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ON THE VANISHING OF A PURE PRODUCT IN A \((G, \sigma)\) SPACE

Abstract

We begin by constructing a vector space over a field \(F\), which we call a \((G, \sigma)\) space of the set \(W = V_1 \times V_2 \times \ldots \times V_n\), a cartesian product, where \(V_1\) is a finite-dimensional vector space over an arbitrary field \(F\), \(G\) is a subgroup of the full symmetric group \(S_n\) and \(\sigma\) is a linear character of \(G\). This space generalizes the spaces called the symmetry class of tensors defined by Marcus and Newman [1]. We can obtain the classical spaces, namely the Tensor space, the Grassman space and the symmetric space, by particularizing the group \(G\) and the linear character \(\sigma\) in our \((G, \sigma)\) space.

If \((v_1, v_2, \ldots, v_n) \in W\), we shall denote the "decomposable" element in our space by \(v_1 \Delta v_2 \Delta \ldots \Delta v_n\) and call it the \((G, \sigma)\) product or the Pure product if there is no confusion regarding \(G\) and \(\sigma\), of the vectors \(v_1, v_2, \ldots, v_n\). This corresponds to the tensor product, the skew symmetric product and the symmetric product in the classical spaces.
The purpose of this thesis is to determine a necessary and sufficient condition for the vanishing of the \((G,\sigma)\) product of the vectors \(v_1, v_2, \ldots, v_n\) in the general case. The results for the classical spaces are well-known and are deduced from our main theorem.

We use the "universal mapping property" of the \((G,\sigma)\) space to prove the necessity of our condition. These conditions are stated in terms of determinant-like functions of the matrices associated with the set of vectors \(v_1, v_2, \ldots, v_n\).

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ON THE VANISHING OF A PURE PRODUCT

IN A \((G,\sigma)\) SPACE

by

KULDIP SINGH

M.A., Punjab University, India, 1954

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Mathematics

We accept this thesis as conforming to the required standard.

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We begin by constructing a vector space over a field $F$, which we call a $(G,\sigma)$ space of the set $W = V_1 \times V_2 \times \ldots \times V_n$, a cartesian product, where $V_i$ is a finite-dimensional vector space over an arbitrary field $F$, $G$ is a subgroup of the full symmetric group $S_n$ and $\sigma$ is a linear character of $G$. This space generalizes the spaces called the symmetry class of tensors defined by Marcus and Newman [1]. We can obtain the classical spaces, namely the Tensor space, the Grassman space and the symmetric space, by particularizing the group $G$ and the linear character $\sigma$ in our $(G,\sigma)$ space.

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INTRODUCTION

Let $V_1, V_2, \ldots, V_n$ be any finite dimensional vector spaces over an arbitrary field $F$ and consider the tensor product $\bigotimes_{i=1}^{n} V_i$. It is well known that the tensor product $v_1 \otimes v_2 \otimes \ldots \otimes v_n$ is zero if and only if $v_i$ is zero for some $i$, $1 \leq i \leq n$. Similarly if $\wedge^n V$, where $V$ is any finite dimensional vector space over an arbitrary field $F$, is the Grassman space, then the skew-symmetric product $v_1 \wedge v_2 \wedge \ldots \wedge v_n$ is zero if and only if $v_1, v_2, \ldots, v_n$ are linearly dependent [3]. Again if $V$ is a finite dimensional unitary space and $V(n)$ is the symmetric product space, then the symmetric product $v_1 \cdot v_2 \cdot \ldots \cdot v_n$ is zero if any only if $v_i$ is zero for some $i$, $1 \leq i \leq n$ [1].

The aim of the present thesis is to define a suitable generalization of these three kinds of products and give a necessary and sufficient condition that a "pure product" vanishes. The general theory contains each of the three foregoing spaces, which we shall call the classical spaces, as special cases.

The starting point is the construction of the $(G, \sigma)$ space of the set $W$, which we denote by $P(W, G, \sigma)$ where $G$ is a subgroup of the full symmetric group $S_n$ of degree $n$, $W = V_1 \times V_2 \times \ldots \times V_n$ is the cartesian product of any finite
dimensional vector spaces $V_1$ over an arbitrary field $F$, such that $V_i = V g(i)$ for $i = 1, 2, \ldots, n$ and for all $g$ in $G$. This space generalizes the classical spaces, as well as the symmetry class of tensors defined by Marcus and Newman [1].

If $(v_1, v_2, \ldots, v_n)$ is in $W$, we denote its image under $\tau$, where $\tau$ is a multilinear and $(G, \sigma)$ function of $W$ into $P(W, G, \sigma)$, by $v_1 \Delta v_2 \Delta \ldots \Delta v_n$ and call it the $(G, \sigma)$ product or the pure product of the vectors $v_1, v_2, \ldots, v_n$. This product corresponds to the tensor product, the skew-symmetric product and the symmetric product respectively in the case of the tensor space, the Grassman space and the symmetric product space. Then we consider the problem of determining a necessary and sufficient condition that the $(G, \sigma)$ product $v_1 \Delta v_2 \Delta \ldots \Delta v_n$ is zero.

The method of approach is as follows: First the Embedding Theorem 3, Chapter II enables us to assume $V_1 = V_2 = \ldots = V_n = V$ (say). We take any basis $v_1, v_2, \ldots, v_m$ of $V$ and associate with any element $(v_1, v_2, \ldots, v_n)$ in $W$, a set $S$ of $n$ tuples $s = (s_1, s_2, \ldots, s_n)$, $1 \leq s_i \leq m$; $1 \leq i \leq n$. We define an equivalence relation on $S$ and denote by $E$, a set of representatives of the equivalence classes. We consider $(v_1, v_2, \ldots, v_n)\phi$, where $\phi$ is a mapping of $W$ into the free space $F(W)$, generated by $\{W\}$. By the Representation Theorem Form I, 5 Chapter II, we write $(v_1, v_2, \ldots, v_n)\phi$ in the
form
\[ (v_1, v_2, \ldots, v_n)\phi = w + \sum_{s \in E} a_s(y_{s_1}, y_{s_2}, \ldots, y_{s_n})\phi \] (*)

where \( w \) is in \( \Omega \), a subspace of \( F(W) \) and \( a_s \) is in \( F \) for all \( s \) in \( E \). In 6, Chapter II, we evaluate the co-efficients \( a_s \) in terms of a determinant like function \( D \) defined on a set of matrices \( M_s \) obtained from a matrix \( M \), where \( M \) is associated with \( (v_1, v_2, \ldots, v_n) \) in \( W \), with respect to the basis \( y_1, y_2, \ldots, y_m \) of \( V \). We rewrite (*) by means of the Representation Theorem Form II, 7, Chapter II, as
\[ (v_1, v_2, \ldots, v_n)\phi = w + \sum_{s \in E} D(M_s)(y_{s_1}, y_{s_2}, \ldots, y_{s_n})\phi . \]

The main results are contained in 2, Chapter III. Lemma 2.1, Chapter III, gives the solution to the problem if \((v_1, v_2, \ldots, v_n)\) satisfies the property \( P \) and Theorem 2.3, Chapter III, gives a necessary and sufficient condition in the general case. In this chapter, we also determine a basis of our \((G, \sigma)\) space, which we use to express \( P(W, G, \sigma) \) as the tensor product of \((G_1, \sigma_1)\) spaces \( P(W_1, G_1, \sigma_1) \) where \( G_1 \) is a "disjoint" factor of \( G \) (4, Chapter III).

Finally in Chapter IV, we deduce some known results about the classical spaces from our main results by a suitable specialization. We also state and prove a condition (4, Chapter IV) which is sufficient for the vanishing of a \((G, \sigma)\)
product $\nu_1 \Delta \nu_2 \Delta \ldots \Delta \nu_n$ in the general case, and is also necessary, if $V$ is unitary and $G$ belongs to a certain class of abelian groups (5, Chapter IV).
The aim of this chapter is to fix the notations and the terminology of some well-known concepts.

1. Vector spaces spanned by sets.

Let $S$ be an arbitrary non-empty set and $F$ an arbitrary field. Let $F(S)$ denote the set of all mappings $\psi$ from $S$ into $F$, such that $(s)\psi \neq 0$ for only a finite number of $s \in S$. If $\psi_1$ and $\psi_2$ are in $F(S)$ and $\alpha \in F$, we can define the mappings $\psi_1 + \psi_2$ and $\alpha \psi_1$ in the usual way by setting

$$(s) (\psi_1 + \psi_2) = (s)\psi_1 + (s)\psi_2 \quad \text{for all } s \in S,$$

and $$(s) (\alpha \psi_1) = \alpha (s)\psi_1 \quad \text{for all } s \in S.$$  

Clearly $\psi_1 + \psi_2$ and $\alpha \psi_1$ again belong to $F(S)$. One can easily verify that these operations of addition and scalar multiplication give $F(S)$ the structure of a vector space over $F$. This vector space is called the free space generated by $S$ over $F$.
1.1 **Basis of $F(S)$**.

For each $s \in S$, consider the mapping

$$e_s : S \longrightarrow F,$$

where $(t)e_s = \delta_{st}$ for all $t \in S$ and $\delta_{st}$ is the Kronecker delta.

Then $e_s \in F(S)$ and the set $B = \{e_s \mid s \in S\}$ is a basis of $F(S)$.

1.2 **Definition**: Let $\phi : S \longrightarrow F(S)$, where

$$(s) \phi = e_s \quad \text{for all} \quad s \in S.$$

$\phi$ is a one-one mapping of $S$ onto $B$, and hence $\dim F(S) = \text{cardinality } S$.

1.3 **Proposition**: Let $F(S)$ be the free space generated by $S$ over $F$, $U$ an arbitrary vector space over $F$ and $\phi$ the mapping defined in 1.2. If $f$ is any mapping from $S$ into $U$, then there exists a unique linear transformation $T$ of $F(S)$ into $U$, which makes the following diagram

\[\begin{array}{ccc}
S & \xrightarrow{\phi} & F(S) \\
\downarrow{f} & & \downarrow{T} \\
& U & \\
\end{array}\]
commutative; i.e. \( \mathcal{O} = f \)

**Proof:** \( B = \{ e_s \mid s \in S \} \) is a basis of \( F(S) \) by l.1.

Define \( \mathcal{T} : B \rightarrow U \), by

\[
(e_s)^T = (s)f,
\]

and extend it linearly to a mapping, to be denoted again by \( \mathcal{T} \), on \( F(S) \) into \( U \). More explicitly

\[
(\sum_{s \in A} a_s e_s) \mathcal{T} = \sum_{s \in A} a_s (e_s)^T,
\]

where \( A \) is a finite non-empty subset of \( S \) and \( a_s \in F \) for all \( s \in A \). Then \( \mathcal{T} \) is the desired mapping.

2. \((G, \sigma)\) spaces.

Let \( G \) be any permutation group on the set \( I = \{1, 2, \ldots, n\} \); i.e., a subgroup of the symmetric group \( S_n \) of degree \( n \). Let \( F \) be any arbitrary field and \( \sigma \) any linear character of \( G \); i.e., \( \sigma : G \rightarrow F^* \) is a (group) homomorphism where \( F^* \) is the multiplicative group of \( F \). For each \( i \in I \), let \( V_i \) be any finite dimensional vector space over \( F \); and let \( W = V_1 \times V_2 \times \cdots \times V_n \), the cartesian product of the \( V_i \).

2.1 **Definition:** \( W \) is called a \( G \)-set if and only if \( V_i = V_g(i) \) for all \( i \in I \) and for all \( g \in G \).

We shall assume in the sequel, that \( W \) is a \( G \)-set, and denote its general elements, \( w \) by \((w_1, w_2, \ldots, w_n)\) where \( w_i \in V_i \) for all \( i \in I \).
2.2 **Definition:** Let $f$ be any mapping of $W$ into $U$, where $U$ is any vector space over $F$. Then

(i) $f$ is called a **multilinear function** if $f$ is linear in each of the $n$-coordinates, that is, if

$$(w_1,w_2,\ldots,aw_i + bw_i',\ldots,w_n)f = \alpha(w_1,\ldots,w_i,\ldots,w_n)f +$$
$$\beta(w_1,\ldots,w_i',\ldots,w_n)f,$$

for any $w_i,w_i'$ in $V_1$, any $\alpha,\beta$ in $F$ and any $i \in I$.

(ii) $f$ is called a **$(G,\sigma)$ function** if

$$(w_1,w_2,\ldots,w_n)f = \sigma(g)(w_{\sigma(1)},w_{\sigma(2)},\ldots,w_{\sigma(n)})f,$$

for any $w_i \in V_1$ and $g \in G$.

2.3 **Definition:** A vector space $T$ over $F$ is called a **$(G,\sigma)$ space of $W$** if and only if there exists a mapping $\tau$ of $W$ into $T$ with the following properties

(i) $\tau$ is a multilinear and $(G,\sigma)$ function.

(ii) $WT = \{ w\tau \mid w \in W \}$ is a spanning set of $T$.

(iii) The space $T$ has a universal mapping property, that is if $U$ is any vector space over $F$ and $f$ is an arbitrary multilinear and $(G,\sigma)$ function from $W$ into $U$, then there exists a unique linear transformation $F: T \rightarrow U$, such that the following diagram
2.4 Theorem: Given $G, \sigma$ and a $G$-set $W$, there exists a $(G, \sigma)$ space of $W$. Any two $(G, \sigma)$ spaces of $W$ are isomorphic as vector spaces.

Proof: We shall prove the latter statement first. Let $T_1$ and $T_2$ be two $(G, \sigma)$ spaces of $W$. Then there exist multilinear and $(G, \sigma)$ functions $\tau_1 : W \to T_1$, and $\tau_2 : W \to T_2$ and unique linear transformations $\overline{\tau}_1 : T_2 \to T_1$ and $\overline{\tau}_2 : T_1 \to T_2$, which make the following diagrams commutative; i.e., $\tau_1 \overline{\tau}_2 = \tau_2$ and $\tau_2 \overline{\tau}_2 = \tau_1$.
Consider the linear transformation

\[ \overline{\tau}_2 \overline{T}_1 : T_1 \rightarrow T_1 \]

We shall show, \( \overline{\tau}_2 \overline{T}_1 = \iota_1 \), the identity map of \( T_1 \). Since \( \overline{\tau}_2 \overline{T}_1 \) and \( \iota_1 \) are linear transformations of \( T_1 \), it is sufficient to show, that \( \overline{\tau}_2 \overline{T}_1 = \iota_1 \) on the spanning set \( W_{T_1} \) of \( T_1 \). For \( w_{T_1} \in W_{T_1} \),

\[ (w_{T_1})(\overline{\tau}_2 \overline{T}_1) = ((w)\overline{T}_2 \overline{T}_1) = (w)\overline{T}_1 = w_{T_1} \]

Therefore \( \overline{\tau}_2 \overline{T}_1 = \iota_1 \). Similarly, \( \overline{T}_1 \overline{\tau}_2 = \iota_2 \) the identity map of \( T_2 \). Hence \( T_1 \) and \( T_2 \) are isomorphic.

We next show the existence of a \((G,\sigma)\) space for \( W \).

Denote by \( \Omega \), the smallest subspace of \( F(W) \), which contains all elements of each of the forms (i) and (ii):

(i) \((w_1, \ldots, aw_i + bw_i', \ldots, w_n) \phi - \alpha(w_1, \ldots, w_i, \ldots, w_n) \phi - \beta(w_1', w_i', \ldots, w_n) \phi\),

where \( \alpha, \beta \in F \), \( w_i \in V_i \) and \( i \in I \).

(ii) \((w_1, w_2, \ldots, w_n) \phi - \sigma(g)(w_\sigma(1), w_\sigma(2), \ldots, w_\sigma(n)) \phi\)

where \( g \in G \), \( w_i \in V_i \), and \( i \in I \).

Such elements will be referred to as elements of type (i) or type (ii) respectively.

Let \( T \) be the quotient space \( F(W)/\Omega \); we shall show that \( T \) is a \((G,\sigma)\) space.
Let \( \eta : F(W) \rightarrow T \) be the natural homomorphism. Set \( \tau = \phi \eta \). Then \( \tau \) is a multilinear and \((G, \sigma)\) function. For if \( \alpha, \beta \in F \), then
\[
(w_1, \ldots, \alpha w_i + \beta w_i', \ldots, w_n) \tau - \alpha(w_1, \ldots, w_i, \ldots, w_n) \tau - \\
\beta(w_1, \ldots, w_i', \ldots, w_n) \tau
\]
\[
= [(w_1, \ldots, \alpha w_i + \beta w_i', \ldots, w_n) \phi - \alpha(w_1, \ldots, w_i, \ldots, w_n) \phi - \\
\beta(w_1, \ldots, w_i', \ldots, w_n) \phi] \eta = 0
\]
since the expression in the square brackets is of type (i). Again if \( g \in G \), then
\[
(w_1, w_2, \ldots, w_n) \tau - \sigma(g)(w_{g(1)}, w_{g(2)}, \ldots, w_{g(n)}) \tau
\]
\[
= [(w_1, w_2, \ldots, w_n) \phi - \sigma(g)(w_{g(1)}, w_{g(2)}, \ldots, w_{g(n)}) \phi] \eta = 0
\]
since the expression in square brackets is of type (ii). Thus \( \tau \) is a multilinear and \((G, \sigma)\) function.

Also \( WT = \{ w \tau \mid w \in W \} \) is a spanning set for \( T \), since given \( z \) in \( T \), there exists \( y \) in \( F(W) \), \( y = \sum a_w e_w \), where \( A \) is a suitable subset of \( S \), such that
\[
z \eta = ( \sum a_w e_w ) \eta = \sum a_w (e_w) \eta = \sum a_w (w) \phi \eta = \sum a_w (w \tau)
\]
Now, if \( U \) is any vector space over \( F \) and \( f \) is any multilinear and \((G, \sigma)\) function of \( W \) into \( U \), then by Proposition 1.3, there exists a unique linear transformation \( f' : F(W) \rightarrow U \), such that \( \phi f' = f \).
Claim: $\Omega \subseteq \ker f'$.

For if $y = (w_1, \ldots, \alpha w_i + \beta w_i', \ldots, w_n) \phi - \alpha(w_1, \ldots, w_i', \ldots, w_n) \phi$

$$- \beta(w_1, \ldots, w_i', \ldots, w_n) \phi$$

is an element of type (i), then since $f'$ is a linear transformation we have

$$y f' = (w_1, \ldots, \alpha w_i + \beta w_i', \ldots, w_n) \phi f' - \alpha(w_1, \ldots, w_i', \ldots, w_n) \phi f'$$

$$- \beta(w_1, \ldots, w_i', \ldots, w_n) \phi f'$$

$$= (w_1, \ldots, \alpha w_i + \beta w_i', \ldots, w_n) f - \alpha(w_1, \ldots, w_i', \ldots, w_n) f$$

$$- \beta(w_1, \ldots, w_i', \ldots, w_n) f$$

$$= 0,$$ since $f$ is multilinear.

Again if $y = (w_1, w_2, \ldots, w_n) \phi - \sigma(g)(w_g(1), w_g(2), \ldots, w_g(n)) \phi$

is an element of type (ii), we have

$$y f' = (w_1, w_2, \ldots, w_n) \phi f' - \sigma(g)(w_g(1), w_g(2), \ldots, w_g(n)) \phi f'$$

$$= (w_1, w_2, \ldots, w_n) f - \sigma(g)(w_g(1), w_g(2), \ldots, w_g(n)) f = 0.$$
since $f$ is $(G,\sigma)$.

Thus elements of type (i) and (ii) are contained in the $\ker f'$ and since $\Omega$ is the subspace generated by the elements of type (i) and (ii), we have $\Omega \subseteq \ker f'$, which proves the claim.

Define $\overline{f}: T \rightarrow U$, as follows.

If $z \in T$, there exists $y \in F(W)$, such that $z = yn$. Set $(z)\overline{f} = (y)f'$. Since $\Omega \subseteq \ker f'$, $\overline{f}$ is well-defined.

$\overline{f}$ is a linear transformation. For if $\alpha, \beta \in F$, and $z_1, z_2 \in T$, we have $z_1 = y_1 n$ and $z_2 = y_2 n$ for some $y_1$ and $y_2$ in $F(W)$. Then $\alpha z_1 + \beta z_2 = (\alpha y_1 + \beta y_2) n$ and therefore $(\alpha z_1 + \beta z_2)\overline{f} = (\alpha y_1 + \beta y_2)f' = \alpha (z_1)\overline{f} + \beta (z_2)\overline{f}$, which shows that $\overline{f}$ is a linear transformation.

Also if $w \in W$, we have $(w)\overline{f} = (w)\overline{f} = (w)\overline{f} = (w)f' = wf$. Thus $\overline{T} = f$.

Now, let $g : T \rightarrow U$ be any linear transformation, such that $\overline{T} = f$. We will show that $g = \overline{f}$. In order to show this, it is sufficient to show, that $g = \overline{f}$ on the spanning set of $W\overline{T}$ of $T$, since $g$ and $\overline{f}$ are both linear transformations. Let $w\overline{T} \in W\overline{T}$, then

$$(w\overline{T})g = (w)\overline{T}g = (w)f = (w)\overline{f} = (w\overline{T})\overline{f}$$

Hence $g = \overline{f}$.

Therefore $T$ is a $(G,\sigma)$ space of $W$ and by the latter statement of the theorem proved already is determined.
uniquely up to isomorphism. We shall denote \( T \), constructed as above, by \( P(W,G,\sigma) \) and call it the \((G,\sigma)\) space of \( W \) determined by \( G \) and \( \sigma \).

2.5 Notation: If \( w = (w_1, w_2, \ldots, w_n) \in W \), we shall denote its image \( w^\varpi \) under \( \varpi \) by \( w_1^\varpi w_2^\varpi \cdots w_n^\varpi \) and call it the \((G,\sigma)\) product or pure product of the vectors \( w_1, w_2, \ldots, w_n \).

3 Some special cases of the \((G,\sigma)\) spaces.

3.1 Let \( G = \{e\} \) where \( e \) is the identity permutation in \( S_n \). Then \( \sigma = 1 \), the trivial character, and \( W \) is obviously a \( G \)-set. The \((G,\sigma)\) space in the case is the tensor product of the vector spaces \( V_1, V_2, \ldots, V_n \), which is denoted usually by \( \otimes V_1 \). If \( (w_1, w_2, \ldots, w_n) \in W \) then we denote \( w_1^\varpi w_2^\varpi \cdots w_n^\varpi \) by the usual notation \( w_1 \otimes w_2 \otimes \cdots \otimes w_n \), which is called the tensor product of the vectors \( w_1, w_2, \ldots, w_n \).

3.2 Let \( G = S_n \) and \( V_1 = V_2 = \ldots = V_n \). Then \( W \) is a \( G \)-set. Let \( \sigma \) be the linear character of \( G \), given by \( \sigma(g) = \text{sign } g \), for all \( g \in G \). Then \( P(W,G,\sigma) \) is the Grassman space which is denoted by \( \wedge V \). If \( (w_1, w_2, \ldots, w_n) \in W \), we denote \( w_1^\varpi w_2^\varpi \cdots w_n^\varpi \) by the usual notation \( w_1 \wedge w_2 \wedge \cdots \wedge w_n \), which is called the skew-symmetric product or the caret product of the vectors \( w_1, w_2, \ldots, w_n \).

3.3 Let \( G \) and \( W \), as in 3.2, and \( \sigma = 1 \), the trivial
character. In this case $P(W,G,\sigma)$ is the symmetric product which is denoted by $V(n)$. If $(w_1, w_2, \ldots, w_n) \in W$ then we denote $w_1 \wedge w_2 \wedge \ldots \wedge w_n$ by the usual notation $w_1 \cdot w_2 \cdot \ldots \cdot w_n$, which is called the symmetric product of the vectors $w_1, w_2, \ldots, w_n$.

We shall call these three kinds of spaces, the classical spaces.
In this chapter, we shall state the problem of the thesis. The concept of "pseudo-determinant" function, a determinant like function is introduced, and this is used in the Representation Theorem Form II, which is the main result of this chapter.

1 Throughout this and the following chapters, we shall assume, unless otherwise stated, the following:

G is any permutation group on \( I = \{1, 2, \ldots, n\} \).

F an arbitrary field.

\( \sigma \) any linear character of G.

V\(_i\) a finite dimensioned vector space over F, for each \( i \in I \).

\( W = V_1 \times V_2 \times \cdots \times V_n \) (cartesian product) is a G-set.

F(W) is the free space over F, generated by the set W.

P(W, G, \( \sigma \)) is the (G, \( \sigma \)) space of W determined by G and \( \sigma \).

\( \phi \) is the mapping of W into F(W), as defined in 1.2, Chapter I.

\( \eta \) is the natural homomorphism of F(W) into P(W, G, \( \sigma \))

\( \tau = \phi \eta \).

Note: We shall see in Remark 3.2, that we may take \( V_1 = V_2 = \cdots = V_n = V \) (say). From then on, \( W = V \times V \times \cdots \times V \) (n copies).
2. Statement of the Problem

Given an arbitrary element \((v_1, v_2, \ldots, v_n) \in W\), find a necessary and sufficient condition that \(v_1 \wedge v_2 \wedge \ldots \wedge v_n = 0\).

The answer to this problem is known in the case of the classical spaces, defined in Chapter I and is the following:

(i) \(v_1 \odot v_2 \odot \ldots \odot v_n = 0\) if and only if \(v_i = 0\) for some \(i\), \(1 \leq i \leq n\).

(ii) \(v_1 \wedge v_2 \wedge \ldots \wedge v_n = 0\) if and only if \(v_1, v_2, \ldots, v_n\) are linearly dependent.

(iii) \(v_1 \cdot v_2 \cdot \ldots \cdot v_n = 0\) if and only if \(v_i = 0\) for some \(i\), \(1 \leq i \leq n\).

See [1],[2],[3],[4] for reference.

2.1 Remark: (iii) is proved in [1], under the restriction, that \(V\) is an \(n\)-dimensional unitary space.

Let \(P(W,G,\sigma)\) be the \((G,\sigma)\) space of \(W\). Let \(U\) be any finite dimensional vector space over \(F\) of dim \(m\), where \(m = \max\{\dim V_i | 1 \leq i \leq n\}\). Let \(W = U \times U \times \ldots \times U\) (\(n\) copies).

Then we have the following

3. Theorem (Embedding): There exists an isomorphism (vector spaces) of \(P(W,G,\sigma)\) into \(P(W',G,\sigma)\), and this isomorphism
carries \ W \gamma into \ W'\gamma', where \ \gamma' \ is a multilinear and 
\ (G,\sigma) \ function of \ W' \ into \ P(W',G,\sigma).

Proof: For each \ i \in \ I, \ let \ f_i : V_i \rightarrow U \ be any non-
singular linear transformation, satisfying \ f_i = f_{g(i)} \ for 
all \ g \in G. \ This \ is \ possible, \ since \ dim \ U = m = \ max 
dim V_i, 1 \leq i \leq n \ and \ V_i = V_{g(i)} \ for \ all \ i \in I \ and \ g \in G. \ Then \ define

\ f: W \rightarrow W', \ by \ setting 
\ (w_1, w_2, \ldots, w_n)f = (w_1f_1, w_2f_2, \ldots, w_nf_n). 

In the following diagram,

\begin{tikzcd}
W' \arrow{r}{\gamma'} \arrow{d}{f} & P(W',G,\sigma) \arrow{d}{f'\gamma'} \\
W \arrow{r}{\gamma} & P(W,G,\sigma)
\end{tikzcd}

consider the mapping \ f\gamma' : W \rightarrow P(W',G,\sigma). \ This
is a multilinear and \ (G,\sigma) \ function.

For if \ \alpha, \beta \in F, \ then for any \ i \in I,
\[(w_1, \ldots, \alpha w_i + \beta w'_i, \ldots, w_n)f\gamma' = (w_1f_1, \ldots, (\alpha w_i + \beta w'_i)f_i, \ldots, w_nf_n)\gamma'
= (w_1f_1, \ldots, \alpha w_if_i + \beta w'_if_i, \ldots, w_nf_n)\gamma', \ \text{since} \ f_i \ \text{is linear}
= \alpha(w_1f_1, \ldots, w_if_i, \ldots, w_nf_n)\gamma'
+ \beta(w_1f_1, \ldots, w'_if_i, \ldots, w_nf_n)\gamma' \ \text{since} \ \gamma' \ \text{is multilinear.}
\[ \begin{align*}
&= \alpha(w_1, w_2, \ldots, w_1, \ldots, w_n) \tau', \\
&\quad + \beta(w_1, w_2, \ldots, w_1', \ldots, w_n) \tau',
\end{align*} \]

which shows that \( \tau' \) is multilinear. Again if \( g \in G \), then

\[ (w_1, w_2, \ldots, w_n) \tau' = (w_1 f_1, w_2 f_2, \ldots, w_n f_n) \tau' \]

\[ = (u_1, u_2, \ldots, u_n) \tau', \quad \text{where } u_i = w_i f_i \text{ for each } i \in I, \]

\[ = \sigma(g)(u_{g(1)}, u_{g(2)}, \ldots, u_{g(n)}) \tau', \quad \text{since } \tau' \]

\[ = \sigma(g)(w_{g(1)} f_{g(1)}, w_{g(2)} f_{g(2)}, \ldots, w_{g(n)} f_{g(n)}) \tau', \]

\[ = \sigma(g)(w_{g(1)} f_1, w_{g(2)} f_2, \ldots, w_{g(n)} f_n) \tau', \quad \]

\[ \text{since } f_i = f_{g(i)} \text{ for all } i \in I. \]

\[ = \sigma(g)(w_{g(1)}, w_{g(2)}, \ldots, w_{g(n)}) \tau', \]

which shows that \( \tau' \) is a \((G,\sigma)\) function.

Therefore, by the universal mapping property, there exists a unique linear transformation \( \overline{\tau}' \) of \( P(W, G, \sigma) \) into \( P(W', G, \sigma) \), such that \( \tau' \overline{\tau}' = \tau' \).

Now for each \( i \in I \), \( U = V_i f_1 \bigoplus U_i' \) (direct sum), where \( U_i' \) is a complement of \( V_i f_1 \) in \( U \).

Define \( g_i : U \rightarrow V_1 \), as follows:

If \( u \in U \), then \( u = v_i f_1 + u_i' \) (uniquely), where \( v_i \in V_i \). 

\[ \text{Define } g_i : U \rightarrow V_1, \text{ as follows:} \]

\[ \text{If } u \in U, \text{ then } u = v_i f_1 + u_i' \text{ (uniquely), where } v_i \in V_i \]
and $u'_i \in U'_i$. Set $u_{g_1} = v_1$. Clearly $g_1$ is a linear transformation and also \( f_i g_1 = i \), the identity map of $V_1$.

Define $g : W' \rightarrow W$, by setting

\[
(u_1, u_2, \ldots, u_n)g = (u_1 g_1, u_2 g_2, \ldots, u_n g_n).
\]

Then $fg = i$ the identity map of $W$. For if $(w_1, w_2, \ldots, w_n) \in W$,

then $(w_1, w_2, \ldots, w_n)fg = (w_1 f_1, w_2 f_2, \ldots, w_n f_n)g$

\[
= (w_1 f_1 g_1, w_2 f_2 g_2, \ldots, w_n f_n g_n)
\]

\[
= (w_1, w_2, \ldots, w_n).
\]

Now in the following diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\tau} & P(W, G, \sigma) \\
\downarrow{g} & & \downarrow{g\tau} \\
W' & \xrightarrow{\tau'} & P(W', G, \sigma)
\end{array}
\]

$g\tau$ can be easily verified as $f\tau'$ in the previous diagram to be multilinear and $(G, \sigma)$. Therefore by the universal mapping property, there exists a unique linear transformation $\overline{g\tau}$ of $P(W', G, \sigma)$, such that $\tau g\tau = g\tau$.

We will show now, that $\overline{f\tau'} \overline{g\tau} = i$, the identity map on $P(W, G, \sigma)$. Let $(w_1 \Delta w_2 \Delta \ldots \Delta w_n) \in P(W, G, \sigma)$, then

\[
(w_1 \Delta w_2 \Delta \ldots \Delta w_n)\overline{f\tau'} \overline{g\tau} = (w_1, w_2, \ldots, w_n)\tau \overline{f\tau'} \overline{g\tau}
\]

\[
= (w_1, w_2, \ldots, w_n)\tau \overline{f\tau'} \overline{g\tau}.
\]
Thus \( \overline{f^T g^T} = \mathbb{1} \) is the identity map on \( W \). But since \( W^\tau \) is a spanning set of \( P(W, G, \sigma) \) and \( \overline{f^T g^T} \) is a linear transformation, we have \( \overline{f^T g^T} = \mathbb{1} \) the identity map on \( P(W, G, \sigma) \). Hence \( f^T' \) is an isomorphism of \( P(W, G, \sigma) \) into \( P(W', G, \sigma) \).

Also if \((w_1, w_2, \ldots, w_n)^\tau \in W^\tau \), then \((w_1, w_2, \ldots, w_n)^{fT'} = (w_1 f_1, w_2 f_2, \ldots, w_n f_n)^\tau \in W'^\tau \). Therefore \( f^T' \) carries \( W^\tau \) into \( W'^\tau \).

3.1 Remark: In Theorem 3, we may take \( U \) to be any vector space over \( F \) of dimension exceeding \( \max \{ \dim V_i | i \in I \} \).

3.2 Remark: In view of the Embedding Theorem 3, we shall assume, to make the proofs simpler, but with no restriction on the generality of the Problem, that \( V_1 = V_2 = \ldots = V_n = V \) (say). So \( W \) becomes \( V \times V \times \ldots \times V \) (\( n \) copies). We shall also assume throughout, that \( \dim V = m \) and \( \{y_1, y_2, \ldots, y_m\} \) is a basis of \( V \).

4. Some Definitions.

(1) \((w_1, w_2, \ldots, w_n)^\phi \in F(W)\) is called a \((G, \sigma)\) element, if and only if there exists \( g \in G \), such that \( \sigma(g) \neq 1 \)
and $w_i', w_g(i)$ are dependent for all $i \in I$; i.e. $\sigma(g) \not\equiv 1$ and $\dim \langle w_i', w_g(i), w_g^2(i), \ldots, w_g^{\text{ord }-1}(i) \rangle = 1$
for all $i \in I$.

(2) $(w_1, w_2, \ldots, w_n) \not\equiv$ and $(w'_1, w'_2, \ldots, w'_n) \not\equiv$ in $F(W)$ are said to be $G$-related, if and only if, there exists $g \in G$, such that $w_i = w_g(i)$ for all $i \in I$.

(3) $(w_1, w_2, \ldots, w_n) \not\equiv$ is said to satisfy the property $P$, if and only if, for each $i \in I$, $i \geq 2$, either $w_i \in \{ w_j \mid 1 \leq j < i \}$
or $w_i$ is independent of the set $\{ w_j \mid 1 \leq j < i \}$.

(4) $(w_1, w_2, \ldots, w_n) \in W$ is called a trivial element, if and only if $w_i = 0$ for some $i \in I$. Otherwise it is a non-trivial element.

5. Theorem (Representation Theorem Form I): If $(w_1, w_2, \ldots, w_n) \in W$ is a non-trivial element, then $(w_1, w_2, \ldots, w_n) \not\equiv$ can be written in the form $(w_1, w_2, \ldots, w_n) \equiv = w + \sum_{i=1}^{k} c_i T_i$, for some positive integer $k$, where $w \in \Omega$, $c_i \in F$, $T_i \in F(W)$.
For each $i$, $1 \leq i \leq k$, $T_i$ satisfies the property $P$ and if $i \not\equiv j$, then $T_i$ and $T_j$ are not $G$-related.

Proof: Let $\{ y_1, y_2, \ldots, y_m \}$ be a basis of $V$. For each $i \in I$, let $w_i = \sum_{j=1}^{m} b_{ij} y_j$, and set $A_i = \{ j \mid 1 \leq j \leq m, b_{i,j} \not\equiv 0 \}$. Since $(w_1, w_2, \ldots, w_n)$ is a non-trivial element, $A_i \not\equiv \emptyset$ for any $i \in I$. Let $S = A_1 \times A_2 \times \ldots \times A_n$, the product of the index sets.
the cartesian product. If \( s \in S \) and \( s = (s_1, s_2, \ldots, s_n) \), let \( b_s = b_1, s_1 b_2, s_2 \ldots b_n, s_n \). Clearly \( b_s \neq 0 \) for any \( s \in S \).

Define a relation "\( \sim \)" on \( S \) as follows:

If \( s = (s_1, s_2, \ldots, s_n) \) and \( t = (t_1, t_2, \ldots, t_n) \) are in \( S \), then \( s \sim t \) if and only if there exists \( g \in G \), such that \( (t_1, t_2, \ldots, t_n) = (s_g(1), s_g(2), \ldots, s_g(n)) \); i.e. \( t_i = s_g(i) \) for all \( i \in I \). Clearly "\( \sim \)" is an equivalence relation on \( S \). If \( s \in S \), let \( A(s) \) denote the equivalence class containing \( s \). Then \( \{ A(s) \mid s \in S \} \) is the set of the equivalence classes. Let \( E \) be a set consisting of a representative of each of the equivalence class \( A(s) \).

Now

\[
(w_1, w_2, \ldots, w_n) \phi = \left( \sum_{j=1}^{w_1} b_1, j y_j, \sum_{j=1}^{w_2} b_2, j y_j, \ldots, \sum_{j=1}^{w_n} b_n, j y_j \right) \phi
\]

\[
= \left( \sum_{s_1 \in A_1} b_1, s_1 y_{s_1}, \sum_{s_2 \in A_2} b_2, s_2 y_{s_2}, \ldots, \sum_{s_n \in A_n} b_n, s_n y_{s_n} \right) \phi
\]

\[
= \left[ \left( \sum_{s_1 \in A_1} b_1, s_1 y_{s_1}, \sum_{s_2 \in A_2} b_2, s_2 y_{s_2}, \ldots, \sum_{s_n \in A_n} b_n, s_n y_{s_n} \right) \phi \right]
\]

\[
- \sum_{s_1 \in A_1} b_1, s_1 (y_{s_1}, \sum_{s_2 \in A_2} b_2, s_2 y_{s_2}, \ldots, \sum_{s_n \in A_n} b_n, s_n y_{s_n}) \phi
\]
since each of the terms within the square brackets is in $\Omega$, we have

$$(w_1, w_2, \ldots, w_n) = w_0 + \sum_{s_1 \in A_1} \sum_{s_1} b_{s_1} \quad \sum_{s_2 \in A_2} \sum_{s_2} y_{s_2} \quad \cdots \quad \sum_{s_n \in A_n} b_{s_n} \quad (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi$$
\[ w_0 + \sum_{s \in A(s)} b_{1} s_1 b_{2} s_2 \ldots b_{n} s_n (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi \]

\[ = w_0 + \sum_{s \in S} b_{s} (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi , \]

where \( w_0 \) is the sum of the terms within the square brackets and is in \( \Omega \).

Now if \( t \in A(s) \), then \( t \sim s \), which implies that there exists \( g \in G \) such that \( (t_1, t_2, \ldots, t_n) = (s_{g(1)}, s_{g(2)}, \ldots, s_{g(n)}) \); i.e. \( t_i = s_{g(i)} \) for all \( i \in I \) or \( t_{g^{-1}(i)} = s_1 \) for all \( i \in I \). Hence

\[ (y_{t_1}, y_{t_2}, \ldots, y_{t_n}) \phi = [(y_{t_1}, y_{t_2}, \ldots, y_{t_n}) \phi - \sigma(g^{-1})(y_{t_{g^{-1}(1)}}, y_{t_{g^{-1}(2)}}, \ldots, y_{t_{g^{-1}(n)}}) \phi ] + \sigma(g^{-1})(y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi \]

where \( w(t, g^{-1}) \) is equal to the term within the square brackets, which being of type \((ii)\) is in \( \Omega \).

Therefore \( (w_1, w_2, \ldots, w_n) \phi = w_0 + \sum_{s \in S} \sum_{t \in A(s)} b_t w(t, g^{-1}) \]

\[ + \sum_{s \in S} \sum_{t \in A(s)} \sigma(g^{-1}) b_t (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi \]

\[ = w_0 + \sum_{s \in S} \sum_{t \in A(s)} b_t w(t, g^{-1}) = \omega \text{ and } \sum_{t \in A(s)} \sigma(g^{-1}) b_t = b' \text{.} \]
Clearly \( w \in \Omega \).

Then \( (w'_1, w'_2, \ldots, w'_n) \phi = w + \sum_{s \in E} b'(s) (y_{s'_1}, y_{s'_2}, \ldots, y_{s'_n}) \phi \)

Now if \( s = (s'_1, s'_2, \ldots, s'_n) \) and \( t = (t'_1, t'_2, \ldots, t'_n) \) are in \( S \), then \( (y_{s'_1}, y_{s'_2}, \ldots, y_{s'_n}) \phi \) and \( (y_{t'_1}, y_{t'_2}, \ldots, y_{t'_n}) \phi \)
are \( G \)-related if and only if there exists \( g \in G \), such that \( (y_{t'_1}, y_{t'_2}, y_{t'_n}) \phi = (y_{g(s'_1)}, y_{g(s'_2)}, \ldots, y_{g(s'_n)}) \phi \);

i.e. if and only if \( (y_{t'_1}, y_{t'_2}, \ldots, y_{t'_n}) = (y_{g(s'_1)}, y_{g(s'_2)}, \ldots, y_{g(s'_n)}) \), since \( \phi \) is one-one; i.e. if

and only if \( y_{t'_i} = y_{g(s'_i)} \) for all \( i \in I \); i.e. if and only if \( t'_i = s'_i \) for all \( i \in I \), since \( y \)'s are the basis elements; i.e. if and only if \( t \sim s \); i.e. if and only if \( t \in A(s) \). Thus if \( s \) and \( t \in E \) and \( s \neq t \), then \( (y_{s'_1}, y_{s'_2}, \ldots, y_{s'_n}) \phi \) and \( (y_{t'_1}, y_{t'_2}, y_{t'_n}) \phi \) are not \( G \)-related. Also for each \( s \in S \), and in particular for each \( s \in E \), \( (y_{s'_1}, y_{s'_2}, \ldots, y_{s'_n}) \phi \) satisfies the property P. Hence

\( (w'_1, w'_2, \ldots, w'_n) \phi = w + \sum_{s \in E} b'(s) T_s \),

where \( T_s = (y_{s'_1}, y_{s'_2}, \ldots, y_{s'_n}) \phi \) has the required form.
Computation of the Co-efficients.

In this section, we shall investigate the co-efficients $b'_{s}$, $s \in E$, appearing in the Representation Theorem 5.

Consider the following $n \times m$ matrix

$$M = \begin{pmatrix}
  b_{1,1} & b_{1,2} & \cdots & b_{1,m} \\
  b_{2,1} & b_{2,2} & \cdots & b_{2,m} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{n,1} & b_{n,2} & \cdots & b_{n,m}
\end{pmatrix}$$

where $w_{i} = \sum_{j=1}^{m} b_{i,j}^{y_{j}}$, for $i = 1, 2, \ldots, n$.

We shall call $M$ the matrix of co-efficients of the element $(w_{1}, w_{2}, \ldots, w_{n}) \in W$.

For each $s = (s_{1}, s_{2}, \ldots, s_{n})$ in $S$, we define an $n \times n$ matrix $M_{s}$, obtained from $M$ as

$$M_{s} = \begin{pmatrix}
  b_{1,s_{1}} & b_{1,s_{2}} & \cdots & b_{1,s_{n}} \\
  b_{2,s_{1}} & b_{2,s_{2}} & \cdots & b_{2,s_{n}} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{n,s_{1}} & b_{n,s_{2}} & \cdots & b_{n,s_{n}}
\end{pmatrix}$$

and a subset $H_{s}$ of $G$ as

$$H_{s} = \{ g \mid g \in G, \sigma(g) = 1 \text{ and } s_{i} = s_{g(i)} \text{ for all } i \in I \}$$

Clearly $H_{s}$ is a subgroup of $G$. 
6.1 Proposition: If \( s = (s_1, s_2, \ldots, s_n) \) and \( t = (t_1, t_2, \ldots, t_n) \) are in \( S \) and \( s \sim t \) then \( H_s \) and \( H_t \) are conjugate. In fact if \( t_i = s g(i) \) for some \( g \in G \) and all \( i \in I \), then \( H_t = g^{-1} H_s g \).

Proof: Since \( s \sim t \), we have \( t_i = s g(i) \) for some \( g \in G \) and all \( i \in I \). Now if \( h \in H_t \), then \( o(h) = 1 \) and
\[
t_i = t h(i) \quad \text{for all } i \in I ,
\]
but since \( t_i = s g(i) \), we have \( o(h) = 1 \) and \( s g(i) = s (g h i) \) for all \( i \in I \), i.e. \( o(h) = 1 \) and \( s_i = s g h^{-1}(i) \) for all \( i \in I \).

Also since \( o(gh^{-1}) = o(g)o(h)o(g)^{-1} = 1 \), we have \( gh^{-1} \in H_s \); i.e. \( h \in g^{-1} H_s g \). Thus \( H_t \subseteq g^{-1} H_s g \) and similarly one can show \( g^{-1} H_s g \subseteq H_t \). Therefore \( H_t = g^{-1} H_s g \).

6.2 Proposition: Let \( s = (s_1, s_2, \ldots, s_n) \) and \( t = (t_1, t_2, \ldots, t_n) \) be in \( S \). Suppose \( s \sim t \) and let \( g, h \in G \) such that \( t_i = s g(i) \) and \( t_i = s h(i) \) for all \( i \in I \). If \( (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi \) is not a \((G, \sigma)\) element, then \( \sigma(g) = \sigma(h) \).

Proof: Since \( t_i = s g(i) = s h(i) \) for all \( i \in I \), we have
\[
s g h^{-1}(i) = s_i \quad \text{for all } i \in I ,
\]
which implies \( y_{s_1} = y_{s h^{-1}(i)} \) for all \( i \in I \). But since \( (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi \) is not a \((G, \sigma)\) element, we must have \( o(gh^{-1}) = 1 \) i.e. \( \sigma(g) = \sigma(h) \).
6.3 For each \( s \in S \), we have associated a matrix \( M_s \) obtained from \( M \) and a subgroup \( H_s \) of \( G \) in 6. Consider the coset decomposition of \( G \) with respect to \( H_s \) and let \( G_s \) be a set of representatives of these cosets. Also let \( \mathcal{S} = \{ M_s \mid s \in S \} \).

Define \( D : \mathcal{S} \rightarrow F \), as

\[
D(M_s) = \sum_{h \in G_s} \sigma(h^{-1}) b_1, s_{h(1)} b_2, s_{h(2)} \ldots b_n, s_{h(n)}.
\]

We must show that \( D \) is well-defined; i.e., it is independent of the choice of the representatives of \( G \) with respect to \( H_s \). If \( h \) and \( h' \) are two representatives of the same coset, then \( hh'^{-1} \in H_s \), which implies that \( \sigma(hh'^{-1}) = 1 \) and \( s_i = s_{hh'^{-1}(i)} \) for all \( i \in I \); i.e., \( \sigma(h) = \sigma(h') \) and \( s_{h(i)} = s_{h'(i)} \) for all \( i \in I \). Therefore

\[
\sigma(h^{-1}) b_1, s_{h(1)} b_2, s_{h(2)} \ldots b_n, s_{h(n)} = \sigma(h'^{-1}) b_1, s_{h'(1)} b_2, s_{h'(2)} \ldots b_n, s_{h'(n)}.
\]

Thus if \( G'_s \) is any other set of the representatives of the cosets of \( G \) with respect to \( H_s \), then

\[
\sum_{h \in G_s} \sigma(h^{-1}) b_1, s_{h(1)} b_2, s_{h(2)} \ldots b_n, s_{h(n)} = \sum_{h' \in G'_s} \sigma(h'^{-1}) b_1, s_{h'(1)} b_2, s_{h'(2)} \ldots b_n, s_{h'(n)}
\]

which shows that \( D \) is well defined.
6.4 **Definition**: The function $\mathcal{D}$ on $\mathcal{J}$ into $\mathcal{F}$, as defined in 6.3 is called a pseudo-determinant function.

6.5 **Proposition**: Following the notation of the Representation Theorem Form I, if $s \in E$, then $b'_s = \mathcal{D}(M_s)$, where $b'_s$ is the co-efficient of $(y_{s_1}, y_{s_2}, \ldots, y_{s_n})$. Further if $(y_{s_1}, y_{s_2}, \ldots, y_{s_n})\phi$ is not a $(G, \sigma)$ element and $t \in A(s)$, then $\mathcal{D}(M_t) = \sigma(g) \mathcal{D}(M_s)$, where $g \in G$, such that $t_i = s_g(i)$ for all $i \in I$.

**Proof**: If $s = (s_1, s_2, \ldots, s_n) \in S$ and $g \in G$, we shall write $s_g = (s_g(1), s_g(2), \ldots, s_g(n))$.

The equivalence class of $S$, containing $s \in S$ is

$A(s) = \{ t \mid t \in S \text{ and } t \sim s \}$

$= \{ s_g \mid s_g \in S \text{ and } g \in G \}$.

But if $s_g$ and $s_h$ are in $A(s)$ and if $g$ and $h$ are in the same coset of $G$ with respect to $H_s$, then $s_g(i) = s_h(i)$ for all $i \in I$; i.e., $s_g = s_h$.

Therefore $A(s) = \{ s_g \mid s_g \in S \text{ and } g \in G_s \}$.

Now $b'_s = \sum_{t \in A(s)} \sigma(g^{-1}) b_t$, where $g \in G$ such that $t = s_g$, by equation (1) (see Representation Theorem Form I)

$= \sum_{t \in A(s)} \sigma(g^{-1}) b_{t_1} b_{t_2} \cdots b_n, t_n$
But $b_1, s_{g(1)} b_2, s_{g(2)} \ldots b_n, s_{g(n)} = 0$, if and only if $b_1, s_{g(1)} = 0$ for some $i \in I$; if and only if $s_{g(1)} \notin A_1$ for some $i$; if and only if $s_1 \notin S$.

Therefore $b'_s = \sum_{g \in G_s} \sigma(g^{-1}) b_1, s_{g(1)} b_2, s_{g(2)} \ldots b_n, s_{g(n)}$.

which proves the first assertion.

Since $t \in A(s)$ we have $t = s_g$ for some $g \in G_s$. Then

$D(M_t) = \sum_{h \in G_t} \sigma(h^{-1}) b_1, t_{h(1)} b_2, t_{h(2)} \ldots b_n, t_{h(n)}$, where $G_t$ is a set of representatives of $H_t$ in $G$. Since $t_i = s_{g(1)}$ for all $i \in I$, we have $t_{h(1)} = s_{gh(1)}$ for all $i \in I$.

Hence $D(M_t) = \sum_{h \in G_t} \sigma(h^{-1}) b_1, s_{gh(1)} b_2, s_{gh(2)} \ldots b_n, s_{gh(n)}$.

By Proposition 6.1, we have $H_s = gH_s g^{-1}$, and therefore $[G : G_s] = [G : G_t]$.

Claim: $G'_s = \{ gh | h \in G_t \}$ is a set of representatives of $H_s$ in $G$.

Clearly $|G'_s| = [G : H_s]$. All that is necessary is to show that two different representatives of $G_t$ give rise to two different representatives of $G'_s$. So if $gh$ and
gh' should belong to the same coset of G with respect to H_s, then \((gh)(gh')^{-1} \in H_s\), which implies \(hh'^{-1} \in g^{-1}H_sg = H_t\) by Proposition 6.1; i.e. h and h' must belong to the same coset of G with respect to H_s, which would contradict the supposition that h and h' belong to G_t.

Therefore (2) becomes

\[
D(M_t) = \sum_{h \in G_t} \frac{g(h^{-1}g^{-1})}{\sigma(g^{-1})} b_1,s_{gh(1)} b_2,s_{gh(2)} \ldots b_n,s_{gh(n)}
\]

\[
= \frac{1}{\sigma(g^{-1})} \sum_{k \in G_s} \sigma(k^{-1}) b_1,s_{k(1)} b_2,s_{k(2)} \ldots b_n,s_{k(n)}
\]

where \(k = gh\)

\[
= \frac{1}{\sigma(g^{-1})} D(M_s), \text{ by the definition of } D \text{ in } 6.3
\]

\[
= \sigma(g) D(M_s), \text{ which completes the proof.}
\]

7. **Representation Theorem Form II:** If \((w_1, w_2, \ldots, w_n)\) is a non-trivial element of W, then \((w_1, w_2, \ldots, w_n)\) can be written in the form

\[
(w_1, w_2, \ldots, w_n) = w + \sum_{s \in E} D(M_s)(y_{s_1}, y_{s_2}, \ldots, y_{s_n})
\]

where \(w \in \Omega\), and for each \(s \in E\), \((y_{s_1}, y_{s_2}, \ldots, y_{s_n})\) satisfies the property P and if \(s, t \in E\), \(s \neq t\), then \((y_{s_1}, y_{s_2}, \ldots, y_{s_n})\) and \((y_{t_1}, y_{t_2}, \ldots, y_{t_n})\) are not G-related.
We shall call \((4)\), a representation of \((w_1, w_2, \ldots, w_n)\) with respect to the basis \(\{y_1, y_2, \ldots, y_m\}\) of \(V\).

7.1 Remark: If \(E'\) is any other set of representatives of the equivalence classes, then

\[
(w_1, w_2, \ldots, w_n) = w' + \sum_{s' \in E} D(M_{s'})(y_{s'_1}, y_{s'_2}, \ldots, y_{s'_n})
\]

is another representation. By Proposition 6.5, if \(s' \in A(s)\) and \((y_{s'_1}, y_{s'_2}, \ldots, y_{s'_n})\) is not a \((G,\sigma)\) element, then \(D(M_s)\) and \(D(M_{s'})\) are related by

\[
D(M_{s'}) = \sigma(g) D(M_s),
\]

where \(s' = s g\) for some \(g \in G\).

7.2 Remark: If \(s = (s_1, s_2, \ldots, s_n) \in S\), then \((y_{s_1}, y_{s_2}, \ldots, y_{s_n})\) is a \((G,\sigma)\) element if and only if, there exists \(g \in G\) such that \(\sigma(g) \neq 1\) and \(s = s_g\).

Proof: Immediate from the definition of a \((G,\sigma)\) element and the facts that \((y_{s_1}, y_{s_2}, \ldots, y_{s_n})\) satisfies the property

\(P\) and \(y_1, y_2, \ldots, y_m\) is an independent set of vectors.
 CHAPTER III

In this chapter, we continue our study of the problem. The main results are Lemma 2.1 and Theorem 2.3. Then we determine a basis of the \((G,\sigma)\) space \(P(W,G,\sigma)\). We shall conclude this chapter, by studying the problem, in case when \(G\) can be written as a direct product of its subgroups; i.e., \(G = G_1 \times G_2 \times \ldots \times G_n\) (direct product), such that this decomposition is "disjoint".

1. **Construction of a multilinear and \((G,\sigma)\) function.**

   Let \((v_1, v_2, \ldots, v_n)\) be a non-trivial element of \(W\). If for each \(i \in I\), \(v_i = \sum a_i j y_j\), consider the sets \(A_i\) and the set \(S = A_1 \times A_2 \times \ldots \times A_n\), as defined in the Representation Theorem Form I. If \(s \in S\), define

   \[
   f_s : W \to F, \quad \text{as follows.}
   \]

   If \((w_1, w_2, \ldots, w_n) \in W\) and \(w_i = \sum b_{ij} y_j\), for \(i = 1, 2, \ldots, n\), then set

   \[
   (w_1, w_2, \ldots, w_n) f_s = \sum_{g \in G_s} \sigma(g^{-1}) b_{1g(1)} b_{2g(2)} \ldots b_{ng(n)}
   \]

   where \(G_s\) is a set of representatives of the cosets of \(H_s\) in \(G\). One can easily show that \(f_s\) is a well-defined function, i.e. independent of the choice of \(G_s\). Then we
have the following

1.1 Lemma: $f_s$ is a multilinear and $(G, \sigma)$ function.

Proof: If $\alpha, \beta \in F$, then for any $i \in I$, if $w_i = \sum_{j=1}^{\ell} b_{ij} y_j$
and $w'_i = \sum_{j=1}^{\ell} b'_{ij} y_j$ we have $(w_1, \ldots, \alpha w_i + \beta w'_i, \ldots, w_n)f_s$

\[= \sum_{g \in G_s} \sigma(g^{-1}) b_1, s_{g(1)}, b_2, s_{g(2)}, \ldots, (\alpha b_i, s_{g(1)} + \beta b'_i, s_{g(1)}) \ldots b_n, s_{g(n)} \]

\[= a \sum_{g \in G_s} \sigma(g^{-1}) b_1, s_{g(1)}, b_2, s_{g(2)}, \ldots b_i, s_{g(1)} \ldots b_n, s_{g(n)} \]

\[+ \beta \sum_{g \in G_s} \sigma(g^{-1}) b_1, s_{g(1)}, b_2, s_{g(2)}, \ldots b'_i, s_{g(1)} \ldots b_n, s_{g(n)} \]

\[= a f_s(w_1, \ldots, w_i, \ldots, w_n)f_s + \beta f_s(w_1, \ldots, w'_i, \ldots, w_n)f_s, \]

which shows $f_s$ is multilinear.

Again, if $h \in G$, then $(w_h(1), w_h(2), \ldots, w_h(n))f_s$

\[= \sum_{g \in G_s} \sigma(g^{-1}) b_h(1), s_{g(1)}, b_h(2), s_{g(2)} \ldots b_h(n), s_{g(n)} \]

\[= \sum_{g \in G_s} \sigma(g^{-1}) b_1, s_{g^{-1}(1)}, b_2, s_{g^{-1}(2)} \ldots b_n, s_{g^{-1}(n)} \]

\[= \sum_{k \in G'_s} \sigma(k^{-1}) b_1, s_{k(1)}, b_2, s_{k(2)} \ldots b_n, s_{k(n)}, \]

where $k = g h^{-1}$, and $G'_s = \{ gh^{-1} \mid g \in G_s \}$ is another set of representatives of the cosets of $G$ with respect to $H_s$. Hence $(w_h(1), w_h(2), \ldots, w_h(n))f_s$

\[= \frac{1}{\sigma(h)} \sum_{k \in G'_s} \sigma(k^{-1}) b_1, s_{k(1)}, b_2, s_{k(2)} \ldots b_n, s_{k(n)} \]
\[ \frac{1}{\sigma(h)} (w_1, w_2, \ldots, w_n) f_s \]
which shows \( f_s \) is \( (G, \sigma) \).

2. **Solution to the Problem**

We now come to our main result. We need the following lemma, which is also a special case of our problem.

2.1 **Lemma:** Let \((v_1, v_2, \ldots, v_n) \in W\) such that \((v_1, v_2, \ldots, v_n)\) satisfies the property \( P \). Then a necessary and sufficient condition for \( v_1 \Delta v_2 \Delta \ldots \Delta v_n = 0 \) is that \((v_1, v_2, \ldots, v_n)\) is a \( (G, \sigma) \) element.

**Proof:** (Sufficiency). If \((v_1, v_2, \ldots, v_n)\) is a \( (G, \sigma) \) element, then there exists \( g \in G \) such that \( \sigma(g) \neq 1 \) and \( v_i, v_{g(i)} \) are dependent for all \( i \in I \). Also since \((v_1, v_2, \ldots, v_n)\) satisfies the property \( P \), we have \( v_i = v_{g(i)} \) for all \( i \in I \). Hence \((v_1, v_2, \ldots, v_n)\)\]
\[ = [(v_1, v_2, \ldots, v_n) - \sigma(g)(v_{g(1)}, v_{g(2)}, \ldots, v_{g(n)})] \]
\[ + \sigma(g)(v_{g(1)}, v_{g(2)}, \ldots, v_{g(n)}) \]
\[ = w + \sigma(g)(v_1, v_2, \ldots, v_n) \]
where \( w \) is the term within the square brackets.
Since \( \sigma(g) \neq 1 \), we have
\[ (v_1, v_2, \ldots, v_n) = \frac{w}{1-\sigma(g)} \in \Omega. \]
Hence \( v_1 \Delta v_2 \Delta \ldots \Delta v_n = 0 \).

(Necessity). Suppose false.
Choose \( a_1 = 1 \) and define \( a \) inductively as follows; \( a_2 \) is the first index \( j \) such that \( v_j \neq v_1 \); \( a \) is the first index \( j \), such that \( v_j \neq \) any one of \( v_1, v_2, \ldots, v_{a-1} \). If these are precisely \( k \) distinct vectors \( v_1 \), we have defined \( 1 = a_1 < a_2 < \ldots < a_k \leq n \).

Clearly \( \{ v_1, v_2, \ldots, v_a \} \) is an independent set of vectors. Extend this to a basis \( \{ y_1, y_2, \ldots, y_m \} \) of \( V \), such that \( y_i = v_{a_i}, \quad i = 1, 2, \ldots, k \leq m \).

Then for each \( i \in I \), if \( i = a_j \) for some \( j \),

\[
 v_i = \sum_{l=1}^{m} a_{i,l} y_{l}, \quad \text{where} \quad a_{i,l} = \begin{cases} 
 1 & \text{if } l = a_j \\
 0 & \text{if } l \neq a_j
\end{cases}
\]

If \( a_j < i < a_{j+1} \), then \( v_i = v_{a_j} \), for some \( j' \leq j \).

In this case \( v_i = \sum_{l=1}^{m} a_{i,l} y_{l} \) where \( a_{i,l} = \begin{cases} 
 1 & \text{if } l = a_j \\
 0 & \text{if } l \neq a_j
\end{cases} \)

And finally if \( a_n < i \leq n \), then \( v_i = v_{a_{j'}} \), for some \( j' \leq k \), then \( v_i = \sum_{l=1}^{m} a_{i,l} y_{l} \), where \( a_{i,l} = \begin{cases} 
 1 & \text{if } l = a_{j'} \\
 0 & \text{if } l \neq a_{j'}
\end{cases} \)

Thus, in every case \( A_1 \) is a singleton, viz.

\[
 A_{i} = \begin{cases} 
 \{j\} & \text{if } i = a_j \\
 \{j'\} & \text{if } a_j < i < a_{j+1} \text{ where } j' \leq j \\
 \{j'\} & \text{if } a_k < i \leq n, \text{ where } j' \leq k.
\end{cases}
\]

Therefore \( S = A_1 \times A_2 \times \ldots \times A_n = \{s\} \) say, where \( s_{a_j} = j \),
j = 1, 2, ..., k. Thus $(v_1, v_2, ..., v_n)\phi = (y_{s_1}, y_{s_2}, ..., y_{s_n})\phi$

and is not a $(G, \sigma)$ element, by our assumption. Therefore by the Remark 7.2, Chapter II, $g \in G$ implies $\sigma(g) = 1$ or $s_i \neq s_{h(i)}$ for some $i \in I$.

Define $f_s : W \rightarrow F$, as in 3, by

$$(w_1, w_2, ..., w_n)f_s = \sum_{h \in G_s} \sigma(h^{-1})b_{1,s_h(1)}b_{2,s_h(2)}...b_{n,s_h(n)},$$

where $w_i = \sum_{j=1}^{W} b_{i,j}y_j$ for all $i \in I$. By Lemma 1.1, $f_s$ is multilinear and $(G, \sigma)$ function. Therefore by the universal mapping property, there exists a unique linear transformation $\bar{f}_s$ of $P(W,G,\sigma)$ into $F$, which makes the following diagram commutative; i.e. $\tau \bar{f}_s = f_s$.

Now

$$(v_1, v_2, ..., v_n)f_s = \sum_{h \in G_s} \sigma(h^{-1})a_{1,s_h(1)}a_{2,s_h(2)}...a_{n,s_h(n)}.$$
Therefore \( s_h(1) \notin A_i \) since \( A_i = \{ s_i \} \). Thus

\[ a_1, s_h(1) = 0 \]

and therefore \( a_1, s_h(1), a_2, s_h(2), \ldots, a_n, s_h(n) = 0 \)

for every \( h, h \in G_s \) and \( h \notin H_s \). Therefore

\[
(v_1, v_2, \ldots, v_n) f_s = \sum_{h \in G_s} \sigma(h^{-1}) a_1, s_h(1), a_2, s_h(2), \ldots, a_n, s_h(n)
\]

\[ = a_1, s_h(1), a_2, s_h(2), \ldots, a_n, s_h(n) \]

\[ = a_1, s_1, a_2, s_2, \ldots, a_n, s_n \]

\[ = 1 \neq 0. \]

But since \( \tau f_s = f_s \), we have \((v_1, v_2, \ldots, v_n) \tau f_s \neq 0 \) and since \( f_s \) is a linear transformation, we have

\[
(v_1, v_2, \ldots, v_n) \tau \neq 0; \text{ i.e., } v_1 \Delta v_2 \Delta \ldots \Delta v_n \neq 0, \]

which is a contradiction.

2.2 **Remark:** If \((v_1, v_2, \ldots, v_n) \in W\) is a trivial element, then

\[ v_1 \Delta v_2 \Delta \ldots \Delta v_n = 0. \]

**Proof:** Let \( v_i = 0 \) for some \( i \in I \). Then

\[
(v_1, \ldots, v_i, \ldots, v_n) \tau = (v_1, \ldots, 0v_i, \ldots, v_n) \tau
\]

where \( 0 \) is the zero element of the field \( F \); and since \( \tau \) is multilinear, we have

\[
(v_1, \ldots, v_i, \ldots, v_n) \tau = 0 (v_1, \ldots, v_i, \ldots, v_n) \tau
\]

\[ = 0 \]

\[ = 0. \]

In this case, we shall say that \( v_1 \Delta v_2 \Delta \ldots \Delta v_n = 0 \) trivially.
The Main Theorem. We can now state and prove the main result of the Thesis.

2.3 Theorem: Suppose \((v_1, v_2, \ldots, v_n) \in W\) is a non-trivial element. Let

\((v_1, v_2, \ldots, v_n) \phi = w + \sum_{s \in E} D(M_s)(y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi \quad (4)\)

be its representation with respect to some basis \(\{y_1, y_2, \ldots, y_m\}\) of \(V\). Then a necessary and sufficient condition for \(v_1 \Delta v_2 \Delta \ldots \Delta v_n\) to be zero is, that for each \(s \in E\), either \((y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi\) is a \((G, \sigma)\) element or \(D(M_s)\) is zero.

Proof: Let \(E' = \{ s \mid s \in E, (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi \text{ is not a } (G, \sigma) \text{ element} \}\).

\(E'\) may be an empty set. Then (4) becomes

\[(v_1, v_2, \ldots, v_n) \phi = w + \sum_{s \in E-E'} D(M_s)(y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi + \sum_{s \in E'} D(M_s)(y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi. \quad (5)\]

We shall prove the sufficiency first.

\(E-E'\) is the index set that selects the non-vanishing terms in the sum (5). Thus

\[(v_1, v_2, \ldots, v_n) \phi = w + \sum_{s \in E-E'} D(M_s)(y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi. \quad (6)\]

Now if \(s \in E-E'\), then \((y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi\) is a \((G, \sigma)\) element. Also it satisfies the property \(P\). Therefore
by the Lemma 2.1 \( y_{s_1} \Delta y_{s_2} \Delta \ldots \Delta y_{s_n} = 0 \). Thus on applying \( \eta \) to (6), we obtain \( v_1 \Delta v_2 \Delta \ldots \Delta v_n = 0 \).

To prove necessity, we assume it to be false; i.e. suppose there exists \( s \in E' \), such that \( D(M_s) \neq 0 \).

Define \( f_s : W \rightarrow F \), as in 1, by

\[
(w_1, w_2, \ldots, w_n)f_s = \sum_{h \in G_s}(\sigma(h^{-1})b_1, s_h(1) b_2, s_h(2) \ldots b_n, s_h(n))
\]

where \( w_i = \sum_{j=1}^{\infty} b_i y_j \), for all \( i \in I \).

Then \( f_s \) is multilinear and \( (G, \sigma) \), by Lemma 1.1 and therefore by the universal mapping property, there exists a unique linear transformation \( \Phi_s : P(W, G, \sigma) \rightarrow F \), which makes the following diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\tau} & P(W, G, \sigma) \\
\downarrow f_s & & \downarrow \Phi_s \\
F & & F
\end{array}
\]

commutative; i.e. \( \Phi_s \tau = f_s \).

Now since in (5), for each \( s \in E-E' \), \( (y_{s_1}, y_{s_2}, \ldots, y_{s_n})\Phi \) is a \( (G, \sigma) \) element, and it also satisfies the property \( P \), therefore, by Lemma 2.1, \( y_{s_1} \Delta y_{s_2} \Delta \ldots \Delta y_{s_n} = 0 \). Hence, on applying \( \eta \) to (5), we obtain

\[
0 = \sum_{s \in E'} D(M_s) (y_{s_1}, y_{s_2}, \ldots, y_{s_n})\tau ,
\]

and which under \( \Phi_s \).
becomes \( \sum_{s \in E'} \text{D}(M_s) \,(y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \bar{t}_s = 0. \) 

Since \( \bar{t}_s = f_s, \) this becomes \( \sum_{s \in E'} \text{D}(M_s) \,(y_{s_1}, y_{s_2}, \ldots, y_{s_n}) f_s = 0 \). \( \quad \) \( \text{(7)} \)

Now we calculate each term of this sum. First, we choose \( s \in E' \), for which \( \text{D}(M_s) \neq 0 \). We know such an \( s \) exists by our assumption. Then

\[
(y_{s_1}, y_{s_2}, \ldots, y_{s_n}) f_s = \sum_{h \in G_s} \sigma(h^{-1}) c_{s_1, s_h(1)} c_{s_2, s_h(2)} \cdots c_{s_n, s_h(n)}
\]

where \( c_{s_i, s_h(i)} = \begin{cases} 1 & \text{if } s_i = s_h(i) \\ 0 & \text{if } s_i \neq s_h(i) \end{cases} \) for all \( i \in I \).

And since \( s \in E' \), \( (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \phi \) is not a \((G, \sigma)\) element we have

\[
c_{s_1, s_h(1)} c_{s_2, s_h(2)} \cdots c_{s_n, s_h(n)} = \begin{cases} 1 & \text{if } h \in H_s \\ 0 & \text{if } h \notin H_s \end{cases}
\]

Remark 7.2, Chapter II.

Therefore \( (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) f_s = 1. \) \( \quad \) \( \text{(8)} \)

Next for any \( t \in E' \) and \( t \neq s \), we have for every \( h \in G_s \)

\( t_i \neq s_h(i) \) for some \( i \in I \). Therefore

\[
(y_{t_1}, y_{t_2}, \ldots, y_{t_n}) f_s = \sum_{h \in G_s} \sigma(h^{-1}) c_{t_1, s_h(1)} c_{t_2, s_h(2)} \cdots c_{t_n, s_h(n)}
\]

\[= 0. \quad \) \( \text{(9)} \)

But \( (8) \) and \( (9) \) contradict \( (7) \), and this completes the proof.
2.4 Thus if \((v_1, v_2, \ldots, v_n) \in W\) and is a trivial element, then 
\[v_1^\Delta v_2^\Delta \ldots v_n^\Delta = 0\] 
trivially by the Remark 2.2, and if 
\((v_1, v_2, \ldots, v_n)\) is not a trivial element, then we apply 
Theorem 2.3 to determine a necessary and sufficient condition 
that \(v_1^\Delta v_2^\Delta \ldots v_n^\Delta = 0\).

3. Basis of \(P(W, G, g)\).

Let \(\{ y_1, y_2, \ldots, y_m \} \) be a basis of \(V\). Let 
\(A = \{1, 2, \ldots, m\}\) and \(S = A \times A \times \ldots \times A\), the cartesian product 
of the \(n\)-copies of \(A\). Define an equivalence relation 
\(\sim\) on \(S\) as in the proof of the Representation Theorem 
Form I, 5, Chapter II, i.e. if \(s = (s_1, s_2, \ldots, s_n)\) and 
\(t = (t_1, t_2, \ldots, t_n)\) are in \(S\), then \(s \sim t\) if and only if 
there exists \(g \in G\), such that \(t = s_g\), where 
\(s_g = (s_g(1), s_g(2), \ldots, s_g(n))\). Let \(A(s)\) be the equiv-
valence class containing \(s \in S\) and let \(E\) be a set of 
representatives of the equivalence classes.

Let \(E' = \{ s \mid s \in E, (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \notin \) is not a 
\((G, \sigma)\) element.\} By the Remark 7.2, Chapter II, if \(s \in S\), then 
\((y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \notin \) is not a \((G, \sigma)\) element if and 
only if \(g \in G\) and \(\sigma(g) \neq 1\) implies \(s_i \neq s_g(i)\) for some 
\(i \in I\).

Hence \(E' = \{ s \mid s \in E, \quad \text{and} \quad \text{if} \quad g \in G \quad \text{such that} \quad \sigma(g) \neq 1, \quad \text{then} \quad s_i \neq s_g(i) \quad \text{for some} \quad i \in I \}\)
Then we have the following

3.1 Theorem: \( B = \{ y_{s_1} \Delta y_{s_2} \Delta \ldots \Delta y_{s_n} \mid s = (s_1, s_2, \ldots, s_n) \in E' \} \) is a basis of the \((G, \sigma)\) space \( P(W, G, \sigma) \).

Proof: We will first show that \( B \) is a generating set of the space \( P(W, G, \sigma) \). Since \( W' \) is a spanning set of \( P(W, G, \sigma) \), it will be sufficient to show that \( B \) generates \( W' \).

Let \((w_1, w_2, \ldots, w_n)\tau \in W'. \) If \((w_1, w_2, \ldots, w_n) \in W\) is a trivial element, then \((w_1, w_2, \ldots, w_n)\tau = 0\) and is therefore generated by the set \( B \). So we assume that \((w_1, w_2, \ldots, w_n) \in W\) is a non-trivial element. Then by (5) (see Theorem 2.3)

\[
(w_1, w_2, \ldots, w_n)\phi = w + \sum_{s \in E-E'} D(M_s)(y_{s_1}, y_{s_2}, \ldots, y_{s_n})\phi + \sum_{s \in E'} D(M_s)(y_{s_1}, y_{s_2}, \ldots, y_{s_n})\phi.
\]

Applying \( \eta \), we get

\[
(w_1, w_2, \ldots, w_n)\tau = \sum_{s \in E'} D(M_s)(y_{s_1}, y_{s_2}, \ldots, y_{s_n})\tau = \sum_{s \in E'} D(M_s)(y_{s_1} \Delta y_{s_2} \Delta \ldots \Delta y_{s_n})
\]

which shows \( B \) is a generating set.

We must also show \( B \) is a linearly independent set.

Suppose \( \sum_{s \in E'} \alpha_s y_{s_1} \Delta y_{s_2} \Delta \ldots \Delta y_{s_n} = 0 \), where \( \alpha_s \in F \).
We wish to show that \( \alpha_s = 0 \) for all \( s \in E' \).

Let \( s \in E' \) be a fixed index.

Define \( f_s : W \rightarrow F \), as in 1, by setting
\[
(w_1, w_2, \ldots, w_n)f_s = \sum_{h \in G_s} \sigma(h^{-1})b_1, s_{h(1)} b_2, s_{h(2)} \ldots b_n, s_{h(n)}
\]
where \( w_i = \sum_{j=1}^{\infty} b_i j y_j \), for each \( i \in I \).

\( f_s \) is a multilinear and \((G, \sigma)\) function by Lemma 1.1.

Therefore by the universal mapping property, there exists a unique linear transformation
\[
\tau : P(W, G, \sigma) \rightarrow F,
\]
such that \( \tau f_s = f_s \).

Now \( \sum_{s \in E'} \alpha_s (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) = 0 \) implies
\[
\sum_{s \in E'} \alpha_s (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \tau f_s = 0.
\]
Then since \( \tau f_s \) is linear,
\[
\sum_{s \in E'} \alpha_s (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) f_s = 0,
\]
and since \( \tau f_s = f_s \), we obtain
\[
\sum_{s \in E'} \alpha_s (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) f_s = 0. \tag{10}
\]
But \( (y_{s_1}, y_{s_2}, \ldots, y_{s_n}) f_s = 0 \) and \( (y_{t_1}, y_{t_2}, \ldots, y_{t_n}) f_s = 0 \)
if \( t \in E' \) and \( t \neq s \), by (8) and (9) (see Theorem 2.3).
Therefore (10) gives \( \alpha_s = 0 \) and since \( s \) is arbitrary, we have \( \alpha_s = 0 \) for all \( s \in E' \). Therefore \( B \) is a linearly independent set and hence is a basis.

4. Decomposition of \( P(W,G,\sigma) \)

Let \( G = G_1 \otimes G_2 \otimes \ldots \otimes G_k \) be a direct product. For each \( 1, i \leq k \), let

\[
I_i = \{ \alpha | \alpha \in I \text{ and } g(\alpha) \neq \alpha \text{ for some } g \in G_i \}.
\]

Then the direct product (11) is said to be disjoint, if and only if \( i_1 \neq i_2 \) implies \( I_{i_1} \cap I_{i_2} = \emptyset \).

Let \( I_0 = \{ \alpha | \alpha \in I, g(\alpha) = \alpha \text{ for all } g \in G \} \).

Clearly \( I = \bigcup_{i=0}^{k} I_i \). For each \( i, 0 < i \leq k \), let

\[
I_i = \{ \alpha_1, \alpha_2, \ldots, \alpha_{n_i} \} \text{ and } W_i = V \times V \times \ldots \times V (n_i \text{ copies}).
\]

Let \( G_0 = \{ e \} \) where \( e \) is the identity permutation.

Then \( W_i \) is a \( G_i \)-set for each \( i, i = 0, 1, 2, \ldots, k \).

If \( \sigma \) is any linear character of \( G \), then \( \sigma_i = \sigma | G_i \), the restriction of \( \sigma \) to \( G_i \) is a linear character of \( G_i \). Under this hypothesis we have the following

4.1 Theorem: \( P(W,G,\sigma) \cong \bigotimes_{i=0}^{k} P(W_i,G_i,\sigma_i) \)

Proof: For each \( i, 0 < i \leq k \), \( \tau_i : W_i \rightarrow P(W_i,G_i,\sigma_i) \)

is a multilinear and \((G_i,\sigma_i)\) function.
Define \( \tau' = \bigotimes_{i=0}^{k} \tau_i : W \rightarrow P(W,G,\sigma) \), as 

\[
(w_1, w_2, \ldots, w_n)\tau' = \bigotimes_{i=0}^{k} (w_{\alpha_1 i}, w_{\alpha_2 i}, \ldots, w_{\alpha_n i})\tau_i .
\]

\( \tau' \) is a multilinear and \((G,\sigma)\) function.

For if \( \beta, \gamma \in F \), then for any \( j \in I, j = \alpha_r, 1 \) for some \( r \) and \( i, 0 \leq r \leq n_1, 0 \leq i \leq k \), we have 

\[
(bw_j + \gamma w_j', \ldots, w_n)\tau' \\
= \bigotimes_{i=0}^{k} (w_{\alpha_1 i}, \ldots, bw_{\alpha_r i}, \ldots, w_{\alpha_n i})\tau_i \\
= \beta \bigotimes_{i=0}^{k} (w_{\alpha_1 i}, \ldots, w_{\alpha_k i}, \ldots, w_{\alpha_n i})\tau_i \\
+ \gamma \bigotimes_{i=0}^{k} (w_{\alpha_1 i}, \ldots, w_{\alpha_k i}, \ldots, w_{\alpha_n i})\tau_i \\
= \beta (w_1, \ldots, w_i, \ldots, w_n)\tau' + \gamma (w_1, \ldots, w_i, \ldots, w_n)\tau'
\]

which shows that \( \tau' \) is multilinear.

Again if \( g \in G, g = g_1 g_2 \cdots g_k \), where \( g_i \in G_1 \), for \( i = 1, 2, \ldots, k \), then 

\[
(w_1, w_2, \ldots, w_n)\tau' \\
= \bigotimes_{i=0}^{k} (w_{\alpha_1 i}, w_{\alpha_2 i}, \ldots, w_{\alpha_n i})\tau_i \\
= \bigotimes_{i=0}^{k} \sigma_i(g_1)(w_{\alpha_1 i}, w_{\alpha_2 i}, \ldots, w_{\alpha_n i})\tau_i \\
= \bigotimes_{i=0}^{k} \sigma_i(g_1) (w_{\alpha_1 i}, w_{\alpha_2 i}, \ldots, w_{\alpha_n i})\tau_i \\
= \sigma(g)(w_{g(1)}, \ldots, w_{g(n)})\tau' ,
\]

which shows that \( \tau' \) is \((G,\sigma)\).
Also
\[ \bigoplus_{i=0}^{k} (w_{a_1,i}w_{a_2,i} \ldots w_{a_n,i}) \tau_i (w_{a_1,i}w_{a_2,i} \ldots w_{a_n,i}) \in W_i \]

\[ = \{(w_1,w_2,\ldots,w_n) \tau' \mid (w_1,w_2,\ldots,w_n) \in W \} = W' \]

is a generating set of \( \bigotimes_{i=0}^{k} P(W_i,G_i,a_i) \).

If \( B_i = \)
\[ \{(y_{s_1,i}y_{s_2,i} \ldots y_{s_{n_i},i}) \mid s^i = (s_{a_1,i},s_{a_2,i} \ldots s_{a_{n_i},i}) \in E_i' \} \]

where \( \Delta_i \) is the \( \Delta \) for the space \( P(W_i,G_i,a_i) \), then
\( B_i \) is a basis for \( P(W_i,G_i,a_i) \) by Theorem 3.1.

Then \( B = \)
\[ \bigoplus_{i=0}^{k} \{(y_{s_1,i}y_{s_2,i} \ldots y_{s_{n_i},i}) \mid s^i = (s_{a_1,i},s_{a_2,i} \ldots s_{a_{n_i},i}) \in E_i' \} \]

is a basis of \( \bigotimes_{i=0}^{k} P(W_i,G_i,a_i) \) (see [4]).

Define \( S \), a set of \( n \) tuples as follows: \( s \in S \) if and only if for each \( i, \ 0 < i < k \), the \( n_i \)-tuples
\( (s_{a_1,i},s_{a_2,i} \ldots s_{a_{n_i},i}) \) are in \( E'_i \), where \( E'_i \) is a set defined similarly as \( E' \) in 3.

Now let \( f : W \rightarrow U \) be any multilinear and \( (G,a) \) function on \( W \) into \( U \), where \( U \) is any vector space over \( F \).
Define \( \tilde{f} : \bigotimes_{i=0}^{k} P(W_i, G_i, \sigma_i) \rightarrow U \), by defining \( \tilde{f} \) on its basis elements as follows:

\[
(\bigotimes_{i=0}^{k} (y_{s_{\alpha_1, i}} \Delta_1 y_{s_{\alpha_2, i}} \Delta_1 \cdots \Delta_1 y_{s_{\alpha_{n_i}, i}})) \tilde{f} = (y_{s_{1}}', y_{s_{2}}', \ldots, y_{s_{n}}') \tilde{f},
\]

and then extend \( \tilde{f} \), linearly to a map on \( \bigotimes_{i=0}^{k} P(W_i, G_i, \sigma_i) \), to be denoted again by \( \tilde{f} \). Clearly \( \forall \tilde{f} = f \) and therefore \( \bigotimes_{i=0}^{k} P(W_i, G_i, \sigma_i) \) is a \((G, \sigma)\) space of \( W \).

Hence, by Theorem 2.4, Chapter I, \( P(W, G, \sigma) \) is isomorphic to \( \bigotimes_{i=0}^{k} P(W_i, G_i, \sigma_i) \).

4.2 Remark: Under the hypothesis of 4,

\[ v_1 \Delta v_2 \Delta \cdots \Delta v_n = 0 \text{ if and only if } \]

\[ v_{\alpha_1, i} \Delta_1 v_{\alpha_2, i} \Delta_1 \cdots \Delta_1 v_{\alpha_{n_i}, i} = 0 \text{ for some } i, 0 \leq i \leq k. \]

Proof: Immediate from Theorem 4.1 and 2(i) Chapter II.
CHAPTER IV

In this chapter, we shall apply our theorems to some special cases. We take particular G and σ and consider special set of vectors \( v_1, v_2, \ldots, v_n \). For the classical spaces, we rederive from our results the necessary and sufficient condition that an element of the space should be zero. We also show how our results lead to the known facts concerning the dimension of these classical spaces. Thus we have included all these spaces in a unified approach.

1. **Particularizing σ**.

1.1 **Proposition**: Let \((v_1, v_2, \ldots, v_n) \in W\) be a non-trivial element. Then we have the following:

(i) If \( σ = 1 \) (trivial Character) then \( v_1 \Delta v_2 \Delta \ldots \Delta v_n \neq 0 \).

(ii) If \( σ \neq 1 \) and \( v_1, v_2, \ldots, v_n \) are linearly independent, then \( v_1 \Delta v_2 \Delta \ldots \Delta v_n \neq 0 \).

(iii) If \( σ \neq 1 \) and \( \dim \langle v_1, v_2, \ldots, v_n \rangle = 1 \), then \( v_1 \Delta v_2 \Delta \ldots \Delta v_n = 0 \).

**Proof**: (i) Let \( \{y_1, y_2, \ldots, y_m\} \) be a basis of \( V \). Then by the Representation Theorem Form II, we have

\[
(v_1, v_2, \ldots, v_n) = \omega + \sum_{s \in E} D(M_s)(y_{s_1}, y_{s_2}, \ldots, y_{s_n})φ .
\]

Since \( σ = 1 \), \((y_{s_1}, y_{s_2}, \ldots, y_{s_n})φ\) is not a \((G, σ)\) element
for any \( s \in S \) and in particular for any \( s \in E \).
Choose \( s \in S \), such that \( s_i = \min \{ j \mid j \in A_i \} \), for any \( i = 1,2,\ldots,n \).

Claim: \( A(s) = \{ s \} \).

For if \( t \in A(s) \) then \( t \sim s \), which implies there exists \( g \in G \) such that \((t_1,t_2,\ldots,t_n) = (s_{g(1)},s_{g(2)},\ldots,s_{g(n)})\); i.e., \( t_i = s_{g(i)} \) for all \( i \in I \). If \( \text{ord}(g) = k \), we have for any \( i \in I \),

\[
t_i = s_{g(i)} \leq t_{g(i)} = s_{g_2(i)} \leq t_{g_2(i)} = \cdots \leq t_{g^{k-1}(i)} = s_{g^k(i)}(=t_1),
\]

which implies that \( s_i = t_i \) for all \( i \in I \); i.e., \( s = t \), and this proves the claim.

\( H_s = \{ g \mid g \in G, \sigma(g) = 1 \text{ and } s_i = s_{g(i)} \text{ for all } i \in I \} \)
in this case is

\( H_s = \{ g \mid g \in G \text{ and } s_i = s_{g(i)} \text{ for all } i \in I \} \).

Therefore if \( h \in G_s \) and \( h \notin H_s \), then \( s_i = s_{h(i)} \) for some \( i \in I \), and since \( A(s) = \{ s \} \), hence

\((s_{h(1)},s_{h(2)},\ldots,s_{h(n)}) \notin S \), which implies that \( s_{h(i)} \notin A_i \) for some \( i \in I \). Therefore if \( v_i = \sum_{j=1}^{m} a_{i,j}y_j \) for \( i \in I \), we have \( a_{i,s_{h(i)}} = 0 \) and therefore

\[ a_1,s_{h(1)}a_2,s_{h(2)} \cdots a_n,s_{h(n)} = 0. \]
Hence \( D(M_s) = \sum_{h \in S} \sigma(h^{-1}) a_1, s_h(1) a_2, s_h(2) \cdots a_n, s_h(n) \)

\[ = a_1, s_g(1) a_2, s_g(2) \cdots a_n, s_g(n) \]

where \( g \) is a representative of the coset \( H_s \) in \( G \).

Therefore \( D(M_s) = a_1, s_1 a_2, s_2 \cdots a_n, s_n \neq 0 \)

Hence by Theorem 2.3 Chapter III, \( v_1 \Delta v_2 \Delta \cdots \Delta v_n \neq 0 \)

(ii) Take \( y_i = v_i \) for \( i = 1, 2, \ldots, n \) and extend \( \{ y_1, y_2, \ldots, y_n \} \) to a basis \( \{ y_1, y_2, \ldots, y_m \} \) of \( V \).

Since \( (v_1, v_2, \ldots, v_n) \neq 0 \) satisfies the property \( P \) and if \( g \in G \), such that \( \sigma(g) \neq 1 \), then \( g \neq e \) (the identity permutation). Thus \( g(1) \neq 1 \) for some \( i \in I \) which implies that \( v_i \) and \( v_g(i) \) are independent for some \( i \in I \). Hence \( (v_1, v_2, \ldots, v_n) \neq 0 \) is not a \( (G, \sigma) \) element, and therefore \( v_1 \Delta v_2 \Delta \cdots \Delta v_n \neq 0 \), by Lemma 2.1 Chapter III.

(iii) Let \( y_1 = v_1 \) and extend it to a basis \( \{ y_1, y_2, \ldots, y_m \} \) of \( V \). Since \( \dim \langle \{ v_1, v_2, \ldots, v_n \} \rangle = 1 \), we have \( A_i = \{ 1 \} \) for all \( i \in I \) and \( S = \{ s \} \), where \( s = (1, 1, \ldots, 1) \)

By the Representation Theorem Form II,

\[ (v_1, v_2, \ldots, v_n) \neq 0 + D(M_s) (y_1, y_1, \ldots, y_1) \neq 0 \]

Since \( \sigma \neq 1 \), \( (y_1, y_1, \ldots, y_1) \) is a \( (G, \sigma) \) element.

Therefore by Theorem 2.3, \( v_1 \Delta v_2 \Delta \cdots \Delta v_n \neq 0 \).

Note that Proposition 1.1 holds for any subgroup \( G \) of \( S_n \).
2. Specialization to the Classical Spaces.

2.1 Theorem: Let $P(W,G,a) = \bigotimes_{i=1}^{n} V_i$.

(i) If $P(W,G,a) = \bigotimes_{i=1}^{n} V_i = 0$, then $v_1 \otimes v_2 \otimes \ldots \otimes v_n = 0$ if and only if $v_i = 0$ for some $i \in I$.

(ii) If $P(W,G,a) = \mathbb{R}^n$, then $v_1 \wedge v_2 \wedge \ldots \wedge v_n = 0$ if and only if $v_1, v_2, \ldots, v_n$ are linearly dependent.

(iii) If $P(W,G,a) = V(n)$, then $v_1 \cdot v_2 \cdot \ldots \cdot v_n = 0$ if only if $v_i = 0$ for some $i \in I$.

Proof: (i) and (iii) follow from Proposition 1.1 (i) and Remark 2.2 Chapter III.

(ii) $(\Rightarrow)$. If not, then $v_1 \wedge v_2 \wedge \ldots \wedge v_n \neq 0$ by Proposition 1.1 (ii), which is a contradiction.

$(\Leftarrow)$ If $(v_1, v_2, \ldots, v_n) \in W$ is a trivial element, then $v_1 \wedge v_2 \wedge \ldots \wedge v_n = 0$ by the Remark 2.2 Chapter III. So we may assume $v_i \neq 0$ for any $i \in I$.

Let $J = \{ 1 \mid i = 1 \text{ or } v_i \text{ is independent of } v_1, v_2, \ldots, v_{i-1} \}$.

Let $J = \{ 1 = j_1, j_2, \ldots, j_r \}$ (say), where $j_1 < j_2 < \ldots < j_r$.

Since $v_1, v_2, \ldots, v_n$ are linearly dependent, we have $r < n$. Since for any $i, j, i \neq j \ 1 \leq i, j \leq n$,

$$v_1 \wedge \ldots \wedge v_i \wedge \ldots v_j \wedge \ldots v_n = \sigma(ij)v_1 \wedge \ldots \wedge v_j \wedge \ldots v_i \wedge \ldots v_n,$$

we can write

$$v_1 \wedge v_2 \wedge \ldots \wedge v_n = c u_1 \wedge u_2 \wedge \ldots \wedge u_r \wedge \ldots u_n \ldots \ldots \ldots \ldots (12)$$
where $c \in F$ and $u_k = \begin{cases} v_{j_t}, & \text{if } k = j_t \in J \\ v_j, & \text{for some } j \in J, \text{ if } k \notin J \end{cases}$

We shall show that $u_1 \wedge u_2 \wedge \ldots \wedge u_n = 0$.

Clearly $\{u_1, u_2, \ldots, u_r\}$ is an independent set. Extend this set to a basis $\{y_1, y_2, \ldots, y_m\}$ of $V$ such that $y_1 = v_{j_1}$, $i = 1, 2, \ldots, r$. For each $i \in I$, $A_i = \{i\}$ if $i \leq r$ and $A_i \subseteq \{1, 2, \ldots, r\}$ if $r < i \leq n$.

Therefore if $s \in E$, then for any $j$, $r < j \leq n$, $s_j = s_i$ for some $i$, $1 \leq i \leq r$. Hence $(y_{s_1}, y_{s_2}, \ldots, y_{s_n})$ is a $(G, \sigma)$ element.

By the Representation Theorem Form II,

$$(u_1, u_2, \ldots, u_n) = w + \sum_{s \in E} D(M_s) (y_{s_1}, y_{s_2}, \ldots, y_{s_n})$$

Since for each $s \in S$, $(y_{s_1}, y_{s_2}, \ldots, y_{s_n})$ is a $(G, \sigma)$ element, we have $u_1 \wedge u_2 \wedge \ldots \wedge u_n = 0$ by Theorem 2.3 Chapter II and therefore from equation (12), we obtain

$v_1 \wedge v_2 \wedge \ldots \wedge v_n = 0$.

3. **Dimension of the classical spaces**

Since our general theory gives a basis of $P(W, G, \sigma)$, (see 3, Chapter III) in every case, we can easily compute the dimension as follows.

3.1 Theorem:

(i) $\dim \bigotimes_{i=1}^{n} V_i = m^n$. 

(ii) \( \dim \frac{n}{V} = \binom{m}{n} \).

(iii) \( \dim \frac{V}{n} = \binom{m+n-1}{n} \).

**Proof:** \( \dim P(W,G,\sigma) = \text{cardinality } E' \), by Theorem 3.1, Chapter III.

(i) If \( s \in S \), then \( A(s) = \{s\} \). Therefore \( E = S \) and since \( \sigma = 1 \), \((y_{s_1}, y_{s_2}, \ldots, y_{s_n})^\phi\) is not a \((G,\sigma)\) element for any \( s \in S \). Therefore \( E' = E = S \).

Hence \( \dim \bigotimes_{i=1}^{n} V_i = \text{cardinality } E' = \text{cardinality } S = m^n \).

(ii) It is easy to see that if \( s = (s_1, s_2, \ldots, s_n) \in E \), then \((y_{s_1}, y_{s_2}, \ldots, y_{s_n})^\phi\) is a \((G,\sigma)\) element, if and only if there exists \( i, j \) in \( I \), \( i \neq j \), such that \( s_i = s_j \). This means that if \((y_{s_1}, y_{s_2}, \ldots, y_{s_n})^\phi\) is not a \((G,\sigma)\) element, then all the \( s_i \)'s are distinct. Also for such an \( s \), cardinality \( A(s) = n! \). Let \( S' = \{ s \mid s \in S, \ s_i \text{ are all distinct} \} \). Then cardinality \( S' = m(m-1)(m-2)\ldots(m-n+1) \).

But \( \sum_{s \in E'} \text{cardinality } A(s) = \text{cardinality } S' \).

Therefore cardinality \( E' = \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!} = \binom{m}{n} \).

Hence \( \dim \bigotimes_{i=1}^{n} V_i = \binom{m}{n} \).

(iii) Clearly cardinality \( E = \sum_{k=1}^{m+n-1} \binom{m}{k} \).

\[ = \binom{m+n-1}{n} \]
Also since \( \sigma = 1 \), \((y_{s_1}, y_{s_2}, \ldots, y_{s_n}) \neq \emptyset \) is not a \((G, \sigma)\)
element for any \( s = (s_1, s_2, \ldots, s_n) \) in \( E \). Therefore \( E = E' \) and hence \( \dim \mathcal{V}(n) = \text{cardinality } E' = \binom{m+n-1}{n} \).

4. A sufficient condition for \( v_1 \Delta v_2 \Delta \cdots \Delta v_n \) to be zero.

4.1 Proposition: If \((v_1, v_2, \ldots, v_n) \neq \emptyset \) is a \((G, \sigma)\) element, then \( v_1 \Delta v_2 \Delta \cdots \Delta v_n = 0 \).

Proof: \((v_1, v_2, \ldots, v_n) \neq \emptyset \) is a \((G, \sigma)\) element implies there is \( g \in G \) such that \( \sigma(g) \neq 1 \) and \( v_i, v_g(i) \) are dependent for all \( i \in I \). Let \( v_g(i) = \lambda_g(i) v_i \), where \( \lambda_g(i) \in F \) for each \( i \in I \). We will first show that \( \lambda_g(1) \lambda_g(2) \cdots \lambda_g(n) = 1 \).

Let \( g = C_1 C_2 \cdots C_k \) be the decomposition of \( g \) into disjoint cycles, including also the cycles of length one, if there are any. Let \( \Delta_j = \text{dom } C_j \), \( j = 1, 2, \ldots, k \).

Then \( \bigcup_{j=1}^{k} \Delta_j = I \) and \( \Delta_j \cap \Delta_{j'} = \emptyset \) for \( j \neq j' \).

Let \( C_j = (\alpha_{j,1}, \alpha_{j,2}, \ldots, \alpha_{j,n_j}) \) for \( j = 1, 2, \ldots, k \).

Then \( n_1 + n_2 + \cdots + n_k = n \).

If \( n_j = 1 \) for some \( j \), then \( C_j = (\alpha_{j,1}) \) and \( v_g(\alpha_{j,1}) = \lambda_g(\alpha_{j,1}) v_{\alpha_{j,1}} \) implies \( \lambda_g(\alpha_{j,1}) = 1 \).

If \( n_j > 1 \), then \( v_{\alpha_{j,1}}^2 = v_g(\alpha_{j,1}) = \lambda_g(\alpha_{j,1}) v_{\alpha_{j,1}} \).
\( \nu_{a_j,3} = \nu_{g(a_j,2)} = \lambda_{g(a_j,2)} \nu_{a_j,2} = \lambda_{g(a_j,2)} \lambda_{g(a_j,1)} \nu_{a_j,1} \)

\[ \cdots \]

\( \nu_{a_j,n_j} = \nu_{g(a_j,n_j-1)} = \lambda_{g(a_j,n_j-1)} \nu_{a_j,n_j-1} \)

\[ = \lambda_{g(a_j,n_j-1)} \lambda_{g(a_j,2)} \lambda_{g(a_j,1)} \nu_{a_j,1} \]

\( \nu_{a_j,1} = \nu_{g(a_j,n_j)} = \lambda_{g(a_j,n_j)} \nu_{a_j,n_j} \)

\[ = \lambda_{g(a_j,n_j)} \lambda_{g(a_j,n_j-1)} \lambda_{g(a_j,2)} \lambda_{g(a_j,1)} \nu_{a_j,1} \]

Therefore \( \lambda_{g(a_j,n_j)} \lambda_{g(a_j,n_j-1)} \lambda_{g(a_j,2)} \lambda_{g(a_j,1)} = 1 \) ;

i.e. \( \prod_{a \in \Delta_j} \lambda_{g(a)} = 1 \) and since \( j \) is arbitrary, we have

\[ \prod_{a \in \Delta_j} \lambda_{g(a)} = 1 \] for \( j = 1, 2, \ldots, k \).

Therefore \( \prod_{a \in I} \lambda_{g(a)} = \prod_{j=1}^{k} \prod_{a \in \Delta_j} \lambda_{g(a)} = 1 \). Now

\( (v_1,v_2,\ldots,v_n)\phi = [(v_1,v_2,\ldots,v_n)\phi - \sigma(g)(v_{g(1)},v_{g(2)},\ldots,v_{g(n)})\phi] \)

\[ + \sigma(g)(v_{g(1)},v_{g(2)},\ldots,v_{g(n)})\phi , \]

and since the terms within the square bracket being of type \((ii)\), is in \( \Omega \) we obtain on applying \( \eta \),

\( \nu_1 \Delta v_2 \Delta \cdots \Delta v_n = (v_1,v_2,\ldots,v_n)\tau \)

\[ = \sigma(g) (v_{g(1)},v_{g(2)},\ldots,v_{g(n)})\tau \]

\[ = \sigma(g) (\lambda_{g(1)}v_1,\lambda_{g(2)}v_2,\ldots,\lambda_{g(n)}v_n)\tau \]
\[= \sigma(g) \prod_{\alpha \in I} \lambda_{g(\alpha)} (v_1, v_2, \ldots, v_n)\]
\[= \sigma(g) v_1^\Delta v_2^\Delta \ldots v_n^\Delta\]

and since \(\sigma(g) \not\equiv 1\), we have

\[v_1^\Delta v_2^\Delta \ldots v_n^\Delta = 0\]

4.2 Remark: One can easily see that the condition of Proposition 4.1 is, however, not necessary.

However, if \(V\) is a unitary space and \(G\) belongs to a certain class of groups \(G\), as we shall define below, then the condition of Proposition 4.1 is both necessary and sufficient.

5. Particularizing \(V\) and \(G\).

Let \(G\) be a subgroup of \(S_n\). If \(\Delta\) is an orbit of \(G\), let \(g^\Delta\) denote the restriction of \(g\) to \(\Delta\). Then \(g^\Delta\) is a permutation of \(\Delta\). Let \(G^\Delta = \{g^\Delta \mid g \in G\}\).

Clearly \(G^\Delta\) is a subgroup of \(S_\Delta\), where \(S_\Delta\) is the full symmetric group on \(\Delta\).

Let \(G = \{G \mid G\) is a subgroup of \(S_n\) and if \(\Delta\) is any orbit of \(G\) then \(G^\Delta\) is cyclic\}

Clearly \(G\) contains every cyclic group. As to the other members of \(G\), they are all abelian.

Let \(G \in G\) and \(W = V \times V \times \ldots \times V\) (\(n\) copies), where \(V\) is
a unitary space of dimension \( m \). Let \( \sigma \) be any linear character of \( G \) and consider the \((G, \sigma)\) space \( P(W, G, \sigma) \) of \( W \).

5.1 **Definition:** If \( \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in W \), then \( \gamma \) is called an indicator of \( \nu \) if and only if 
\[ \gamma : I \rightarrow I \], such that \( \gamma_i = \gamma_j \) where \( \gamma_i = \gamma(1) \), when and only when \( \nu_i \) and \( \nu_j \) are dependent.

Let \( G_\gamma = \{ g \mid g \in G, \gamma_i = \gamma_{g(i)}(1) \text{ for all } i \in I \} \).

Then it is proved by Stanley Gill Williamson in his Ph.D. thesis [5], that \( \nu_1 \Delta \nu_2 \Delta \ldots \Delta \nu_n = 0 \) if and only if
\[ \sum_{g \in G_\gamma} \sigma(g) = 0 \], for any indicator \( \gamma \) of \( \nu \).

5.2 **Theorem:** With \( G \) and \( W \) as in 5.1, \( \nu_1 \Delta \nu_2 \Delta \ldots \Delta \nu_n = 0 \) if and only if \((\nu_1, \nu_2, \ldots, \nu_n)_\emptyset\) is a \((G, \sigma)\) element.

**Proof:** \((\Leftarrow)\) It is a particular case of Proposition 4.1.

\((\Rightarrow)\) Let \( \gamma \) be any indicator of \( \nu \). Then
\[ \sum_{g \in G_\gamma} \sigma(g) = 0 \].

This implies there exists \( g \in G \), such that \( \sigma(g) \neq 1 \).

Also \( g \in G_\gamma \) implies \( \gamma_i = \gamma_{g(i)} \) for all \( i \in I \) and this implies that \( \nu_i \) and \( \nu_{g(i)} \) are dependent for all \( i \in I \). Hence there exists \( g \in G \), such that \( \sigma(g) \neq 1 \) and \( \nu_i, \nu_{g(i)} \) are dependent for all \( i \in I \), which means that \((\nu_1, \nu_2, \ldots, \nu_n)_\emptyset\) is a \((G, \sigma)\) element.


