pi_j[C]].$$

Then

$$g(K_n) = \mu_i(\pi_i^{-1}[K_n]) \leq (1 - \varepsilon)^n.$$

Hence for any $\delta > 0$ there exists $m \in \omega$ such that $g(K_m) < \delta$, and since $C \subset K_m$, $\mu(C) = 0$.

Thus for any compact set $K$ and $\pi$-limit outer measure $\nu$, $\nu(K) = 0$, and thus no $\pi$-limit outer measure can be Radón for the product topology, and also no inverse limit outer measure can be Radón for the product topology (in this case the inverse limit measure is a product measure).

The next example shows that an i.s.o.m. may be such that no $\pi$-limit outer measure or inverse limit outer measure may be Radon w.r.t. the product topology even though the spaces $X_i$ are topological (and locally compact), the measures $\mu_i$
are Radón, the functions $p_{ij}$ are continuous except at one point, and the system has in fact an upper bound.

However, if we use the relativized topology of complements of compact sets, we can check that the restriction of $\tilde{\mu}$ to the inverse limit set is an inverse limit outer measure which is Radón w.r.t. this topology. Moreover this topology is essentially the topology we naturally expect on the inverse limit set, namely that of the upper bound space. Also, theorem 1.6, Ch.IV shows that a $\tau$-limit outer measure exists which is Radón w.r.t. the topology of complements of compact sets on \( \prod_{i \in I} X_i \).

6. **Example.** Let \( S = [0,1) \) and \( \lambda \) be Lebesgue outer measure on \( S \). Define the i.s.o.m. \( (X,p,\mu;I) \) as follows: let \( I = (0,1] \) with the usual ordering and for each \( j \in I \) let

\[ X_j = S, \]
\[ \mu_j = \lambda, \]
\[ \mathcal{J}_j \] be the usual topology on \( X, \)
\[ \mathcal{C}_i \] the closed compact subsets of \( S, \)
\[ p_{ij} \] be the function defined by

\[ p_{ij}(x) = x + j - i \pmod{1}, \text{ whenever } i \leq j. \]

Note that $p_{ij}$ is continuous except at $1 - (j - 1)$.

Since the system is also an inverse system of spaces, and has an upper bound, we expect \((S, \lambda)\) to be the "limit". It is clear that the inverse limit set, \( L \), can be identified with \( X_1 = S \) by the 1:1 mapping $\varphi : L \to X_1$ defined by

\[ \varphi(x) = \pi_1(x) = x_1 \] for every \( x \in L \), since
\[ L = \{ x \in \prod_{i \in I} X_i : x_i = p_{ij}(x_j) \text{ for all } i, j \in I \text{ with } i < j \} \]

\[ = \{ x \in \prod_{i \in I} X_i : x_i = p_{i1}(x_1) \text{ for all } i \in I \}. \]

Furthermore, it is not difficult to see that the outer measure \( \nu \) on \( \prod_{i \in I} X_i \), given by \( \nu(A) = \lambda(\varphi[A \cap L]) \) for every \( A \subseteq \prod_{i \in I} X_i \), is carried by \( L \) and is an inverse limit outer measure, which again is as we would expect (in fact \( \nu = \widetilde{\mu} \mid L \)). However the topology induced on \( L \) by the product topology is much finer than the ordinary topology on \( S \). It is in fact equivalent to the "half-open" interval topology on \( S \). To see this let \( s \in S \) and let \( i = s \).

Then \( p_{i1}(s) = 0 \), and for \( 0 < h < s \), \( [0, h) \) is a neighborhood of \( 0 \), so that

\[ \varphi^{-1}[p_{i1}^{-1}([0, h])] = \varphi^{-1}[[s, s+h)] \]

is a neighborhood of \( \varphi^{-1}[[s]] \). This topology has a very restricted class of compact sets (in fact no compact set can be uncountable) and no continuous non-zero measure can be Radon w.r.t. this topology.

If, on the other hand, we use the topology \( \mathcal{J} \) with \( \mathcal{C} \) as a base (\( \mathcal{C} = \delta_1(\sigma(\text{Rect}(\mathcal{C}))) \)), we induce a topology which corresponds to a more natural topology on \( S \). For each \( i \in I \), in the topology of complements of compact subsets of \( X_i \), sets of the form \( [0, h) \cup (1-h, 1) \) form a base for the neighborhood system at \( 0 \). Thus for \( s \in S \), \( s \neq 0 \), if we let \( i = s \) and choose \( h \) sufficiently small, we...
obtain
\[ L \cap \pi_1^{-1}[[0, h) \cup (1-h, 1)] = \varphi^{-1}[p_{11}^{-1}[[0, h) \cup (1-h, 1)]] \]
\[ = \varphi^{-1}[(s-h, s+h)] \]
as a neighborhood of \( \varphi^{-1}(s) \) in the relative topology of \( \mathcal{J} \) to \( L \). Thus, standard neighborhoods of \( s \) in the ordinary topology on \( S \) are lifted into neighborhoods of \( \varphi^{-1}(s) \) in the new topology on \( L \). For \( \varphi^{-1}(0) \), we obtain neighborhoods of the form \( \varphi^{-1}[[0, h) \cup (1-h, 1)] \) for \( h > 0 \), which are sufficiently different from the usual neighborhoods to make the topology compact (and still Hausdorff). Let \( \mathcal{E} \) be the family of open neighborhoods on \( L \) described above.

We next check that \( \mathcal{E} \) actually forms a base for the topology induced on \( L \) by \( \mathcal{J} \). From lemma A which follows this example we see that \( \mathcal{J}_C = \mathcal{E} \), hence
\[ \mathcal{J}_C = \mathcal{C}(C) = \sigma_1(\text{Rect}(C)), \]
so that \( \mathcal{J} \) has a subbase \( \text{Cyl}\mathcal{J} \), where for each \( i \in I \), \( \mathcal{J}_i \) is any base for the topology \( \mathcal{C}(C_i) \) on \( X_i \). In particular we shall take for \( \mathcal{J}_i \) the family of sets of the form
\[ [0, a) \cup (b, c) \cup (c, 1), \]
where \( 0 \leq a \leq b \leq c \leq 1 \). For any such set \( G \), and \( i \in I \), it is clear that \( p_{11}^{-1}[G] \) is open in the usual topology on \( X_i = S \) and that if \( 0 \in p_{11}^{-1}[G] \), then \( p_{11}^{-1}[G] \) contains a set of the form \( [0, h) \cup (1-h, 1) \) for some \( h > 0 \). Consequently \( \varphi^{-1}[p_{11}^{-1}[G]] \) is contained in the topology generated by the family \( \mathcal{E} \) described above, hence \( \mathcal{E} \) is a base (it is clearly closed under finite intersections) for the topology \( \mathcal{J}' \) induced on \( L \) by \( \mathcal{J} \).
If we now define the topology $\mathfrak{J}''$ on $\tilde{X}$ by

$$\mathfrak{J}'' = \mathfrak{J}' \cup \{\tilde{X} \sim L\}$$

and extend $\nu$ to all of $\tilde{X}$ by letting $\nu(\tilde{X} \sim L) = 0$, then $\nu$ is an inverse limit outer measure which is Radon w.r.t. $\mathfrak{J}''$.

Note also that since $L$ is a closed set in the product topology, any $\pi$-limit outer measure which is Radon w.r.t the product topology must be supported by $L$, (theorem 3.4, Ch.IV). Since this is impossible no such $\pi$-limit outer measure exists. However, theorem 1.5, Ch.IV shows that there exists a $\pi$-limit outer measure which is Radon w.r.t. $\mathfrak{J}$.

The next lemma shows that in many product space situations we do not increase the number of sets under consideration by using $\mathfrak{J}_C$ instead of $C$.

**Lemma A.** Let $(X,I)$ be a system of spaces with $I$ uncountable and let $C$ be a system of families of sets w.r.t. $(X,I)$ such that for every $i \in I$, $C_i$ is $\aleph_0$-compact, $\{x\} \in C_i$ for every $x \in X_i$, and there exists a sequence $C_0, C_1, \ldots \in C_i$ such that $X_i = \bigcup_{n \in \omega} C_i$. Then if we let

$$C = \delta_1(\sigma_0(\text{Rect}(C))),$$

we have $\mathfrak{J}_C = C$.

**Proof.** Let $F \in \mathfrak{J}_C$, $F \neq \emptyset$. Then since for $i \in I$, $X_i = \bigcup_{n \in \omega} C_i$ for some sequence $C_0, C_1, \ldots$ in $C_i$, if we let for each $n \in \omega$,
\[ \mathcal{K}_n = F \setminus \pi_1^{-1}([c_n]), \]
then \( K_n \in \mathcal{C} \) for each \( n \in w \) and \( F = \bigcup K_n \). Then for every \( n \in w \) there exist finite families \( \mathcal{A}_{(n,m)} \subset \text{Rect}(\mathcal{C}) \) for every \( m \in w \) such that
\[ K_n = \bigcap_{m \in w} \mathcal{A}_{(n,m)}. \]
Let
\[ T = \bigcup_{m,n \in w} \bigcup_{A \in \mathcal{A}_{(n,m)}} J_A, \]
\( i \in I \sim T \) and \( x_i \in X_i \). Then \( \pi_1^{-1}([x_i]) \in \mathcal{C} \), hence there exist finite families \( \mathcal{B}_m \subset \text{Rect}(\mathcal{C}) \) for every \( m \in w \) such that
\[ \pi_1[B] = \{x_i\} \text{ for every } m \in w \text{ and } B \in \mathcal{B}_m, \text{ and} \]
\[ \pi_1^{-1}([x_i]) \cap F = \bigcap_{m \in w} \bigcup \mathcal{B}_m. \]
For every \( m \in w \) and \( B \in \mathcal{B}_m \) let
\[ B' = \bigcap_{j \in (J_B \setminus J)} \pi_1^{-1}([\pi_j[B]]) \]
and \( \mathcal{B}'_m = \{B' : B \in \mathcal{B}_m\} \). Since for every \( m \in w \) and \( B \in \mathcal{B}_m \) we have \( B' \in \text{Rect}(\mathcal{C}) \), we will have shown that \( F \in \mathcal{C} \) if we show that
\[ F = \bigcap_{m \in w} \bigcup \mathcal{B}'_m. \]
Let \( y \in F \). Then for some \( n \in w \), \( y \in K_n \), hence for each \( m \in w \) there exists \( A_m \in \mathcal{A}_{(n,m)} \) with \( y \in A_m \). Let \( z \in X \) be such that
\[ z_j = y_j \text{ for } j \neq i \text{ and } z_i = x_i. \]
Then since \( i \notin J_{A_m} \), \( z \in \mathcal{A}_m \) for every \( m \in w \) and thus \( z \in F \), hence \( z \in \pi_1^{-1}([x_i]) \subset F \). Consequently for every \( m \in w \) there exists \( B_m \in \mathcal{B}_m \) with \( z \in B_m \), hence \( z \in B'_m \).
But since \( i \notin J_{B_m} \), for any \( m \in w \), we must have \( y \in B'_m \) for every \( m \in w \), hence
\[ y \in \bigcap_{m \in w} B'_m. \]
Now let \( y \in \bigcap_{m \in w} \bigcup_{m \in w} B_m' \). Then there exists for every \( m \in w \), \( B_m' \in B_m' \) with \( y \in B_m' \). Hence if again we let \( z \in X \) be such that \( z_j = y_j, j \neq i \), and \( z_i = x_i \), we must have \( z \in \bigcup_{m \in w} [(x_i)_{m \in w}] \).

\[ B_m' = B_m \text{ for every } m \in w. \]

Hence \( z \in \bigcap_{m \in w} B_m' \) and \( z \in F \). Then again there exists \( K_n \) such that \( z \in K_n \), and hence for every \( m \in w \), \( A_m \in A_{(n,m)} \) with \( z \in A_m \). Since \( i \notin J_m \) for any \( m \in w \), \( y \in A_m \). Hence \( y \in K_n \), and therefore \( y \in F \). Thus \( F = \bigcap_{m \in w} B_m' \), hence \( F \in \mathcal{C} \), and hence \( \mathcal{I} \mathcal{C} \subseteq \mathcal{C} \). Since \( \mathcal{C} \) is closed under intersection, \( \mathcal{C} \subseteq \mathcal{I} \mathcal{C} \) and so \( \mathcal{I} \mathcal{C} = \mathcal{C} \).
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