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HAUSDORFF MEASURES ON TOPOLOGICAL SPACES

ABSTRACT

Given a non-negative set function τ on a family $\mathcal{A}$ of subsets of a metric space X, an outer measure ν can be generated on X as follows:

for $B \subset X$ and $\delta > 0$,

$$\nu_\delta B = \inf \left\{ \sum_{i \in \omega} \tau A_i \; : \; B \subset \bigcup_{i \in \omega} A_i \quad \text{and for } i \in \omega, \quad A_i \in \mathcal{A} \text{ and diam } A_i \leq \delta \right\}$$

and

$$\nu B = \lim_{\delta \to 0} \nu_\delta B.$$

The Hausdorff s-dimensional and h-measures are special cases of this measure. A number of processes have been suggested for generating a measure on an arbitrary topological space, which generalize this Hausdorff measure process in a metric space.

In this thesis we introduce and study a process for generating a measure on an arbitrary space, which abstracts the essential idea behind all the Hausdorff measures and their generalizations, and contains them as special cases.

In chapter I the concept of a measure generated on a space by a gauge and a filterbase is introduced. We show that with any such filterbase is automatically associated a topology for the space, the filterbase topology. We then impose different conditions on the filterbase and deduce resulting properties of the filterbase topology and of the measure. Measurability and approximation properties of the measure are obtained for sets defined in terms of the filterbase, and then for sets defined in terms of the filterbase topology, such as closed, compact, etc.

In chapter II we consider measures generated on a topological space. We show that previous measures are special cases of our measure and that known measurability and approximation results can be obtained for them from our general theory. The relationship between the given topology and the topologies of the filterbases used to generate the various measures is examined. A number of additional processes for generating a measure on a topological space are investigated and relations among the

various measures are studied.

In chapter III we consider several processes for generating measures on a quasi-uniform space, showing that a number of the previously studied measures are included. In particular, we study the measure generated on a uniform space, and obtain some measurability properties by applying our general theory.

In chapter IV we work in a compact Hausdorff space and generate a measure using the uniformity for the space and the process of the previous chapter. For the first time, restrictions are placed on the generating set function τ. We examine some consequences of this restriction and then introduce a partial ordering on the family of such functions which generalizes the usual ordering on the h-functions in Hausdorff h-measure theory. This ordering has been used in connection with studies of non-σ- finiteness. We show here that its interest is essentially limited to the metric case.

GRADUATE STUDIES

Field of Study: Mathematics

Measure theory	M. Sion C. A. Rogers
Point Set Topology	S. Kobayashi
Functional Analysis	C. Clark
Complex Variables	Z. Melzak
Group Theory	R. Ree

HAUSDORFF MEASURES IN TOPOLOGICAL SPACES

by

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Supervisor: Dr. M. Sion.

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The Hausdorff s-dimensional and h-measures are special cases of this measure. A number of processes have been suggested for generating a measure on an arbitrary topological space, which generalize this Hausdorff measure process in a metric space.

In this thesis we introduce and study a process for generating a measure on an arbitrary space, which abstracts the essential idea behind all the Hausdorff measures and their generalizations, and contains them as special cases.

In chapter I the concept of a measure generated on a space by a gauge and a filterbase is introduced. We show that with any such filterbase is automatically associated a topology for the space, the filterbase topology. We then impose different conditions on the filterbase and deduce

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CONTENTS

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Introduction

Given a non-negative set function τ on a family $\mathcal{A}$ of subsets of a metric space X, an outer measure ν can be generated on X as follows:

for $B \subset X$ and $\delta > 0$,

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and

$$\nu B = \lim_{\delta \to 0} \nu_\delta B.$$

F. Hausdorff [6] introduced this abstract measure (a generalization of the linear measure of C. Carathéodory [5]), and proved a few basic results for it. He considered in some detail the measures obtained when various restrictions were placed on the set function τ, in particular when $\tau B = h(\operatorname{diam} B)$ for some continuous increasing function $h: R_+ \to R_+$, with $h(0) = 0$ and $h(t) > 0$ for $t > 0$. The measure generated using this function is called the Hausdorff h-measure, and in the case that $h(t) = t^s$, the Hausdorff s-dimensional measure. In these forms it has been studied extensively. Two recent papers by W. W. Bledsoe and A. P. Morse [2], and by C. A. Rogers and M. Sion [12], have suggested processes for defining a measure on a topological space which generalize the Hausdorff measure process in a metric space. They obtain some (in general, different) measurability and approximation results for these measures.

In this thesis we introduce a process for generating a measure on an arbitrary space, which abstracts the essential idea behind all of the above Hausdorff measures and generalizations. Results are obtained which can be specialized to give many of the known results, and which throw some light on the relation between measures introduced before. In a secondary study, using some results from the abstract approach, we extend some specific theorems first obtained in a metric space to a compact Hausdorff space.

In chapter I we introduce the concept of a measure generated by a gauge and a filterbase. We show that with any such filterbase is automatically associated a topology for the space, the filterbase topology, independent of any existing topology. We then impose different conditions on the filterbase and deduce resulting properties of the filterbase topology and of the measure. Measurability and approximation properties of the measure are first obtained in terms of the filterbase. Additional conditions on the filterbase are then applied to give results, stated in terms of the filterbase topology, on measurability of closed, closed $\mathcal{G}_\delta$, and compact $\mathcal{G}_\delta$ sets, and on approximation by $\mathcal{G}_\delta$, $\mathcal{F}_\sigma$, open and closed sets.

In chapter II we consider measures generated on a topological space. We show that the Hausdorff measure in a metric space and the measures of Bledsoe and Morse [2], and of Rogers and Sion [12] are encompassed by the general theory of chapter I and that some of the measurability and approximation results can be specialized to yield existing results for these measures. The latter two of the above measures are defined in an arbitrary topological space; we examine in each case the relation between the given topology and the topology associated with the filterbase used to generate the measure, and point out some consequences of their equality or difference.

A number of additional processes are suggested for generating a measure on a topological space, some variations of processes already studied, and one a different approach. Again, all come under the theory of chapter I, and results from it are applied to give properties of these measures. Relations among the various measures of the chapter are examined.

In chapter III, we consider several processes for generating measures on a quasi-uniform space. We show that these measures include a number of those studied in chapter II. In particular, we study the measure generated on a uniform space, and obtain some measurability properties for it by applying results from chapter I.

Chapter IV is devoted to an examination of the possibility of extending some specific results obtained in a compact metric space by M. Sion and D. Sjerve [13] to a compact Hausdorff space, considered as a uniform space, using the measure of chapter III. For the first time, restrictions are put on the generating set function τ. We examine some consequences of this restriction and then introduce a partial ordering on such functions τ which generalizes the usual ordering in a metric space (see Sion and Sjerve [13], section 6). This ordering has been used in Hausdorff h-measure theory in connection with studies of non-σ- finiteness. We show here (theorem 21.3) that its interest is essentially limited to the metric case.

CHAPTER 0

PRELIMINARIES

In chapter 0 we collect definitions, notation, and known or elementary results in set theory, topology, and measure theory which will be needed later. The only new idea is the concept of property Q (2.2.4).

1. Set theoretic definitions and notation.

.1 $\emptyset$ denotes the empty set.

.2 ω denotes the set of natural numbers.

.3 $A \sim B = \{x : x \in A \text{ and } x \notin B\}$.

Let $\mathcal{B}$ be a family of sets. Then

.4 $\pi\mathcal{B} = \bigcap_{A\in\mathcal{B}} A$;

.5 $\sigma\mathcal{B} = \bigcup_{A\in\mathcal{B}} A$;

.6 $\mathcal{B}^{\sim} = \{A : A = \sigma\mathcal{B} \sim B \text{ for some } B \in \mathcal{B}\}$;

.7 $\mathcal{B}_\sigma = \{A : A = \bigcup_{n\in\omega} B_n$ for some sequence B of sets in $\mathcal{B}\}$;

.8 $\mathcal{B}_\delta = \{A : A = \bigcap_{n\in\omega} B_n$ for some sequence B of sets in $\mathcal{B}\}$;

.9 $\mathcal{B}_{\sigma\delta} = (\mathcal{B}_\sigma)_\delta$; $\mathcal{B}_{\delta\sigma} = (\mathcal{B}_\delta)_\sigma$;

.10 $\mathcal{B}$ is a cover of A iff $A \subset \sigma\mathcal{B}$;

.11 $\mathcal{B}$ is an $\mathcal{A}$- cover of A iff $\mathcal{B}$ is a cover of A and $\mathcal{B} \subset \mathcal{A}$;

.12 $\mathcal{A}$ refines $\mathcal{B}$ or $\mathcal{A}$ is a refinement of $\mathcal{B}$ iff for each $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $A \subset B$;

.13 $\mathcal{B}$ is a σ-field iff $\mathcal{B}^{\sim} \subset \mathcal{B}$ and $\mathcal{B}_{\sigma} \subset \mathcal{B}$; and

.14 Borel $\mathcal{B} = \pi\{\mathcal{A} : \mathcal{A}$ is a σ-field and $\mathcal{B} \subset \mathcal{A}\}$ is the smallest σ-field containing $\mathcal{B}$.

.15 If ρ is a non-negative function on $X \times X$ and $A \subset X$, then

$$\operatorname{diam}\rho A = \begin{cases} \sup\{\rho(x,y) : x \in A,\ y \in A\} \text{ if } A \neq \emptyset \\ 0 \text{ if } A = \emptyset . \end{cases}$$

.16 $\mathcal{H}$ is a filterbase iff $\mathcal{H}$ is a non-empty family of sets such that for every $M \in \mathcal{H}$ and $N \in \mathcal{H}$, there exists $H \in \mathcal{H}$ such that $\emptyset \neq H \subset M \cap N$.

$\mathcal{H}$ is a filterbase in X iff $\mathcal{H}$ is a filterbase and for every $H \in \mathcal{H}$, H is a family of subsets of X and $\emptyset \in H$.

If $\mathcal{H}$ is a filterbase in X, then $\mathcal{M}$ is a subfilterbase of $\mathcal{H}$ iff $\mathcal{M}$ is a filterbase in X and for some $\mathcal{A}$,

$$\mathcal{M} = \{H \cap \mathcal{A} : H \in \mathcal{H}\} .$$

.17 $(A_{\alpha}, \alpha \in D)$ denotes a net. The ordering directing D will be denoted by $\gg$. (see Kelley [7], chapter 2)

2. General topological concepts.

Most of our topological concepts are based closely on those of Kelley [7] (hereafter referred to simply as Kelley).

2.1 NOTATION. Suppose $(X, \mathcal{G})$ is a topological space and $A \subset X$. Then

.1 $\mathcal{G}$ of course denotes the family of open sets, $\mathcal{F}$ will denote the family of closed sets, and $\mathcal{D}$ the family of differences of open sets, i.e.

$$\mathcal{D} = \{A \subset X : A = G_1 \sim G_2 \text{ for some } G_1, G_2 \in \mathcal{G}\} ;$$

.2 $\overline{A}$ or ClA denotes the closure of A ;

.3 A° denotes the interior of A ; and

.4 bdry A = $\overline{A} \sim A^{\circ}$ is the boundary of A.

2.2 DEFINITIONS. Conditions on a topology.

.1 If $\mathcal{B}$ is a family of subsets of a space X, and $x \in X$, then the <u>star</u> at x of $\mathcal{B}$ is the union of the members of $\mathcal{B}$ to which x belongs.

A cover $\mathcal{A}$ of X is a <u>star-refinement</u> of $\mathcal{B}$ iff the family of stars of $\mathcal{A}$ at points of X is a refinement of $\mathcal{B}$.

A topological space is <u>fully normal</u> iff for each open cover $\mathcal{B}$, there exists an open cover $\mathcal{A}$ which is a star-refinement of $\mathcal{B}$. (Tukey [16])

.2 A family of subsets of X is <u>point finite</u> iff no point of X belongs to more than a finite number of members of the family.

A topological space is <u>metacompact</u> iff for each open cover $\mathcal{B}$, there is an open cover which is a point finite refinement of $\mathcal{B}$. (Kelley, p. 171)

.3 A family $\mathcal{A}$ of subsets of a topological space is <u>locally finite</u> iff each point of the space has a neighborhood which intersects only finitely many members of $\mathcal{A}$.

A topological space is <u>paracompact</u> iff it is regular and for each open cover $\mathcal{B}$, there is an open cover which is a locally finite refinement of $\mathcal{B}$. (Kelley, p. 156)

.4 A topological space has <u>property Q</u> iff for any open cover $\mathcal{A}$ of X, there exists an open cover $\mathcal{B}$ refining $\mathcal{A}$ and such that for every $x \in X$, $\pi\{G \in \mathcal{B} : x \in G\}$ is open.

2.3 REMARKS. Relations between conditions on a topology.

.1 Metric spaces are fully normal. (Tukey [16])

.2 A regular space is paracompact iff it is fully normal. (Stone [14])

.3 A paracompact space is metacompact and a metacompact space has property Q.

.4 A topological space may have property Q without being metacompact.

Let $X = R_+$, $\mathcal{J} = \{[0,a) : a > 0\}$. Then $(X, \mathcal{J})$ is a topological space which clearly is not metacompact. To see that it has property Q, let $\mathcal{A}$ be any open cover of X. Set $\mathcal{B} = \{[0,n) : n \in \omega\}$. Then $\mathcal{B}$ is an open cover of X which refines $\mathcal{A}$, and for any $x \in X$,

$$\pi\{G \in \mathcal{B} : x \in G\} = [0,n) \text{ for some } n \in \omega.$$

3. Quasi-uniformities and uniformities.

We consider now concepts associated with quasi-uniformities and uniformities. For a fuller exposition see the two papers of W. J. Pervin [10,11] and chapter 6 of Kelley.

3.1 DEFINITIONS.

.1 $A \circ B = \{(x,z) : \text{for some } y,\ (x,y) \in B \text{ and } (y,z) \in A\}$.

.2 If X is a space,

$$\Delta = \{(x,x) : x \in X\}$$

is the <u>diagonal</u> of the space $X \times X$.

The following are immediate consequences of the definitions.

3.2 LEMMAS

.1 $(A \times A) \circ (A \times A) = A \times A$.

.2 If $A \subset X \times X$, then $\Delta \circ A = A \circ \Delta = A$.

.3 If I and J are any index sets and

$$A = \bigcup_{i \in I} A_i \ , \ B = \bigcup_{j \in J} B_j$$

then

$$A \circ B = \bigcup_{i \in I} \bigcup_{j \in J} A_i \circ B_j = \bigcup_{j \in J} \bigcup_{i \in I} A_i \circ B_j \ .$$

3.3 DEFINITION. If X is a space, $\mathcal{U}$ a family of subsets of $X \times X$ such that for every $U \in \mathcal{U}$ and $V \in \mathcal{U}$,

.1 $\Delta \subset U$,

.2 $W \supset U$ and $W \subset X \times X \Rightarrow W \in \mathcal{U}$,

.3 $U \cap V \in \mathcal{U}$, and

.4 there exists $W \in \mathcal{U}$ such that $W \circ W \subset U$,

then $\mathcal{U}$ is a <u>quasi-uniformity</u> for X.

If, in addition to the above requirements, for every $U \in \mathcal{U}$,

.5 $U^{-1} = \{(x,y) : (y,x) \in U\} \in \mathcal{U}$,

then $\mathcal{U}$ is a <u>uniformity</u> for X.

.6 $(X, \mathcal{U})$ is a (<u>quasi</u>-) <u>uniform</u> space iff $\mathcal{U}$ is a (quasi-) uniformity for X.

3.4 DEFINITIONS. If U is an element of a quasi-uniformity,

.1 $U[A] = \{y : (x,y) \in U \text{ for some } x \in A\}$,

.2 $U[x] = U[\{x\}]$.

The following lemmas are immediate consequences of the definitions.

3.5 LEMMAS. Suppose $\mathcal{U}$ is a quasi-uniformity, $U \in \mathcal{U}$, $V \in \mathcal{U}$, and for each $i \in I$, $V_i \in \mathcal{U}$. Then

.1 for each A,

$$U[A] = \bigcup_{x \in A} U[x];$$

.2 for each A,

$$(U \cap V)[A] \subset U[A] \cap V[A] :$$

.3 for each A,

$$U[V]A]] = (U \circ V)[A]; \text{ and}$$

.4 for each x

$$\left(\bigcap_{i \in I} V_i\right)[x] = \bigcap_{i \in I} V_i[x] .$$

3.6 REMARK. A quasi-uniformity $\mathcal{U}$ for X generates a topology $\mathcal{T}_{\mathcal{U}}$ on X consisting of all subsets G of X such that for each $x \in G$, there exists $U \in \mathcal{U}$ such that $U[x] \subset G$. For $x \in X$, $\{U[x] : U \in \mathcal{U}\}$ is a neighborhood system for x. (see Pervin [11])

3.7 DEFINITION. A topological space $(X, \mathcal{J})$ is (<u>quasi-</u>) <u>uniformizable</u> iff there exists a (quasi-) uniformity $\mathcal{U}$ for X such that $\mathcal{T}_{\mathcal{U}} = \mathcal{J}$.

3.8 THEOREM. Every topological space is quasi-uniformizable.

<u>Proof</u>: (see Pervin [11]) Let $(X, \mathcal{J})$ be a topological space. For each $G \in \mathcal{J}$ let

$$S_G = (G \times G) \cup ((X \sim G) \times X),$$

and let $\mathcal{A} = \{S_G : G \in \mathcal{J}\}$. Pervin shows that $\mathcal{A}$ is a subbase for a quasi-uniformity $\mathcal{U}$ for X (hereafter referred to as <u>Pervin's quasi-uniformity</u>), and that $\mathcal{T}_{\mathcal{U}} = \mathcal{J}$.

3.9 REMARK. Pervin points out that non-comparable quasi-uniformities for the same space X may induce identical topologies. The same is true for uniformities. (see, for example, 19.5)

3.10 REMARK. For a given topological space $(X, \mathcal{J})$ there is a maximal quasi-uniformity $\mathcal{U}$ such that $\mathcal{T}_{\mathcal{U}} = \mathcal{J}$. We take as a subbase for $\mathcal{U}$ the union of all quasi-uniformities $\mathcal{V}$ such that $\mathcal{T}_{\mathcal{V}} = \mathcal{J}$. Then $\mathcal{U}$ is a quasi-uniformity by theorem 6.3 of Kelley. To see that $\mathcal{T}_{\mathcal{U}} = \mathcal{J}$:

Suppose $G \in \mathcal{J}$ and $x \in G$. Choose U from Pervin's quasi-uniformity such that $U[x] \subset G$. Then $U \in \mathcal{U}$ and so $G \in \mathcal{T}_{\mathcal{U}}$.

Now suppose $G \in \mathcal{T}_{\mathcal{U}}$ and $x \in G$. Then for some $U \in \mathcal{U}$, $U[x] \subset G$. But there exists $W \in \mathcal{U}$, $W \subset U$, such that

$$W = \bigcap_{i=1}^{n} V_i ,$$

where $V_i[x]$ is a neighborhood in $\mathcal{J}$ of x for $i = 1, \ldots, n$. Hence by lemma 3.5.4,

$$W[x] = \left(\bigcap_{i=1}^{n} V_i\right)[x] = \bigcap_{i=1}^{n} V_i[x] ,$$

and so $W[x]$ is a neighborhood in $\mathcal{J}$ of x. Since

$$W[x] \subset U[x] \subset G,$$

we conclude that $G \in \mathcal{J}$.

3.11 REMARKS on uniformities. (see chapter 6 of Kelley)

.1 A topological space is uniformizable iff it is completely regular.

.2 There may be non-comparable uniformities inducing the same topology on a space.

.3 If $\mathcal{U}$ is a uniformity for $(X, \mathcal{J})$ such that $\mathcal{T}_{\mathcal{U}} = \mathcal{J}$, then $U \in \mathcal{U}$ implies that U is a neighborhood of Δ in the product topology on $X \times X$.

.4 U is <u>symmetric</u> iff $U = U^{-1}$. For any uniformity $\mathcal{U}$, there is a base of open symmetric members of $\mathcal{U}$, and a base of closed symmetric members of $\mathcal{U}$.

5. Suppose $\mathcal{U}$ is a uniformity, $U \in \mathcal{U}$, $V \in \mathcal{U}$, and V is symmetric. Then for any A, if $A \times A \subset U$, then

$$V[A] \times V[A] \subset V \circ U \circ V .$$

.6 A family $\mathcal{B}$ of subsets of $X \times X$ is a base for some uniformity for X iff

a) $U \in \mathcal{B} \Rightarrow \Delta \subset U$;

b) if $U \in \mathcal{B}$, then U^{-1} contains a member of $\mathcal{B}$;

c) if $U \in \mathcal{B}$, then for some $V \in \mathcal{B}$, $V \circ V \subset U$; and

d) the intersection of two members of $\mathcal{B}$ contains a member.

.7 A uniformity $\mathcal{U}$ is characterized by the <u>gage</u> of $\mathcal{U}$, i.e. the family of pseudo-metrics on X which are uniformly continuous on $X \times X$ relative to the product uniformity derived from $\mathcal{U}$.

.8 For a given completely regular space $(X, \mathcal{J})$ there is a maximal uniformity $\mathcal{U}$ such that $\mathcal{T}_{\mathcal{U}} = \mathcal{J}$. The demonstration is analogous to that in 3.10 for quasi-uniformities (or see Kelley, problem 6G). Note that if a uniformity consists of all neighborhoods of Δ, then it is the maximal uniformity by remark 3.11.3.

.9 A paracompact space is completely regular and the maximal uniformity consists of all neighborhoods of Δ. (Kelley, problem 6L)

.10 If $(X, \mathcal{U})$ is a uniform space and $(X, \mathcal{T}_{\mathcal{U}})$ is compact, then $\mathcal{U}$ is unique and consists of all neighborhoods of Δ.

.11 If $(X, \mathcal{U})$ is a uniform space and $(X, \mathcal{T}_{\mathcal{U}})$ is compact, then each neighborhood of a compact subset A of X contains a neighborhood of the form $U[A]$ for some $U \in \mathcal{U}$.

.12 If $(X, \mathcal{U})$ is a uniform space and $A \subset X$, then the closure of A in the uniform topology,

$$\bar{A} = \bigcap_{U \in \mathcal{U}} U[A] .$$

.13 If $(X, \mathcal{U})$ is a uniform space and $M \subset X \times X$, then the closure of M in the product uniform topology on $X \times X$,

$$\bar{M} = \bigcap_{U \in \mathcal{U}} U \circ M \circ U .$$

4. Measure theoretic concepts.

4.1 DEFINITIONS.

.1 μ is an outer measure on X iff μ is a function on the family of subsets of X such that

i) $\mu \varnothing = 0$, and

ii) $0 \leqslant \mu A \leqslant \sum_{n \in \omega} \mu B_n$ whenever $A \subset \bigcup_{n \in \omega} B_n \subset X$.

As all measures discussed in this thesis will be outer measures we will hence forth drop the qualifying word 'outer'.

.2 For μ a measure on X, a set A is μ-measurable iff $A \subset X$ and for every $B \subset X$.

$$\mu B = \mu(B \cap A) + \mu(B \sim A).$$

.3 For μ a measure on X,

$$\mathcal{M}_\mu = \{A \subset X : A \text{ is } \mu\text{-measurable}\}.$$

.4 $\mu|A$, the <u>restriction</u> of μ to A, is the function ν having the same domain as μ such that for every B in the domain of μ, $\nu B = \mu(B \cap A)$.

.5 ν is a <u>finite submeasure</u> of μ iff for some A with $\mu A < \infty$, $\nu = \mu|A$.

.6 For μ a measure on X, X is μ-σ-<u>finite</u>, or σ-<u>finite</u>, iff there exists a sequence A such that $X = \bigcup_{n\in\omega} A_n$, where for each $n \in \omega$, $\mu A_n < \infty$.

.7 If $\mathcal{B}$ is a family of sets, τ is a <u>gauge</u> on $\mathcal{B}$ iff τ is a function on $\mathcal{B} \cup \{\emptyset\}$ to the extended non-negative real line, such that $\tau\emptyset = 0$.

.8 For μ a measure on X, μ is a <u>regular</u> measure iff for every $A \subset X$, there exists $B \in \mathcal{M}_\mu$ such that $A \subset B$ and $\mu A = \mu B$.

The following theorem is well known. (See, for example, corollary 12.1.1 in Monroe [9].)

4.2 THEOREM. If μ is a regular measure on X and A is an ascending sequence of subsets of X, then

$$\mu\left(\bigcup_{n\in\omega} A_n\right) = \lim_{n\to\infty} \mu A_n .$$

The following is a form of the well known lemma of Carathéodory.

4.3 LEMMA. Suppose μ is a measure on X, and $A \subset X$. If for every $\varepsilon > 0$ and every $T \subset X$ such that $\mu T < \infty$ there exists a sequence D of subsets of X such that

1) $D_{n+1} \subset D_n$ for every $n \in \omega$;

2) $\bigcap_{n\in\omega} D_n \subset A$;

3) $\mu(T \cap A) \leqslant \mu(T \cap D_n) + \varepsilon$ for every $n \in \omega$; and

4) for every $P \subset T$ and $n \in \omega$,

$$\mu(\,(P \cap D_{n+1}) \cup (P \sim D_n)) = \mu(P \cap D_{n+1}) + \mu(P \sim D_n),$$

then A is μ-measurable.

Proof: Let $\varepsilon > 0$, $T \subset X$, $\mu T < \infty$, $B = \bigcap_{n\in\omega} D_n$. We show

$$\mu(T \cap A) + \mu(T \sim A) \leqslant \mu T + 2\varepsilon,$$

which implies that A is μ-measurable.

We obtain first

5) There exists $N \in \omega$ such that

$$\mu(T \sim B) \leqslant \mu(T \sim D_N) + \varepsilon .$$

Setting $P = T \cap D_n$ we have

$$\begin{aligned} \mu(T \cap D_n) &= \mu P \geqslant \mu(\,(P \cap D_{n+2}) \cup (P \sim D_{n+1})) && \text{by 1)} \\ &= \mu(P \cap D_{n+2}) + \mu(P \sim D_{n+1}) && \text{by 4)} \\ &= \mu(T \cap D_{n+2}) + \mu(T \cap D_n \sim D_{n+1}) && \text{by 1).} \end{aligned}$$

Hence for any $M \in \omega$,

$$\sum_{n=0}^{M} \mu(T \cap D_n \sim D_{n+1}) \leqslant \sum_{n=0}^{M} (\mu(T \cap D_n) - \mu(T \cap D_{n+2}))$$

$$= \mu(T \cap D_0) + \mu(T \cap D_1) - \mu(T \cap D_{M+1}) - \mu(T \cap D_{M+2})$$

$$\leqslant 2\mu(T \cap D_0) < \infty ,$$

and

$$\sum_{n=0}^{\infty} \mu(T \cap D_n \sim D_{n+1}) = \lim_{M\to\infty} \sum_{n=0}^{M} \mu(T \cap D_n \sim D_{n+1})$$

$$\leqslant 2\mu(T \cap D_0) < \infty .$$

Choose $N \in \omega$ so that

$$\sum_{n=N}^{\infty} \mu(T \cap D_n \sim D_{n+1}) < \varepsilon .$$

Since

$$(T \cap D_N \sim B) = \bigcup_{n \geqslant N} (T \cap D_n \sim D_{n+1}) \quad \text{by 1)},$$

we have

$$\mu(T \cap D_N \sim B) < \varepsilon .$$

But

$$\mu(T \sim B) \leqslant \mu(T \sim D_N) + \mu(T \cap D_N \sim B) \leqslant \mu(T \sim D_N) + \varepsilon ,$$

which establishes 5). Now

$$\begin{aligned} \mu(T \cap A) + \mu(T \sim A) &\leqslant \mu(T \cap A) + \mu(T \sim B) && \text{since } B \subset A, \\ &\leqslant \mu(T \cap D_{N+1}) + \varepsilon + \mu(T \sim D_N) + \varepsilon && \text{by 3) and 5)}, \\ &= \mu((T \cap D_{N+1}) \cup (T \sim D_N)) + 2\varepsilon && \text{by 4)}. \\ &\leqslant \mu T + 2\varepsilon . \end{aligned}$$

CHAPTER I

THE MEASURE GENERATED BY A GAUGE AND A FILTERBASE

In this chapter we start with an abstract space X, a filterbase $\mathcal{H}$ in X (see 1.16) and a gauge τ on some family $\mathcal{A}$ of subsets of X such that $\emptyset \in \mathcal{A}$ (see 4.1.7). From these we generate a measure and a topology on X, and then investigate properties of the measure and of the topology. In particular we obtain conditions under which certain topological sets, such as closed, closed $\mathcal{G}_\delta$, and compact $\mathcal{G}_\delta$ sets, are measurable (section 9), and also results on the approximation of a given set from above and below by measurable sets or by topological sets (section 10). The proof of theorem 9.5 was suggested by the development in section 2 of Bledsoe and Morse [2]; theorems 10.3 and 10.4 are based on theorem 1 and its corollary in Rogers and Sion [12]; and the proofs of theorems 10.9, 10.10, and 10.11 are essentially contained in those of theorems 13.5 - 13.7 of Monroe [9]. The topology itself is studied first (section 8) and the key result, used repeatedly later, is theorem 8.1.2, which establishes conditions under which a certain natural family forms a base for the neighborhood system of a point. From this we determine when the topology is regular (8.1.4), Hausdorff (8.1.5), or generated by a uniformity (8.2).

5. The measure ν.

We now introduce the measure generated on X by the filterbase $\mathcal{H}$ in X and the gauge τ on $\mathcal{A}$. We may assume without any loss of generality that $\mathcal{A} \subset \sigma\mathcal{H}$.

5.1 DEFINITION. For $H \in \mathcal{H}$ and $A \subset X$ let

.1 $\nu_H^{(\mathcal{H},\tau)} A = \inf \{ t : t = \sum_{B \in \mathcal{B}} \tau B$ for some countable $\mathcal{B} \subset H \cap \mathcal{A}$ such that $A \subset \sigma\mathcal{B} \}$.

(note: $\inf \emptyset = \infty$)

.2 $\nu^{(\mathcal{H},\tau)} A = \sup_{H \in \mathcal{H}} \nu_H^{(\mathcal{H},\tau)} A$.

If no ambiguity can arise as a result, we will drop one or both superscripts on ν.

5.2 THEOREM. ν is a measure on X.

Proof: ν_H is constructed by Method I of Monroe [9], pp. 90,91, and so, by theorem 11.3 in Monroe, is a measure. Since ν is the supremum of such measures, it is again one.

5.3 REMARK. $\mathcal{H}$ is a set directed by inclusion, so $(\nu_H A,\ H \in \mathcal{H})$ is a net. It is an increasing net, i.e. $H, N \in \mathcal{H}$ and $H \subset N$ implies $\nu_H A \geqslant \nu_N A$, so we have

$$\nu A = \sup_{H \in \mathcal{H}} \nu_H A = \lim_{H \in \mathcal{H}} \nu_H A.$$

6. The filterbase topology.

We now use the filterbase $\mathcal{H}$ in X to introduce a topology on X, closely related to the measure ν.

6.1 DEFINITIONS

.1 For $H \in \mathcal{H}$, $x \in X$,

$$H[x] = \{x\} \cup \sigma\{h \in H : x \in h\}.$$

.2 For $H \in \mathcal{H}$, $A \subset X$,

$$H[A] = \bigcup_{x \in A} H[x] = A \cup \sigma\{h \in H\ ;\ h \cap A \neq \emptyset\}.$$

.3 The $\mathcal{H}$-topology, $\mathcal{G}_{\mathcal{H}} = \{G \subset X$: for every $x \in G$, there exists $H \in \mathcal{H}$ such that $H[x] \subset G\}$. The subscript $\mathcal{H}$ may be dropped if no ambiguity can result.

6.2 THEOREM. The $\mathcal{H}$-topology is a topology for X.

<u>Proof</u>: Clearly $\mathcal{G}_{\mathcal{H}}$ is closed under arbitrary unions. Suppose $B, G \in \mathcal{G}_{\mathcal{H}}$ and $x \in B \cap G$. Then there exist $H, N \in \mathcal{H}$ such that $H[x] \subset B$ and $N[x] \subset G$. Since $\mathcal{H}$ is a filterbase, there exists $M \in \mathcal{H}$ such that $M \subset H \cap N$. Referring to definition 6.1.1 we see

$$M[x] \subset (H[x] \cap N[x]) \subset B \cap G,$$

so $B \cap G \in \mathcal{G}_{\mathcal{H}}$. Finally, $\emptyset$, $X \in \mathcal{G}_{\mathcal{H}}$.

We note that if for a point $x \in X$ there is $H \in \mathcal{H}$ such that $x \notin \sigma H$, i.e. no element of H covers x, then $\{x\}$ is both open and closed in the $\mathcal{H}$-topology.

<u>Remark</u>. Throughout the remainder of this chapter all topological concepts refer to the $\mathcal{H}$-topology.

The following lemmas will be needed later.

6.3 LEMMAS. If H, H_1, $H_2 \in \mathcal{H}$; for each $i \in I$, $A_i \subset X$; and $A \subset X$, $B \subset X$, then

.1 $H[\bigcup_{i \in I} A_i] = \bigcup_{i \in I} H[A_i]$,

.2 $H_1[H_2[A]] = \bigcup_{x \in A} H_1[H_2[x]]$, and

.3 $H[A] \cap B = \emptyset$ iff $A \cap H[B] = \emptyset$.

<u>Proof</u> <u>of</u> .1: Let $x \in \bigcup_{i \in I} H[A_i]$. Then $x \in H[A_i]$ for some $i \in I$. But $A_i \subset \bigcup_{i \in I} A_i$, whence $H[A_i] \subset H[\bigcup_{i \in I} A_i]$, and so

$$H[\bigcup_{i \in I} A_i] \supset \bigcup_{i \in I} H[A_i].$$

On the other hand,

$$H[\bigcup_{i\in I}A_i]\subset\bigcup_{i\in I}H[A_i],$$

for

$$x\in H[\bigcup_{i\in I}A_i] \Rightarrow \text{for some } y\in\bigcup_{i\in I}A_i,\ x\in H[y]$$

$$\Rightarrow \text{for some } i\in I, \text{ there exists } y\in A_i \text{ such that } x\in H[y]$$

$$\Rightarrow \text{for some } i\in I,\ x\in H[A_i]$$

$$\Rightarrow x\in\bigcup_{i\in I}H[A_i].$$

<u>Proof of</u> .2:

$$H_1[H_2[A]] = H_1[\bigcup_{x\in A}H_2[x]] = \bigcup_{x\in A}H_1[H_2[x]],$$

by definition 6.1.2 and lemma 6.3.1.

<u>Proof of</u> .3: By definition 6.1.2,

$$H[A]\cap B = \emptyset \text{ iff } A\cap B = \emptyset \text{ and there exists no } f\in H \text{ such that } f\cap A \neq \emptyset \text{ and } f\cap B \neq \emptyset$$

$$\text{iff } A\cap H[B] = \emptyset .$$

7. <u>Conditions on a filterbase in X.</u>

We now introduce conditions on $\mathcal{H}$ which will allow us to draw conclusions about the $\mathcal{H}$-topology and about properties of the measure ν .

(7I) Given $x\in X$ and $H\in\mathcal{H}$, there exist H_1, $H_2\in\mathcal{H}$ such that $H_1[H_2[x]]\subset H[x]$.

(7II) Given $H\in\mathcal{H}$, there exist H_1, $H_2\in\mathcal{H}$ such that for every $x\in X$,

$$H_1[H_2[x]]\subset H[x].$$

(We note that by 6.1.2 an equivalent statement would be that for every $A\subset X$, $H_1[H_2[A]]\subset H[A]$.)

(7III) If A is closed, B is open and $A \subset B$, then there exists $H \in \mathcal{H}$ such that $H[A] \subset B$.

(7IV) There exists a sequence H in $\mathcal{H}$ such that for every $N \in \mathcal{H}$, there exists $n \in \omega$ such that $H_n \subset N$.

(7V) Given an open cover of X, there exists $H \in \mathcal{H}$ which refines this cover.

7.1 REMARKS.

.1 If $\mathcal{H}$ satisfies (7II), then it satisfies (7I).

.2 If $\mathcal{H}$ satisfies (7V), then it satisfies (7III).

Proof: Suppose A is closed, B is open and $A \subset B$. Then $\mathcal{E} = \{B, X \sim A\}$ is an open cover of X. By (7V), there exists $H \in \mathcal{H}$ which refines $\mathcal{E}$. Now any element of $\mathcal{E}$, and hence also of H, which intersects A is contained in B so $H[A] \subset B$.

8. Properties of the $\mathcal{H}$-topology.

In this section we deduce properties of the $\mathcal{H}$-topology which result from imposing conditions (7I) and (7II) on $\mathcal{H}$.

8.1 THEOREM. Suppose $\mathcal{H}$ satisfies (7I). Then

.1 If $H \in \mathcal{H}$, $A \subset X$, then there exists an open G such that $A \subset G \subset H[A]$.

.2 For $x \in X$, $\{H[x] : H \in \mathcal{H}\}$ is a base for the neighborhood system of x. (H[x] itself may not be open. See example 8.4.)

.3 for $A \subset X$, the closure of A,

$$\overline{A} = \bigcap_{H \in \mathcal{H}} H[A],$$

and if for some sequence H in $\mathcal{H}$,

$$A = \bigcap_{n \in \omega} H_n[A],$$

then A is closed.

.4 The $\mathcal{H}$-topology is regular.

.5 The $\mathcal{H}$-topology is Hausdorff iff

$$\bigcap_{H\in\mathcal{H}} H[x] = \{x\} \quad \text{for each } x\in X.$$

<u>Proof of</u> .1: Given $x\in X$ and $H\in\mathcal{H}$, we show there exists an open set G such that $x\in G\subset H[x]$. Let

$$G = \{y\in X; \ \text{for some } N\in\mathcal{H},\ N[y]\subset H[x]\}.$$

Clearly $G\subset H[x]$. Let $y\in G$. Then for some $N\in\mathcal{H}$, $N[y]\subset H[x]$. Choose N_1, $N_2\in\mathcal{H}$ such that

$$N_1[N_2[y]]\subset N[y].$$

Then for any $z\in N_2[y]$,

$$N_1[z]\subset N_1[N_2[y]]\subset H[x],$$

so $N_2[y]\subset G$. Hence G is open.

.2 follows immediately from .1 and the definition of the $\mathcal{H}$-topology.

<u>Proof of</u> .3: $\overline{A}\subset\bigcap_{H\in\mathcal{H}} H[A]$: Given $H\in\mathcal{H}$, suppose $x\notin H[A]$. Then $\{x\}\cap H[A] = \emptyset$, whence by lemma 6.3.3, $H[x]\cap A = \emptyset$. By 8.1.1 there exists a neighborhood of x free of points of A and so $x\notin\overline{A}$. We conclude that $\overline{A}\subset H[A]$ for every $H\in\mathcal{H}$.

$\overline{A}\supset\bigcap_{H\in\mathcal{H}} H[A]$: Suppose $x\notin\overline{A}$. Then since $X\sim A$ is open, there exists $H\in\mathcal{H}$ such that $H[x]\cap\overline{A} = \emptyset$, by definition 6.1.3. Again using lemma 6.3.3 we have $x\notin H[\overline{A}]$. But

$$H[\overline{A}]\supset H[A]\supset\bigcap_{H\in\mathcal{H}} H[A],$$

and hence $x\notin\bigcap_{H\in\mathcal{H}} H[A]$.

If for some sequence H in $\mathcal{H}$,

$$A = \bigcap_{n\in\omega} H_n[A],$$

then

$$A \subset \bigcap_{H \in \mathcal{H}} H[A] \subset \bigcap_{n \in \omega} H_n[A] = A,$$

and $A = \overline{A}$.

<u>Proof of</u> .4: Let A be closed, $x \notin A$. By definition there exists $H \in \mathcal{H}$ such that $H[x] \cap A = \emptyset$. Choose $H_1, H_2 \in \mathcal{H}$ such that

$$H_1[H_2[x]] \subset H[x].$$

Then

$$H_1[H_2[x]] \cap A = \emptyset,$$

and so by lemma 6.3.3,

$$H_2[x] \cap H_1[A] = \emptyset.$$

By 8.1.1 there exist disjoint open sets G_2 and G_1 such that

$$x \in G_2 \subset H_2[x] \quad \text{and} \quad A \subset G_1 \subset H_1[A].$$

<u>Proof of</u> .5: Suppose the $\mathcal{H}$-topology is Hausdorff and $x \in X$. For any $y \neq x$, there exists $H \in \mathcal{H}$ such that $y \notin H[x]$. Hence $y \notin \bigcap_{H \in \mathcal{H}} H[x]$. (This does not use condition (7I).)

Now suppose $\bigcap_{H \in \mathcal{H}} H[x] = \{x\}$ for each $x \in X$. Then by .3 and .4, the $\mathcal{H}$-topology is T_1 and regular, and hence Hausdorff.

8.2 THEOREM. If $\mathcal{H}$ satisfies (7II), then there is a uniformity for X such that the uniform topology is the $\mathcal{H}$-topology and hence the $\mathcal{H}$-topology is completely regular.

<u>Proof</u>: Let $M = \{\{x\} : x \in X\}$ and set

$$U_H = \sigma\{h \times h : h \in H \cup M\}$$

and

$$\mathcal{U} = \{U_H : H \in \mathcal{H}\}.$$

We now check:

a) If $U \in \mathcal{U}$, then $\Delta \subset U$.

b) If $U \in \mathcal{U}$, then $U = U^{-1}$ since every U_H is symmetric.

c) If $U \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$.

Suppose $U \in \mathcal{U}$. Then for some $H \in \mathcal{H}$,

$$U = \sigma\{h \times h : h \in H \cup M\}.$$

Choose N_1, $N_2 \in \mathcal{H}$ such that for every $x \in X$,

$$N_1[N_2[x]] \subset H[x].$$

Now let $N \in \mathcal{H}$, $N \subset N_1 \cap N_2$ to get

$$N[N[x]] \subset H[x] \text{ for every } x \in X,$$

and set

$$V = \sigma\{f \times f : f \in N \cup M\}.$$

Suppose $(x,y) \in V \circ V$. Then for some z, $(x,z) \in V$ and $(z,y) \in V$. By definition of V, there exist f_1, $f_2 \in N \cup M$ such that $x,z \in f_1$ and $z,y \in f_2$, whence $x \in N[z]$ and $z \in N[y]$. But $\{z\} \subset N[y]$ implies

$$N[z] \subset N[N[y]] \subset H[y]$$

and hence $x \in H[y]$. By definition then, there exists $h \in H \cup M$ such that $x,y \in h$, and so $(x,y) \in h \times h \subset U$, and $V \circ V \subset U$.

d) If $U,V \in \mathcal{U}$, then for some $W \in \mathcal{U}$, $W \subset U \cap V$.

Suppose $U,V \in \mathcal{U}$. Then there exist H_1, $H_2 \in \mathcal{H}$ such that

$$U = \sigma\{h \times h : h \in H_1 \cup M\}$$

and

$$V = \sigma\{h \times h : h \in H_2 \cup M\}.$$

Choose $H_3 \in \mathcal{H}$, $H_3 \subset H_1 \cap H_2$ and set

$$W = \sigma\{h \times h : h \in H_3 \cup M\}.$$

Now let $(x,y) \in W$. Then for some $h \in H_3 \cup M$, $(x,y) \in h \times h$. But $h \in H_3 \cup M \subset (H_1 \cap H_2) \cup M$, so $h \in H_1 \cup M$ and $h \in H_2 \cup M$. Hence $(x,y) \in U$ and $(x,y) \in V$, so $(x,y) \in U \cap V$. We conclude that $W \subset U \cap V$.

By theorem 6.2 of Kelley, $\mathcal{U}$ is a base for a uniformity for X. We show now that the uniform topology is just the

$\mathcal{H}$-topology. Let $G \subset X$. Then

G is open in the uniform topology

iff for each $x \in G$, there exists $U \in \mathcal{U}$ such that $U[x] \subset G$

iff for each $x \in G$, there exists $U \in \mathcal{U}$ such that

$$\{y : (x,y) \in U\} \subset G$$

iff for each $x \in G$, there exists $H \in \mathcal{H}$ such that

$$\{y : (x,y \in h \times h \text{ for some } h \in H \cup M\} \subset G$$

iff for each $x \in G$, there exists $H \in \mathcal{H}$ such that

$$\{y : x,y \in h \text{ for some } h \in H \cup M\} \subset G$$

iff for each $x \in G$, there exists $H \in \mathcal{H}$ such that $H[x] \subset G$

iff G is open in the $\mathcal{H}$-topology.

8.3 LEMMA. If $\mathcal{H}$ satisfies (7I) and (7IV), and A is closed, then there exists a sequence H in $\mathcal{H}$ such that

$$A = \bigcap_{n \in \omega} H_n[A].$$

Proof: Using (7IV) let H be a sequence in $\mathcal{H}$ such that for every $N \in \mathcal{H}$, there exists $n \in \omega$ such that $H_n \subset N$. Then since A is closed we have by 8.1.3

$$A = \overline{A} = \bigcap_{N \in \mathcal{H}} N[A] \supset \bigcap_{n \in \omega} H_n[A] \supset A.$$

8.4 EXAMPLE. Let $X = R$, $H_r = \{\{x,y\} : |x-y| \leq r\} \cup \{\emptyset\}$, $\mathcal{H} = \{H_r : r>0\}$. For $A \in \sigma\mathcal{H}$ let

$$\tau A = \begin{cases} \text{diam } A \text{ if } A \neq \emptyset \\ 0 \text{ if } A = \emptyset . \end{cases}$$

Then $\mathcal{H}$ is a filterbase in X; the $\mathcal{H}$-topology is the usual topology; for any $x \in X$, $r>0$,

$$H_r[x] = [x-r, x+r],$$

a closed neighborhood of x; $\mathcal{H}$ satisfies the four conditions

(7I) - (7IV) but not (7V); τ is a gauge on $\sigma\mathcal{H}$; and for $A \subset X$,

$$\nu^{(\mathcal{H},\tau)} A = \begin{cases} 0 & \text{if } A \text{ is countable} \\ \infty & \text{if } A \text{ is uncountable.} \end{cases}$$

8.5 REMARK. Even for a given fixed gauge τ, the filterbase $\mathcal{H}$, the measure $\nu^{(\mathcal{H},\tau)}$ and the $\mathcal{H}$-topology are not necessarily in one-to-one correspondence. We will see later examples of

i) different filterbases in X giving the same topology and measure (14.5),

ii) different filterbases yielding the same topology but different measures (14.6), and

iii) different filterbases inducing different topologies but the same measure (14.7).

9. Measurability theorems.

The following definition and lemma are taken from a paper by Bledsoe and Morse [2].

9.1 DEFINITION. For φ a measure on X, A is φ-compact iff $A \subset X$ and given any $\varepsilon > 0$, finite submeasure θ of φ, and open cover $\mathcal{B}$ of A, there is a finite subfamily $\mathcal{E}$ of $\mathcal{B}$ such that

$$\theta A \leqslant \theta(A \cap \sigma\mathcal{E}) + \varepsilon .$$

9.2 LEMMA. A closed subset of a φ-compact set is φ-compact.

We first state two theorems and a corollary on ν-measurability of sets characterized in terms of the filterbase $\mathcal{H}$.

9.3 THEOREM. If for some sequence B,

$$A = \bigcap_{n\in\omega} B_n ,$$

where for each $n\in\omega$ there exists $M_{n+1}\in\mathcal{H}$ such that

$$M_{n+1}[B_{n+1}]\subset B_n\subset X,$$

then A is ν-measurable.

9.4 COROLLARY. If $\mathcal{H}$ satisfies (7II), $A\subset X$, and for some sequence H in $\mathcal{H}$,

$$A = \bigcap_{n\in\omega} H_n[A],$$

then A is ν-measurable.

9.5 THEOREM. If $\mathcal{H}$ satisfies (7I), A is ν-compact, and for some sequence H in $\mathcal{H}$,

$$A = \bigcap_{n\in\omega} H_n[A],$$

then A is ν-measurable.

We now relate the restrictions on A in the above theorems to topological properties of A and, using additional conditions on $\mathcal{H}$, we obtain a number of theorems on the measurability of purely topological sets.

9.6 THEOREM. If $\mathcal{H}$ satisfies (7I), then compact $\mathcal{G}_\delta$ sets are ν-measurable.

9.7 THEOREM. If $\mathcal{H}$ satisfies (7II) and (7III), then closed $\mathcal{G}_\delta$ sets are ν-measurable.

9.8 THEOREM. If $\mathcal{H}$ satisfies (7II) and (7IV), then closed sets are ν-measurable.

9.9 THEOREM. If $\mathcal{H}$ satisfies (7I) and (7V), then closed $\mathcal{G}_\delta$ sets are ν-measurable.

9.10 THEOREM. If $\mathcal{H}$ satisfies (7I), (7IV), and (7V), then closed sets are ν-measurable.

9.11 REMARKS. We note that if there is any subspace $X' \subset X$ which is such that for any $x \in X'$, there is some $H \in \mathcal{H}$ such that no element of H covers x, i.e. $x \notin \sigma H$, then for every $A \subset X'$, $\nu A = \infty$; and by the comment at the end of theorem 6.2, $\mathcal{G}_{\mathcal{H}}$ is discrete on X'. Thus the discrete topology on X' reflects the fact that all subsets of X' are ν-measurable.

Now it may happen as a result of the nature of the family $\mathcal{a}$ that the class of measurable sets is larger than that given us by any of the theorems 9.6 to 9.10, using the filterbase $\mathcal{H}$. (For example, if $\mathcal{a}$ is the family of singletons, then all subsets of X are ν-measurable, a result which is independent of the filterbase $\mathcal{H}$.) In this case, it may be of some advantage to consider the subfilterbase of $\mathcal{H}$,

$$\eta = \{H \cap \mathcal{a} \ : \ H \in \mathcal{H}\}.$$

Evidently the measure $\nu^{(\eta,\tau)} = \nu^{(\mathcal{H},\tau)}$, but the η-topology, $\mathcal{G}_\eta$, may be strictly larger than $\mathcal{G}_{\mathcal{H}}$. If this is the case, and if η satisfies the requisite conditions, we may be able to apply one of the theorems 9.6 to 9.10 with the filterbase η to obtain a stronger result than that obtained using $\mathcal{H}$. (For example, if in the case above of $\mathcal{a}$ the family of singletons, we form the filterbase η, then trivially η satisfies (7II) and (7IV), and $\mathcal{G}_\eta$ is the discrete topology. Then by theorem 9.8, all subsets of X are ν-measurable.) However, η may not satisfy enough conditions to allow us to apply any theorems from chapter I (see example 11.6), so we cannot automatically use $\mathcal{H}$ to get stronger measurability results.

Again, it may happen that although $\mathcal{H}$ itself does not satisfy enough conditions, another filterbase $\mathcal{M}$ can be found such that

i) $\mathcal{H}$ is a subfilterbase of $\mathcal{M}$, so that $\nu^{(\mathcal{M})} = \nu^{(\mathcal{H})} = \nu^{(\mathcal{H})}$,

ii) $\mathcal{G}_{\mathcal{M}}$ is strictly larger than $\mathcal{G}_{\mathcal{H}}$, and

iii) $\mathcal{M}$ satisfies conditions allowing application of some theorem giving a stronger result than that obtained using $\mathcal{H}$. (see example 11.6) Unfortunately, we know of no general method, in such a case, of choosing a filterbase in X, optimum in the sense that using it we obtain the largest possible class of measurable sets.

We note also that the nature of $\mathcal{A}$ may result in a large class of measurable sets at the same time that the $\mathcal{H}$-topology is no larger than the $\mathcal{H}$-topology, i.e. the $\mathcal{H}$-topology may not be large enough to reflect the class of measurable sets. (For instance, if in example 8.4 H_r consisted of all sets of diameter $\leq r$, while $\mathcal{A}$ consisted of all doubletons, the same measure would be obtained, under which all subsets of X are measurable, while both the $\mathcal{H}$ and $\mathcal{H}$-topologies would be the usual topology, and the best theorem obtainable would be 9.8, giving closed sets measurable.)

PROOFS

9.12 LEMMA. If $A \subset X$, $B \subset X$, and there exists $M \in \mathcal{N}$ such that $M[A] \cap B = \emptyset$, then $\nu(A \cup B) = \nu A + \nu B$.

Proof: Suppose $M \in \mathcal{N}$, $M[A] \cap B = \emptyset$, and $\nu(A \cup B) < \infty$.

Let $N \in \mathcal{N}$, $N \subset M$. Then also $N[A] \cap B = \emptyset$. By definition 6.1.2 no $h \in N$ can intersect both A and B, so any cover of $A \cup B$ by elements of $N \cap \mathcal{a}$ can be separated into disjoint covers of A and B. Checking 5.1.1 we see that

$$\nu_N (A \cup B) \geqslant \nu_N A = \nu_N B.$$

Since ν_N is a measure, we have the inequality the other way also, whence

$$\nu_N (A \cup B) = \nu_N A + \nu_N B$$

for every $N \in \mathcal{N}$ such that $N \subset M$. Hence by remark 5.3

$$\nu(A \cup B) = \lim_{N \in \mathcal{N}} \nu_N(A \cup B) = \lim_{N \in \mathcal{N}} (\nu_N A + \nu_N B)$$

$$= \lim_{N \in \mathcal{N}} \nu_N A + \lim_{N \in \mathcal{N}} \nu_N B = \nu A + \nu B.$$

Proof of 9.3: We use lemma 4.3 with $D_n = B_n$ for each $n \in \omega$. Let $\varepsilon > 0$ and $T \subset X$ such that $\nu T < \infty$. To check 4) note

$$M_{n+1}[B_{n+1}] \cap (X \sim B_n) = \emptyset \text{ for each } n \in \omega,$$

from which it follows that for each $n \in \omega$ and $P \subset T$,

$$M_{n+1}[P \cap B_{n+1}] \cap (P \sim B_n) = \emptyset .$$

Applying lemma 9.12 we obtain

$$\nu((P \cap B_{n+1}) \cup (P \sim B_n)) = \nu(P \cap B_{n+1}) + \nu(P \sim B_n)$$

for all $n \in \omega$ and $P \subset T$.

Proof of 9.4: We show first that A can be put in the form

$$A = \bigcap_{n \in \omega} N_n[A],$$

where N is a sequence in $\mathcal{N}$ and for each $n \in \omega$

$$N_{n+1}[N_{n+1}[A] \subset N_n[A].$$

We construct the sequence N by recursion.

Let $N_o = H_o$ and suppose we have $N_i \in \mathcal{N}$ for $i = 1, \ldots, n$ such that

$$N_i[A] \subset H_i[A] \quad \text{for } i = 0, \ldots, n,$$

and

$$N_{i+1}[N_{i+1}[A]] \subset N_i[A] \quad \text{for } i = 1, \ldots, n-1.$$

We choose N_{n+1} as follows:

using (7II) choose $M \in \mathcal{N}$ such that

$M[M[A]] \subset N_n[A]$.

Then choose $N_{n+1} \in \mathcal{N}$ such that $N_{n+1} \subset M \cap H_{n+1}$. We have

i) $N_{n+1}[A] \subset H_{n+1}[A]$, and

ii) $N_{n+1}[N_{n+1}[A]] \subset N_n[A]$.

Now i) and ii) will be true for all $n \in \omega$. From i) we have

$$A \subset \bigcap_{n \in \omega} N_n[A] \subset \bigcap_{n \in \omega} H_n[A] = A$$

and so

$$A = \bigcap_{n \in \omega} N_n[A].$$

Setting $B_n = N_n[A]$, the conclusion follows by application of theorem 9.3.

Proof of 9.5: Let $T \subset X$, $\nu T < \infty$, $\varepsilon > 0$, and $\theta = \nu | T$. We employ lemma 4.3 to show that A is ν-measurable. Sequences C, D, M, and N are constructed by recursion. To start we set $C_o = C_1 = A$; $M_o = M_1 = H_o$; $H_o \supset N_o = N_1 \in \mathcal{N}$; $D_o = X$; and $D_1 = M_1[C_1] = H_o[A]$. Having obtained C_i, D_i, $M_i \in \mathcal{N}$ and $N_i \in \mathcal{N}$

satisfying

a) C_i is closed for $i = 0,\ldots,n$, ($C_0 = A$ is closed by 8.1.3)

b) $C_{i+1} \subset C_i \subset A$ for $i = 0,\ldots,n-1$,

c) $\theta C_{i-1} \leqslant \theta C_i + \varepsilon/2^{i-1}$ for $i = 1,\ldots,n$,

d) $D_i = M_i[C_i] \subset H_{i-1}[A]$ for $i = 1,\ldots,n$, and

e) $N_i[D_i] \subset D_{i-1}$ for $i = 1,\ldots,n$,

we construct C_{n+1}, D_{n+1}, M_{n+1}, and N_{n+1} as follows:
For each $x \in C_n$ choose, using (7I), H_{x1}, H_{x2}, H_{x3}, and $H_{x4} \in \mathcal{H}$ such that

$$H_{x1}[H_{x2}[H_{x3}[H_{x4}[x]]]] \subset M_n[x] \subset M_n[C_n] \subset D_n.$$

By a) and lemma 9.2, C_n is ν-compact. Since $\mathcal{H}$ satisfies (7I), for each $x \in C_n$ there is by 8.1.1 open G_x such that

$$x \in G_x \subset H_{x4}[x] .$$

Hence $\{G_x : x \in C_n\}$ is an open cover of C_n and by definition 9.1 there is a finite subset $Q \subset C_n$ such that

$$\theta C_n \leqslant \theta(C_n \cap \bigcup_{x\in Q} G_x) + \varepsilon/2^n$$

and so

$$\theta C_n \leqslant \theta(C_n \cap \bigcup_{x\in Q} H_{x4}[x]) + \varepsilon/2^n .$$

Now set

$$C_{n+1} = \bigcap_{H\in\mathcal{H}} H[C_n \cap \bigcup_{x\in Q} H_{x4}[x]] = \mathrm{Cl}(C_n \cap \bigcup_{x\in Q} H_{x4}[x]),$$

and choose $M_{n+1} \in \mathcal{H}$, $N_{n+1} \in \mathcal{H}$ such that

$$M_{n+1} \subset (\bigcap_{x\in Q} H_{x2}) \cap H_n ,$$

$$N_{n+1} \subset \bigcap_{x\in Q} H_{x1} ,$$

and set

$$D_{n+1} = M_{n+1}[C_{n+1}].$$

We now check:

a) C_{n+1} is closed.

b) $C_{n+1} \subset C_n \subset A$ since $C_{n+1} \subset \overline{C}_n = C_n \subset A$.

c) $\theta C_n \leqslant \theta C_{n+1} + \varepsilon/2^n$ since $C_{n+1} \supset (C_n \cap \bigcup_{x\in Q} H_{x4}[x])$.

d) $D_{n+1} = M_{n+1}[C_{n+1}] \subset H_n[A]$ since $C_{n+1} \subset A$, $M_{n+1} \subset H_n$.

d) $N_{n+1}[D_{n+1}] \subset D_n$: first,

$$C_{n+1} \subset \bigcap_{H\in\mathcal{H}} H[\bigcup_{x\in Q} H_{x4}[x]] = \bigcup_{x\in Q} \bigcap_{H\in\mathcal{H}} H[H_{x4}[x]]$$

$$\subset \bigcup_{x\in Q} H_{x3}[H_{x4}[x]].$$

The equality is obtained using theorem 8.1.3 and the fact that the closure of a finite union is the union of the individual closures. Now

$$\begin{aligned} N_{n+1}[D_{n+1}] &= N_{n+1}[M_{n+1}[C_{n+1}]] \\ &\subset N_{n+1}[M_{n+1}[(\bigcup_{x\in Q} H_{x3}[H_{x4}[x]])]] \\ &= \bigcup_{x\in Q} N_{n+1}[M_{n+1}[H_{x3}[H_{x4}[x]]]] \\ &\subset \bigcup_{x\in Q} H_{x1}[H_{x2}[H_{x3}[H_{x4}[x]]]] \subset D_n. \end{aligned}$$

The second to last inclusion follows from the choice of M_{n+1} and N_{n+1}, and the equality from lemma 6.3.1.

The completed sequences satisfy a), b), c), d) and e) for each $n\in\omega$. We now check that the sequence D satisfies the hypothesis of lemma 4.3.

1) $D_{n+1} \subset D_n$ by e).

2) $\bigcap_{n\in\omega} D_n \subset \bigcap_{n\in\omega} H_n[A] = A$ follows from d).

3) Using $A = C_1$, c) and induction, we have

$$\theta A \leqslant \theta C_n + \varepsilon(1 - 1/2^{n-1}) < \theta C_n + \varepsilon \quad \text{for every } n\in\omega,$$

or

$$\nu(T\cap A) \leqslant \nu(T\cap C_n) + \varepsilon \quad \text{for every } n\in\omega.$$

Since $C_n \subset D_n$ by d), we have finally

$$\nu(T\cap A) \leqslant \nu(T\cap D_n) + \varepsilon \quad \text{for every } n\in\omega.$$

4) It follows from e) that

$$N_n[P\cap D_n] \cap (P\sim D_{n-1}) = \emptyset$$

for any $P\subset T$ and $n \geqslant 1$. Lemma 9.12 then gives us

$$\nu((P\cap D_n)\cup(P\sim D_{n-1})) = \nu(P\cap D_n) + \nu(P\sim D_{n-1})$$

for every $P\subset T$ and $n\geqslant 1$.

Proof of 9.6: We show that for any compact $\mathcal{G}_\delta$ set A, there exists a sequence H in $\mathcal{H}$ such that

$$A = \bigcap_{n\in\omega} H_n[A],$$

where for each $n\in\omega$,

$$H_{n+1}[H_{n+1}[A]] \subset H_n[A].$$

Then setting $B_n = H_n[A]$, we apply theorem 9.3 to obtain the conclusion.

Suppose A is compact, and

$$A = \bigcap_{n\in\omega} G_n ,$$

where for each $n\in\omega$, G_n is open.

We assume $G_0 = X$, set $H_0 = \{X\} \cup \sigma\mathcal{H}$ and construct H_n recursively as follows:

For each $x \in A$, using (7I) choose $H_{nx} \in \mathcal{H}$ such that

$$H_{nx}[H_{nx}[x]] \subset G_n \text{ and}$$

$$H_{nx}[H_{nx}[H_{nx}[x]]] \subset H_{n-1}[x].$$

Now each $H_{nx}[x]$ contains an open set containing x by theorem 8.1.1 and since A is compact, a finite number of these open sets and hence of the sets $H_{nx}[x]$ covers A, i.e. there exists finite $Q \subset A$ such that

$$A \subset \bigcup_{x \in Q} H_{nx}[x].$$

Now choose $H_n \subset \bigcap_{x \in Q} H_{nx}$, $H_n \in \mathcal{H}$. Then

$$H_n[A] \subset H_n[\bigcup_{x \in Q} H_{nx}[x]] = \bigcup_{x \in Q} H_n[H_{nx}[x]]$$

by lemma 6.3.1. Since $H_n \subset H_{nx}$ for each $x \in Q$,

$$H_n[A] \subset \bigcup_{x \in Q} H_{nx}[H_{nx}[x]] \subset G_n.$$

Similarly,

$$H_n[H_n[A]] \subset \bigcup_{x \in Q} H_{nx}[H_{nx}[H_{nx}[x]]] \subset H_{n-1}[A].$$

Then

$$A \subset \bigcap_{n \in \omega} H_n[A] \subset \bigcap_{n \in \omega} G_n = A$$

and so

$$A = \bigcap_{n \in \omega} H_n[A].$$

<u>Proof of</u> 9.7: We show that for any closed $\mathcal{G}_\delta$ set A, there is a sequence H in $\mathcal{H}$ such that

$$A = \bigcap_{n \in \omega} H_n[A],$$

and apply corollary 9.4.

Suppose A is closed,

$$A = \bigcap_{n\in\omega} G_n,$$

where G_n is open for each $n\in\omega$. Using (7III), choose $H_n \in \mathcal{H}$ such that

$$H_n[A] \subset G_n \text{ for each } n\in\omega.$$

Then again

$$A \subset \bigcap_{n\in\omega} H_n[A] \subset \bigcap_{n\in\omega} G_n = A$$

and

$$A = \bigcap_{n\in\omega} H_n[A].$$

<u>Proof of</u> 9.8: We know by lemma 8.3 that for every closed set A, there exists a sequence H in $\mathcal{H}$ such that

$$A = \bigcap_{n\in\omega} H_n[A].$$

The conclusion follows from corollary 9.4.

<u>Proof of</u> 9.9: i) By 7.1.2 $\mathcal{H}$ satisfies (7III).

ii) If A is a closed $\mathcal{G}_\delta$ set, then for some sequence H in $\mathcal{H}$,

$$A = \bigcap_{n\in\omega} H_n[A].$$

The proof is contained in that of theorem 9.7.

iii) X is ν-compact.

Let $T \subset X$, $\nu T < \infty$, $\theta = \nu|T$, $\varepsilon > 0$, and $\mathcal{L}$ be an open cover of X. Using (7V), choose $N \in \mathcal{H}$, N refining $\mathcal{L}$. Now choose $M \in \mathcal{H}$ such that

$$\nu T \leqslant \nu_M T + \varepsilon/2.$$

Choose $H\in\mathcal{H}$, $H\subset N\cap M$, so by remark 5.3,

a) $\nu T\leqslant \nu_H T + \varepsilon/2$.

Since $\nu_H T<\infty$, choose countable $\mathcal{B}\subset H\cap\mathcal{A}$, $\mathcal{B}=\{B_i\}_{i\in\omega}$,

such that $T\subset\sigma\mathcal{B}$ and

$$\nu_H T \leqslant \sum_{i\in\omega}\tau B_i < \infty.$$

Now choose $K\in\omega$ such that

$$\sum_{i=K+1}^{\infty}\tau B_i < \varepsilon/2.$$

Since N is a refinement of $\mathcal{L}$ and $H\subset N$, for each $i\leqslant K$ choose $G_i\in\mathcal{L}$ such that $B_i\subset G_i$ and let

$$\mathcal{E}=\{G_i : i\leqslant K\}.$$

Now

$$(T\sim\sigma\mathcal{E})\subset\sigma\{B_i : i>K\}$$

and so

$$\nu_H(T\sim\sigma\mathcal{E})\leqslant\varepsilon/2.$$

Hence

$$\begin{aligned}\nu_H T\leqslant\nu_H(T\cap\sigma\mathcal{E}) + \nu_H(T\sim\sigma\mathcal{E})&\leqslant\nu_H(T\cap\sigma\mathcal{E}) + \varepsilon/2\\ &\leqslant\nu(T\cap\sigma\mathcal{E}) + \varepsilon/2,\end{aligned}$$

and by a),

$$\nu T\leqslant\nu(T\cap\sigma\mathcal{E}) + \varepsilon .$$

Hence

$$\theta X\leqslant\theta(X\cap\sigma\mathcal{E}) + \varepsilon$$

and by definition 9.1, X is ν-compact.

The desired conclusion now follows from ii), iii), lemma 9.2 and theorem 9.5.

Proof of 9.10: We know from the proof of theorem 9.9 that if $\mathcal{H}$ satisfies (7V), then closed sets are ν-compact, and from lemma 8.3 that if $\mathcal{H}$ satisfies (7I) and (7IV), then for every closed set A, there is a sequence H in $\mathcal{H}$ such that

$$A = \bigcap_{n\in\omega} H_n[A].$$

The conclusion follows by application of theorem 9.5.

10. Approximation theorems.

We consider first several theorems on approximation from outside in which the only restriction on the set to be approximated is that its measure be finite. The restriction that elements of $\mathcal{A}$ be ν-measurable sets is necessary in all the theorems of this section but the first.

10.1 THEOREM. Suppose $\mathcal{H}$ satisfies (7IV) and $A \subset X$. If for every $H \in \mathcal{H}$ there is a countable subfamily of $H \cap \mathcal{A}$ which covers A (in particular if $\nu A < \infty$), then there exists $B \in \mathcal{A}_{\sigma\delta}$ such that $B \supset A$ and $\nu B = \nu A$.

10.2 COROLLARY. If $\mathcal{H}$ satisfies (7IV) and $\mathcal{A} \subset \mathcal{M}_\nu$, then ν is a regular measure.

10.3 THEOREM. Suppose $\mathcal{A} \subset \mathcal{M}_\nu$, $\nu A < \infty$, and $E \subset A$. Then given $\varepsilon > 0$, there exists $B \in \mathcal{A}_\sigma$ such that $E \subset B$ and $\nu(A \cap B) \leqslant \nu E + \varepsilon$.

10.4 COROLLARY. Suppose $\mathcal{A} \subset \mathcal{M}_\nu$, $\nu A < \infty$, and $E \subset A$. Then there exists $D \in \mathcal{A}_{\sigma\delta}$ such that $E \subset D$ and $\nu(A \cap D) = \nu E$.

10.5 COROLLARY. If $X = \bigcup_{n\in\omega} A_n$ where for each $n\in\omega$, $A_n \in \mathcal{M}_\nu$ and $\nu A_n < \infty$, and $\mathcal{A} \subset \mathcal{M}_\nu$, then ν is a regular measure.

By putting further restrictions on the approximated set, we can get the following results on approximation from inside.

10.6 THEOREM. Suppose $\mathcal{A} \subset \mathcal{B} \subset \mathcal{M}_\nu$, $A \in (\mathcal{B}_\sigma)^\sim$, (see 1.6) $\nu A < \infty$, $E \subset A$, and $E \in \mathcal{M}_\nu$. Then given $\varepsilon > 0$ there exists $C \in (\mathcal{B}_\sigma)^\sim$ such that $C \subset E$ and $\nu(E \sim C) < \varepsilon$.

10.7 THEOREM. Suppose $\mathcal{A} \subset \mathcal{B} \subset \mathcal{M}_\nu$, $A \in (\mathcal{B}_{\sigma\delta})^\sim$, $\nu A < \infty$, $E \subset A$, and $E \in \mathcal{M}_\nu$. Then there exists $C \in (\mathcal{B}_{\sigma\delta})^\sim$ such that $C \subset E$ and $\nu(E \sim C) = 0$.

10.8 THEOREM. Suppose $\mathcal{A} \subset \text{Borel}\,\mathcal{B} \subset \mathcal{M}_\nu$, $A \in \text{Borel}\,\mathcal{B}$ (see 1.14), and $\nu A < \infty$. Then for each $E \subset A$ there exists $B \in \text{Borel}\,\mathcal{B}$ such that $E \subset B$ and $\nu E = \nu B$; and for each ν-measurable $E \subset A$, there exists $C \in \text{Borel}\,\mathcal{B}$ such that $C \subset E$ and $\nu(E \sim C) = 0$.

If it happens that the sets of $\mathcal{A}$ have some topological properties and are ν-measurable (e.g. the open sets in the classical Hausdorff measure theory), we obtain in the above theorems approximating sets which also have topological properties. If in our hypotheses we restrict $\mathcal{A}$ to open sets, require that open sets be ν-measurable and put additional restrictions on $\mathcal{H}$ and X, we obtain some sharper results. (Recall that $\mathcal{G}$ and $\mathcal{F}$ denote respectively the families of open and closed sets.)

10.9 THEOREM. Suppose $\mathcal{A}\subset\mathcal{G}\subset\mathfrak{M}_\nu$, $\mathcal{H}$ satisfies (7I) and (7IV), $E\subset X$, $\nu E<\infty$, and $E\in\mathfrak{M}_\nu$. Then there exist $A\in\mathcal{G}_\delta$ such that $A\supset E$ and $\nu(A\sim E)=0$, and $C\in\mathcal{F}_\sigma$ such that $C\subset E$ and $\nu(E\sim C)=0$.

10.10 COROLLARY. Suppose $\mathcal{A}\subset\mathcal{G}\subset\mathfrak{M}_\nu$, $\mathcal{H}$ satisfies (7I) and (7IV), $E\subset X$, $E\in\mathfrak{M}_\nu$, and X is σ-finite. Then the conclusions of theorem 10.9 still hold.

It is not the case that the existence of a $\mathcal{G}_\delta$ set covering $E\subset X$ and having the same measure implies that given $\varepsilon>0$, there exists $G\in\mathcal{G}$ such that $G\supset E$ and $\nu G<\nu E+\varepsilon$. It may happen that all non-empty open sets have infinite measure (as, for example, with counting measure on R or on the rationals, and Hausdorff $\frac{1}{2}$-dimensional measure on R). To obtain this conclusion we need an additional restriction on the space.

10.11 THEOREM. Suppose $\mathcal{A}\subset\mathcal{G}\subset\mathfrak{M}_\nu$, $\mathcal{H}$ satisfies (7I) and (7IV), $E\subset X$, $E\in\mathfrak{M}_\nu$, and $X=\bigcup_{n\in\omega}A_n$, where for each $n\in\omega$, $\nu A_n<\infty$ and $A_n\in\mathcal{G}$. Then given $\varepsilon>0$, there exist open $G\supset E$ such that $\nu(G\sim E)<\varepsilon$ and closed $F\subset E$ such that $\nu(E\sim F)<\varepsilon$.

PROOFS

Proof of 10.1: Using (7IV), choose a sequence H in $\mathcal{H}$ such that for every $N\in\mathcal{H}$, there exists $n\in\omega$ such that $H_n\subset N$. For each $n\in\omega$ choose countable $\mathcal{B}_n\subset H_n\cap\mathcal{A}$ such that $A\subset\sigma\mathcal{B}_n$ and

$$\nu_{H_n}(\sigma\mathcal{B}_n)\leqslant\sum_{D\in\mathcal{B}_n}\tau D\leqslant\nu_{H_n}A+1/n.$$

Let

$$B = \bigcap_{n=0}^{\infty}(\sigma\mathcal{B}_n) \in \mathcal{A}_{\sigma\delta}.$$

Then $B \supset A$ and

$$\nu_{H_n} B \leqslant \nu_{H_n}(\sigma\mathcal{B}_n) \leqslant \nu_{H_n} A + 1/n \text{ for every } n\in\omega.$$

By remark 5.3, taking the limit as $n \to \infty$ gives $\nu B \leqslant \nu A$. Since $B \supset A$, we have $\nu B \geqslant \nu A$, and so $\nu B = \nu A$.

<u>Proof of</u> 10.2: 10.2 is a direct consequence of 10.1.

<u>Proof of</u> 10.3: Choose $H \in \mathcal{H}$ such that

$$\nu A \leqslant \nu_H A + \varepsilon/2.$$

Suppose $B \subset X$ is ν-measurable. Then

$$\begin{aligned}\nu(A\cap B) + \nu(A\sim B) = \nu A &\leqslant \nu_H A + \varepsilon/2\\ &\leqslant \nu_H (A\cap B) + \nu_H(A\sim B) + \varepsilon/2\\ &\leqslant \nu_H (A\cap B) + \nu(A\sim B) + \varepsilon/2.\end{aligned}$$

Cancelling $\nu(A\sim B)$ in the first and last expressions gives

a) $\nu(A\cap B) \leqslant \nu_H (A\cap B) + \varepsilon/2$ for ν-measurable B.

Now given $E \subset A$, choose countable $\mathcal{B} \subset H \cap \mathcal{A}$ such that $E \subset \sigma\mathcal{B} = B$ and

$$\sum_{D\in\mathcal{B}} \tau D < \nu_H E + \varepsilon/2.$$

But $B \in \mathcal{A}_\sigma$ and so is ν-measurable, whence by a),

$$\begin{aligned}\nu(A\cap B) &\leqslant \nu_H (A\cap B) + \varepsilon/2 \leqslant \nu_H B + \varepsilon/2\\ &\leqslant \sum_{D\in\mathcal{B}} \tau D + \varepsilon/2 \qquad \text{(since } \mathcal{B} \text{ is a cover of B)}\\ &\leqslant \nu_H E + \varepsilon \leqslant \nu E + \varepsilon.\end{aligned}$$

10.4 follows immediately from 10.3, and 10.5 directly from 10.4.

Proof of 10.6: By theorem 10.3 there exists $B \in \mathcal{A}_\delta$ such that $A \sim E \subset B$ and

$$\nu(A \cap B) \leqslant \nu(A \sim E) + \varepsilon .$$

Now $A \sim E$ is ν-measurable so

$$\nu(E \cap B) = \nu((A \cap B) \sim (A \sim E)) = \nu(A \cap B) - \nu(A \sim E) < \varepsilon.$$

Setting $C = A \sim B$ we have by definition 1.6, $C \in (\mathcal{B}_\sigma)^\sim$, and since $E \cap B = E \sim C$,

$$\nu(E \sim C) < \varepsilon.$$

Proof of 10.7: The proof is identical to that of 10.6 except that the result of theorem 10.4 is used instead of that of 10.3.

Proof of 10.8: We obtain B from corollary 10.4 and C from theorem 10.7.

Proof of 10.9: i) Use theorem 10.1 to choose $A \in \mathcal{A}_{\sigma\delta} \subset \mathcal{D}_\delta$ such that $A \supset E$ and $\nu A = \nu E$. Since E is ν-measurable and $\nu E < \infty$, we have $\nu(A \sim E) = 0$.

ii) We show now that if $B \in \mathcal{D}_\delta$ and $\nu B < \infty$, there exists $D \in \mathfrak{F}_\sigma$ such that $D \subset B$ and $\nu(B \sim D) = 0$.

By lemma 8.3 and theorem 8.1.1 we have $\mathfrak{F} \subset \mathcal{D}_\delta$, so $\mathcal{D} \subset \mathfrak{F}_\sigma$ and we may set, for $B \in \mathcal{D}_\delta$,

$$B = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} F(n,i),$$

where for each $n \in \omega$ and $i \in \omega$, $F(n,i) \in \mathfrak{F}$.

We may assume that $F(n,i+1) \supset F(n,i)$ for each $i \in \omega$. By corollary 10.2, ν is a regular measure, so by theorem 4.2 we have for each $n \in \omega$,

$$\nu B = \nu(B \cap \bigcup_{i=1}^{\infty} F(n,i)) = \lim_i \nu(B \cap F(n,i)).$$

Hence for each $n\in\omega$ there exists a sequence i_n such that for each $k\in\omega$,

$$\nu(B\sim[B\cap F(n,i_{n_k})]) = \nu B - \nu(B\cap F(n,i_{n_k})) \leq \frac{1}{k2^n},$$

since $\nu B<\infty$ and $B\cap F(n,i_{n_k})$ is ν-measurable.

Let

$$F(k) = \bigcap_{n=1}^{\infty} F(n,i_{n_k}) \quad \text{for each } k\in\omega.$$

Then $F(k)\in\mathscr{F}$ and

$$F(k)\subset\bigcup_{i=1}^{\infty} F(n,i) \text{ for every } n\in\omega,$$

whence $F(k)\subset B$ and

$$\begin{aligned}\nu B - \nu F(k) &= \nu(B\sim F(k)) = \nu(B\sim\bigcap_{n=1}^{\infty} F(n,i_{n_k}))\\ &= \nu(\bigcup_{i=1}^{\infty}[B\sim(B\cap F(n,i_{n_k}))]) \text{ by de Morgan's law,}\\ &\leq \sum_{n=1}^{\infty}\frac{1}{k2^n} = 1/k.\end{aligned}$$

Set

$$D = \bigcup_{k=1}^{\infty} F(k).$$

Then $D\in\mathscr{F}_\sigma$, $D\subset B$, and

$$\nu(B\sim D) = \nu B - \nu D \leq 1/k \text{ for every } k\in\omega.$$

Hence

$$\nu(B\sim D) = 0.$$

Now let $E\in\mathfrak{M}_\nu$, $\nu E<\infty$. Using i), choose $B\in\mathscr{G}_\delta$ such that $B\supset E$ and $\nu B = \nu E$. Since $\nu(B\sim E) = 0$, we choose $Q\in\mathscr{G}_\delta$ such that $Q\supset B\sim E$ and $\nu Q = 0$. Using ii), choose $D\in\mathscr{F}_\sigma$ such that $D\subset B$ and $\nu(B\sim D) = 0$. Now

$$\nu E = \nu B = \nu D = \nu D - \nu Q = \nu(D\sim Q).$$

Set $C = D \sim Q$. Then $C \in \mathcal{F}_\sigma$, $C \subset E$, $\nu C = \nu E$ and so

$$\nu(E \sim C) = 0.$$

Proof of 10.10: Let $X = \bigcup_{n\in\omega} A_n$ where for each $n\in\omega$, $\nu A_n < \infty$. By theorem 10.1, choose for each $n\in\omega$, $B_n \in \mathcal{G}_\delta$ such that $B_n \supset A_n$ and $\nu B_n = \nu A_n$. Let $E_n = E \cap B_n$, so $E_n \in \mathcal{M}_\nu$ and $\nu E_n < \infty$. By 10.9 choose for each $n\in\omega$, $C_n \in \mathcal{F}_\sigma$ such that $C_n \subset E_n$ and $\nu(E_n \sim C_n) = 0$. Set

$$C = \bigcup_{n\in\omega} C_n .$$

Then $C \in \mathcal{F}_\sigma$, $C \subset E$ and

$$E \sim C \subset \bigcup_{n\in\omega} (E_n \sim C_n),$$

whence

$$\nu(E \sim C) \leqslant \sum_{n=0}^{\infty} \nu(E_n \sim C_n) = 0.$$

We now apply this result to $X \sim E$ to obtain $A \in \mathcal{G}_\delta$ such that $A \supset E$ and $\nu(A \sim E) = 0$.

Proof of 10.11: For each $n\in\omega$ let $E_n = E \cap A_n$ and using 10.9 choose $B_n \in \mathcal{G}_\delta$ such that $B_n \supset E_n$ and $\nu(B_n \sim E_n) = 0$. Let

$$B_n = \bigcap_{i\in\omega} B_{n_i} ,$$

where for each $i\in\omega$, $B_{n_i} \in \mathcal{G}$ and $B_{n_i} \supset B_{n_{i+1}}$. Then

$$\infty > \nu E_n = \nu B_n \geqslant \nu(A_n \cap B_n) = \lim_{i\to\infty} \nu(A_n \cap B_{n_i}) .$$

Choose $i\in\omega$ such that

$$\nu(A_n \cap B_{n_i}) < \nu E_n + \varepsilon/2^{n+1}$$

and set

$$G_n = A_n \cap B_{n_i} \in \mathcal{G}.$$

Since E_n is ν-measurable and $\nu E_n < \infty$,

$$\nu(G_n \sim E_n) < \varepsilon/2^{n+1} .$$

Set

$$G = \bigcup_{n\in\omega} G_n \in \mathcal{G}.$$

Then $E \subset G$,

$$(G \sim E) \subset \bigcup_{n\in\omega} (G_n \sim E_n),$$

and

$$\nu(G \sim E) \leqslant \sum_{n\in\omega} \nu(G_n \sim E_n) < \varepsilon.$$

To obtain $F \in \mathcal{F}$, $F \subset E$, $\nu(E \sim F) < \varepsilon$, apply the above result to $X \sim E$.

CHAPTER II

MEASURES ON TOPOLOGICAL SPACES

In this chapter we start with a topological space $(X, \mathcal{G})$ and a gauge τ on some family $\mathcal{A}$ of subsets of X such that $\emptyset \in \mathcal{A}$. Our aim is to study measures on X generated by τ and $\mathcal{G}$ through processes which are generalizations of the well known Hausdorff process in a metric space (see Method II of Monroe [9], p. 105).

We first consider the Hausdorff process itself, showing that the standard results can be obtained by application of the general theory developed in chapter I. Generalizations of the process were introduced by Bledsoe and Morse [2] and by Rogers and Sion [12]. We show that each of these cases can be obtained as an application of the theory in chapter I. More specifically, in each case we consider a filterbase $\mathcal{H}$ in X and see that the known measure is $\nu^{(\mathcal{H},\tau)}$. Since properties of $\nu^{(\mathcal{H},\tau)}$ are stated in terms of the $\mathcal{H}$-topology, it is important to study the relation between the given topology $\mathcal{G}$ and the $\mathcal{H}$-topology. In particular we determine how conditions on $\mathcal{G}$ affect the $\mathcal{H}$-topology and its relation to $\mathcal{G}$, thereby throwing some light on the role played by such conditions.

We suggest variations of the processes used by Bledsoe and Morse, and Rogers and Sion and determine some relations between various of these measures.

Finally, an approach from a somewhat different point of view is introduced, and shown to include all the measures studied previously, as well as yielding another measure.

11. <u>The measure</u> $\mathcal{J}$ <u>in a metric space</u>.

In this section we suppose that the topology $\mathcal{G}$ is induced by some metric ρ on X. All metric concepts refer to ρ.

The standard metric measure $\mathcal{J}$ generated by a gauge τ on a family $\mathcal{A}$ (Method II of Monroe [9], p. 105), is given by

11.1 DEFINITION. For $A \subset X$ and $\delta > 0$

$$\mathcal{J}_\delta A = \inf\{\sum_{i\in\omega} \tau B_i : A \subset \bigcup_{i\in\omega} B_i, \text{ for each } i\in\omega, \; B_i \in \mathcal{A} \text{ and } \operatorname{diam} B_i \leqslant \delta\}.$$

$$\mathcal{J}A = \lim_{\delta\to 0} \mathcal{J}_\delta A.$$

To see that the theory of chapter I applies to $\mathcal{J}$, let

11.2 DEFINITIONS.

$$H_r = \{A \subset X: \operatorname{diam} A \leqslant r\},$$

$$\mathcal{H} = \{H_r : r > 0\},$$

$$\mathfrak{h} = \{H \cap \mathcal{A} : H \in \mathcal{H}\}.$$

Then $\mathcal{H}$ and $\mathfrak{h}$ are filterbases in X, $\mathfrak{h}$ is a subfilterbase of $\mathcal{H}$, and $\mathcal{J} = \nu^{(\mathfrak{h},\tau)} = \nu^{(\mathcal{H},\tau)}$. The well known properties of $\mathcal{J}$ will follow from the results of chapter I and the following easily verified lemmas.

11.3 LEMMAS.

.1 The $\mathcal{H}$-topology is the metric topology, i.e. $\mathcal{G}_{\mathcal{H}} = \mathcal{T}$.

.2 $\mathcal{H}$ satisfies (7II) and (7IV).

Specifically, we have the following theorems.

11.4 THEOREM. If A is closed in $\mathcal{G}$, then A is $\mathcal{J}$-measurable.

We note that stronger measurability results may be available (see remarks 9.11 and example 11.6)

11.5 THEOREMS. Suppose $\mathcal{a} \subset \mathcal{G}$. Then

.1 $\mathcal{J}$ is a regular measure.

.2 If $\mathcal{J}E < \infty$ and $E \in \mathcal{M}_{\mathcal{J}}$, then there exists $D \in \mathcal{G}_\delta$ such that $D \supset E$ and $\mathcal{J}(D \sim E) = 0$, and $C \in \mathcal{F}_\sigma$ such that $C \subset E$ and $\mathcal{J}(E \sim C) = 0$.

.3 If X is $\mathcal{J}$-σ-finite, $E \in \mathcal{M}_{\mathcal{J}}$, then there exist $D \in \mathcal{G}_\delta$ such that $D \supset E$ and $\mathcal{J}(D \sim E) = 0$, and $C \in \mathcal{F}_\sigma$ such that $C \subset E$ and $\mathcal{J}(E \sim C) = 0$.

.4 If $X = \bigcup_{n \in \omega} G_n$, where for each $n \in \omega$, $G_n \in \mathcal{G}$ and $\mathcal{J}G_n < \infty$, $E \in \mathcal{M}_{\mathcal{J}}$ and $\varepsilon > 0$, then there exist open $G \supset E$ such that $\mathcal{J}(G \sim E) < \varepsilon$ and closed $F \subset E$ such that $\mathcal{J}(E \sim F) < \varepsilon$.

PROOFS

Proof of 11.4: Use 11.3 and theorem 9.8.

Proof of 11.5: .1: Use 11.3, 11.4 and corollary 10.2.

.2, .3, and .4: Use 11.3 and then respectively theorems 10.9, 10.10, and 10.11.

11.6 EXAMPLE. On obtaining a stronger measurability result by choice of an appropriate filterbase.

Let $X = R^2$,

$$\mathcal{A} = \{A\text{: for some } x_0 \in R,\ y \in R, \text{ and } K \in \omega,$$
$$A = \{(x,y) : x = x_0 \text{ or for some } n > k,$$
$$x = x_0 \pm 1/2^n\}\}.$$

Let $\mathcal{H}$, η be defined as in 11.2. Then by 9.8, closed sets in the usual topology are $\mathcal{f}$-measurable. Now η does not satisfy (7I), so we cannot get any measurability results using η. Let

$$\mathcal{B} = \{A\text{: for some } x_0 \in R \text{ and } y \in R,\ A \subset \{(x,y)\text{: for some } s > 0,\ |x - x_0| < s\}\},$$

and

$$\mathcal{M} = \{H \cap \mathcal{B} : H \in \mathcal{H}\}.$$

Then η is a subfilter base of $\mathcal{M}$, $\mathcal{f} = \nu^{(\mathcal{M})}$, $\mathcal{M}$ satisfies (7II) and (7IV), and so closed sets in $\mathcal{G}_{\mathcal{M}}$ are $\mathcal{f}$-measurable. Clearly $\mathcal{G}_{\mathcal{M}}$ strictly contains $\mathcal{G}_{\mathcal{H}}$.

12. <u>The measures</u> φ, φ_1, <u>and</u> φ_2 <u>in a topological space</u>.

The measures φ and φ_2 below were introduced and studied by Bledsoe and Morse [2] and by C. A. Rogers and M. Sion (unpublished) respectively.

12.1 DEFINITIONS

.1 Families of open covers.

$$\text{cover } \mathcal{G} = \{\mathcal{B} : \mathcal{B} \subset \mathcal{G},\ \sigma\mathcal{B} = X, \text{ and } \emptyset \in \mathcal{B}\}.$$

$$\text{cover}_1 \mathcal{G} = \{\mathcal{B} : \mathcal{B} \subset \mathcal{G},\ \mathcal{B} \text{ is countable, } \sigma\mathcal{B} = X, \text{ and } \emptyset \in \mathcal{B}\}.$$

$$\text{cover}_2 \mathcal{G} = \{\mathcal{B} : \mathcal{B} \subset \mathcal{G},\ \mathcal{B} \text{ is finite, } \sigma\mathcal{B} = X, \text{ and } \emptyset \in \mathcal{B}\}.$$

.2 For $A \subset X$, $\mathcal{B}$ a cover of X.

$$\varphi_{\mathcal{B}} A = \inf \{ \sum_{B \in \mathcal{E}} \tau B\text{: } \mathcal{E} \text{ is a countable refinement of } \mathcal{B},\ \mathcal{E} \subset \mathcal{A}, \text{ and } A \subset \sigma\mathcal{E}\}.$$

.3 For $A \subset X$,

$$\varphi A = \sup_{\mathcal{B} \in \text{cover}\,\mathcal{J}} \varphi_{\mathcal{B}} A.$$

$$\varphi_1 A = \sup_{\mathcal{B} \in \text{cover}_1 \mathcal{J}} \varphi_{\mathcal{B}} A.$$

$$\varphi_2 A = \sup_{\mathcal{B} \in \text{cover}_2 \mathcal{J}} \varphi_{\mathcal{B}} A.$$

To apply the theory of chapter I we set

12.2 DEFINITIONS.

$$H_{\mathcal{B}} = \{A\colon A \subset B \text{ for some } B \in \mathcal{B}\}.$$

$$\mathcal{H} = \{H_{\mathcal{B}}\colon \mathcal{B} \in \text{cover}\ \mathcal{J}\}.$$

$$\mathcal{H}_1 = \{H_{\mathcal{B}}\colon \mathcal{B} \in \text{cover}_1 \mathcal{J}\}.$$

$$\mathcal{H}_2 = \{H_{\mathcal{B}}\colon \mathcal{B} \in \text{cover}_2 \mathcal{J}\}.$$

$$\mathcal{J}^{\circ} = \mathcal{H}\text{-topology}.$$

$$\mathcal{J}^{1} = \mathcal{H}_1\text{-topology}.$$

$$\mathcal{J}^{2} = \mathcal{H}_2\text{-topology}.$$

Then $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ are filterbases in X, and

$$\varphi = \nu(\mathcal{H}, \tau),\ \varphi_1 = \nu(\mathcal{H}_1, \tau),\ \varphi_2 = \nu(\mathcal{H}_2, \tau).$$

The relations between the given topology $\mathcal{J}$ and the induced topologies $\mathcal{J}^{\circ}$, $\mathcal{J}^{1}$, $\mathcal{J}^{2}$, and properties of the filterbases $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ are indicated in the following theorem.

12.3 THEOREM.

.1 $\mathcal{J}^2 \subset \mathcal{J}^1 \subset \mathcal{J}^{\circ} \subset \mathcal{J}$.

If $\mathcal{J}$ is regular, then

.2 $\mathcal{J}^2 = \mathcal{J}^1 = \mathcal{J}^{\circ} = \mathcal{J}$,

.3 $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ satisfy condition (7I), and

.4 $\mathcal{H}$ satisfies condition (7V).

In general, $\mathcal{G} \neq \mathcal{G}^{\circ}$ as we show in 12.6. On the other hand, regularity of $\mathcal{G}$ is not needed for $\mathcal{G} = \mathcal{G}^2$, as we show in the proof. In view of 8.1.4, $\mathcal{G} = \mathcal{G}^{\circ}$ and $\mathcal{H}$ satisfies (7I) iff $\mathcal{G}$ is regular.

Applying the results of chapter I we then get the following measurability theorems (already known for φ and φ_2).

12.4 THEOREMS.

.1 If $\mathcal{G}$ is regular then closed $\mathcal{G}_\delta$ sets are φ-measurable and compact $\mathcal{G}_\delta$ sets are φ_1, φ_2-measurable.

.2 If $\mathcal{G}$ is normal, then closed $\mathcal{G}_\delta$ sets in $\mathcal{G}$ are φ, φ_1, φ_2-measurable.

(Since singletons are not assumed closed, normality does not imply regularity.)

Again, stronger results may be available, as indicated in the discussion in remarks 9.11.

To obtain approximation results we require that $\mathcal{a} \subset \mathcal{m}_\varphi$ or $\mathcal{a} \subset \mathcal{m}_{\varphi_1}$ or $\mathcal{a} \subset \mathcal{m}_{\varphi_2}$. In any of these cases we can apply directly theorems 10.3 to 10.8.

In general the three measures φ, φ_1, φ_2 are distinct, as is shown in 12.7. It follows immediately from the definitions, however, that we always have

12.5 THEOREM. $\varphi_2 \leqslant \varphi_1 \leqslant \varphi$.

PROOFS AND EXAMPLES

Proof of 12.3: .1: Let $G \in \mathcal{G}^{\circ}$ and let $x \in G$. Then for some $H \in \mathcal{H}$, $H[x] \subset G$. But for some $\mathcal{B} \in$ cover $\mathcal{G}$, $H = H_{\mathcal{B}}$ and

$$H[x] = \sigma\{G \in \mathcal{B} : x \in G\} \in \mathcal{G}.$$

Hence $G \in \mathcal{G}$, and $\mathcal{G}^o \subset \mathcal{G}$.

Clearly $\mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}$ and so $\mathcal{G}^2 \subset \mathcal{G}^1 \subset \mathcal{G}^o$.

.2: We need only show that $\mathcal{G} \subset \mathcal{G}^2$.

Let $G \in \mathcal{G}$ and $x \in G$. By regularity choose closed C such that $x \in C \subset G$, and set

$$\mathcal{B} = \{G, X \sim C\} \in \text{cover}_2 \mathcal{G}.$$

Then letting $H = H_{\mathcal{B}}$, we have $H[x] = G$. Thus, for each $x \in G$, there exists $H \in \mathcal{H}_2$ such that $H[x] \subset G$, i.e. G is open in the $\mathcal{H}_2$-topology.

Note that in this proof we need only that $Cl\{x\} \subset G$. Thus, if $\mathcal{G}$ is a T_1 topology, then $\mathcal{G} = \mathcal{G}^2$.

.3: Suppose $x \in X$ and $H \in \mathcal{H}$. Then for some $\mathcal{B} \in \text{cover}\mathcal{G}$, $H = H_{\mathcal{B}}$. Now $x \in G_o$ for some $G_o \in \mathcal{B}$, and

$$G_o \subset H[x] = \sigma\{G \in \mathcal{B} : x \in G\}.$$

By regularity choose G_1, $G_2 \in \mathcal{G}$ such that

$$x \in G_2 \subset \overline{G}_2 \subset G_1 \subset \overline{G}_1 \subset G_o$$

and let

$$\mathcal{B}_1 = \{G_o, X \sim \overline{G}_1\} \in \text{cover}_2 \mathcal{G},$$

$$\mathcal{B}_2 = \{G_1, X \sim \overline{G}_2\} \in \text{cover}_2 \mathcal{G},$$

$$H_1 = H_{\mathcal{B}_1} \in \mathcal{H}_2, \text{ and}$$

$$H_2 = H_{\mathcal{B}_2} \in \mathcal{H}_2.$$

Then

$$H_2[x] = G_1,$$

$$H_1[G_1] = G_o,$$

and so

$$H_2[H_1[x]] \subset H[x].$$

.4: By .2, any cover $\mathcal{B}$ consisting of sets open in the $\mathcal{H}$-topology is a cover of sets open in $\mathcal{G}$, i.e. $\mathcal{B} \in$ cover$\mathcal{G}$, and so $H_{\mathcal{B}} \in \mathcal{H}$ and $H_{\mathcal{B}}$ refines $\mathcal{B}$.

<u>Proof of</u> 12.4: .1: Use 12.3 and theorems 9.6 and 9.9.

.2: The result follows from theorem 9.3 after it has been shown that if A is a closed $\mathcal{G}_\delta$ set in $\mathcal{G}$, then there exists a sequence B of subsets of X such that

$$A = \bigcap_{n\in\omega} B_n,$$

and for each $n\in\omega$, there exists $N_{n+1} \in \mathcal{H}_2$ such that

$$N_{n+1}[B_{n+1}] \subset B_n.$$

Suppose A is closed in $\mathcal{G}$ and

$$A = \bigcap_{n\in\omega} G_n,$$

where for each $n\in\omega$, $G_n \in \mathcal{G}$. The sequences N in $\mathcal{H}_2$ and B are defined recursively. To start, set $\mathcal{B}_0 = \{G_0, X\sim A\}$, $N_0 = H_{\mathcal{B}_0}$, and $B_0 = G_0$. Having obtained B_i and N_i such that

a) $N_i[B_i] \subset B_{i-1}$ for $i=0,\ldots,n$, (take $B_{-1} = X$)

b) B_i is open for $i=0,\ldots,n$, and

c) $A \subset B_i \subset G_i$ for $i=0,\ldots,n$,

we construct B_{n+1} and N_{n+1} as follows: Let

$$D_{n+1} = G_{n+1} \cap B_n \in \mathcal{G}.$$

Using normality choose open B_{n+1} such that

$$A \subset B_{n+1} \subset \overline{B}_{n+1} \subset D_{n+1}.$$

Let

$$\mathcal{B}_{n+1} = \{D_{n+1}, X\sim\overline{B}_{n+1}\} \in \text{cover}_2\mathcal{G},$$

and $N_{n+1} = H_{\mathcal{B}_{n+1}} \in \mathcal{H}_2$. Then the only element of N_{n+1} which intersects B_{n+1} is D_{n+1}, so

$$N_{n+1}[B_{n+1}] = D_{n+1} \subset B_n.$$

For the sequences B and N, a) and c) hold for every $n \in \omega$. c) assures us that

$$A = \bigcap_{n \in \omega} B_n.$$

12.6 EXAMPLE. $\mathcal{J}^{\circ}$ does not always coincide with $\mathcal{J}$.

Let $X = R_+$, $\mathcal{J} = \{[0,a) : a > 0\}$. Then for any open cover $\mathcal{B}$ of X, and $x \in X$, $H_{\mathcal{B}}[x] = X$, so $\mathcal{J}^{\circ}$ is the trivial topology.

12.7 EXAMPLES. We can have $\varphi_2 \neq \varphi_1 \neq \varphi$.

.1 A case where $\varphi_2 \neq \varphi_1$.

Let $X = R_+$; $\mathcal{J} = \{[0,a); a > 0\}$; for $E \subset X$,

$$\tau E = \begin{cases} 0 \text{ if } E = X \text{ or } E = \emptyset \\ 1 \text{ otherwise} \end{cases}.$$

Then for any $A \subset X$, $\mathcal{B} \in \text{cover}_2 \mathcal{J}$, we have $\varphi_{\mathcal{B}} A = 0$, since any finite open cover of X must include X as an element. Hence φ_2 is just the zero measure.

On the other hand, for any unbounded $A \subset X$, and $\mathcal{B} \in \text{cover}_1 \mathcal{J}$ such that $X \notin \mathcal{B}$, we have $\varphi_{\mathcal{B}} A = \infty$ and so $\varphi_1 A = \infty$.

.2 A case where $\varphi_1 \neq \varphi$.

Let X be any uncountable space with the discrete topology and $\tau E = 0$ for any $E \subset X$. Then for any $A \subset X$ and $\mathcal{B} \in \text{cover}_1 \mathcal{J}$, A can be covered by a countable refinement of $\mathcal{B}$ and so $\varphi_{\mathcal{B}} A = 0$ and φ_1 is the zero measure. (By 12.5 φ_2 is also the zero measure.)

On the other hand, if $A \subset X$ is uncountable and $\mathcal{B}$ is the open cover consisting of singletons, then $\varphi_{\mathcal{B}} A = \infty$ and so too $\varphi A = \infty$.

13. The measure λ in a topological space.

The measure λ_2 defined below was studied by Rogers and Sion [12]. We recall that $\mathcal{D}$ is the family of differences of open sets (2.1.1).

13.1 DEFINITIONS.

.1 $\text{cover}_1 \mathcal{D} = \{\mathcal{B} : \mathcal{B}$ is a countable disjoint cover of X consisting of elements of $\mathcal{D}\}$.

$\text{cover}_2 \mathcal{D} = \{\mathcal{B} : \mathcal{B}$ is a finite disjoint cover of X consisting of elements of $\mathcal{D}\}$.

.2 For $A \subset X$, $\mathcal{B}$ a cover of X,

$$\lambda_{\mathcal{B}} A = \inf\{\sum_{B \in \mathcal{E}} \tau B : \mathcal{E} \text{ is a countable refinement of } \mathcal{B},\ \mathcal{E} \subset \mathcal{A}, \text{ and } A \subset \sigma\mathcal{E}\}.$$

.3 For $A \subset X$,

$$\lambda_1 A = \sup_{\mathcal{B} \in \text{cover}_1 \mathcal{D}} \lambda_{\mathcal{B}} A,$$

$$\lambda_2 A = \sup_{\mathcal{B} \in \text{cover}_2 \mathcal{D}} \lambda_{\mathcal{B}} A.$$

The process above breaks down if we attempt to use arbitrary $\mathcal{D}$ covers. If the topology is T_1, then a cover consisting of singletons would be of this kind and the resulting measure would be infinite on any uncountable set, regardless of what gauge τ was used.

To apply the theory of chapter I we set

13.2 DEFINITIONS.

$H_{\mathcal{B}} = \{A : A \subset B \text{ for some } B \in \mathcal{B}\}$.

$\mathcal{H}_1 = \{H_{\mathcal{B}} : \mathcal{B} \in \text{cover}_1 \mathcal{D}\}$.

$\mathcal{H}_2 = \{H_{\mathcal{B}} : \mathcal{B} \in \text{cover}_2 \mathcal{D}\}$.

$\mathcal{J}^1 = \mathcal{H}_1$-topology.

$\mathcal{J}^2 = \mathcal{H}_2$-topology.

Then $\mathcal{H}_1$ and $\mathcal{H}_2$ are filterbases in X (the intersection of two sets in $\mathcal{D}$ is again a set in $\mathcal{D}$), and $\lambda_1 = \nu^{(\mathcal{H}_1,\tau)}$, $\lambda_2 = \nu^{(\mathcal{H}_2,\tau)}$.

13.3 THEOREM. $\lambda_1 = \lambda_2$.

In view of this theorem, subscripts on λ will be dropped.

13.4 THEOREMS.

.1 For any $H \in \mathcal{H}_1$ and $x \in X$, $H[H[x]] = H[x]$, and so $\mathcal{H}_1$, $\mathcal{H}_2$ satisfy (7II).

.2 $\mathcal{J} \subset \mathcal{J}^2 = \mathcal{J}^1$.

.3 $\mathcal{J}^2$ is completely regular.

.4 If A is closed in $\mathcal{J}$, then A is both open and closed in $\mathcal{J}^2$.

.5 If $\mathcal{J}$ is T_1, then $\mathcal{J}^2$ is the discrete topology.

Theorem 13.5.2 following was obtained by Rogers and Sion [12].

13.5 THEOREMS. Measurability.

.1 Compact $\mathcal{J}^2_\delta$ sets are λ-measurable.

.2 If $G \in \mathcal{G}$, then G is λ-measurable.

Again, as discussed in remarks 9.11, stronger results may be available.

To obtain results on approximation we require that $\mathcal{A} \subset \mathcal{M}_\lambda$. (For example, suppose $\mathcal{A} \subset \mathcal{D}$. Sets in $\mathcal{D}$ will be λ-measurable by theorem 13.5.2.) In this case we can apply theorems 10.3 to 10.8.

PROOFS

Proof of 13.3: Lemma: Given a topological space $(X, \mathcal{G})$. If $D_1, \ldots, D_N$ are disjoint sets in $\mathcal{D}$, then $X \sim \bigcup_{i=1}^{N} D_i$ can be covered by a finite number of disjoint sets in $\mathcal{D}$.

Proof: We note first that we may assume for any set in $\mathcal{D}$, $D = A \sim B$; $A, B \in \mathcal{G}$, that $B \subset A$.

The proof is by induction.

$n = 1$: $D_1 = A_1 \sim B_1$, $A_1, B_1 \in \mathcal{G}$. Then $X \sim D_1$ is covered by B_1 and $X \sim A_1$, both elements of $\mathcal{D}$.

Now suppose that for every set $D_1, \ldots, D_r$ or disjoint sets in $\mathcal{D}$, $X \sim \bigcup_{i=1}^{r} D_i$ can be covered by a finite number of disjoint sets in $\mathcal{D}$, and let $D_1, \ldots, D_{r+1}$ be disjoint sets in $\mathcal{D}$. By hypothesis we can form the following finite cover of disjoint sets in $\mathcal{D}$:

$$\{D_1, \ldots, D_r, C_1, \ldots, C_k\}.$$

Consider now the family $D_1, \ldots, D_r$, $D_{r+1} = A_{r+1} \sim B_{r+1}$, and $C_j \cap B_{r+1}$, $C_j \cap (X \sim A_{r+1})$ for $j = 1, \ldots, k$.

i) It is obviously finite.

ii) It is a cover, for any point not in D_1 to D_{r+1} must be in some C_j and in either B_{r+1} or $X \sim A_{r+1}$.

iii) It is a disjoint family since the D_i, $i = 1,\dots,r+1$ are disjoint, and for any j, $1 \leqslant j \leqslant k$, $C_j \cap B_{r+1}$ is disjoint from $C_j \cap (X \sim A_{r+1})$ (since $B_{r+1} \subset A_{r+1}$), and either of them is disjoint from any D_i, $i=1,\dots,r$, or C_m, $m \neq j$ by hypothesis, and from D_{r+1} by construction.

iv) Finally, it consists only of sets in $\mathcal{D}$ for, since

$$(G_1 \sim G_2) \cap (G_3 \sim G_4) = (G_1 \cap G_3) \sim (G_2 \cup G_4),$$

the intersection of two sets in $\mathcal{D}$ is again a set in $\mathcal{D}$. This completes the proof of the lemma.

Now $\lambda_1 \geqslant \lambda_2$ since $\text{cover}_1\mathcal{D} \supset \text{cover}_2\mathcal{D}$.

$\lambda_1 \leqslant \lambda_2$: Let $A \subset X$, $a < \lambda_1 A$. Choose $\mathcal{B} \in \text{cover}_1\mathcal{D}$ such that

$$\lambda_{\mathcal{B}} A > a.$$

Let $\mathcal{B} = \{D_j\}_{j \in \omega}$. Then

$$\lambda_{\mathcal{B}} A = \sum_{j\in\omega} \lambda_{\mathcal{B}}(A \cap D_j)$$

since $A \subset \bigcup_{j\in\omega}(A \cap D_j)$ and any cover of A which is a refinement of $\mathcal{B}$ can be broken into disjoint families covering the $D_j \cap A$, $j \in \omega$. Hence for some $N \in \omega$,

$$a < \sum_{j=1}^{N} \lambda_{\mathcal{B}}(A \cap D_j).$$

Consider now

$$\mathcal{E} = \{D_1,\dots,D_N\} \cup \{X \sim \bigcup_{i=1}^{N} D_i\}.$$

By the lemma above, choose $\mathcal{L} \in \text{cover}_2\mathcal{D}$, $\mathcal{L}$ refining $\mathcal{E}$, and

$$\mathcal{L} = \{F_i\}_{i=1,\dots,k}, \text{ where } F_i = D_i \text{ for } i = 1,\dots,N.$$

Now as in the case of $\lambda_{\mathcal{B}}A$ above, we have

$$\lambda_{\mathcal{L}}A = \sum_{j=1}^{K}\lambda_{\mathcal{L}}(A \cap F_j).$$

Also, for $1 \leqslant j \leqslant N$, $F_j = D_j$ so $A \cap F_j = A \cap D_j$ and

$$\lambda_{\mathcal{L}}(A \cap F_j) = \lambda_{\mathcal{B}}(A \cap D_j)$$

since any set contained in an element of $\mathcal{L}$ or $\mathcal{B}$ and intersecting $F_j = D_j$ is contained in $F_j = D_j$. Hence

$$\lambda_{\mathcal{L}}A \geqslant \sum_{j=1}^{N}\lambda_{\mathcal{L}}(A \cap F_j) = \sum_{j=1}^{N}\lambda_{\mathcal{B}}(A \cap D_j) > a.$$

Therefore

$$\lambda_2 A \geqslant \lambda_{\mathcal{L}}A > a,$$

and so

$$\lambda_2 A \geqslant \lambda_1 A.$$

<u>Proof of</u> 13.4.1: Let $x \in X$ and $H \in \mathcal{H}_1$. For some $\mathcal{B} \in \text{cover}_1\mathcal{D}$, $H = H_{\mathcal{B}}$, and for some $B \in \mathcal{B}$, $x \in B$. Now $B \in \mathcal{B}$ implies $B \in H_{\mathcal{B}}$ so by definition 6.1.1

$$B \subset H[x].$$

But any element of H containing x must be contained in B, since H refines $\mathcal{B}$ and the elements of $\mathcal{B}$ are disjoint, so we have $H[x] \subset B$. Hence $H[x] = B$ and

$$H[H[x]] = H[B].$$

But $H[B] = B$ by definition 6.1.2 and an argument similar to the one above. Hence

$$H[H[x]] = H[x] \text{ for each } x \in X.$$

Proof of 13.4.2: Let $G \in \mathcal{G}$ and set $\mathcal{B} = \{G, X \sim G\}$. Then $\mathcal{B} \in \text{cover}_2\mathcal{D}$ and $H_{\mathcal{B}} \in \mathcal{H}_2$. Further, for every $x \in G$, $H_{\mathcal{B}}[x] = G$. Hence $G \in \mathcal{G}^2$.

Clearly $\mathcal{G}^2 \subset \mathcal{G}^1$.

Suppose $G \in \mathcal{G}^1$ and $x \in G$. Then for some $H \in \text{cover}_1\mathcal{D}$, $H[x] \subset G$. But for some $A \sim B = D \in H$, $H[x] = D$. Then

$$\mathcal{B} = \{D, B, X \sim A\} \in \text{cover}_2\mathcal{D},$$

$H_{\mathcal{B}} \in \mathcal{H}_2$, and $H_{\mathcal{B}}[x] = D \subset G$. Hence $G \in \mathcal{G}^2$.

Proof of 13.4.3: Use 13.4.1 and theorem 8.2.

Proof of 13.4.4: A is closed in $\mathcal{G}^2$ because $\mathcal{G} \subset \mathcal{G}^2$. Let $\mathcal{B} = \{A, X \sim A\}$. Then $\mathcal{B} \in \text{cover}_2\mathcal{D}$, and $H_{\mathcal{B}} \in \mathcal{H}_2$. For every $x \in A$, $H_{\mathcal{B}}[x] = A$. Hence $A \in \mathcal{G}^2$.

Proof of 13.4.5: If $\mathcal{G}$ is T_1, then points are closed in $\mathcal{G}$ and hence by .4 above, open and closed in $\mathcal{G}^2$.

Proof of 13.5.1: Use 13.4.1 and theorem 9.6.

Proof of 13.5.2: Suppose $G \in \mathcal{G}$ and let

$$\mathcal{B} = \{G, X \sim G\}.$$

Then $\mathcal{B} \in \text{cover}_2\mathcal{D}$ and $H_{\mathcal{B}} \in \mathcal{H}_2$. As in the argument in the proof of 13.4.1, we show $H_{\mathcal{B}}[G] = G$. The conclusion follows from theorem 9.3, setting $B_n = G$ for every $n \in \omega$.

13.6 EXAMPLE. $\mathcal{G}^2$ may be strictly larger than $\mathcal{G}$. Let $X = R_+$, $\mathcal{G} = \{[0,a) : a > 0\}$. Then $\mathcal{G}^2$ is the half-open interval topology, which is not only larger than $\mathcal{G}$, but larger than the usual topology on R_+ as well. (Compare 12.6)

13.7 EXAMPLE. Theorem 13.5.1 cannot be strengthened to closed $\mathcal{G}_\delta^2$ sets.

It was shown by Rogers and Sion ([12], theorem 8) that the measure λ defined on the real line, with the gauge τ on the subsets of R defined by τA = diam A is just the measure $\mathcal{Y}$, which in this case is known to be the same as Lebesque measure. But the $\mathcal{H}_2$-topology in this case is discrete by 13.4.5 and so all subsets of R are closed $\mathcal{G}_\delta^2$ sets.

14. <u>Relations</u> <u>between</u> <u>measures</u>, <u>examples</u>.

In this section we first establish some relations among some of the measures we have studied. We then provide examples showing the lack of one-to-one correspondence between the filterbase $\mathcal{H}$, the measure $\nu^{(\mathcal{H})}$, and the $\mathcal{H}$-topology mentioned in remark 8.5.

The following result was obtained by Bledsoe and Morse [2].

14.1 THEOREM. If $(X, \mathcal{G})$ is a metric space, then $\varphi = \mathcal{Y}$.

14.2 REMARK. It was shown by Rogers and Sion ([12], theorem 8, and in some unpublished work) that if $(X, \mathcal{G})$ is a separable metric space, and τ is well behaved in a certain sense, then $\lambda = \mathcal{Y} = \varphi_2$. In this case then (which includes Lebesgue measure and the classical Hausdorff measures) $\mathcal{Y} = \lambda = \varphi_2 = \varphi_1 = \varphi$ by theorems 12.5 and 14.1.

14.3 REMARK. In the example in 12.7.2, the space is metric and it is easy to see that $\varphi = \mathcal{Y}$, as is assured by theorem 14.1. On the other hand, by 12.5, φ_2 is the zero measure, and since $\text{cover}_2 \mathcal{D} = \text{cover}_2 \mathcal{G}$, we have also that $\lambda = \varphi_2$. We have then

$$\Upsilon = \varphi \neq \lambda = \varphi_1 = \varphi_2.$$

In this sense, φ is the most successful of these measures in generalizing from the metric case.

14.4 REMARK. We note that in example 12.7.1, λ is counting measure, different from both φ_1 and φ_2.

We now consider examples of different filterbases, measures and topologies.

14.5 EXAMPLE. Different filterbases may yield the same topology and measure. Let $X = R$; for $A \subset X$, $\tau A = \text{diam } A$. Let $H_r = \{A \subset X\colon \text{diam } A \leq r\}$, $M_r = \{(a,b) : b-a \leq r\}$ and $\mathcal{H} = \{H_r : r > 0\}$, $\mathcal{M} = \{M_r : r > 0\}$. It is well known that $\nu^{(\mathcal{H})} = \nu^{(\mathcal{M})}$ is Lebesgue measure, and it is clear that $\mathcal{G}_{\mathcal{H}} = \mathcal{G}_{\mathcal{M}}$.

14.6 EXAMPLE. Different filterbases may give the same topology but different measures. In certain cases of Hausdorff s-dimensional measures Besicovitch [1] has shown that unlike the case above, a different measure is obtained if the covering sets are restricted to open spheres. Again it is clear that the topology is the usual one. (Another example may be constructed easily from the case in example 8.4.)

14.7 EXAMPLE. That different filterbases may induce the same measure but different topologies is just the point of the remarks in 9.11. For another example, let X = R; for $A \subset X$, $\tau A = \operatorname{diam} A$. Then by theorem 8 of [12], $\lambda = \mathcal{J}$, which in this case is Lebesgue measure. By 11.3.1 the $\mathcal{H}$-topology associated with $\mathcal{J}$ is the usual topology while by 13.4.5 the $\mathcal{H}$-topology associated with λ is discrete.

15. <u>Measures generated using non-negative functions</u>.

A weak metric on X (sometimes called a quasi-metric) is a non-negative function ρ on $X \times X$ satisfying $\rho(x,x) = 0$ and the triangle inequality $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for $x,y,z \in X$. Any topological space $(X, \mathcal{G})$ has a canonical family of weak metrics $\{\rho_G : G \in \mathcal{G}\}$ associated with it (see definition 15.1.1). This family in fact generates a quasi-uniformity which induces the topology $\mathcal{G}$ (see Pervin [11]). It is natural therefore to try to generate measures using this family and the standard Hausdorff process.

In this section we explore several avenues along this approach and consider measures generated using non-negative functions on $X \times X$. We obtain again the measures of sections 12 and 13 and an additional one, η, which is very similar to the measure φ.

15.1 DEFINITIONS.

.1 For $E \subset X$,

$$\rho_E(x,y) = \begin{cases} 1 \text{ if } x \in E \text{ and } y \in X \sim E \\ 0 \text{ otherwise.} \end{cases}$$

.2 For $\mathcal{B}$ a family of subsets of X,

$$r_{\mathcal{B}}(x,y) = \sup_{E\in\mathcal{B}} \rho_E(x,y).$$

.3 For G a sequence of subsets of X,

$$s_G(x,y) = \sum_{n\in\omega} 1/2^n \rho_{G_n}(x,y).$$

We note that ρ_E, $r_{\mathcal{B}}$, and s_G are all weak metrics which may fail to be symmetric. In general, $r_{\mathcal{B}}$ is not likely to be very interesting. For example, if X is the real line and $\mathcal{B}$ is the family of all open intervals with rational end-points, then $r_{\mathcal{B}}(x,y) = 1$ iff $x \neq y$ and the measure generated by the Hausdorff process using sets of $r_{\mathcal{B}}$-diameter less than 1 will be infinite for any uncountable set, regardless of the function τ. On the other hand, if $\mathcal{B}$ is finite and G is any ordering of the elements of $\mathcal{B}$ then, for sufficiently small $\delta > 0$, $\operatorname{diam}_{r_{\mathcal{B}}} A < \delta$ iff $\operatorname{diam}_{s_G} A < \delta$. For this reason, we use only s_G to generate measures.

We note also that taking the infimum rather than the supremum over a family of the weak metrics ρ_E does not lead to anything of interest, as increasing the number of sets in the family increases the family of sets of zero diameter. In fact, if two sets of the family $\mathcal{B}$ were disjoint, then all subsets of X would have zero diameter.

15.2 DEFINITION. For any countable family $\mathcal{B}$, let G be any ordering of the elements of $\mathcal{B}$, and for $A \subset X$ and $\delta > 0$,

$$\Psi_{G,\delta} A = \inf\{ \sum_{i\in\omega} \tau E_i \ : \ A \subset \bigcup_{i\in\omega} E_i \text{ and for } i\in\omega,\ E_i \in \mathcal{A} \text{ and } \operatorname{diam}_{s_G} E_i < \delta \}.$$

$$\Psi_G A = \lim_{\delta\to 0} \Psi_{G,\delta} A.$$

15.2.1 REMARK. If D is any other ordering of the elements of $\mathcal{B}$, we have $\Psi_G = \Psi_D$.

Hence, we shall write $\Psi_{\mathcal{B}}$ instead of Ψ_G.

15.3 DEFINITIONS. For any $A \subset X$,

$$\Psi_1 A = \sup_{\mathcal{B}\in \mathrm{cover}_1 \mathcal{S}} \Psi_{\mathcal{B}} A$$

$$\Psi_2 A = \sup_{\mathcal{B}\in \mathrm{cover}_2 \mathcal{S}} \Psi_{\mathcal{B}} A.$$

We then have

15.4 THEOREM. $\Psi_1 = \Psi_2 = \lambda$.

Next, we consider another family of non-negative functions on $X \times X$ which may fail to be weak metrics.

15.5 DEFINITIONS

.1 For $B \subset X$,

$$\partial_B(x,y) = \begin{cases} 0 \text{ if } x,y \in B \\ 1 \text{ otherwise} \end{cases}$$

.2 For $\mathcal{B}$ a family of subsets of X, $A \subset X$,

$$\mathfrak{J}_{\mathcal{B}} A = \inf\{ \sum_{i\in\omega} \tau E_i \ : \ A \subset \bigcup_{i\in\omega} E_i \text{ and for } i\in\omega,\ E_i \in \mathcal{A} \text{ and } \inf_{B\in\mathcal{B}} \operatorname{diam}\partial_B E_i = 0 \}.$$

$$\mathfrak{S}A = \sup_{\mathcal{B} \in \text{cover}\,\mathcal{S}} \mathfrak{S}_{\mathcal{B}}A$$

$$\mathfrak{S}_1A = \sup_{\mathcal{B} \in \text{cover}_1\mathcal{S}} \mathfrak{S}_{\mathcal{B}}A \cdot$$

$$\mathfrak{S}_2A = \sup_{\mathcal{B} \in \text{cover}_2\mathcal{S}} \mathfrak{S}_{\mathcal{B}}A .$$

In this case, taking the supremum rather than the infimum over a family of the functions ∂_B does not lead to anything interesting. If we have two disjoint sets in the family $\mathcal{B}$, then

$$\sup_{B \in \mathcal{B}} \operatorname{diam} \partial_B(x,y) = 1 \quad \text{if } x \neq y,$$

and again we automatically get infinite measure on uncountable sets.

15.6 THEOREM. $\mathfrak{S} = \varphi$, $\mathfrak{S}_1 = \varphi_1$, $\mathfrak{S}_2 = \varphi_2$.

In definition 15.2,

$$\sum_{n \in \omega} 1/2^n \operatorname{diam} \rho_{G_n} E_1 < \delta$$

can be substituted for $\operatorname{diam}_{s_G} E_1 < \delta$ (without affecting the measure $\Psi_1 = \Psi_2$). We have already used the fact that the former implies the latter. Although the converse does not necessarily hold, it is not hard to show that the measure obtained using the first is less than or equal to that obtained using the second; the proof is almost identical to that in 15.2.1.

In the case of the function ∂_B, using the infimum, this situation does not hold. If we change the position where the infimum is taken, we may get a different measure.

15.7 DEFINITIONS

.1 For $\mathcal{B}$ a family of subsets of X,

$$d_{\mathcal{B}}(x,y) = \inf_{B\in\mathcal{B}} \delta_B(x,y).$$

.2 For $A\subset X$, $\mathcal{B}$ a family of subsets of X,

$$\eta_{\mathcal{B}}A = \inf\{\sum_{i\in\omega}\tau E_i : A\subset\bigcup_{i\in\omega}E_i, \text{ for } i\in\omega,\ E_i\in\mathcal{A} \text{ and } \operatorname{diam}_{d_{\mathcal{B}}}E_i = 0\}.$$

$$\eta A = \sup_{\mathcal{B}\in\text{cover}\,\mathcal{G}} \eta_{\mathcal{B}}A$$

$$\eta_1 A = \sup_{\mathcal{B}\in\text{cover}_1\mathcal{G}} \eta_{\mathcal{B}}A$$

$$\eta_2 A = \sup_{\mathcal{B}\in\text{cover}_2\mathcal{G}} \eta_{\mathcal{B}}A.$$

We compare the η-measures and the φ-measures (section 12).

15.8 THEOREMS.

.1 $\eta\leq\varphi$, $\eta_1\leq\varphi_1$, $\eta_2\leq\varphi_2$.

.2 If $(X,\mathcal{G})$ is fully normal (2.2.1) then $\eta=\varphi$. Since metric spaces are fully normal (2.3.1), $\eta=\varphi$ in any metric space, and hence $\eta=\mathcal{f}$ in any metric space, by theorem 14.1. Although there are cases in which $\eta\neq\varphi$ (see example 15.12 below), η has many of the same properties as φ. To see this we let $\mathcal{H}_\varphi$ be the $\mathcal{H}$ of 12.2 and consider the filterbase $\mathcal{H}_\eta$ associated with η.

15.9 DEFINITION. For $\mathcal{B}\in\text{cover}\,\mathcal{G}$,

$$M_{\mathcal{B}} = \{A\subset X : \operatorname{diam}_{d_{\mathcal{B}}}A = 0\}$$

$$\mathcal{H}_\eta = \{M_{\mathcal{B}} : \mathcal{B}\in\text{cover}\,\mathcal{G}\}.$$

15.10 LEMMA. Given $x \in X$, $\mathcal{B} \in$ cover $\mathcal{S}$, $M = M_{\mathcal{B}} \in \mathcal{H}_\eta$, $H = H_{\mathcal{B}} \in \mathcal{H}_\varphi$, then $H[x] = M[x]$.

It follows from 15.10 that all results which depend on the concept $H[x]$ are the same for the measures η and φ. Thus results analogous to those in 12.3 and 12.4 hold also for η, and the $\mathcal{H}_\eta$-topology is the same as the $\mathcal{H}_\varphi$-topology.

PROOFS

Proof of 15.2.1: Let G and D be different orderings of the same countable family. We show that for $E \subset X$,

$\Psi_G E = \Psi_D E$.

Let $\delta > 0$, and choose $N \in \omega$ so that

$$1/2^N < \delta \text{ and } \sum_{i=N+1}^{\infty} 1/2^i < \delta.$$

Let

$$\mathcal{L} = \{G_1, \ldots, G_N\}$$

and choose $M \in \omega$ such that

$$\{D_1, \ldots, D_M\} \supset \mathcal{L}.$$

Clearly $M \geq N$. Let $\varepsilon < 1/2^M$. Then for $E \subset X$,

$$\begin{aligned}
\operatorname{diam}_{s_D} E < \varepsilon &\Rightarrow \sup_{x,y \in E} \Big(\sum_{j=1} 1/2^j \rho_{D_j}(x,y)\Big) < \varepsilon \\
&\Rightarrow \sup_{x,y \in E} 1/2^j \rho_{D_j}(x,y) < \varepsilon \quad \text{for } j \geq 1 \\
&\Rightarrow 1/2^j \operatorname{diam}\rho_{D_j} E < \varepsilon \text{ for } j \geq 1 \\
&\Rightarrow \operatorname{diam}\rho_{D_j} E = 0 \quad \text{for } j = 1, \ldots, M \\
&\Rightarrow \operatorname{diam}\rho_{G_i} E = 0 \quad \text{for } i = 1, \ldots, N \\
&\Rightarrow \sum_{i=1}^{\infty} 1/2^i \operatorname{diam}\rho_{G_i} E < \delta \\
&\Rightarrow \operatorname{diam}_{s_G} E < \delta,
\end{aligned}$$

the last implication following from the fact that the sum of suprema is greater than or equal to the supremum of the sums. We have then

$$\Psi_{D,\varepsilon} E \geq \Psi_{G,\delta} E,$$

since by the above result, any covering set used to define the first can be used for the second, and so in the latter case the infimum is taken over a larger set. It follows that

$$\Psi_D E \geq \Psi_G E$$

since δ was arbitrary.

The reverse inequality is proved in an analogous manner.

Proof of 15.4: Clearly $\Psi_1 \geq \Psi_2$. We show $\Psi_2 \geq \lambda_2 \geq \Psi_1$.

$\Psi_2 \geq \lambda_2$: We show that given $\mathcal{E} \in \mathrm{cover}_2 \mathcal{D}$, there exists $\mathcal{B} \in \mathrm{cover}_2 \mathcal{S}$ and $\delta > 0$ such that for some ordering G of $\mathcal{B}$,

$$\Psi_{G,\delta} \geq \lambda_{\mathcal{E}}.$$

Suppose $\mathcal{E} \in \mathrm{cover}_2 \mathcal{D}$,

$$\mathcal{E} = \{G_i \sim G_{i+n} : i=1, \ldots, n\}.$$

Let $\delta < 1/2^{2n}$ and

$$\mathcal{B} = G = \{G_i : i=1,\ldots,2n\} \in \mathrm{cover}_2 \mathcal{S}.$$

Now suppose $\mathrm{diam}_{s_G} B < \delta$. Then for $i=1,\ldots,2n$, $\mathrm{diam}_{\rho_{G_i}} B = 0$ and so $B \subset G_i$ iff $B \cap G_i \neq \emptyset$. Hence for some $D \in \mathcal{E}$, $D \supset B$. Thus for any family $\mathcal{J}$ of elements of s_G-diameter less than δ, $\mathcal{J}$ refines $\mathcal{E}$ and we have

$$\Psi_{G,\delta} \geq \lambda_{\mathcal{E}}.$$

$\lambda_2 \geqslant \Psi_1$: Let $\mathcal{E} \in \text{cover}_1 \mathcal{S}$, $\mathcal{E} = G = \{G_i\}_{i\in\omega}$, $A \subset X$, and $a < \Psi_{\mathcal{E}} A$. Choose $\delta > 0$ such that

$$\Psi_{G,\delta}\ A > a,$$

and $N \in \omega$ such that

$$1/2^N < \delta \text{ and } \sum_{i=N+1}^{\infty} 1/2^i < \delta.$$

Let

$$\mathcal{J} = \{G_1, \ldots G_N, X\} \in \text{cover}_2 \mathcal{S},$$

and

$$\mathcal{B} = \{A \sim B : A = \pi\mathcal{L} \text{ for some } \mathcal{L} \subset \mathcal{J},\ \mathcal{L} \neq \varnothing, \text{ and } B = \sigma(\mathcal{J} \sim \mathcal{L})\}.$$

Then $\mathcal{B} \in \text{cover}_2 \mathcal{D}$ for

1) $\mathcal{B}$ is clearly finite,

2) $\mathcal{B}$ is a cover of X since $\mathcal{J}$ is a cover of X,

3) for any $(A \sim B) \in \mathcal{B}$, A and B are both open,

and

4) elements of $\mathcal{B}$ are disjoint. To see this, suppose $x \in A \sim B$ for some $(A \sim B) \in \mathcal{B}$. Then x is a member of each element in the intersection forming A and of none of the elements in the union forming B. This is clearly the only decomposition of $\mathcal{J}$ into two families for which this is true, i.e. x can be in no other element of $\mathcal{B}$.

Now suppose $D \subset A \sim B$ for some $(A \sim B) \in \mathcal{B}$. Then for $i=1,\ldots,N$, $D \subset G_i$ iff $D \cap G_i \neq \varnothing$ and so

$$\operatorname{diam}_{\rho_{G_i}} D = 0 \text{ for } i=1,\ldots,N.$$

Hence

$$\sum_{i=0}^{\infty} 1/2^i \operatorname{diam}_{\rho_{G_i}} D < \delta$$

and so

$$\operatorname{diam}_{s_G} D < \delta.$$

Thus for any family which refines $\mathcal{B}$, every element of that family has s_G-diameter less than δ, whence

$$\lambda_{\mathcal{B}} \geq \Psi_{G,\delta} > a.$$

Since $\mathcal{E}$ was an arbitrary element of cover$_1 \mathcal{G}$, we then have

$$\lambda_2 \geq \Psi_1 .$$

<u>Proof of</u> 15.6: The conclusion follows from the fact that $\inf_{G \in \mathcal{B}} \operatorname{diam}_{\partial_G} B = 0$ iff $B \subset G$ for some $G \in \mathcal{B}$.

15.11 LEMMA. For $A \subset X$, $\mathcal{B}$ a cover of X, $\operatorname{diam}_{d_{\mathcal{B}}} A = 0$ iff $A \subset \sigma\{G \in \mathcal{B} : x \in G\}$ for each $x \in A$.

<u>Proof</u>: $\operatorname{diam}_{d_{\mathcal{B}}} A = 0$

iff $\sup_{x,y \in A} d_{\mathcal{B}}(x,y) = 0$

iff $d_{\mathcal{B}}(x,y) = 0$ for every $x,y \in A$

iff $\inf_{G \in \mathcal{B}} \partial_G(x,y) = 0$ for every $x,y \in A$

iff $x,y \in G$ for some $G \in \mathcal{B}$ for every $x,y \in A$

iff $A \subset \sigma\{G \in \mathcal{B} : x \in G\}$ for every $x \in A$.

<u>Proof of</u> 15.8.1: If $\mathcal{B} \in$ cover $\mathcal{G}$ or cover$_1 \mathcal{G}$ or cover$_2 \mathcal{G}$, then $B \subset G$ for some $G \in \mathcal{B}$ implies $\operatorname{diam}_{d_{\mathcal{B}}} B = 0$, by lemma 15.11.

<u>Proof</u> <u>of</u> 15.8.2: Let $\mathcal{B} \in \text{cover}\mathcal{S}$ and choose an open cover $\mathcal{E}$, a star-refinement of $\mathcal{B}$. Then by lemma 15.11, $\text{diam}_{d_\mathcal{E}} B = 0$ implies B is contained in the star at x of $\mathcal{E}$ for any $x \in B$. Since $\mathcal{E}$ is a star-refinement of $\mathcal{B}$, $B \subset G$ for some $G \in \mathcal{B}$. Hence

$$\eta_\mathcal{E} \geq \varphi_\mathcal{B}$$

and since $\mathcal{B}$ was an arbitrary element of cover$\mathcal{S}$,

$$\eta \geq \varphi.$$

<u>Proof</u> <u>of</u> 15.10: Suppose $\mathcal{B} \in \text{cover}\mathcal{S}$ and $H_\mathcal{B} \in \mathcal{H}_\varphi$, $M_\mathcal{B} \in \mathcal{H}_\eta$, and $x \in X$. We know from the proof of 12.3.3 that

$$H_\mathcal{B}[x] = \sigma\{G \in \mathcal{B} : x \in G\}.$$

By definitions 6.1.1 and 15.9, and lemma 15.11,

$$M_\mathcal{B}[x] = \sigma\{A \subset X : x \in A \text{ and } A \subset \sigma\{G \in \mathcal{B} : x \in G\}\}$$

$$= \sigma\{G \in \mathcal{B} : x \in G\} = H_\mathcal{B}[x].$$

15.12 EXAMPLE. A case in which $\varphi A > \eta A$.

Let G_1, G_2, G_3 be subsets of X such that

$$X = G_1 \cup G_2 \cup G_3;$$

$$G_1 \sim (G_2 \cup G_3) \neq \varnothing, \quad G_2 \sim (G_1 \cup G_3) \neq \varnothing, \quad G_3 \sim (G_1 \cup G_2) \neq \varnothing;$$

$$G_1 \cap G_2 \sim G_3 \neq \varnothing, \quad G_1 \cap G_3 \sim G_2 \neq \varnothing, \quad G_2 \cap G_3 \sim G_1 \neq \varnothing.$$

Let $\{G_1, G_2, G_3\}$ be the subbase for a topology for X.

Let

$$\tau B = \begin{cases} 0 \text{ if } B = \varnothing \\ 1 \text{ if } B \neq \varnothing, \end{cases}$$

and

$$A = (G_1 \cap G_2) \cup (G_2 \cap G_3) \cup (G_3 \cap G_1).$$

Now any open cover of X must have G_1, G_2, and G_3 as elements. Since two of these are necessary and sufficient to cover A, we have

$$\varphi_{\mathcal{B}} A = 2 \text{ for each } \mathcal{B} \in \text{cover } \mathcal{G} ,$$

and hence

$$\varphi A = 2.$$

On the other hand, by lemma 15.11, $\operatorname{diam}_{d_{\mathcal{B}}} A = 0$, and so

$$\eta_{\mathcal{B}} A = 1 \text{ for each } \mathcal{B} \in \text{cover } \mathcal{G}$$

and

$$\eta A = 1.$$

CHAPTER III

MEASURES ON QUASI-UNIFORM SPACES

In this chapter we consider three methods of generating a measure on a quasi-uniform space, and so, since every topological space is quasi-uniformizable (3.8), on any topological space. We show that these measures include $\mathcal{J}$, λ, and φ_2, and in certain cases φ. When we restrict the quasi-uniformity to a uniformity, the three methods result in the same measure μ and we apply some theorems from chapter I to obtain measurability properties of μ. Theorem 18.3.2, on measurability of closed $\mathcal{G}_\delta$ sets, is an important result for the development in chapter IV.

Throughout this chapter we assume given $(X, \mathcal{U})$, a quasi-uniform space, and τ, a gauge on $\mathcal{a}$, a family of subsets of X such that $\emptyset \in \mathcal{a}$.

16. <u>The measures</u> μ, $\mu^{\dagger}$, <u>and</u> $\mu^{\#}$.

We now define three types of covering sets in terms of the quasi-uniformity $\mathcal{U}$, and use them to generate the measures μ, $\mu^{\dagger}$, and $\mu^{\#}$.

Recall the definitions in section 3 of chapter 0.

16.1 DEFINITIONS.

For $U \subset X \times X$,

.1 $U^* = \{A \subset X : A \times A \subset U\}$,

$U^\dagger = \{A \subset X :$ for some $x \in X$, $A \subset U[x]\}$,

$U^\# = \{A \subset X :$ for some $V \in \mathcal{U}$, $A \times V[A] \subset U\}$.

For $A \subset X$,

.2 $\mu_U A = \inf\{\sum_{B \in \mathcal{B}} \tau B : \mathcal{B}$ is countable, $\mathcal{B} \subset U^* \cap \mathcal{A}$, and $A \subset \sigma\mathcal{B}\}$.

$\mu A = \sup_{U \in \mathcal{U}} \mu_U A$.

.3 $\mu^\dagger_U A = \inf\{\sum_{B \in \mathcal{B}} \tau B : \mathcal{B}$ is countable, $\mathcal{B} \subset U^\dagger \cap \mathcal{A}$, and $A \subset \sigma\mathcal{B}\}$.

$\mu^\dagger A = \sup_{U \in \mathcal{U}} \mu_U A$.

.4 $\mu^\#_U A = \inf\{\sum_{B \in \mathcal{B}} \tau B : \mathcal{B}$ is countable, $\mathcal{B} \subset U^\# \cap \mathcal{A}$, and $A \subset \sigma\mathcal{B}\}$.

$\mu^\# A = \sup_{U \in \mathcal{U}} \mu^\#_U A$.

16.2 REMARK. The same measures are obtained if the supremum is taken over a base for $\mathcal{U}$, i.e. if $\mathcal{V}$ is a base for the quasi-uniformity $\mathcal{U}$, and for $A \subset X$,

$$\theta A = \sup_{V \in \mathcal{V}} \mu_V A,$$

then $\theta = \mu$.

<u>Proof</u>: Since $\mathcal{V} \subset \mathcal{U}$, we have $\mu \geqslant \theta$. Given any $U \in \mathcal{U}$, there exists $V \in \mathcal{V}$ such that $V \subset U$. Hence $V^* \subset U^*$ and so

$$\theta \geqslant \mu_V \geqslant \mu_U ,$$

and since U was an arbitrary element of $\mathcal{U}$,

$$\theta \geqslant \mu .$$

16.3 REMARK. Pervin [11] points out that two non-comparable quasi-uniformities for X may give identical topologies for X. They may at the same time yield different measures. In the case in 16.5 below, we note that $\mathcal{T}_{\mathcal{U}}$ (3.6) is just the metric topology. In the case in theorem 17.1 below, applied in a metric space, $\mathcal{T}_{\mathcal{U}}$ is by construction again the metric topology, but we have seen that γ and λ do not always agree on a metric space (see remark 14.3).

16.4 REMARK. The same situation is true for uniformities. Let $(X, \mathcal{J})$ be the set of all ordinals less than the first uncountable ordinal, with the discrete topology. Let

$$\mathcal{U}_1 = \{\Delta\},$$

and

$$\mathcal{U}_2 = \{U : U = \Delta \cup (X \sim \{x : x < a\}) \times (X \sim \{x : x < a\}) \text{ for some } a \in X\}.$$

It is easy to check that $\mathcal{U}_1$ is the base for a uniformity for X. Clearly $\mathcal{U}_2$ satisfies a), b), and d) of 3.11.6. By lemmas 3.2, for any $U \in \mathcal{U}_2$, $U \circ U = U$, so c) is satisfied also.

The topology induced by each of these uniformities is obviously the discrete topology. Now for $B \subset X$, let

$$\tau B = \begin{cases} 0 \text{ if } B \text{ is countable} \\ 1 \text{ otherwise} \end{cases}.$$

Let μ^1 be generated using $\mathcal{U}_1$ and μ^2 using $\mathcal{U}_2$.

Now suppose $A \subset X$ is uncountable. Then $\mu^1_\Delta A = \infty$ and so $\mu^1 A = \infty$. On the other hand, for any $U \in \mathcal{U}_2$, $\mu^2_U A = 1$, since

$U = \Delta \cup (X \sim \{x : x < a\}) \times (X \sim \{x : x < a\})$ for

some $a \in X$, and A can be covered by $\{(X \sim \{x : x < a\})\} \cup \{x : x < a\}$. Hence $\mu^2 A = 1$.

16.5 REMARK. μ is a direct generalization of $\mathcal{f}$.

Let X be a metric space with metric d. If we set

$$U_r = \{(x,y) : d(x,y) \leqslant r\},$$

and

$$\mathcal{U} = \{U_r : r > 0\},$$

then $\mathcal{U}$ is clearly a base for a (quasi-) uniformity for X. Since $A \times A \subset U_r$ iff $\operatorname{diam}_d A \leqslant r$, we have $\mathcal{f}_r = \mu_{U_r}$ for $r > 0$. Using remark 5.3 we conclude

$$\mathcal{f} = \mu.$$

17. <u>Properties of μ, $\mu^\dagger$, and $\mu^\#$.</u>

We consider first some properties of these measures when $\mathcal{U}$ is only a quasi-uniformity.

17.1 THEOREM. If $\mathcal{U}$ is Pervin's quasi-uniformity, (see 3.8) for a topological space $(X, \mathcal{G})$, then $\mu = \lambda$ (section 13).

17.2 THEOREM. If $\mathcal{U}$ is Pervin's quasi-uniformity for a topological space $(X, \mathcal{G})$, then $\mu^\dagger = \varphi_2$ (section 12).

17.3 THEOREM. If $\mathcal{U}$ is a quasi-uniformity for a topological space $(X, \mathcal{G})$ such that $\mathcal{T}_{\mathcal{U}} = \mathcal{G}$, then $\mu^\dagger \leqslant \varphi$

(section 12). If $\mathcal{U}$ is the maximal quasi-uniformity inducing $\mathcal{J}$ on X, and $(X, \mathcal{J})$ has property Q (2.2.4), then $\mu^{\dagger} = \varphi$.

17.4 REMARK. We can have $\mu^{\dagger} \neq \varphi$ in a space having property Q. If $\mu^{\dagger}$ is obtained using Pervin's quasi-uniformity for the space in example 12.7.1 (see also 2.3.4), then $\mu^{\dagger} = \varphi_2 \neq \varphi$.

We now consider some properties of these measures when $\mathcal{U}$ is a uniformity.

17.5 REMARK. Unlike the case for a quasi-uniformity, the same measure is obtained in the uniformity case whether we use covers from U^* or from $U^{\dagger}$, i.e. $\mu = \mu^{\dagger}$.

The following theorem is analogous to a similar well known result for $\mathcal{f}$-measure (section 11) : $\mathcal{f}A$ is the same whether we require of the covering sets in definition 11.1 that diam $B_i \leqslant \delta$ or diam $B_i < \delta$. (See, for example, the first sentence on p. 147 of Sion and Sjerve [13].)

17.6 THEOREM. If $\mathcal{U}$ is a uniformity, then $\mu = \mu^{\#}$.

We have seen that a uniformity $\mathcal{U}$ is characterized by a family of pseudo-metrics, the gage of $\mathcal{U}$ (3.11.7). The definition of $\mathcal{f}$ in section 11 is valid for a pseudo-metric. The measures generated using the pseudo-metrics in the gage of $\mathcal{U}$ can be used to obtain μ.

17.7 DEFINITIONS. Let $\mathcal{U}$ be a uniformity for X, G the gage of $\mathcal{U}$. For $\rho \in G$, $A \subset X$,

$$\alpha_{\rho,n}A = \inf\{\sum_{i\in\omega}\tau B_i : A\subset\bigcup_{i\in\omega}B_i, \text{ and for each } i\in\omega,$$
$$B_i\in\mathcal{A} \text{ and } \operatorname{diam} B_i\leqslant 1/n\}.$$

$$\alpha_\rho A = \sup_{n\in\omega}\alpha_{\rho,n}A.$$

$$\alpha A = \sup_{\rho\in G}\alpha_\rho A.$$

17.8 THEOREM. If G is the gage of $\mathcal{U}$, then $\alpha = \mu$.

We now examine the relations between μ, and η and φ of sections 15 and 12.

17.9 THEOREM. If $\mathcal{U}$ is a uniformity for a topological space $(X, \mathcal{G})$ such that $\mathcal{T}_{\mathcal{U}} = \mathcal{G}$, then $\mu\leqslant\eta$. If further, $\mathcal{U}$ consists of all neighborhoods of Δ, then $\mu = \eta$.

17.10 THEOREM. If $(X, \mathcal{G})$ is a paracompact topological space, and $\mathcal{U}$ is the maximal uniformity inducing $\mathcal{G}$ on X, then $\varphi = \eta = \mu$.

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Proof of 17.1: Applying theorem 15.4, we show $\mu = \varphi_2$, the measure defined in 15.3.

$\mu\geqslant\varphi_2$: Let $\mathcal{B} \in \text{cover}_2\mathcal{G}$, $\mathcal{B} = G = \{G_1,\ldots,G_n\}$, $G_i\in\mathcal{G}$ for $i = 1,\ldots,n$. Let

$$U = \bigcap_{i=1}^{n} S_{G_i}$$

where $S_G = (G\times G) \cup ((X\sim G)\times X)$. (see 3.8) Then $U\in\mathcal{U}$. Now for $G\in\mathcal{G}$,

$$A\times A \subset S_G \text{ iff } A\subset G \text{ or } A\cap G = \varnothing.$$

Hence $A \times A \subset U$ ($A \in U^*$) iff A is contained in each $G_i \in \mathcal{B}$ which it intersects. But then

$$\operatorname{diam}\rho_{G_i} A = 0 \text{ for } i=1,\dots,n,$$

whence

$$\operatorname{diam}_{s_G} A = 0,$$

and so

$$\mu_U \geqslant \Psi_{G,0} = \Psi_{\mathcal{B}}.$$

Since $\mathcal{B}$ was an arbitrary member of $\text{cover}_2 \mathcal{G}$, we have

$$\mu \geqslant \Psi_2.$$

$\Psi_2 \geqslant \mu$: Let $U \in \mathcal{U}$. Then there exists $V \in \mathcal{U}$, $V \subset U$ such that

$$V = \bigcap_{j=1}^{m} s_{G_j}$$

where $G_j \in \mathcal{G}$ for $j=1,\dots,m$. Let

$$\mathcal{E} = G = \{G_j\}_{j=1,\dots,m} \in \text{cover}_2 \mathcal{G}.$$

Let $\delta < 1/2^m$, and $\operatorname{diam}_{s_G} B < \delta$. Then for $j=1,\cdots,m$. $\operatorname{diam}\rho_{G_j} B = 0$ and so $B \subset G_i$ iff $B \cap G_i \neq \emptyset$. Hence

$$B \times B \subset \bigcap_{j=1}^{m} s_{G_j},$$

or $B \in V^*$. Thus any set of s_G-diameter less than δ is an element of V^* and we have

$$\Psi_{\mathcal{E}} \geqslant \mu_V \geqslant \mu_U,$$

and since U was an arbitrary element of $\mathcal{U}$,

$$\Psi_2 \geqslant \mu.$$

<u>Proof</u> <u>of</u> 17.2 : $\mu^\dagger \geqslant \varphi_2$: Let $\mathcal{B} \in \text{cover}_2 \mathcal{G}$, $\mathcal{B} = \{G_1,\dots,G_n\}$, where $G_i \in \mathcal{G}$ for $i=1,\dots,n$. Let

$$U = \bigcap_{i=1}^{n} S_{G_i} \in \mathcal{U}.$$

Suppose $A \in U^{\dagger}$. Then for some $x \in X$, $A \subset U[x]$. Now $x \in G_j$ for some $G_j \in \mathcal{B}$, and since $U \subset S_{G_j}$, we have

$$U[x] \subset S_{G_j}[x] = G_j .$$

Hence $A \subset G_j$ and so $\mathcal{E} \subset U^{\dagger}$ implies $\mathcal{E}$ refines $\mathcal{B}$. Therefore

$$\mu_U^{\dagger} \geq \varphi_{\mathcal{B}}$$

and

$$\mu^{\dagger} \geq \varphi_2.$$

$\mu^{\dagger} \leq \varphi_2$: Suppose $U \in \mathcal{U}$. Chose $V \in \mathcal{U}$, $V \subset U$,

$$V = \bigcap_{j=1}^{m} S_{G_j} \text{ where } G_j \in \mathcal{G} \text{ for } j=1,\ldots,m.$$

Let

$$\mathcal{B} = \{V[x] = x \in X\}.$$

Now for $x \in X$, either $V[x] = X$, or

$$V[x] = \bigcap_{i=1}^{k} G_{j_i} \text{ for some } k,\ 1 \leq k \leq m,$$

and some function j on $\{1,\ldots,m\}$ onto $\{1,\ldots,m\}$, for:
Suppose $x \notin G_i$ for $i=1,\ldots,m$. Then

$$\begin{aligned} V[x] &= \left(\bigcap_{i=1}^{m} S_{G_i}\right)[x] \\ &= \{y: (x,y) \in (G_i \times G_i) \cup ((X \sim G_i) \times X) \text{ for } i=1,\ldots,m\} \\ &= X. \end{aligned}$$

On the other hand, suppose $x \in G_{j_i}$ for $i=1,\ldots,k$, $1 \leq k \leq m$, and $x \notin G_{j_i}$ for $i = k+1,\ldots m$, for some function j on $\{1,\ldots,m\}$ onto $\{1,\ldots,m\}$. Then

$$(x,y)\in V \text{ iff } (x,y) \in G_{j_i} \times G_{j_i} \text{ for } i=1,\dots,k$$

$$\text{iff } y\in G_{j_i} \text{ for } i=1,\dots,k$$

$$\text{iff } y\in \bigcap_{i=1}^{k} G_{j_i},$$

i.e.

$$V[x] = \bigcap_{i=1}^{k} G_{j_i}.$$

We conclude that each element of $\mathcal{B}$ is open and $\mathcal{B}$ is finite. Clearly $\mathcal{B}$ is a cover of X, so $\mathcal{B} \in \text{cover}_2\mathcal{J}$. Trivially, if $A\subset B$ for some $B\in\mathcal{B}$, then $A\subset V[x]$ for some $x\in X$, so $\mathcal{E}$ refines $\mathcal{B}$ implies $\mathcal{E} \subset U^{\dagger}$ and hence

$$\varphi_{\mathcal{B}} \geq \mu_V^{\dagger} \geq \mu_U^{\dagger}$$

and

$$\varphi_2 \geq \mu^{\dagger}.$$

<u>Proof</u> <u>of</u> 17.3: Suppose $\mathcal{U}$ is a quasi-uniformity for $(X,\mathcal{J})$ and $\mathcal{T}_{\mathcal{U}} = \mathcal{J}$. Let $U\in\mathcal{U}$. For each $x\in X$, $U[x]$ is a neighborhood of x so there exists open G_x such that

$$x\in G_x \subset U[x].$$

Let

$$\mathcal{B} = \{G_x : x\in X\}.$$

Then $\mathcal{B} \in \text{cover}\mathcal{J}$ and if $\mathcal{E}$ refines $\mathcal{B}$, $\mathcal{E}\subset U^{\dagger}$, so

$$\varphi_{\mathcal{B}} \geq \mu_U^{\dagger}$$

and

$$\varphi \geq \mu^{\dagger}.$$

Suppose now $\mathcal{U}$ is the maximal quasi-uniformity inducing $\mathcal{J}$ on X, and $(X,\mathcal{J})$ has property Q. Let $\mathcal{E}\in \text{cover}\mathcal{J}$

and let $\mathcal{B} \in \text{cover}\mathcal{J}$, $\mathcal{B}$ refining $\mathcal{E}$ and such that $\pi\{G \in \mathcal{B} : x \in G\}$ is open for every $x \in X$.

Set

$$U = \bigcap_{G \in \mathcal{B}} S_G$$

where again $S_G = (G \times G) \cup ((X \sim G) \times X)$. Then

1) for every $x \in X$, $U[x]$ is a neighborhood of x, and

2) $U \circ U = U$.

1): We show $U[x] = \pi\{G \in \mathcal{B} : x \in G\}$.

$y \in U[x]$ iff $(x,y) \in U$

iff $(x,y) \in S_G$ for every $G \in \mathcal{B}$

iff $(x,y) \in G \times G$ or $(x,y) \in (X \sim G) \times X$ for every $G \in \mathcal{B}$

iff $x,y \in G$ or $x \notin G$ for every $G \in \mathcal{B}$

iff $y \in G$ for each $G \in \mathcal{B}$ such that $x \in G$

iff $y \in \pi\{G \in \mathcal{B} : x \in G\}$.

2): By definition,

$$U \circ U = \{(x,y) : \text{for some } z, (x,z) \in U \text{ and } (z,y) \in U\}.$$

Now if $(x,y) \in U$, then since $(x,x) \in U$, we have $(x,y) \in U \circ U$, and so $U \subset U \circ U$.

Suppose now $(x,y) \in U \circ U$. Then for some z, $(x,z) \in U$ and $(z,y) \in U$. Hence $(x,z) \in S_G$ and $(z,y) \in S_G$ for every $G \in \mathcal{B}$. Let $G \in \mathcal{B}$.

a) if $(x,z) \in G \times G$ and $(z,y) \in G \times G$, then $(x,y) \in G \times G \subset S_G$.

b) if $(x,z) \in (X \sim G) \times X$ and $(z,y) \in (X \sim G) \times X$, then $(x,y) \in (X \sim G) \times X \subset S_G$.

c) If $(x,z) \in (X \sim G) \times X$, $(z,y) \in G \times G$, then $(x,y) \in (X \sim G) \times X \subset S_G$.

d) $(x,z) \in G \times G$ and $(z,y) \in (X \sim G) \times X$ is impossible.

Hence $(x,y) \in S_G$ for every $G \in \mathcal{B}$ and so $(x,y) \in U$ and we have $U \circ U \subset U$.

Now 1) implies that $\Delta \subset U$, and this with 2) implies that $\{U\}$ is the base for a quasi-uniformity for X. Hence by theorem 6.3 of Kelley, $\mathcal{U} \cup \{U\}$ is the subbase for a quasi-uniformity $\mathcal{V}$ for X. But $\mathcal{T}_{\mathcal{V}} = \mathcal{J}$ (the proof, using 1), is essentially the same as that in 3.10), and so since $\mathcal{U}$ is the maximal quasi-uniformity inducing $\mathcal{J}$ on X, we have $\mathcal{V} = \mathcal{U}$ and $U \in \mathcal{U}$.

Now if $A \subset U[x]$ for some $x \in X$ (i.e. $A \in U^{\dagger}$), then by the proof of 1), $A \subset G$ for each $G \in \mathcal{B}$ such that $x \in G$, and hence $\mathcal{L} \subset U^{\dagger}$ implies $\mathcal{L}$ is a refinement of $\mathcal{B}$. Therefore

$$\mu_U^{\dagger} \geqslant \varphi_{\mathcal{B}} \geqslant \varphi_{\varepsilon},$$

and we have

$$\mu^{\dagger} \geqslant \varphi.$$

<u>Proof</u> <u>of</u> 17.5: $\mu^{\dagger} \leqslant \mu$: Suppose $U \in \mathcal{U}$, and let $A \times A \subset U$, and $x,y \in A$. Then $(x,y) \in U$ and so $y \in U[x]$. Hence $A \subset U[x]$ for every $x \in A$ and $U^* \subset U^{\dagger}$, and we have $\mu_U^{\dagger} \leqslant \mu_U$ and

$$\mu^{\dagger} \leqslant \mu.$$

$\mu \leqslant \mu^{\dagger}$: Given $U \in \mathcal{U}$, using 3.11.4 and 3.11.6, c), choose symmetric $V \in \mathcal{U}$ such that $V \circ V \circ V \subset U$. Applying 3.11.5 we conclude $V^{\dagger} \subset U^*$. Hence

$$\mu_V^\dagger \geqslant \mu_U$$

and

$$\mu^\dagger \geqslant \mu .$$

Proof of 17.6: $\mu \leqslant \mu^\#$: Given $U \in \mathcal{U}$, $U^\# \subset U^*$, so $\mu_U \leqslant \mu_U^\#$.

$\mu \geqslant \mu^\#$: Let $U \in \mathcal{U}$. Chose $V \in \mathcal{U}$ such that V is symmetric and $V \circ V \circ V \subset U$. Suppose $A \in V^*$. Then $A \times A \subset V$ and by 3.11.5,

$$V[A] \times V[A] \subset U.$$

Hence $A \in U^\#$ and we have

$$\mu_V \geqslant \mu_U^\#$$

and

$$\mu \geqslant \mu^\# .$$

Proof of 17.8: For $\rho \in G$, let

$$V_{\rho,n} = \{(x,y) : \rho(x,y) \leqslant 1/n\}.$$

Then by theorem 6.19 of Kelley

$$\mathcal{V} = \{V_{\rho,n} : \rho \in G, \ n \in \omega\}$$

is a base for $\mathcal{U}$. By remark 16.2 μ may be defined using $\mathcal{V}$. Now for $A \subset X$,

$$\begin{aligned}
\alpha A &= \sup_{\rho \in G} \ \sup_{n \in \omega} \alpha_{\rho,n} A = \sup_{(\rho,n) \in G \times \omega} \alpha_{\rho,n} A \\
&= \sup_{(\rho,n) \in G \times \omega} \inf\Big\{ \sum_{i \in \omega} \tau B_i \ : A \subset \bigcup_{i \in \omega} B_i \text{ and for } i \in \omega, \\
&\qquad\qquad B_i \in \mathcal{a} \text{ and } \operatorname{diam}_\rho B_i \leqslant 1/n \Big\} \\
&= \sup_{(\rho,n) \in G \times \omega} \inf\Big\{ \sum_{i \in \omega} \tau B_i \ : A \subset \bigcup_{i \in \omega} B_i \text{ and for } i \in \omega, \\
&\qquad\qquad B_i \in \mathcal{a} \text{ and } B_i \times B_i \subset V_{\rho,n} \Big\} \\
&= \sup_{V \in \mathcal{V}} \mu_V A = \mu A.
\end{aligned}$$

<u>Proof of</u> 17.9: Suppose $\mathcal{U}$ is a uniformity inducing $\mathcal{J}$ on X. Let $U \in \mathcal{U}$. For each $x \in X$, using remark 3.11.3 choose an open neighborhood G_x of x such that $G_x \times G_x \subset U$, and let

$$\mathcal{B} = \{G_x : x \in X\} \in \text{cover}\,\mathcal{J} .$$

If $\operatorname{diam}_{d_{\mathcal{B}}} A = 0$, then $A \times A \subset U$, for suppose $\operatorname{diam}_{d_{\mathcal{B}}} A = 0$. Then

$$\begin{aligned}(x,y) \in A \times A &\Rightarrow d_{\mathcal{B}}(x,y) = 0\\ &\Rightarrow \text{for some } G \in \mathcal{B},\ \partial_G(x,y) = 0\\ &\Rightarrow \text{for some } G \in \mathcal{B},\ x,y \in G\\ &\Rightarrow \text{for some } G \in \mathcal{B},\ (x,y) \in G \times G\\ &\Rightarrow (x,y) \in U.\end{aligned}$$

Hence

$$\eta_{\mathcal{B}} \geqslant \mu_U.$$

and

$$\eta \geqslant \mu.$$

Now suppose further that $\mathcal{U}$ consists of all neighborhoods of Δ. Let $\mathcal{B} \in \text{cover}\,\mathcal{J}$. Let

$$U = \sigma\{G \times G : G \in \mathcal{B}\}.$$

Then $U \in \mathcal{U}$. Suppose $A \times A \subset U$. Suppose $x \in A$ and let $y \in A$. Then $(x,y) \in A \times A$ and so $(x,y) \in G \times G$ for some $G \in \mathcal{B}$. Hence

$$y \in \sigma\{G \in \mathcal{B} : x \in G\},$$

and we have

$$A \subset \sigma\{G \in \mathcal{B} : x \in G\} \text{ for each } x \in A.$$

By lemma 15.11, $\operatorname{diam}_{d_{\mathcal{B}}} A = 0$. We conclude that

$$\mu_U \geqslant \eta_{\mathcal{B}},$$

and

$$\mu \geqslant \eta.$$

<u>Proof of</u> 17.10: A paracompact space is completely regular and the maximal uniformity consists of all neighborhoods of Δ (3.11.9). By theorem 17.9, $\mu = \eta$.

A regular space is paracompact iff it is fully normal (2.3.2). By theorem 15.8.2 then, $\varphi = \eta$. (We could replace 'paracompact' by 'regular and fully normal' in the hypothesis of this theorem.)

17.11 EXAMPLE. A case where $\varphi = \eta \neq \mu$. In the example of 16.4, $\mathcal{U}_1$ is clearly a base for the maximum uniformity for X. Also the space is metric, and hence paracompact, so we have $\varphi = \eta = \mu^1 = \gamma$. However, we saw there that $\mu^2 \neq \mu^1$, showing that the hypothesis that $\mathcal{U}$ consist of all neighborhoods of Δ is required for theorem 17.10. In this light we note also that different uniformities for a metric space may result in different measures.

18. <u>Measurability theorems</u>.

In this section we restrict our attention to the uniformity case and obtainsome measurability theorems for μ. To apply the theory of chapter I we let

18.1 DEFINITIONS.

$$H_U = \{A \subset X : A \times A \subset U\},$$

$$\mathcal{H} = \{H_U : U \in \mathcal{U}\}.$$

Then $\mathcal{H}$ is a filterbase in X and $\mu = \nu^{(\mathcal{H},\tau)}$. Properties of $\mathcal{H}$ and the $\mathcal{H}$-topology are indicated in the following theorems.

18.2 THEOREMS.

.1 $\mathcal{H}$ satisfies (7II).

.2 If $\mathcal{U}$ consists of all neighborhoods of Δ, then $\mathcal{H}$ satisfies (7III).

.3 The $\mathcal{H}$-topology is the uniform topology, $\mathcal{T}_{\mathcal{U}}$.

Applying the results of chapter I we then get the following measurability theorems.

18.3 THEOREMS.

.1 Compact $\mathcal{G}_\delta$ sets are μ-measurable.

.2 If the uniform topology is compact, then closed $\mathcal{G}_\delta$ sets are μ-measurable.

.3 If $\mathcal{U}$ consists of all the neighborhoods of Δ, then closed $\mathcal{G}_\delta$ sets are μ-measurable.

.4 If the uniform topology is paracompact, and $\mathcal{U}$ is the maximal uniformity, then closed $\mathcal{G}_\delta$ sets are μ-measurable.

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18.4 LEMMA. If $U \in \mathcal{U}$, U is symmetric, and $x \in X$, then

$H_U[x] = U[x]$.

Proof: Using definitions 6.1.1 and 3.4.2,

$y \in H_U[x]$

iff $y \in A$ for some $A \in H_U$ such that $x \in A$

iff $y \in A$ for some A such that $A \times A \subset U$ and $x \in A$

iff $\{x,y\} \times \{x,y\} \subset U$

iff $(x,y) \in U$

iff $y \in U[x]$.

<u>Proof of</u> 18.2.1: Suppose $H \in \mathcal{H}$. Then $H = H_U$ for some $U \in \mathcal{U}$. Choose symmetric $V, W \in \mathcal{U}$ such that $V \subset U$ and $W \circ W \subset V$. Suppose $A \subset X$. Then by lemmas 6.1.2, 18.4, and 3.5.1,

$$H_V[A] = \bigcup_{x \in A} H_V[x] = \bigcup_{x \in A} V[x] = V[A].$$

Similarly $H_W[A] = W[A]$ and $H_U[A] = U[A]$. But by 3.5.3,

$$W[W[A]] = (W \circ W)[A] \subset V[A].$$

Hence

$$H_W[H_W[A]] \subset H_V[A] \subset H_U[A].$$

<u>Proof of</u> 18.2.2: Suppose $A \subset X$ is closed, $B \subset X$ is open and $A \subset B$. Let

$$U = B \times B \cup (X \sim A) \times (X \sim A).$$

Then U is a neighborhood of Δ and so $U \in \mathcal{U}$. It is clear that $U[A] = B$.

<u>Proof of</u> 18.2.3: Definitions 6.1.3 and remark 3.6 show that these two topologies are defined in the same way, one using $H_U[x]$ and the other $U[x]$. The desired conclusion follows from remark 3.11.4 and 18.4.

<u>Proof of</u> 18.3.1: Use 18.2 and theorem 9.6.

<u>Proof of</u> 18.3.2: The conclusion is an immediate corollary of 18.3.1.

<u>Proof of</u> 18.3.3: Use 18.2 and theorem 9.7.

<u>Proof of</u> 18.3.4: It has already been observed in the proof of 17.10 that in this case the maximal uniformity consists of all neighborhoods of Δ. The conclusion is then a colollary of 18.3.3.

18.5 EXAMPLE. A non-measurable closed $\mathcal{G}_\delta$ set. The space of 16.4 is metric and so paracompact. It is not compact. $\mathcal{U}_2$ is a base for a uniformity which does not consist of all neighborhoods of Δ. $\mathcal{U}_1$ is a base for the uniformity consisting of all neighborhoods of Δ. We saw that if $A \subset X$ was uncountable, then $\mu^2 A = 1$. If the complement of A is also uncountable, then A is not μ^2-measurable. But any subset of X is a closed $\mathcal{G}_\delta$ set. Any compact subset of X is finite and so has μ^2-measure zero and is μ^2-measurable. μ^1 is a 0-∞ measure and all subsets of X are μ^1-measurable.

CHAPTER IV.

MEASURES ON A COMPACT HAUSDORFF SPACE

The purpose of this chapter is to investigate the possibility of extending certain results obtained in a compact metric space by Sion and Sjerve [13] (hereafter referred to simply as Sion and Sjerve). We work in a compact Hausdorff space and generate the measure μ (16.2) using the uniformity for the space and a gauge τ restricted as in Sion and Sjerve. We could assume a compact regular space but the Hausdorff assumption simplifies the development and can be made with little loss of generality (see remark 20.3).

After introducing the restriction on τ, we obtain some properties of μ, a major one being regularity (theorem 20.6). In section 21 the partial ordering $\succ$ on gauges is introduced. A good deal of work in Hausdorff h-measure theory has been associated with this concept and non-σ-finiteness. Theorem 21.3 shows that its usefulness is essentially restricted to the metric case. We conclude with a theorem and example which begin an investigation of a theorem on non-σ-finiteness which does not involve the ordering $\succ$.

19. Preliminaries.

Given the uniformity $\mathcal{U}$ for X, we now introduce the topology induced by $\mathcal{U}$ on $\mathcal{S}$, the family of subsets of X, and obtain some simple facts about it.

19.1 DEFINITIONS. For $A \in \mathcal{S}$ and $U \in \mathcal{U}$,

.1 $N_U(A) = \{B \subset X : A \subset U[B] \text{ and } B \subset U[A]\}$,

.2 $\mathcal{V}_A = \{W \subset \mathcal{S} : \text{for some } U \in \mathcal{U},\ W \supset N_U(A)\}$,

.3 $\mathcal{R} = \{W : W \in \mathcal{V}_A \text{ for each } A \in W\}$.

19.2 THEOREM. $\mathcal{R}$ is a topology for $\mathcal{S}$, and for $A \in \mathcal{S}$, $\mathcal{V}_A$ is the neighborhood system of A relative to $\mathcal{R}$.

19.3 REMARK. Michael [8] induces a topology on the subsets of a uniform space by the same process as that above, except that he sets

$$N_U(A) = \{B \subset X : B \subset U[A] \text{ and } B \cap U[x] \neq \emptyset \text{ for all } x \in A\}.$$

19.4 LEMMA. Michael's topology on $\mathcal{S}$ is just $\mathcal{R}$.

19.5 REMARK. In the non-compact situation, it is possible to have two non-comparable uniformities which induce the same topology on X, but non-comparable topologies on $\mathcal{S}$.

In the compact case the uniformity is unique and the topology on $\mathcal{S}$ is determined by that on X.

19.6 LEMMA. If X is compact in $\mathcal{T}_{\mathcal{U}}$, then the family $\mathcal{F}$ of sets closed in $\mathcal{T}_{\mathcal{U}}$ is a compact subset of $\mathcal{S}$ in $\mathcal{R}$.

We now introduce some results on a particular kind of convergence in $\mathcal{S}$.

19.7 DEFINITION. Suppose $(B_\alpha, \alpha \in D)$ is a net in $\mathcal{S}$. Then

$$B = \mathcal{R}\text{-}\lim_{\alpha} B_\alpha \text{ iff } B \in \mathfrak{F} \text{ and for every } U \in \mathcal{U}, \text{ there exists } \beta \in D \text{ such that for } \alpha \gg \beta,\ B_\alpha \in N_U(B).$$

(see 1.17)

19.8 THEOREM. If X is compact in $\mathcal{T}_{\mathcal{U}}$, then for every net $(B_\alpha, \alpha \in D)$ in $\mathcal{S}$, there is a subnet $(B_\beta, \beta \in E)$ and $B \in \mathfrak{F}$ such that

$$B = \mathcal{R}\text{-}\lim_{\beta} B_\beta.$$

19.9 LEMMA. Suppose $(A_\alpha, \alpha \in D)$ is a net in $\mathcal{S}$, $A = \mathcal{R}\text{-}\lim_{\alpha} A_\alpha$, $U \in \mathcal{U}$, U is closed in the product topology, and $A_\alpha \times A_\alpha \subset U$ for every $\alpha \in D$. Then $A \times A \subset U$.

PROOFS

Proof of 19.2: The conclusion follows from proposition 2, chapter I of Bourbaki [3] after we have checked the following:

1) If $W \in \mathcal{V}_A$, then $A \in W$, since $A \in N_U(A)$ for every $U \in \mathcal{U}$.

2) If $W \in \mathcal{V}_A$, and $Y \supset W$, then $Y \in \mathcal{V}_A$ by definition of $\mathcal{V}_A$.

3) If $W, Y \in \mathcal{V}_A$, then $W \cap Y \in \mathcal{V}_A$. Suppose $W, Y \in \mathcal{V}_A$. Then there exist $U \in \mathcal{U}$ such that $W \supset U[A]$ and $V \in \mathcal{U}$ such that $Y \supset V[A]$. Hence

$$W \cap Y \supset U[A] \cap V[A] \supset (U \cap V)[A]$$

by lemma 3.5.2. But $U \cap V \in \mathcal{U}$, so $W \cap Y \in \mathcal{V}_A$.

4) If $W \in \mathcal{V}_A$, then there exists $Y \in \mathcal{V}_A$ such that $Y \subset W$ and for each $B \in Y$, $W \in \mathcal{V}_B$.

Since $W \in V_A$, there exists $U \in \mathcal{U}$ such that $W \supset N_U(A)$. Choose symmetric $V \in \mathcal{U}$, $V \circ V \subset U$, and set $Y = N_V(A)$.

a) $Y \subset W$: for let $B \in N_V(A)$. Then $B \subset V[A]$ and $A \subset V[B]$. But $V \subset U$, so

$$B \subset V[A] \subset U[A], \quad A \subset V[B] \subset U[B].$$

Hence $B \in N_U(A) \subset W$.

b) For $B \in Y + N_V(A)$, we have $W \in \mathcal{V}_B$:

Suppose $B \in N_V(A)$. Then $B \subset V[A]$, $A \subset V[B]$. Let $C \in N_B(B)$. Then $C \subset V[B]$ and $B \subset V[C]$. Hence

$$C \subset V[B] \subset V[V[A]] = (V \circ V)[A] \subset U[A],$$

and

$$A \subset V[B] \subset V[V[C]] = (V \circ V)[C] \subset U[C].$$

Hence $C \in N_U(A)$ and so

$$N_V(B) \subset N_U(A) \subset W.$$

Therefore $W \in \mathcal{V}_B$.

<u>Proof of</u> 19.4: If U is symmetric, then

$$A \subset U[B]$$

iff for all $x \in A$ there exists $y \in B$ such that $(y,x) \in U$

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iff for all $x \in A$ there exists $y \in B$ such that $y \in U[x]$

iff for all $x \in A$, $B \cap U[x] \neq \emptyset$,

and the two definitions of $N_U(A)$ are equivalent. But the symmetric members of $\mathcal{U}$ form a base for $\mathcal{U}$ (3.11.4) and hence using only symmetric members, we get the same base for the neighborhood system of $A \in \mathcal{S}$ in each case, and the topologies are the same.

<u>Example for</u> 19.5: Let $X = (0,\infty)$ with the usual

topology, $\mathcal{U}$ be the uniformity having as a base all elements of the form

$$\{(x,y) : |x-y| < r\} \text{ for some } r > 0,$$

$\mathcal{V}$ the uniformity having as a base all elements of the form

$$\{(x,y) : |x/y - 1| < r\} \text{ for some } r > 0.$$

Then $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_{\mathcal{V}}$ is the usual topology. We show, however, that there is a neighborhood $\mathcal{B}$ of $X \in \mathcal{S}$ in $\mathcal{R}_{\mathcal{V}}$ (the topology induced on $\mathcal{S}$ by $\mathcal{V}$) such that every neighborhood of X in $\mathcal{R}_{\mathcal{U}}$ (the topology induced by $\mathcal{U}$) contains points in $\mathcal{S}$ not in $\mathcal{B}$, and similarly for a neighborhood of X in $\mathcal{R}_{\mathcal{U}}$.

Let $\mathcal{B} = N_V(X)$, where

$$V = \{(x,y) : |x/y - 1| < r\} \text{ for some } r > 0.$$

Now $V[X] = X$ for any $V \in \mathcal{V}$, so

$$N_V(X) = \{B \subset X : X \subset V[B]\}.$$

Suppose $\mathcal{P}$ is a neighborhood of X in $\mathcal{R}_{\mathcal{U}}$. Then for some U in the above base for $\mathcal{U}$,

$$\mathcal{P} \supset N_U(X) = \{B \subset X : X \subset U[B]\}.$$

But

$$U = \{(x,y) : |x-y| < \delta\} \quad \text{for some } \delta > 0.$$

Choose $A = (\delta, \infty)$. Then $U[A] = X$ and $A \in N_U(X)$, but

$$V[A] = \left(\frac{\delta}{1+r}, \infty\right) \neq X.$$

Hence $A \in N_U(X)$, $A \notin N_V(X)$.

The construction for the other case, given $\mathcal{B} = N_U(X)$,

$$U = \{(x,y) : |x - y| < r\} \text{ for some } r > 0, \text{ and}$$

any neighborhood $\mathcal{P}$ of X in $\mathcal{R}_{\mathcal{V}}$ involves picking a set A whose points get arbitrarily far apart as their numerical values increase, while A still is an element of $\mathcal{P}$. It cannot, of course, be an element of $\mathcal{B}$.

<u>Proof</u> <u>of</u> 19.6: Theorems 3.3 and 4.2 of Michael [8] prove the lemma for his topology on $\mathcal{S}$. The conclusion follows by using the fact that our topology agrees with his (19.4).

<u>Proof</u> <u>of</u> 19.8: By 19.6 the theorem holds for any net in $\mathcal{F}$. Let $(B_\alpha, \alpha \in D)$ be a net in $\mathcal{S}$ and consider the net $(\overline{B}_\alpha, \alpha \in D)$. Let $(\overline{B}_\beta, \beta \in E)$ be a subnet and $B \in \mathcal{F}$ such that

$$B = \mathcal{R}\text{-}\lim_\beta \overline{B}_\beta .$$

Now suppose $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$, $V \circ V \subset U$. Then there exists $\gamma \in E$ such that for all $\beta \gg \gamma$.

$$B \subset V[\overline{B}_\beta] \quad \text{and} \quad \overline{B}_\beta \subset V[B].$$

Then $B_\beta \subset U[B]$ and using 3.11.12,

$$B \subset V[\overline{B}_\beta] = V[\bigcap_{U \in \mathcal{U}} U[B_\beta]] \subset V[V[B_\beta]]$$

$$= (V \circ V)[B_\beta] \subset U[B_\beta].$$

Hence $B = \mathcal{R}\text{-}\lim_\beta B_\beta$.

<u>Proof</u> <u>of</u> 19.9: Suppose $(x,y) \notin U$. By 3.11.13

$$U = \bigcap_{V \in \mathcal{U}} V \circ U \circ V,$$

so for some symmetric $V \in \mathcal{U}$, $(x,y) \notin V \circ U \circ V$.

Now $A_\alpha \times A_\alpha \subset U$ for every $\alpha \in D$. Hence by 3.11.5

$$V[A_\alpha] \times V[A_\alpha] \subset V \circ U \circ V \quad \text{for every } \alpha \in D.$$

Therefore for every $\alpha \in D$,

$$(x,y) \notin V[A_\alpha] \times V[A_\alpha],$$

and so by definition 19.7, $(x,y) \notin A \times A$.

20. The family T.

In this section $(X, \mathcal{J})$ is a compact Hausdorff space, $\mathcal{U}$ is the uniformity for X which induces $\mathcal{J}$, $\mathcal{S}$ is the family of all subsets of X, τ is a gauge on $\mathcal{S}$, and μ is the measure of chapter III generated by $\mathcal{U}$ and τ.

We now introduce the family of set functions to which we will restrict τ, and examine some sequences of this restriction.

20.1 DEFINITION. T is the set of all functions τ' on $\mathcal{S}$ such that

.1 $\tau'\emptyset = 0$;

.2 if $A \subset B$, then $\tau' A \leqslant \tau' B \leqslant \infty$;

.3 $\tau' A = 0 \Rightarrow A \times A \subset \Delta$;

.4 τ' is continuous in $\mathcal{S}$ with respect to the topology $\mathcal{R}$ (19.1.3); and

.5 τ' is bounded on $\mathcal{S}$.

20.2 REMARKS.

.1 If $(X, \mathcal{J})$ is a compact metric space then our family T and measure μ are just the family T and measure $\mu^{(\tau)}$ of Sion and Sjerve.

.2 The fact that the topology is compact guarantees both that the family T is determined by the topology on X (see 19.5) and that given $\tau \in T$, the measure μ is determined by the topology on X, since there is a unique uniformity.

20.3 REMARK. Suppose $(X, \mathcal{G})$ is a compact, regular, but not Hausdorff, and $\tau \in T$. Then a consequence of corollary 20.10 is that for $A \subset X$,

$$\mu A = \mu(A \cup \bigcup_{x \in A} \mathrm{Cl}\{x\}).$$

Thus the measure does not distinguish between points and their closures, and so without any real loss of generality we may identify each point with its closure, giving a T_1, and hence Hausdorff space since the space is regular. For this reason we assume from the start that the space is Hausdorff.

The next two theorems extend similar theorems for Hausdorff h-measures.

20.4 THEOREM. If $\tau \in T$, then the restriction of τ to the family of open sets generates the same measure μ as τ.

20.5 THEOREM. If $\tau \in T$, then the restriction of τ to the family of closed $\mathcal{G}_\delta$ sets generates the same measure μ as τ.

20.6 THEOREM. If $\tau \in T$ and $X = \bigcup_{n \in \omega} A_n$, where for each $n \in \omega$, $A_n \in \mathfrak{M}_\mu$ and $\mu A_n < \infty$, then μ is a regular measure.

The property of the pre-measure μ_U given in the next theorem is a partial extension of theorem 5.2 in Sion and Sjerve. A direct extension would contain no reference to V, but V entered as a consequence of the fact that closed sets are not necessarily closed $\mathcal{G}_\delta$ sets, and we were not able to eliminate it.

20.7 THEOREM. If $\tau \in T$; $X = \bigcup_{n\in\omega} B_n$, where for each $n \in \omega$, $B_n \in \mathcal{M}_\mu$ and $\mu B_n < \infty$; A is an ascending sequence of subsets of X; and $U, V \in \mathcal{U}$, then

$$\mu_{V\circ U\circ V} \bigcup_{n\in\omega} A_n \leq \lim_n \mu_U A_n \ .$$

PROOFS

Proof of 20.2.1: For the equality of the families T, we need only check that our topology $\mathcal{R}$ is the same as the subset topology in Sion and Sjerve. This follows from the fact that the uniformity is unique and that there is a base consisting of elements of the form

$$\{(x,y) : |x-y| < r\} \text{ for some } r > 0.$$

For the equality of the measures we note that our measure $\mathcal{T}$ is just the measure $\mu^{(\tau)}$ of Sion and Sjerve, and by remark 16.5, $\mathcal{T} = \mu$.

20.8 LEMMA. If $\tau \in T$, $A \subset X$, and $\varepsilon > 0$, then there exists $U \in \mathcal{U}$ such that $\tau U[A] \leq \tau A + \varepsilon$.

Proof: If $\tau A = \infty$, the conclusion is trivial. Suppose $\tau A < \infty$. By 20.1.4 there is a neighborhood V of A in $\mathcal{S}$ such that for $B \in V$,

$$|\tau A - \tau B| < \varepsilon \ .$$

By definition 19.1.2 and theorem 19.2, there exists $U \in \mathcal{U}$ such that $V \supset N_U(A)$. But

$$U[A] \in N_U(A),$$

whence, since τ is increasing (20.1.2),

$$\tau U[A] - \tau A \leqslant \varepsilon .$$

20.9 COROLLARY. If $\tau \in T$, $A \subset X$, then there exists a sequence U in $\mathcal{U}$ such that

$$\tau A = \tau \bigcap_{n\in\omega} U_n[A].$$

<u>Proof</u>: If $\tau A = \infty$, the conclusion follows from the fact that τ is increasing (20.1.2). Suppose $\tau A < \infty$. Using lemma 20.8, choose a sequence U in $\mathcal{U}$ such that

$$\tau U_n[A] \leqslant \tau A + 1/n \quad \text{for each } n \geqslant 1.$$

Then using the fact that τ is increasing, we have

$$\tau A = \tau \bigcap_{n\in\omega} U_n[A].$$

20.10 COROLLARY. If $\tau \in T$, $A \subset X$, then $\tau A = \tau \overline{A}$.

<u>Proof</u>: Using 20.9 choose a sequence U such that

$$\tau A = \tau \bigcap_{n\in\omega} U_n[A].$$

But

$$A \subset \overline{A} = \bigcap_{U\in\mathcal{U}} U[A] \subset \bigcap_{n\in\omega} U_n[A].$$

Again since τ is increasing we have $\tau A = \tau \overline{A}$.

<u>Proof of</u> 20.4: We use the fact that $\mu = \mu^{\#}$ (theorem 17.6) and definition 16.1.4. Let $\mathcal{V}$ be a base of symmetric members of $\mathcal{U}$. Let $U \in \mathcal{V}$, $A \subset X$, and suppose $\mu^{\#}_U A < \infty$, the argument being trivial otherwise. Let $\varepsilon > 0$. Choose $B_i \in U^{\#}$ for $i \in \omega$ such that

$$A \subset \bigcup_{i\in\omega} B_i$$

and

$$\sum_{i\in\omega} \tau B_i \leqslant \mu_U^{\#} A + \varepsilon/2.$$

Now for each $i \in \omega$, by definition 16.1.1 and symmetry of U, there exists $U_i \in \mathcal{V}$ such that

$$U_i[B_i] \times U_i[B_i] \subset U.$$

Using lemma 20.8 choose for each $i \in \omega$, $V_i \in \mathcal{V}$ such that $V_i \circ V_i \subset U_i$ and

$$\tau V_i[B_i] \leqslant \tau B_i + \varepsilon/_{2^{i+2}}.$$

Then for each $i \in \omega$, choose $G_i \in \mathcal{G}$ such that

$$B_i \subset G_i \subset V_i[B_i].$$

Then

$$A \subset \bigcup_{i\in\omega} G_i,$$

and since τ is increasing,

$$\tau G_i \leqslant \tau B_i + \varepsilon/2^{i+2} \text{ for each } i \in \omega.$$

Hence

$$\sum_{i\in\omega} \tau G_i \leqslant \sum_{i\in\omega} \tau B_i + \varepsilon/2 \leqslant \mu_U^{\#} A + \varepsilon .$$

Also for each $i \in \omega$,

$$\begin{aligned} V_i[G_i] \times V_i[G_i] &\subset V_i[V_i[B_i]] \times V_i[V_i[B_i]] \\ &\subset U_i[B_i] \times U_i[B_i] \subset U, \end{aligned}$$

and so $G_i \in U^{\#} \cap \mathcal{G}$ for each $i \in \omega$. We conclude that $\mu_U^{\#}$ is unaffected by restricting τ to $\mathcal{G}$, and hence the same is true for $\mu^{\#}$.

<u>Proof</u> <u>of</u> 20.5: We show first that for $A \subset X$, U a sequence in $\mathcal{U}$, there exists a closed $\mathcal{G}_\delta$ set E such that

$$A \subset E \subset \bigcap_{n \in \omega} U_n[A].$$

To see this we construct a sequence V in $\mathcal{U}$ by recursion such that for $n \in \omega$, V_n is symmetric, $V_n \subset U_n$, and $V_n \circ V_n \subset V_{n-1}$. Set

$$E = \bigcap_{n \in \omega} V_n[A].$$

Then

1) $A \subset E \subset \bigcap_{n \in \omega} U_n[A]$ is obvious.

2) E is closed. If $x \notin E$, then for some $n \in \omega$, $x \notin V_n[A]$, so $x \notin V_{n+1}[V_{n+1}[A]]$. By lemmas 18.4 and 6.3.3,

$$V_{n+1}[x] \cap V_{n+1}[A] = \varnothing .$$

But $V_{n+1}[x]$ is a neighborhood of x and $E \subset V_{n+1}[A]$ so x is contained in an open set not intersecting E.

3) $E \in \mathcal{G}_\delta$. For each $n \in \omega$ choose $G_n \in \mathcal{G}$ such that

$$E \subset V_{n+1}[A] \subset G_n \subset V_{n+1}[V_{n+1}[A]] \subset V_n[A].$$

Then

$$E \subset \bigcap_{n \in \omega} G_n \subset \bigcap_{n \in \omega} V_n[A] = E,$$

$$E = \bigcap_{n \in \omega} G_n.$$

Now suppose $U \in \mathcal{U}$, U symmetric, and $B \in U^{\#}$. Then for some $V \in \mathcal{U}$,

$$V[B] \times V[B] \subset U.$$

Choose symmetric $W \in \mathcal{U}$, $W \circ W \subset V$. By corollary 20.9 let $\{U_n\}_{n \in \omega}$ be a sequence in $\mathcal{U}$ such that $U_1 \subset W$ and

$$\tau B = \tau \bigcap_{n \in \omega} U_n[B].$$

Using the result above, construct a closed $\mathcal{G}_\delta$ set E such that

$$B \subset E \subset \bigcap_{n \in \omega} U_n[B].$$

Then $\tau B = \tau E$ and $E \in U^{\#}$ since $E \subset W[B]$ implies

$$W[E] \subset W[W[B]] = (W \circ W)[B] \subset V[B],$$

and so

$$W[E] \times W[E] \subset V[B] \times V[B] \subset U.$$

Hence for any set $B \in U^{\#}$, there is a closed $\mathcal{G}_\delta$ set $E \in U^{\#}$ such that $\tau B = \tau E$, and so $\mu^{\#}_U$ is unaffected by the restriction of τ to closed $\mathcal{G}_\delta$ sets. Since $\mu^{\#}$ may be defined using a base of symmetric elements of $\mathcal{U}$, the same is then true of $\mu^{\#} = \mu$ (17.6).

<u>Proof of</u> 20.6: By theorem 20.5, μ can be obtained by restricting τ to closed $\mathcal{G}_\delta$ sets. But by 18.3.2, closed $\mathcal{G}_\delta$ sets are μ-measurable, and the conclusion follows from corollary 10.5.

20.11 LEMMA. If $\tau \in T$, $(A_\alpha, \alpha \in D)$ is a net in $\mathcal{S}$, and $\lim_\alpha \tau A_\alpha = 0$, then given $U \in \mathcal{U}$, there exists $\beta \in D$ such that for $\alpha \gg \beta$, $A_\alpha \times A_\alpha \subset U$.

<u>Proof</u>: Suppose there exists $U \in \mathcal{U}$ such that for every $\beta \in D$, there exists $\alpha \gg \beta$ such that $A_\alpha \times A_\alpha \not\subset U$. Then

$$\{A_\alpha : A_\alpha \times A_\alpha \not\subset U\}$$

is a subnet of $(A_\alpha, \alpha \in D)$, say $(A_\gamma, \gamma \in C)$. By theorem 19.8 there exists $B \in \mathcal{F}$ and a subnet $(A_\beta, \beta \in E)$ of $(A_\gamma, \gamma \in C)$ and so also of $(A_\alpha, \alpha \in D)$ such that

$$B = \mathcal{R}\text{-}\lim_\beta A_\beta.$$

By the continuity of τ,

$$\tau B = \lim_{\alpha} \tau A_{\alpha} = 0.$$

Hence by definition 20.1.3,

$$B \times B \subset \Delta.$$

Now choose symmetric $V \in \mathcal{U}$ such that $V \circ V \subset U$. Then by definition 19.7, there exists $\beta' \in E$ such that for $\beta \gg \beta'$, $A_{\beta} \subset V[B]$. Using 3.11.5 and lemma 3.2.2 we have

$$A_{\beta} \times A_{\beta} \subset V[B] \times V[B] \subset V \circ \Delta \circ V = V \circ V \subset U.$$

But for every $\beta \in E$, $A_{\beta} \times A_{\beta} \not\subset U$. This contradiction establishes the lemma.

20.12 LEMMA. Suppose the sets $E, P(i,\alpha) \subset X$, and V_i for $i \in \omega$ and $\alpha \in D$, a directed set, satisfy the following conditions:

i) There exists $\xi \in D$ such that for $\alpha \gg \xi$,
$$E \subset \bigcup_{i \in \omega} P(i,\alpha);$$

ii) $\tau P(i,\alpha) \geqslant \tau P(i+1,\alpha)$; and

iii) $V_i = \mathcal{R}\text{-}\lim_{\alpha} P(i,\alpha)$.

If $\tau \in T$ and X is μ-measurably σ-finite, then given $Y \in \mathcal{U}$, for each $i \in \omega$ there exists B_i such that

$$V_i \subset B_i \subset Y[V_i]$$
$$\tau B_i = \tau V_i,$$

and

$$\sum_{i \in \omega} \tau V_i + \mu(E \sim \bigcup_{i \in \omega} B_i) \leqslant \underline{\lim}_{\alpha} \sum_{i \in \omega} \tau P(i,\alpha).$$

<u>Proof</u>: Let $a = \underline{\lim}_{\alpha} \sum_{i \in \omega} \tau P(i,\alpha)$,

and suppose $a < \infty$. For each $i \in \omega$, $n \in \omega$, choose open $W_{i,n}$ such that

$$V_i \subset W_{i,n} \subset Y[V_i],$$

and

$$\tau W_{i,n} \leq \tau V_i + 1/n+1,$$

and such that for each $n \in \omega$,

$$W_{i,n+1} \subset W_{i,n} \text{ ,}$$

as is done in the proof of theorem 20.5 except for the descending requirement, which can be met by taking intersections. Let

$$B_i = \bigcap_{n \in \omega} W_{i,n} \text{ .}$$

Given $\varepsilon > 0$, choose a net $(\alpha_j, j \in E)$ in D such that for every $j, k \in E$, $\alpha_k \gg \alpha_j$ if $k \gg j$, and for every $j \in E$,

$$\text{iv)} \quad \sum_{i \in \omega} \tau P(i, \alpha_j) \leq a + \varepsilon \text{ .}$$

Then for each $i \in \omega$,

$$(P(i,\alpha_j),\ j \in E) = (P(i, \beta),\ \beta \in C)$$

is a subnet of $(P(i,\alpha), \alpha \in D)$ so by the continuity of τ

$$\tau V_i = \lim_{\alpha \in D} \tau P(i,\alpha) = \lim_{\beta \in C} \tau P(i,\beta) \text{ for } i \in \omega.$$

Hence

$$\sum_{i \in \omega} \tau V_i = \sum_{i \in \omega} \lim_{\beta \in C} \tau P(i, \beta) \leq \lim_{\beta \in C} \sum_{i \in \omega} \tau P(i,\beta) \leq a + \varepsilon,$$

and since ε was arbitrary,

$$\sum_{i \in \omega} \tau V_i \leq a.$$

Let

$$b = a - \sum_{i \in \omega} \tau V_i \text{ .}$$

For each $i \in \omega$, choose $n_i \in \omega$ and $W_i = W_{i,n_i}$ such that

$$\tau W_i \leq \tau V_i + \varepsilon/2^i$$

Then $\sum_{i \in \omega} \tau V_i \leq a$ implies $\lim_i \tau V_i = 0$, whence

$\lim_{i} \tau W_i = 0$.

Now let $U \in \mathcal{U}$. Then

a) There exists $K_1 \in \omega$ such that for $i \geqslant K_1$, $W_i \in U^*$. This follows from $\lim_{i} \tau W_i = 0$ and lemma 20.11.

b) There exists $K_2 \in \omega$ such that for all $\beta \in C$ and $i \geqslant K_2$, $P(i, \beta) \in U^*$. Suppose not, then given $n \in \omega$ there exist $i_n \geqslant 2^n$ and $\beta_n \in C$ such that

$$P(i_n, \beta_n) \notin U^*.$$

But for each $n \in \omega$,

$$\tau P(i_n, \beta_n) \leqslant \frac{a+\varepsilon}{2^n}$$

since

$$\sum_{i=1}^{i_n} \tau P(i, \beta_n) \leqslant a + \varepsilon,$$

$i_n \geqslant 2^n$ and the $P(i, \beta_n)$ are ordered by decreasing τ values. Hence

$$\lim_{n} \tau P(i_n, \beta_n) = 0, \text{ which by lemma 20.11,}$$

contradicts the fact that for each $n \in \omega$,

$$P(i_n, \beta_n) \notin U^*.$$

c) There exists $K_3 \in \omega$ such that

$$\sum_{i > K_3} \tau V_i < \varepsilon.$$

Let $K = \max\{K_1, K_2, K_3\}$.

Now for $i \in \omega$, there exists $\beta_i \in C$ such that for $\beta \gg \beta_i$

$$P(i, \beta) \subset W_i.$$

For by theorem 6.33 of Kelley, since V_i is closed and hence compact, $W_i \supset V_i$ implies there exists $M \in \mathcal{U}$ such that

$M[V_i] \subset W_i$; and

$$V_i = \mathcal{R}\text{-}\lim_{\beta} P(i,\beta)$$

implies by definition 19.7 that there exists $\beta_i \in C$ such that for $\beta \gg \beta_i$,

$$P(i,\beta) \subset M[V_i].$$

Choose $\gamma \in C$ such that $\gamma \gg \xi$, $\gamma \gg \beta_i$ for $i=1,\ldots K$, and such that

$$\sum_{i=0}^{K} \left|\tau V_i - \tau P(i,\gamma)\right| < \varepsilon,$$

using

$$V_i = \mathcal{R}\text{-}\lim_{\beta} P(i,\beta)$$

and the continuity of τ. Now

$$E \sim \bigcup_{i=1}^{K} W_i \subset \bigcup_{i>K} P(i,\gamma)$$

and for $i > K$, $P(i,\gamma) \in U^*$ by b). Hence

$$\mu_U (E \sim \bigcup_{i\in\omega} W_i) \leqslant \mu_U (E \sim \bigcup_{i=1}^{K} W_i) \leqslant \sum_{i>K} \tau P(i,\gamma) =$$

$$= \sum_{i\in\omega} \tau P(i,\gamma) - \sum_{i\in\omega} \tau V_i + \sum_{i=0}^{K} \tau V_i - \sum_{i=0}^{K} \tau P(i,\gamma) + \sum_{i>K} \tau V_i$$

$$\leqslant a + \varepsilon - \sum_{i\in\omega} \tau V_i + \varepsilon + \varepsilon \leqslant b + 3\varepsilon .$$

Since U was arbitrary, we have

$$\mu(E \sim \bigcup_{i\in\omega} W_i) \leqslant b + 3\varepsilon .$$

Now for each $i \in \omega$, the restrictions on W_i used in obtaining the expression above are satisfied by $W_{i,n}$ for $n \geqslant n_i$. Letting

$$N = \max\{n_i : i \leqslant K\}$$

we have then for any $n \geqslant N$,

$$\mu(E \sim \bigcup_{i \leqslant K} W_{i,n} \sim \bigcup_{i>K} W_i) \leqslant b + 3\varepsilon .$$

Letting

$$D_n = E \sim \bigcup_{i \leqslant K} W_{i,n} \sim \bigcup_{i>K} W_i ,$$

and applying theorems 20.6 and 4.2 we have

$$\mu(E \sim \bigcup_{i=0}^{K} B_i \sim \bigcup_{i>K} W_i) = \mu(\bigcup_{n\in\omega} D_n) = \lim_{n\to\infty} \mu D_n \leqslant b + 3\varepsilon$$

whence

$$\mu_U(E \sim \bigcup_{i=0}^{K} B_i \sim \bigcup_{i>K} W_i) \leqslant b + 3\varepsilon .$$

Since for $i > K$ we have $W_i \in U^*$, we conclude

$$\mu_U(E \sim \bigcup_{i=0}^{K} B_i) \leqslant b + 3\varepsilon + \mu_U \bigcup_{i>K} W_i$$

$$\leqslant b + 3\varepsilon + \sum_{i>K} \tau W_i \leqslant b + 5\varepsilon ,$$

and so

$$\mu_U (E \sim \bigcup_{i\in\omega} B_i) \leqslant b + 5\varepsilon .$$

Taking the supremum over $U \in \mathcal{U}$ and letting $\varepsilon \to 0$, we have

$$\mu(E \sim \bigcup_{i\in\omega} B_i) \leqslant b.$$

<u>Proof</u> <u>of</u> 20.7: Let $a = \lim_n \mu_U A_n$ and suppose $a < \infty$. Choose sets $P(i,n)$ such that for $i \in \omega$, $0 < n \in \omega$,

i) $A_n \subset \bigcup_{i\in\omega} P(i,n)$;

ii) $\tau P(i,n) \geqslant \tau P(i+1,n)$;

iii) $\sum_{i\in\omega} \tau P(i,n) \leqslant \mu_U A_n + 1/n$; and

iv) $P(i,n) \in U^*$.

In connection with ii), we note that for any $P(i,n)$ with $\tau P(i,n) = 0$, by definition 20.1.3 and the fact that the space is Hausdorff, $P(i,n)$ is a singleton or is empty, and has μ_U and μ-measure zero. Thus the countable family of all such $P(i,n)$ also has measure zero and so we may without loss of generality assume that for every $i\in\omega$ and $n\in\omega$, $\tau P(i,n) > 0$. Then for any $n\in\omega$, only a finite number of the $P(i,n)$ can have the same τ-value and we can carry out theordering by non-increasing τ-values.

From iii) we have

v) $\lim\limits_{n} \sum\limits_{n\in\omega} \tau P(i,n) = a.$

We now apply a diagonal process to our sets $P(i,n)$. Let $(f_\alpha, \alpha\in D)$ be a universal subnet of the net $(n, n\in\omega)$ (see Kelley, problem 2J(d)). Then from i) and the fact that A is an ascending sequence we have

a) For each $n\in\omega$ there exists $\beta_n\in D$ such that for all $\alpha \gg \beta_n$,

$$A_n \subset \bigcup_{i\in\omega} P(i,f_\alpha),$$

and from v),

b) $\lim\limits_{\alpha} \sum\limits_{i\in\omega} \tau P(i,f_\alpha) = a.$

Clearly for all $i\in\omega$, $\alpha\in D$,

c) $\tau P(i,f_\alpha) \geqslant \tau P(i+1,f_\alpha)$;

d) $P(i,f_\alpha)\in U^*$.

For each $i\in\omega$, by theorem 19.8 there is a $V_i\in \mathfrak{F}$ which is a limit point of the sequence $(P(i,n), n\in\omega)$. Then V_i is a limit point of the subnet $(P(i,f_\alpha), \alpha\in D)$ and since $(f_\alpha, \alpha\in D)$ is universal, by problem 2J, (b) and (a), of Kelley,

$$\text{e)}\ V_i = \mathcal{R}\text{-}\lim_{\alpha} P(i,f_\alpha) \text{ for each } i\in\omega.$$

Now choose symmetric $W\in\mathcal{U}$, $W\circ W\subset V$. By lemma 20.12 for each $i\in\omega$ there exists B_i such that $\tau B_i = \tau V_i$, $V_i\subset B_i\subset W[V_i]$, and since the B_i are independent of n, for every $n\in\omega$,

$$\sum_{i\in\omega}\tau V_i + \mu(A_n\sim\bigcup_{i\in\omega}B_i) \leqslant \underline{\lim_{\alpha}}\ \sum_{i\in\omega}\tau P(i,f_\alpha) = a.$$

It follows from theorems 20.6 and 4.2 that

$$\sum_{i\in\omega}\tau V_i + \mu(\bigcup_{n\in\omega}A_n\sim\bigcup_{i\in\omega}B_i) \leqslant a.$$

By d), e) and lemma 19.9,

$$V_i\times V_i\subset\overline{U}\subset W\circ U\circ W$$

and so by 3.11.5,

$$B_i\times B_i\subset W[V_i]\times W[V_i]\subset W\circ W\circ U\circ W\circ W\subset V\circ U\circ V \quad \text{for } i\in\omega$$

and

$$B_i\in(V\circ U\circ V)^* \quad \text{for } i\in\omega.$$

Hence

$$\begin{aligned}\mu_{V\circ U\circ V}\bigcup_{n\in\omega}A_n &\leqslant \mu_{V\circ U\circ V}(\bigcup_{n\in\omega}A_n\sim\bigcup_{i\in\omega}B_i) + \mu_{V\circ U\circ V}\bigcup_{i\in\omega}B_i\\ &\leqslant \mu_{V\circ U\circ V}(\bigcup_{n\in\omega}A_n\sim\bigcup_{i\in\omega}B_i) + \sum_{i\in\omega}\tau B_i\\ &\leqslant \mu(\bigcup_{n\in\omega}A_n\sim\bigcup_{i\in\omega}B_i) + \sum_{i\in\omega}\tau V_i \leqslant a.\end{aligned}$$

21. Sets of non-σ- finite measure.

In this section we define a partial order $\succ$ on the functions in T and examine some consequences of this ordering. The definition extends that in 6.1 of Sion and Sjerve to compact Hausdorff spaces.

Again we assume $(X,\mathcal{G})$ is a compact Hausdorff space, $\mathcal{U}$ is the uniformity for X which induces $\mathcal{G}$, $\mathcal{S}$ is the family of subsets of X. For any gauge τ on $\mathcal{S}$, μ or $\mu^{(\tau)}$ is the measure of chapter III generated by $\mathcal{U}$ and τ.

21.1 DEFINITION. Suppose $\tau_1, \tau_2 \in T$. Then $\tau_1 \succ \tau_2$ iff given $\varepsilon > 0$, there exists $U \in \mathcal{U}$ such that if $A \times A \subset U$, then $\tau_1 A \leqslant \varepsilon \tau_2 A$.

21.2 DEFINITION. If $(X, \mathcal{G})$ is a topological space, then A is <u>analytic</u> in X iff $A = f[\alpha]$ for some $\alpha \in K'_{\sigma\delta}$ and some continuous function f on α to X, where K' is the family of closed, compact sets in a topological space $(X', \mathcal{G}')$. (For a good resumé of the theory of analytic sets, see Bressler and Sion [4].)

In connection with these concepts, Sion and Sjerve proved the following theorems.

THEOREM (6.4). Suppose for every $n \in \omega$, $\tau_{n+1} \succ \tau_n \in T$, and $\mathcal{B}$ is the family of all sets of the form $\bigcup_{n \in \omega} \beta_n$, where $\mu^{(\tau_n)} \beta_n = 0$ for $n \in \omega$. If E is analytic in X and $E \notin \mathcal{B}$, then there exists $\tau' \in T$ such that $\tau' \succ \tau_n$ for $n \in \omega$, and E has non-σ- finite $\mu^{(\tau')}$-measure.

THEOREM (6.5). Suppose $\tau \in T$, E is analytic in X and has non-σ- finite $\mu^{(\tau)}$-measure. Then there exists a $\tau' \in T$ such that $\tau' \succ \tau$, and E has non-σ- finite $\mu^{(\tau')}$-measure.

THEOREM (6.6) Suppose $\tau \in T$, E is analytic in X and has non-σ- finite $\mu^{(\tau)}$-measure. Then there exists a compact $C \subset E$, such that C has non-σ- finite $\mu^{(\tau)}$-measure.

The following theorem, along with lemma 6.2 in Sion and Sjerve, shows that the existence of functions in T ordered by $\succ$ is equivalent to metrizability in a compact Hausdorff space.

21.3 THEOREM. If there exist τ_1, $\tau_2 \in T$ such that $\tau_1 \succ \tau_2$, then $(X, \mathcal{S})$ is metrizable.

21.4 REMARK. By the above theorem, the hypothesis in theorem (6.4) of Sion and Sjerve implies that the space is metric, for which case that theorem was proved. However, if the compact Hausdorff space is non-metrizable then theorem 21.3 shows that theorem (6.5) is false in this space.

The question remains of whether theorem (6.6) can be generalized to the compact Hausdorff case. An essential step in the proof of that theorem in a metric space was the construction, given $\tau_1 \in T$, of a function $\tau_2 \in T$ such that $\tau_2 \succ \tau_1$. This cannot be done in a non-metrizable space, as theorem 21.3 shows, so that the proof does not generalize.

The following theorem and example are related to the question.

21.5 THEOREM. Suppose $\tau \in T$. If X is of σ-finite μ-measure, then X is metrizable.

21.6 REMARK. The attractive hypothesis that every non-metrizable analytic subset of a compact Hausdorff space contains a non-metrizable compact subset is false (see concluding example).

PROOFS AND EXAMPLE

21.7 LEMMA. If there exists a descending sequence U of symmetric members of $\mathcal{U}$ closed in the product topology on $X \times X$, and $\tau \in T$ such that for each $n \in \omega$,

$$A \times A \subset U_n \Rightarrow \tau A \leqslant 1/n,$$

then $\{U_n\}_{n \in \omega}$ is a base for $\mathcal{U}$ and $(X, \mathcal{J})$ is metrizable.

<u>Proof</u>: Since U_n is symmetric for each $n \in \omega$, $\bigcap_{n \in \omega} U_n$ is symmetric. Suppose

$$\bigcap_{n \in \omega} U_n \neq \Delta .$$

Then there exists $A \subset X$ such that $A \times A \not\subset \Delta$ and

$$A \times A \subset \bigcap_{n \in \omega} U_n .$$

Since $A \times A \subset U_n$ for each $n \in \omega$, we have $\tau A = 0$. On the other hand, since $\tau \in T$ and $A \times A \not\subset \Delta$, we have $\tau A > 0$ by definition 20.1.3. We conclude that

$$\bigcap_{n \in \omega} U_n = \Delta .$$

Now since the space is compact, $\mathcal{U}$ consists of all neighborhoods of Δ. Let $U \in \mathcal{U}$. Then there exists G, open in $X \times X$, such that

$$\Delta \subset G \subset U.$$

But Δ is the intersection of a descending sequence of sets

U_n closed in the product topology on $X \times X$, which is compact. Hence for some $n \in \omega$,

$$U_n \subset G \subset U.$$

Hence $\{U_n\}_{n\in\omega}$ is a base for $\mathcal{U}$ and by theorem 6.13 of Kelley, $(X, \mathcal{G})$ is metrizable.

Proof of 21.3: Suppose $\tau_1, \tau_2 \in T$, $\tau_1 \succ \tau_2$. By definition 20.1.5 there exists K such that for all $A \subset X$,

$$\tau_2 A \leqslant K.$$

Using definition 21.1, choose for each $n \geqslant 1$, closed symmetric $U_n \in \mathcal{U}$ (the closed symmetric members of $\mathcal{U}$ form a base for $\mathcal{U}$ (3.11.4)) such that $U_{n+1} \subset U_n$ and

$$A \times A \subset U_n \Rightarrow \tau_1 A \leqslant \frac{1}{nK} \cdot \tau_2 A \leqslant 1/n.$$

By lemma 21.7, $(X, \mathcal{G})$ is metrizable.

Proof of 21.5: Suppose X is of σ-finite μ-measure.

i) There exist only countably many points $x \in X$ such that $\tau\{x\} > 0$. Although singletons may not be μ-measurable, since the space is Hausdorff, any two points x, y can be separated by disjoint closed $\mathcal{G}_\delta$ sets, and since closed $\mathcal{G}_\delta$ sets are μ-measurable (theorem 18.3.2),

$$\mu\{x, y\} = \mu\{x\} + \mu\{y\}.$$

Since X has σ-finite measure, there can be only a countable number of points x with $\mu\{x\} > 0$. But $\mu\{x\} = \tau\{x\}$.

ii) Let $F = \{x \in X : \tau\{x\} = 0\}$. We show that F is closed and hence compact.

If $(x_\alpha, \alpha \in D)$ is a net in F which converges to $y \in X$ in the topology $\mathcal{G}$, then the net $(\{x_\alpha\}, \alpha \in D)$ in $\mathcal{S}$ converges to $\{y\}$ in the topology $\mathcal{R}$. By continuity of τ in $\mathcal{R}$,

$$\tau\{x_\alpha\} = 0 \text{ for } \alpha \in D \Rightarrow \tau\{y\} = 0$$

and so $y \in F$. Hence F is closed.

iii) F with the relative topology $\mathcal{S}_F$ is metrizable. Let $\mathcal{V}$ be the relativization of $\mathcal{U}$ to F, so $\mathcal{S}_F = \mathcal{T}_{\mathcal{V}}$ (see Kelley, p. 182). Then for every $\varepsilon > 0$ there exists $U \in \mathcal{V}$ such that

$$B \times B \subset U \Rightarrow \tau B < \varepsilon,$$

for otherwise suppose that for each $U \in \mathcal{V}$, there exists $A_U \subset F$ such that $A_U \times A_U \subset U$ and

$$\tau A_U > \varepsilon.$$

Then, since F is compact, given the net $(A_U, U \in \mathcal{V})$, by theorem 19.8 there is a subnet $(A_\alpha, \alpha \in D)$ and $A \subset F$ such that

$$A = \mathcal{R}\text{-}\lim_\alpha A_\alpha.$$

By the continuity of τ,

$$\tau A = \lim_\alpha \tau A_\alpha \geqslant \varepsilon.$$

Also $A \times A \subset \Delta$, for let $U \in \mathcal{V}$ and choose symmetric $V \in \mathcal{V}$ such that $V \circ V \circ V \subset U$. By definition 19.7 there exists $\gamma \in D$ such that for $\beta \gg \gamma$,

$$A \subset V[A_\beta].$$

Since $A_U \times A_U \subset U$ for each $U \in \mathcal{V}$ and $(A_\alpha, \alpha \in D)$ is a subnet of $(A_U, U \in \mathcal{V})$ we can and do choose $\beta \gg \gamma$ such that

$$A_\beta \times A_\beta \subset V.$$

Then by 3.11.5,

$$A \times A \subset V[A_\beta] \times V[A_\beta] \subset V \circ V \circ V \subset U,$$

and so $A \times A \subset U$ for every $U \in \mathcal{V}$. Hence

$$A \times A \subset \Delta.$$

Therefore $A = \{x\}$ for some $x \in F$ and so $\tau A = 0$, contradicting

$$\tau A \geqslant \varepsilon > 0.$$

We conclude that given $\varepsilon > 0$, there exists $U \in \mathcal{V}$ such that for $A \times A \subset U$, $\tau A < \varepsilon$. Now choose a sequence U in $\mathcal{V}$ such that for $0 < n \in \omega$, U_n is closed and symmetric, $U_{n+1} \subset U_n$, and

$$A \times A \subset U_n \Rightarrow \tau A \leqslant 1/n .$$

Then by lemma 21.7, $(F, \mathcal{J}_F)$ is metrizable. We note that $(F, \mathcal{J}_F)$ is separable since it is compact.

We have then that X is a compact Hausdorff space which is a union of countably many separable metrizable spaces (F and $\{x\}$ for each $x \in X \sim F$). Hence by a theorem of Smirnov (see Stone [15], proposition (B)), X is metrizable.

<u>Example for</u> 21.6: This example was suggested by that in the remark following corollary 2 in Stone [15]. Let

$$X = \{(0,0)\} \cup \{(x,y) \in R \times R : x \neq 0\}.$$

For $(x, y) \neq (0, 0)$ take as a neighborhood system the neighborhood system of (x,y) in the plane relativized to X. For $(0,0)$ take as a base for the neighborhood system the family of open sets in the usual topology on the plane which contain the y-axis. Let $\mathcal{J}$ be the resulting topology on X. Then

i) $\mathcal{J}$ is clearly Hausdorff.

ii) $(X, \mathcal{J})$ is not metrizable, since $\mathcal{J}$ does not satisfy the first axiom of countability. Given any countable family of open sets in the plane which contain the y-axis, we can construct an open set containing the y-axis which excludes points of every member of the countable family, i.e. the neighborhood system of (0,0) in $\mathcal{J}$ does not have a countable base.

iii) If $A \subset X$ is compact, then A is metrizable. X is the union of a countable number of separable metrizable spaces, and so the same is true of any $A \subset X$. But since A is compact, it is metrizable by the theorem of Smirnov (Stone [15], proposition (B)).

iv) X is analytic since it is the union of a countable number of compact sets (see 21.2).

v) $(X, \mathcal{J})$ is normal, hence completely regular since it is Hausdorff, and so by theorem 5.15 in Kelley, can be embedded in a compact Hausdorff space. To see that it is normal, consider the map $f : R^2 \to X$ defined by

$$f((x,y)) = \begin{cases} (x,y) & \text{if } x \neq 0. \\ (0,0) & \text{if } x = 0. \end{cases}$$

The verification that f is continuous is trivial. It is also easy to see that the image of an open set in R^2 containing the y-axis is open, and the image of an open set not intersecting the y-axis is open. Let A, B be closed in $\mathcal{J}$, $A \cap B = \emptyset$. If (0,0) is a member of one of them, say A, then $f^{-1}[A]$, $f^{-1}[B]$ are disjoint closed sets in R^2 and $f^{-1}[A]$ contains the y-axis. Since R^2 is normal, there exist open sets C, G in R^2 such that

$$f^{-1}[A] \subset C, \quad f^{-1}[B] \subset G, \text{ and } C \cap G = \emptyset.$$

Now C contains the y-axis, so f[C] and f[G] are open, disjoint and

$$A \subset f[C], \quad B \subset f[G].$$

If (0,0) is a member of neither A nor B, then $f^{-1}[A]$, $f^{-1}[B]$ and the y-axis are disjoint closed sets in R^2 and we can find disjoint open sets C and G in R^2 which do not intersect the y-axis and which contain respectively $f^{-1}[A]$ and $f^{-1}[B]$. But then f[C] are f[G] and open, disjoint and

$$A \subset f[C], \quad B \subset f[G].$$

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