DOMAIN-PERTURBED PROBLEMS FOR ORDINARY LINEAR DIFFERENTIAL OPERATORS

by

JOHN FROESE
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M.A. Queen's University, 1961

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JOHN FROESE

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COMMITTEE IN CHARGE

Chairman: J.A. Jacobs
Douglas Derry       Earl D. Rogak
Charlotte Froese   C.A. Swanson
Elod Macskasy      Roy Westwick

External Examiner: R.R.D. Kemp
Queen's University
Kingston, Ontario

Research Supervisor: C.A. Swanson
The variation of the eigenvalues and eigenfunctions of an ordinary linear self-adjoint differential operator \( L \) is considered under perturbations of the domain of \( L \). The basic problem is defined as a suitable singular eigenvalue problem for \( L \) on the open interval \( w_- < s < w_+ \) and is assumed to have at least one real eigenvalue \( \lambda \) of multiplicity \( k \). The perturbed problem is a regular self-adjoint problem defined for \( L \) on a closed subinterval \([a, b] \) of \((w_-, w_+)\). It is proved under suitable conditions on the boundary operators of the perturbed problem that exactly \( k \) perturbed eigenvalues \( \mu_{ab}^1 - \lambda \) as \( a, b \to w_-, w_+ \). Further, asymptotic estimates are obtained for \( \mu_{ab}^1 - \lambda \) as \( a, b \to w_-, w_+ \). The other results are refinements which lead to asymptotic estimates for the eigenfunctions and variational formulae for the eigenvalues.

The conditions on the limiting behaviour of the boundary operators depend strongly on the nature of the singularities \( w_-, w_+ \). If for some number \( \ell_0, \ell_0 \) not an eigenvalue, linearly independent solutions of \( Lx = \ell_0 x \) exist which are asymptotically ordered at \( w_- \), then \( w_- \) is called a class 1 singularity. In the case
that both \( \omega_-, \omega_+ \) are class 1 singularities, very general boundary operators permit the convergence of \( \mu_{ab}^1 \) to \( \lambda \). Class 2 singularities are defined as follows: If all solutions of \( Lx = \ell_0 x \) are square-integrable on \( (\omega_-, c] \) for any \( c \) satisfying \( \omega_- < c < \omega_+ \), then \( \omega_- \) is called a class 2 singularity. An asymptotic ordering of the solutions is not assumed in this case.

Since the behaviour of the solutions of \( Lx = \ell_0 x \) is essentially arbitrary when both \( \omega_-, \omega_+ \) are class 2 singularities, the generality of the boundary operators has to be sacrificed to ensure that \( \mu_{ab}^1 \rightarrow \lambda \). Certain one end perturbation problems and examples also are considered.

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**GRADUATE STUDIES**

Asymptotic Analysis \hspace{1cm} C.A. Swanson
Complex Variable \hspace{1cm} R.A. Cleveland
Functional Analysis \hspace{1cm} A.H. Cayford
Measure and Integration \hspace{1cm} S.W. Nash
Point Set Topology \hspace{1cm} P.S. Bullen
ABSTRACT

The variation of the eigenvalues and eigenfunctions of an ordinary linear self-adjoint differential operator \( L \) is considered under perturbations of the domain of \( L \). The basic problem is defined as a suitable singular eigenvalue problem for \( L \) on the open interval \( \omega_\text{-} < s < \omega_+ \) and is assumed to have at least one real eigenvalue \( \lambda \) of multiplicity \( k \). The perturbed problem is a regular self-adjoint problem defined for \( L \) on a closed subinterval \([a, b]\) of \((\omega_\text{-}, \omega_+)\). It is proved under suitable conditions on the boundary operators of the perturbed problem that exactly \( k \) perturbed eigenvalues \( \mu_{ab}^{-1} \rightarrow \lambda \) as \( a, b \rightarrow \omega_\text{-}, \omega_+ \). Further, asymptotic estimates are obtained for \( \mu_{ab}^{-1} \rightarrow \lambda \) as \( a, b \rightarrow \omega_\text{-}, \omega_+ \). The other results are refinements which lead to asymptotic estimates for the eigenfunctions and variational formulae for the eigenvalues.

The conditions on the limiting behaviour of the boundary operators depend strongly on the nature of the singularities \( \omega_-, \omega_+ \). If for some number \( \ell_0 \), \( \ell_0 \) not an eigenvalue, linearly independent solutions of \( Lx = \ell_0 x \) exist which are asymptotically ordered at \( \omega_- \), then \( \omega_- \) is called a class 1 singularity. In the case that both \( \omega_-, \omega_+ \) are class 1 singularities, very general boundary operators permit the convergence of \( \mu_{ab}^{-1} \) to \( \lambda \). Class 2 singularities are defined as follows: If all solutions of \( Lx = \ell_0 x \) are square-integrable on \((\omega_-, c)\) for any \( c \) satisfying \( \omega_- < c < \omega_+ \), then \( \omega_- \) is called a class 2 singularity. An
asymptotic ordering of the solutions is not assumed in this case. Since the behaviour of the solutions of \( Lx = \lambda_0 x \) is essentially arbitrary when both \( w_-, w_+ \) are class 2 singularities, the generality of the boundary operators has to be sacrificed to ensure that \( \mu^1_{ab} \to \lambda \). Certain one end perturbation problems and examples also are considered.
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INTRODUCTION

Eigenvalue problems will be considered for the n-th order, ordinary, linear differential operator \( L \) defined by

\[
Lx = \frac{1}{k(s)} \sum_{i=0}^{n} p_i(s) x^{(n-i)}(s)
\]

on the open interval \( w_- < s < w_+ \) where \( k \) and \( p_i \), \( i = 0,1,\ldots,n \) are real-valued functions on this interval with the properties that:

(i) \( p_i(s) \in C^{n-i}(w_-,w_+) \), \( i = 0,1,\ldots,n \);

(ii) \( k(s) \) is piecewise continuous on \((w_-,w_+)\); and

(iii) \( p_0(s) \neq 0 \) and \( k(s) > 0 \) on \((w_-,w_+)\).

Furthermore we assume that the operator

\[
k(s) \cdot Lx = \sum_{i=0}^{n} p_i(s) x^{(n-i)}(s)
\]

is formally self-adjoint, i.e., \( k(s) \cdot Lx \) coincides with its Lagrangian adjoint \( [k(s) \cdot Lx]^+ \) where

\[
[k(s) \cdot Lx]^+ = \sum_{i=0}^{n} (-1)^{n-i} [p_i x]^{(n-i)}.
\]

The points \( w_- \) and \( w_+ \) are in general singularities of \( L \); the possibility that they are \( + \infty \) is not excluded.

It will be convenient to use the following notations:
\[
(x,y)^t_s = \int_s^t x(u)y(u)k(u)du, \quad \omega_- \leq s < t \leq \omega_+ ;
\]
\[
(x,y)_a = (x,y)^{\omega_+}_a ;
\]
\[
(x,y)^b = (x,y)^{\omega_-}_a ;
\]
\[
(x,y) = (x,y)^{\omega_+}_a ;
\]
\[
[x(y)](s) = \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^j x(k)(s)[p_{n-m}(s)y(s)](j); \quad j>0, k>0
\]
\[
[x(y)](\omega_+) = \lim_{s \to \omega_+} [x(y)](s).
\]

Since the operator \( k \cdot L \) is formally self-adjoint Green's symmetric formula has the form
\[
(Lx,y)^t_s - (x,Ly)^t_s = [x(y)](t) - [x(y)](s). \tag{0.5}
\]

Let \( H, H[a,b] \) denote the Hilbert spaces which are the Lebesgue spaces with respective inner products \((x,y), (x,y)_a\) and norms
\[
\| x \| = (x,x)^{1\over 2}, \quad \| x \|_a = [(x,x)^a_1]^{1\over 2}, \quad \omega_- \leq a < b \leq \omega_+ .
\]

For \( c \) any intermediate point, \( \omega_- < c < \omega_+ \), we likewise define \( H(\omega_-, c), H[c, \omega_+] \) to be the Hilbert spaces which are the Lebesgue spaces with respective inner products \((x,y)^c, (x,y)_c\) and norms \( \| x \|_c = [(x,x)^c]^{1\over 2}, \quad \| x \|_c = [(x,x)_c]^{1\over 2} \). From (0.5) it is clear that \([x(y)](\omega_+)\) (or \([x(y)](\omega_+)\)) exists provided \( x,y,Lx, \) and \( Ly \) are in \( H(c, \omega_+) \) (or \( x,y,Lx \) and \( Ly \) are in \( H(\omega_-, c) \)).

Let \( a_o \) and \( b_o \) be fixed numbers satisfying \( \omega_- < a_o < b_o < \omega_+ \) and let \( R_o \) be the rectangle in the \( a-b \)-plane described by the
inequalities \( w_- < a \leq a_0, \ b_0 \leq b < w_+ \). Every closed, bounded interval \([a, b]\), \( w_- < a \leq a_0, \ b_0 \leq b < w_+ \), can be associated in a one-to-one manner with a point of \( \mathbb{R}_0 \). For every such \([a, b]\) we shall consider the eigenvalue problem

\[
(0.6) \quad L_y = \mu y, \quad U^i_a y = 0, \ i = 1, 2, \ldots, m, \\
                U^i_b y = 0, \ i = 1, 2, \ldots, n-m
\]

where \( U^i_a \) and \( U^i_b \) are the linear boundary operators

\[
(0.7) \quad \begin{cases}
U^i_a y = \sum_{k=0}^{n-1} a_{ik}(a)y^{(k)}(a), \ i = 1, 2, \ldots, m \\
U^i_b y = \sum_{k=0}^{n-1} \beta_{ik}(b)y^{(k)}(b), \ i = 1, 2, \ldots, n-m
\end{cases}
\]

where \( a_{ik}(a), \beta_{ik}(b) \) are real-valued functions on the respective intervals \( w_- < a \leq a_0, \ b_0 \leq b < w_+ \) and are such that the conditions \( U^i_a y = U^j_b y = 0, \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n-m \) form a linearly independent self-adjoint set of boundary conditions for \( L \) (See [3], pp. 288 - 291). Our problem is to obtain estimates for each eigenvalue \( \mu = \mu_{ab} \) of (0.6) for \( a, b \) near \( w_-, w_+ \) under hypothesis that will ensure that the limits of \( \mu_{ab} \) as \( a, b \to w_-, w_+ \) will exist. Accordingly, we shall assume that eigenvalues \( \lambda \) of suitable singular eigenvalue problems for \( L \) on \((w_-, w_+)\) exist. If the eigenspace of \( \lambda \) is \( k \)-dimensional our first theorem shows in particular that at least \( k \) eigenvalues of (0.6) converge to \( \lambda \) as \( a, b \to w_-, w_+ \). Our other results are refinements of this which lead to asymptotic estimates for eigenfunctions. The method of estimation used here is due to H.F. Bohnenblust (see [11], p. 1553).
Results like these have been previously obtained for the second order, ordinary case by C.A. Swanson. In [11] he considers the case when, in Weyl's classification, both \( w_ - \) and \( w_ + \) are limit circle singularities, and in [13] he considers the cases (i) when both \( w_ - \) and \( w_ + \) are limit point singularities; (ii) when \( w_ - \) is a limit circle singularity and \( w_ + \) is a limit point singularity. Swanson makes strong use of two well-known theorems of Weyl ([9], p. 35 and p. 45) in setting up suitable singular eigenvalue problems. However, for higher order cases \( (n > 2) \) these theorems are no longer valid so that assumptions will have to be made in particular about the behaviour of the solutions of \( Lx = \lambda x, \text{Im} \lambda \neq 0 \), at \( w_ - \) and at \( w_ + \). Also we will always require the existence of at least one real eigenvalue for the singular eigenvalue problem on \( (w_-, w_+) \) at hand.

It is easily seen in [11] and [13] that the "limit point, limit circle" classification of singularities for the second order differential operator is not a natural classification in relation to domain-perturbed problems. Moreover these terms "limit point, limit circle" have little or no meaning in reference to singularities of higher order differential operators. Consequently, we shall use the following classification of the singularities \( w_ - \) and \( w_ + \) of \( L \); let \( \lambda_0 \) be any complex number, \( \text{Im} \lambda_0 \neq 0 \), and let \( \varphi_1, i=1,2,\ldots,n \), be linearly independent solutions (hereafter to be referred to as basic solutions) of \( L_0x = 0 \) where \( L_0 = L - \lambda_0 \).

If there exist basic solutions \( \varphi_1, \varphi_j, i,j=1,2,\ldots,n \), such that

\[
\lim_{s \to w_+} \frac{\varphi_1(s)}{\varphi_j(s)} = 0 \text{ or } \infty
\]

for each pair \( \varphi_1, \varphi_j, i,j=1,2,\ldots,n, i \neq j \), then \( w_+ \) will be
referred to as class 1 singularities. Note that in this case \( w_\pm \) cannot be accumulation points of zeros for the basic solutions.

It will be seen in chapter 1 that for this case only slight restrictions are needed on the limiting behaviour of the boundary operators (0.7) to obtain convergence of the eigenvalues of (0.6) as \( a, b \to w_-, w_+ \) (see (2.7) - (2.12)). On the other hand \( w_- \) (or \( w_+ \)) will be called a class 2 singularity when the behaviour of the basic solutions is essentially arbitrary as \( s \to w_- \) (or \( s \to w_+ \)). In particular this includes cases for which (0.8) does not hold or for which the basic solutions oscillate infinitely often as \( s \to w_- \) (or \( s \to w_+ \)). It will be seen in chapter 2 that in this case more restrictive conditions are needed on the limiting behaviour of the boundary operators (0.7) as \( a \to w_- \) (or \( b \to w_+ \)) to obtain convergence of the eigenvalues of (0.6) (see (10.3)).

The singularities \( w_- \) (or \( w_+ \)) will be further characterized by the number of basic solutions that are in \( H(w_-, c) \) (or in \( H(c, w_+) \)) where \( c \) is any number satisfying \( w_- < c < w_+ \). Thus for \( n=2 \), \( w_- \) is a "limit circle" singularity if both basic solutions are in \( H(w_-, c) \); otherwise \( w_- \) is a "limit point" singularity.

The plan is as follows: Chapter I will be devoted to perturbation problems where both \( w_- \) and \( w_+ \) are class 1 singularities. In Chapter II perturbation problems are considered for which both \( w_- \) and \( w_+ \) are class 2 singularities and all the basic solutions are in \( H \). Finally examples of perturbation problems will be given in Chapter III which will illustrate the material in Chapters I and II.
CHAPTER I

ASYMPTOTIC ESTIMATES FOR PROBLEMS WITH \( w_- , w_+ \) CLASS 1 SINGULARITIES

1. Description of the basic and perturbed problems.

One type of singular problem on \( (w_- , w_+) \) to be considered in this chapter is the case that no solution of \( L_0x = 0 \) is in \( H \). More precisely we shall assume the existence of basic solutions \( \varphi_1, \varphi_2, \ldots, \varphi_n \) such that for any number \( c \), \( w_- < c < w_+ \);

\[
(i) \quad \varphi_j \in H[c, w_+), \varphi_j \notin H(w_-, c], j = 1, 2, \ldots, m, \\
\varphi_j \in H(w_-, c), \varphi_j \notin H[c, w_+), j = m+1, \ldots, n;
\]

\[
(1.1) \quad (ii) \lim_{a \to w_-} \frac{\varphi_{i+1}(a)}{\varphi_i(a)} = 0, \quad i = 1, 2, \ldots, m-1, \\
\lim_{b \to w_+} \frac{\varphi_i(b)}{\varphi_{i+1}(b)} = 0, \quad i = m+1, \ldots, n-1.
\]

Let \( D \) be the set of all \( x \in H \) such that \( x \in C^{n-1}(w_-, w_+) \) and \( x^{(n-1)} \) is absolutely continuous on every closed bounded sub-interval of \( (w_-, w_+) \). Then the singular eigenvalue problem on \( (w_-, w_+) \)

\[
(1.2) \quad Lx = \lambda x, \quad x \in D
\]

is called the basic problem. Our main assumption is that there exists at least one eigenvalue \( \lambda \) of this problem. In [6], K. Kodaira shows that the dimension of the solution space of \( Lx = \lambda x \) in \( H \) is independent of \( \lambda \), provided only that \( \text{Im } \lambda \neq 0 \). Since \( Lx = \lambda_0 x \) has no solutions in \( H \), it follows that all eigenvalues of \( (1.2) \) are necessarily real. In particular, \( \lambda_0 \)
is not an eigenvalue and \( \lambda \) is necessarily real.

For each \([a,b] \in \mathbb{R}_0\), let \( D[a,b] \) denote the set of all \( y \in H[a,b] \) which satisfy the following conditions:

1. \( y \in C^{n-1}[a,b] \), \( y^{(n-1)} \) is absolutely continuous on \([a,b]\);
2. \( Ly \in H[a,b] \); and
3. \( y \) satisfies the homogeneous boundary conditions of (0.6).

Then the set \( D[a,b] \) will be referred to as the perturbed domain and the eigenvalue problem

\begin{equation}
Ly = \mu y, \quad y \in D[a,b]
\end{equation}

as the perturbed problem. Since the set of boundary conditions for (1.3) is self-adjoint, it follows from the boundary-formula (\([3], p. 288\)) and (0.5) that the perturbed problem (1.3) is self-adjoint. It is well-known (\([3\), chapter 7\)) that for such a self-adjoint problem there exists a countable set of real eigenvalues accumulating only at \( \infty \) and a set of (real) eigenfunctions complete in \( H[a,b] \).

2. **Comparison of the basic and perturbed problems.**

To obtain convergence of the eigenvalues of (1.3) to those of (1.2), restrictions will have to be imposed on the behaviour of the boundary operators \( U^i_a \), \( U^i_b \) as \( \alpha \to \omega^- \), \( \beta \to \omega^+ \). Let \( A(a,b) \) denote the \( n \)-by-\( n \) matrix \( (A_{ij}(a,b)) \) where

\[
A_{ij}(a,b) = \begin{cases} 
U^i_a \phi_j & \text{if } i = 1, \ldots, m; \quad j = 1, 2, \ldots, n \\
U^{i-m} b \phi_j & \text{if } i = m+1, \ldots, n; \quad j = 1, 2, \ldots, n.
\end{cases}
\]
The symbols \((i^r_i, i^s_j)\) and \((j^r_i, j^s_j)\) will be referred to as sequences and will represent any increasing sequences of \(k\) integers selected from the set \(\{1, 2, \ldots, n\}\). We adopt the following notations:

\[
(2.1) \quad a(j^r_i, j^s_j) = \det \left( U^r_{a_j} \varphi^r_{j^s_j} \right), \quad i, k = 1, 2, \ldots, m; \\
q(j^r_i, j^s_j) = \det \left( U^s_{j^r_j} \varphi^s_{j^s_j} \right), \quad i, k = 1, 2, \ldots, n-m; \\
a = a(1, m); \\
b = a(m+1, n).
\]

For \(x\) any normalized eigenfunction of \((1.2)\) and for any sequence \((i^r_i, i^s_{m-1})\) we let \(\delta_a(i^r_i, i^s_{m-1})\) denote the determinant of the matrix

\[
\begin{bmatrix}
U^1_{a_1} x & U^1_{a_i} \varphi^1_{i^1_{m-1}} & \cdots & U^1_{a_i} \varphi^1_{i^1_{m-1}} \\
U^2_{a_1} x & U^2_{a_i} \varphi^2_{i^2_{m-1}} & \cdots & U^2_{a_i} \varphi^2_{i^2_{m-1}} \\
\vdots & \vdots & \ddots & \vdots \\
U^m_{a_1} x & U^m_{a_i} \varphi^m_{i^m_{m-1}} & \cdots & U^m_{a_i} \varphi^m_{i^m_{m-1}}
\end{bmatrix}
\]

Similarly for any sequence \((j^r_i, j^s_{n-m-1})\) we let \(\delta_b(j^r_i, j^s_{n-m-1})\) denote the determinant of the matrix

\[
\begin{bmatrix}
U^1_{b_1} x & U^1_{b_i} \varphi^1_{j^1_{n-m-1}} & \cdots & U^1_{b_i} \varphi^1_{j^1_{n-m-1}} \\
U^2_{b_1} x & U^2_{b_i} \varphi^2_{j^2_{n-m-1}} & \cdots & U^2_{b_i} \varphi^2_{j^2_{n-m-1}} \\
\vdots & \vdots & \ddots & \vdots \\
U^{n-m}_{b_1} x & U^{n-m}_{b_i} \varphi^{n-m}_{j^{n-m}_{n-m-1}} & \cdots & U^{n-m}_{b_i} \varphi^{n-m}_{j^{n-m}_{n-m-1}}
\end{bmatrix}
\]
The following notations will be used:

\[ (2.2) \quad \sigma_a(i,j) = \frac{\varphi_j(a)}{\varphi_i(a)} ; \quad \sigma_b(i,j) = \frac{\varphi_i(b)}{\varphi_j(b)} ; \]

\[ (2.3) \quad s_a(i,j) = \sigma_a(i,j) \left\| \varphi_i \right\|_a ; \quad s_b(i,j) = \sigma_b(i,j) \left\| \varphi_j \right\|_b ; \]

\[ (2.4) \quad \Theta_a^i(j_1, i_{m-1}) = \frac{\delta_a(i_1, i_{m-1})}{\eta_a} \left\| \varphi_i \right\|_a ; \]

\[ \Theta_b^j(j_1, j_{n-m-1}) = \frac{\delta_b(j_1, j_{n-m-1})}{\eta_b} \left\| \varphi_j \right\|_b ; \]

\[ (2.5) \quad \Theta_a^i(j_1, i_m) = \frac{\eta_a(i_1, i_m)}{\eta_a} \left\| \varphi_i \right\|_a ; \]

\[ \Theta_b^j(j_1, j_{n-m}) = \frac{\eta_b(j_1, j_{n-m})}{\eta_b} \left\| \varphi_j \right\|_b ; \]

Finally for \( \lambda \) an eigenvalue of (1.2) and \( A_\lambda \) the corresponding eigenspace of dimension \( k \) we let

\[ (2.6) \quad \Theta_a = \sup_{x \in A_\lambda} \left\| \Theta_a^1(2, m) \right\| , \quad \Theta_b = \sup_{x \in A_\lambda} \left\| \Theta_b^1(m+1, n-1) \right\| \]

The assumptions below turn out to be sufficient to obtain convergence of the eigenvalues of (1.3) to those of (1.2) as

\[ a, b \rightarrow w_- , w_+ . \]
Assumptions:

(a) The singularities $\omega_-$ and $\omega_+$ are not accumulation points of the zeros of $\varphi_j, j = 1,2,\ldots,n$.

(b) There exists a positive continuous function $h(s)$ defined on $(\omega_-, \omega_+)$ such that $\lim_{s \to \omega_+} h(s) = +\infty$ and such that the following conditions are satisfied:

\begin{align*}
\text{(2.7) } & (i) \quad h(a) \varphi_a(i,j) = o(1) \text{ as } a \to \omega_- \text{ and } \\
\text{(2.8) } & h(b) \varphi_b(i,j) = o(1) \text{ as } b \to \omega_+ \\
& \text{for } i = 1,2,\ldots,m, \quad j = m+1,\ldots,n;
\end{align*}

(ii) There exist integers $i_o, j_o$ where $1 \leq i_o \leq m$, $m+1 \leq j_o \leq n$ such that

\begin{align*}
\text{(2.9) } & \frac{n_a(i_l,i_m)\varphi_a(a)}{h(a) \varphi_j(a)} \\
& \text{is bounded on } (\omega_-, a_o) \text{ for } i = 1,2,\ldots,m \text{ and for all sequences } (i_l,i_m) \text{ for which } i \neq i_k, \quad k = 1,2,\ldots,m; \text{ and}
\end{align*}

\begin{align*}
\text{(2.10) } & \frac{n_b(j_l,j_{n-m})\varphi_j(b)}{h(b) \varphi_j(a)} \\
& \text{is bounded on } [b_o, \omega_+) \text{ for } j = m+1,\ldots,n \text{ and all sequences } (j_l,j_{n-m}) \text{ for which } j \neq j_k, \quad k = m+1,\ldots,n.
\end{align*}

\begin{align*}
\text{(2.11) } & (c) \quad \varphi_a^i(i_l,i_{m-1}) = 0[\varphi_a^i(2,m)] = o(\varphi_a) = o(1) \\
& \text{as } a \to \omega_- \text{ for } i = 1,2,\ldots,m \text{ and all sequences } (i_l,i_{m-1}) \text{ for which } i \neq i_k, \quad k = 1,2,\ldots,m-1; \text{ and}
\end{align*}

\begin{align*}
\text{(2.12) } & \varphi_b^j(j_l,j_{n-m-1}) = 0[\varphi_b^j(m+1,n-1)] = o(\varphi_b) = o(1)
\end{align*}
as \( b \to \omega_+ \) for \( j = m+1, \ldots, n \) and all sequences \((j_1', j_{n-m-1})\) for which \( j \neq j_k, k = 1, 2, \ldots, n-m-1\).

These assumptions imply the following when \( a \to \omega_, b \to \omega_+ \):

\[
\begin{align*}
(2.13) \quad h(a)\sigma_a(i,j) &= o(1), \quad h(b)\sigma_b(i,j) = o(1), \\
&\quad i = 1, 2, \ldots, m, \quad j = m+1, \ldots, n; \\
(2.14) \quad \frac{n_a(i_1, i_m)}{n_a} &= o(1) \text{ for } (i_1, i_m) \neq (1, m); \\
(2.15) \quad \frac{n_b(j_1, j_{n-m})}{n_b} &= o(1) \text{ for } (j_1, j_{n-m}) \neq (m+1, n); \\
(2.16) \quad \Phi^i_a(i_1, i_m) &= o(1) \text{ for } i = 1, 2, \ldots, m \text{ and all sequences } (i_1, i_m) \text{ for which } i \neq i_k, \quad k = 1, 2, \ldots, m; \\
(2.17) \quad \Phi^j_b(j_1, j_{n-m}) &= o(1) \text{ for } j = m+1, \ldots, n \text{ and all sequences } \\
&\quad \text{for which } j \neq j_k, k = 1, 2, \ldots, n-m.
\end{align*}
\]

Also (2.9) and (2.10) imply that there exists neighbourhoods 
\((\omega_, a_0), [b_0, \omega_+]\) of \(\omega_, \omega_+\) respectively and a constant \(C\) such that

\[
(2.18) \quad \left| \frac{n_a(i_1, i_m)}{n_a} \right| \leq C |\sigma_a(i, j_o)| h(a)
\]

whenever \(\omega_- < a < a_0\) for \(i = 1, 2, \ldots, m\) and all sequence \((i_1, i_m)\) for which \(i_k \neq i, \quad k = 1, 2, \ldots, m\) and

\[
\left| \frac{n_b(j_1, j_{n-m})}{n_b} \right| \leq C |\sigma_b(i_0, j)| h(b)
\]

whenever \(b_0 \leq b < \omega_+\) for \(j = m+1, \ldots, n\) and all sequences
\((j_1, j_{n-m})\) such that \(j_k \neq j, \quad k = 1, 2, \ldots, n-m.\)

Conditions (2.11) - (2.17) are actually sufficient to obtain convergence of the eigenvalues of (1.3) to those of (1.2).
However, the stronger assumptions (2.7) - (2.12) will be needed to obtain the uniform estimates of section 3. We have the following theorem:

**Theorem 1.** Let \( \omega_- \) and \( \omega_+ \) be singularities of \( L \) as described in section 1. Let \( \lambda \) be an eigenvalue of (1.2) possessing \( k \) orthonormal eigenfunctions. Then under the assumptions (2.7) - (2.12) (or under the weaker conditions (2.11)-(2.17)) there exists a rectangle \( R_0 \) and a constant \( C \) on \( R_0 \) such that at least \( k \) perturbed eigenvalues \( \mu_{ab}^j \) of (1.3) satisfy

\[
|\mu_{ab}^j - \lambda| \leq C(\varphi_a + \varphi_b)
\]

whenever \([a,b] \in R_0\).

**Proof:** Let \( G_{ab}(s,t) \) be the Green's function for the operator \( kL_0 \) associated with the boundary conditions of (0.6) and let \( G_{ab} \) be the linear transformation on \( H[a,b] \) defined by

\[
G_{ab}y = \int_a^b G_{ab}(s,t)y(t)k(t) \, dt, \quad y \in H[a,b].
\]

It is well-known ([3], p. 192) that for any function \( y \in H[a,b] \), the function \( w = G_{ab}y \) is the unique solution in \( D[a,b] \) of the differential equation \( L_0w = y \). For \( \lambda \) an eigenvalue and \( x \) any corresponding normalized eigenfunction of (1.2), define a function \( f \) on \([a,b]\) by

\[
(2.19) \quad f = x - \gamma G_{ab}x \quad \text{where} \quad \gamma = \lambda - \ell_0.
\]

It is easily verified because of the linearity of all the operators involved that \( f \) is a solution of the boundary value
problem

(2.20) \[ L_0 f = 0, \quad U^i_a f = U^i_a x, \quad i = 1, 2, \ldots, m, \]
\[ U^i_b f = U^i_b x, \quad i = 1, 2, \ldots, n-m. \]

We can find the solution \( f \) of (2.20) in terms of the basic solutions in the following way: We apply the boundary conditions of (2.20) to the equation
\[ f = \sum_{k=1}^{n} A_k \phi_k \]
to obtain the non-homogeneous system of linear equations
\[
\begin{cases}
U^i_a x = \sum_{k=1}^{n} A_k U^i_a \phi_k, & i = 1, 2, \ldots, m \\
U^i_b x = \sum_{k=1}^{n} A_k U^i_b \phi_k, & i = 1, 2, \ldots, n-m
\end{cases}
\]
and apply Cramer's rule. If for each \( k, \quad k = 1, 2, \ldots, n, \) the determinant corresponding to \( A_k \) is expanded by the complementary minors \( \eta_a(\quad), \quad \eta_b(\quad), \quad \delta_a(\quad), \quad \delta_b(\quad) \) (See [1], chapter 4), then (except for the + signs as indicated) \( f \) has a representation of the form

(2.21) \[ f(s) = \frac{1}{K(a,b)} \left\{ \sum_{a} \frac{\delta_a(i_1, i_{m-1})}{\eta_a} \left( \sum_{b} \frac{\eta_b(j_1, j_{n-m})}{\eta_b} \right) \varphi_k(s) \right\} + \sum_{c} \frac{\delta_b(j_1, j_{n-m-1})}{\eta_b} \left( \sum_{d} \frac{\eta_a(i_1, i_m)}{\eta_a} \right) \varphi_k(s) \]

where:

(1) \[ K(a,b) = \frac{\det A(a,b)}{\eta_a \eta_b} = 1 + \sum_{+} \frac{\eta_a(i_1, i_m)}{\eta_a} \frac{\eta_b(j_1, j_{n-m})}{\eta_b} \]
where \( \Sigma \) indicates summation over all possible disjoint sequences \((i_1, i_m)\) and \((j_1, j_{n-m})\) such that \((i_1, i_m) \neq (1, m)\);

(ii) \( \Sigma \) indicates summation over all possible sequences \((i_1, i_{m-1})\);

(iii) \( \Sigma \) indicates summation over all \(k\), \(k = 1, 2, \ldots, n\) and all possible sequences \((j_1, j_{n-m})\) such that under the summation \(\Sigma\), \(k \neq i_r \neq j_s\) for \(r = 1, 2, \ldots, m-1\), \(s = 1, 2, \ldots, n-m\);

(iv) \( \Sigma \) indicates summation over all possible sequences \((j_1, j_{n-m-1})\);

(v) \( \Sigma \) indicates summation over all \(k\), \(k = 1, 2, \ldots, n\) and all possible sequences \((i_1, i_m)\) such that under the summation \(\Sigma\), \(k \neq i_r \neq j_s\) for \(r = 1, 2, \ldots, m\), \(s = 1, 2, \ldots, n-m-1\).

The solution \(f\) of (2.20) as given by (2.19) or (2.21) is unique. In fact, if \(g\) is any solution of (2.20), the function \(h = g - f\) satisfies \(L_0 h = 0\), \(U_1 h = 0\), \(i = 1, 2, \ldots, m\), \(U_b h = 0\), \(i = 1, 2, \ldots, n-m\). Since \(\lambda_0\) is non-real and hence not an eigenvalue of (1.3), it follows that \(h\) must be the zero function or \(g = f\). Using (2.14) and (2.15) we can find numbers \(a_0, b_0\) (which we may suppose coincide with the original choices of \(a_0, b_0\)) such that \(K(a, b)\) is bounded away from zero on \(w_- < a < a_0\), \(b_0 < b < w_+\). Now using (2.4) and (2.5) in (2.21) and the fact that \(K(a, b)\) is bounded away from zero, we can find a constant \(C\) such that for \(w_- < a < a_0\), \(b_0 < b < w_+\),
\[
\|f\|_a^b \leq C \left\{ \sum_{(1)} \left| \Phi_a^{i}(i_1, i_{m-1}) \frac{\delta_a(i_1, i_{m-1})}{\delta_b(j_1, j_{n-m})} \right| \\
+ \sum_{(2)} \left| \Phi_b^{j}(j_1, j_{n-m}) \frac{\delta_a(i_1, i_{m-1})}{\delta_b(j_1, j_{n-m})} \right| \\
+ \sum_{(3)} \left| \Phi_b^{j}(j_1, j_{n-m-1}) \frac{\delta_a(i_1, i_m)}{\delta_b(j_1, j_{n-m-1})} \right| \\
+ \sum_{(4)} \left| \Phi_a^{i}(i_1, i_m) \frac{\delta_b(j_1, j_{n-m-1})}{\delta_b(j_1, j_{n-m})} \right| \right\}
\]

where:

(i) \( \sum \) indicates summation over all \( i, i = 1,2,\ldots,m \) and all possible sequences \((i_1, i_{m-1})\) and \((j_1, j_{n-m})\) such that \( i_r \neq j_s \neq i \) for \( r = 1,2,\ldots,m-1, s = 1,2,\ldots,n-m \);

(ii) \( \sum \) indicates summation over all \( j, j = m+1,\ldots,n \) and all possible sequences \((i_1, i_{m-1})\) and \((j_1, j_{n-m})\) such that \( i_r \neq j_s \neq j \) for \( r = 1,2,\ldots,m-1, s = 1,2,\ldots,n-m \);

(iii) \( \sum \) indicates summation over all \( j, j = m+1,\ldots,n \) and all possible sequences \((i_1, i_m)\) and \((j_1, j_{n-m-1})\) such that \( i_r \neq j_s \neq j \) for \( r = 1,2,\ldots,m, s = 1,2,\ldots,n-m-1 \);

(iv) \( \sum \) indicates summation over all \( i, i = 1,2,\ldots,m \) and all possible sequences \((i_1, i_m)\) and \((j_1, j_{n-m-1})\) such that \( i_r \neq j_s \neq i \) for \( r = 1,2,\ldots,m, s = 1,2,\ldots,n-m-1 \).

Clearly from (2.11) and (2.15) each term in the sum \( \sum \) of (2.22) will be such that

\[
\Phi_a^{i}(i_1, i_{m-1}) \frac{\delta_a(i_1, i_{m-1})}{\delta_b(j_1, j_{n-m})} = O[\Phi_a^{i}(2,m)]
\]
as \( a,b \to w_-,w_+ \) except when \( i=1, \ (i_1,i_{m-1})=(2,m) \) and 
\( (j_1,j_{n-m})=(m+1,n) \) in which case the term is exactly \( \Theta^1_a(2,m) \).
Hence we have

\[
(2.23) \quad \sum_{(1)} \left| \Theta^1_a(i_1,i_{m-1}) \frac{n_a(j_1,j_{n-m})}{n_b} \right| = o[\Theta^1_a(2,m)]
\]
as \( a,b \to w_-,w_+ \).

We now show that the second sum, \( \Sigma \), in (2.22) satisfies

\[
(2.24) \quad \sum_{(2)} \left| \Theta^j_b(j_1,j_{n-m}) \frac{\delta_a(i_1,i_{m-1})}{n_a} \right| = o[\Theta^1_a(2,m)]
\]
as \( a,b \to w_-,w_+ \). For a given sequence \( (i_1,i_{m-1}) \) select an integer
\( i, \ i \leq i \leq m \), such that \( i \neq i_1, \ldots, i_{m-1} \). Then by (2.4) and (2.11)

\[
\frac{\delta_a(i_1,i_{m-1})}{n_a} \| \varphi^j_a \| = \Theta^1_a(i_1,i_{m-1}) = o[\Theta^1_a(2,m)]
\]
as \( a \to w_- \). Since \( \varphi^j_a \notin H(w_-,c) \), it follows that

\[
\frac{\delta_a(i_1,i_{m-1})}{n_a} = o[\Theta^1_a(2,m)]
\]
as \( a \to w_- \). Using this result and (2.17), (2.24) now follows.

The same reasoning used in establishing (2.23) and (2.24) can be used to obtain

\[
(2.25) \quad \sum_{(3)} \left| \Theta^j_b(j_1,j_{n-m-1}) \frac{n_a(i_1,i_m)}{n_a} \right| = o[\Theta^1_b(m+1,n-1)]
\]
and

\[
(2.26) \quad \sum_{(4)} \left| \Theta^1_a(i_1,i_m) \frac{\delta_b(j_1,j_{n-m-1})}{n_b} \right| = o[\Theta^1_b(m+1,n-1)]
\]
as \( a,b \to w_-,w_+ \).
We now apply (2.6), (2.23)-(2.26) to (2.22) to obtain the estimate

\[(2.27) \quad \|r\|_a^b \leq C(\Psi_a + \Phi_b)\]

for \(w \leq a \leq a_0, b_0 \leq b < w_+\). It follows from (2.19) and (2.27) that for any eigenfunction \(x\) associated with the eigenvalue \(\lambda\),

\[(2.28) \quad \|x - \gamma G_{ab}x\|_a^b \leq C(\Psi_a + \Phi_b)\|x\|_a^b\]

whenever \([a,b] \in R\). 

Let \(\mu^j = \mu^{j}_{ab}\) denote the \(j\)-th eigenvalue (counting multiplicities) of the regular problem (1.3), \(|\mu^j| \leq |\mu^2| \leq \ldots\) and let \(y_j\) denote the corresponding eigenfunction, chosen so that \(\{y_j\}\) is an orthonormal basis in \(H[a,b]\).

Lemma 1. Let \(P(\delta)\) be the projection mapping from the Hilbert space \(H[a,b]\) onto its subspace \(H_\delta[a,b]\) (\(\delta > 0\)) spanned by all the eigenfunctions \(y_j\) of (1.3) such that the corresponding eigenvalues \(\mu^j\) satisfy \(|\mu^j - \lambda| \leq \delta\). Then for any \(w \in H[a,b]\),

\[\|w - P(\delta)w\|_a^b \leq (1 + \frac{1}{\delta})\|w - \gamma G_{ab}w\|_a^b.\]

The proof is given in ([11], p. 1554).

From (2.28) and lemma 1, we see that there exists a constant \(C\) on \(R\) such that

\[(2.29) \quad \|x - P(\delta)x\|_a^b \leq \frac{C}{2\delta}(\Psi_a + \Phi_b)\|x\|_a^b.\]

With the choice \(\delta = \delta_{ab} = C(\Psi_a + \Phi_b)\) we obtain

\[(2.30) \quad \|x - P(\delta)x\|_a^b \leq \frac{1}{2}\|x\|_a^b.\]
and hence \( P(\delta)x = 0 \) implies \( x = 0 \) on \([a,b]\). Since the eigenspace \( A_\lambda \) corresponding to the eigenvalue \( \lambda \) of the basic problem has dimension \( k \), it follows immediately that \( \dim P(\delta)A_\lambda \geq k \). This implies that there exist at least \( k \) eigenvalues \( \mu_{ab}^j \) (counting multiplicities) of the perturbed problem (1.3) such that \( |\mu_{ab}^j - \lambda| \leq \delta_{ab} \) holds whenever \([a,b] \in R_0\). This proves the theorem.

Theorem 1 shows, in particular, that if \( \lambda \) is an eigenvalue of the basic problem of multiplicity \( k \), then (using (2.11) and (2.12)) there are at least \( k \) eigenvalues \( \mu_{ab}^j \) of the perturbed problem such that \( \mu_{ab}^j - \lambda \) as \( a,b \to a_0, b_0 \). In addition, the estimate of the theorem is valid uniformly on \( w_- < a \leq a_0, b_0 \leq b < w_+ \).

The question that comes up next is under what conditions does theorem 1 yield exactly \( k \) perturbed eigenvalues \( \mu_{ab}^j \) near the basic eigenvalue \( \lambda \) when \( a \) is near \( w_- \) and \( b \) near \( w_+ \)? This result is obtained if we require that the absolute value of the \( i \)-th eigenvalue of problem (1.2) is not larger than the absolute value of the \( i \)-th eigenvalue of problem (1.3). This property will be referred to as the monotonicity property.

Theorem 2. If in addition to the hypotheses of theorem 1, the monotonicity property holds, then for the basic eigenvalue \( \lambda \) of multiplicity \( k \), there exists a rectangle \( R_0 \) and a constant \( C \) on \( R_0 \) such that exactly \( k \) perturbed eigenvalues \( \mu_{ab}^j \) satisfy
whenever \([a, b] \in \mathbb{R}_+\).

\[ (2.31) \quad |\mu_{ab}^j - \lambda| \leq C(\Theta_a + \Theta_b) \]

**Proof.** Suppose first that \(\lambda\) is the smallest non-negative basic eigenvalue and has multiplicity \(k\). Then since \(\delta_{ab} \to 0\) as \(a, b, \omega_-\), \(\omega_+\) by (2.11), (2.12) and theorem 1, we can find points \(a_0, b_0\) such that \(\delta_{ab}\) is less than the minimum of all the differences \(|\lambda^i - \lambda^j|\), \(\lambda^i, \lambda^j\) distinct \((i, j = 1, 2, \ldots)\), whenever \(\omega_- < a \leq a_0, b_0 \leq b < \omega_+\). By theorem 1 and the monotonicity property of (1.2), at least \(k\) perturbed eigenvalues \(\mu_{ab}^j\) lie in the interval \([\lambda, \lambda + \delta_{ab}]\). Since \(\mu_{ab}^j \geq \lambda^j\) for each \(j\) (including multiplicities of both the \(\mu_{ab}^j\) and \(\lambda^j\)) there are at most \(k\) eigenvalues \(\mu_{ab}^j\) on \([\lambda, \lambda + \delta_{ab}]\) and hence exactly \(k\). A similar statement applies to the case that \(\lambda\) is negative.

Let \(k_j\) denote the multiplicity of the \(j\)-th distinct basic eigenvalue \(\lambda^j\). In order to prove by induction that there are exactly \(k_j\) perturbed eigenvalues \(\mu_{ab}^j\) which satisfy

\[ |\mu_{ab} - \lambda^j| \leq \delta_{ab} \quad (j = 1, 2, \ldots) \]

assume that this is true for each integer \(j \leq n\). In the case that \(|\lambda^{n+1}| < |\lambda^{n+2}|\) there are at most \(\sum_{j=1}^{n+1} k_j\) eigenvalues \(\mu_{ab}^j\) which satisfy

\[ |\mu_{ab}| \leq |\lambda^{n+1}| + \delta_{ab}. \]

It then follows from the induction assumption that there are at most \(k_{n+1}\) eigenvalues \(\mu_{ab}\) satisfying

\[ |\mu_{ab} - \lambda^{n+1}| \leq \delta_{ab}. \]
and hence exactly $k_{n+1}$ by theorem 1. In the other case

$$\lambda^{n+2} = -\lambda^{n+1},$$

there are $k_{n+1} + k_{n+2}$ eigenvalues $\mu_{ab}$ satisfying

$$|\lambda^{n+1}| \leq |\mu_{ab}| \leq |\lambda^{n+1}| + \delta_{ab}$$

and again by theorem 1 there are exactly $k_{n+1}$ eigenvalues $\mu_{ab}$ near $\lambda^{n+1}$ and $k_{n+2}$ eigenvalues $\mu_{ab}$ near $\lambda^{n+2}$. This completes the proof of theorem 2.

**Theorem 3.** Let the hypotheses of theorem 2 be satisfied.

Then corresponding to the eigenvalues $\lambda$ and $\mu_{ab}^j$ of theorem 2, there are orthonormal eigenfunctions $x^j$ on $[a,b]$ associated with $\lambda$ and $y^j$ associated with the $\mu_{ab}^j$ such that

$$\|y_{ab}^j - x^j\|_a^b \leq C(\varphi_a + \varphi_b), \quad j = 1, 2, \ldots, k,$$

whenever $[a,b] \in R_o$.

**Proof:** Let $\{y^j\}$ be a set of orthonormal eigenfunctions on $[a,b]$ corresponding to the set of eigenvalues $\{\mu_{ab}^j\}$ in theorem 2. Then $H_0[a,b]$ is $k$-dimensional by theorem 2 and $P(\delta)x = 0$ implies $x = 0$ by (2.30). Hence there exists $k$ unique linearly independent eigenfunctions $z^j$ corresponding to $\lambda$ which $P(\delta)$ maps into the orthonormal eigenfunctions $y^j$ and by (2.29)

$$\|z^j - y^j\|_a^b = O(\varphi_a + \varphi_b), \quad [a,b] \in R_o .$$

Since

$$|(z^i, z^j)_a^b - (y^i, y^j)_a^b| \leq \|y^i\|_a^b \|z^j - y^j\|_a^b$$

$$+ \|z^j\|_a^b \|z^i - y^i\|_a^b$$
by the Schwarz inequality,

\[(z^i, z^j)^b_a = \delta_{ij} \circ (\varphi_a + \varphi_b)\]

for \(i, j = 1, 2, \ldots, k\) where \(\delta_{ij}\) denotes the Kronecker delta.

Since the \(z^j\) are linearly independent, an orthonormal sequence \(x^j\) can be constructed by the Schmidt process as linear combinations of the \(z^j\) and it is easily verified that

\[\|x^j - z^j\|^b_a = 0(\varphi_a + \varphi_b).\]

Hence

\[\|x^j - y^j\|^b_a = 0(\varphi_a + \varphi_b), \ j = 1, 2, \ldots, k.\]

3. **Uniform estimates for eigenfunctions on \([a, b]\).**

The purpose of this section is to obtain uniform estimates on \([a, b]\) for the eigenfunctions of theorem 3 when \(\omega_+\) and \(\omega_-\) are singularities for \(L\) as described in section 1. For the second order case when \(\omega_-, \omega_+\) are both limit point singularities (according to Weyl's classification ([3], p. 225)) such uniform estimates are given in [13]. The central conditions used are such that the positive function \(g_{ab}(s)\) defined by

\[(3.1) \quad g_{ab}^2(s) = \int^b_a |G_{ab}(s, t)|^2 k(t) dt\]

is uniformly bounded on \(a \leq s \leq b\) provided \(\omega_- < a < a_0,\)
\(b_0 < b < \omega_+\).

For the \(n\)-th order case we proceed in a similar manner.
First sufficient conditions will be stated for the uniform boundedness of (3.1) on \( a \leq s \leq b \) when \( a \leq a_o, b_o \leq b \). These will be applied to a suitable form of the Green's function, \( G_{ab}(s,t) \), which will also be developed in this section. The boundedness of (3.1) will be obtained as a result and finally the desired uniform estimates.

Let \( W(s) \) be the "Wronskian" determinant, i.e., \( W(s) = \det (\dot{\varphi}^1_j(s)), i, j = 1, 2, \ldots, n \) and let \( C_i(s) \) denote the (signed) cofactor of \( \varphi_i^{(n-1)}(s) \) in \( W(s) \). Let

\[
(3.2) \quad \sigma^*_a(i,j) = \frac{C_i(a)}{C_j(a)} \quad \text{and} \quad \sigma^*_b(i,j) = \frac{C_j(b)}{C_i(b)}.
\]

For \( (i_1, i_{m-1}) \) any sequence and \( p \) any integer we let \( \eta^i_a(i_1, i_{m-1}, p) \) denote the determinant of the matrix which has \( U_1^i \varphi_{i_k} \) in the \( i \)-th row and \( k \)-th column, \( i = 1, 2, \ldots, m, k = 1, 2, \ldots, m-1 \) and \( U_p^i \varphi_{i_p} \) in the \( i \)-th row and \( m \)-th column. Similarly for \( (j_1, j_{n-m-1}) \) any sequence we let \( \eta^i_b(j_1, j_{n-m-1}, q) \) denote the determinant of the matrix which has \( U_q^i \varphi_{j_k} \) in the \( i \)-th row and \( k \)-th column, \( i = 1, 2, \ldots, n-m, k = 1, 2, \ldots, n-m-1 \) and \( U_{n-m}^i \varphi_{j_{n-m}} \) in the \( i \)-th row and \( n-m \)-th column.

**Assumptions:** In addition to (2.7) - (2.12) we make the following assumptions:

(i) \( \omega_-, \omega_+ \) are not accumulation points of the zeros for \( C_i(s) \), \( i = 1, 2, \ldots, n \), and for \( i = 1, 2, \ldots, m, j = m+1, \ldots, n \)

\[
(3.3) \quad h(a)\left|\sigma_a(i,j)\right|, h(a)\left|\sigma^*_a(i,j)\right| \text{ are non-decreasing on } \omega_- < a \leq a_o, \text{ and}
\]
(3.4) \( h(b)|\sigma_b(i,j)|, h(b)|\sigma^*_b(i,j)| \) are non-increasing on \\[ b_0 \leq b < w_+ \].

(ii) The following functions are bounded on \( w_- < s < w_+ \):

(3.5) \[ h(s) \varphi_i(s) \left\| \frac{h(t) C_j(t)}{p(t) W(t)} \right\|_s, \ i, j = 1, 2, \ldots, m; \]

\[ h(s) \varphi_i(s) \left\| \frac{h(t) C_j(t)}{p(t) W(t)} \right\|_s, \ i, j = m+1, \ldots, n. \]

(iii) There exist integers \( i'_0, j'_0 \) where \( 1 \leq i'_0 \leq m, m+1 \leq j'_0 \leq n \) such that

(3.6) \[ \frac{h(a) C_{j'_0}(a)}{h(a)} \frac{h(a)}{C_{i'_0}(a)} \]

is bounded on \( w_- < a < a_0 \) for \( j = m+1, \ldots, n \) and all sequences \((i'_1, i'_m)\), and

(3.7) \[ \frac{h(b) C_{j'_0}(b)}{h(b)} \frac{h(b)}{C_{j'_0}(b)} \]

is bounded on \( b_0 \leq b < w_+ \) for \( i = 1, 2, \ldots, m \) and all sequences \((j'_1, j'_n)\).

Since there exists a positive constant \( c \) such that \( h(s) \geq c, w_- < s < w_+ \), we have the following obvious inequalities on \( w_- < s < w_+ \):

(3.8) \[ |\varphi_i(s)| \leq C h(s)|\varphi_i(s)|, \ i = 1, \ldots, n; \]
We now prove some fundamental inequalities which will be needed to obtain the boundedness of (3.1).

Lemma 2. For $k = 1, 2, \ldots, n$ let

$$\hat{\varphi}_k(s) = \left\{ \begin{array}{ll} \varphi_k(s) & \text{if } w_0 < s < a_0, b_0 < s < w_1 \\
1 & \text{if } a_0 < s < b_0, \end{array} \right. $$

$$\hat{C}_k(s) = \left\{ \begin{array}{ll} C_k(s) & \text{if } w_0 < s < a_0, b_0 < s < w_1 \\
1 & \text{if } a_0 < s < b_0. \end{array} \right. $$

Then there exists a constant $C$, independent of $a, b$ as well as $s$, such that

$$\left| \frac{h_a(i, l, i, m)}{h_a} \varphi_l(s) \right| \leq C \cdot \hat{\varphi}_0(s) h(s)$$

on $a < s < b$, $a < a_0$ for all integers $i, 1 < i < m$ and all sequences $(i, l, i, m)$ such that $i \neq i_k$, $k = 1, 2, \ldots, m$;

$$\left| \frac{h_b(j, l, j, n-m)}{h_b} \varphi_j(s) \right| \leq C \cdot \hat{\varphi}_0(s) h(s)$$

on $a < s < b$, $b_0 < b$ for all integers $j$, $m+1 < j < n$ and all sequences $(j, l, j, n-m)$ such that $j \neq j_k$, $k = 1, 2, \ldots, n-m$.
on $a < s < b$ for all integers $j$, $m+1 < j < n$ and all sequences $(i_1, i_{m-1})$;

(3.14) \[
\left| \frac{n_b(j_1, j_{n-m-1}, i)}{\eta_a} \right| c_1(s) \leq C \cdot \hat{C}_j(s) h(s)
\]
on $a < s < b$, $b < b_0$ for all integers $i$, $1 < i < m$ and all sequences $(j_1, j_{n-m-1})$.

**Proof.** We first note that by (3.3) and (3.4)

(3.15)

\[
|\sigma_a(i,j)| h(a) \leq |\sigma_s(i,j)| h(s), \quad |\sigma^*(i,j)| h(a) \leq |\sigma^*_s(i,j)| h(s)
\]
when $w_0 < a < s < a_0$, and

\[
|\sigma_b(i,j)| h(b) \leq |\sigma_s(i,j)| h(s), \quad |\sigma^*_b(i,j)| h(b) \leq |\sigma^*_s(i,j)| h(s)
\]
when $b_0 < s < b < w_+$ for $i = 1, 2, \ldots, m$, $j = m+1, \ldots, n$.

From the fact that $h(s) \geq c$ for some positive constant $c$, inequalities (3.11)-(3.14) are clearly valid on $[a_0, b_0]$. To prove (3.11) on $w_0 < a < s < a_0$ we have by (2.18) and (3.15) that

\[
\left| \frac{n_a(i_1, i_{m-1})}{\eta_a} \varphi_1(s) \right| \leq C|\sigma_a(i,j_0)| \varphi_1(s)|h(a)
\]

\[
\leq C|\sigma_s(i,j_0)| \varphi_1(s)|h(s)
\]

\[
= C|\varphi_{j_0}(s)| h(s).
\]

Thus (3.11) holds on $w_0 < a < s < a_0$ as well. Finally (2.13) and (3.4) yield $|\sigma_s(i,j_0)| h(s) \leq C$ on $b_0 < s < w_+$ hence (3.11) is valid on $[b_0, w_+)$. Thus (3.11) is valid on the whole interval $a < s < w_+$, $a < a_0$. The proof of (3.12) is completed in the same way and will be omitted.
To prove (3.13) on $w_- < a \leq s \leq a_0$ we deduce from (3.6) that

$$\left| \frac{n_{a}(i,j)}{n_{a}} \right| \leq C|\sigma_{a}^{*}(i,j)|h(a)$$

and hence by (3.15) that

$$\left| \frac{n_{a}(i,j)}{n_{a}} \right| c_{j}(s) \leq C|\sigma_{s}^{*}(i,j)|c_{j}(s)|h(s)| = C|c_{1_{o}}(s)|h(s).$$

Hence (3.13) is valid on $w_- < a \leq s \leq a_0$. Finally (3.13) holds for $b_0 \leq s < w_+$ as well since $|\sigma_{s}^{*}(i,j)| < C$ on $b_0 \leq s < w_+$ by (3.4).

Thus (3.13) is valid on the whole interval, $a \leq s < w_+$, $a \leq a_0$. The proof of (3.14) is completed in the same way and will be omitted.

We now construct the Green's function for $k \cdot L_0$ associated with the boundary conditions of (0.6). The method used will be the one given in [3], page 192. Let $\Omega(a,b)$ denote the determinant of $A(a,b)$ and $\Omega_{a}^{j}(a,b)$ the determinant of the matrix obtained from $A(a,b)$ by replacing the $j$-th column by

$$\sum_{i=1}^{n} \frac{C_{i}(t) U_{a_{i}}}{{p}_{o}(t) W(t)}, \ldots, \sum_{i=1}^{n} \frac{C_{i}(t) U_{a_{i}^{m}}}{p_{o}(t) W(t)}, 0, 0, \ldots, 0.$$ (n-m zeros)

Similarly let $\Omega_{b}^{j}(a,b)$ be the determinant of the matrix obtained from $A(a,b)$ by replacing the $j$-th column by

$$\overbrace{0,0,\ldots,0}^{(m \text{ zeros})}, \sum_{i=1}^{n} \frac{C_{i}(t) U_{b_{i}}}{{p}_{o}(t) W(t)}, \ldots, \sum_{i=1}^{n} \frac{C_{i}(t) U_{b_{i}^{n-m}}}{p_{o}(t) W(t)}.$$
We set
\[
K_{ab}(s,t) = \begin{cases} 
\frac{1}{p_0(t) W(t)} \sum_{i=1}^{n} \varphi_i(s) C_i(t), & a \leq t \leq s \leq b \\
0 & , a \leq s < t \leq b 
\end{cases}
\]
and determine constants \( A_j, j = 1,2,\ldots,n \) such that the function

(3.16) \[ G_{ab}(s,t) = K_{ab}(s,t) + \sum_{j=1}^{n} A_j \varphi_j(s) \]
satisfies the boundary conditions of (0.6) (as a function of \( s \)). The resulting function \( G_{ab}(s,t) \) is then the required Green's function.

**Lemma 3.** The Green's function \( G_{ab}(s,t) \) for \( k \cdot L_0 \) associated with the boundary conditions of (0.6) can be expressed in the form:

(3.17) \[ G_{ab}(s,t) = \begin{cases} 
\frac{1}{\Omega(a,b)} \sum_{j=1}^{n} \Omega_j^{a}(a,b) \varphi_j(s), & a \leq t \leq s \leq b \\
-\frac{1}{\Omega(a,b)} \sum_{j=1}^{n} \Omega_j^{b}(a,b) \varphi_j(s), & a \leq s < t \leq b 
\end{cases} \]

**Proof.** Applying the boundary conditions of (0.6) to (3.16), we obtain the following non-homogeneous system of \( n \) linear equations:

\[
\begin{cases} 
\sum_{j=1}^{n} A_j U^i_a \varphi_j = 0, & i = 1,2,\ldots,m \\
\sum_{j=1}^{n} A_j U^i_b \varphi_j = -\sum_{k=1}^{n} C_k(t) U^i_b \varphi_k & , i = 1,2,\ldots,n-m.
\end{cases}
\]

Then Cramer's rule yields

\[ A_j = -\frac{\Omega_j^{a}(a,b)}{\Omega(a,b)}, & j = 1,2,\ldots,n. \]
Thus for $s < t$ (for which $K_{ab}(s,t) = 0$) it follows immediately that

\[(3.18) \quad G_{ab}(s,t) = -\frac{1}{\Omega(a,b)} \sum_{j=1}^{n} \Omega_{b}^{j}(a,b) \varphi_{j}(s).\]

For $t \leq s$,

\[G_{ab}(s,t) = K_{ab}(s,t) - \frac{1}{\Omega(a,b)} \sum_{j=1}^{n} \Omega_{b}^{j}(a,b) \varphi_{j}(s).\]

Subtracting and adding

\[\frac{1}{\Omega(a,b)} \sum_{j=1}^{n} \Omega_{a}^{j}(a,b) \varphi_{j}(s)\]

to the right side we obtain

\[(3.19) \quad G_{ab}(s,t) = K_{ab}(s,t) - \frac{1}{\Omega(a,b)} \sum_{j=1}^{n} (\Omega_{b}^{j}(a,b) + \Omega_{a}^{j}(a,b)) \varphi_{j}(s)\]

\[+ \frac{1}{\Omega(a,b)} \sum_{j=1}^{n} \Omega_{a}^{j}(a,b) \varphi_{j}(s).\]

Consider the second term on the right of (3.19);

By elementary operations on determinants one may show that

\[\frac{1}{\Omega(a,b)} \left( \Omega_{b}^{j}(a,b) + \Omega_{a}^{j}(a,b) \right) \varphi_{j}(s) = \frac{C_{j}(t) \varphi_{j}(s)}{P_{0}(t) W(t)} .\]

Hence summing over $j$ (as in (3.19)) we obtain that

\[\frac{1}{\Omega(a,b)} \sum_{j=1}^{n} \left( \Omega_{b}^{j}(a,b) + \Omega_{a}^{j}(a,b) \right) \varphi_{j}(s) = \sum_{j=1}^{n} \frac{C_{j}(t) \varphi_{j}(s)}{P_{0}(t) W(t)}\]

\[= \frac{1}{p_{0}(t) W(t)} \sum_{j=1}^{n} C_{j}(t) \varphi_{j}(s) = K_{ab}(s,t).\]
Using this result in (3.19), it is clear that

\[ G_{ab}(s,t) = \frac{1}{\Omega(a,b)} \sum_{j=1}^{n} \Omega_a^j(a,b) \varphi_j(s), \quad t \leq s. \]

This and (3.18) prove the lemma.

If the determinants in (3.17) are expanded by complementary minors (see [1], chapter 4) using the \( n_a() \), \( n_b() \) notation then it can be shown that \( G_{ab}(s,t) \) has the following form:

For \( a \leq t < s \leq b \),

\[ (3.20) \quad G_{ab}(s,t) = \]

\[ \frac{1}{K(a,b)} \sum_{(i_1, i_{m-1})} \left( \sum_{q} (\pm) \frac{h_a(i_1, i_{m-1}, q)}{n_a p_o(t) w(t)} \right) \left( \sum_{*} (\pm) \frac{h_b(j_1, j_{n-m})}{h_b} \varphi_p(s) \right) \]

where:

(i) \( \sum \) indicates summation over all possible sequences \( (i_1, i_{m-1}) \);

(ii) \( \sum \) indicates summation over all \( q \), \( q = 1, 2, \ldots, n \) such that \( q \neq i_k \), \( k = 1, 2, \ldots, m-1 \);

(iii) \( \sum \) indicates summation over all \( p \), \( p = 1, 2, \ldots, n \) and all possible sequences \( (j_1, j_{n-m}) \) such that \( p \neq j_r \neq i_k \) for \( r = 1, 2, \ldots, n-m \), \( k = 1, 2, \ldots, m-1 \);

(iv) The signs in front of \( n_a() \) and \( n_b() \) may be determined by the relative position of the corresponding minors in (3.17) and also by the arrangement of the columns in
these minors. For our purposes it is unimportant to know which sign to use (as will be seen later).

For \( a \leq s < t \leq b \),

\[
(3.21) \quad G_{ab}(s,t) = \frac{1}{K(a,b)} \sum_{(j_1, j_{n-m-1})} \left( \sum_q (\pm) \frac{n_b(j_1 j_{n-m-1}, q) G_q(t)}{n_b p_0(t) W(t)} \right) \left[ \sum_q (\pm) \frac{n_a(i_1 i_m)}{n_a} \varphi_p(s) \right]
\]

where:

(i) \( \sum \) indicates summation of all possible sequences \((j_1, j_{n-m-1}) \); \( (j_1, j_{n-m-1}) \);

(ii) \( \sum \) indicates summation over all \( q \), \( q = 1, 2, \ldots, n \) such that \( q \neq j_k \), \( k = 1, 2, \ldots, n-m-1 \);

(iii) \( \sum \) indicates summation over all \( p \), \( p = 1, 2, \ldots, n \) and all possible sequences \((i_1, i_m) \) such that \( p \neq i_r \neq j_k \) for \( r = 1, 2, \ldots, m \), \( k = 1, 2, \ldots, n-m-1 \);

(iv) The signs in front of \( n_a(\quad) \) and \( n_b(\quad) \) may be determined as in (3.20) but are unimportant for our purposes.

**Proof of 3.20:** For any sequence \((i_1, i_{m-1}) \) let \( n_a^*(i_1, i_{m-1}) \) denote the following determinant:
Then if each determinant \( \Omega_j^j(a,b) \), \( j = 1, 2, \ldots, n \) is expanded by complementary minors it is clear that the sum

\[
\sum_{j=1}^{n} \Omega_j^j(a,b) \varphi_j(s)
\]

will consist only of terms of the form \( \eta_a^*(a,b) \eta_b(\cdot) \varphi_k(s) \).

Let \( (i_1, i_{m-1}) \) be any fixed sequence. Then except for interchange of columns, the cofactor \( \eta_a^*(i_1, i_{m-1}) \) will occur in exactly \( n - m + 1 \) determinants \( \Omega_j^j(a,b) \). In fact the cofactor \( \eta_a^*(i_1, i_{m-1}) \) will occur in exactly those determinants \( \Omega_j^j(a,b) \) for which \( j \neq i_k \), \( k = 1, 2, \ldots, m-1 \). Factoring this cofactor \( \eta_a^*(i_1, i_{m-1}) \) out of the determinants \( \Omega_j^j(a,b) \), \( j \neq i_k \), \( k = 1, 2, \ldots, m-1 \), in the sum (3.22) we obtain the product

\[
\left\{ \sum_{*} (\pm) \eta_b(j_1, j_{n-m}) \varphi_p(s) \right\} \eta_a^*(i_1, i_{m-1})
\]

where the summation \( \sum_{*} \) is described by (iii) under (3.20).

Note that for each \( p \neq i_k \) there is exactly one sequence \( (j_1, j_{n-m}) \)
satisfying these conditions.

We now expand the cofactor \( h_a^*(i_1, i_{m-1}) \) to obtain

\[
(3.24) \quad h_a^*(i_1, i_{m-1}) = \sum_q (\pm) h_a(i_1, i_{m-1}, q) \frac{C_q(t)}{p_0(t) w(t)}
\]

where the summation \( \sum_q \) is described by (ii) under (3.20). From (3.23) and (3.24) it now follows that the term in (3.22) corresponding to the cofactor \( h_a^*(i_1, i_{m-1}) \) is

\[
\left( \sum_q (\pm) h_a(i_1, i_{m-1}, q) \frac{C_q(t)}{p_0(t) w(t)} \right) \left( \sum_\star (\pm) h_b(j_1, j_{n-m}) \varphi_p(s) \right).
\]

Since every term in the expansion of (3.22) involves a cofactor \( h_a^*(i_1, i_{m-1}) \), we obtain for \( t < s \) that

\[
(3.25) \quad G_{ab}(s,t) = \frac{1}{\Omega(a,b)} \sum_{(i_1, i_{m-1})} \left( \sum_q (\pm) h_a(i_1, i_{m-1}, q) \frac{C_q(t)}{p_0(t) w(t)} \right) \left( \sum_\star (\pm) h_b(j_1, j_{n-m}) \varphi_p(s) \right)
\]

where the summation \( \sum_{(i_1, i_{m-1})} \) is described by (i) under (3.20).

To complete the proof of (3.20) we divide numerator and denominator of the right side of (3.25) by \( h_a \cdot h_b \). A similar proof holds for (3.21) and will be omitted.

**Lemma 4.** Assume that conditions (2.7) - (2.10) and (3.3) - (3.7) are satisfied. Then the positive function \( g_{ab}(s) \) defined by
is uniformly bounded on \(a \leq s \leq b\) provided \(a \leq a_o, b_o \leq b\).

**Proof.** We first express \(g_{ab}(s)\) in the form

\[
g_{ab}(s) = \left( \int_a^b |G_{ab}(s,t)|^2 k(t) dt \right)^{1/2}.
\]

from which follows that

\[
(3.26) \quad g_{ab}(s) \leq \left( \int_a^b |G_{ab}(s,t)|^2 k(t) dt \right)^{1/2} + \left( \int_s^b |G_{ab}(s,t)|^2 k(t) dt \right)^{1/2}.
\]

Using (3.20) and the Minkowski inequality we obtain that

\[
(3.27) \quad \left( \int_a^b |G_{ab}(s,t)|^2 k(t) dt \right)^{1/2} \leq
\]

\[
\frac{1}{|K(a,b)|} \sum_{(i_1, i_{m-1})} \left( \sum_{p} \frac{n_b(j_1, j_{n-m})}{n_b} \varphi_p(s) \right) \left( \sum_{q} \frac{n_a(i_1, i_{m-1}, j_q) C_q(t)}{p_0(t) W(t)} \right)
\]

By (3.12) each term in the sum \(\Sigma\) involving \(\varphi_p(s), p > m\), can be replaced by a term involving \(\hat{\varphi}_{i_1}(s) h(s)\) and by (3.13) each term in \(\Sigma\) involving \(C_q(t), q > m\), can be converted into a term involving \(\hat{C}_{i_1}(t) h(t)\). Having done this, we obtain, using (3.8) and (3.9) that there exists a constant \(C\) such that

\[
\left( \int_a^b |G_{ab}(s,t)|^2 k(t) dt \right)^{1/2} \leq \frac{C}{|K(a,b)|} \sum_{i,j=1}^m \left| \varphi_{ij} \right| h(s) \left\| \hat{\varphi}_{i}(s) \right\|_a \left\| \frac{\hat{C}_i(t) h(t)}{p_0(t) W(t)} \right\|_a
\]

where \(w_- < a \leq a_o, b_o \leq b < w_+\) and \(\varphi_{ij}\) are bounded functions of \(a\) and \(b\) on \(w_- < a \leq a_o, b_o \leq b < w_+\). From (3.5) and the fact that \(K(a,b)\) is bounded away from 0 on \(w_- < a \leq a_o, b_o \leq b < w_+\)
follows that
\[
\left( \int_a^b |G_{ab}(s,t)|^2 k(t) dt \right)^{1/2}
\]
is uniformly bounded on \(a \leq s \leq b, \quad a \leq a_o, \quad b_o \leq b\). The proof that the second term on the right of (3.26) is uniformly bounded on \(a \leq s \leq b, \quad a \leq a_o, \quad b_o \leq b\) is similar and will be omitted. This gives the desired result.

**Theorem 4.** Assume that \(\omega_-, \omega_+\) are singularities for \(L\) as described in section 1 and that assumptions (2.7) - (2.12), (3.3) - (3.7) are satisfied. If in addition the monotonicity property is satisfied, then the eigenfunctions \(x^j\) corresponding to \(\lambda\) and \(y^j_{ab}\) corresponding to \(\mu^j_{ab}\) in theorem 3 are such that

\[
y^j_{ab} = x^j(s) - f^j(s) + 0(\Omega_a) + 0(\Omega_b), \quad j = 1, 2, \ldots, k
\]
for \(a \leq s \leq b, \quad \omega_- < a \leq a_o, \quad b_o \leq b < \omega_+\) where \(f^j(s)\) is the unique solution of

\[
(3.29) \quad Lf = \lambda_o f, \quad U^j_a f = U^j_a x^j, \quad i = 1, 2, \ldots, m,
\]
\[
U^j_b f = U^j_b x^j, \quad i = 1, 2, \ldots, n-m.
\]

**Proof.** The Schwarz inequality for \(H[a,b]\) yields

\[
|y^j_{ab}(s) - (\lambda - \lambda_o)G_{ab}x^j(s)|
\]
\[
= \left| G_{ab}[(\mu^j_{ab} - \lambda_o)y^j_{ab}(s) - (\lambda - \lambda_o)x^j(s)] \right|
\]
\[
\leq g_{ab}(s) \left( |\mu^j_{ab} - \lambda_o| \|y^j_{ab} - x^j\|^b_a + |\mu^j_{ab} - \lambda| \|x^j\|^b_a \right)
\]
Hence lemma 4 and theorems 2 and 3 show that there exists a constant C such that

\[(3.30) \quad |y_{ab}^j(s) - (\lambda - \mu)G_{ab}x^j(s)| \leq C(\Phi_a + \Phi_b)\]
on a \leq s \leq b, whenever a \leq a_0, b_0 \leq b.

The solution \(f^j(s)\) of the boundary-value problem (3.29) is given by (2.19) or (2.21) with \(x\) replaced by \(x^j\). The function \(F^j\) defined by

\[F^j(s) = (\lambda - \mu)G_{ab}x^j(s) - x^j(s) + f^j(s)\]
satisfies

\[L(F^j) = \mu F^j, \quad \mu^i_{a} F^j = 0, \quad i = 1, 2, \ldots, m,\]
\[\mu^i_{b} F^j = 0, \quad i = 1, 2, \ldots, n-m\]

and hence \(F^j\) is the zero solution on \(a \leq s \leq b\) for \(j = 1, 2, \ldots, k\). This with (3.30) immediately gives the uniform estimates (3.28).

4. **Asymptotic variational formulae for eigenvalues.**

The purpose here is to derive formulae for the change \(\mu_{ab}^j - \lambda\) of eigenvalues under the perturbation \(D \rightarrow D[a,b]\), valid for \(a, b\) in neighbourhoods of \(w_-, w_+\) respectively. Let \(x^j, y^j\) denote the normalized eigenfunctions associated with \(\lambda\) and \(\mu_{ab}^j = \mu_{ab}^j\) as described in theorem 3 and let \(f^j\) be the unique solution of (3.29). We then have the following theorem:
Theorem 5. Under the assumptions of theorem 4 the following asymptotic variational formulae for the eigenvalues \( \lambda, \mu_{ab}^j \) are valid:

\[
(4.1) \quad \lambda - \mu_{ab}^j = \left[ f^j x^j \right](b) - \left[ f^j x^j \right](a) + (\ell_0 - \lambda)(f^j, f^j)_a^b + (\phi_a + \phi_b)(f^j, 1)_a^b + o(1)
\]

as \( a, b \to \omega_-, \omega_+ \).

Proof. Let \( U_y = 0 \) denote the self-adjoint set of boundary conditions given by (0.6) and (0.7). Then by [3], chapter 11, there exist boundary forms \( U_c, U_c^+ \) of rank \( n \) such that

\[
[u, v](b) - [u, v](a) = U_u U_c^+ v + U_c u \cdot U v
\]

for any pair \( u, v \in C^{n-1}[a, b] \), where \( \cdot \) represents the "dot" product.

Now \( U_y^j = 0 \) by (0.6) and (1.3) and \( U_x^j = U_f^j \) by (3.29), hence (dropping the superscripts \( j \))

\[
[x, y](b) - [x, y](a) = U_x U_c^+ y
\]

\[
= [f, y](b) - [f, y](a).
\]

Then application of Green's formula (0.5) to the differential equations \( Lx = \lambda x \), \( Lf = \ell_0 f \) and \( Ly = \mu y \) on \([a, b]\) leads to

\[
(4.2) \quad (\lambda - \mu)(x, y)_a^b = (\ell_0 - \mu)(f, y)_a^b;
\]

\[
(4.3) \quad [f, x](b) - [f, x](a) = (\ell_0 - \lambda)(f, x)_a^b.
\]

We obtain as a consequence of theorems 1, 2 and 3 that \( \mu = \lambda + o(1) \) and

\[
|(x, y)_a^b - (x, x)_a^b| \leq \|x\|_a^b \|y - x\|_a^b = o(1)
\]

as \( a, b \to \omega_-, \omega_+ \).
Hence
\[(x,y)^b_a = 1 + o(1), \quad a,b \to \omega_- , \omega_+\]
and (4.2) yield
\[(4.4) \quad \lambda - \mu = (\lambda_0 - \lambda)(f,y)^b_a [1 + o(1)].\]

We now appeal to the uniform estimate (3.2b) to obtain
\[(4.5) \quad (f,y)^b_a = (f,x)^b_a - (f,f)^b_a + (\Theta_a + \Theta_b)(f,1)^b_a o(1).\]

Then applying (4.3) and (4.5) to (4.4) the result (4.1) follows easily.

In conclusion we point out that in many examples conditions
(2.7) - (2.10), (3.3) - (3.7) are satisfied when \(h(s) = 1\) on
\(\omega_- < s < \omega_+\). Setting \(h(s) = 1\) we obtain the simpler conditions
((2.7) - (2.10), (3.3) - (3.7) with \(h(s) = 1\)) which are actually
sufficient for the boundedness of (3.1). Except for minor
simplifications the proof is the same as that of lemma 4. Also
in some cases, \(\lambda = 0\) is not an eigenvalue and it is permissible
to replace \(\lambda_0\) by 0. Then \(f\) can be taken as a real-valued
solution of \(L f = 0\). Finally for various problems the first two
terms on the right of (4.1) dominate the other terms and the
asymptotic relation
\[(4.6) \quad \lambda - \mu^j_{ab} \sim [f^j x^j](b) - [f^j x^j](a)\]
is valid for \(a,b \to \omega_- , \omega_+\) (See chapter III).
5. **The second order case: \( w_-, w_+ \) limit point singularities**

In this section we shall show how our theory applies to the second order differential operator \( L = L_2 \) defined by

\[
L_2 x = \frac{1}{k(s)} \left\{ - \frac{d}{ds}(p(s)\frac{dx}{ds}) + q(s)x \right\}
\]

on the open interval \( w_- < s < w_+ \) where \( k, p, q \) are real-valued functions on this interval with the properties that

(i) \( p \) is differentiable;
(ii) \( k, q \) are piecewise continuous; and
(iii) \( k, p \) are positive-valued.

The points \( w_- \) and \( w_+ \) will be limit point singularities for \( L \) according to Weyl's classification ([3], p. 225). Clearly this operator is a particular case of (0.1) with \( n = 2 \).

For this operator \( L \) we can appeal to a theorem of Weyl ([9], p. 45) to obtain the existence of basic solutions \( \varphi_1, \varphi_2 \) of \( L_0 x = 0 \) such that

\[
(5.1) \quad \varphi_1 \in H(c, w_+) \quad , \quad \varphi_2 \in H(w_-, c) \\
\varphi_1 \notin H(w_-, c) \quad , \quad \varphi_2 \notin H(c, w_+)
\]

for any \( c \) satisfying \( w_- < c < w_+ \) and such that \( [\varphi_1 \varphi_2](s) = 1 \) on \( w_- < s < w_+ \).

It will be assumed in the sequel that \( w_-, w_+ \) are not accumulation points of the zeros of \( \varphi_1 \) and \( \varphi_2 \).

Let \( D_2 \) be the set of all \( x \in H \) such that \( x \in C^1(w_-, w_+) \) and
x' is absolutely continuous on every closed bounded subinterval of \((w_-, w_+)\). Then the basic problem corresponding to (1.2) is

\[(5.2) \quad Lx = \lambda x, \quad x \in D_2.\]

Let \(D_2[a,b]\) be the set of all \(y \in H[a,b]\) such that

(i) \(y \in C^1[a,b]\) and \(y'\) is absolutely continuous on \([a,b]\);
(ii) \(Ly \in H[a,b]\); and
(iii) \(U_a y = U_b y = 0\)

where

\[U_a y = \alpha_0(a)y(a) + \alpha_1(a)y'(a)\]
\[U_b y = \beta_0(b)y(b) + \beta_1(b)y'(b)\]

with \(\alpha_0, \alpha_1\) real-valued functions not both zero for any \(a\) on \((w_-, w_+)\), and with \(\beta_0, \beta_1\) real-valued and not both zero on \([b_0, b_1]\). Then the perturbed problem corresponding to (1.3) becomes the regular self-adjoint eigenvalue problem

\[(5.3) \quad Ly = \mu y, \quad y \in D_2[a,b].\]

The problem of obtaining estimates for eigenvalues and eigenfunctions of (5.3) for \(a, b\) near \(w_-, w_+\) has already been considered by C.A. Swanson in [13]. We shall show that our assumptions used for the \(n\)-th order case specialize with little variation to his assumptions when \(n = 2\) (See [13], pp. 306-307).

For the present case exactly one basic solution \(\varphi_1 \in H(c, w_+)\) and the other basic solution \(\varphi_2 \in H(w_-, c)\),
\( w_- < c < w_+ \), hence condition (1.1) is satisfied trivially.

The notation used in section 2 now specializes to the following:

\[
\begin{align*}
\eta_a &= U_a \varphi_1; & \eta_b &= U_b \varphi_2; \\
\eta_a(1) &= U_a \varphi_1; & \eta_b(1) &= U_b \varphi_1; \\
\delta_a(1) &= U_a x; & \delta_b(1) &= U_b x.
\end{align*}
\]

Hence assumptions (2.7) - (2.12) reduce to the following:

\[
\begin{align*}
(5.4) & \quad h(a) \frac{\varphi_2(a)}{\varphi_1(a)} \|\varphi_1\|_a = o(1); \quad h(b) \frac{\varphi_1(b)}{\varphi_2(b)} \|\varphi_2\|_b = o(1); \\
(5.5) & \quad \frac{U_a x}{U_a \varphi_1} \|\varphi_1\|_a = o(1); \quad \frac{U_b x}{U_b \varphi_2} \|\varphi_2\|_b = o(1)
\end{align*}
\]

as \( a, b \to w_-, w_+ \). Also

\[
\begin{align*}
(5.6) & \quad \frac{U_a \varphi_2(a)}{h(a) U_a \varphi_1} \frac{\varphi_1(a)}{\varphi_1(b)}; \quad \frac{U_b \varphi_1(b)}{h(b) U_b \varphi_2} \frac{\varphi_2(b)}{\varphi_2(a)}
\end{align*}
\]

are bounded on neighbourhoods \( w_- < a \leq a_0, b_0 \leq b < w_+ \) of \( w_-, w_+ \) respectively. Finally it follows from the maximum-minimum principle for eigenvalues \([4], [10]\) that the monotonicity property for (5.2) holds. Since it is known that each eigenvalue of (5.2) has multiplicity 1, theorems 1, 2, and 3 hold with

\[
\begin{align*}
\Theta_a &= \left| \frac{U_a x}{U_a \varphi_1} \right| \|\varphi_1\|_a, & \Theta_b &= \left| \frac{U_b x}{U_b \varphi_2} \right| \|\varphi_2\|_b.
\end{align*}
\]

Considering now the assumptions in section 3 we first note that

\[
(5.7) \quad C_1(s) = -\varphi_2(s), \quad C_2(s) = \varphi_1(s).
\]

Hence assumptions (3.3) and (3.4) reduce to
(5.8) \[ h(a) \left| \frac{\varphi_2(a)}{\varphi_1(a)} \right| \leq h(s) \left| \frac{\varphi_2(s)}{\varphi_1(s)} \right| , \quad w_\text{a} \leq a \leq s \leq a_0; \]

(5.9) \[ h(b) \left| \frac{\varphi_1(b)}{\varphi_2(b)} \right| \leq h(s) \left| \frac{\varphi_1(s)}{\varphi_2(s)} \right| , \quad b_0 \leq s \leq b \leq w_+. \]

From (5.7) it is clear that assumptions (3.6) and (3.7) are a restatement of (5.6) for the present case. Finally, assumption (3.5) requires the boundedness of

(5.10) \[ h(s) \varphi_1(s) \left\| \varphi_2(t) h(t) \right\|^s , \quad h(s) \varphi_2(s) \left\| \varphi_1(t) h(t) \right\| \]

on \( w_- < s < w_+ \). (We are making use of the fact that \( p(s)\bar{W}(s) = 1 \) on \( w_- < s < w_+ \).) We summarize the results in the following theorem:

**Theorem 6.** Let \( w_\text{a}, w_+ \) be limit point singularities for \( L \) and let \( \lambda \) be an eigenvalue of (5.2) and \( x \) the corresponding normalized eigenfunction. Then, if conditions (5.4) - (5.6) are satisfied there exists a rectangle \( R_\text{a} \) and a constant \( C \) on \( R_\text{a} \) such that exactly one perturbed eigenvalue \( \mu_{ab} \) of (5.3) satisfies

\[ |\mu_{ab} - \lambda| \leq C(\theta_a + \theta_b) \]

whenever \([a,b] \in R_\text{a}\). Further, there exists a normalized eigenfunction \( y_{ab} \) corresponding to \( \mu_{ab} \) such that

\[ \|y_{ab} - x\|^b_a \leq C(\theta_a + \theta_b) \]

whenever \([a,b] \in R_\text{a}\). If in addition conditions (5.8) - (5.10) are satisfied, the uniform representation

\[ y_{ab}(s) = x(s) - f(s) + o(\theta_a) + o(\theta_b), \]
is valid on \( a \leq s \leq b, \omega_- < a < a_0, b_0 < b < \omega_+ \) where \( f(s) \) is the unique solution of the boundary problem

\[
L_0 f = 0, \quad U_a f = U_a x, \quad U_b f = U_b x.
\]

Also the following variational formula is valid as \( a, b \rightarrow \omega_-, \omega_+; \)

\[
\lambda - \mu_{ab} = [f_x](b) - [f_x](a) + (t_0 - \lambda)(f, f)_a^b + (\varphi_a + \varphi_b)(f, l)_a^b 0(l).
\]

**Proof.** The different parts of theorem 6 follow as particular cases of theorems 1-5.

In many second order examples, conditions (5.4), (5.6) and (5.10) hold with \( h(s) = 1 \) (see [13] for examples). In this case assumptions (5.8) and (5.9) may be omitted; these are then consequences of (5.4) and the fact that \( [\varphi_1 \varphi_2](s) = 1 \).

Then the new conditions ((5.4) - (5.6), (5.10) with \( h(s) = 1 \)) are sufficient for the estimates of theorem 6.

6. One end perturbation problems.

As a special case of the material in sections 1-4, we consider the operator \( L \) given by (0.1) on \( (\omega_, b], \) \( b \) fixed, \( b_0 < b < \omega_+. \) We assume that the differential equation \( L_0 x = 0 \) has basic solutions \( \varphi_j, j = 1, 2, \ldots, n \) such that \( \omega_- \) is not an accumulation point of zeros for \( \varphi_j, j = 1, 2, \ldots, n \) and such that
Let \( D(\omega_-, b) \) be the set of all \( x \in H(\omega_-, b] \) such that:

(i) \( x \in C^{n-1}(\omega_-, b] \) and \( x^{(n-1)} \) is absolutely continuous on every closed bounded subinterval of \((\omega_-, b] \); and

(ii) \( x \) satisfies the boundary conditions \( U^1_b x = 0, i = 1, 2, \ldots, n-m \)

where the boundary operators \( U^1_b \) are given by (0.7).

Then the eigenvalue problem

\[
(6.2) \quad Lx = \lambda x, \quad x \in D(\omega_-, b]
\]

will be referred to as the semi-perturbed problem. As before we shall suppose that (6.2) has at least one real eigenvalue \( \lambda \) and that \( \lambda_0 \) is not an eigenvalue. We shall (as in the previous cases) obtain estimates for eigenvalues and eigenfunctions of (1.3) for a near \( \omega_- \). This will be done by comparing problems (1.3) and (6.2) with (1.3) regarded as a perturbation of (6.2).

Let \( \lambda \) be any eigenvalue and \( x \) a corresponding normalized eigenfunction of (6.2). To obtain convergence of the eigenvalues of (1.3) to those of (6.2) we shall require that conditions (2.7), (2.9) and (2.11) are satisfied. At the fixed endpoint \( b \) one deduces from (0.7) and the fact that \( \lambda_0 \) is not an eigenvalue of (1.3) that \( h_b(j_1, j_{n-m}) \) will be constant for each sequence
Finally the monotonicity property that will be required is that the absolute value of the $i$-th eigenvalue of (6.2) is not greater than the absolute value of the $i$-th eigenvalue of (1.3). We then have the following theorem:

**Theorem 7.** Let $\lambda$ be an eigenvalue of (6.2) of multiplicity $k$ and assume that the monotonicity property holds. Then, if conditions (2.7), (2.9) and (2.11) are satisfied, there exists an interval $(w_-, a_o]$ and a constant $C$ such that exactly $k$ eigenvalues $\mu^j_a$ of (1.3) satisfy

$$|\lambda - \mu^j_a| \leq C\Theta_a$$

whenever $w_- < a \leq a_o$. There exist $k$ orthonormal eigenfunctions $x^j$ of (6.2) corresponding to $\lambda$ and $k$ orthonormal eigenfunctions $y^j_a$ of (1.3) corresponding to $\mu^j_a$ such that

$$\|x^j - y^j_a\|^b_a \leq C\Theta_a, \quad j = 1, 2, \ldots, k$$

whenever $w_- < a \leq a_o$.

**Proof:** Let $G_a(s, t)$ denote the Green's function for the operator $kL_0$ associated with the boundary conditions of (0.6) and let $G_a$ be the linear operator on $H[a, b]$ defined by

$$G_a v(s) = \int_a^b G_a(s, t)v(t)k(t)dt, \quad v \in H[a, b].$$

For $x$ any normalized eigenfunction of (6.2) corresponding to the eigenvalue $\lambda$ we define a function $f$ on $[a, b]$ by the equation

$$f = x - \gamma G_a x$$

where $\gamma = \lambda - l_0$. 

(j, j_{n-m})$ and that $h_b(m+1,n) \neq 0$. Finally the monotonicity property that will be required is that the absolute value of the $i$-th eigenvalue of (6.2) is not greater than the absolute value of the $i$-th eigenvalue of (1.3). We then have the following theorem:
Then \( L_0^i f = 0 \), \( U_a^i f = U_a^i x \), \( i = 1, 2, \ldots, m \),
\( U_b^i f = 0 \), \( i = 1, 2, \ldots, n-m \).

In terms of the basic solutions \( f \) has a representation of the form:

\[
(6.4) \quad f = \frac{1}{K(a, b)} \sum_{k=1}^{n} \left( \sum (\pm) \frac{\delta_a(i_1, i_{m-1})}{h_a} \frac{h_b(j_1, j_{n-m})}{h_b} \right) \varphi_k(s)
\]

where \( \Sigma \) indicates summation over all possible sequences \( (i_1, i_{m-1}) \) and \( (j_1, j_{n-m}) \) such that \( i_p \neq j_q \neq k \) for \( p = 1, 2, \ldots, m-1 \), \( q = 1, 2, \ldots, n-m \).

Now \( \frac{h_b(j_1, j_{n-m})}{h_b} \) is constant for all sequences \( (j_1, j_{n-m}) \), and
and by (2.14) \( \frac{h_a(i_1, i_m)}{h_a} = o(1) \) as \( a \to \omega_- \) for all sequences
\( (i_1, i_m) \neq (1, m) \), hence there exists a number \( \omega_-, \omega_- < \omega_+ < b \)
such that \( K(a, b) \) is bounded away from 0 for \( \omega_- < a \leq \omega_+ \). This
enables us to find a constant \( C \) on \( (\omega_-, \omega_+) \) such that

\[
\|f\|^b_a \leq C \sum_{k=1}^{n} \left( \sum (\pm) \frac{\delta_a(i_1, i_{m-1})}{h_a} \right) \|\varphi_k\|^b_a
\]

whenever \( \omega_- < a \leq \omega_+ \), where \( \Sigma \) indicates summation over all
possible sequences \( (i_1, i_{m-1}) \) such that \( i_r \neq k, r = 1, 2, \ldots, m-1 \).

But for each sequence \( (i_1, i_{m-1}, k) \), \( m+1 \leq k \leq n \), there exists a
sequence \( (i_1, i_{m-1}, k') \) with \( 1 \leq k' \leq m \). Then by using
\( \|\varphi_k\|^b_a = o(\|\varphi_k\|^b_a) \) as \( a \to \omega_- \) by (6.1) and (2.4), we obtain that

\[
\frac{\delta_a(i_1, i_{m-1})}{h_a} \|\varphi_k\|^b_a = o\left(\frac{\delta_a(i_1, i_{m-1})}{h_a}\right)
\]
as \( a \to \omega_- \). This implies with the help of (2.11) that there
exists a constant \( C \) and a number \( \omega_+ \) (which may be pre-supposed
to be our original choice) such that
whenever \( w - < a \leq a_0 \). The remainder of the proof that there exist at least \( k \) eigenvalues \( \mu_a^j \) of (1.3) such that

\[
(6.5) \quad |\mu_a^j - \lambda| \leq c \theta_a
\]

is similar to that of theorem 1 and will be omitted. The proof that there are exactly \( k \) perturbed eigenvalues \( \mu_a^j \) satisfying (6.5) using the monotonicity property is the same as that of theorem 2 and also will be omitted. The proof of the existence of \( k \) orthonormal eigenfunctions \( x^j \) corresponding to \( \lambda \) and \( k \) orthonormal eigenfunctions \( y_a^j \) corresponding to \( \mu_a^j \) satisfying (6.3) is the same as that of theorem 3 except that \( b \) is fixed for the present case.

In order to obtain uniform estimates of \( y_a^j(s) - x^j(s) \) on \( a \leq s \leq b \), following the method of section 4, we need stronger assumptions than (2.7), (2.9) and (2.11). It will be supposed in addition that conditions (3.3) and (3.6) are satisfied and that (3.5) holds on \( w - < s \leq b \).

To prove the boundedness of (3.1) on \( w - < s \leq b \) by lemma 4 we need only show that inequalities (3.11) and (3.12) are valid on \( a \leq s \leq b \), \( a \leq a_0 \) and that inequalities (3.12) and (3.14) are valid on \( a \leq s \leq b \), \( w - < a < b \). Let

\[
(6.6) \quad \hat{\varphi}^j(s) = \begin{cases} |\varphi^j(s)| & \text{if } a \leq s < a_0 \\ 1 & \text{if } a_0 \leq s \leq b \end{cases}
\]

\[
(6.6) \quad \hat{\psi}^j(s) = \begin{cases} |\psi^j(s)| & \text{if } a \leq s < a_0 \\ 1 & \text{if } a_0 \leq s \leq b \end{cases}
\]
Then it is obvious that the proofs of (3.11) and (3.12) (with $b_0$ replaced by $b$) hold for the present case. To prove (3.12) and (3.14) for the present case we note that since $b$ is fixed, $\frac{\eta_b(j_1', j_{n-m})}{\eta_b}$ is constant, hence from (6.6) inequalities (3.12) and (3.14) are obvious on $a_o \leq s \leq b$. The proof that these inequalities hold on $a \leq s < a_o$ follows easily from (3.3). These inequalities in addition to (3.5) imply the boundedness of (3.1) by lemma 4. Hence we have the following results:

**Theorem 8.** Assume that $\omega_-$ is a singularity for $L$ as described in this section and that conditions (2.7), (2.9), (2.11), (3.3), (3.5) and (3.6) are satisfied. If, in addition, the monotonicity property of this section is satisfied then the eigenfunctions $x^j$ corresponding to $\lambda$ and $y^j_a$ corresponding to $\mu^j_a$ of theorem 7 have the uniform representation

$$y^j_a(s) = x^j(s) - f^j(s) + o(a_a), \ j = 1, 2, \ldots, k$$

on $a \leq s < b$ provided $\omega_- < a \leq a_o$ where $f^j(s)$ is the unique solution of the boundary problem

$$L_0 \ f = 0, \quad U^i_a \ f = U^i_a x^j, \ i = 1, 2, \ldots, m$$

$$U^i_b \ f = 0, \ i = 1, 2, \ldots, n-m.$$

The following variational formulae for $\lambda - \mu^j_a$ as $a \to \omega_-$ are immediate consequences of theorem 5.

**Theorem 9.** Under the assumptions of theorem 8 the following variational formulae for the eigenvalues $\mu^j_a$ of theorem 7 are valid:
(6.8) \[ \lambda - \mu^j_a = [f^j x^j](b) - [f^j x^j](a) + (\nu_0 - \lambda)(f^j, f^j)^b_a + \Theta_a(f, l)^b_a \circledast(1) \]

as \( a \to w_- \) where \( f^j \) is the unique solution of (6.7), \( j = 1, 2, \ldots, k \).

7. **Class 1 singular problems for which all basic solutions are in \( H \).**

Perturbation problems will be considered for the case that the differential operator \( L \) given by (0.1) has singularities \( w_-, w_+ \) both of class 1 variety and that the basic solutions are all in \( H \). More precisely, we shall assume that the differential equation \( L_0 x = 0 \) has basic solutions \( \varphi_i, i = 1, 2, \ldots, n \) in \( H \) such that \( w_-, w_+ \) are not accumulation points of the zeros of \( \varphi_i \) for each \( i \), and such that

\[(7.1) \lim_\mathclap{s \to w_-} \frac{\varphi_{i+1}(s)}{\varphi_i(s)} = 0, \quad i = 1, 2, \ldots, m-1; \]
\[ \lim_\mathclap{s \to w_+} \frac{\varphi_i(s)}{\varphi_{i+1}(s)} = 0, \quad i = m+1, \ldots, n+1. \]

The treatment in this section is designed to include cases for which some of the basic solutions may be unbounded at \( w_- \) or \( w_+ \). As in the previous cases we shall define a basic eigenvalue problem on \((w_-, w_+)\) and obtain estimates for the eigenvalues and eigenfunctions of the perturbed problem (1.3) for \( a, b \) near \( w_-, w_+ \). We shall make use of the following lemma, which is a generalization of Weyl's first theorem ([9], p.31).
Lemma 5. Assume for some complex number \( t \) all solutions of 
\[ Lx = t_1 x \] are in \( H[a, \omega_+), \omega_- < a < \omega_+ \). Then for any complex number \( t \) all solutions of \( Lx = tx \) are in \( H[a, \omega_+). \)

**Proof:** The proof depends on the use of a variation-of-constants formula which differs slightly from that used by Coddington in ([2], p. 195). From (0.4) it is clear that \( [xy](s) \) may be written in the form

\[
[xy](s) = \sum_{i,j=0}^{n-1} B_{ij}(s) x^{(1)}(s) y^{(j)}(s)
\]

with

\[
B_{ij}(s) = \begin{cases} 
(-1)^{i+j} p_i(s), & i + j = n - 1 \\
0, & i + j > n - 1
\end{cases}
\]

(7.2)

Let \( B \) denote the \( n \)-by-\( n \) matrix which has the element \( B_{ij} \) in the \( i+1 \)-th row and \( j+1 \)-th column, \( i,j = 0,1,\ldots,n-1 \). Then (7.2) implies that \( B \) is non-singular on \( [a, \omega_+] \).

Let \( \varphi_1, \varphi_2, \ldots, \varphi_n \) be any \( n \) linearly independent solutions of \( Lx = t_1 x \). Using Green's formula (0.5) we see that \( [\varphi_\alpha \bar{\varphi}_\beta](s) \) is a constant \( [\varphi_\alpha \bar{\varphi}_\beta] \) independent of \( s \). Let \( S \) denote the matrix with element \( [\varphi_\alpha \bar{\varphi}_\beta] \) in the \( \alpha \)-th row and \( \beta \)-th column, \( \alpha, \beta = 1,2,\ldots,n \). Then it is easily verified that

(7.3)

\[
S = Y^t B Y
\]

where \( Y \) denotes the Wronskian matrix \( (\varphi_j^{(i-1)}(s)) \), \( i,j = 1,2,\ldots,n \) and \( Y^t \) the transpose of the matrix \( Y \). Since the matrices \( B, Y \) (and hence \( Y^t \)) are non-singular on \( [a, \omega_+] \), it follows that \( S \) is a non-singular constant matrix.
Let $S^{-1} = (\gamma_{\alpha\beta})$ denote the matrix inverse to $S$, and consider the function $K$ of $(s,t)$ defined by

$$K(s,t) = \sum_{\alpha, \beta=1}^{n} \gamma_{\alpha\beta} \varphi_{\alpha}(s) \varphi_{\beta}(t). \quad (7.4)$$

For any closed subinterval $[a,b]$ of $[a,\omega_+)$ let $v \in H[a,b]$. We shall show that the function

$$u(s) = \int_{a}^{b} K(s,t)v(t)k(t)dt \quad (7.5)$$

is such that $u^{(n-1)}$ is absolutely continuous on $[a,b]$, and $u$ satisfies the differential equation

$$Lu = \ell_1 u + v. \quad (7.6)$$

From (7.3) we have

$$(Y^*)^{-1} S Y^{-1} = B \quad (7.7)$$

and hence

$$Y S^{-1} Y^* = B^{-1}. \quad (7.7)$$

Let $B^{-1}_{ij}$ denote the element in the $i+1$-th row and $j+1$-th column of $B^{-1}$, $i,j = 0,1,\ldots,n-1$. Then (7.2) clearly implies that

$$B^{-1}_{ij} = \begin{cases} 0, & i + j < n - 1 \\ p_{0}(s)^{-1} \frac{(-1)^i}{p_{0}^{(i)}(s)}, & i + j = n - 1. \end{cases} \quad (7.8)$$

Consequently from (7.7) and (7.8) it follows that

$$\sum_{\alpha, \beta=1}^{n} \gamma_{\alpha\beta} \varphi_{\alpha}(s) \varphi_{\beta}(j)(s) = 0 \quad (7.9)$$

for $j = 0,1,\ldots,n-2$, and
It is now a straightforward calculation to show from (7.4), (7.5) that \( u, u^1, \ldots, u^{(n-1)} \) exist and that
\[
\begin{align*}
(7.11) \quad u^{(i)}(s) &= \sum_{\alpha, \beta=1}^{n} \gamma_{\alpha \beta} \phi_{\alpha}^{(i)}(s) \int_{a}^{s} \phi_{\alpha}(t) v(t) k(t) dt, \\
& \quad i = 1, 2, \ldots, n-1. \end{align*}
\]
Also from (7.10) we have
\[
(7.12) \quad u^{(n)}(s) &= \sum_{\alpha, \beta=1}^{n} \gamma_{\alpha \beta} \phi_{\alpha}^{(n)}(s) \int_{a}^{s} \phi_{\alpha}(t) v(t) k(t) dt + \frac{v(s) k(s)}{P_0(s)}. 
\]
From (7.11) and (7.12) it is now clear that \( u \) satisfies (7.6).

Let \( x \) be any solution of the differential equation
\[
Lx = lx \text{ which may be written in the form } \\
Lx = \ell_+ x + (\ell - \ell_+) x.
\]
Then the variation-of-constants formula given by (7.4) and (7.5) yields
\[
(7.13) \quad x(s) = \sum_{i=1}^{n} c_i \phi_i(s) + (\ell - \ell_+) \int_{a}^{s} \left( \sum_{\alpha, \beta=1}^{n} \gamma_{\alpha \beta} \phi_{\alpha}(t) \phi_{\alpha}(t) \right) x(t) k(t) dt,
\]
where \( c, c_i, \gamma_{\alpha \beta} \) are constants. Let \( M \) be a constant such that
\[
\gamma_0 \|\phi\|_c \leq M, \beta = 1, 2, \ldots, n \text{ where } \gamma_0 = \max_{1 \leq \alpha, \beta \leq n} |\gamma_{\alpha \beta}|.
\]
The Schwarz inequality gives
\[
\int_{a}^{s} \left( \sum_{\alpha, \beta=1}^{n} \gamma_{\alpha \beta} \phi_{\alpha}(t) \phi_{\alpha}(t) \right) x(t) k(t) dt \leq M \sum_{\beta=1}^{n} |\phi_{\beta}(s)| \|x\|_c^2.
\]
Using this in (7.13), the Minkowski inequality yields
\[
\|x\|_c^2 \leq \left( \sum_{i=1}^{n} |c_i| \right) M + n |\ell - \ell_+| M^2 \|x\|_c^2.
\]
If \( c \) is chosen large enough so that \( n|\ell-\ell_1|M^2 < \frac{1}{2} \) then

\[
\|x\|_c^s \leq 2 \sum_{i=1}^{n} |c_i| |M|
\]

Since the right side of this inequality is independent of \( s \) it follows that \( x \in \mathcal{H}(a,\omega_+) \). This completes the proof of the lemma.

From lemma 5 it is clear that \( n \) independent end conditions are required to obtain a reasonable eigenvalue problem on \((\omega_-,\omega_+)\). To obtain suitable end conditions we shall basically follow the method suggested by Kodaira in [6]. Our method will also resemble those used by Coddington [2] and Weyl in [9].

Let \( D \) be the set of all \( x \in \mathcal{H} \) such that \( x \in C^{n-1}(\omega_-,\omega_+) \) and \( x^{(n-1)} \) is absolutely continuous on every closed bounded sub-interval of \((\omega_-,\omega_+)\). By an end condition at \( \omega_- \) we shall mean a condition for \( x \in D \) of the form

\[
[x\psi](\omega_-) = 0
\]

where \( \psi \) is a fixed function in \( D \) such that \( L\psi \in \mathcal{H}(\omega_-,a) \) for any \( a \) satisfying \( \omega_- < a < \omega_+ \). A finite number of end conditions at \( \omega_- \)

\[
[x_1\psi_1](\omega_-) = 0, [x_2\psi_2](\omega_-) = 0, \ldots, [x_k\psi_k](\omega_-) = 0
\]

will be called **linearly independent at** \( \omega_- \) whenever

\[
\sum_{j=1}^{k} a_j [x_j\psi_j](\omega_-) = 0
\]

identically in \( x \) implies \( a_j = 0, j = 1,2,\ldots,k \).
End conditions at \( \omega_+ \) and their linear independence are to be defined similarly.

Let \( \{ \psi_1, \ldots, \psi_n \} \) be any set of real functions in \( D \) such that \( L\psi_j \in H, \ j = 1,2,\ldots, n \) and such that the set
\[
\{ [x\psi_j](\omega_-) = 0, \ j = 1,2,\ldots, \frac{n}{2} \}
\]
is linearly independent at \( \omega_- \), and the set
\[
\{ [x\psi_j](\omega_+) = 0, \ j = \frac{n}{2} + 1,\ldots, n \}
\]
is linearly independent at \( \omega_+ \). Then Kodaira's method applied to the present case gives rise to the eigenvalue problem;
\[
Lx = \lambda x \text{ for all } x \in D \text{ such that } Lx \in H \text{ and }
\]
\[
[x\psi_j](\omega_-) = 0, \ j = 1,2,\ldots, \frac{n}{2},
\]
\[
[x\psi_j](\omega_+) = 0, \ j = \frac{n}{2} + 1,\ldots, n.
\]
Actually Kodaira puts additional restrictions on the \( \psi_j \) to obtain a self-adjoint problem but we do not need the self-adjointness for our purposes.

For our purpose we shall choose a set of end conditions which is more general than those demonstrated by Kodaira or Weyl. We shall assume that there exists \( n \) functions \( \chi_1, \ldots, \chi_n \) in \( D \) such that \( L\chi_j \in H, \ j = 1,2,\ldots, n \) and such that the set
\[
\{ [x\chi_j](\omega_-) = 0, \ j = 1,2,\ldots, m \}
\]
is linearly independent at \( \omega_- \) and the set
\[
\{ [x\chi_j](\omega_+) = 0, \ j = m+1,\ldots, n \}
\]
is linearly independent at $w_+, 1 \leq m < n$. Then the eigenvalue problem

\begin{equation}
Lx = \lambda x, \quad x \in D, \quad [xx_j](w_-) = 0, \quad j = 1, \ldots, m, \\
[xx_j](w_+) = 0, \quad j = m+1, \ldots, n
\end{equation}

will be referred to as the basic problem and $(w_-, w_+)$ as the basic interval. Again we stress the point that (7.14) is to be a reasonable problem, i.e. eigenvalues are supposed to exist.

Finally we require that all eigenvalues of (7.14) are real. This implies that $\ell_0$ is not an eigenvalue of (7.14).

For the perturbed problem on $[a, b], w_- < a < b < w_+$ we choose the regular self-adjoint problem (1.3). The first theorem will provide conditions under which the eigenvalues of (1.3) converge to those of (7.14) as $a, b \to w_-, w_+$. The remaining theorems are refinements which lead to asymptotic estimates of the eigenvalues and eigenfunctions.

Let $\lambda$ be an eigenvalue and \{x_j, j = 1, 2, \ldots, k\} a corresponding orthonormal set of eigenfunctions for (7.14). Let $A_\lambda$ denote the space spanned by \{x_j, j = 1, 2, \ldots, k\}. To obtain convergence of the eigenvalues of (1.3) to those of (7.14) we assume that conditions (2.7) - (2.10) hold and that for any $x \in A_\lambda$:

\begin{equation}
\delta_a(i_1, i_{m-1}) = o(1) \\
\delta_b(j_1, j_{n-m-1}) = o(1)
\end{equation}

as $a, b \to w_-, w_+$ for all sequences $(i_1, i_{m-1})$ and $(j_1, j_{n-m-1})$. 
Let

\[ (7.17) \quad \rho_a = \sup_{x \in A, \|x\| = 1} \left| \frac{\delta_a(h, m-l)}{h_a} \right| \]
\[ (7.18) \quad \rho_b = \sup_{x \in A, \|x\| = 1} \left| \frac{\delta_b(m+2, n)}{h_b} \right| . \]

Then (7.15) - (7.17) clearly imply that

\[ (7.18) \quad \rho(a) = o(1), \quad \rho(b) = o(1) \]
as \(a, b \to w_-, w_+\). Assumptions (2.7) - (2.10) imply for the present case that

\[ (7.19) \quad h(a)\sigma_a(i, j) = o(1), \quad h(b)\sigma_b(i, j) = o(1) \]
for \(i = 1, 2, \ldots, m, j = m+1, \ldots, n\) and

\[ (7.20) \quad \frac{h_a(i_1, i_m)}{h_a} = o(1) \text{ for } (i_1, i_m) \neq (l, m), \]
\[ (7.21) \quad \frac{h_b(j_1, j_{n-m})}{h_b} = o(1) \text{ for } (j_1, j_{n-m}) \neq (m+1, n) \]
as \(a, b \to w_-, w_+\).

The weaker conditions (7.15), (7.16), (7.20), (7.21) are actually sufficient to obtain the convergence of the eigenvalues of (1.3) to those of (7.14) while the stronger assumptions (2.7)-(2.10), (7.15), (7.16) will be required to obtain the uniform estimates in theorem 12.

**Theorem 10.** Let \(w_-\) and \(w_+\) be singularities for \(L\) as described in this section. Let \(\lambda\) be an eigenvalue of (7.14) possessing \(k\) orthonormal eigenfunctions. Then under assumptions (2.7)-(2.10),
\( (7.15), (7.16) \) (or the weaker conditions \( (7.15), (7.16), (7.20), \)
\( (7.21) \)) there exists a rectangle \( R_o \) and a constant \( C \) on \( R_o \),
such that at least \( k \) perturbed eigenvalues \( \mu_{ab}^j \) of \( (1.3) \) satisfy

\[
|\mu_{ab}^j - \lambda| \leq C(\rho(a) + \rho(b))
\]
whenever \( [a,b] \in R_o \).

**Proof.** Let \( x \) be any normalized function in \( A_\chi \). Then, proceeding
as in theorem 1, we define a function \( f \) on \([a,b]\) by

\[
f = x - \gamma G_{ab}x \quad \text{where} \quad \gamma = \lambda - \lambda_0.
\]

Then \( f \) is the unique solution of the boundary problem

\[
(7.24) \quad L_0 f = 0, \quad U_a^i f = U_a^i x, \quad i = 1,2,\ldots,m,
\]
\[
U_b^i f = U_b^i x, \quad i = 1,2,\ldots,n-m.
\]

If the \( h_a(\quad), h_b(\quad), \delta_a(\quad), \delta_b(\quad) \) notation is used one
may find a representation of \( f \) in terms of the basic solutions
as follows

\[
(7.25) \quad f(s) = \frac{1}{K(a,b)} \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} (\pm) \frac{\delta_a(i_1,i_m-l)}{h_a} - \frac{\delta_b(j_1,j_{n-m}-l)}{h_b} \right) \varphi_k(s)
\]

where

\[
(1) \quad K(a,b) = 1 + \sum (\pm) \frac{\delta_a(i_1,i_m)}{h_a} - \frac{\delta_b(j_1,j_{n-m})}{h_b}
\]

where \( \Sigma \) indicates summation over all possible sequences \((i_1,i_m)\)
and \((j_1,j_{n-m})\) such that \((i_1,i_m) \neq (1,m)\) and such that \( i_r \neq j_s \),
\( r = 1,2,\ldots,m, s = 1,2,\ldots,n-m; \)
(ii) $\sum (1)$ indicates summation over all possible sequences $(i_1, i_{m-1})$ and $(j_1, j_{n-m})$ such that $i_r \neq j_s \neq k$, $r = 1, 2, \ldots, m-1$, $s = 1, 2, \ldots, n-m$;

(iii) $\sum (2)$ indicates summation over all possible sequences $(i_1, i_m)$ and $(j_1, j_{n-m-1})$ such that $i_r \neq j_s \neq k$, $r = 1, 2, \ldots, m$, $s = 1, 2, \ldots, n-m-1$.

That this representation (7.25) of $f$ is valid follows in the same way as (2.21).

From (7.20) and (7.21) we can find numbers $a_0, b_0, w_-$ such that $K(a, b)$ is bounded away from 0 whenever $w_- < a < b < w_+$. Also by (7.15) - (7.17), (7.20), (7.21) there exist numbers $a_0, b_0$ (which may be presupposed to be the previous choices) such that

$$\frac{\delta_a(i_1, i_{m-1})}{n_a} \frac{n_b(j_1, j_{n-m})}{n_b} = O(p_a)$$

for all sequences $(i_1, i_{m-1})$ and $(j_1, j_{n-m-1})$, and such that

$$\frac{n_a(i_1, i_m)}{n_a} \frac{\delta_b(j_1, j_{n-m-1})}{n_b} = O(p_b)$$

for all sequences $(i_1, i_m)$ and $(j_1, j_{n-m-1})$ whenever $w_- < a < a_0$, $b_0 < b < w_+$. These considerations in addition to the fact that $\omega_j \in \Omega$, $j = 1, 2, \ldots, n$ permit us to deduce that there exists a rectangle $R_0$, and a constant $C$ on $R_0$, such that

$$\|f\|_a \leq C(p_a + p_b)$$

whenever $[a, b] \in R_0$. 
By (7.18) we have that \( \|f\|^b_a = o(1) \) as \( a, b \to w_-, w_+ \). Hence an application of lemma 1 (as in theorem 1) shows that there exists a constant \( C \) and a rectangle \( R_o \), such that at least \( k \) eigenvalues \( \mu^{j}_{ab} \) (counting multiplicities) of (1.3) satisfy

\[
|\mu^{j}_{ab} - \lambda| \leq C(\rho_a + \rho_b)
\]

whenever \([a, b] \in R_o\). This completes the proof of the theorem.

Theorem 10 and (7.18) show in particular that for each basic eigenvalue \( \lambda \) of multiplicity \( k \) there exist at least \( k \) perturbed eigenvalues \( \mu^{j}_{ab} \) (counting multiplicities) such that \( \mu^{j}_{ab} \to \lambda \) when \( a, b \to w_-, w_+ \). To obtain the stronger result that exactly \( k \) perturbed eigenvalues satisfy (7.22) in theorem 10 we require that (7.14) satisfy the following monotonicity property: The absolute value of the \( i \)-th eigenvalue of problem (7.14) is not larger than the absolute value of the \( i \)-th eigenvalue of (1.3). The following theorem is then obtained analogously to theorems 2 and 3.

Theorem 11. If the monotonicity property holds in addition to the hypotheses of theorem 10, then for every eigenvalue \( \lambda \) of (7.14), of multiplicity \( k \), there exists a rectangle \( R_o \) and a constant \( C \) on \( R_o \) such that exactly \( k \) perturbed eigenvalues \( \mu^{j}_{ab} \) of (1.3) satisfy

\[
|\mu^{j}_{ab} - \lambda| \leq C(\rho_a + \rho_b), \quad j = 1, 2, \ldots, k
\]

whenever \([a, b] \in R_o\). There exist orthonormal eigenfunctions \( x^j \) associated with \( \lambda \) and \( y^j_{ab} \) associated with \( \mu^{j}_{ab} \) such that
\[ \| y^j_{ab} - x^j \|_{a}^b \leq C(\rho_a + \rho_b) , \quad j = 1,2,\ldots,k \]

whenever \([a,b] \in R_0\).

To obtain uniform estimates of \( y^j_{ab}(s) - x^j(s) \) in theorem 11, stronger assumptions are needed on the behaviour of the basic solutions at \( u_0^- \) and \( u_0^+ \). In addition to the hypotheses of theorem 11, we shall require that conditions (3.3) - (3.7) are satisfied. Then the hypotheses of lemma 4 are clearly satisfied and one can obtain the following theorems:

**Theorem 12.** If in addition to the hypotheses of theorem 11, conditions (3.3) - (3.7) are satisfied, then the eigenfunctions \( x^j \) corresponding to \( \lambda \) and \( y^j_{ab} \) corresponding to \( \mu^j_{ab} \) of theorem 11 have the uniform representation:

\[ y^j_{ab}(s) = x^j(s) - f^j(s) + O(\rho_a + \rho_b), \quad j = 1,2,\ldots,k, \]

for \( a \leq s \leq b, \quad w^- < a < a_0, \quad b_0 < b < w^+ \) where \( f^j(s) \) is the unique solution of the boundary problem

\[ L f = 0, \quad U^j_a f = U^j_a x^j, \quad i = 1,2,\ldots,m, \]
\[ U^j_b f = U^j_b x^j, \quad i = 1,2,\ldots,n-m. \]

**Theorem 13.** Under the assumptions of theorem 12, the following variational formulae hold for the eigenvalues \( \lambda \) and \( \mu^j_{ab} \) of theorem 10:

\[ \lambda - \mu^j_{ab} = [f^j x^j](b) - [f^j x^j](a) + (k_0 - k)(f^j, f^j)^b_a + (\rho_a + \rho_b)(f^j, 1)^b_a 0(1) \]

as \( a, b \to w^-, w^+ \) for \( j = 1,2,\ldots,k \).
8. A one-end perturbation problem; all basic solutions in $H(w_-, b)$

The material in the preceding section applies easily to the case for which the basic problem is defined on $(w_-, b)$, $b$ fixed, $b_0 \leq b < w_+$. Hence we shall consider the case that $w_-$ is a class 1 singularity for $L$ and all the basic solutions are in $H(w_-, b)$. Let $D(w_-, b)$ be the set of all $x \in H(w_-, b)$ such that

(i) $x \in C^{n-1}(w_-, b]$ and $x^{(n-1)}$ is absolutely continuous on every closed bounded subinterval of $(w_-, b]$;

(ii) $x$ satisfies the regular boundary conditions

$$U^i_b x = 0, \quad i = 1, 2, \ldots, n-m$$

where $U^i_b$ is given by (0.7), and the end conditions

$$[x' x_j](w_-) = 0, \quad j = 1, 2, \ldots, m$$

of (7.14).

Then the basic problem to be considered is the eigenvalue problem

(8.1) \[ Lx = \lambda x, \quad x \in D(w_-, b). \]

Again we require that at least one real eigenvalue exists and that $\lambda_0$ is not an eigenvalue. We shall compare problems (1.3) and (8.1) with (1.3) regarded as a perturbation of (8.1). The regular endpoint $b$ is to remain fixed for this case.

To obtain the results corresponding to theorems 10 and 11, we require that conditions (2.7), (2.9), (7.15) (or even the weaker conditions (7.15) and (7.20)) hold and that (8.1) satisfies the monotonicity property described in section 6. We then have the following result:
Theorem 14. Let $w_-$ be a singularity for $L$ as described in this section and let $\lambda$ be an eigenvalue of (8.1) possessing $k$ orthonormal eigenfunctions. Then, if conditions (2.7), (2.9) and (7.15) (or even the weaker conditions (7.15) and (7.20)) are satisfied and the monotonicity property holds, there exists an interval $(w_-, a_o]$ and a constant $C$ such that exactly $k$ perturbed eigenvalues $\mu^j_a$ of (1.3) satisfy

$$|\mu^j_a - \lambda| \leq C \rho_a, \quad j = 1, 2, \ldots, k$$

whenever $w_- < a \leq a_o$. There exist orthonormal eigenfunctions $x^j$ corresponding to $\lambda$ and $y^j_a$ corresponding to $\mu^j_a$ such that

$$\|y^j_a - x^j\|_a \leq C \rho_a$$

whenever $w_- < a \leq a_o$.

To obtain uniform estimates on $[a, b]$ and the variational formulae for eigenvalues we assume in addition to (2.7), (2.9), (7.15) and the monotonicity property that conditions (3.3), (3.5) and (3.6) hold. Then the boundedness of (3.1) on $a \leq s \leq b$, $a \leq a_o$ follows in exactly the same way as in section 6.

Theorem 15. If, in addition to the assumptions of theorem 14, we require conditions (3.3), (3.5) and (3.6) to be valid then the eigenfunctions $x^j$ corresponding to $\lambda$ and $y^j_a$ corresponding to $\mu^j_a$ of theorem 14 have the uniform representation

$$y^j_a(s) = x^j(s) - f^j(s) + O(\rho_a), \quad j = 1, 2, \ldots, k$$

for $a \leq s \leq b$, $w_- < a \leq a_o$, where $f^j(s)$ is the unique solution of the boundary problem.
The following variational formulae are also valid:

\[(8.2) \quad \lambda - \mu_j^{(a)} = [f_j^fx^j](b) - [f_j^fx^j](a) + (t_0 - \lambda)(f_j^j, f_j^j)^b_a + \rho_a (f_j^j, l)^b_0(1)\]

as \(a \to w_-\) for \(j = 1, 2, \ldots, k\).

9. **The second order case; \(w_-, w_+\) class 1 limit circle singularities.**

The assumptions and results of section 7 will be specialized to the second order case; i.e. to the operator \(L\) defined at the beginning of section 5. The case to be considered is that for which both \(w_-\) and \(w_+\) are class 1 singularities and of the limit circle variety ([3], p. 225). A theorem of Weyl ([9], p. 39) states that there exist linearly independent solutions \(\varphi_1, \varphi_2 \in H\) of \(L_0x = 0\) such that

\[ [\varphi_1, \varphi_1](w_+) = [\varphi_2, \varphi_2](w_-) = 0 \]

and

\[ [\varphi_1, \varphi_2](s) = 1, \ w_- < s < w_+ . \]

It will be assumed that \(w_-, w_+\) are not accumulation points for the zeros of \(\varphi_1\) and \(\varphi_2\).

Let \(D_2\) denote the set of all \(x \in H\) such that

1. \(x\) is differentiable on \((w_, w_+)\) and \(x'\) is absolutely continuous on every closed bounded subinterval of \((w_, w_+)\).
(11) \( x \) satisfies the end conditions
\[
[x\varphi_1](w_+)[x\varphi_2](w_-) = 0.
\]

Then the eigenvalue problem (to be referred to as the **basic** problem)

(9.1) \[ Lx = \lambda x, \quad x \in D_2 \]

is known to have a denumerable set of real eigenvalues \( \lambda^j \) and a corresponding orthonormal set of eigenfunctions complete in \( H \).

It is also known that each eigenvalue \( \lambda^j \) of (9.1) has multiplicity 1. The **perturbed** problem on \([a,b], w_- < a < b < w_+\) to be considered is the regular self-adjoint eigenvalue problem given by (5.3).

The stronger assumptions of theorem 10 when applied to the present case are:

(9.2) \[ h(a) \frac{\varphi_2(a)}{\varphi_1(a)} = o(1); \quad h(b) \frac{\varphi_1(b)}{\varphi_2(b)} = o(1); \]

(9.3) \[ \frac{U_a x}{U_a \varphi_1} = o(1); \quad \frac{U_b x}{U_b \varphi_2} = o(1) \]

as \( a, b \to w_- , w_+ \), and that

(9.4) \[ \frac{U_a \varphi_2 \varphi_1(a)}{h(a) U_a \varphi_1 \varphi_2(a)} , \quad \frac{U_b \varphi_1 \varphi_2(b)}{h(b) U_b \varphi_2 \varphi_1(b)} \]

are bounded on neighbourhoods \( w_- < a < a_o \), \( b_o < b < w_+ \) of \( w_-, w_+ \) respectively. Assumptions (9.2) and (9.4) imply that

(9.5) \[ \frac{\varphi_2(a)}{\varphi_1(a)} = o(1) , \quad \frac{\varphi_1(b)}{\varphi_2(b)} = o(1) \]
as \( a, b \to \omega_-, \omega_+ \). The weaker conditions of theorem 10 are precisely (9.3) and (9.6). Also a monotonicity property of (9.1) is known to hold (from the maximum-minimum principle for eigenvalues, [4], [10]) hence theorems 10 and 11 give the following results with

\[
\rho_a = \left| \frac{u_a x}{u_a \varphi_1} \right|, \quad \rho_b = \left| \frac{u_b x}{u_b \varphi_2} \right|
\]

Theorem 16. Let \( \omega_- \) and \( \omega_+ \) be singularities for \( L \) as described in this section. Let \( \lambda \) and \( x \) be an eigenvalue and the corresponding normalized eigenfunction of (9.1). If conditions (9.2), (9.3) and (9.4) (or even the weaker conditions (9.5) and (9.6)) are satisfied then there exists a rectangle \( Q_0 \) and a constant \( C \) on \( Q_0 \) such that exactly one eigenvalue \( \mu_{ab} \) of (5.3) satisfies

\[
|\mu_{ab} - \lambda| \leq C \left( \frac{|u_a x|}{|u_a \varphi_1|} + \frac{|u_b x|}{|u_b \varphi_2|} \right)
\]

whenever \([a, b] \in Q_0\). There exists a normalized eigenfunction \( y_{ab} \) of (5.3) corresponding to \( \mu_{ab} \) such that

\[
\|y_{ab} - x\|^b_a \leq C \left( \frac{|u_a x|}{|u_a \varphi_1|} + \frac{|u_b x|}{|u_b \varphi_2|} \right)
\]

whenever \([a, b] \in Q_0\).

The assumptions in theorem 12 when applied to the present case yield, in addition to (9.2)-(9.4), that

\[
(9.7) \quad h(a) \frac{\varphi_2(a)}{\varphi_1(a)} \leq h(s) \frac{\varphi_2(s)}{\varphi_1(s)}, \quad \omega_- < a \leq s \leq a_0,
\]

\[
(9.8) \quad h(b) \frac{\varphi_1(b)}{\varphi_2(b)} \leq h(s) \frac{\varphi_1(s)}{\varphi_2(s)}, \quad b_0 \leq s \leq b < \omega_+
\]
and that
\[
(9.9) \quad h(s) \varphi_1(s) \parallel h(t) \varphi_2(t) \parallel^s, \quad h(s) \varphi_2(s) \parallel h(t) \varphi_1(t) \parallel^s
\]
are bounded on \( w_- < s < w_+ \). We then have the following results:

**Theorem 17.** Under assumptions (9.2)-(9.4), (9.7)-(9.9) the eigenfunctions \( x \) and \( y_{ab} \) of theorem 16 have the uniform representation:

\[
y_{ab}(s) = x(s) - f(s) + O \left( \frac{U_a x}{U_a \varphi_1} + \frac{U_b x}{U_b \varphi_2} \right)
\]
on \( a \leq s \leq b \), \( w_- < a < a_o \), \( b_o < b < w_+ \) where \( f \) is the unique solution of the boundary problem

\[
L_0 f = 0, \quad U_a f = U_a x, \quad U_b f = U_b x.
\]
The following variational formula for the eigenvalues \( \lambda, \mu_{ab} \) is valid as \( a, b \to w_-, w_+ \):

\[
\lambda - \mu_{ab} = [fx](b) - [fx](a) + (t_0 - \lambda)(f,f)_a^b
\]
\[
+ \left( \frac{U_a x}{U_a \varphi_1} + \frac{U_b x}{U_b \varphi_2} \right)(f,1)_a^b 0(1).
\]
CHAPTER II

CLASS 2 SINGULAR PROBLEMS

Preliminaries

The variation of the eigenvalues and eigenfunctions of the regular self-adjoint eigenvalue problem (1.3) will be considered for \( a, b \) near \( w_-, w_+ \) in the case that all the basic solutions \( \varphi_1 \) are in \( H \) and \( w_-, w_+ \) are singularities for \( L \) both of class 2 variety. Again suitable singular eigenvalue problems will be defined on \( w_- < s < w_+ \) and conditions will be obtained such that the eigenvalues of (1.3) converge to those of the singular eigenvalue problem as \( a \to w_-, b \to w_+ \).

Since all the basic solutions are assumed to be in \( H \), it follows from lemma 5 that \( n \) linearly independent end conditions will be required to obtain a singular eigenvalue problem on \( (w_-, w_+) \). The corresponding situation for the second order operator is that for which both \( w_- \) and \( w_+ \) are limit circle singularities according to Weyl's classification ([3], p. 225).

The theory of class 2 perturbations will be developed in sections 10-13. The results will be specialized in section 14 to second order operators with limit circle singularities at \( w_+ \).
10. **Description of the basic and perturbed problems.**

To establish an eigenvalue problem on \((\omega_-, \omega_+)\) for \(L\) as given by (0.1), we shall basically follow the method suggested by Kodaira in [6]. (See also [2] where mixed conditions are used.)

Let \(D\) be the set of all \(x \in H\) such that \(x \in C^{n-1}(\omega_-, \omega_+)\) and \(x^{(n-1)}\) is absolutely continuous on every closed bounded sub-interval of \((\omega_-, \omega_+)\). Let \(\chi_i\), \(i = 1, 2, \ldots, n\) be a set of functions (to remain fixed) in \(D\) such that \(L\chi_i \in H\), \(i = 1, 2, \ldots, n\), and such that the end conditions \([x\chi_i]_{\omega_-} = 0\), \(i = 1, 2, \ldots, m\) are linearly independent at \(\omega_-\) and \([x\chi_i]_{\omega_+} = 0\), \(i = m+1, \ldots, n\) are linearly independent at \(\omega_+\). Then the **basic problem** is the singular eigenvalue problem

\[
(10.1) \quad L x = \lambda x, \quad x \in D_0
\]

where \(D_0\) is the set of all \(x \in D\) such that

\[
(10.2) \quad \begin{cases} 
[x\chi_i]_{\omega_-} = 0, & i = 1, 2, \ldots, m \\
[x\chi_i]_{\omega_+} = 0, & i = m+1, \ldots, n.
\end{cases}
\]

Again we stress that (10.1) is to be a reasonable eigenvalue problem, i.e. at least one eigenvalue is supposed to exist. Also we require that all eigenvalues of (10.1) are real.

The **perturbed problem** is the regular self-adjoint eigenvalue problem given by (1.3), and is defined for each \([a,b] \in R^0\).
For the class of perturbation problems to be considered, the basic solutions are not necessarily ordered according to their asymptotic behaviour at $\omega_{\pm}$. Consequently stronger conditions have to be imposed on the limiting behaviour of the boundary operators $U_a^i, U_b^i$ as $a, b \to \omega_{\pm}$. In particular we shall require that for every $n-1$ times differentiable function $y$

\begin{align}
(10.3) \quad \begin{cases}
U_a^i y &= [yx_i](a)(1 + o(1)) \text{ as } a \to \omega_{\pm}, \ i = 1, 2, \ldots, m \\
U_b^i y &= [yx_{m+i}](b)(1 + o(1)) \text{ as } b \to \omega_{\pm}, \ i = 1, 2, \ldots, n-m.
\end{cases}
\end{align}

Let $A$ denote the matrix $(A_{ij})$ where

$$A_{ij} = \begin{cases}
[\varphi_i x_j](\omega_{\pm}), & i = 1, 2, \ldots, n; \ j = 1, 2, \ldots, m \\
[\varphi_i x_j](\omega_{\pm}), & i = 1, 2, \ldots, n; \ j = m+1, \ldots, n
\end{cases}$$

and let $\Omega = \det A$. Then since $\Omega = \det A^t$ where $A^t$ is the transpose of $A$ and since $t_o$ is non-real it follows immediately that $\Omega \neq 0$ (otherwise $t_o$ would be an eigenvalue of (10.1)). Also for each $j$, $j = 1, 2, \ldots, n$, $\varphi_j$, $\Lambda x_j$, $\Lambda x_j$, are in $H$ hence (0.5) implies that each limit $[\varphi_1 x_j](\omega_{\pm})$ exists (and is finite) for $i, j = 1, 2, \ldots, n$. This implies that $\Omega$ is equal to some non-zero constant.

Let $A(a, b)$ denote the matrix $(A_{ij}(a, b))$ where

$$A_{ij}(a, b) = \begin{cases}
U_{a}^i \varphi_j, & i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n \\
U_{b}^{i-m} \varphi_j, & i = m+1, \ldots, n; \ j = 1, 2, \ldots, n
\end{cases}$$
and let \( \Omega(a,b) = \det A(a,b) \). Since the limits of \([ \varphi_i \chi_j ](a)\) and \([ \varphi_i \chi_j ](b)\) are finite as \( a \to \omega_- \) and \( b \to \omega_+ \) for \( i, j = 1, 2, \ldots, n \), it follows from (10.3) that we can select numbers \( a_0, b_0 \) (which may be pre-supposed to be the original choices) and a constant \( C \) such that

\[
(10.4) \quad |U^i_a \varphi_j| \leq C, \quad |U^i_b \varphi_j| \leq C
\]

for \( i, j = 1, 2, \ldots, n \) whenever \( \omega_- < a \leq a_0, \quad b_0 \leq b < \omega_+ \). Also by (10.3) the element in the \( i \)-th row and \( j \)-th column in \( A(a,b) \) approaches the element in the \( i \)-th row and \( j \)-th column in \( A^t \) as \( a, b \to \omega_-, \omega_+ \). This implies that

\[
(10.5) \quad \Omega(a,b) \to \Omega \neq 0
\]

as \( a, b \to \omega_-, \omega_+ \) and hence we can assume by (10.4) and (10.5) that the numbers \( a_0, b_0 \) previously chosen are such that \( \Omega(a,b) \) is bounded above and away from zero whenever \( \omega_- < a \leq a_0, \quad b_0 \leq b < \omega_+ \).

11. Comparison of the basic and perturbed problems.

The two problems (10.1) and (1.3) will be compared, with (1.3) regarded as a perturbation of (10.1). We are going to estimate the variation of the eigenvalues and eigenfunctions under the perturbation \( D_0 \to D[a,b] \) and show that this variation has the limit \( 0 \) as \( a, b \to \omega_-, \omega_+ \). Let \( \lambda \) be an eigenvalue of (10.1) and let \( A_\lambda \) denote the eigenspace of dimension \( k \) corresponding to \( \lambda \). Let \( x_j, \; j = 1, 2, \ldots, k \) be an orthonormal basis for \( A_\lambda \) and let \( \tau^i_a(x) \) and \( \tau^i_b(x) \) be defined by
Then (10.2) and (10.3) clearly imply that

\[(11.2) \quad \pi_i^a(x) = o(1), \quad i = 1, 2, \ldots, m, \]

\[(11.3) \quad \pi_i^b(x) = o(1), \quad i = 1, 2, \ldots, n-m \]
as \(a \to \omega_-, b \to \omega_+\). The following theorem proves the convergence of the eigenvalues of (1.3) to those of (10.1).

**Theorem 18.** Let \(\omega_-\) and \(\omega_+\) be singularities for \(L\) as described in section 10. Let \(\lambda\) be an eigenvalue of (10.1) possessing \(k\) orthonormal eigenfunctions. Then under assumption (10.3) there exists a rectangle \(R_0\), and a constant \(C\) on \(R_0\), such that at least \(k\) perturbed eigenvalues \(\mu_{ab}^j\) of (1.3) satisfy

\[(11.3) \quad |\mu_{ab}^j - \lambda| \leq C \left( \sum_{i=1}^{m} \pi_i^a(x) + \sum_{i=1}^{n-m} \pi_i^b(x) \right) \]

whenever \([a,b] \in R_0\).

**Proof.** Proceeding as in the proof of theorem 1 we define a function \(f\) on \([a,b]\) by

\[(11.4) \quad f = x - \gamma G_{ab}x\]

where \(\gamma = \lambda - \lambda_0\) and \(x \in A_\lambda\). Then \(f\) is the unique solution of the boundary problem.
\[ L f = 0, \quad U^i_a f = U^i_a x, \quad i = 1, 2, \ldots, m \]
\[ U^i_b f = U^i_b x, \quad i = 1, 2, \ldots, n - m. \]

Let \( K^j(a, b) \) denote the determinant of the matrix obtained from \( A(a, b) \) by replacing the \( j \)-th column by
\[
U^1_a x, \ U^2_a x, \ldots, U^m_a x, \ U^1_b x, \ldots, U^{n-m}_b x.
\]

Then \( f \) has the following representation in terms of the basic solutions:

\[
(11.5) \quad f(s) = \frac{1}{\Omega(a, b)} \sum_{j=1}^{n} K^j(a, b) \varphi_j(s).
\]

Consider now the determinant \( K^j(a, b) \) on the rectangle, \( w_- < a < a_0, \ b_0 < b < w_+ \). Since each element of \( A(a, b) \) is bounded on this rectangle by (10.4), there exists a constant \( C \) such that
\[
|K^j(a, b)| \leq C \left( \sum_{i=1}^{m} |U^i_a x| + \sum_{i=1}^{n-m} |U^i_b x| \right)
\]
for each \( j, \ j = 1, 2, \ldots, n \) whenever \( w_- < a < a_0, \ b_0 < b < w_+ \).

This applied to (11.5) yields a constant \( C \) on \( R_o \) such that
\[
\|f\|^b_a \leq \frac{C}{\Omega(a, b)} \sum_{j=1}^{n} \left( \sum_{i=1}^{m} |U^i_a x| + \sum_{i=1}^{n-m} |U^i_b x| \right) \|\varphi_j(s)\|^b_a
\]
But \( \Omega(a, b) \) is bounded away from 0 on \( R_o \) by (10.5) and \( \varphi_j \in H, \ j = 1, 2, \ldots, n \); hence there exists a constant \( C \) such that

\[
(11.6) \quad \|f\|^b_a \leq C \left( \sum_{i=1}^{m} |U^i_a x| + \sum_{i=1}^{n-m} |U^i_b x| \right)
\]
holds uniformly on \( w_- < a < a_0, \ b_0 < b < w_+ \). By (11.1), (11.4) and (11.6) one may deduce that there exists a constant \( C \) such that
for any normalized \( x \in A^\lambda \)

\[
(11.7) \quad \| x - \gamma G_{ab} x \|^b_a \leq C \left( \sum_{i=1}^{m} \tau_a^i(x) + \sum_{i=1}^{n-m} \tau_b^i(x) \right) \| x \|^b_a
\]

whenever \( \omega_- < a \leq a_o, b_o \leq b < \omega_+ \). It follows from (11.7) and lemma 1 that there exists a constant \( C \) on \( R_o \) such that

\[
\| x - P(\delta) x \|^b_a \leq \frac{C}{2\delta} \left( \sum_{i=1}^{m} \tau_a^i(x) + \sum_{i=1}^{n-m} \tau_b^i(x) \right) \| x \|^b_a.
\]

With the choice

\[
\delta = C \left( \sum_{i=1}^{m} \tau_a^i(x) + \sum_{i=1}^{n-m} \tau_b^i(x) \right)
\]

we conclude that \( P(\delta) = 0 \) implies \( x = 0 \) on \([a,b]\). But \( \dim A^\lambda = k \); hence there exist at least \( k \) perturbed eigenvalues \( \mu_{ab}^j \) (counting multiplicities) of (1.3) such that

\[
| \mu_{ab}^j - \lambda | \leq C \left( \sum_{i=1}^{m} \tau_a^i(x) + \sum_{i=1}^{n-m} \tau_b^i(x) \right)
\]

if \([a,b] \in R_o\). This completes the proof of the theorem.

Theorem 18 and (11.2) show in particular that if \( \lambda \) is a basic eigenvalue of multiplicity \( k \) there exist at least \( k \) perturbed eigenvalues \( \mu_{ab}^j \) (counting multiplicities) such that \( \mu_{ab}^j - \lambda \) when \( a,b \rightarrow \omega_-, \omega_+ \). To obtain the stronger result that exactly \( k \) perturbed eigenvalues \( \mu_{ab}^j \) satisfy (11.3) in theorem 18, we require the monotonicity property that the absolute value of the \( i \)-th eigenvalue of (10.1) is not larger than the absolute value of the \( i \)-th eigenvalue of (1.3). We then have the following theorem:
Theorem 19. If in addition to the hypotheses of theorem 18 the monotonicity property holds, then for every basic eigenvalue $\lambda$ of (10.1), of multiplicity $k$, there exists a rectangle $R_\varnothing$ and a constant $C$ on $R_\varnothing$, such that exactly $k$ perturbed eigenvalues $\mu_{ab}^j$ (counting multiplicities) of (1.3) satisfy

$$|\mu_{ab}^j - \lambda| \leq C \left( \sum_{i=1}^{m} \tau_a^i(x) + \sum_{i=1}^{n-m} \tau_b^i(x) \right)$$

whenever $[a,b] \in R_\varnothing$. There exist $k$ orthonormal eigenfunctions $x^j$ associated with $\lambda$ and $k$ orthonormal eigenfunctions $y_{ab}^j$ associated with $\mu_{ab}^j$ such that

$$\|x^j - y_{ab}^j\|^b_a \leq C \left( \sum_{i=1}^{m} \tau_a^i(x) + \sum_{i=1}^{n-m} \tau_b^i(x) \right),$$

$j = 1, 2, \ldots, k$ whenever $[a,b] \in R_\varnothing$.

12. Uniform estimates and asymptotic variational formulae.

To obtain uniform estimates for $y_{ab}^j(s) - x^j(s)$ in theorem 19, additional restrictions are needed on the basic solutions $\varphi_j, j = 1, 2, \ldots, n$. In particular we shall require that all the basic solutions are bounded on $(\omega_-, \omega_+)$.

Lemma 6. Let $G_{ab}(s,t)$ be the Green's function for $k \cdot L_\varnothing$ associated with the boundary conditions of (0.6). Then the positive function $g_{ab}(s)$ defined by

$$(12.1) \quad g_{ab}^2(s) = \int_a^b |G_{ab}(s,t)|^2 k(t)dt$$
is uniformly bounded on $a \leq s \leq b$ provided $a \leq a_0, b_0 \leq b$.

**Proof.** We first construct the Green's function $G_{ab}(s,t)$ for $k \cdot L_0$ associated with the boundary conditions of (0.6) following the method used in lemma 5. Define a function $K_{ab}(s,t)$ of $(s,t)$ on $[a,b]$ by

$$K_{ab}(s,t) = \begin{cases} K(s,t), & a \leq t \leq s \leq b \\ 0, & a \leq s < t \leq b \end{cases}$$

where $K(s,t)$ is given by (7.4). It was shown in lemma 5 that for any $v \in H[a,b]$ the function $u$ given by

$$u(s) = \int_a^b K_{ab}(s,t)v(t)k(t)dt = \int_a^b K(s,t)v(t)k(t)dt$$

is such that $u^{(n-1)}(s)$ is absolutely continuous on $[a,b]$ and $u$ satisfies the differential equation $L_0 u = v$.

Let $G_{ab}(s,t)$ be the function of $(s,t)$ on $[a,b]$ defined by

$$G_{ab}(s,t) = K_{ab}(s,t) + \sum_{k=1}^n A_k \varphi_k(s)$$

where the $A_k$ are chosen in such a way that $G_{ab}(s,t)$ as a function of $s$ satisfies the boundary conditions of (0.6). Then clearly $G_{ab}(s,t)$ is the Green's function for $k \cdot L_0$ associated with the boundary conditions of (0.6). Applying these boundary conditions to (12.3) and using Cramer's rule we obtain that

$$A_k = A_k(t) = \frac{\Omega_{ab}(t)}{\Omega(a,b)}$$

where $\Omega_{ab}(t)$ denotes the determinant of the matrix obtained from $A(a,b)$ by replacing the $k$-th column by
Since \( \varphi_k \in H, k = 1, 2, \ldots, n \) we obtain immediately from (10.6) and (10.7) that there exists a constant \( C \) such that

\[
(12.5) \quad \| A_k(t) \|_a^b \leq C, \quad k = 1, 2, \ldots, n
\]

whenever \( a \leq a_o, b_o \leq b \).

It follows from (12.1) that

\[
(12.6) \quad g_{ab}(s) \leq \left( \int_a^s |G_{ab}(s,t)|^2 k(t) dt \right)^{1/2} + \left( \int_s^b |G_{ab}(s,t)| k(t) dt \right)^{1/2}.
\]

By (12.2), (12.3) and the triangle inequality we obtain that

\[
\left( \int_a^s |G_{ab}(s,t)|^2 k(t) dt \right)^{1/2} \leq \sum_{i, j=1}^n \gamma_{ij} \varphi_j(s) \| \varphi_i(t) \|_a^s + \sum_{j=1}^n \| \varphi_j(s) \| A_j(t) \|_a^s.
\]

But \( \varphi_j \) is bounded on \((w_-, w_+)\) and \( \varphi_j \in H \) for each \( j, j = 1, 2, \ldots, n; \) hence by (12.4) and (12.5) the first quantity on the right in (12.6) is uniformly bounded on \( a \leq s \leq b \) provided \( a \leq a_o, b_o \leq b \). A similar proof shows that the second integral on the right in (12.6) is also uniformly bounded on \( a \leq s \leq b \) provided \( a \leq a_o, b_o \leq b \). This gives the desired result. We now state a theorem which gives uniform estimates for the eigenfunctions of theorem 19.

**Theorem 20.** If in addition to the hypotheses of theorem 19, \( \varphi_j \) is bounded on \( w_- < s < w_+ \), \( j = 1, 2, \ldots, n \), then the eigenfunctions \( x_j^+ \) corresponding to \( \lambda \) and \( y_{ab}^j \) corresponding to \( \mu_{ab}^j \) of theorem 19 are such that
(12.7) \[ y_{ab}^j(s) = x_j(s) - f_j(s) + O\left(\sum_{i=1}^{m} \tau_a^i(x)\right) + O\left(\sum_{i=1}^{n-m} \tau_b^i(x)\right), \]

\[ j = 1, 2, \ldots, k, \]

where \( f_j(s) \) is the unique solution of the boundary problem

(12.8) \[ Lf = \ell_o f, \quad U_a^i f = U_a^i x_j, \quad i = 1, 2, \ldots, m, \]

\[ U_b^i f = U_b^i x_j, \quad i = 1, 2, \ldots, n-m. \]

**Theorem 21.** Under the hypotheses of theorem 20 the following variational formulae are valid for the eigenvalues \( \mu_{ab}^j \) and \( \lambda \) of theorem 19:

\[ \lambda - \mu_{ab}^j = [f_j x^j(b)] - [f_j x^j(a)] + (\ell_o - \lambda)(f_j^a, f_j^b) + \sum_{i=1}^{m} \tau_a^i(x) + \sum_{i=1}^{n-m} \tau_b^i(x)(f_j^a, f_j^b) 0(1) \]

as \( a, b \to \omega_- \omega_+ \) for \( j = 1, 2, \ldots, k \) where \( f_j^a \) is the unique solution of (12.8).

13. **One end perturbation problems.**

As special cases of the perturbation problem discussed in section 10-12 we shall consider problems where only one end is perturbed while the other end remains fixed. For this purpose we shall define an eigenvalue problem for \( L \) given by (0.1) on the interval \([a, \omega_+], \omega_- < a < \omega_+\).
Let $D[a, w_+]$ denote the set of all $z \in H[a, w_+]$ having the following properties:

(i) $z \in C^{n-1}[a, w_+]$ and $z^{(n-1)}$ is absolutely continuous on every closed bounded subinterval of $[a, w_+]$; and

(ii) $z$ satisfies the boundary conditions of (0.6) at $a$ and also the end conditions in (10.2) at $w_+$.

The eigenvalue problem

$$Lz = \nu z, \quad z \in D[a, w_+]$$

will be referred to as the semi-perturbed problem and may be regarded as intermediate to (10.1) and (1.3).

The basic assumptions regarding the singularities $w_-$ and $w_+$ are as before, i.e. we assume that $w_\pm$ are class 2 singularities for $L$ and that all basic solutions are in $H$. Also we assume that the complex number $\ell_0$, $\text{Im} \ell_0 \neq 0$, is not an eigenvalue of (13.1) for $a$ in some neighbourhood of $w_-$, say $w_- < a < a_0$. Note that an ordering of the basic solutions is not assumed; hence we suppose that $U^i_a, U^i_b$ have the limiting behaviour given by (10.3).

(a) **Comparison of problems (1.3) and (13.1).**

The eigenvalue problems (1.3) and (13.1) will be compared when $w_- < a \leq a_0$, a fixed, with (13.1) regarded as "basic" and (1.3) regarded as a perturbation of (13.1). We assume that at least one real eigenvalue $\nu$ of (13.1) exists possessing $k$ orthonormal eigenfunctions $z_j$. Let
\[ \tau^1_b(z) = \sum_{j=1}^{k} |U^1_b z_j|. \]

Then by (10.2) and (10.3) it follows that

(13.2) \[ \tau^i_b(z) = o(1) \]

as \( b \to \omega_+ \) for \( i = 1, 2, \ldots, n-m \).

Let \( A_\gamma \) denote the space spanned by \( z_j, j = 1, 2, \ldots, k \). Then for any \( z \in A_\gamma \) the function \( f \) defined on \([a, b]\) by

\[ f = z - \gamma G_{ab} z, \quad \gamma = \nu - t_0. \]

satisfies the boundary problem

\[ L^i_0 f = 0, \quad U^i_a f = 0, \quad i = 1, 2, \ldots, m, \]

\[ U^i_0 f = U^i_b z, \quad i = 1, 2, \ldots, n-m. \]

In terms of the basic solutions, \( f \) has the representation

\[ f = \sum_{j=1}^{n} K^j_{a, b} \varphi_j(s) \]

where \( K^j_{a, b} \) denotes the determinant of the matrix obtained from \( A(a, b) \) by replacing the \( j \)-th column by

\[ 0, 0, \ldots, 0, U^1_b z, U^2_b z, \ldots, U^{n-m}_b z. \]

Then an argument similar to that used in theorem 18 shows that there exists a constant \( C \) and an interval \( b_0 \leq b < \omega_+ \) such that

\[ \|f\|_a^b \leq C \left( \sum_{i=1}^{n-m} \tau^i_b(z) \right). \]
An application of lemma 1 now implies that at least \( k \) eigenvalues \( \mu_\nu^j \) (counting multiplicities) of (1.3) lie on the interval

\[
|\nu - \mu_\nu^0| \leq C \left( \sum_{i=1}^{n-m} \tau_\nu^i(z) \right)
\]

whenever \( b_0 \leq b < \omega_+ \). In particular (13.2) implies that there exist at least \( k \) eigenvalues \( \mu_\nu^0 \rightarrow \nu \) as \( b \rightarrow \omega_+ \).

If, in addition, we require that (13.1) satisfies the monotonicity property, then we obtain exactly \( k \) eigenvalues \( \mu_\nu^j \) (counting multiplicities) of (1.3) satisfying (13.3). In this case one also obtains orthonormal eigenfunctions \( z_\nu^j \) associated with \( \nu \) and \( y_\nu^j \) associated with the \( \mu_\nu^j \) such that

\[
\|y_\nu^j - z_\nu^j\|_a^b \leq C \left( \sum_{i=1}^{n-m} \tau_\nu^i(z) \right)
\]

whenever \( b_0 \leq b < \omega_+ \), \( j = 1, 2, \ldots, k \).

To obtain uniform estimates of \( y_\nu^j(s) - z_\nu^j(s) \) on \( a \leq s \leq b \) and variational formulae for the eigenvalues \( \mu_\nu^j \) as \( b \rightarrow \omega_+ \) we need the additional assumption that each \( \varphi_j(s), j = 1, 2, \ldots, n \) is bounded on some neighbourhood of \( \omega_+ \), say \( b_0 \leq s < \omega_+ \). This assumption implies that each \( \varphi_j \) is bounded on \([a, \omega_+]\). Then it is easily shown that the positive function \( g_{ab}(s) \) defined by (12.1) is uniformly bounded on \( a \leq s \leq b \), provided \( b_0 \leq b \). The proof of this is the same as that of lemma 6 except for obvious simplifications. The following uniform estimate of \( y_\nu^j(s) - z_\nu^j(s) \) on \( a \leq s \leq b \), \( b_0 \leq b \), is then a direct consequence of theorem 20:
\[ y_j^*(s) = z^*(s) - f_j^*(s) + \sum_{i=1}^{n-m} \tau_i^*(z), \ j = 1, 2, \ldots, k \]

where \( f_j^*(s) \) is the unique solution of the boundary problem

\[ L_0 f = 0, \quad U^i_a = 0, \ i = 1, 2, \ldots, m \]
\[ U^i_b = U^i_b z^j, \ i = 1, 2, \ldots, n-m. \]

Also we obtain as a consequence of theorem 21 the following variational formulae:

\[ v - \mu_j^b = [f_j^b z^j]^b - [f_j^b z^j]^a + (t_0 - \lambda)(f_j^b, f_j^b)_a \]
\[ + \sum_{i=1}^{n-m} \tau_i^b(z)(f_j^b, l)_a^b o(1) \]

as \( b \to \omega_+ \), \( j = 1, 2, \ldots, k \).

(b) **Comparison of problems (10.1) and (13.1).**

We now compare problems (10.1) and (13.1), with (13.1) regarded as a perturbation of (10.1). Since the "perturbed" problem (13.1) is a singular problem (on the half open interval \([a, \omega_+]\)) special conditions need to be imposed to obtain the desired estimates.

Let \( Ux \) denote the vector

\[ [U^1_a x, \ldots, U^m_a x, [x x_{m+1}]^a(\omega_+), \ldots, [x x_n]^a(\omega_+)] \]

We shall assume that there exists a number \( a_0, \omega_- < a_0 < \omega_+ \), such that the set of boundary conditions \( Ux = 0 \) is self-adjoint
whenever \( w_- < a \leq a_0 \), i.e. we require that there exist boundary forms \( U_c, U_c^+ \) of rank \( n \) (See [3], Chapter 11) such that

\[
(13.4) \quad [u v](w_+) - [u v](a) = U_u \cdot U_c^+ v + U_c u \cdot U v
\]

holds identically in \( u \) and \( v \) (here \( \cdot \) represents the "dot" product). Then (0.5) and (13.4) clearly imply that for any pair \( x, y \in D[a, w_+] \) such that \( Lx, Ly \in H[a, w_+] \)

\[
(Lx, y)_a - (x, Ly)_a = 0.
\]

We shall also suppose that (13.1) has real eigenvalues only and a corresponding set of eigenfunctions complete in \( H[a, w_+] \). This will be needed in order that lemma 1 will apply. Since \( w_- \) is a class 2 singularity for \( L \) we require that the boundary operators \( U^+ \) have the limiting behaviour given by (10.3) as \( a \to w_+ \).

Let \( G_a(s, t) \) denote the Green's function for \( kL_o \) associated with the boundary conditions of (0.6) at \( a \) and the end conditions (10.2) at \( w_+ \) and let \( G_a \) be the linear transformation on \( H[a, w_+] \) defined by

\[
G_a u(s) = \int_a^{w_+} G_a(s, t) u(t) k(t) dt, \quad u \in H[a, w_+].
\]

Let \( \lambda \) be an eigenvalue of (10.1) possessing \( k \) orthonormal eigenfunctions \( x_j, j = 1, 2, \ldots, k \) and let \( A_\lambda \) denote the space spanned by these eigenfunctions \( x_j \). For any normalized \( x \in A_\lambda \) we define a function \( f \) on \( [a, w_+] \) by

\[
(13.5) \quad f(s) = x(s) - \gamma G_a x(s) \quad \text{where} \quad \gamma = \lambda - \ell_o.
\]
Proceeding as in section 11 we obtain the following representation of (13.5) in terms of the basic solutions:

\[ f(s) = \sum_{k=1}^{n} \frac{\Omega^k(a)}{\Omega(a)} \varphi_k(s) \]

where \( \Omega(a) \) is the determinant of the matrix \( A(a) \) defined by

\[ A(a) = (A_{ij}) \text{ where} \]

\[ A_{ij} = \begin{cases} U_{a}^i \tau_j & \text{if } i = 1,2,\ldots,m; j=1,2,\ldots,n \\ [\varphi_j \chi_i](w_+) & \text{if } i = m+1,\ldots,n; j=1,2,\ldots,n \end{cases} \]

and where \( \Omega^k(a) \) is the determinant of the matrix obtained from \( A(a) \) by replacing the \( k \)-th column by

\[ U_{a}^1 x, \ldots, U_{a}^m x, [x\chi_{m+1}](w_+), \ldots, [x\chi_n](w_+) \].

From (10.3) and the fact that \( \Omega \neq 0 \) follows that there exists a point \( a_0, w_- < a_0 < w_+ \) (which may be pre-supposed to be the original choice) such that \( \Omega(a) \) is bounded away from 0 on \( w_- < a \leq a_0 \). Also each element of \( A(a) \) is bounded on \( w_- < a \leq a_0 \) by (10.4) hence

\[ \Omega^k(a) = O\left( \sum_{i=1}^{m} \tau_a^i(x) \right) \]

whenever \( w_- < a \leq a_0 \). Since \( \varphi_k \in H, k = 1,2,\ldots,n \) one deduces that there exists an interval \( (w_-, a_0] \) and a constant \( C \) such that for any normalized eigenfunction \( x \in A_\lambda \)

\[ \|f\|_a \leq C \left( \sum_{i=1}^{m} \tau_a^i(x) \right) \]

whenever \( w_- < a \leq a_0 \).
Since (13.1) has a set of eigenfunctions complete in $H[a, w_+]$ and since $\sum_{i=1}^{m} \tau_i^a(x) \to 0$ as $a \to w_-$ by (10.2), (10.3) and (11.1) we may apply lemma 1 to deduce the existence of an interval $(a, a_0]$ and a constant $C$ on this interval such that at least $k$ eigenvalues $\nu^j_a$ (counting multiplicities) of (13.1) satisfy

\[(13.6) \quad |\lambda - \nu^j_a| \leq C \left( \sum_{i=1}^{m} \tau_i^a(x) \right)\]

whenever $w_- < a \leq a_0$. If in addition the monotonicity property of (10.1) (given in section 11) is assumed, then one can show that exactly $k$ eigenvalues $\nu^j_a$ (counting multiplicities) of (13.1) satisfy (13.6). Under this assumption one also obtains normalized eigenfunctions $x^j$ corresponding to $\lambda$ and $z^j$ corresponding to $\nu^j_a$ such that

$$\|x^j - z^j\|_a \leq C \left( \sum_{i=1}^{m} \tau_i^a(x) \right), \quad j = 1, 2, \ldots, k$$

whenever $w_- < a \leq a_0$.

To obtain uniform estimates for $z^j_a - x^j$ on $[a, b]$ and variational formulae we shall need the stronger assumptions that $\varphi_j(s), j = 1, 2, \ldots, n$ is bounded on $w_- < s < w_+$. The explicit form of $G_a(s, t)$ can then be obtained by applying the boundary conditions $U_i^a x = 0, i = 1, 2, \ldots, m$ and the end conditions $[x x_1](w_+) = 0, i = m+1, \ldots, n$ to

$$G_a(s, t) = K_a(s, t) + \sum_{j=1}^{n} A_j \varphi_j(s)$$

as a function of $s$ where $K_a(s, t)$ is given by (12.2) with $b$
replaced by \( w_+ \). Since \( \varphi_j \in H \) and \( \varphi_j \) is bounded on \((w_-, w_+)\), \( j = 1, 2, \ldots, n \), a proof similar to that of lemma 6 shows that the positive function \( g_a(s) \) defined by

\[
g_a^2(s) = \int_a^{w_+} |G_a(s, t)|^2 k(t) \, dt
\]
is uniformly bounded on \( a < s < w_+ \), provided \( a < a_0 \).

Then one may obtain the following uniform estimates on 
\( a < s < w_+ \), \( a < a_0 \):

\[
z_j^a(s) = x_j(s) - f_j^a(s) + \left( \sum_{i=1}^{m} \tau_i^a(x) \right), \quad j = 1, 2, \ldots, k
\]
where \( f_j^a(s) \) is the unique solution of the boundary problem

\[
L_0 f = 0, \quad U_a^i f = U_a^i x_j, \quad i = 1, 2, \ldots, m,
\]
\[
[f x_i^a](w_+) = 0, \quad i = m+1, \ldots, n.
\]

The following variational formulae can also be shown to be valid as \( a \to w_- \):

\[
\lambda - \lambda_a^j = [f_a^j x_j^a](w_+) - [f_a^j x_j^a](a) + (\tau_0 - \lambda)(f_a^j, f_a^j)_a
\]
\[
+ \left( \sum_{i=1}^{m} \tau_i^a(x) \right) (f_a^j, 1)_a O(1),
\]
\( j = 1, 2, \ldots, k. \)

14. **The second order case**: \( w_-, w_+ \) class 2 limit circle singularities.

We consider as a particular case of (0.1) the operator
L = L_2 defined in section 5. The points \( w_- \) and \( w_+ \) are in general limit circle singularities for \( L \); the possibility that they be \( \pm \infty \) is not excluded. The notations (0.3) and (0.4) will be adhered to; in particular (0.4) takes the form

\[
(14.1) \quad [xy](s) = p(s)(x(s)y'(s) - x'(s)y(s)).
\]

The basic problem corresponding to (10.1) is described as follows: choose a complex number \( \iota_0 \), \( \text{Im} \iota_0 \neq 0 \), and let \( L_0 \) be the differential operator \( L - \iota_0 \). A theorem of Weyl ([9], pp. 35-44) states that there exist linearly independent solutions \( \varphi_1, \varphi_2 \in H \) of \( L_0 x = 0 \) such that

\[
(14.2) \quad [\varphi_1 \varphi_1](w_-) = [\varphi_2 \varphi_2](w_+) = 0, \quad [\varphi_1 \varphi_2](s) = 1,
\]

\( w_- < s < w_+ \).

For our purposes here we shall assume only the boundedness of \( \varphi_1 \) and \( \varphi_2 \) on \( (w_-, w_+) \). A condition like (5.4) regarding the "ordering" of \( \varphi_1(s) \) and \( \varphi_2(s) \) as \( s \to w_\pm \) is not assumed.

Since \( [\varphi_2 \varphi_1](w_-) \neq 0 \) and \( [\varphi_1 \varphi_2](w_+) \neq 0 \) by (14.1) and (14.2) we can choose \( x_1 \) and \( x_2 \) (described in section 10) to be \( \varphi_1 \) and \( \varphi_2 \) respectively. Let \( D_2 \) denote the set of all \( x \in H \) which have the following properties:

(i) \( x \) is differentiable on \( (w_-, w_+) \) and \( x' \) is absolutely continuous on every closed bounded subinterval of \( (w_-, w_+) \); and

(ii) \( [x\varphi_1](w_-) = [x\varphi_2](w_+) = 0 \).
The basic eigenvalue problem

\[(14.3) \quad Lx = \lambda x, \quad x \in D_2\]

is known to have a denumerable set of real eigenvalues \(\{\lambda^i\}\) and a corresponding set of eigenfunctions \(\{x_i\}\) complete in \(H\) \((i = 1, 2, \ldots)\). It is also known that each eigenvalue \(\lambda^i\) has multiplicity 1. The perturbed problem on \([a, b]\) corresponding to \((1.3)\) is the regular self-adjoint eigenvalue problem given by \((5.3)\).

According to \((10.3)\) we obtain convergence of the eigenvalues and eigenfunctions of \((5.3)\) to those of \((14.3)\) if we require that \(U_a, U_b\) have the limiting behaviour

\[(14.4) \quad \begin{cases} U_a y = [y\varphi_1](a)[1 + o(1)] & \text{as } a \to -\infty \\ U_b y = [y\varphi_2](b)[1 + o(1)] & \text{as } b \to +\infty \end{cases} \]

for every differentiable function \(y\).

Let \(\lambda\) be an eigenvalue for \((14.3)\) and \(x\) the corresponding normalized eigenfunction. Since exactly one boundary condition is used at each point \(a\) and \(b\) in \((5.3)\) we can replace the quantities

\[\sum_{i=1}^{m} \tau^1_a(x), \quad \sum_{i=1}^{n-m} \tau^1_b(x)\]

in \((11.1)\) by \(|U_a x|\), \(|U_b x|\) respectively. Finally a monotonicity property for \((14.3)\) is known to hold \([4], [10]\), which leads to the following theorem:
Theorem 22.

(i) Let $w_-$ and $w_+$ be limit circle singularities for $L_2$. If
the boundary operators $U_a$, $U_b$ satisfy (14.4) then for each
eigenvalue $\lambda$ of (14.3) there exists a rectangle $R_o$ and a constant
$C$ on $R_o$ such that a unique eigenvalue $\mu_{ab}$ of (5.3) lies in the
interval

$$|\mu_{ab} - \lambda| \leq C(|U_a x| + |U_b x|)$$

whenever $[a,b] \in R_o$. There exist normalized eigenfunctions
$y_{ab}$ and $x$ associated with $\mu_{ab}$ and $\lambda$ respectively such that the
estimate

$$\|y_{ab} - x\|^b_a \leq C(|U_a x| + |U_b x|)$$

is valid on $R_o$.

(ii) If, in addition to the hypotheses of (i), $\varphi_1$ and $\varphi_2$ are
bounded on $(w_-, w_+)$ then the following uniform estimate is valid
for $a \leq s \leq b$, $[a,b] \in R_o$;

$$y_{ab}(s) = x(s) - f(s) + O(|U_a x| + |U_b x|)$$

where $f(s)$ is the unique solution of the boundary problem

$$L_0 f = 0, \quad U_a f = U_a x, \quad U_b f = U_b x.$$ 

(iii) Under the hypotheses of (ii) the following variational
formula is valid as $a, b \to w_-, w_+$;

$$\lambda - \mu_{ab} = [fx](b) - [fx](a) + (t_o - \lambda)(f,f)^b_a$$

$$+ (|U_a x| + |U_b x|)(f,l)^b_a 0(1).$$
CHAPTER III
EXAMPLES

15. Preliminary remarks and lemmas.

In this chapter examples will be given to illustrate the theory of chapter I. The operators to be considered will all be of the fourth order with class 1 singularities. Examples of problems involving second order operators are found in [13]. We shall need the following lemmas:

Lemma 7. Let \( \varphi_1, \varphi_2, \ldots, \varphi_n \) be linearly independent solutions of class \( C^{2n} \) of \( Lx = \lambda x \) where \( L \) is the operator (0.1) and \( \lambda \neq 0 \), and let \( \chi_1, \chi_2, \ldots, \chi_n \) be linearly independent solutions of class \( C^{2n} \) of \( Lx = -\lambda x \). Then \( \varphi_1, \varphi_2, \ldots, \varphi_n, \chi_1, \chi_2, \ldots, \chi_n \) are linearly independent solutions of \( LLx = \lambda^2 x \).

Proof: For \( i = 1, 2, \ldots, n \), \( \varphi, \chi \) are solutions of \( LLx = \lambda^2 x \) since \( LL\varphi = L(\varphi_1) = \lambda^2 \varphi_1 \) and \( LL\chi = L(-\lambda\chi_1) = \lambda^2 \chi_1 \).

If \( \sum_{i=1}^{n}(A_i \varphi_i + B_i \chi_i) = 0 \), then

\[
\sum_{i=1}^{n}(A_i L\varphi_i + B_i L\chi_i) = \lambda \sum_{i=1}^{n}(A_i \varphi_i - B_i \chi_i) = 0.
\]

Hence \( \sum_{i=1}^{n} A_i \varphi_i = \sum_{i=1}^{n} B_i \chi_i = 0 \), which implies \( A_i = B_i = 0 \), \( i = 1, 2, \ldots, n \) by hypothesis. Thus \( \varphi_1, \varphi_2, \ldots, \varphi_n, \chi_1, \chi_2, \ldots, \chi_n \) are linearly independent.

Lemma 8. If \( \varphi_1, \varphi_2, \ldots, \varphi_n \) are linearly independent solutions of \( Lx = 0 \) of class \( C^{2n} \) and if \( \chi_1, \chi_2, \ldots, \chi_n \) are corresponding solutions of class \( C^{2n} \) of \( Lx = \varphi_1 \), \( i = 1, 2, \ldots, n \), then
\( \varphi_1, \varphi_2, \ldots, \varphi_n, \lambda_1, \lambda_2, \ldots, \lambda_n \) are linearly independent solutions for \( LLx = 0 \).

16. The modified Hermite operator.

As an example to illustrate the theory of sections 1–4, consider the operator \( L = L_0L_0 \) on the interval \((-\infty, \infty)\) where \( L_0 \) is the modified Hermite operator given by

\[
L_0x = -x'' + (s^2 + 2)x.
\]

Then

\[
Lx = x^{(4)} - 2(s^2 + 2)x'' - 4sx' + (s^4 + 4s^2 + 2)x
\]
or, in self-adjoint form,

\[
Lx = (x'')^n - 2[(s^2 + 2)x']' + (s^4 + 4s^2 + 2)x.
\]

By (0.4) one obtains for \( L \) that

\[
(16.1) \quad [xy](s) = x^{(3)}(s) y(s) - x''(s) y'(s) + x'(s) y''(s)
- x(s) y^{(3)}(s) + 2(s^2 + 2)[x(s) y'(s) - x'(s) y(s)].
\]

It is known ([15], pp. 347–348) that there exist linearly independent solutions \( u \) and \( v \) of \( L_0x = 0 \) such that

\[
(16.2) \quad u(s) \sim Ce^{-s^2/2}, \quad v(s) \sim Ce^{-s^2/2} s^{-3/2}
\]
as \( s \to -\infty \), and

\[
(16.3) \quad u(s) \sim Ce^{-s^2/2} s^{-3/2}, \quad v(s) \sim Ce^{s^2/2} s^{1/2}
\]
as \( s \to \infty \).
Since the coefficient of the first derivative term is zero for
\( L_0 \), it follows that the Wronskian for \( L_0 x = 0 \) is constant and
hence we can assume without loss of generality that

\[
(16.4) \quad u(s)v'(s) - u'(s)v(s) = 1.
\]

Obviously \( u \) and \( v \) are linearly independent solutions of \( Lx = 0 \).
By lemma 8 two other linearly independent solutions may be found
by solving the differential equations \( Lx = u \) and \( Lx = v \). The
respective solutions are

\[
(16.5) \quad u(s) \int_0^s u(t)v(t)dt - v(s) \int_0^s [u(t)]^2 dt,
\]

and

\[
(16.6) \quad -v(s) \int_0^s u(t)v(t)dt + u(s) \int_0^s [v(t)]^2 dt.
\]

Let \( \varphi_1 \) denote the function (16.5), \( \varphi_2 = u \), \( \varphi_3 = v \) and let \( \varphi_4 \)
denote the function (16.6). Then from (16.2)-(16.6) one obtains
that

\[
(16.7) \quad \varphi_1 \sim C e^{s^{2/2} |s|^{1/2} \log |s|}, \quad \varphi_4 \sim C e^{-s^{2/2} |s|^{-3/2} \log |s|}
\]
as \( s \to -\infty \), and

\[
(16.8) \quad \varphi_1 \sim C e^{-s^{2/2} s^{-3/2} \log s}, \quad \varphi_4 \sim C e^{s^{2/2} s^{1/2} \log s}
\]
as \( s \to \infty \). Clearly for any number \( c, -\infty < c < \infty \),

\[
\varphi_1, \varphi_2 \in H[c, \infty), \quad \varphi_1, \varphi_2 \notin H(-\infty, c],
\]

\[
\varphi_3, \varphi_4 \in H(-\infty, c], \quad \varphi_3, \varphi_4 \notin H[c, \infty)
\]

and condition (1.1) is satisfied. By the asymptotic behaviour
of \( \varphi_1, i = 1, 2, 3, 4 \) at \( \pm \infty \) one easily sees that the differential
equation \( Lx = 0 \) has no solution in \( H = H(-\infty, \infty) \). Hence \( t_0 \) is replaced by 0 and the solutions \( \varphi_i, \ i = 1, 2, 3, 4 \) are regarded as the basic solutions.

The basic problem on \( (-\infty, \infty) \) is the eigenvalue problem

\[
(16.9) \quad Lx = \lambda x \, , \, x \in D
\]

where \( D \) is the set of all \( x \in H \) such that \( x \in C^3(-\infty, \infty) \) and \( x^{(3)} \) is absolutely continuous on every closed bounded subinterval of \( (-\infty, \infty) \). By consideration of the operator \( L_0 \) on \( (-\infty, \infty) \) one can deduce that the eigenvalue problem (16.9) has eigenvalues \( \lambda_n = (2n + 3)^2, \ n = 0, 1, 2, \ldots \), and corresponding normalized eigenfunctions

\[
(16.10) \quad x_n(s) = \pi^{-1/4} 2^{-(n+1)/2} (n!)^{-1} \exp(-s^2/2) H_n(s),
\]

\( n = 0, 1, \ldots \), where \( H_n(s) \) denotes a Hermite polynomial. The well-known asymptotic behaviour of \( x_n(s) \) as \( s \to \pm \infty \) is

\[
(16.11) \quad x_n(s) \sim \pi^{-1/4} 2^{(n+1)/2} (n!)^{-1/2} s^n \exp(-s^2/2).
\]

We now show that \( \lambda_n = (2n+3)^2, \ n = 0, 1, \ldots \), are the only eigenvalues of (16.9). Since \( L \) is formally self-adjoint, all eigenvalues of (16.9) are necessarily real. Let \( n \) be any complex number such that \( n \neq 0, 1, 2, \ldots \). Then it is known ([15] pp. 347-348) that \( L_0 x = (2n+3)x \) has linearly independent solutions \( \psi_1, \psi_2 \) such that

\[
(16.12) \quad \psi_1 \sim C e^{s^2/2} s^{-n-1}, \quad \psi_2 \sim C e^{-s^2/2} s^n
\]

as \( s \to \infty \), and
(16.13) \( \psi_1 \sim \text{e}^{-s^2/2}|s|^n, \quad \psi_2 \sim \text{e}^{s^2/2}|s|^{-n-1} \)
as \( s \to -\infty \).

Similarly the equation \( L_0 x = -(2n+3)x \) has linearly independent solutions \( x_1, x_2 \) such that
\[
(16.14) \quad x_1 \sim \text{e}^{-s^2/2}s^{-n-3}, \quad x_2 \sim \text{e}^{s^2/2}s^{n+2}
\]
as \( s \to -\infty \), and
\[
(16.15) \quad x_1 \sim \text{e}^{s^2/2}s^{n+2}, \quad x_2 \sim \text{e}^{-s^2/2}s^{-n-3}
\]
as \( s \to -\infty \). By lemma 7, \( \psi_1, \psi_2, x_1, x_2 \) are linearly independent solutions of \( Lx = (2n+3)^2x \) and hence by (16.12) - (16.15) one can deduce that no solution of \( Lx = (2n+3)^2x \) is in \( H \). This implies that a number \( k \) is an eigenvalue of (16.9) if and only if \( k = (2n+3)^2 \) for some non-negative integer \( n \). A similar procedure shows that these eigenvalues \( \lambda_n = (2n+3)^2, \ n = 0, 1, \ldots \), all have multiplicity 1.

Let \( D[a,b] \) be the set of all \( y \in H[a,b], \ -\infty < a < b < \infty \) such that

(i) \( y \in C^3[a,b] \) and \( y^{(3)}(s) \) is absolutely continuous on \([a,b]\); 
(ii) \( Ly \in H[a,b] \); and 
(iii) \( y \) satisfies the boundary conditions 
\[
U^1_ey = U^1_by = 0, \quad i = 1, 2
\]
where
\[
(16.16) \quad U^1_sy = y^{(3)}(s) - 2(s^{2+2})y'(s) \
U^2_sy = y''(s) + s^3y'(s).
\]
Then the perturbed problem corresponding to (1.3) is the self-adjoint eigenvalue problem

(16.17) \[ Ly = \mu y, \quad y \in D[a,b]. \]

By actual calculation using (16.2) - (16.8) one can show that

(16.18) \[
\begin{align*}
\eta_a(1,2) & \sim C e^{2a} a^6; \\
\eta_a(2,3) & \sim a^4; \\
\eta_a(1,3) & \sim a^4 \log|a|; \\
\eta_a(2,4) & \sim a^4 \log|a|; \\
\eta_a(1,4) & \sim a^4 (\log|a|)^2; \\
\eta_a(3,4) & \sim C e^{-a} a^4
\end{align*}
\]
as \( a \to -\infty \), and

(16.19) \[
\begin{align*}
\eta_b(1,2) & \sim C e^{-2b} b^4; \\
\eta_b(2,3) & \sim b^4; \\
\eta_b(1,3) & \sim b^4 \log b; \\
\eta_b(2,4) & \sim b^4 \log b; \\
\eta_b(1,4) & \sim b^4 (\log b)^2; \\
\eta_b(3,4) & \sim C b^2 b^6
\end{align*}
\]
as \( b \to -\infty \). Also, it is easily verified that

(16.20) \[
\begin{align*}
C_1(s) &= \varphi_3(s); \\
C_3(s) &= -\varphi_1(s); \\
C_2(s) &= \varphi_4(s); \\
C_4(s) &= -\varphi_2(s).
\end{align*}
\]

By (16.10) and (16.16) one obtains for the eigenfunction \( x_n \) of (16.9) that

(16.21) \[
\begin{align*}
U_1^s x_n & \sim x(s)[s^3 + (n+7)s], \\
U_2^s x_n & \sim x(s)[-s^4 + (n+1)s^2]
\end{align*}
\]
as \( s \to \pm \infty \) and hence that conditions (2.11) and (2.12) are satisfied with
\( (16.22) \quad \Theta_a = O[e^{-a^{2/2}} |a|^{n-1/2} \log |a|], \quad a \to -\infty \),
\( \Theta_b = O[e^{-b^{2/2}} b^{n-1/2} \log b], \quad b \to \infty. \)

Also by (16.2) - (16.8), (16.18), (16.19) and the asymptotic behaviour of the \( \varphi_i \), \( i = 1,2,3,4 \) it is easily seen that conditions (2.7) - (2.10), (3.3) - (3.7) are satisfied if we choose

\[
h(s) = \begin{cases} 
|s|^{1/2} & \text{if } s \leq -1, \quad l \leq s \\
1 & \text{if } -1 < s < 1
\end{cases}
\]

and \( 1_o = 2, \quad 1'_o = 1, \quad j_o = 4, \quad j'_o = 3. \)

From (16.1) and the fact that

\[
U^i_a f = U^i_a x, \quad U^i_b f = U^i_b x
\]

\( i = 1,2, \) one can easily verify that

\[
(16.23) \quad [fx](b) - [fx](a) = U^1_a [f(a) - x(a)] + U^2_a [x'(a) - f'(a)]
\]

\[
+ U^1_b [x(b) - f(b)] + U^2_b [f'(b) - x'b].
\]

Then using the representation (2.21) of \( f(s) \) and (16.2) - (16.8), (16.16) in (16.23), it can be verified that for \( x = x_n \)

\[
(16.24) \quad [fx](b) - [fx](a) \sim 2^3(n + 3)\left[a[x(a)]^2 - b[x(b)]^2\right]
\]

as \( a, b \to -\infty, \infty. \)
We now show that (16.9) satisfies the monotonicity property of section 2, i.e. that the \( j \)-th eigenvalue \( \mu_j = \mu_{ab}^j \) of (16.17) is not less than the \( j \)-th eigenvalue \( \lambda_j \) of (16.9) \( j = 0, 1, \ldots \). Let \( D^* \) be the set of all \( x \in D \) such that \( Lx \in H \) and

\[
(16.25) \quad \lim_{s \to \pm \infty} x^{(1)}(s) B_s^{i+1}[x] = 0, \quad i = 0, 1
\]

where

\[
B_s^1[x] = x^{(3)}(s) - 2(s^2 + 2)x'(s), \\
B_s^2[x] = x''(s).
\]

Then by (16.10) every eigenfunction \( x \) of (16.9) is in \( D^* \).

Let

\[
(16.26) \quad \|x\|_a^b = \int_a^b [(x'')^2 + 2(s^2 + 2)(x')^2 + (s^4 + 4s^2 + 2)x^2] ds; \\
\|x\| = \|x\|_{-\infty}^{\infty}.
\]

We shall show that there exists \( x_0 \in D^* \) such that

\[
\frac{\|x_0\|^2}{\|x_0\|} = \inf_{x \in D^*} \frac{\|x\|^2}{\|x\|^2}
\]

and that \( x_0 \) is in fact the eigenfunction of (16.9) corresponding to the smallest eigenvalue \( \lambda_0 \).

Obviously \( L \) is positive-bounded-below on \( D^* \) since for any \( x \in D^* \) we obtain by the integration-by-parts formula that

\[
(Lx, x) = \|x\|^2 + \lim_{a \to -\infty} \left( x(s)B_s^1[x] - x'(s)B_s^2[x]\right)_{a}^{b}
\]

and hence by (16.25) and (16.26) that

\[
(16.27) \quad (Lx, x) = \|x\|^2 \geq 2\|x\|^2.
\]
We next assert that any infinite set \( S \subset D^* \) which is bounded in the \( \| \cdot \| \) norm has a convergent subsequence in the \( L^2 \) norm. Let \( S \) be any infinite set of functions \( x \in D^* \) such that \( \|x\|^2 \leq C \) for some positive constant \( C \). Then for any positive number \( a \) and any \( x \in S \)

\[
\int_a^\infty |x(s)|^2 ds \leq \frac{1}{a^2} \int_a^\infty s^2 |x(s)|^2 \leq \frac{1}{a^2} \|x\|^2 \leq \frac{C}{a^2}.
\]

Let \( \epsilon > 0 \) be given and set \( a = \sqrt{6C/\epsilon} \). Then

(16.28) \[
\int_a^\infty |x(s)|^2 ds \leq \frac{C}{a^2} = \frac{\epsilon}{6}.
\]

Similarly

(16.29) \[
\int_{-a}^a |x(s)|^2 ds \leq \frac{\epsilon}{6}.
\]

For any \( x \in S \), \( x(s) = x(a) + \int_a^s v(t) dt \), where \( v = x' \). Hence by (16.26), \( \|v\| \leq C \).

Since

\[
|x(s) - x(r)| = \left| \int_r^s v(t) dt \right| \leq \sqrt{s-r} \|v\| \leq C\sqrt{s-r},
\]

\( S \) is equicontinuous on \(-a \leq r, s \leq a\) where \( a = \sqrt{\frac{6C}{\epsilon}} \). Also \( S \) is uniformly bounded on this interval, since

\[
|x(s)| \leq \sqrt{\frac{C}{2a}} + C\sqrt{2a}.
\]

By Ascoli's theorem there exists a uniformly convergent subsequence \( \{x_n\} \) on \([-a, a]\), which implies that there exists an integer \( N \) such that

\[
\int_{-a}^a |x_m(s) - x_n(s)|^2 ds < \frac{\epsilon}{3}
\]

provided \( n, m > N \). But (16.28) and (16.29) are independent of \( x; \)
so for all \( n, m, \)
\[
\int_{a}^{\infty} |x_m - x_n|^2 \, dx \leq \frac{\epsilon}{3}, \quad \int_{-\infty}^{a} |x_m - x_n|^2 \, ds \leq \frac{\epsilon}{3}.
\]

Thus
\[
\int_{-\infty}^{\infty} |x_m - x_n|^2 \, ds = \|x_m - x_n\|^2 < \epsilon,
\]
provided \( m, n > N. \) Since \( H \) is complete there exists \( x \in H \) such that \( \|x_n - x\| \to 0, \) as \( n \to \infty. \)

By theorem 3 ([7], pages 222-226) it follows that the eigenfunctions \( x_n, \) \( n = 0, 1, 2, \ldots \) of (16.9) form a system which is complete with respect to both the \( || \) norm and the \( L^2 \) norm. Further,
\[
(16.30) \quad \lambda_{o} = \frac{\|x_{o}\|^2}{\|x_{o}\|^2} \leq \frac{\|x\|^2}{\|x\|^2}
\]
for all \( x \in D^{*}, \) and
\[
(16.31) \quad \lambda_{n} = \frac{\|x_{n}\|^2}{\|x_{n}\|^2} \leq \frac{\|x\|^2}{\|x\|^2}
\]
for all \( x \in D^{*} \) such that
\[
(x, x_i) = 0, \quad i = 0, 1, 2, \ldots, n-1.
\]

Let \( G \) be the set of all \( x \in H \) satisfying (16.25) and such that
\[(i) \quad Lx \in H\]
\[(ii) \quad \text{For } i = 0, 1, 2, 3, 4, \ x^{(i)}(s) \text{ is continuous on } -\infty < s < \infty \text{ except possibly at a finite set of points; at points of discontinuity } x^{(i)}(s^{\pm}) \text{ exists (and is finite) for } i = 0, 1, 2, 3, 4.\]

By completeness of the eigenfunctions \( x_n \) it follows that (16.30)
and (16.31) are valid for all \( x \in \Omega \) as well.

Let \( \mu_0 \) be the smallest eigenvalue of (16.17) and \( y_0 \) a corresponding normalized eigenfunction. Extend \( y_0 \) to a function \( y_0^* \in \Omega \) by letting

\[
y_0^*(s) = \begin{cases} 
y(s) & \text{if } \alpha \leq s \leq \beta \\
0 & \text{if } -\infty < s < \alpha \text{, } \beta < s < \infty.
\end{cases}
\]

Then the integration-by-parts formula and (16.16) and (16.26) lead to

\[
\mu_0 = (Ly_0, y_0)_a^b = \left[ y_0^b_a + \left( s^3 y_0'(s) \right)_a^b \right] \geq \left\| y_0^* \right\|_{\Omega}
\geq \inf_{x \in \Omega} \frac{\|x\|_2^2}{\|x\|_2^2} = \lambda_0.
\]

Thus \( \mu_0 \geq \lambda_0 \). A simple proof by induction shows that \( \mu_i \geq \lambda_i \), \( i = 1, 2, \ldots \).

Theorem 5 and (16.24) yield the following variational formula for the eigenvalues \( \mu_n = \mu_{ab}^n \) of (16.17):

\[
\mu_{ab}^n \sim (2n + 3)^2 + (2n+3)\pi^{-1/2}2n^4(n!)^{-1}\left[ b^{2n+1}e^{-b^2} - a^{2n+1}e^{-a^2} \right]
\]

as \( a, b \to -\infty, \infty \) for \( n = 0, 1, 2, \ldots \).

17. Example 2.

The following example is derived from Bessel's equation and will illustrate the material in section 6. Let \( L \) be the fourth order differential operator defined on the half-open interval \((0,1]\) by \( Lx = L_0L_0x \) where
\[
L_0 x = x'' + \frac{1}{s}x' - \frac{n^2}{s^2}
\]

n fixed, \(n \geq 3\). Then

\[L x = \frac{1}{s} \left[ \left( \frac{2n^2 + 1}{s} \right) x' \right]' + \left( \frac{n^4 - 4n^2}{s^2} \right) x.\]

We have \(p_0(s) = s\), \(k(s) = s\) and from (0.4)

\[\begin{align*}
(17.2) \ [uv] &= \frac{2n^2 + 1}{s} (u'v'' - u''v) + (u''v'' - u''v') \\
&\quad + s(u^3v'' - u''v'' + u''v - uv'(3)).
\end{align*}\]

Let \(D(0,1]\) be the set of all \(x \in H(0,1]\) such that

(i) \(x \in C^3(0,1]\) and \(x(3)\) is absolutely continuous on every closed subinterval of \((0,1]\); and

(ii)

\[(17.3) \quad x(1) = x'(1) = 0.\]

Then the basic problem is the eigenvalue problem

\[(17.4) \quad L x = \lambda x, \quad x \in D(0,1].\]

The differential equation \(L x = 0\) has the following linearly independent solutions:

\[(17.5) \quad \varphi_1(s) = s^{-n}; \quad \varphi_3(s) = s^n;\]

\[\varphi_2(s) = s^{-n+2}; \quad \varphi_4(s) = s^{n+2}.\]

From (17.3) it follows easily that \(0\) is not an eigenvalue of (17.4). Hence we choose \(\tau_0 = 0\) and \(\varphi_i, \ i = 1,2,3,4\) as the basic solutions. By constructing the Green's function for \(L\) (using (17.5) associated with the boundary conditions (17.3), one can easily
verify that (17.4) has a countable set of real eigenvalues and a corresponding set of orthonormal eigenfunctions complete in \( H(0,1) \). Since 0 is a regular singularity for L, one obtains by consideration of the indicial equation ([3], pp. 122-127) that each eigenfunction \( x \) of (17.4) satisfies

\[
(17.6) \quad \lim_{s \to 0} x^{(1)}(s) = 0, \quad i = 0, 1, 2;
\]
\[
\lim_{s \to 0} x^{(3)}(s) = C.
\]

Then since

\[
(Lx, x)_0^1 = \int_0^1 \left\{ [s x'']^2 + \left( \frac{2n^2 + 1}{s} \right) [x']^2 + \left( \frac{4n^2 - 4n^2}{s^2} \right) x^2 \right\} ds
\]
\[
+ \lim_{a \to 0} \left\{ (sx'')'x - sx''x' - \left( \frac{2n^2 + 1}{s} \right) xx' \right\} \bigg|_a^1
\]
by the integration-by-parts formula one obtains by (17.3) and (17.6) that for any eigenfunction \( x \) of (17.4),

\[
(Lx, x)_0^1 \geq (\|x\|_0^1)^2
\]

and hence that all eigenvalues of (17.4) are positive.

Let \( \lambda \) be any eigenvalue of (17.4) and let \( t = \lambda^{1/4} \), \( t > 0 \). Then any corresponding eigenfunction \( x \) has the form

\[
x(s) = A J_n(\lambda s) + B J_n(it s)
\]

where \( A, B \) are constants and \( J_n \) denotes a Bessel function of the first kind. By (17.3) one can easily deduce that for the eigenvalue \( \lambda \) there exists exactly one linearly independent eigenfunction \( x \). Hence all eigenvalues of (17.4) have multiplicity 1.
For \( 0 < a < 1 \), let \( D[a, l] \) denote the set of all \( y \in H[a, l] \) such that

(i) \( y \in C^3[a, l] \) and \( y(3) \) is absolutely continuous on \([a, l]\);

(ii) \( Ly \in H[a, l] \); and

(iii) \( y(a) = y'(a) = y(1) = y'(1) = 0 \).

Then the perturbed problem is the regular self-adjoint eigenvalue problem

\[
\begin{align*}
Ly &= \mu y, \quad y \in D[a, l].
\end{align*}
\]

Let \( \lambda_j \) be the \( j \)-th eigenvalue of (17.4),

\( \lambda_1 < \lambda_2 < \ldots < \lambda_j < \ldots \). The corresponding eigenfunction \( x_j(s) \)

can be expressed in the form:

\[
\begin{align*}
(17.8) \quad x_j(s) &= i^{3n} C_j \left[ J_n(\ell_j s) - J_n(\ell_j s) \right] \\
&\quad \text{where } C_j \text{ is the normalization constant and } \ell_j = \lambda_j^{1/4}, \; \ell_j > 0. \\
\end{align*}
\]

By inspecting the series representation of \( J_n(s) \) one obtains that

\[
\begin{align*}
(17.9) \quad x_j(a) &\sim \frac{C_j(\ell_j)^n}{n!2^n} |J_n(1\ell_j) - i^n J_n(\ell_j)|a^n \\
x_j'(a) &\sim na^{-1} x_j(a); \\
x_j''(a) &\sim n(n - 1)a^{-2} x_j(a); \\
x_j'''(a) &\sim n(n - 1)(n - 2)a^{-3} x_j(a)
\end{align*}
\]

as \( a \to 0 \).

It is easily verified that conditions (2.7), (2.9), (3.3), (3.5) and (3.6) are satisfied if we choose \( h(s) = 1 \),
\( \lambda_0^2 = 2 \) and \( j_0 = 3 \). In particular, \( \varphi_a \sim C a^{n+1} \) as \( a \to 0 \).

By actual calculation one may obtain that

\[
\begin{align*}
\varphi(a) & \sim x(a), \\
\varphi''(a) & \sim -3n(n-1)a^{-2}x(a), \\
\varphi'(a) & \sim na^{-1}x(a), \\
\varphi^{(3)}(a) & \sim n(n-1)(5n+2)a^{-3}x(a)
\end{align*}
\]
as \( a \to 0 \). Hence by (17.2) and (17.9)

(17.10) \[ [\varphi_x](a) \sim 8n^2(n-1)a^{-2}[x(a)]^2. \]

Also by (17.2) and (17.3) and the fact that \( \varphi(1) = \varphi'(1) = x(1) = x'(1) = 0 \) one obtains

(7.11) \[ [\varphi_x](1) = 0. \]

By consideration of the biharmonic operator (iterated Laplacian) \( L = \Delta \Delta \) (See [12]) on the unit disc one can deduce that (17.4) satisfies the monotonicity property of section 6. Then theorem 9 and (17.9) - (17.11) yield the variational formula

\[
\mu_a^j \sim \lambda_j + 8n^2(n-1)a^{-2}[x_j(a)]^2.
\]
as \( a \to 0 \), where \( C_j \) is the normalization constant defined by (17.8).

It is easily seen that the remaining terms on the right of (6.8), namely

\[
(t_0 - \lambda)(f, f)_a^{b}, \quad \varphi_a(f, l)_a^{b}0(1)
\]
are of smaller asymptotic order for this example and hence may be disregarded.
18. Example 3.

As an example of the theory in section 8 consider the operator \( L \) defined on the half-open interval \((0,1]\) by (17.1) with \( n = 0 \). Then

\[
Lx = \frac{1}{s} \left\{ (sx'')'' - \left( \frac{1}{s}x' \right)' \right\}.
\]

Let \( \varphi_i, i = 1,2,3,4 \) denote the following linearly independent solutions of \( Lx = 0 \):

\[
\begin{align*}
\varphi_1(s) &= \log s ; \\
\varphi_2(s) &= 1 ; \\
\varphi_3(s) &= s^2 \log s ; \\
\varphi_4(s) &= s^2 .
\end{align*}
\]

Then for \( i = 1,2,3,4 \), \( \varphi_i \in H(0,1] \) where the integrals representing the inner product and norm for \( H(0,1] \) are taken with respect to the weight function \( k(s) = s \). Setting \( n = 0 \) in (17.2) we obtain

\[
[uv](s) = \frac{1}{s} (uv'' - uv') + (u''v - uv'') \\
+ s(u(3)v - u''v' + u'v'' - uv(3)).
\]

By Green's formula (0.5), \( [\varphi_i \varphi_j](s) \) is constant on \( 0 < s \leq 1 \), \( i,j = 1,2,3,4 \), and by actual calculation

\[
[\varphi_i \varphi_j](s) = \begin{bmatrix}
0 & 0 & 4 & 4 \\
0 & 0 & -4 & 0 \\
-4 & 4 & 0 & 0 \\
-4 & 0 & 0 & 0
\end{bmatrix}.
\]

Let \( \chi_i, i = 1,2,3,4 \), be defined by:
\[ (18.5) \]
\[
\begin{align*}
\chi_1(s) &= \varphi_3(s); \\
\chi_2(s) &= \varphi_4(s); \\
\chi_3(s) &= -\frac{1}{4} \left\{ \varphi_1(s) + \varphi_2(s) + \varphi_3(s) - \varphi_4(s) \right\}; \\
\chi_4(s) &= -\frac{1}{4} \left\{ 2\varphi_1(s) + \varphi_2(s) - \varphi_4(s) \right\}.
\end{align*}
\]

Then, with the help of \((18.4)\), it is easily verified that
\[
\begin{align*}
[x_i x_j](0) &= 0, \quad i, j = 1, 2; \\
[x_i x_j](1) &= 0, \quad i, j = 1, 2.
\end{align*}
\]

From \((18.4)\) it also follows that
\[ (18.6) \]
\[
\begin{align*}
[\varphi_i x_j](0) &\neq 0, \quad i, j = 1, 2, i \neq j; \\
[\varphi_j x_j](1) &\neq 0, \quad i, j = 3, 4, i \neq j.
\end{align*}
\]

Let \(D(0,1]\) be the set of all \(x \in H(0,1]\) such that
(1) \(x \in C^3(0,1]\) and \(x(3)\) is absolutely continuous on closed subintervals of \((0,1]\);

(ii) \(x\) satisfies the end conditions
\[ (18.7) \]
\[
\begin{align*}
[x x_j](0) &= 0, \quad j = 1, 2, \\
[x x_j](1) &= 0, \quad j = 3, 4.
\end{align*}
\]

(Note that the above end conditions at \(s = 1\) are \(x(1) = x'(1) = 0\).) The basic problem (corresponding to \((8.1)\)) is
\[ (18.8) \]
\[ Lx = \lambda x, \quad x \in D(0,1]. \]

An immediate consequence of \((18.6)\) is that \(\lambda = 0\) is not an eigenvalue of \((18.8)\). Hence we choose \(\ell_0 = 0\) and \(\varphi_i, i = 1, 2, 3, 4, \)
to be the basic solutions. By constructing the Green's function
for \( L \) (using (18.2)) associated with the boundary conditions (18.7), one can easily verify that (18.8) has a countable set of real eigenvalues accumulating only at \( \infty \) and a corresponding set of orthonormal eigenfunctions complete in \( H(0,1) \). By consideration of the indicial equation for \( Lx = \lambda x \) one obtains that each eigenfunction of (18.8) is such that

\[
(18.9) \quad x^{(i)}(s) = 0(s^{2-i}\log s), \quad i = 0, 1, 2, 3.
\]

Then since

\[
(Lx, x)^{1}_0 = \int_0^1 \left\{ s[x'']^2 + \frac{1}{s}[x']^2 \right\} ds
+ \lim_{a \to 0} \left\{ (sx'')'x - sx''x' - \frac{1}{s} x' \right\} \bigg|_0^1
\]

by the integration-by-parts formula, one obtains by (18.7) and (18.9) that for any normalized eigenfunction \( x \) of (18.8), \( (Lx, x) > 0 \). This implies that the eigenvalues of (18.8) are all positive.

Let \( \lambda \) be any eigenvalue of (18.8) and let \( \lambda = k^{1/4} \), \( k > 0 \). Then linearly independent solutions of \( Lx = \lambda x \) are

\[ J_0(ks), J_0(iks), Y_0(ks) \text{ and } Y_0(iks) \]
where \( J_0 \) and \( Y_0 \) are Bessel functions of the first and second kind respectively. By inspecting the series representations of \( J_0(s) \) and \( Y_0(s) \) (See [15]) one obtains that the normalized eigenfunctions \( x \) corresponding to \( \lambda = k^4 \) must be of the form

\[
(18.10) \quad x(s) = C_\lambda \left\{ A_\lambda [Y_0(iks) - Y_0(ks)]
+ B_\lambda [J_0(iks) - J_0(ks)] \right\},
\]

where \( A_\lambda, B_\lambda, C_\lambda \) are constants given by
Then it follows easily from (18.10) that each eigenvalue of (18.8) has multiplicity 1.

For $0 < a < 1$, let $D[a,l]$ denote the set of all $y \in H[a,l]$ such that

(i) $y \in C^3[a,l]$ and $y^{(3)}$ is absolutely continuous on $[a,l]$;
(ii) $Ly \in H[a,l]$; and
(iii) $y(a) = y'(a) = y(l) = y'(l) = 0$.

Then the perturbed problem is the regular self-adjoint eigenvalue problem

(18.11) $Ly = \mu y$, $y \in D[a,l]$.

It is easily verified that conditions (2.7), (2.9), (3.3), (3.5) and (3.6) are satisfied if we choose $i_o' = 2$, $J_o = 3$ and

$$h(s) = \begin{cases} 
\log s & \text{if } 0 < s < e^{-1} \\
1 & \text{if } e^{-1} \leq s \leq 1.
\end{cases}$$

Condition (7.15) is also satisfied with

$$\rho_a \sim Ca^2(\log a)^2 = o(1)$$
as $a \to 0$. By (18.10) one obtains that
(18.12) \[ x'(a) \sim \frac{2}{a} x(a) \; ; \; x''(a) \sim \frac{2}{a^2} x(a) \; ; \]
\[ x^{(3)}(a) \sim \frac{2}{a^3} \log a \]
as \( a \to 0 \). By actual calculation one obtains:

(18.13) \[ f(a) \sim x(a) \; ; \; f''(a) \sim -\frac{2}{a^2} x(a) \; ; \]
\[ f'(a) \sim \frac{2}{a} x(a) \; ; \; f^{(3)}(a) \sim \frac{4}{a^3} x(a) \]
as \( a \to 0 \). Hence (18.3), (18.12), and (18.13) yield

(18.14) \[ [fx](a) \sim 8a^{-2}[x(a)]^2 \; , \; a \to 0 \cdot \]
Also since \( f(1) = f'(1) = x(1) = x'(1) = 0 \), (18.3) implies that

(18.15) \[ [fx](1) = 0. \]

Since a monotonicity property is known to hold (in the same way as in example 2), theorem 15 together with (18.10), (18.14) and (18.15) yield the following variational formula for the eigenvalues \( \mu_1^a \) of (18.11), \( i = 1,2,... \):

\[ \mu_1^a - \lambda_1 \sim 4C_1 A_1 \lambda_1 a^2(\log a)^2 \]
as \( a \to 0 \), \( i = 1,2,... \). Note that the remaining terms on the right of (8.2) are all \( 0[a^4(\log a)^4] \) as \( a \to 0 \) and hence may be disregarded.
BIBLIOGRAPHY


