ENCLOSURE THEOREMS FOR EIGENVALUES OF
ELLIPTIC OPERATORS
by
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Enclosure theorems for the eigenvalues and representational formulae for the eigenfunctions of a linear, elliptic, second order partial differential operator will be established for specific domain perturbations to which the classical theory cannot be applied. In particular, the perturbation of \( n \)-dimensional Euclidean space \( \mathbb{E}^n \) to an \( n \)-disk \( D_a \) of radius \( a \) is considered in Chapter I and the perturbation of the upper half-space \( \mathbb{H}^n \) of \( \mathbb{E}^n \) to the upper half of \( D_a, S_a \), is discussed in Chapter II. In each case a general self-adjoint boundary condition is adjoined on the bounding surface of the perturbed domain.
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INTRODUCTION

Let $L$ be the linear, elliptic, self-adjoint partial differential operator defined by

$$Lu = - \sum_{i,j=1}^{n} D_1(a_{ij} D_j u) + bu$$

where $D_1$ denotes partial differentiation with respect to the variable $x_1$. The assumptions to be made on the coefficients are as follows:

(i) the coefficients $a_{ij}$ and $b$ are continuous real-valued functions of $x = (x_1, \ldots, x_n)$ in $n$-dimensional Euclidean space $E^n$ and $b(x) > 0$ for all $x \in E^n$.

(ii) $a_{ij} = a_{ji}$ for every $i$ and $j$, and the $a_{ij}$ possess uniformly continuous first partial derivatives in $E^n$.

(iii) $\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq c_1 \sum_{i=1}^{n} |\xi_i|^2$ for some positive real number $c_1$ and for every $\xi \in E^n$.

Our purpose is to establish variational formulae for the eigenvalues and eigenfunctions of $L$ for two specific domain perturbations. These are: the perturbation of $E^n$ to an $n$-disk $D_a$ of radius $a$, considered in Chapter I, and the perturbation of the upper half-space $H^n$ of $E^n$ to the upper half of $D_a$, $S_a$, discussed in Chapter II. The boundary condition adjoined on the bounding surface of the perturbed domain is
(1) \( U v = 0 \)

where \( U \) is the linear boundary operator defined by

\[
U v = \sigma_1 \sum_{i,j=1}^{n} a_{ij} D_j v \cos(v, x_i) + \sigma_2 v
\]

and \( v \) denotes the outer normal to the bounding surface. It is assumed that the functions \( \sigma_1(x) \) and \( \sigma_2(x) \) are defined on the bounding surface, are piecewise continuous and nonnegative, and that the sum \( \sigma_1(x) + \sigma_2(x) \) has a positive lower bound. It may be noted that additional, compatibility conditions for the special case \( \sigma_2 = 0 \) are not needed here since the coefficient \( b(x) \) of \( L \) is required to be positive, ([5], p. 95).

A solution \( u \) of \( Lu = 0 \) is assumed to be of class \( C^1 \) and all derivatives involved in \( L \) are supposed to exist, be continuous, and satisfy \( Lu = 0 \) at every point.

Let \( H_1, H_2, H_a \) and \( H_s \) denote the Hilbert spaces which are the Lebesgue spaces with respective inner products

\[
(u, v)_1 = \int_{E^n} u(x) \bar{v}(x) dx
\]

\[
(u, v)_2 = \int_{H^n} u(x) \bar{v}(x) dx
\]

\[
(u, v)_a = \int_{D_a} u(x) \bar{v}(x) dx
\]

\[
(u, v)_s = \int_{S_a} u(x) \bar{v}(x) dx
\]
and respective norms \( \|u\|_1, \|u\|_2, \|u\|_a \) and \( \|u\|_s \). For \( x \in \mathbb{E}^n \), \( |x| \) denotes the usual Euclidean norm.

The eigenvalue problem for \( L \) on \( \mathbb{E}^n \)

\[
Lu = \lambda u \\
u \in \mathcal{X}_1
\]

will be called the basic problem in Chapter I. The eigenvalue problem for \( L \) on \( \mathbb{H}^n \)

\[
Lu = \kappa u \\
u \in \mathcal{X}_2
\]

\( Uu = 0 \) on the \((n-1)\) hyperplane \( P = \{x | x \in \mathbb{E}^n, x_n = 0\} \) will be called the basic problem in Chapter II. The corresponding perturbed eigenvalue problems will be defined in Chapters I and II.

In general it is not true that the eigenvalues of the perturbed problems tend to limits as \( a \to \infty \), even when the spectrum of the basic problem is entirely discrete, for example, in the case \( n = 1 \) and \( \sigma_1 = 0 \) in (1), when the singularity at \( \infty \) is of the limit circle type in Weyl's classification [9]. The only assumption required in order to obtain the enclosure theorems and representational formulae is that there exists at least one eigenvalue of the basic problem whose corresponding eigenfunctions satisfy some limit property. For example, Theorem 1 in Chapter I shows that if the eigenfunctions corres-
ponding to the basic eigenvalue \( \lambda \) of multiplicity \( m \) satisfy condition (1.2), then at least \( m \) eigenvalues of the perturbed problem (1.1) converge to \( \lambda \) as the radius, \( a \), of the \( n \)-disk \( D_a \) tends to infinity.

The principle difficulty in this estimation problem is in establishing a reasonable condition on the basic eigenfunctions in terms of a simple solution, \( g \), of \( Lg = 0 \) so that the norm of the function \( f = R_a u - au \), constructed in Lemma 2, remains small even though \( f(x) \) may become large for large \( x \). For example, in Chapter I this condition is characterized in terms of the "L - measure" \( (1.3) \) which is independent of the basic eigenfunctions.

The method employed for the treatment of this estimation problem involving the boundary condition (1) follows almost directly from that used by C. A. Swanson [7] for the special case \( q_1 = 0 \).

This problem of estimating eigenvalues and eigenfunctions for large domains has its physical origin in certain models of enclosed quantum mechanical systems \[2], [3] and [6]. In the case that the Schrödinger equation is separable, (a special case of \( L \)), the problem reduces to a domain perturbation problem for a singular second order ordinary differential operator. In particular, the example considered in section 1.5 reduces to two ordinary differential equations each of which have a singularity of the limit point type at infinity.
CHAPTER I

THE PERTURBATION OF $\mathbb{E}^n$ TO AN n-DISK

1.1 Introduction. Our purpose in this chapter is to obtain variational formulae for the eigenvalues and corresponding eigenfunctions of the operator $L$ when $\mathbb{E}^n$ is perturbed to an n-disk, $D_a$, of large radius, $a$, and condition (1) is adjoined on the bounding (n-1) hypersphere, $B_a$.

The perturbed eigenvalue problem to be considered is

$$Lv = \mu v \quad \text{in} \quad D_a = \{x \mid |x| < a, a > 0\}$$

where the perturbed domain, $D_a$, is defined as the set of all complex valued functions $v$ with the following properties

(i) $v$ is twice continuously differentiable in $D_a$.

(ii) $v$ and $v'$ are continuous at those points of the boundary $B_a$, at which $\sigma_1$ and $\sigma_2$ are continuous.

(iii) $v$ satisfies (1) on $B_a = \{x \mid |x| = a\}$.

The only assumption to be made here is that there exists at least one eigenvalue $\lambda$ of the basic problem (2) whose corresponding eigenfunctions, $u$, satisfy

$$\max_{B_a} |u| \|g\|_a / \|u\|_a = o(1) \quad \text{as} \quad a \to \infty,$$
is the "L measure"

\[ L_{g} = 0 \text{ in } D_{a} \]

\[ U_{g} = 1 \text{ on } B_{a} \]

It is known [5], [4] that for the perturbed problem (1.1) there exists a denumerable sequence of eigenvalues \( \mu_{1} , 0 < \mu_{1} \leq \mu_{2} \leq \mu_{3} \leq \ldots \) , and a complete orthonormal sequence of eigenfunctions \( \{ v_{1} \} \) such that for some Robin function, \( R_{a}(x,y) \) , (Green's function of the third kind),

\[ v_{1}(y) = \mu_{1} R_{a} v_{1}(y) = \mu_{1} \int_{D_{a}} R_{a}(x,y) v_{1}(x) \, dx \]

and any basic eigenfunction, \( u \), satisfies \( L_{R_{a}} u = u \text{ in } D_{a} \).

Here \( R_{a} \) denotes the integral operator whose kernel is \( R_{a}(x,y) , y \in D_{a} \). It may be noted that for \( \sigma_{2} = 0 \), the kernel function \( R_{a}(x,y) \) is replaced by a Neumann function, \( N_{a}(x,y) \), and for \( \sigma_{1} = 0 \) by a Green's function, \( G_{a}(x,y) \).

Clearly, the results of this chapter apply to these special cases even though the representations of the solutions of (1.5) may be slightly different from (1.6). \( R_{a}(x,y) \) is constructed in the usual way as the sum of a fixed fundamental solution, \( \gamma(x,y) \), and the solution of a particular Robin problem, \( r(x,y) \).

That is,

\[ R_{a}(x,y) = \gamma(x,y) + r(x,y) \]

where:
(i) \( \gamma(x,y) \), regarded as a function of \( x \), is a regular solution of \( L \gamma = 0 \) except when \( x = y \), where it has a singularity of order \( |x-y|^{2-n} \) for \( n \geq 3 \).

(ii) \( r(x,y) \), regarded as a function of \( x \), is a solution of

\[
Lr = 0 \text{ in } D_a \\
Ur = -U\gamma \text{ on } B_a
\]

and \( R_a(x,y) \) satisfies the boundary condition

(1.4) \( UR_a(x,y) = 0 \) on \( B_a \).

\( R_a(x,y) \) is unique, symmetric and non-negative for all \( x \) and \( y \) in \( D_a \), ([4], p. 161).

In addition, any solution of the Robin problem,

(1.5) \( Lf = 0 \text{ in } D_a \\
Uf = h \text{ on } B_a \)

has the representation

(1.6) \( f(y) = \int_{B_a} R_a(x,y)h(x)dS_x \) for every \( y \in D_a \)

I.2 Enclosure Theorems for The Perturbed Eigenvalues. The following notation will be used

\[
\varphi_a[u] = \{ \max_{B_a} |UU| \} \| g \|_a / \| u \|_a \quad (u \neq 0) \\
\varphi_a = \sup_{u \in G_\lambda} \varphi_a[u]
\]
\[ \rho_a = \lambda \varphi_a/(1 - \varphi_a) \]

where \( G \) is the eigenspace associated with the basic eigenvalue \( \lambda \) of (2).

Lemma 1. \( \omega_a = o(1) \) and \( \rho_a = o(1) \) as \( a \to \infty \).

Proof. Let \( u \) be that function in \( G_\lambda \) such that \( \varphi_a[u] = \varphi_a \).

Since every \( u \in G_\lambda \) has the representation \( u = \sum_{i=1}^{m} \alpha_i u_i \) in terms of an orthonormal basis \( \{u_i\} \), where the basic eigenvalue \( \lambda \) is of multiplicity \( m \),

\[
\omega_a[u] = \left\{ \max_{B_a} \left| \sum_{i=1}^{m} \alpha_i U u_i \right| \right\} \|g\|_a / \left\{ \sum_{i=1}^{m} \alpha_i u_i \right\} (u \neq 0)
\]

\[
\leq \sum_{i=1}^{m} \left( \frac{|\alpha_i|}{\|g\|_a} \max_{B_a} \left| U u_i \right| \right) \|g\|_a
\]

\[
\leq m \max_{1 \leq i \leq m} \omega_a[u_i]
\]

and the proof follows from condition (1.2) and the fact that the \( u_i \), \( i = 1, \ldots, m \), are the basic eigenfunctions corresponding to \( \lambda \).

Lemma 2. For \( a = 1 / \lambda \)

(2.1) \( \|R_a u - \alpha u\|_a \leq \alpha \varphi_a \|u\|_a \) for every \( u \in G_\lambda \)

Proof. The eigenvalues of (2) are positive. In fact, if \( \lambda \leq 0 \) for some \( \lambda \), the maximum principle ([1], p. 325)
implies that the eigenfunction \( u \) corresponding to \( \lambda \) approaches its maximum as \( |x| \to \infty \). This contradicts \( u \in L^2(\mathbb{E}^n) \) and \( u \neq 0 \).

Let \( \alpha = 1/\lambda \) and define the function \( f = R_{\alpha}u - \alpha u \). Then for every \( u \in G_{\lambda} \), \( f \) is a solution of the Robin problem

\[
\begin{align*}
Lf &= 0 \text{ in } D_a \\
Uf &= -\alpha Uu \text{ on } B_a
\end{align*}
\]

For, \( Lf = L(R_{\alpha}u - \alpha u) = LR_{\alpha}u - \alpha Lu \)

\[
= u - (1/\lambda) \lambda u = 0 \text{ in } D_a
\]

and \( Uf = U(R_{\alpha}u - \alpha u) = UR_{\alpha}u - \alpha Uu \)

\[
= \sigma_1(x) \sum_{i,j=1}^{n} a_{ij}(x) D_j \int_{D_a} R_{\alpha}(x,y)u(y)dy \cos(y,x_i) \\
+ \sigma_2(x) \int_{D_a} R_{\alpha}(x,y)u(y)dy - \alpha Uu
\]

Since the order of the singularity of the kernel function, \( R_{\alpha}(x,y) \), is less than \( |x-y|^{1-n} \) and since \( u(x) \) is continuous, the integral

\[
\int_{D_a} D_j(R_{\alpha}(x,y))u(y)dy
\]

converges uniformly and the derivative

\[
D_j \int_{D_a} R_{\alpha}(x,y)u(y)dy = \int_{D_a} D_j(R_{\alpha}(x,y))u(y)dy
\]

exists and is continuous, ([5], p. 56). Hence

\[
Uf = \int_{D_a} (\sigma_1(x) \sum_{i,j=1}^{n} a_{ij}(x) D_j R_{\alpha}(x,y) \cos(y,x_i) + \sigma_2(x) R_{\alpha}(x,y))u(y)dy - \alpha Uu
\]
= \int_{D_a} (UR_a(x,y))u(y)dy - \alpha Uu \\
= -\alpha Uu \text{ on } B_a, \text{ by condition (1.4)}.

That is, \( f \) is a solution of the Robin problem (1.5) with \( h = -\alpha Uu \) and hence has the representation (1.6),

\[
f(y) = -\alpha \int_{B_a} R_a(x,y)Uu \text{d}S_x \text{ for every } y \in D_a.
\]

Then

\[ |f(y)| \leq \alpha \{ \max_{B_a} |Uu| \} \int_{B_a} R_a(x,y) \text{d}S_x \]

since \( R_a(x,y) \) is non-negative and \( \alpha \) is positive, and

\[ |f(y)| \leq \alpha \{ \max_{B_a} |Uu| \} g(y) \text{ for every } y \in D_a \text{ since } g \text{ is the solution of (1.5) with } h = 1. \]

Therefore

\[
\|f\|_a = \|R_a u - \alpha u\|_a \leq \alpha \{ \max_{B_a} |Uu| \} (\|g\|_a / \|u\|_a) \|u\|_a = \alpha \varphi_a[u] \|u\|_a \\
\leq \alpha \varphi_a \|u\|_a
\]

for every \( u \in C_\lambda \).

Theorem 1. Let \( \lambda \) be an \( m \)-fold degenerate eigenvalue of (2) whose corresponding eigenfunctions satisfy condition (1.2).

Then there exists a positive number \( \alpha_0 \) such that at least \( m \) perturbed eigenvalues \( \mu_1(a) \) of (1.1) are enclosed in the
interval \([\lambda, \lambda + \rho_a]\) whenever \(a > a_0\) and converge to \(\lambda\) as \(a \to \infty\).

Proof. Since \(\rho_a = o(1)\) as \(a \to \infty\) by Lemma 1, there exists an \(a_0\) such that \(\rho_a < 1\) for every \(a > a_0\) and \(\rho_a\) is well defined for large \(a\).

Let \(\mathcal{F}_{a\varepsilon}\) be the subspace of \(\mathcal{H}_a\) generated by all the eigenfunctions of \(R_a\) whose eigenvalues \(\beta_i = 1/\mu_i\) lie in the interval \(|\beta - \alpha| < \epsilon\). Let \(P(\epsilon)\) be the projection of \(\mathcal{H}_a\) onto \(\mathcal{F}_{a\varepsilon}\). Then

\[
\|u - P(\epsilon)u\|_a \leq \epsilon^{-1}\|R_a u - \alpha u\|_a
\]

for every \(u \in \mathcal{F}_a\) by ([8], p. 33), since the integral operator \(R_a\) is a self-adjoint linear transformation on \(\mathcal{H}_a\).

From (2.1)

(2.2) \(\|u - P(\epsilon)u\|_a \leq \alpha \omega_a \epsilon^{-1} \|u\|_a\) for every \(u \in \mathcal{F}_a\). Thus, by ([8], p. 35), since \(R_a\) is completely continuous on \(\mathcal{F}_a\), there are at least \(m\) eigenvalues \(\beta_i\) contained in the interval

\(|\beta_i - \alpha| < \alpha \varphi_a, \ i = 1, 2, \ldots, \), or, more precisely, the interval

\(|\mu_i - \lambda| < \mu_i \omega_a\). Since \(D_a \subset \mathbb{E}^n\), it follows by the minimax principle for eigenvalues [1] that \(\mu_i > \lambda\) for every \(i\) and

\[
\lambda \leq \mu_i \leq \lambda + \mu_i \omega_a
\]

or

\[
\lambda \leq \mu_i \leq \lambda / (1 - \varphi_a) = \lambda + \lambda \omega_a / (1 - \varphi_a) = \lambda + \rho_a, \ a \geq a_0
\]

for every \(i = 1, 2, \ldots\).
Then at least \( m \) eigenvalues \( \mu_1 \) of (1.1) are in \([ \lambda, \lambda + \rho_a ]\) whenever \( a > a_\circ \) and, since \( \rho_a = o(1) \) as \( a \to \infty \), converge to \( \lambda \) as \( a \to \infty \).

Theorem 2. Let \( \lambda \) be as in Theorem 1. If there exists a basic eigenvalue exceeding \( \lambda \), then there is a positive number \( a_\triangleright a_\circ \) such that exactly \( m \) perturbed eigenvalues \( \mu_i \) are enclosed in the interval \([ \lambda, \lambda + \rho_a ]\) whenever \( a > a_\triangleright a_1 \).

Proof. Let \( \lambda' \) be the smallest basic eigenvalue and \( \lambda \) the smallest eigenvalue exceeding \( \lambda \). Then since \( \varphi_a = o(1) \) as \( a \to \infty \), there exists a number \( a_\triangleright a_\circ \) such that \( \varphi_a < (\lambda' - \lambda) / \lambda' \) for every \( a > a_\triangleright a_1 \). This implies \( \lambda + \rho_a < \lambda' \), \( a > a_\triangleright a_1 \), since

\[
\lambda + \rho_a = \lambda + \lambda \varphi_a / (1 - \varphi_a) = \lambda / (1 - \varphi_a) < \lambda / (1 - ((\lambda' - \lambda) / \lambda')) = \lambda'
\]

and by Theorem 1 at least \( m \) perturbed eigenvalues, \( \mu_i \), are enclosed in the subinterval \([ \lambda, \lambda + \rho_a ] \subset [\lambda, \lambda'] \).

Since \( \mu_i \geq \lambda_i \) for every \( i \), by the minimax property, and \( \lambda \) is \( m \)-fold degenerate by hypothesis:

\[
\lambda = \lambda_1 = \ldots = \lambda_m
\]

\[
\mu_{m+1} \triangleright \lambda' = \lambda_{m+1}
\]

and

\( \mu_{m+1} \notin [\lambda, \lambda + \rho_a], a \geq a_\triangleright a_1 \).
Hence, at most $m$ perturbed eigenvalues, $\mu_1$, are in $[\lambda, \lambda + \rho_a]$ for $a > a_1$. Therefore exactly $m$ are in $[\lambda, \lambda + \rho_a]$ whenever $a > a_1$. An easy induction proof establishes the same result if $\lambda = \lambda^1$ is the $i$th distinct eigenvalue $\lambda^1 < \lambda^2 < \lambda^3 < \ldots$.

### 1.3 Uniform Estimates for The Perturbed Eigenfunctions

Let $p = p(n)$ be a positive number satisfying $p(2) = 0$, $p(3) = 0$, and $0 < n - 2p < 4$. Because the fundamental singularity of $R_a(x, y)$ is of order $|x - y|^{2-n}$ for $n \geq 3$, the function

$$k_a(x) = \left( \int_{D_a} |x - y|^{2p} R_a^2(x, y) \, dy \right)^{1/2}$$

is well defined in $D_a$. It is assumed for the next theorem that

$$\varphi_a^q k_a(x) = o(1) \quad \text{as } a \to \infty \quad (q = (n-2p)/n)$$

uniformly for all $x \in D_a$.

**Theorem 3.** Let $u_i$ be the orthonormal eigenfunctions corresponding to the $m$-fold degenerate eigenvalue $\lambda$ of Theorem 2, and $v_i$ those corresponding to the $m$ perturbed eigenvalues $\mu_i$, $i = 1, \ldots, m$. Then

$$(3.1) \quad v_i(x) = u_i(x) - f_i(x) + o(\varphi_a^q k_a(x)) \quad i = 1, \ldots, m, \quad x \in D_a, \quad a > a_1$$

where $f_i$ is the solution of (1.5) with $h_1 = U u_i$.

**Proof.** Let $\epsilon = \alpha - \alpha'$ in (2.2), where $\alpha = 1/\lambda, \alpha' = 1/\lambda'$. It follows from Theorem 2 that $\alpha \varphi_a < \alpha (\lambda' - \lambda)/\lambda'$, $a > a_1$. That is,

$$\alpha \varphi_a < \alpha (\lambda' - \lambda)/\lambda' = \alpha - \alpha' = \epsilon \quad a > a_1.$$
Then $\mathcal{F}_a\varepsilon$ is $m$-dimensional by Theorem 2 and $\|u-P(\varepsilon)\|_a < \|u\|_a$ implies that $u=0$ if $P(\varepsilon)u=0$, $u \in \mathcal{O}_\lambda$. Therefore, $m$ uniquely determined linearly independent eigenfunctions $Z_i$ corresponding to $\alpha$ are mapped by $P(\varepsilon)$ into the orthonormal functions $v_i$. By (2.2) $\|Z_i-P(\varepsilon)Z_i\|_a = \|Z_i-v_i\|_a \leq \alpha \varphi_a \|Z_i\|_a$

and

$\|Z_i-v_i\| = O(\varphi_a)$.

Since, by the Schwarz inequality

$|\langle Z_i, Z_j \rangle_a - \langle v_i, v_j \rangle_a| \leq \|Z_i\|_a \|Z_j - v_j\|_a + \|v_j\|_a \|Z_i - v_i\|_a$

$= O(\varphi_a) + O(\varphi_a) = O(\varphi_a)$

Let $\{u_i\}$ be the orthonormal sequence constructed by the Schmidt orthonormalization process as linear combinations of the $Z_i$. Then $u_i = \sum_{j=1}^m \gamma_{ij} Z_j$ for some $\gamma_{ij} \in \mathbb{C}$, and

$\|u_i - Z_i\|_a^2 = \|\sum_{j=1}^m \gamma_{ij} Z_j - Z_i\|_a^2$

$= (\sum_{j=1}^m \gamma_{ij} Z_j - Z_i)(\sum_{j=1}^m \gamma_{ij} Z_j - Z_i)_a$

$= O(\varphi_a)$ for $i = 1, \ldots, m$ by (3.2).

and

(3.3) $\|u_i - v_i\|_a = \|u_i - Z_i + Z_i - v_i\|_a$

$\leq \|u_i - Z_i\|_a + \|Z_i - v_i\|_a$

$= O(\varphi_a) + O(\varphi_a) = O(\varphi_a)$ for $i = 1, \ldots, m$. 

Let $u$ be an element of the set $\{u_1\}$ and $v$ the corresponding element in $\{v_1\}$. Then, by Theorem 2 and (3.3)

$$
\mu - \lambda = O(\varphi_a) \quad \text{and} \quad \|u - v\|_a = O(\varphi_a).
$$

Hence

$$
\|\mu v - \lambda u\|_a \leq \|v - u\|_a + (\mu - \lambda) \|u\|_a = O(\varphi_a).
$$

Define

$$
w_a(x) = \left( \int_{D_a} \left| x-y \right|^{-2p} |\mu v(y) - \lambda u(y)|^2 \, dy \right)^{1/2}
$$

$$
w_a^2(x) = \sum_{D_a - d_\delta} \int_{d_\delta} \left| x-y \right|^{-2p} |\mu v(y) - \lambda u(y)|^2 \, dy
$$

$$
+ \int_{d_\delta} \left| x-y \right|^{-2p} |\mu v(y) - \lambda u(y)|^2 \, dy
$$

$$
\leq \delta^{-2p} \|\mu v - \lambda u\|_a^2 + O(\delta^{-2p} + (n-1)+1)
$$

where $d_\delta$ is the $n$-disk with centre $x$ and radius $\delta$. If we choose $\delta = \varphi_a^{2/n}$ we obtain the uniform estimate $w_a(x) = O(\varphi_a^q)$, where $0 < q = (n-2p)/n < 4/n$. In particular, $w_a(x) = O(\varphi_a)$ if $n=2$ or 3.

It is asserted that $\lambda R_a u(x)$ gives a uniform estimate for $v(x)$ since

$$
|v(x) - \lambda R_a u(x)| = |R_a(\mu v(x) - \lambda u(x))|
$$
\[
\int_{D_a} R_a(x,y) |x-y|^{-p+1} \left( \mu v(y) - \lambda u(y) \right) dy \\
= \int_{D_a} \left( R_a(x,y) |x-y|^p, (\mu v(y) - \lambda u(y)) |x-y|^{-p} \right)_a \\
\leq \| R_a(x,y) |x-y|^p \|_a \cdot \| (\mu v(y) - \lambda u(y)) |x-y|^{-p} \|_a \\
= \left( \int_{D_a} R_a(x,y) |x-y|^{2p} dy \right)^{1/2} \cdot \left( \int_{D_a} |\mu v(y) - \lambda u(y)|^2 |x-y|^{-2p} dy \right)^{1/2} \\
= k_a(x) \cdot w_a(x) = O(\varphi^q_a) k_a(x)
\]

and

\[(3.4) \quad |v(x) - \lambda R_a u(x)| = O(\varphi^q_a) k_a(x). \]

Define the function

\[(3.5) \quad \psi(x) = \lambda R_a u(x) - u(x) + f(x). \]

Then, \( \psi(x) \) is the solution of the Robin problem

\[(3.6) \quad \begin{aligned}
L\psi &= 0 \quad \text{in } D_a \\
U\psi &= 0 \quad \text{on } B_a
\end{aligned} \]

For, \( L\psi = L(\lambda R_a u(x) - u(x) + f(x)) \)
\[(\lambda R_a u(x) - Lu(x) + Lf(x)) \]
\[= \lambda u(x) - \lambda u(x) = 0 \quad \text{in } D_a, \]
and \( U\psi = U(\lambda R_a u(x)) - Uu(x) + Uf(x) \)
\[= 0 \quad -Uu(x) + Uu(x) = 0 \quad \text{on } B_a. \]
By ([5], p. 97), the operator $L$ is positive definite on the domain $D_a$ which is dense in $H_a=L^2(D_a)$. That is, $(Lv,v)_a \geq \delta^2 \|v\|^2_a$ for every $v \in D_a$ and for some positive real number $\delta$. Now $\psi(x) \in D_a$ since $\psi$ is a solution of (3.5), hence

$$(L\psi,\psi)_a = 0 = \delta^2 \|\psi\|^2_a$$

and $\psi = 0$ in $D_a$.

Therefore, combining (3.4) and (3.5) we obtain

$$v(x) = u(x) - f(x) + O(\varphi_a^q)k_a(x) \quad a \geq a_1, \ x \in D_a.$$  

Since $u(x)$ was an arbitrary element of the set $\{u_i\}$ the theorem is proved.

I.4 Asymptotic Formulae for The Perturbed Eigenvalues.

Let $u$ and $v$ be as described in Theorem 3. The following asymptotic estimate will be based on Green's symmetric identity ([5], p. 76),

$$(Lu,v)_a - (u,Lv)_a = \sum_{i,j=1}^{n} a_{ij} D_j \tilde{v} \cos(v,x_i) - \tilde{v} \sum_{i,j=1}^{n} a_{ij} D_j u \cos(v,x_i) dS.$$  

In view of the boundary condition (1) and the condition $\sigma_1(x) + \sigma_2(x) > c_2 > 0$ for some real number $c_2$, we can put

$$\sum_{i,j=1}^{n} a_{ij} D_j \tilde{v} \cos(v,x_i) = -(\sigma_1/\sigma_2) \tilde{v} \text{ on } B_1$$

where $B_1$ is the set of all points of $B_a$ on which
\( \sigma_1(x) > c_2/2 \), and

\[ \tilde{v} = -\left(\sigma_1/\sigma_2\right) \sum_{j=1}^{n} a_{ij} D_j \tilde{v} \cos(\nu, x_1) \quad \text{on } B_2 \]

where \( B_2 \) is the set of all points of \( B_a \) on which \( \sigma_2(x) > c_2/2 \). Clearly \( B_a = B_1 \cup B_2 \) because of the above inequality and we can write Green's identity in the form

\[
(4.1) \quad (Lu, v)_a\! - (u, Lv)_a
\]

\[
= \int_{B_1} (1/\sigma_2) \sum_{j} a_{ij} D_j \tilde{v} \cos(\nu, x_1) Uu \, dS - \int_{B_1} (\tilde{v}/\sigma_1) Uu \, dS = [uv]_a
\]

Since \( u \) and \( v \) are as in Theorem 3,

\[
(4.2) \quad [u, v]_a = \lambda(u, v)_a - \mu(u, v)_a = (\lambda - \mu)(u, v)_a
\]

From (3.3) and application of the Schwarz inequality we obtain

\[
|u, v]_a - (v, v]_a| = |u, v]_a - (u, v]_a| \leq \|u-v\|_a \cdot \|v\|_a
\]

\[
= \|u-v\|_a = O(\varphi_a)
\]

and \( (u, v]_a = 1+O(\varphi_a) \). Then, using (4.2),

\[
(4.3) \quad \lambda - \mu = \frac{(u, v]_a}{(u, v]_a} = \frac{(u, v]_a}{(1+O(\varphi_a))}
\]

\[
= [uv]_a \left[1+O(\varphi_a)\right].
\]

Let \( f \) be a solution of the Robin problem (1.5) with \( h = Uu \). Application of (4.1) to the differential equations \( Lf = 0 \), \( Lv = \mu u \) and \( Lf = 0 \), \( Lu = \lambda u \) yields, respectively,

\[
(4.4) \quad (Lf, v)_a - (f, Lv)_a = -\mu(f, v)_a = [f, v]_a = [uv]_a
\]
and

\[(4.5) \quad (L_f, u)_a - (f, Lu)_a = -\lambda(f, u)_a = [fu]_a\]

It follows from (4.3) and (4.4) and the fact that \(u = \lambda + O(\varphi_a)\), that

\[\lambda - \mu = \lambda(f, v)_a [1 + O(\varphi_a)]\]

In addition, application of (3.1)' and (4.5) gives

\[\lambda - \mu = -\lambda(f, u - f + O(\varphi_a^q)k_a)_a [1 + O(\varphi_a)]\]

\[= (-\lambda(f, u)_a + \lambda(f, f)_a - \lambda O(\varphi_a^q)(f, k_a)_a) \cdot [1 + O(\varphi_a)]\]

\[= ([fu]_a + \lambda(f, f)_a) \cdot [1 + O(\varphi_a)] + O(\varphi_a^q)(f, k_a)_a\]

In some cases the first term dominates the others and we obtain the asymptotic formula

\[\mu(a) - \lambda \sim [fu]_a \quad \text{as} \quad a \to \infty.\]

The results of Theorem 1-3 are then sharpened accordingly.

I.5 A Typical Example.

Consider the elliptic operator in \(E^2\) defined by

\[Lu = -\Delta u + (x_1^2 + x_2^2 + 2)u\]

\(L\) is an operator of the Schrödinger type with the potential function \(V = (x_1^2 + y_2^2 + 2)\) and satisfies all the requirements stated in the introduction. The basic eigenvalue problem to be
considered is

\[ Lu = \lambda u \]  
\[ u \in L^2(\mathbb{R}^2) \]

and the basic spectrum is entirely discrete since the potential function \( V \) tends to infinity as \( (x_1^2 + x_2^2)^{1/2} \to \infty \), ([9], p. 150).

The perturbed eigenvalue problem is

\[ Lv = \mu v \quad \text{in} \quad D_a \]
\[ \frac{dv}{d\nu} + v = 0 \quad \text{on} \quad B_a \]

where \( D_a = \{(x_1, x_2) | (x_1^2 + x_2^2)^{1/2} < a, \ a > 0 \} \), and \( B_a = \{(x_1, x_2) | (x_1^2 + x_2^2)^{1/2} = a \} \). That is, we are considering the special case \( \sigma_1 = 1 \), \( \sigma_2 = 1 \) for (1).

In order to apply the results of this chapter it must be shown that there exists at least one eigenvalue \( \lambda \) of (5.1) whose corresponding eigenfunctions satisfy condition (1.2).

In fact, it will be shown for this example that every eigenvalue has multiplicity two and every eigenfunction satisfies (1.2).

Since (5.1) is separable we obtain, by the method of separation of variables, the orthonormal eigenfunctions

\[ u_{nm} = (\pi n ! m ! 2^{n+m})^{-1/2} \exp[-(x_1^2 + x_2^2)/2] H_n(x_1)H_m(x_2) \]
corresponding to the eigenvalues \( \lambda = 2(n+m)^{1/2} \), \((n,m=0,1,2,\ldots)\),
where \( H_n(x_1) \) denotes the Hermite polynomial of degree \( n \) in \( x_1 \)
\[
H_n(x_1) = (-1)^n \exp[x_1^2] d^n/dx_1^n (\exp[-x_1^2]) \quad ([1], p. 375).
\]

Then every eigenvalue has multiplicity two and
\[
\|u_{nm}\|_a \leq 1 \quad \text{for every } a \text{ and all } m \text{ and } n.
\]

The "L-measure" \( g \) defined by (1.3) is a solution of
\[
Lg = 0 \quad \text{in } D_a
\]
\[
dg/dv + g = 1 \quad \text{on } B_a
\]

After a routine transformation to polar coordinates \((r,\theta)\) and
separation of variables we obtain the unique solution
\[
g(r,\theta) = (a+1)^{-1} \exp[(r^2-a^2)/2]
\]
and it follows easily that
\[
\|g\|_a \leq \pi^{1/2} (a+1)^{-1} \quad \text{for every } a.
\]

Application of (5.3), (5.4) and (5.5) gives
\[
\max_{B_a} \left| \frac{du_{nm}}{dv} + u_{nm} \right| \|g\|_a/\|u_{nm}\|_a
\leq (2a)^{n+m} \exp[-a^2/2] (a+1)^{-1}
\]
\[= o(1) \quad \text{as } a \to \infty.
\]
and every basic eigenfunction satisfies condition (1.2). Hence, by Theorem 2, exactly two perturbed eigenvalues \( \mu \) of (5.2) converge to each eigenvalue of (5.1) as \( a \to \infty \).

In order to obtain the uniform estimates for the perturbed eigenfunctions, it is assumed that

\[
\phi_a k_a(x_1, x_2) = o(1) \quad \text{as} \quad a \to \infty,
\]

uniformly for all \( x \in \Omega_a \), where \( k_a(x) \) is defined in section I.3 and

\[
\phi_a = \sup_{u \in \Omega_a} \phi_a[u] \leq (2a)^{n+m} (a+1)^{-1} \exp[-a^2/2] (u+0)
\]

by (5.6) and Lemma 1. Let \( f_{nm} \) be the solution of

\[
L f_{nm} = 0 \quad \text{in} \quad \Omega_a
\]

\[
df_{nm}/dv + f_{nm} = du_{nm}/dv + u_{nm} \quad \text{on} \quad \Omega_a.
\]

Since \( f_{nm} \) attains its maximum on \( \Omega_a \) and at those points
\[
df_{nm}/dv > 0 \quad \text{, (} [1], \text{ p. 326) ,}
\]

\[
f_{nm} \leq \max_{\Omega_a} \left| du_{nm}/dv + u_{nm} \right| \quad \text{for all} \quad x \in \Omega_a \quad \text{and by (5.6)}
\]

\[
f_{nm} = O(\phi_a) [a+1] \quad \text{for all} \quad x \in \Omega_a.
\]

Let \( v_{nm} \) be the orthonormal eigenfunctions corresponding to
the perturbed eigenvalues $\mu_{nm}$ of (5.2). Then from Theorem 3 we have the uniform estimate

$$v_{nm}(x_1, x_2) = (\pi n! m! 2^{n+m})^{-1/2} \exp[-(x_1^2 + x_2^2)/2] H_n(x_1) H_m(x_2)$$

$$+ O(\varphi_a) [k_a(x_1, x_2) - (a+1)]$$

for all $x=(x_1, x_2) \in D_a$, for all $n$ and $m$ and for every $a$ for which

$$(2a)^{n+m} (n+m+3) (a+1)^{-1} \exp[-a^2/2] < 1.$$ 

Similarly, one can obtain sharper estimates for the perturbed eigenvalues using the results of section I.4.
CHAPTER II

THE PERTURBATION OF $H^n$ TO $S_a$

II.1 Introduction. Our purpose here is to obtain variational formulae for the eigenvalues and eigenfunctions of $L$ when $H^n$ is perturbed to $S_a$ and condition (1) is adjoined on the bounding surface, $\partial S_a$. $H^n$ is the upper half-space of $E^n$, $H^n = \{x|x=(x_1, \ldots, x_n) \in E^n, x_n > 0\}$. $S_a = \{x|x \in D_a \cap H^n\}$ and $\partial S_a$ can be expressed as the union of two disjoint sets, $A_a$ and $C_a$, where $A_a = \{x|x \in E^n, x_n = 0, |x| < a\}$ and $C_a = \{x|x \in B_a \cap H^n\}$. For example, in the plane $E^2$, $H^2$ is the upper half-plane and $S_a$ is the upper half of the disk centred at the origin and having radius $a$.

It may be noted that it is not necessary to restrict the domains to half-spaces and half-disks. It can easily be verified that the results obtained in this chapter would also apply to any solid $n$-cone, $C^n$ in $E^n$, perturbed to the solid spherical cone $C^n_a = C^n \cap D_a$.

The perturbed eigenvalue problem to be considered here is

$$Lw = \gamma w,$$

(6.1)

$w \in D_s$.
and the perturbed domain, $D_s$, is defined as the set of all complex valued functions $w$ which satisfy

(i) $w$ is twice continuously differentiable in $S_a$.
(ii) $w$ and $w'$ are continuous at those points of $\partial S_a$ at which $\sigma_1$ and $\sigma_2$ are continuous.
(iii) $w$ satisfies condition (1) on $\partial S_a$.

For $n>1$ in this domain perturbation it is not possible to characterize the condition on the basic eigenfunctions, $u$, in terms of the "L-measure" (1.3) since $Uu=0$ on the $(n-1)$ hyperplane $P$. That is, the "L-measure" would have to be the solution of $Lg=0$ in $S_a$, $Ug=1$ on $C_a$, $Ug=0$ on $A_a$ and hence would have to satisfy discontinuous boundary conditions. Therefore, it is assumed here that there exists at least one eigenvalue $\lambda$ of the basic problem (3) whose corresponding eigenfunctions satisfy

(6.2) $\| h \|_g / \| u \|_g = o(1)$ as $a \to \infty$

where $h$ is the solution of

\[
Lh = 0 \quad \text{in} \quad S_a \\
Uh = Uu \quad \text{on} \quad C_a \\
Uh = 0 \quad \text{on} \quad A_a
\]

As in Chapter I, it is known [5], [4] that for the perturbed problem (6.1) there exists a denumerable sequence of eigenvalues, $\gamma_1$, $0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \ldots$, and a complete ortho-
normal sequence of eigenfunctions \{w_i\} such that for some Robin function, \(R_s(x,y)\),

\[ w_1(y) = \gamma_1 R_s w_1(y) = \gamma_1 \int_{S_a} R_s(x,y) w_1(x) dx \]

and every basic eigenfunction satisfies \(LR_s u = u\) in \(S_a\). Here \(R_s\) denotes the integral operator whose kernel is \(R_s(x,y), y \in S_a\).

II.2 Enclosure Theorems and Representational Formulae.

The following notation will be used

\[ \phi_s[u] = \frac{\|h\|_s}{\|u\|_s} \quad (u \neq 0) \]

\[ \varphi_s = \sup_{u \in G_\kappa} \phi_s[u] \]

\[ \rho_s = \kappa \varphi_s / (1 - \varphi_s) \]

where \(G_\kappa\) is the eigenspace associated with the basic eigenvalue \(\kappa\) of (3). It is easily verified that \(\varphi_s = o(1)\) and \(\rho_s = o(1)\) as \(a \to \infty\) and that for \(\tau = 1/\kappa\)

\[ (7.1) \|R_s u - \tau u\|_S \leq \tau \varphi_s \|u\|_S \quad \text{for every } u \in G_\kappa. \]

Theorem 4. Let \(\kappa\) be the eigenvalue of multiplicity \(m\) of (3) whose corresponding eigenfunctions satisfy condition (6.2). Then there exists a positive number \(a_2\) such that at least \(m\) perturbed eigenvalues \(\gamma_1(a)\) of (6.1) are enclosed in the interval \([\kappa, \kappa + \rho_s]\) whenever \(a \geq a_2\) and converge to \(\kappa\) as \(a \to \infty\).
Proof. This follows directly from the proof of Theorem 1 in Chapter I and (7.1).

Theorem 5. Let $\kappa$ be as in Theorem 1. If there exists a basic eigenvalue exceeding $\kappa$, then there is a positive number $a > a_2$ such that exactly $m$ perturbed eigenvalues $\gamma_1$ are enclosed in the interval $[\kappa, \kappa + p]$ whenever $a > a_3$.

Proof. The proof follows almost without change from that of Theorem 2 in Chapter I.

As before, the fundamental singularity of $R_s(x,y)$ is of order $|x-y|^{2-n}$ for $n \geq 3$, the function

$$k_s(x) = \left( \int_{S_a} R_s^2(x,y) |x-y|^{2p} dy \right)^{1/2}$$

is well defined in $S_a$, and it is assumed that

$$\varphi_s^q k_s(x) = o(1) \quad \text{as} \quad a \to \infty \quad (q = (n - sp)/n)$$

uniformly for all $x \in S_a$.

Theorem 6. Let $u_i$ be the orthonormal eigenfunctions corresponding to the $m$-fold degenerate eigenvalue $\kappa$ of Theorem 2, and $w_i$ those corresponding to the $m$ perturbed eigenvalues $\gamma_i$, $i = 1, \ldots, m$. Then

$$w_i(x) = u_i(x) - h_i(x) + O(\varphi_s^q) k_s(x)$$

$$i = 1, \ldots, m \quad x \in S_a \quad a > a_3$$
where \( h_1(x) \) is the solution of

\[
\begin{align*}
Lh_1 &= 0 \quad \text{in } S_a \\
U_{h_1} &= U_{u_1} \quad \text{on } C_a \\
U_{h_1} &= 0 \quad \text{on } A_a
\end{align*}
\]

II.3 Asymptotic Formulae.

As in section I.4 of Chapter I, Green's symmetric identity has the form

\[
(Lu, w)_s - (u, Lw)_s = \int_{C_a \cup C_1} \left[ (1/\sigma_2) \sum_{i,j=1}^{n} a_{ij} D_j \bar{w} \cos(v, x_1) \right] U_u dS
\]

\[
- \int_{C_1} (\bar{w}/\sigma_1) U_u dS = \{uw\}_s
\]

for \( u \) and \( w \) as in Theorem 3. Here \( C_1 \) is the set of all points of \( C_a \) on which \( \sigma_1(x) > \sigma_2/2 \). The asymptotic formula obtained is

\[
\gamma(a) - \kappa \sim \{hu\}_s \quad \text{as } a \to \infty.
\]
REFERENCES


