ENCLOSURE THEOREMS FOR EIGENVALUES OF ELLIPTIC OPERATORS

by

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF ARTS

in the Department

of

MATHEMATICS

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

August, 1966

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ABSTRACT

Enclosure theorems for the eigenvalues and representational formulae for the eigenfunctions of a linear, elliptic, second order partial differential operator will be established for specific domain perturbations to which the classical theory cannot be applied. In particular, the perturbation of n-dimensional Euclidean space E^n to an n-disk D_a of radius a is considered in Chapter I and the perturbation of the upper half-space H^n of E^n to the upper half of D_a, S_a, is discussed in Chapter II. In each case a general self-adjoint boundary condition is adjoined on the bounding surface of the perturbed domain.

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ACKNOWLEDGEMENTS

This paper was developed to explore special applications of original work by Dr. C. A. Swanson in the field of functional analysis. His guidance and direction are gratefully acknowledged.

The cooperation of the University of British Columbia and the Department of Mathematics in making available funds and facilities is greatly appreciated.

INTRODUCTION

Let L be the linear, elliptic, self-adjoint partial differential operator defined by

$$Lu = - \sum_{i,j=1}^{n} D_i(a_{ij}\, D_j u) + bu$$

where D_i denotes partial differentiation with respect to the variable x_i. The assumptions to be made on the coefficients are as follows:

(i) the coefficients a_{ij} and b are continuous real-valued functions of $x = (x_1 , \ldots\ldots , x_n)$ in n-dimensional Euclidean space E^n and $b(x) > 0$ for all $x \in E^n$.

(ii) $a_{ij} = a_{ji}$ for every i and j, and the a_{ij} possess uniformly continuous first partial derivatives in E^n.

(iii) $\sum_{i,j=1}^{n} a_{ij}\, \xi_i \bar{\xi}_j \geq c_1 \sum_{i=1}^{n} |\xi_i|^2$ for some positive real number c_1 and for every $\xi \in E^n$.

Our purpose is to establish variational formulae for the eigenvalues and eigenfunctions of L for two specific domain perturbations. These are: the perturbation of E^n to an n-disk D_a of radius a, considered in Chapter I, and the perturbation of the upper half-space H^n of E^n to the upper half of D_a, S_a, discussed in Chapter II. The boundary condition adjoined on the bounding surface of the perturbed domain is

(1) $Uv = 0$

where U is the linear boundary operator defined by

$$Uv = \sigma_1 \sum_{i,j=1}^{n} a_{ij} D_j v \cos(\nu, x_i) + \sigma_2 v$$

and ν denotes the outer normal to the bounding surface. It is assumed that the functions $\sigma_1(x)$ and $\sigma_2(x)$ are defined on the bounding surface, are piecewise continuous and nonnegative , and that the sum $\sigma_1(x) + \sigma_2(x)$ has a positive lower bound. It may be noted that additional, compatibility conditions for the special case $\sigma_2 \equiv 0$ are not needed here since the coefficient $b(x)$ of L is required to be positive, ([5], p. 95) .

A solution u of $Lu = 0$ is assumed to be of class C^1 and all derivatives involved in L are supposed to exist, be continuous, and satisfy $Lu = 0$ at every point.

Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_a$ and $\mathcal{H}_s$ denote the Hilbert spaces which are the Lebesgue spaces with respective unner products

$$(u,v)_1 = \int_{E^n} u(x) \bar{v}(x) dx$$

$$(u,v)_2 = \int_{H^n} u(x) \bar{v}(x) dx$$

$$(u,v)_a = \int_{D_a} u(x) \bar{v}(x) dx$$

$$(u,v)_s = \int_{S_a} u(x) \bar{v}(x) dx$$

and respective norms $\|u\|_1$, $\|u\|_2$, $\|u\|_a$ and $\|u\|_s$. For $x \in E^n$, $|x|$ denotes the usual Euclidean norm.

The eigenvalue problem for L on E^n

$$(2) \quad \begin{aligned} &Lu = \lambda u \\ &u \in \mathcal{H}_1 \end{aligned}$$

will be called the basic problem in Chapter I. The eigenvalue problem for L on H^n

$$(3) \quad \begin{aligned} &Lu = \varkappa u \\ &u \in \mathcal{H}_2 \\ &Uu = 0 \quad \text{on the } (n-1) \text{ hyperplane } P = \{x \mid x \in E^n , x_n = 0\} \end{aligned}$$

will be called the basic problem in Chapter II. The corresponding perturbed eigenvalue problems will be defined in Chapters I and II.

In general it is not true that the eigenvalues of the perturbed problems tend to limits as $a \to \infty$, even when the spectrum of the basic problem is entirely discrete, for example, in the case $n = 1$ and $\sigma_1 \equiv 0$ in (1) , when the singularity at ∞ is of the limit circle type in Weyl's classification [9]. The only assumption requred in order to obtain the enclosure theorems and representational formulae is that there exists at least one eigenvalve of the basic problem whose corresponding eigenfunctions satisfy some limit property. For example, Theorem 1 in Chapter I shows that if the eigenfunctions corresponding to the basic eigenvalue λ of multiplicity m satisfy condition (1.2), then at least m eigenvalues of the perturbed problem (1.1) converge to λ as the radius, a, of the n-disk D_a tends to infinity.

The principle difficulty in this estimation problem is in establishing a reasonable condition on the basic eigenfunctions in terms of a simple solution, g, of $Lg = 0$ so that the norm of the function $f = R_a u - \alpha u$, constructed in Lemma 2, remains small even though $f(x)$ may become large for large x. For example, in Chapter I this condition is characterized in terms of the "L - measure" (1.3) which is independent of the basic eigenfunctions.

The method employed for the treatment of this estimation problem involving the boundary condition (1) follows almost directly from that used by C. A. Swanson [7] for the special case $\sigma_1 \equiv 0$.

This problem of estimating eigenvalues and eigenfunctions for large domains has its physical origin in certain models of enclosed quantum mechanical systems [2], [3] and [6]. In the case that the Schrödinger equation is separable, (a special case of L), the problem reduces to a domain perturbation problem for a singular second order ordinary differential operator. In particular, the example considered in section I.5 reduces to two ordinary differential equations each of which have a singularity of the limit point type at infinity.

CHAPTER I

THE PERTURBATION OF E^n TO AN n-DISK

I.1 Introduction. Our purpose in this chapter is to obtain variational formulae for the eigenvalues and corresponding eigenfunctions of the operator L when E^n is perturbed to an n-disk, D_a, of large radius, a, and condition (1) is adjoined on the bounding (n-1) hypersphere, B_a.

The perturbed eigenvalue problem to be considered is

$$(1.1) \quad \begin{matrix} Lv = \mu v \\ v \in \mathcal{D}_a \end{matrix} \quad \text{in } D_a = \{x \mid |x| < a,\ a > 0\}$$

where the perturbed domain, $\mathcal{D}_a$, is defined as the set of all complex valued functions v with the following properties

(i) v is twice continuously differentiable in D_a.

(ii) v and v' are continuous at those points of the boundary B_a, at which σ_1 and σ_2 are continuous.

(iii) v satisfies (1) on $B_a = \{x \mid |x| = a\}$.

The only assumption to be made here is that there exists at least one eigenvalue λ of the basic problem (2) whose corresponding eigenfunctions, u, satisfy

(1.2) $\{\max_{B_a} |Uu|\} \, \|g\|_a / \|u\|_a = o(1)$ as $a \to \infty$, where g is the "L measure"

$$(1.3) \quad \begin{aligned} Lg &= 0 \quad \text{in } D_a \\ Ug &= 1 \quad \text{on } B_a \end{aligned}$$

It is known [5], [4] that for the perturbed problem (1.1) there exists a denumerable sequence of eigenvalues μ_i, $0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots\ldots$, and a complete orthonormal sequence of eigenfunctions $\{v_i\}$ such that for some Robin function, $R_a(x,y)$, (Green's function of the third kind),

$$v_i(y) = \mu_i \; R_a \; v_i(y) = \mu_i \int_{D_a} R_a(x,y) \; v_i(x)dx$$

and any basic eigenfunction, u, satisfies $L\, R_a u = u$ in D_a. Here R_a denotes the integral operator whose kernel is $R_a(x,y)$, $y \in D_a$. It may be noted that for $\sigma_2 \equiv 0$, the kernel function $R_a(x,y)$ is replaced by a Neümann function, $N_a(x,y)$, and for $\sigma_1 \equiv 0$ by a Green's function, $G_a(x,y)$. Clearly, the results of this chapter apply to these special cases even though the representations of the solutions of (1.5) may be slightly different from (1.6). $R_a(x,y)$ is constructed in the usual way as the sum of a fixed fundamental solution, $\gamma(x,y)$, and the solution of a particular Robin problem, $r(x,y)$. That is,

$$R_a(x,y) = \gamma(x,y) + r(x,y)$$

where:

(i) $\gamma(x,y)$, regarded as a function of x, is a regular solution of $Lv = 0$ except when $x = y$, where it has a singularity of order $|x-y|^{2-n}$ for $n \geq 3$.

(ii) $r(x,y)$, regarded as a function of x , is a solution of

$$Lr = 0 \text{ in } D_a$$
$$Ur = -U\gamma \text{ on } B_a$$

and $R_a(x,y)$ satisfies the boundary condition

$$UR_a(x,y) = 0 \text{ on } B_a . \tag{1.4}$$

$R_a(x,y)$ is unique, symmetric and non-negative for all x and y in D_a , ([4] , p. 161) .

In addition, any solution of the Robin problem,

$$\begin{aligned} Lf &= 0 \text{ in } D_a \\ Uf &= h \text{ on } B_a \end{aligned} \tag{1.5}$$

has the representation

$$f(y) = \int_{B_a} R_a(x,y)h(x)dS_x \quad \text{for every } y \in D_a \tag{1.6}$$

I.2 Enclosure Theorems for The Perturbed Eigenvalues . The following notation will be used

$$\varphi_a[u] = \{\max_{B_a}|Uu|\} \, \|g\|_a \,/\, \|u\|_a \qquad (u \neq 0)$$

$$\varphi_a = \sup_{u \in G_\lambda} \varphi_a[u]$$

$\rho_a = \lambda\varphi_a/(1-\varphi_a)$

where G_λ is the eigenspace associated with the basic eigenvalue λ of (2) .

Lemma 1. $\varphi_a = o(1)$ and $\rho_a = o(1)$ as $a \to \infty$.

Proof. Let u be that function in G_λ such that $\varphi_a[u] = \varphi_a$. Since every $u \in G_\lambda$ has the representation $u = \sum_{i=1}^{m} \alpha_i u_i$ in terms of an orthonormal basis $\{u_i\}$, where the basic eigenvalue λ is of multiplicity m ,

$$\begin{aligned} \varphi_a[u] &= \{\max_{B_a} |\sum_{i-1}^{m} \alpha_i \, Uu_i|\} \, \|g\|_a / \|\sum_{i=1}^{m} \alpha_i u_i\|_a \qquad (u \neq 0) \\ &\leq \sum_{i=1}^{m} (|\alpha_i| \,/\, \|\sum_{i=1}^{m} \alpha_i u_i\|) \, \{\max_{B_a} |Uu_i|\} \, \|g\|_a \\ &\leq \sum_{i=1}^{m} \|u_i\|_a^{-1} \{\max_{B_a} |Uu_i|\} \, \|g\|_a \\ &\leq m \max_{1 \leq i \leq m} \varphi_a[u_i] \end{aligned}$$

and the proof follows from condition (1.2) and the fact that the u_i , $i=1$, , m , are the basic eigenfunctions corresponding to λ .

Lemma 2. For $\alpha = 1/\lambda$

(2.1) $\|R_a u - \alpha u\|_a \leq \alpha\varphi_a \|u\|_a$ for every $u \in G_\lambda$

Proof. The eigenvalues of (2) are positive. In fact, if $\lambda \leq 0$ for some λ , the maximum principle ([1], p. 326) implies that the eigenfunction u corresponding to λ approaches its maximum as $|x| \to \infty$. This contradicts $u \in L^2(E^n)$ and $u \neq 0$.

Let $\alpha = 1 / \lambda$ and define the function $f = R_a u - \alpha u$. Then for every $u \in G_\lambda$, f is a solution of the Robin problem

$$Lf = 0 \quad \text{in} \quad D_a$$

$$Uf = -\alpha Uu \quad \text{on} \quad B_a$$

For, $Lf = L(R_a u - \alpha u) = LR_a u - \alpha Lu$

$$= u - (1 / \lambda) \lambda u = 0 \quad \text{in} \quad D_a$$

and $Uf = U(R_a u - \alpha u) = UR_a u - \alpha Uu$

$$= \sigma_1(x) \sum_{i,j=1}^{n} a_{ij}(x) \, D_j \int_{D_a} R_a(x,y)u(y)dy \quad \cos(\nu, x_i)$$

$$+ \sigma_2(x) \int_{D_a} R_a(x,y)u(y)dy - \alpha Uu$$

Since the order of the singularity of the kernel function, $R_a(x,y)$, is less than $|x-y|^{1-n}$ and since $u(x)$ is continuous, the integral

$$\int_{D_a} D_j(R_a(x,y))u(y)dy$$

converges uniformly and the derivative

$$D_j \int_{D_a} R_a(x,y)u(y)dy = \int_{D_a} D_j(R_a(x,y))u(y)dy$$

exists and is continuous, ([5], p. 56) . Hence

$$Uf = \int_{D_a} (\sigma_1(x) \sum_{i,j=1}^{n} a_{ij}(x) D_j R_a(x,y) \cos(\nu, x_i) + \sigma_2(x) R_a(x,y))u(y)dy - \alpha Uu$$

$$= \int_{D_a} (UR_a(x,y))u(y)dy - \alpha Uu$$

$$= -\alpha Uu \quad \text{on } B_a \text{ , by condition (1.4) .}$$

That is, f is a solution of the Robin problem (1.5) with $h = -\alpha Uu$ and hence has the representation (1.6) ,

$$f(y) = -\alpha \int_{B_a} R_a(x,y)UudS_x \quad \text{for every } y \in D_a \text{ .}$$

Then

$$|f(y)| \leq \alpha \{\max_{B_a}|Uu|\} \int_{B_a} R_a(x,y)dS_x$$

since $R_a(x,y)$ is non-negative and α is positive, and

$$|f(y)| \leq \alpha \{\max_{B_a}|Uu|\} g(y) \quad \text{for every } y \in D_a$$ since g is the solution of (1.5) with $h = 1$.

Therefore

$$\|f\|_a = \|R_a u - \alpha u\|_a \leq \alpha \{\max_{B_a} |Uu|\} (\|g\|_a / \|u\|_a) \|u\|_a = \alpha\varphi_a[u] \|u\|_a$$

$$\leq \alpha\varphi_a \|u\|_a$$

for every $u \in G_\lambda$.

Theorem 1. Let λ be an m-fold degnerate eigenvalue of (2) whose corresponding eigenfunctions satisfy condition (1.2) . Then there exists a positive number a_o such that at least m perturbed eigenvalues $\mu_i(a)$ of (1.1) are enclosed in the interval $[\lambda,\lambda+\rho_a]$ whenever $a\geq a_o$ and converge to λ as $a \to \infty$.

Proof. Since $\rho_a = o(1)$ as $a \to \infty$ by Lemma 1, there exists an a_o such that $\rho_a < 1$ for every $a\geq a_o$ and ρ_a is well defined for large a.

Let $\mathfrak{F}_{a\epsilon}$ be the subspace of $\mathcal{H}_a$ generated by all the eigenfunctions of R_a whose eigenvalues $\beta_i = 1/\mu_i$ lie in the interval $|\beta-\alpha|<\epsilon$. Let $P(\epsilon)$ be the projection of $\mathcal{H}_a$ onto $\mathfrak{F}_{a\epsilon}$. Then

$\|u-P(\epsilon)u\|_a \leq \epsilon^{-1}\|R_a u-\alpha u\|_a$ for every $u\epsilon G_\lambda$ by ([8], p. 33), since the integral operator R_a is a self-adjoint linear transformation on $\mathcal{H}_a$.

From (2.1)

(2.2) $\|u-P(\epsilon)u\|_a \leq \alpha\varphi_a\epsilon^{-1}\|u\|_a$ for every $u\epsilon G_\lambda$. Thus, by ([8], p. 35), since R_a is completely continuous on $\mathcal{H}_a$, there are at least m eigenvalues β_i contained in the interval $|\beta_i-\alpha|\leq\alpha\varphi_a$, $i=1,2,\ldots\ldots$, or, more precisely, the interval $|\mu_i-\lambda|\leq\mu_i\varphi_a$. Since $D_a\subset E^n$, it follows by the minimax principle for eigenvalues [1] that $\mu_i\geq\lambda$ for every i and

$$\lambda \leq \mu_i \leq \lambda + \mu_i\varphi_a$$

or

$$\lambda \leq \mu_i \leq \lambda / (1 - \varphi_a) = \lambda + \lambda\varphi_a / (1-\varphi_a) = \lambda + \rho_a \,, a \geq a_o$$

for every $i = 1,2, \ldots$.

Then at least m eigenvalues μ_i of (1.1) are in $[\lambda,\lambda+\rho_a]$ whenever $a \geq a_o$ and, since $\rho_a = o(1)$ as $a \to \infty$, converge to λ as $a \to \infty$.

Theorem 2. Let λ be as in Theorem 1. If there exists a basic eigenvalue exceeding λ, then there is a positive number $a_1 \geq a_o$ such that exactly m perturbed eigenvalues μ_i are enclosed in the interval $[\lambda,\lambda+\rho_a]$ whenever $a \geq a_1$.

Proof. Let λ be the smallest basic eigenvalue and λ' the smallest eigenvalue exceeding λ. Then since $\varphi_a = o(1)$ as $a \to \infty$, there exists a number $a_1 \geq a_o$ such that $\varphi_a < (\lambda'-\lambda)/\lambda'$ for every $a \geq a_1$. This implies $\lambda+\rho_a < \lambda'$, $a \geq a_1$, since

$$\begin{aligned} \lambda + \rho_a &= \lambda + \lambda\varphi_a/(1-\varphi_a) = \lambda/(1-\varphi_a) \\ &< \lambda/(1-[(\lambda'-\lambda)/\lambda']) = \lambda' \end{aligned}$$

and by Theorem 1 at least m perturbed eigenvalues, μ_i, are enclosed in the subinterval $[\lambda,\lambda+\rho_a] \subset [\lambda,\lambda']$.

Since $\mu_i \geq \lambda_i$ for every i, by the minimax property, and λ is m-fold degenerate by hypothesis:

$$\lambda = \lambda_1 = \cdots\cdots = \lambda_m$$
$$\mu_{m+1} \geq \lambda' = \lambda_{m+1}$$

and

$$\mu_{m+1} \notin [\lambda,\ \lambda + \rho_a],\ a \geq a_1 .$$

Hence, at most m perturbed eigenvalues, μ_i, are in $[\lambda,\lambda+\rho_a]$ for $a\geq a_1$. Therefore exactly m are in $[\lambda,\lambda+\rho_a]$ whenever $a\geq a_1$. An easy induction proof establishes the same result if $\lambda=\lambda^i$ is the i^{th} distinct eigenvalue $\lambda^1<\lambda^2<\lambda^3<\ldots$.

I.3 Uniform Estimates for The Perturbed Eigenfunctions. Let $p=p(n)$ be a positive number satisfying $p(2)=0$, $p(3)=0$, and $0<n-2p<4$. Because the fundamental singularity of $R_a(x,y)$ is of order $|x-y|^{2-n}$ for $n\geq 3$, the function

$$k_a(x) = \left(\int_{D_a} |x-y|^{2p} R_a^2(x,y)\, dy \right)^{1/2}$$

is well defined in D_a. It is assumed for the next theorem that

$$\varphi_a^q k_a(x) = o(1) \quad \text{as} \quad a\to\infty \qquad (q = (n-2p)/n)$$

uniformly for all $x\in D_a$.

Theorem 3. Let u_i be the orthonormal eigenfunctions corresponding to the m-fold degenerate eigenvalue λ of Theorem 2, and v_i those corresponding to the m perturbed eigenvalues μ_i, $i=1\ldots, m$. Then

(3.1) $v_i(x)=u_i(x)-f_i(x)+O(\varphi_a^q)k_a(x) \qquad i=1,\ldots, m, \quad x\in D_a, \quad a\geq a_1$

where f_i is the solution of (1.5) with $h_i=Uu_i$.

Proof. Let $\epsilon=\alpha-\alpha'$ in (2.2), where $\alpha=1/\lambda, \alpha'=1/\lambda'$. It follows from Theorem 2 that $\alpha\varphi_a<\alpha(\lambda'-\lambda)/\lambda'$, $a\geq a_1$. That is,

$$\alpha\,\varphi_a < \alpha\,(\lambda'-\lambda)/\lambda' = \alpha - \alpha' = \epsilon \qquad a\geq a_1 .$$

Then $\mathfrak{F}_{a\epsilon}$ is m-dimensional by Theorem 2 and $\|u-P(\epsilon)u\|_a<\|u\|_a$ implies that $u=0$ if $P(\epsilon)u=0$, $u\in G_\lambda$. Therefore, m uniquely determined linearly independent eigenfunctions z_i corresponding to α are mapped by $P(\epsilon)$ into the orthonormal functions v_i . By (2.2) $\|z_i-P(\epsilon)\ z_i\|_a = \|z_i-v_i\|_a \leq \alpha\varphi_a\|z_i\|_a$ and

$$\|z_i-v_i\| = O(\varphi_a) \ .$$

Since, by the Schwarz inequality

$$|(z_i,z_j)_a - (v_i,v_j)_a| \leq \|z_i\|_a\|z_j - v_j\|_a + \|v_j\|_a\|z_i - v_i\|_a$$
$$= O(\varphi_a) + O(\varphi_a) = O(\varphi_a)$$

Let $\{u_i\}$ be the orthonormal sequence constructed by the Schmidt orthonormalization process as linear combinations of the z_i . Then $u_i=\sum_{j=1}^{m} \gamma_{ij}z_j$ for some $\gamma_{ij}\in\mathbb{C}$, and

$$\|u_i - z_i\|_a^2 = \|\sum_{j-1}^{m} \gamma_{ij}\ z_j - z_i\|_a^2$$

$$= (\sum_{j=1}^{m} \gamma_{ij}\ z_j - z_i\ , \sum_{j=1}^{m} \gamma_{ij}\ z_j - z_i)_a$$

$$= O(\varphi_a) \quad i = 1, \ldots , m \text{ by } (3.2) \ .$$

and

$$(3.3) \qquad \|u_i - v_i\|_a = \|u_i - z_i + z_i - v_i\|_a$$
$$\leq \|u_i - z_i\|_a + \|z_i - v_i\|_a$$
$$= O(\varphi_a) + O(\varphi_a) = O(\varphi_a) \qquad i = 1, \ldots , m \ .$$

Let u be an element of the set $\{u_i\}$ and v the corresponding element in $\{v_i\}$. Then, by Theorem 2 and (3.3)

$$\mu - \lambda = O(\varphi_a) \qquad \text{and} \qquad \|u - v\|_a = O(\varphi_a) .$$

Hence

$$\|\mu v - \lambda u\|_a \leq \mu\|v - u\|_a + (\mu - \lambda)\ \|u\|_a = O(\varphi_a) .$$

Define

$$w_a(x) = \left(\int_{D_a} |x-y|^{-2p} |\mu v(y) - \lambda u(y)|^2 dy \right)^{1/2}$$

$$w_a^2(x) = \int_{D_a - d_\delta} |x-y|^{-2p} |\mu v(y) - \lambda u(y)|^2 dy$$

$$+ \int_{d_\delta} |x-y|^{-2p} |\mu v(y) - \lambda u(y)|^2 dy$$

$$\leq \delta^{-2p} \|\mu v - \lambda u\|_a^2 + O(\delta^{-2p+(n-1)+1})$$

where d_δ is the n-disk with centre x and radius δ. If we choose $\delta = \varphi_a^{2/n}$ we obtain the uniform estimate $w_a(x) = O(\varphi_a^{q})$, where $0<q=(n-2p)/n<4/n$. In particular, $w_a(x) = O(\varphi_a)$ if $n=2$ or 3.

It is asserted that $\lambda R_a u(x)$ gives a uniform estimate for $v(x)$ since

$$|v(x) - \lambda R_a u(x)| = |R_a(\mu v(x) - \lambda u(x))|$$

$$= \left| \int_{D_a} R_a(x,y) \; |x-y|^{-p+p} \Big(\mu v(y) - \lambda u(y)\Big) \, dy \right|$$

$$= \left| \Big(R_a(x,y) \; |x-y|^p \; , \; (\mu v(y) - \lambda u(y)\Big) \; |x-y|^{-p} \Big)_a \right|$$

$$\leq \| R_a(x,y) \; |x-y|^p \|_a \cdot \| \Big(\mu v(y) - \lambda u(y)\Big) \; |x-y|^{-p} \|_a$$

$$= \left(\int_{D_a} R_a^2(x,y) \; |x-y|^{2p} dy \right)^{1/2} \cdot \left(\int_{D_a} |\mu v(y) - \lambda u(y)|^2 |x-y|^{-2p} dy \right)^{1/2}$$

$$= k_a(x) \cdot w_a(x) = O(\varphi_a^q) \; k_a(x)$$

and

(3.4) $$|v(x) - \lambda \, R_a u(x)| = O(\varphi_a^q) \; k_a(x) \; .$$

Define the function

(3.5) $$\psi(x) = \lambda \, R_a u(x) - u(x) + f(x) .$$

Then, $\psi(x)$ is the solution of the Robin problem

(3.6) $$L\psi = 0 \quad \text{in } D_a$$
$$U\psi = 0 \quad \text{on } B_a$$

For, $$\begin{aligned} L\psi &= L(\lambda R_a u(x) - u(x) + f(x)) \\ &= \lambda L R_a u(x) - Lu(x) + Lf(x) \\ &= \lambda u(x) - \lambda u(x) = 0 \quad \text{in } D_a \, , \end{aligned}$$

and $$\begin{aligned} U\psi &= U(\lambda R_a u(x)) - Uu(x) + Uf(x) \\ &= 0 \; - Uu(x) + Uu(x) = 0 \quad \text{on } B_a \; . \end{aligned}$$

By ([5], p. 97) , the operator L is positive definite on the domain $\mathcal{D}_a$ which is dense in $\mathcal{H}_a = L^2(D_a)$. That is, $(Lv,v)_a \geq \delta^2 \|v\|_a^2$ for every $v \in \mathcal{D}_a$ and for some positive real number δ. Now $\psi(x) \in \mathcal{D}_a$ since ψ is a solution of (3.6) , hence

$$(L\psi,\psi)_a = 0 = \delta^2\|\psi\|_a^2$$

and $\psi \equiv 0$ in D_a .

Therefore, combining (3.4) and (3.5) we obtain

$$v(x) = u(x) - f(x) + O(\varphi_a^q)k_a(x) \quad a \geq a_1 \text{ , } x \in D_a \text{ .}$$

Since $u(x)$ was an arbitrary element of the set $\{u_i\}$ the theorem is proved.

I.4 Asymptotic Formulae for The Perturbed Eigenvalues.

Let u and v be as described in Theorem 3. The following asymptotic estimate will be based on Green's symmetric identity ([5], p. 76) ,

$$(Lu,v)_a-(u,Lv)_a=\int_{B_a} u\sum_{i,j=1}^{n} a_{ij}D_j \bar{v} \cos(\nu,x_i)-\bar{v} \sum_{i,j=1}^{n} a_{ij}D_j u \cos(\nu,x_i)dS.$$

In view of the boundary condition (1) and the condition $\sigma_1(x)+\sigma_2(x)>c_2>0$ for some real number c_2 , we can put

$$\sum_{i,j=1}^{n} a_{ij}D_j\bar{v} \cos(\nu,x_i) = -(\sigma_1/\sigma_2)\bar{v} \quad \text{on } B_1$$

where B_1 is the set of all points of B_a on which

$\sigma_1(x) > c_2/2$, and

$$\bar{v} = -(\sigma_1/\sigma_2) \sum_{i,j=1}^{n} a_{ij} D_j \bar{v} \cos(\nu, x_i) \quad \text{on } B_2$$

where B_2 is the set of all points of B_a on which $\sigma_2(x) > c_2/2$. Clearly $B_a = B_1 \cup B_2$ because of the above inequality and we can write Green's identity in the form

(4.1) $(Lu,v)_a - (u,Lv)_a$

$$= \int_{B_a \sim B_1} (1/\sigma_2) \sum_{i,j=1}^{n} a_{ij} D_j \bar{v} \cos(\nu, x_i)) Uu \, dS - \int_{B_1} (\bar{v}/\sigma_1) Uu \, dS = \{uv\}_a$$

Since u and v are as in Theorem 3,

(4.2) $\{u,v\}_a = \lambda(u,v)_a - \mu(u,v)_a = (\lambda-\mu)\ (u,v)_a$.

From (3.3) and application of the Schwarz inequality we obtain

$$|(u,v)_a - (v,v)_a| = |(u-v,v)_a| \le \|u-v\|_a \cdot \|v\|_a$$
$$= \|u-v\|_a = O(\varphi_a)$$

and $(u,v)_a = 1 + O(\varphi_a)$. Then, using (4.2)

(4.3) $$\lambda - \mu = \{u,v\}_a \,/\, (u,v)_a = \{u,v\}_a \,/\, (1+O(\varphi_a))$$
$$= \{uv\}_a \, [1+O(\varphi_a)] \, .$$

Let f be a solution of the Robin problem (1.5) with $h = Uu$. Application of (4.1) to the differential equations $Lf = 0$, $Lv = \mu v$ and $Lf = 0$, $Lu = \lambda u$ yields, respectively,

(4.4) $(Lf,v)_a - (f,Lv)_a = -\mu(f,v)_a = \{f,v\}_a = \{uv\}_a$

and

$$(4.5) \quad (Lf,u)_a - (f,Lu)_a = - \lambda(f,u)_a = \{fu\}_a$$

It follows from (4.3) and (4.4) and the fact that $\mu=\lambda+O(\varphi_a)$, that

$$\lambda - \mu = \lambda(f,v)_a \, [1+O(\varphi_a)] \, .$$

In addition, application of (3.1) and (4.5) gives

$$\begin{aligned} \lambda - \mu &= - \lambda(f, u-f + O(\varphi_a^q)k_a)_a \, [1+O(\varphi_a)] \\ &= (-\lambda(f,u)_a + \lambda(f,f)_a - \lambda O(\varphi_a^q) \, (f,k_a)_a) \cdot [1+O(\varphi_a)] \\ &= (\{fu\}_a + \lambda(f,f)_a) \cdot [1+O(\varphi_a)] + O(\varphi_a^q) \, (f,k_a)_a \end{aligned}$$

In some cases the first term dominates the others and we obtain the asymptotic formula

$$\mu(a) - \lambda \sim \{fu\}_a \quad \text{as} \quad a \to \infty \, .$$

The results of Theorem 1-3 are then sharpened accordingly.

I.5 A Typical Example.

Consider the elliptic operator in E^2 defined by

$$Lu = - \Delta u + (x_1^2+x_2^2+2)u \, .$$

L is an operator of the Schrödinger type with the potential function $V = (x^2+y^2+2)$ and satisfies all the requirements stated in the introduction. The basic eigenvalue problem to be considered is

$$(5.1)\quad \begin{aligned} &Lu = \lambda u \\ &u \in L^2(E^2) \end{aligned}$$

and the basic spectrum is entirely discrete since the potential function V tends to infinity as $(x_1^2+x_2^2)^{1/2} \to \infty$, ([9], p. 150). The perturbed eigenvalue problem is

$$(5.2)\quad \begin{aligned} &Lv = \mu v \quad \text{in } D_a \\ &dv/d\nu + v = 0 \quad \text{on } B_a \end{aligned}$$

where $D_a = \{(x_1,x_2) | \ (x_1^2 + x_2^2)^{1/2} < a \ , \ a>0\}$, and $B_a=\{(x_1,x_2) | \ (x_1^2+x_2^2)^{1/2}=a\}$. That is, we are considering the special case $\sigma_1 \equiv 1$, $\sigma_2 \equiv 1$ for (1) .

In order to apply the results of this chapter it must be shown that there exists at least one eigenvalue λ of (5.1) whose corresponding eigenfunctions satisfy condition (1.2) . In fact, it will be shown for this example that every eigenvalue has multiplicity two and every eigenfunction satisfies (1.2) .

Since (5.1) is separable we obtain, by the method of separation of variables, the orthonormal eigenfunctions

$$(5.3)\quad u_{nm} = (\pi n \ ! \ m \ ! \ 2^{n+m})^{-1/2} \exp[-(x_1^2+x_2^2)/2] \ H_n(x_1) H_m(x_2)$$

corresponding to the eigenvalues $\lambda=2(n+m)+4$, $(n,m=0,1,2,\ldots)$, where $H_n(x_1)$ denotes the Hermite polynomial of degree n in x_1

$$H_n(x_1) = (-1)^n \exp[x_1^2]\, d^n/dx_1^n\, (\exp[-x_1^2]) \quad ([1], \text{p. } 375).$$

Then every eigenvalue has multiplicity two and

(5.4) $\|u_{nm}\|_a \leq 1$ for every a and all m and n .

The "L-measure" g defined by (1.3) is a solution of

$$Lg = 0 \quad \text{in } D_a$$

$$dg/d\nu + g = 1 \quad \text{on } B_a$$

After a routine transformation to polar coordinates (r,θ) and separation of variables we obtain the unique solution

$$g(r,\theta) = (a+1)^{-1} \exp[(r^2-a^2)/2]$$

and it follows easily that

(5.5) $\|g\|_a \leq \pi^{1/2}(a+1)^{-1}$ for every a .

Application of (5.3), (5.4) and (5.5) gives

(5.6) $$\{\max_{B_a} |\, du_{nm} / d\nu + u_{nm} \,|\} \|g\|_a/\|u_{nm}\|_a$$

$$\leq (2a)^{n+m} \exp[-a^2/2]\, (a+1)^{-1}$$

$$= o(1) \quad \text{as} \quad a \to \infty$$

and every basic eigenfunction satisfies condition (1.2) . Hence, by Theorem 2, exactly two perturbed eigenvalues μ of (5.2) converge to each eigenvalue of (5.1) as $a \to \infty$.

In order to obtain the uniform estimates for the perturbed eigenfunctions, it is assumed that

$$\varphi_a \, k_a(x_1, x_2) = o(1) \qquad \text{as} \quad a \to \infty ,$$

uniformly for all $x \in D_a$, where $k_a(x)$ is defined in section I.3 and

$$\varphi_a = \sup_{u \in G_\lambda} \varphi_a [u] \leq (2a)^{n+m} (a+1)^{-1} \exp[-a^2/2] \qquad (u \neq 0)$$

by (5.6) and Lemma 1 . Let f_{nm} be the solution of

$$Lf_{nm} = 0 \quad \text{in } D_a$$

$$df_{nm}/d\nu + f_{nm} = du_{nm}/d\nu + u_{nm} \quad \text{on } B_a \, .$$

Since f_{nm} attains its maximum on B_a and at those points $df_{nm}/d\nu > 0$, ([1], p. 326) ,

$$f_{nm} \leq \max_{B_a} | \, du_{nm}/d\nu + u_{nm} \, | \qquad \text{for all } x \in D_a \text{ and by (5.6)}$$

$$f_{nm} = O(\varphi_a) \; [a+1] \qquad \text{for all } x \in D_a .$$

Let v_{nm} be the orthonormal eigenfunctions corresponding to the perturbed eigenvalues μ_{nm} of (5.2) . Then from Theorem 3 we have the uniform estimate

$$v_{nm}(x_1,x_2) = (\pi\, n!\, m!\, 2^{n+m})^{-1/2} \exp[-(x_1^2+x_2^2)/2]\, H_n(x_1)H_m(x_2) + O(\varphi_a)\,[k_a(x_1,x_2) - (a+1)]$$

for all $x=(x_1,x_2) \in D_a$, for all n and m and for every a for which

$$(2a)^{n+m}\,(n+m+3)\,(a+1)^{-1} \exp[-a^2/2] \le 1 .$$

Similarly, one can obtain sharper estimates for the perturbed eigenvalues using the results of section I.4 .

CHAPTER II

THE PERTURBATION OF H^n TO S_a

II.1 Introduction. Our purpose here is to obtain variational formulae for the eigenvalues and eigenfunctions of L when H^n is perturbed to S_a and condition (1) is adjoined on the bounding surface, ∂S_a . H^n is the upper half-space of E^n , $H^n = \{x | x=(x_1, \ ,x_n) \in E^n, x_n>0\}$. $S_a=\{x | x\in D_a \cap H^n\}$ and ∂S_a can be expressed as the union of two disjoint sets, A_a and C_a, where $A_a=\{x | x\in E^n, x_n=0 , |x| \leq a\}$ and $C_a=\{x | x\in B_a \cap H^n\}$. For example, in the plane E^2 , H^2 is the upper half-plane and S_a is the upper half of the disk centred at the origin and having radius a .

It may be noted that it is not necessary to restrict the domains to half-spaces and half-disks. It can easily be verified that the results obtained in this chapter would also apply to any solid n-cone, C^n in E^n , perturbed to the solid spherical cone $C_a^n=C^n \cap D_a$.

The perturbed eigenvalue problem to be considered here is

$$(6.1)\qquad \begin{aligned} &Lw = \gamma w \\ &w\in \mathcal{D}_s \end{aligned}$$

and the perturbed domain, $\mathcal{D}_s$, is defined as the set of all complex valued functions w which satisfy

(i) w is twice continuously differentiable in S_a .

(ii) w and w' are continuous at those points of ∂S_a at which σ_1 and σ_2 are continuous.

(iii) w satisfies condition (1) on ∂S_a .

For n>1 in this domain perturbation it is not possible to characterize the condition on the basic eigenfunctions, u, in terms of the "L-measure" (1.3) since Uu=0 on the (n-1) hyperplane P . That is, the "L-measure" would have to be the solution of Lg=0 in S_a , Ug=1 on C_a , Ug=0 on A_a and hence would have to satisfy discontinuous boundary conditions. Therefore, it is assumed here that there exists at least one eigenvalue $\varkappa$ of the basic problem (3) whose corresponding eigenfunctions satisfy

(6.2) $\quad \| h \|_s / \| u \|_s = o(1) \quad \text{as} \quad a \to \infty$

where h is the solution of

$$(6.3) \qquad \begin{aligned} Lh &= 0 \quad \text{in } S_a \\ Uh &= Uu \quad \text{on } C_a \\ Uh &= 0 \quad \text{on } A_a \end{aligned}$$

As in Chapter I, it is known [5], [4] that for the perturbed problem (6.1) there exists a denumerable sequence of eigenvalues, γ_i , $0<\gamma_1\leq\gamma_2\leq\gamma_3\leq\cdots\cdots$, and a complete orthonormal sequence of eigenfunctions $\{w_i\}$ such that for some Robin function, $R_s(x,y)$,

$$w_i(y) = \gamma_i R_s w_i(y) = \gamma_i \int_{S_a} R_s(x,y)\, w_i(x)dx$$

and every basic eigenfunction satisfies $LR_s u=u$ in S_a . Here R_s denotes the integral operator whose kernel is $R_s(x,y)$, $y\epsilon S_a$.

II.2 Enclosure Theorems and Representational Formulae.

The following notation will be used

$$\varphi_s[u] = \|h\|_s \,/\, \|u\|_s \qquad (u\neq 0)$$

$$\varphi_s = \sup_{u\epsilon G_\varkappa} \varphi_s[u]$$

$$\rho_s = \varkappa\varphi_s \,/\, (1-\varphi_s)$$

where $G_\varkappa$ is the eigenspace associated with the basic eigenvalue $\varkappa$ of (3) . It is easily verified that $\varphi_s=o(1)$ and $\rho_s=o(1)$ as $a\to\infty$ and that for $\tau=1/\varkappa$

(7.1) $\|R_s u - \tau u\|_s \leq \tau\, \varphi_s\, \|u\|_s$ for every $u\epsilon G_\varkappa$.

Theorem 4. Let $\varkappa$ be the eigenvalue of multiplicity m of (3) whose corresponding eigenfunctions satisfy condition (6.2) . Then there exists a positive number a_2 such that at least m perturbed eigenvalues $\gamma_i(a)$ of (6.1) are enclosed in the interval $[\varkappa,\varkappa+\rho_s]$ whenever $a\geq a_2$ and converge to $\varkappa$ as $a\to\infty$.

Proof. This follows directly from the proof of Theorem 1 in Chapter I and (7.1) .

Theorem 5. Let $\varkappa$ be as in Theorem 1 . If there exists a basic eigenvalue exceeding $\varkappa$, then there is a positive number $a_3 \geq a_2$ such that exactly m perturbed eigenvalues γ_i are enclosed in the interval $[\varkappa, \varkappa+\rho_s]$ whenever $a \geq a_3$.

Proof. The proof follows almost without change from that of Theorem 2 in Chapter I .

As before, the fundamental singularity of $R_s(x,y)$ is of order $|x-y|^{2-n}$ for $n \geq 3$, the function

$$k_s(x) = \left(\int_{S_a} R_s^2(x,y) \; |x-y|^{2p} dy \right)^{1/2}$$

is well defined in S_a , and it is assumed that

$$\varphi_s^q \, k_s(x) = o(1) \qquad \text{as} \quad a \to \infty \quad (q=(n-sp/n))$$

uniformly for all $x \in S_a$.

Theorem 6. Let u_i be the orthonormal eigenfunctions corresponding to the m-fold degenerate eigenvalue $\varkappa$ of Theorem 2, and w_i those corresponding to the m perturbed eigenvalues γ_i , i=1 , ..., m . Then

$$w_i(x) = u_i(x) - h_i(x) + O(\varphi_s^q) k_s(x)$$

$$i = 1, \ldots, m \quad , \; x \in S_a \quad , \; a \geq a_3$$

where $h_i(x)$ is the solution of

$$\begin{aligned} Lh_i &= 0 && \text{in } S_a \\ Uh_i &= Uu_i && \text{on } C_a \\ Uh_i &= 0 && \text{on } A_a \end{aligned}$$

II.3 Asymptotic Formulae.

As in section I.4 of Chapter I , Green's symmetric identity has the form

$$(Lu,w)_s - (u,Lw)_s = \int_{C_a \sim C_1} [(1/\sigma_2) \sum_{i,j=1}^{n} a_{ij} D_j \bar{w} \cos(\nu,x_i)]Uu \, dS$$

$$- \int_{C_1} (\bar{w}/\sigma_1)Uu \, dS = \{uw\}_s$$

for u and w as in Theorem 3 . Here C_1 is the set of all points of C_a on which $\sigma_1(x)>C_2/2$. The asymptotic formula obtained is

$$\gamma(a) - \varkappa \sim \{hu\}_s \qquad \text{as} \quad a \to \infty .$$

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